Newtonian Mechanics

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Preface

Chapter 1

Newton's Laws of Motion

1.1 Introduction

Mechanics is the study of how things move; from the Earth orbiting the Sun, to a ball rolling down a hill, to an electron in a cathode ray tube. The modern formulation of classical mechanics was first developed in the 17^{th} and 18^{th} centuries by European natural philosophers like Galileo and Newton, who based their ideas on previous theories like those from ancient Greece. It was then reformulated in the 19^{th} century by French mathematicians such as Lagrange, Poisson and Liouville, as well as Hamilton, whose developments paved the way for the innovations of the 20^{th} century, when it was realised that classical mechanics cannot accurately describe objects travelling close to the speed of light or objects that are extremely small (on an atomic scale). These discoveries led to the development of relativistic mechanics and quantum mechanics, for which the advanced Lagrangian and Hamiltonian formalisms serve as a mathematical basis.

In this text we will be studying the Newtonian formulation of classical mechanics, which is still used today not only as a teaching method, but also as a tool in research. The Lagrangian and Hamiltonian formalisms come into their own for advanced problems, but can be quite unwieldy for the simple systems that we will be describing the motion of. One may wonder why we still study classical mechanics if it has been proven to be obsolete in some areas. The answer is that there are still many real-life systems which are best described using a classical description. It is also a great opportunity to become familiar with the language of vector calculus while studying examples which are relevant to real life. Without further ado, let us dive in to how we describe the world in mechanics.

1.2 Motion in One Dimension

No doubt you will have noticed that the world is three dimensional. However it is certainly beneficial to study motion in a simplified setting before extending our ideas to the full 3D application. In Newtonian mechanics, we label each point in space with a **con-**

tinuous variable x, and we define an origin where $x = 0$. This defines our **coordinate system**. We can describe the motion of an object by writing its position x as a function of a continuous parameter t, which describes the passage of time from a reference point $t = 0$.

Definition 1.1. The **trajectory** of an object is a function $x(t)$ where $t, x(t) \in \mathbb{R}$. $x(t_0) = x_0$ describes the position of the object x_0 at some time t_0.

In classical mechanics, time evolves as the same rate for the whole universe. Our choice of coordinate system together with our choice of reference point for time is called a **reference frame**. By choosing our reference frame cleverly, we can often simplify problems. For example, if we were studying a block sliding down a slope, we could simplify the problem by rotating our coordinate system so that the x axis lies parallel to the slope. In general, it is often best to align the x axis with the direction of motion.

Definition 1.2. The **displacement** of an object is the difference in positions between two times t_1 and $t_2 > t_1$. In one dimension:

$$\Delta x = x(t_2) - x(t_1) = x_2 - x_1. \tag{1.1}$$

If $x(0) = x_0$, then the **total displacement** of an object as a function of time is given by

$$s(t) = x(t) - x_0. \tag{1.2}$$

The **distance** travelled by an object between t_1 and t_2 is given by the magnitude of displacement,

$$\text{distance} = |\Delta x| = |x(t_2) - x(t_1)| = |x_2 - x_1|. \tag{1.3}$$

In general, distance $\neq$ displacement. This is because displacement is a **vector** quantity, meaning it has direction and magnitude, whereas distance is a **scalar** quantity. In one dimension, the only difference between vector and scalar quantities is that vector quantities are **signed**. The distinction becomes greater in more dimensions when vector quantities are actually represented as vectors.

Definition 1.3. The **instantaneous velocity**, or just the **velocity** of an object is defined as the rate of change of the object's position with respect to time, or simply the derivative.

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}. \tag{1.4}$$

Velocity is also a vector quantity. The corresponding scalar quantity is **speed**, defined as the magnitude of velocity,

$$\text{speed} = |v(t)|. \tag{1.5}$$

The **average velocity** of an object in a time interval $\Delta t = t_2 - t_1$ is given by the displacement over the time interval, or also the time average of the velocity (which must be computed with an integral since it is a continuous property).

$$\bar{v} = \frac{\text{total displacement}}{\text{total time}} = \frac{\Delta x}{\Delta t} = \frac{1}{\Delta t}\int_{t_1}^{t_2} v(t)\mathrm{d}t. \tag{1.6}$$

Meanwhile, the **average speed** is given by

$$\overline{\text{speed}} = \frac{\text{total distance}}{\text{total time}} = \frac{1}{\Delta t}\int_{t_1}^{t_2} |v(t)|\mathrm{d}t. \tag{1.7}$$

Here, we may identify

$$\Delta x = \int_{t_1}^{t_2} v(t)\mathrm{d}t, \quad \text{total distance} = \int_{t_1}^{t_2} |v(t)|\mathrm{d}t. \tag{1.8}$$

If the object is travelling at constant velocity, then the integral is trivial and we recover (setting $t_1 = 0$) the equation that you probably learned in school,

$$\text{distance} = \text{speed} \times \text{time}. \tag{1.9}$$

Definition 1.4. The **instantaneous acceleration**, or just the **acceleration** of an object is defined as the rate of change of the object's velocity with respect to time.

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}. \tag{1.10}$$

Likewise, the **average acceleration** of an object in a time interval Δt is

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{1}{\Delta t}\int_{t_1}^{t_2} a(t)\mathrm{d}t. \tag{1.11}$$

1.3 Constant Acceleration

When the acceleration of an object is constant, we can derive some useful equations for simple motion. Starting from equation 1.11 above,

$$\begin{aligned} \Delta v &= \int_{t_1}^{t_2} a\mathrm{d}t \\ \implies v_2 - v_1 &= a \cdot (t_2 - t_1), \end{aligned}$$

now setting $t_1 = 0, t_2 = t, v_1 = v(0) = u, v_2 = v(t)$, we get

$$v(t) = u + at, \tag{1.12}$$

where u is the initial velocity of the object. Now we substitute equation 1.12 into equation 1.8 to get

$$\Delta x = \int_{t_1}^{t_2} (u + at)\mathrm{d}t$$
$$\implies x_2 - x_1 = u \cdot (t_2 - t_1) + \frac{1}{2}a \cdot (t_2^2 - t_1^2).$$

Setting $t_1 = 0, t_2 = t, x_1 = x(0) = x_0, x_2 = x(t)$ like before and recalling $s(t) = x(t) - x_0$, we get

$$s(t) = ut + \frac{1}{2}at^2. \tag{1.13}$$

Finally, squaring equation 1.12 and substituting in equation 1.13 gives

$$v^2(t) = u^2 + 2as(t). \tag{1.14}$$

Equations 1.12, 1.13, and 1.14 are known as the SUVAT equations, you probably learned them in school. They are the **equations of motion** for a object under constant acceleration, i.e. all problems involving constant acceleration in a straight line are solved by them.

Example 1.1. Ball thrown in the air

Now let's do some examples where we do not have constant acceleration.

Example 1.2. Car accelerating then decelerating

$$v(t) = -\frac{1}{2}t^4 + 3t^3$$

Example 1.3. Block sliding down a hill (no friction)

1.4 Forces

Now we know how to describe the motion of an object and changes in the motion (kinematics), but we still don't know how to describe *why* the motion of an object changes (dynamics). This is the focus of this section.

A force is some influence on an object that changes the object's motion. In classical mechanics, we describe forces as vectors and denote a generic force with the symbol F. The exact dynamics of how forces affect motion are described in Newton's laws of motion, which we will write now. In the following the symbol F stands for the sum of all forces acting on the object, otherwise known as the **net force**.

Definition 1.5 (Newton's First Law)**.** An object moving with constant velocity v, will stay at the same constant velocity unless acted upon by a force. In other words:

$$v = \text{const.} \iff F = 0. \tag{1.15}$$

Note that this includes an object at rest, which has velocity $v = 0$.

Mass is defined as an objects resistance to acceleration. In order words, a more massive object will accelerate slower relative to a less massive object when under the influence of identical forces. This is quantified by Newton's Second Law.

Definition 1.6 (Newton's Second Law)**.** The net force on the object is equal to the object's mass times the object's acceleration.

$$F = ma. \tag{1.16}$$

Note that force is always parallel to acceleration.

Since acceleration is the second derivative of position, we can write Newton's Second Law as

$$F(t) = m\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}, \tag{1.17}$$

which is a differential equation for $x(t)$. This is known as an **equation of motion**, and all classical mechanics problems boil down to solving the equation of motion to obtain the trajectory of the object.

Example 1.4. Suppose an object is acted upon by a constant force F_0, then we align the x axis with the direction of the force and the equation of motion is

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = \frac{F_0}{m}.$$

This is a very easy differential equation to solve. Integrating twice, we get

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \int \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = v_0 + \frac{F_0}{m}t \\ x(t) &= \int \frac{\mathrm{d}x(t)}{\mathrm{d}t} = x_0 + v_0 t + \frac{F_0}{2m}t^2, \end{aligned}$$

where $v_0 = v(0)$ and $x_0 = x(0)$ as before. Note that comparing to equation 1.13, we can identify $a = \frac{F_0}{m}$ i.e. acceleration is constant, which is consistent with what we developed before.

It is important to note that Newton's Laws of Motion are only valid in **inertial reference frames**, which are reference frames travelling at a constant velocity v. If we are in a nonintertial reference frame, i.e. one that is accelerating, and we try to apply Newton's Laws, we will encounter odd things such as phantom forces which have no source. One way to test if we are in an inertial frame or not us by using Newton's First Law. If an object accelerates while under the influence of no forces, then our reference frame must be noninertial.

Definition 1.7 (Newton's Third Law)**.** When two objects interact with each other, the forces on each object due to the other are **equal in magnitude** and **opposite in direction**. In other words, if object A exerts a force $F_{A\to B}$ on object B, then object B exerts a force $F_{B\to A}$ on object A and we may write

$$F_{A\to B} = -F_{B\to A}. \tag{1.18}$$

These two forces are then known as a "Newton (III) pair".

We can now solve any problem in classical mechanics.

1.5 Motion in Two and Three Dimensions

Armed with the principles of Newtonian mechanics for one dimension, we will now generalise everything we have learned to two and three dimensions. It turns out that this will be very simple because the set of all points in 3D space forms a **vector space**, $\mathbb{R}^3$, so we can pick 3 **orthogonal** axes and an origin to use as our coordinate system and define three basis vectors to span our space. In **cartesian** coordinates, which are the most commonly used system to label points in 3D space, we label the axes x, y and z, and choose the unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ to point along each axis respectively.

Note that it is important that our basis vectors are **orthonormal**, meaning both orthogonal and of unit length, as this means we have the relations

$$\begin{aligned}
|\hat{\boldsymbol{\imath}}| = |\hat{\boldsymbol{\jmath}}| = |\hat{\boldsymbol{k}}| = 1 \\
\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1 \\
\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\imath}} = 0.
\end{aligned}$$

If these results were not true, then some of the mathematics we will do later on (involving dot and cross products) would become much more complicated than it needs to be.

Definition 1.8. The **trajectory** of an object in three dimensions is a vector-valued function $\vec{\boldsymbol{r}}(t)$ where $t \in \mathbb{R}$, $\vec{\boldsymbol{r}}(t) \in \mathbb{R}^3$. We write

$$\vec{\boldsymbol{r}}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}} \tag{1.19}$$

in cartesian coordinates, where $x(t)$, $y(t)$ and $z(t)$ are the 1D trajectories of the object along each axis. For example, if $\vec{\boldsymbol{r}}(t_0) = \vec{\boldsymbol{r}}_0 = x_0\hat{\boldsymbol{\imath}} + y_0\hat{\boldsymbol{\jmath}} + z_0\hat{\boldsymbol{k}}$, then the object is located at position x_0 along the x axis, y_0 along the y axis and z_0 along the z axis. $\vec{\boldsymbol{r}}(t)$ is known as the **position vector**.

At this point it is worth introducing a new notation which will simplify our expressions going forward. We will represent a time derivative of a quantity by simply writing a dot above the letter, and put two dots for a second derivative.

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \ddot{x} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2},$$

i.e. $\dot{x}$ represents velocity and $\ddot{x}$ represents acceleration. The notation is due to Newton and so it is fitting that we use it a lot in mechanics.

Definition 1.9. Just as in one dimension. The velocity is defined as the time derivative of position.

$$\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}} = \frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\boldsymbol{\jmath}} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\boldsymbol{k}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}}. \tag{1.20}$$

Sometimes we denote $\dot{x}$, $\dot{y}$ and $\dot{z}$ as v_x, v_y and v_z respectively.

Definition 1.10. Likewise, acceleration is the time derivative of velocity, or the second time derivative of position.

$$\vec{\boldsymbol{a}} = \ddot{\vec{\boldsymbol{r}}} = \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{\boldsymbol{r}}}{\mathrm{d}t^2} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}. \tag{1.21}$$

Sometimes $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$ are called a_x, a_y and a_z.

Note that although we have stopped notating time dependence explicitly for brevity, all quantities will still depend on time in general.

Other vector quantities that we have defined are generalised to 3D in the same way, for example, Newton's Second Law becomes

$$\begin{aligned}\vec{\boldsymbol{F}}(t) &= F_x(t)\hat{\boldsymbol{\imath}} + F_y(t)\hat{\boldsymbol{\jmath}} + F_z(t)\hat{\boldsymbol{k}} \\ &= ma_x(t)\hat{\boldsymbol{\imath}} + ma_y(t)\hat{\boldsymbol{\jmath}} + ma_z(t)\hat{\boldsymbol{k}} \\ &= m\vec{\boldsymbol{a}} = m\ddot{\vec{\boldsymbol{r}}}.\end{aligned}$$

The key insight is that motion in 3D Cartesian coordinates is simply a superposition of three one-dimensional motions.

1.6 Projectile Motion

Let's now look at a concrete example of an experiment which will bring together everything we have looked at in this chapter. Suppose we have a cannon situated at the origin which shoots a projectile with a fixed initial velocity $\vec{v}_0$ which makes an initial angle θ relatitive to the ground. Can we work out how far the projectile will fly and what its flight time is?

This is going to be a 2D problem as we have two axes of motion. We shall label the horizontal direction that the cannon is shooting along the x axis and the vertical direction the y axis. As stated in the problem, the cannon is located at the origin. We can now write the initial velocity as

$$\vec{v}_0 = v_0 \cos(\theta)\hat{\boldsymbol{\imath}} + v_0 \sin(\theta)\hat{\boldsymbol{\jmath}}, \tag{1.22}$$

where $v_0 = |\vec{v}_0|$. Ignoring air resistance, there are no forces acting on the projectile in the x direction. By Newton's second law this means that $a_x = 0$ and we can immediately write

$$x(t) = v_0 \cos(\theta)t, \tag{1.23}$$

using the SUVAT equation 1.13. In the y direction, the only force acting on the projectile is the constant force of gravity, so Newton's second law tells us

$$F_y = -mg = ma_y. \tag{1.24}$$

Hence, we have

$$y(t) = v_0 \sin(\theta)t - \frac{1}{2}gt^2, \tag{1.25}$$

again by equation 1.13.

Now, the time of flight t_f will be given when $y(t_f) = 0$. Solving for this, we get

$$y(t_f) = v_0 \sin(\theta)t_f - \frac{1}{2}gt_f^2 = 0 \tag{1.26}$$

$$v_0 \sin(\theta)t_f = \frac{1}{2}gt_f^2 \tag{1.27}$$

$$t_f = \frac{2v_0 \sin(\theta)}{g}. \tag{1.28}$$

Let's consider briefly if this answer makes physical sense. If the initial speed of the projectile v_0 was higher, then the flight time would be longer. Additionally, for a fixed initial speed, a projectile fired at a higher angle would have a longer flight time because more of the initial velocity was aimed along the vertical direction. On the other hand, if gravity was stronger, the projectile would fall to the ground faster and the flight time would be shorter.

Finally, the range of the projectile is given by

$$x_f = x(t_f) = v_0 \cos(\theta) t_f \tag{1.29}$$
$$= \frac{2v_0^2 \sin(2\theta)}{g}. \tag{1.30}$$

So the maximum range is given when $\theta = \frac{\pi}{4}$.

Example 1.5. A motorcyclist is doing a stunt jump between two buildings. If the buildings are separated by a distance d and have a vertical height difference h, what is the minimum velocity the motorcyclist needs to make the jump?

1.7 Gravitation

Aside from inventing classical mechanics and calculus, Isaac Newton's most well-known contribution to science is his discovery of the universal law of gravitation. If we consider two objects, one of which is located at the origin, then the gravitational force between the two bodies has magnitude

$$F = \frac{Gm_1m_2}{r^2}. \tag{1.31}$$

In vectorial form, these forces form a Newton III pair.

$$\vec{\boldsymbol{F}}_{1\to 2} = -\frac{Gm_1m_2}{|\vec{\boldsymbol{r}}_{1\to 2}|^3}\vec{\boldsymbol{r}}_{1\to 2} \tag{1.32}$$
$$\vec{\boldsymbol{F}}_{2\to 1} = -\frac{Gm_1m_2}{|\vec{\boldsymbol{r}}_{2\to 1}|^3}\vec{\boldsymbol{r}}_{2\to 1} = \frac{Gm_1m_2}{|\vec{\boldsymbol{r}}_{1\to 2}|^3}\vec{\boldsymbol{r}}_{1\to 2} = -\vec{\boldsymbol{F}}_{2\to 1}, \tag{1.33}$$

where $\vec{\boldsymbol{r}}_{1\to 2} = \vec{\boldsymbol{r}}_{2\to 1} = \vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1$.

Consider an object of mass m near the surface on the Earth. The radius of the Earth is very large so we can approximate the distance between the Earth and the object as simply the radius of the Earth. The gravitational force which the Earth exerts on the object is given by

$$F_G \approx \frac{GM_{\text{Earth}}}{R_{\text{Earth}}^2} m = mg, \tag{1.34}$$

where $g = \frac{GM_{\text{Earth}}}{R_{\text{Earth}}^2}$. Thus we have recovered the weight force that we have been using for the force due to gravity. If the height of the object is so large that the approximation no longer holds, then g depends on the height $R_{\text{Earth}} + h = r$. This is the same as the general case, the force only depends on the distance between the centre of the Earth and the object. Now consider the case where the object is *below* the surface of the Earth. In this case, the mass of the Earth depends on the distance between the centre and the object. The mass enclosed within a radius r is given by the density multiplied by the enclosed volume:

$$M(r) = \frac{4}{3}\rho\pi r^3. \tag{1.35}$$

If we assume that the density of the Earth is constant, then it is given by the total mass divided by total volume:

$$\rho = \frac{M_{\text{Earth}}}{\frac{4}{3}\pi R_{\text{Earth}}^3}. \tag{1.36}$$

Thus, $g(r)$ is given by

$$g(r) = \frac{GM(r)}{r^2} = \frac{G}{r^2}\frac{M_{\text{Earth}}}{R_{\text{Earth}}^3}r^3 \tag{1.37}$$

$$= \frac{GM_{\text{Earth}}}{R_{\text{Earth}}^3}r. \tag{1.38}$$

Example 1.6. Calculate g at the surface of Earth.

Chapter 2

Linear Momentum

2.1 Conservation of Momentum

Consider two objects interacting with each other via some forces. They could be two electrons repelling each other because of the electrostatic force or two planets falling together due to gravity. By Newton's third law,

$$\vec{F}_{A\to B}(t) = -\vec{F}_{B\to A}(t), \tag{2.1}$$

and so by Newton's second law we can write

$$m_A\vec{a}_A(t) + m_B\vec{a}_B(t) = 0. \tag{2.2}$$

We can integrate this equation over some arbitrary time period $t_1 < t_2$ to get

$$\int_{t_1}^{t_2} \left(m_A\vec{a}_A(t) + m_B\vec{a}_B(t)\right) \mathrm{d}t = m_A\left(\vec{v}_A(t_2) - \vec{v}_A(t_1)\right) + m_B\left(\vec{v}_B(t_2) - \vec{v}_B(t_1)\right) \tag{2.3}$$

$$= 0 \tag{2.4}$$

$$\implies m_A\vec{v}_A(t_1) + m_B\vec{v}_B(t_1) = m_A\vec{v}_A(t_2) + m_B\vec{v}_B(t_2). \tag{2.5}$$

Thus, we have discovered that Newton's third law implies that the quantity $m_A\vec{v}_A + m_B\vec{v}_B$ is **conserved**. This means it is constant for all time. We call this quantity the **linear momentum**.

2.2 Centre of Mass

Some times it is useful to consider the **centre of mass** of a system, which is defined as the *average* position of all the objects in the system. For the system of two objects, this is calculated as

$$\vec{r}_{\text{COM}} = \frac{m_A\vec{r}_A + m_B\vec{r}_B}{m_A + m_B}. \tag{2.6}$$

This defines a position vector which points to the centre of mass of the system. By differentiating this vector with respect to time, we can get the velocity of the centre of mass

$$\vec{v}_{\text{COM}} = \frac{\mathrm{d}\vec{r}_{\text{COM}}}{\mathrm{d}t} = \frac{m_A \frac{\mathrm{d}\vec{r}_A}{\mathrm{d}t} + m_B \frac{\mathrm{d}\vec{r}_B}{\mathrm{d}t}}{m_A + m_B} = \frac{m_A \vec{v}_A + m_B \vec{v}_B}{m_A + m_B}. \tag{2.7}$$

Hence the conserved quantity that we found before is the centre of mass momentum,

$$M\vec{v}_{\text{COM}} = m_A \vec{v}_A + m_B \vec{v}_B = \text{constant}. \tag{2.8}$$

From these definitions we can see that equation 2.2 is the accleration of the centre of mass multiplied by the total mass, which by Newton's second law is the force on the centre of mass. This implies that if the resultant force on the centre of mass is 0, then the centre of mass moves with constant velocity, just like a single object following Newton's first law, and the total linear momentum is conserved.

Example 2.1. For a system of two objects, show that the centre of mass is always located on the line that connects the two objects.

For a general system of N objects, we define the centre of mass position, velocity and acceleration as follows.

Definition 2.1. Consider a system of N objects. Write the total mass of the system as $M = \sum_{i=1}^{N} m_i$, then the **centre of mass** is is a vector function defined as

$$\vec{r}_{\text{COM}} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i. \tag{2.9}$$

The **centre of mass velocity** is defined as

$$\vec{v}_{\text{COM}} = \frac{\mathrm{d}\vec{r}_{\text{COM}}}{\mathrm{d}t} = \frac{1}{M} \sum_{i=1}^{N} m_i \frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{v}_i, \tag{2.10}$$

and the **centre of mass acceleration** is defined as

$$\vec{a}_{\text{COM}} = \frac{\mathrm{d}^2\vec{r}_{\text{COM}}}{\mathrm{d}t^2} = \frac{1}{M} \sum_{i=1}^{N} m_i \frac{\mathrm{d}^2\vec{r}_i}{\mathrm{d}t^2} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{a}_i, \tag{2.11}$$

If we consider a system of N objects, each under the influence of forces from every other object and also external forces, we can write the net force on each object as

$$\vec{F}_{i,\text{net}} = \vec{F}_{i,\text{ext}} + \sum_{j \neq i}^{N} \vec{F}_{j \to i}. \tag{2.12}$$

Let's now add all the forces together to get the net force on the centre of mass:

$$\vec{F_{\text{net}}} = \sum_{i=1}^{N} \vec{F}_{i,\text{net}} = \underbrace{\sum_{i=1}^{N} \vec{F}_{i,\text{ext}}}_{=\vec{F}_{\text{ext}}} + \underbrace{\sum_{i=1}^{N} \sum_{j \neq i}^{N} \vec{F}_{j \to i}}_{=0} . \tag{2.13}$$

The second term on the far right-hand side is 0 by Newton's third law (you can prove this by induction). Hence,

$$M \vec{a}_{\text{COM}} = \vec{F}_{\text{ext}}. \tag{2.14}$$

This result is actually quite profound because it is what allows us to treat systems of particles as point masses themselves while ignoring all of the internal forces between the particles since they all cancel out. Without this equation, we could not apply the laws of mechanics as we have been learning them to macroscopic bodies!

If $\vec{F}_{\text{ext}} = 0$, i.e. the system is isolated and there are no external forces, then the centre of mass moves in a straight line with a constant velocity and the total linear momentum is conserved.

2.3 Impulse

We have seen that an object has a linear momentum given by $\vec{p} = m\vec{v}$. How does the momentum change under the action of a force? Notice that

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} = m\vec{a} = \vec{F}, \tag{2.15}$$

so we have a new way to write Newton's second law. Let's now integrate this equation over an arbitrary time interval:

$$\int_{t_1}^{t_2} \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \mathrm{d}t = \vec{p}(t_2) - \vec{p}(t_1) \tag{2.16}$$

$$= \Delta\vec{p} = \int_{t_1}^{t_2} \vec{F}(t) \mathrm{d}t. \tag{2.17}$$

We call the integral of a force over a time interval the **impulse**.

Definition 2.2. The **impulse** of a force $\vec{F}$ over a time interval $t_1 \leq t_2$ is defined as

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}(t) \mathrm{d}t. \tag{2.18}$$

It has units of Newton second (N·s). As we have seen, the impulse is equal to the change in momentum.

$$\Delta\vec{p} = \vec{J}. \tag{2.19}$$

This result is sometimes called the **impulse-momentum theorem**.

If we had a constant force, then the impulse would be $\vec{J} = \vec{F}\Delta t$ ($\Delta t = t_2 - t_1$). In most problems we want to solve this will not be the case. However, we can define the **average force** such that

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}(t)\mathrm{d}t = \vec{F}_{\text{avg}}\Delta t. \tag{2.20}$$

This is quite useful because if we have a short interaction, we can simply consider the average force over the interval which is a good approximation.

For a general system of N objects, the total impulse on the system over a time interval is

$$\vec{J} = \int_{t_1}^{t_2} \sum_{i=1}^{N} \vec{F}_i(t)\mathrm{d}t = \int_{t_1}^{t_2} \vec{F}_{\text{ext}}\mathrm{d}t. \tag{2.21}$$

By equation 2.14 in the previous section,

$$\begin{aligned}
\vec{J} = \int_{t_1}^{t_2} \vec{F}_{\text{ext}}\mathrm{d}t &= \int_{t_1}^{t_2} M\vec{a}_{\text{COM}}\mathrm{d}t && (2.22)\\
&= M\vec{v}_{\text{COM}}(t_2) - M\vec{v}_{\text{COM}}(t_1) && (2.23)\\
&= \sum_{i=1}^{N} \left(m_i\vec{v}_i(t_2) - m_i\vec{v}_i(t_1)\right) && (2.24)\\
&= \sum_{i=1}^{N} \left(\vec{p}_i(t_2) - \vec{p}_i(t_1)\right) && (2.25)\\
&= \vec{P}(t_2) - \vec{P}(t_1) = \Delta\vec{P}, && (2.26)
\end{aligned}$$

where $\vec{P}$ denotes the total momentum of the system. So the impulse-momentum theorem still holds for composite systems.

Example 2.2. Consider a baseball of mass m =0.3kg being thrown at a speed of 15ms^{-1}. If the batter bats the ball at a speed of 25ms^{-1} and the bat is in contact with the ball for 0.005s, what is the impulse imparted to the ball? What is the average force exerted on the ball? What is the average acceleration of the ball?

2.4 Transforming Between Reference Frames

To transform between one frame of reference to another, we subtract the constant velocity between the frames from the position vector.

$$\vec{r}'(t) = \vec{r}(t) - vt. \tag{2.27}$$

We can then transform the velocity and acceleration as

$$\vec{v}'(t) = \frac{\mathrm{d}\vec{r}'(t)}{\mathrm{d}t} = \frac{\mathrm{d}\vec{r}(t)}{\mathrm{d}t} - v = \vec{v}(t) - v \tag{2.28}$$

$$\vec{a}'(t) = \frac{\mathrm{d}^2\vec{r}'(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\vec{r}(t)}{\mathrm{d}t^2} = \vec{a}(t). \tag{2.29}$$

One of the most useful reference frames to transform into is the centre of mass frame.

Chapter 3

Energy & Work

3.1 The Conservation of Energy

Consider the gravitational force $\vec{F} = -mg\hat{\boldsymbol{k}}$. Then

$$\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right)\mathrm{d}t = \frac{1}{2}m(v(t_2)^2 - v(t_1)^2) \tag{3.1}$$

$$= \int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\mathrm{d}t \tag{3.2}$$

$$= -\int_{t_1}^{t_2} mgv_z\mathrm{d}t \tag{3.3}$$

$$= -mg(z(t_2) - z(t_1)), \tag{3.4}$$

and hence,

$$\frac{1}{2}mv_1^2 + mgz_1 = \frac{1}{2}mv_2^2 + mgz_2. \tag{3.5}$$

So this quantity is **constant** over the path of the object (since t_1 and t_2 were arbitrary). If we write

$$K = \frac{1}{2}mv^2, \quad U_g = mgz, \tag{3.6}$$

then we have

$$E = K + U_g. \tag{3.7}$$

Example 3.1. Consider a pendulum on the end of a string. What is the maximum speed that the pendulum attains as it swings?

3.2 Work-Energy Theorem

Lets calculate the rate of change of velocity squared.

$$\frac{\mathrm{d}v^2}{\mathrm{d}t} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\cdot\vec{v} + \vec{v}\cdot\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \tag{3.8}$$

$$= 2\vec{v}\cdot\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \tag{3.9}$$

$$= 2\vec{v}\cdot\frac{\vec{F}}{m}, \tag{3.10}$$

where in the last line we have used Newton II. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right) = \vec{F}\cdot v. \tag{3.11}$$

We define the quantity in parentheses as the kinetic energy.

Definition 3.1. **Kinetic energy** is given by

$$K = \frac{1}{2}mv^2. \tag{3.12}$$

Let's see how the kinetic energy changes for a given force. We can find the change in kinetic energy by integrating:

$$\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right)\mathrm{d}t = \frac{1}{2}m(v(t_2)^2 - v(t_1)^2) \tag{3.13}$$

$$= \Delta K = \int_{t_1}^{t_2}\vec{F}\cdot\vec{v}\mathrm{d}t. \tag{3.14}$$

For a constant force, we can take $\vec{F}$ out of the integral.

$$\Delta K = \vec{F}\cdot\int_{t_1}^{t_2}\vec{v}\mathrm{d}t \tag{3.15}$$

$$= \vec{F}\cdot\vec{d}, \tag{3.16}$$

where $\vec{d} = \vec{r}_2 - \vec{r}_1$. We call this quantity $\vec{F}\cdot\vec{d}$ the **work**, and give it the symbol W.

This result that $\Delta K = W$ is known as the work-energy theorem and can be extended to a general force which changes with time. In this case, we writ ethe work as the integral over the infinitesimal work done over an infinitesimal part of the path, $\mathrm{d}W = \vec{F}\cdot\mathrm{d}\vec{r}$.

Theorem 3.1 (Work-Energy Theorem) *The net work on an object is equal to the* *change in its kinetic energy.*

$$W = \int_{\text{path}} \mathrm{d}W \tag{3.17}$$
$$= \int_{r_1}^{r_2} \vec{F} \cdot \mathrm{d}\vec{r} \tag{3.18}$$
$$= \int_{t_1}^{t_2} \vec{F} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\mathrm{d}t \tag{3.19}$$
$$= \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\mathrm{d}t \tag{3.20}$$
$$= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right)\mathrm{d}t = \Delta K. \tag{3.21}$$

The kinetic energy depends on the speed of the object, so if the net work is > 0 then the object must have sped up. Likewise, if the net work is < 0, the object has slowed down. If the net work is 0, the object must be at the same speed that it started at.

Example 3.2. Consider a chain of length L and total mass m hanging over the edge of a frictionless table.

Now let's look at a mass on a spring. The restoring force on the mass always acts opposite to displacement. Hooke's law says

$$F_s = -k(x - x_0), \tag{3.22}$$

where k is the spring constant, x_0 is the equilibrium position of the spring. Then the work done by the spring force is

$$W_s = \int_{x_0}^{x_f} F_s(x)\mathrm{d}x = -\int_{x_0}^{x_f} k(x - x_0)\mathrm{d}x \tag{3.23}$$
$$= -\frac{1}{2}k(x_f - x_0)^2. \tag{3.24}$$

Note that the work done is always negative no matter if the displacement is positive or negative because the force points in the opposite direction. Now we define the spring potential energy as

$$U_s = -W_s = \frac{1}{2}k(x_f - x_0)^2, \tag{3.25}$$

Then by the work-energy theorem we have

$$\Delta K = W_s = -\Delta U_s, \tag{3.26}$$

so

$$K + U_s = E \tag{3.27}$$

is constant.

3.3 Friction

Friction is a very complicated process which occurs on a miroscopic scale, so in order to model in on a macroscopic scale we must use simplified empirical laws. In general, friction is a force which opposes change in motion. Hence if a force is applied parallel to a surface, then the frictional force will be antiparallel to this, perpendicular to the normal force.

Satic friction appears when two objects are motionless with respect to one another. If a force is applied between the two objects and they don't move, there must be a frictional force opposing the motion.

$$\vec{f}_s = -\vec{F}_{app}. \tag{3.28}$$

As the applied force gets larger, the static friction must get larger to preserve equilibrium, until a maximum limit is reached and the object starts moving.

Definition 3.2. The maximum magnitude of **static friction** is given by

$$f_{s,max} = \mu_s |\vec{N}|. \tag{3.29}$$

Hence,

$$0 \leq |\vec{f}_s| \leq f_{s,max}. \tag{3.30}$$

Kinetic friction opposes the motion of two surfaces sliding against each other.

Definition 3.3. **Kinetic friction** is given by

$$\vec{f}_k = -\mu_k |\vec{N}| \hat{v}. \tag{3.31}$$

Example 3.3. Find the stopping distance of a block sliding down a slope.

3.4 Conservative Forces

We have seen that in some cases the total energy is conserved and in others it is not. Consider the definition of work:

$$W = \int_A^B \vec{F} \cdot \mathrm{d}\vec{r}. \tag{3.32}$$

This is a line integral, so in general the value of the integral depends on the path chosen for integration, which in this case corresponds to the path of the object through space. However, we have seen for the case of gravity and the spring force that the work done depends *only* on the initial and final positions; the integral is *path independent*. We also saw that in this case we can define a potential energy function for which

$$W = \int_A^B \vec{F} \cdot \mathrm{d}\vec{r} = U(A) - U(B), \tag{3.33}$$

where $F = -\frac{\mathrm{d}U}{\mathrm{d}\vec{r}}$. This is actually consequence of a generalisation of the fundamental theorem of calculus. We call forces which have this property **conservative forces**. They are called this because by the work-energy theorem:

$$W = U(A) - U(B) = -\Delta U = \Delta K, \tag{3.34}$$

and hence the total energy is conserved.

Example 3.4. Consider a force $\vec{F}$ then the work done by this force along an infinitesimal displacement $\mathrm{d}\vec{r} = \mathrm{d}x\hat{\boldsymbol{\imath}} + \mathrm{d}y\hat{\boldsymbol{\jmath}} + \mathrm{d}z\hat{\boldsymbol{k}}$ is

$$\mathrm{d}W = \vec{F} \cdot \mathrm{d}\vec{r} = F_x\mathrm{d}x + F_y\mathrm{d}y + F_z\mathrm{d}z. \tag{3.35}$$

If $\vec{F}$ is conservative, then $\mathrm{d}W = -\mathrm{d}U$. By expanding the full differential of U as

$$\mathrm{d}U = \frac{\partial U}{\partial x}\mathrm{d}x + \frac{\partial U}{\partial y}\mathrm{d}y + \frac{\partial U}{\partial z}\mathrm{d}z, \tag{3.36}$$

we see by comparing coefficients that

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}, \tag{3.37}$$

so

$$\vec{F} = -\nabla U. \tag{3.38}$$

For a non-conservative force where the line integral depends on the path taken, for example friction, there is no potential energy function and total energy is not conserved. Another way of viewing this is that under the action of conservative forces, the work done along a path is equal to the negative of the work done by reversing along the path. So we get back the energy that we put in. But in the case of a non-conservative force like friction, we don't regain energy we put in by moving an object along a path.

In a general system, an object may be under the influence of multiple forces which can be conservative or non-conservative. If we split the resultant force on the system into a conservative part and a non-conservative part: $\vec{F} = \vec{F}_{\text{conservative}} + \vec{F}_{\text{non-conservative}}$, then using the work energy theorem again we can write

$$\Delta K = W = \int_A^B \vec{F} \cdot \mathrm{d}\vec{r} \tag{3.39}$$

$$= \int_A^B \vec{F}_{\text{conservative}} \cdot \mathrm{d}\vec{r} + \int_A^B \vec{F}_{\text{non-conservative}} \cdot \mathrm{d}\vec{r} \tag{3.40}$$

$$= -\Delta U + W_{\text{non-conservative}}. \tag{3.41}$$

If we call the sum $K + U$ — the sum of kinetic energy and the potential energy from conservative forces — the **mechanical energy** E_{mech}, then we get

$$\Delta E_{\text{mech}} = W_{\text{non-conservative}}. \tag{3.42}$$

In the case where the non-conservative force is friction, this work done is converted to heat, or **thermal energy**. So $W_{\text{non-conservative}} = -\Delta E_{\text{thermal}}$. This implies that we can write energy conservation as

$$\Delta E_{\text{mech}} + \Delta E_{\text{thermal}} = 0. \tag{3.43}$$

This implies that $E_{\text{mech}} \leq 0$, so the total mechanical energy in a closed system can only decrease. This is related to the second law of thermodynamics. If our system is not isolated and is acted on by an external force, we can say

$$\Delta E_{\text{mech}} + \Delta E_{\text{thermal}} = W_{\text{ext}} \tag{3.44}$$

Example 3.5. A 2000kg elevator cable snaps at a height of 20m above a spring with $k = 10,000\text{Nm}^{-1}$. Taking into consideration that the friction of the shaft walls exert a constant force of 15,000N to resist the fall of the elevator, what is the maximum compression of the spring?

Consider the work done by the the gravitational force between two bodies as we move them closer or further apart. The gravitational force acts in the opposite direction to the displacement, so the infinitesimal amount of work done for an infinitesimal displacement is

$$\mathrm{d}W = \vec{\boldsymbol{F}}_G \cdot \mathrm{d}\vec{\boldsymbol{r}} \tag{3.45}$$

$$= -F_G \mathrm{d}r \tag{3.46}$$

$$= -\frac{Gm_1m_2}{r^2}\mathrm{d}r. \tag{3.47}$$

Then the total work done on an object by the gravitational force is

$$W = \int \mathrm{d}W = -\int_{r_1}^{r_2} \frac{Gm_1m_2}{r^2}\mathrm{d}r \tag{3.48}$$

$$= \left.\frac{Gm_1m_2}{r}\right|_{r_1}^{r_2} = \frac{Gm_1m_2}{r_2} - \frac{Gm_1m_2}{r_1}. \tag{3.49}$$

Since this only depends on the initial and final positions, the law of universal gravitation is a conservative force and we can define the change in **gravitational potential energy** as

$$\Delta U_G = -W = \frac{Gm_1m_2}{r_1} - \frac{Gm_1m_2}{r_2}. \tag{3.50}$$

If we choose U_G to be 0 at $r = \infty$, then

$$U_G(r) = -\frac{Gm_1m_2}{r}. \tag{3.51}$$

If we differentiate this with respect to r, we get the law of universal gravitation as expected.

3.5 Collisions

A collision is an interaction between two objects over a short time interval. To solve these problems, we can use the concept of momentum conservation and energy conservation that we have been studying in the last two chapters. Consider two blocks sliding along a frictionles surface towards each other (1-dimensional problem). The blocks have masses m_1, m_2 and velocities v_1, v_2 respectively. What we want to find is the velocities of the blocks after the collision. To do this, we write the total momentum and kinetic energy before and after as

$$\text{Before:} \quad P = m_1v_1 + m_2v_2 \tag{3.52}$$

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \tag{3.53}$$

$$\text{After:} \quad P' = m_1v_1' + m_2v_2' \tag{3.54}$$

$$K' = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2. \tag{3.55}$$

Total momentum is always conserved in collisions. On the other hand, depending on the forces involved during the collision, total kinetic energy may or may not be conserved. We call the case where it is conserved "**elastic**" and the case where it is not "**inelastic**".

In the case of elastic collisions, where total kinetic energy is conserved, we can write

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2' \tag{3.56}$$

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \tag{3.57}$$

This is a system of two equations for two unknowns, v_1' and v_2'. We can solving this system using algebra, and the solution is

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2}v_1 + \frac{2m_2}{m_1 + m_2}v_2 \tag{3.58}$$

$$v_2' = \frac{2m_1}{m_1 + m_2}v_1 + \frac{m_2 - m_1}{m_1 + m_2}v_2. \tag{3.59}$$

Does this make sense? To examine whether this answer makes physical sense we can take some limits and see what happens to the solution. Set $v_2 = 0$ and then consider the limit where $m_1 \gg m_2$. In this case, $v_1' \to v_1$ and $v_2' \to 2v_1$. This is like a bowling ball colliding with a ping-pong ball, the bowling ball keeps on going and the ping ball gets deflected in the same direction with twice the speed. If the two masses are equal, $v_1' = 0$ and $v_2' = v_1$, which is like a perfect billiard ball collision. On the other hand, if $m_1 \ll m_2$, $v_1' \to -v_1$ and $v_2' \to 0$. This corresponds to a ping-pong ball hitting a bowling ball at rest. It bounces off with the same speed in the opposite direction while the bowling ball stays still.

If we transform the velocities into the centre of mass frame we get

$$v_{1,\mathrm{COM}} = \frac{m_2(v_1 - v_2)}{m_1 + m_2} \tag{3.60}$$

$$v_{2,\mathrm{COM}} = \frac{m_1(v_2 - v_1)}{m_1 + m_2} = -\frac{m_1}{m_2} v_{1,\mathrm{COM}} \tag{3.61}$$

$$v_{1,\mathrm{COM}}' = -v_{1,\mathrm{COM}} \tag{3.62}$$

$$v_{2,\mathrm{COM}}' = -v_{2,\mathrm{COM}}. \tag{3.63}$$

So in the centre of mass frame, the two objects approach each other from opposite directions with velocities antiproportional to thei masses. After the collision, the magnitude of the velocities remains the same but they switch sign.

In an inelastic collision, we only have conservation of momentum since some energy is lost to non-conservative forces in the collision. To solve the system, we need another constraint on the velocities after the collision. In the case where the *maximum* kinetic energy is lost, which is when the objects stick together and move as a single body with velocity $v' = v_1' = v_2'$. This reduces the two equations for two unknowns that we had to solve before to one equation for one unknown.

$$m_1 v_1 + m_2 v_2 = m_1 v' + m_2 v' \tag{3.64}$$

$$\implies v' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}. \tag{3.65}$$

Notice that v' is simply the centre of mass velocity. So, if we transform into the centre of mass frame the final velocity is 0

$$v_{\mathrm{COM}}' = v' - v_{\mathrm{COM}} = 0. \tag{3.66}$$

This means that in a **perfectly inelastic** collision seen from the centre of mass frame, the objects approach each other with the same velocities as in the elastic case, but then come together at rest at the origin.

Example 3.6. Golf ball on a basketball.

Chapter 4

Angular Motion

4.1 Circular Motion

We have studied in great detail the mechanics of objects travelling in straight lines. Now we want to extend this to more general situations where objects can move along curved paths. The simplest case of a curved path is circular motion. Suppose we have an object moving in a circular path. Instead of describing its trajectory as a 2D vector in cartesian coordinates, it is much simpler to describe its trajectory in **polar coordinates**. The position of the object in 2D space is described by the distance from the origin and the angle which the position vector makes with the x axis. In circular motion, the distance from the centre is constant so the 2D motion is reduced to a 1D problem. We can then define an *angular displacement* which is given in terms of the angle.

Definition 4.1. The **angular displacement** of an object is the difference in angle to the x axis between two times t_1 and $t_2 > t_1$.

$$\Delta\theta = \theta(t_2) - \theta(t_1) = \theta_2 - \theta_1. \tag{4.1}$$

Now consider the velocity of the object, analogously to the linear case, we can define the angular velocity as the rate of change of angular displacement.

Definition 4.2. The **instananeous angular velocity** of an object is defined as the time derivative of angular displacement.

$$\omega(t) = \lim_{\Delta t \to 0} \frac{\theta(t + \Delta t) - \theta(t)}{\Delta t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}. \tag{4.2}$$

Likewise, the angular acceleration is given by the rate of change of angular velocity.

Definition 4.3. The **instaneous angular acceleration** of an object is defined as the time derivative of angular velocity.

$$\alpha(t) = \lim_{\Delta t \to 0} \frac{\omega(t + \Delta t) - \omega(t)}{\Delta t} = \frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2}. \tag{4.3}$$

Again analogously to the linear case, we define the average angular velocity and angular acceleration as an integral.

$$\bar{\omega}(t) = \frac{\Delta\theta}{\Delta t} = \frac{1}{\Delta t}\int_{t_1}^{t_2} \omega(t)\mathrm{d}t \tag{4.4}$$

$$\bar{\alpha}(t) = \frac{\Delta\omega}{\Delta t} = \frac{1}{\Delta t}\int_{t_1}^{t_2} \alpha(t)\mathrm{d}t. \tag{4.5}$$

4.2 Constant Angular Acceleration

In the case where we have a constant angular acceleration, we can derive a set of equation analogous to the SUVAT equations for linear motion. From the equation for average acceleration above, we get

$$\Delta\omega = \omega(t) - \omega_0 = \alpha t. \tag{4.6}$$

Then from the definition of angular displacement,

$$\Delta\theta = \int_{t_1}^{t_2} (\omega_0 + \alpha t)\mathrm{d}t \tag{4.7}$$

$$= \omega_0 t + \frac{1}{2}\alpha t^2 \tag{4.8}$$

$$\implies \theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2. \tag{4.9}$$

Squaring the first equation and substituting it into the second gives the last equation:

$$\omega^2(t) = \omega_0^2 + 2\alpha\theta(t). \tag{4.10}$$

4.3 Relating Linear and Angular Quantities

The arc length along a circle is given by $s = r\theta$, where r is the radius of the circle and θ is the angle. For an object travelling in a circular path, the arc length is the displacement $s(t)$. Thus, the velocity along the path is given as

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = r\omega. \tag{4.11}$$

Similarly, the acceleration is given by

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = r\frac{\mathrm{d}\omega}{\mathrm{d}t} = r\alpha. \tag{4.12}$$

Let's look at things from a vector perspective in the case where the object is moving with a constant speed, i.e. $\alpha = 0$. In this case, ω is constant and so we have $\theta = \omega t$. Using some trigonometry, we can see that the position vector, velocity and acceleration in cartesian coordinates is given by

$$\vec{\boldsymbol{r}} = r\cos(\omega t)\hat{\boldsymbol{\imath}} + r\sin(\omega t)\hat{\boldsymbol{\jmath}} \tag{4.13}$$

$$\vec{\boldsymbol{v}} = \frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} = -r\omega\sin(\omega t)\hat{\boldsymbol{\imath}} + r\omega\cos(\omega t)\hat{\boldsymbol{\jmath}} \tag{4.14}$$

$$\vec{\boldsymbol{a}} = \frac{\mathrm{d}\vec{\boldsymbol{v}}}{\mathrm{d}t} = -r\omega^2\cos(\omega t)\hat{\boldsymbol{\imath}} - r\omega^2\sin(\omega t)\hat{\boldsymbol{\jmath}}. \tag{4.15}$$

As a sanity check, we can see that

$$|\vec{\boldsymbol{v}}| = \sqrt{v_x^2 + v_y^2} = \sqrt{(-r\omega\sin(\omega t))^2 + (r\omega\cos(\omega t))^2} \tag{4.16}$$

$$= \sqrt{r^2\omega^2(\sin^2(\omega t) + \cos^2(\omega t))} = r\omega, \tag{4.17}$$

which is what we found before. Let's look at the dot product of the vectors.

$$\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{r}} = (-r\omega\sin(\omega t)\hat{\boldsymbol{\imath}} + r\omega\cos(\omega t)\hat{\boldsymbol{\jmath}}) \cdot (r\cos(\omega t)\hat{\boldsymbol{\imath}} + r\sin(\omega t)\hat{\boldsymbol{\jmath}}) \tag{4.18}$$

$$= -r^2\omega\sin(\omega t)\cos(\omega t) + r^2\omega\sin(\omega t)\cos(\omega t) \tag{4.19}$$

$$= 0. \tag{4.20}$$

Hence the velocity and position vectors are perpendicular. This makes sense because as we know from before, the velocity is always tangent to the trajectory, which corresponds to being perpendicular to the position vector in the case of a circle. Finally, notice that

$$\vec{\boldsymbol{a}} = -\omega^2\vec{\boldsymbol{r}}. \tag{4.21}$$

The acceleration is antiparallel to the position vector. The magnitude of the acceleration is

$$|\vec{\boldsymbol{a}}| = \sqrt{a_x^2 + a_y^2} = \sqrt{(-r\omega^2\cos(\omega t))^2 + (-r\omega^2\sin(\omega t))^2} \tag{4.22}$$

$$= \sqrt{r^2\omega^4(\sin^2(\omega t) + \cos^2(\omega t))} \tag{4.23}$$

$$= r\omega^2 = \frac{v^2}{r}. \tag{4.24}$$

Thus the magnitude of the acceleration does not change over time, but the direction constantly changes as the object moves on its circular path. By Newton's second law, a nonzero acceleration implies an unbalanced force. This force which has constant magnitude and constantly changing direction is what keeps the object moving on its circular path and is known as the **centripetal force**. It is given by

$$\vec{\boldsymbol{F}}_{\text{centripetal}} = m\vec{\boldsymbol{a}} = -\frac{mv^2}{r}\hat{r}. \tag{4.25}$$

Example 4.1. Consider a child on a merry-go-round. If the platform is rotating at 60rpm and the child is holding on, what is the force on the child's arm?

Example 4.2. Consider a conical pendulum. If the bob of mass of 200g on a string of length 50cm is swinging around at a frequency of 1 rotation per second, what is the angle that the pendulum makes with the vertical?

Example 4.3. Consider a car going round a circular bend. Find the maximum velocity that the car can take the bend at without skidding.

Example 4.4. Consider a ball rolling on a circular banked curve. What is the speed required to maintain a constant height on the curve as a function on the banking angle?

This next example uses the concept of energy conservation in combination with the traditional method of comparing forces to solve the problem.

Example 4.5. Consider a mass on a string. The mass starts hanging vertically downwards, then it gets projected sideways at a speed v_0. When the angle between the string and the vertical is 120°, the string becomes slack and the mass falls. Find the initial speed v_0 in terms of the length of the string. First use energy conservation to relate the change in kinetic energy to the change in gravitational potential energy. Then evaluate the forces at the top of the path to get a formula for the final velocity. Finally use the conservation of energy to solve for v_0.

How fast would the mass haves to be projected to get to the top of the loop?

4.4 Non-uniform Angular Acceleration

Suppose now that the angular acceleration α changes with time.

Chapter 5

Angular Momentum

5.1 Rigid Body Rotation

A rigid body is a system of particles which are fixed together such that they move together, no matter the forces applied. We have seen in previous chapters that it is possible to treat rigid bodies as point masses when considering linear motion. However, when considering rotational effects as well, we need to consider the macroscopic extent of an object. Consider a rigid body in motion and look at the total kinetic energy, which is given by the sum of kinetic energy of each constituent particle:

$$K = \sum_i \frac{1}{2} m_i v_i^2. \tag{5.1}$$

We can split the velocity of each particle into the sum of velocity around the centre of rotation and velocity along the line of motion.

$$\begin{aligned} K &= \sum_i \frac{1}{2} m_i (\vec{\boldsymbol{v}}_{i,\text{rot}} + \vec{\boldsymbol{v}}_{i,\text{tan}})^2 && (5.2) \\ &= \sum_i \frac{1}{2} m_i (v_{i,\text{rot}}^2 + v_{i,\text{tan}}^2 + 2\vec{\boldsymbol{v}}_{i,\text{rot}} \cdot \vec{\boldsymbol{v}}_{i,\text{tan}}) && (5.3) \\ &= \underbrace{\sum_i \frac{1}{2} m_i v_{i,\text{rot}}^2}_{K_\text{rotational}} + \underbrace{\sum_i \frac{1}{2} m_i v_{i,\text{tan}}^2}_{K_\text{linear}}. && (5.4) \end{aligned}$$

The cross-term in the square cancels out (why?). This rotational kinetic energy can be written in terms of the angular velocity ω since it is the same for each particle in a rigid body.

$$K_{\text{rot}} = \sum_i \frac{1}{2} m_i v_{i,\text{rot}}^2 = \sum_i \frac{1}{2} m_i (r_i \omega)^2 \tag{5.5}$$

$$= \frac{1}{2}\omega^2 \sum_i m_i r_i^2 \tag{5.6}$$

$$= \frac{1}{2} I \omega^2. \tag{5.7}$$

I is called the **moment of inertia**, and it is kind of an angular equivalent of mass. Notice how it is calculated in a similar way to the centre of mass except with the square of r. Also notice how in the formula for K_{rot}, I and ω play the role of m and v respectively, which shows how they are analogous to the linear quantities. However I, just like the centre of mass, depends on how the mass is distributed in an object. For example, consider a solid disc and a hoop of the same mass. From the formula for I, we can see that since all of the mass in the hoop is concentrated further out, the moment of inertia will be larger than the disc. This means that for the same angular velocity, the rotational kinetic energy of the hoop will be larger than the disc. I also depends on the axis of rotation.

5.2 Torque

Torque is defined as the **moment** of force, that is, the product of the distance from a reference point and the force. In terms of vectors, this is given by the cross product

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}. \tag{5.8}$$

The magnitude of $\vec{\boldsymbol{\tau}}$ is given by

$$\tau = |\vec{\boldsymbol{r}}||\vec{\boldsymbol{F}}| \sin\theta \tag{5.9}$$

$$= r F_{\text{tan}}. \tag{5.10}$$

This formula tells us that for a fixed radius, the torque has maximum magnitude when $\theta = \pm\frac{\pi}{2}$ i.e. the force acts *perpendicular* to the radius. It also tells us that if the force acts along the same line as the radius ($\theta = 0$ or $\theta = \pi$), then the torque is equal to 0. We can also see from the formula that the magnitude of torque depends on the radius. For the same force, if it is applied further away from the centre, the torque will be greater.

Using Newton's second law on the equation above, we can write the equation for torque above as

$$\tau = \sum_i \tau_i = \sum_i m_i r_i a_i = \sum_i m_i r_i^2 \alpha = I\alpha. \tag{5.11}$$

This works because for a rigid body, the angular acceleration is the same for all parts of the body, just like angular velocity. We can see from this we have an perfect angular analogue of Newton's second law for torques.

Example 5.1. Consider two connected masses on a massless pulley with $m_1 > m_2$. Suppose the system starts from rest and assume the string is massless, inextensible, and lies vertically. Find an expression for the magnitude of acceleration of the masses. To do this, we have to analyse the forces acting on the blocks and also the torques acting on the pulley.

5.3 Static Equilibrium

In the net force and the net torque on a rigid body are both 0, then it is in **static equilibrium**.

$$\sum_i \vec{\boldsymbol{F}}_i = 0 \quad \text{and} \quad \sum_i \vec{\boldsymbol{\tau}}_i = 0. \tag{5.12}$$

Note that we are free to choose any point as the origin to make finding the net torqe easier.

Example 5.2. Consider a beam of mass 10kg and length 4m. It sits on a fulcrum placed 1m from one end of the beam, and is supported from the other end by a string. Find the tension in the string and the force of the beam on the fulcrum.

Example 5.3. A ladder weighing 10kg rests on a smooth wall. Find the the static friction force between the floor and the ladder.

Example 5.4. Consider a sign hanging from a bar attached to a wall supported by a string. Find the force between the bar and the wall.

Chapter 6

Oscillations

6.1 Simple Harmonic Motion

Simple harmonic motion is defined as the motion determined by a force which is directly proportional to the displacement from some equilibrium position. Consider a mass on a horizontal spring. This is a system which we have seen before, but we have not studied the full motion of the mass in detail. The only force acting on the mass is the spring restoring force

$$F = -kx = ma, \tag{6.1}$$

so by Newton II, we can write

$$a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{k}{m}x. \tag{6.2}$$

The general solution to this differential equation is

$$x(t) = A\cos(\omega t) + B\sin(\omega t), \tag{6.3}$$

where

$$\omega = \sqrt{\frac{k}{m}}. \tag{6.4}$$

This can be rewritten as

$$x(t) = A\cos(\omega t + \phi), \tag{6.5}$$

where ϕ is the initial phase at $t = 0$. Notice that A is the maximum amplitude of the mass. Working out the derivatives of $x(t)$, We see that

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v(t) = -A\omega\sin(\omega t + \phi) \tag{6.6}$$

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = a(t) = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x(t), \tag{6.7}$$

and so we have recovered the equation of motion.

Let's work out the period of oscillation. It should be equal to the time taken for the phase to change by 2π. Hence

$$\omega t_0 + \phi = \omega(t_0 + T) + \phi \tag{6.8}$$

$$\implies \omega T = 2\pi \tag{6.9}$$

$$T = \frac{2\pi}{\omega}. \tag{6.10}$$

Since $f = 1/T$, the frequency of the oscillation is related to ω, the **angular frequency**, by

$$\omega = 2\pi f. \tag{6.11}$$

Now consider the energy of the mass on the spring. From before, we know this is given by

$$E = K + U_s \tag{6.12}$$

$$= \frac{1}{2}mv^2 = \frac{1}{2}kx^2. \tag{6.13}$$

Since we now know that $x(t) = A\cos(\omega t + \phi)$ and $v(t) = -A\omega\sin(\omega t + \phi)$, we can show that

$$E = \frac{1}{2}m(-A\omega\sin(\omega t + \phi))^2 + \frac{1}{2}k(A\cos(\omega t + \phi))^2 \tag{6.14}$$

$$= \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi) \tag{6.15}$$

$$= \frac{1}{2}mA^2\left(\frac{k}{m}\right)\sin^2(\omega t + \phi) + \frac{1}{2}kA^2\cos^2(\omega t + \phi) \tag{6.16}$$

$$= \frac{1}{2}kA^2\left(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right) \tag{6.17}$$

$$= \frac{1}{2}kA^2. \tag{6.18}$$

Where in the last line we have used the identity $\sin^2(\theta) + \cos^2(\theta) = 1$. Note that this is independent of time, so the total energy is conserved as we found before.

Let's look at a pendulum on a string of length L now. We are assuming that the string is massless and ignoring the effects of air resistance. Given that the arc length of the pendulum is related to the angular displacement by $s = L\theta$ and that the restoring force is $F = -mg\sin(\theta)$, we have the equation of motion:

$$F = m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -mg\sin(\theta). \tag{6.19}$$

This is a nonlinear differential equation. We will simplify this by assuming that the angle θ is small so $\sin(\theta) \approx \theta$ (small-angle approximation). Then the equation of motion becomes

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -g\theta = -\frac{g}{L}s. \tag{6.20}$$

Note that this has the same form as equation 6.2, so the trajectory will have the same form!

$$s(t) = A\cos(\omega t + \phi), \tag{6.21}$$

where ω in this case is given by

$$\omega = \sqrt{\frac{g}{L}}. \tag{6.22}$$

As an aside, think back to uniform circular motion. An object moving with UCM has a trajectory of the form

$$\theta(t) = \omega t + \phi, \tag{6.23}$$

where ω is the angular velocity and ϕ is the angular displacement at $t = 0$. If we describe this motion in cartesian coordinates, the horizontal motion takes the form

$$x(t) = r\cos(\omega t + \phi), \tag{6.24}$$

which is exactly the same form as an object moving under SHM!

Chapter 7

Waves

7.1 The Wave Equation

A wave is a periodic variation or disturbance which travels at a well defined speed through space. How do we describe waves mathematically? Suppose $f(\chi)$ is some periodic function which takes a phase χ measured in radians (fractions of 2π). Then we can describe the variation in space at a specific point in time as a snapshot:

$$y(x) = Af\left(\frac{2\pi}{\lambda}x + \delta\right), \tag{7.1}$$

where λ is the **wavelength** of the wave (the spatial period). We can also describe the variation in amplitude at a single point in space over time:

$$y(t) = Af\left(\frac{2\pi}{T}t + \theta\right), \tag{7.2}$$

where T is the temporal period.

To put these pictures together, we can consider the snapshot picture with a shift $x - vt$ where v is the speed of the wave. Then we have

$$\begin{aligned} y(x,t) &= Af\left(\frac{2\pi}{\lambda}(x - vt) + \phi\right) && (7.3) \\ &= Af\left(\frac{2\pi}{\lambda}x - \frac{2\pi v}{\lambda}t + \phi\right) && (7.4) \\ &= Af\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t + \phi\right) && (7.5) \\ &= Af(kx - \omega t + \phi). && (7.6) \end{aligned}$$

Where we have defined the **wavenumber** (spatial frequency measured in radians/m) $k = 2\pi/\lambda$ and recalled $v = f\lambda$, $f = 1/T$ and $\omega = 2\pi f$.

Example 7.1. Suppose we have a sinusoidal wave given by $y(x,t) = A\cos(kx - \omega t + \phi)$. What is the particle velocity at fixed position x?
To solve this, we take the *partial derivative* with respect to time (to keep x constant)

$$\frac{\partial y(x,t)}{\partial t} = A\omega \sin(kx - \omega t + \phi). \tag{7.7}$$

Note that this is different to the propagation speed of the wave itself, which is constant.

For simplicity, let's consider a general right-travelling wave $f(x - vt)$. We can make a substitution $u = x - vt$. Then we get

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} &(7.8)\\
&= \frac{\partial f}{\partial u} &(7.9)\\
\implies \frac{\partial^2 f}{\partial x^2} &= \frac{\partial^2 f}{\partial u^2}, &(7.10)\\
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} &(7.11)\\
&= -v\frac{\partial f}{\partial u} &(7.12)\\
\implies \frac{\partial^2 f}{\partial t^2} &= v^2\frac{\partial^2 f}{\partial u^2}. &(7.13)
\end{aligned}$$

If we do this same calculation with a left-travelling wave $f(x + vt)$, we get the same relation. Thus, by construction, the general solution to the differential equation

$$\frac{\partial^2 f}{\partial t^2} = v^2\frac{\partial^2 f}{\partial x^2}, \tag{7.14}$$

is $f(x,t) = f_l(x + vt) + f_r(x - vt)$. This linear differential equation is known as the **wave equation**. All functions which satisfy our definition of a wave solve this equation.

7.2 Superposition of Waves

Since the wave equation is linear, the solutions follow the principle of **linear superposition**. This means that when multiple waves come together, the amplitude at every point in space and time is determined by the sum of all the waves at that point.

Example 7.2. Consider two sinusoidal waves with the travelling with the same frequency and direction. Then the superposition is given by

$$y(x,t) = A\cos(kx - \omega t) + A\cos(kx - \omega t) \tag{7.15}$$
$$= 2A\cos(kx - \omega t). \tag{7.16}$$

So, the resultant wave has the same frequency and direction but double the amplitude.

Now consider what happens if one of the waves has a phase shift of π radians. The resultant wave is

$$y(x,t) = A\cos(kx - \omega t) + A\cos(kx - \omega t + \pi) \tag{7.17}$$
$$= A\cos(kx - \omega t) - A\cos(kx - \omega t) \tag{7.18}$$
$$= 0. \tag{7.19}$$

The two waves cancel each other out completely.

In the general case with a phase shift Ω, we get

$$y(x,t) = A\cos(kx - \omega t) + A\cos(kx - \omega t + \Omega) \tag{7.20}$$
$$= 2A\cos\left(kx - \omega t + \frac{\Omega}{2}\right)\cos\left(-\frac{\Omega}{2}\right) \tag{7.21}$$
$$= \underbrace{2A\cos\left(\frac{\Omega}{2}\right)}_{\text{Amplitude } \leq 2A}\underbrace{\cos\left(kx - \omega t + \frac{\Omega}{2}\right)}_{\text{Time-dependent part}}. \tag{7.22}$$

Note that we have used the identity $\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$.

Now consider what happens if the waves still have the same frequency but are moving in opposite directions. In this case the superposition is

$$y(x,t) = A\cos(kx - \omega t) + A\cos(kx + \omega t) \tag{7.23}$$
$$= 2A\cos\left(\frac{2kx}{2}\right)\cos\left(-\frac{2\omega t}{2}\right) \tag{7.24}$$
$$= \underbrace{2A\cos(kx)}_{A(x)}\cos(\omega t). \tag{7.25}$$

So we have a spatially varying amplitude $A(x)$ multiplied by a time-dependent variation. This is known as a **standing wave**.

Example 7.3. In the case where one of the waves has a phase shift Ω. The relation above becomes

$$
\begin{aligned}
y(x,t) &= A\cos(kx - \omega t) + A\cos(kx + \omega t + \Omega) && (7.26)\\
&= 2A\cos\left(\frac{2kx+\Omega}{2}\right)\cos\left(-\frac{w\omega t+\Omega}{2}\right) && (7.27)\\
&= 2A\cos\left(kx+\frac{\Omega}{2}\right)\cos\left(\omega t+\frac{\Omega}{2}\right). && (7.28)
\end{aligned}
$$

The standing waves allowed in a one-dimensional region of length L are given by

$$\lambda_{\frac{2L}{p}}, \quad f_p = p\frac{v}{2L} = pf_1, \tag{7.29}$$

where $p \in \mathbb{Z}$.

Note that sometimes it is more convenient to express waves in terms of complex exponential functions according to Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{7.30}$$

so a general sinusoidal wave like above would be written as

$$y(x,t) = Ae^{i(kx-\omega t+\phi)}. \tag{7.31}$$

We recover the trigonometric form (cosine in this case) by taking the real part of this function.

Example 7.4. Given a periodic wave, what is the phase difference between two points on the wave separated by a distance Δx?

$$\Delta\phi = 2\pi\frac{\Delta x}{\lambda} = k\Delta k. \tag{7.32}$$

What is the phase difference between a single point over a interval of time Δt?

$$\Delta\phi = 2\pi\frac{\Delta t}{T} = \omega\Delta t. \tag{7.33}$$

7.3 Phase Velocity & Group Velocity

As we have seen, the speed you need to keep up with a point of constant phase along the wave is given by

$$v_\phi = f\lambda = \frac{\omega}{k}. \tag{7.34}$$

This is known as the **phase velocity**. The dependence of ω on k (or vice-versa) is called the **dispersion relation**. If the relationship is linear, i.e. if v_ϕ is constant, the wave is said to be dispersionless. Otherwise, the wave will undergo dispersion as different frequencies will travel at difference speeds.

The **group velocity** is defined as

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{7.35}$$

So if a wave is dispersionless, the phase velocity and group velocity will be the same. In the case where $v_g \neq v_\phi$, the group velocity is the speed that the wave envelope propagates.

7.4 Transverse Waves on a String

Consider an infinite string under constant tension T. We will now show that the equation of motion of the string is the wave equation and derive the wave speed. Consider a short section of the string of length Δx. We are assuming that the string has linear density μ, zero stiffness, and we are ignoring the effects of gravity. Then assuming that there are only small displacements on the string, then $\frac{\partial y}{\partial X}$ is small, so the angles θ_1 and θ_2 are also small. Hence we use the small angle approximation and say that $\cos\theta_1 \approx \cos\theta_2 \approx 1$.

$$\sum F_x = -|\vec{\boldsymbol{T}}_1|\cos\theta_1 + |\vec{\boldsymbol{T}}_2|\cos\theta_2 = 0 \tag{7.36}$$

$$\implies |T_{1,x}| \approx |T_{2,x}| \approx T. \tag{7.37}$$

From these we get that $T_{1,x} \approx -T$ and $T_{2,x} \approx T$.

Now, using some trigonometry, notice that

$$\left.\frac{\partial y}{\partial x}\right|_x = \frac{T_{1,y}}{T_{1,x}} \approx -\frac{T_{1,y}}{T} \tag{7.38}$$

$$\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} = \frac{T_{2,y}}{T_{2,y}} \approx \frac{T_{2,y}}{T}. \tag{7.39}$$

Thus the net force in the y-direction is given by

$$F_y = T_{1,y} + T_{2,y} \tag{7.40}$$

$$= T\left(-\left.\frac{\partial y}{\partial x}\right|_x + \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}\right). \tag{7.41}$$

Using Newton's second law, we get

$$F_y = ma_y \tag{7.42}$$

$$= \mu\Delta x a_y \tag{7.43}$$

$$= \mu\Delta x \left.\frac{\partial^2 y}{\partial t^2}\right|_{x+\frac{\Delta x}{2}} = \left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_x\right)T. \tag{7.44}$$

Finally we divide by Δx on both sides and take the limit as $\Delta x \to 0$ to get

$$\lim_{\Delta x\to 0}\left(\mu \left.\frac{\partial^2 y}{\partial t^2}\right|_{x+\frac{\Delta x}{2}}\right) = \lim_{\Delta x\to 0}\left[\frac{\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right)T}{\Delta x}\right] \tag{7.45}$$

$$\mu\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2} \tag{7.46}$$

$$\implies \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}. \tag{7.47}$$

This is the wave equations and we can see that for waves on a string, $v = \sqrt{\frac{T}{\mu}}$.

What is the mechanical energy stored in a wave on a string? It will have two contributions, potential energy which depends on the displacement of every point and kinetic energy which depends on the velocity of every point. Consider a segment of the string of length $\mathrm{d}x$, mass $\mathrm{d}m = \mu\mathrm{d}x$. The infinitesimal contribution to the kinetic energy of the wave is given by

$$\mathrm{d}K = \frac{1}{2}\mathrm{d}mv_y^2 = \frac{1}{2}\mu\mathrm{d}x\left(\frac{\partial y}{\partial t}\right)^2. \tag{7.48}$$

To get a value for this, we integrate it over some length L.

$$K = \frac{1}{2}\mu\int_0^L\left(\frac{\partial y}{\partial t}\right)^2\mathrm{d}x. \tag{7.49}$$

The potential energy is due to the stretching of the string. A segment of length $\mathrm{d}x$ stretches to a length $\mathrm{d}s$, and we can calculate the relationship between the two as follows:

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} \tag{7.50}$$

$$= \sqrt{\mathrm{d}x^2 + \mathrm{d}x^2\left(\frac{\partial y}{\partial x}\right)^2} \tag{7.51}$$

$$= \mathrm{d}x\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \tag{7.52}$$

$$\approx \mathrm{d}x\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right), \tag{7.53}$$

where in the last line we have used the Taylor expansion $\sqrt{(1+u^2)} \approx 1 + \frac{1}{2}u^2$ when u is small. This means we can calculate the potential energy as

$$\mathrm{d}U = T(\mathrm{d}s - \mathrm{d}x) \tag{7.54}$$

$$\approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2\mathrm{d}x \tag{7.55}$$

$$\implies U = \frac{1}{2}T\int_0^L\left(\frac{\partial y}{\partial x}\right)^2\mathrm{d}x. \tag{7.56}$$

Example 7.5. Consider a sinusoidal wave $y = A\cos(kx - \omega t)$. What is the energy per unit wavelength? The partial derivatives are given by

$$\frac{\partial y}{\partial t} = A\omega \sin(kx - \omega t) \tag{7.57}$$
$$\frac{\partial y}{\partial x} = -Ak \sin(kx - \omega t), \tag{7.58}$$

so the infinitesimal contribution to the total energy is

$$\mathrm{d}E = \mathrm{d}K + \mathrm{d}U \tag{7.59}$$
$$= \frac{1}{2}\left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x \tag{7.60}$$
$$= \frac{1}{2}A^2\sin^2(kx - \omega t)(\mu\omega^2 + Tk^2)\mathrm{d}x. \tag{7.61}$$

Note that $v = \frac{\omega}{k} = \sqrt{T}\mu$, so $Tk^2 = \mu\omega^2$. Hence for a sinudoidal wave, the kinetic and potential energies are the same. The energy per unit wavelength is then

$$E_\lambda = \mu A^2\omega^2 \int_0^\lambda \sin^2(kx)\mathrm{d}x \tag{7.62}$$
$$= \frac{1}{2}\lambda\mu A^2\omega^2. \tag{7.63}$$

Note that we choose to write the energy in terms of μ rather than T because linear density is an easily measurable property whereas the tension is not. One important thing to mention is that the dependence on A^2 is actually general to all forms of waves, not just sinusoidal. We can calculate the power transmitted through a single point by the wave as

$$P = E_\lambda f = \frac{1}{2}\lambda f\mu A^2\omega^2 \tag{7.64}$$
$$= \frac{1}{2}v\mu A^2\omega^2 \tag{7.65}$$
$$= \frac{1}{2}\sqrt{\mu T}A^2\omega^2. \tag{7.66}$$

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