Classical Mechanics

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Contents

Preface

Chapter 1

Newton's Laws of Motion

1.1 Introduction

Mechanics is the study of how things move; from the Earth orbiting the Sun, to a ball rolling down a hill, to an electron in a cathode ray tube. The modern formulation of classical mechanics was first developed in the 17^{th} and 18^{th} centuries by European natural philosophers like Galileo and Newton, who based their ideas on previous theories like those from ancient Greece. It was then reformulated in the 19^{th} century by French mathematicians such as Lagrange, Poisson and Liouville, as well as Hamilton, whose developments paved the way for the innovations of the 20^{th} century, when it was realised that classical mechanics cannot accurately describe objects travelling close to the speed of light or objects that are extremely small (on an atomic scale). These discoveries led to the development of relativistic mechanics and quantum mechanics, for which the advanced Lagrangian and Hamiltonian formalisms serve as a mathematical basis.

In this text we will be studying the Newtonian formulation of classical mechanics, which is still used today not only as a teaching method, but also as a tool in research. The Lagrangian and Hamiltonian formalisms come into their own for advanced problems, but can be quite unwieldy for the simple systems that we will be describing the motion of. One may wonder why we still study classical mechanics if it has been proven to be obsolete in some areas. The answer is that there are still many real-life systems which are best described using a classical description. It is also a great opportunity to become familiar with the language of vector calculus while studying examples which are relevant to real life. Without further ado, let us dive in to how we describe the world in mechanics.

1.2 Motion in One Dimension

No doubt you will have noticed that the world is three dimensional. However it is certainly beneficial to study motion in a simplified setting before extending our ideas to the full 3D application. In Newtonian mechanics, we label each point in space with a **continuous variable** x, and we define an origin where $x = 0$. This defines our **coordinate system**. We can describe the motion of an object by writing its position x as a function of a continuous parameter t, which describes the passage of time from a reference point $t = 0$.

Definition 1.1. The **trajectory** of an object is a function $x(t)$ where $t, x(t) \in \mathbb{R}$. $x(t_0) = x_0$ describes the position of the object x_0 at some time t_0.

In classical mechanics, time evolves as the same rate for the whole universe. Our choice of coordinate system together with our choice of reference point for time is called a **reference frame**. By choosing our reference frame cleverly, we can often simplify problems. For example, if we were studying a block sliding down a slope, we could simplify the problem by rotating our coordinate system so that the x axis lies parallel to the slope. In general, it is often best to align the x axis with the direction of motion.

Definition 1.2. The **displacement** of an object is the difference in positions between two times t_1 and $t_2 > t_1$. In one dimension:

$$\Delta x = x(t_2) - x(t_1) = x_2 - x_1. \tag{1.1}$$

If $x(0) = x_0$, then the **total displacement** of an object as a function of time is given by

$$s(t) = x(t) - x_0. \tag{1.2}$$

The **distance** travelled by an object between t_1 and t_2 is given by the magnitude of displacement,

$$\text{distance} = |\Delta x| = |x(t_2) - x(t_1)| = |x_2 - x_1|. \tag{1.3}$$

In general, distance $\neq$ displacement. This is because displacement is a **vector** quantity, meaning it has direction and magnitude, whereas distance is a **scalar** quantity. In one dimension, the only difference between vector and scalar quantities is that vector quantities are **signed**. The distinction becomes greater in more dimensions when vector quantities are actually represented as vectors.

Definition 1.3. The **instantaneous velocity**, or just the **velocity** of an object is defined as the rate of change of the object's position with respect to time, or simply the derivative.

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}. \tag{1.4}$$

Velocity is also a vector quantity. The corresponding scalar quantity is **speed**, defined as the magnitude of velocity,

$$\text{speed} = |v(t)|. \tag{1.5}$$

The **average velocity** of an object in a time interval $\Delta t = t_2 - t_1$ is given by the displacement over the time interval, or also the time average of the velocity (which must be computed with an integral since it is a continuous property).

$$\bar{v} = \frac{\text{total displacement}}{\text{total time}} = \frac{\Delta x}{\Delta t} = \frac{1}{\Delta t} \int_{t_1}^{t_2} v(t)\,\mathrm{d}t. \tag{1.6}$$

Meanwhile, the **average speed** is given by

$$\overline{\text{speed}} = \frac{\text{total distance}}{\text{total time}} = \frac{1}{\Delta t} \int_{t_1}^{t_2} |v(t)|\,\mathrm{d}t. \tag{1.7}$$

Here, we may identify

$$\Delta x = \int_{t_1}^{t_2} v(t)\mathrm{d}t, \quad \text{total distance} = \int_{t_1}^{t_2} |v(t)|\,\mathrm{d}t. \tag{1.8}$$

If the object is travelling at constant velocity, then the integral is trivial and we recover (setting $t_1 = 0$) the equation that you probably learned in school,

$$\text{distance} = \text{speed} \times \text{time}. \tag{1.9}$$

Definition 1.4. The **instantaneous acceleration**, or just the **acceleration** of an object is defined as the rate of change of the object's velocity with respect to time.

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}. \tag{1.10}$$

Likewise, the **average acceleration** of an object in a time interval Δt is

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{1}{\Delta t} \int_{t_1}^{t_2} a(t)\,\mathrm{d}t. \tag{1.11}$$

1.3 Constant Acceleration

When the acceleration of an object is constant, we can derive some useful equations for simple motion. Starting from equation 1.11 above,

$$\Delta v = \int_{t_1}^{t_2} a\,\mathrm{d}t$$
$$\implies v_2 - v_1 = a \cdot (t_2 - t_1),$$

now setting $t_1 = 0, t_2 = t, v_1 = v(0) = u, v_2 = v(t)$, we get

$$v(t) = u + at, \tag{1.12}$$

where u is the initial velocity of the object. Now we substitute equation 1.12 into equation 1.8 to get

$$\Delta x = \int_{t_1}^{t_2} (u + at)\,\mathrm{d}t$$
$$\implies x_2 - x_1 = u \cdot (t_2 - t_1) + \frac{1}{2} a \cdot (t_2^2 - t_1^2).$$

Setting $t_1 = 0, t_2 = t, x_1 = x(0) = x_0, x_2 = x(t)$ like before and recalling $s(t) = x(t) - x_0$, we get

$$s(t) = ut + \frac{1}{2}at^2. \tag{1.13}$$

Finally, squaring equation 1.12 and substituting in equation 1.13 gives

$$v^2(t) = u^2 + 2as(t). \tag{1.14}$$

Equations 1.12, 1.13, and 1.14 are known as the SUVAT equations, you probably learned them in school. They are the **equations of motion** for a object under constant acceleration, i.e. all problems involving constant acceleration in a straight line are solved by them.

Example 1.1. Ball thrown in the air

Now let's do some examples where we do not have constant acceleration.

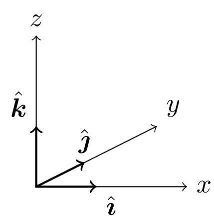
Example 1.2. Car accelerating then decelerating

$$v(t) = -\frac{1}{2}t^4 + 3t^3$$

Example 1.3. Block sliding down a hill (no friction)

1.4 Motion in More than One Dimension

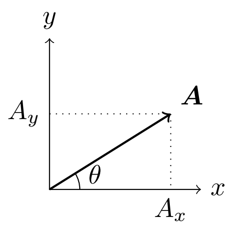
As alluded to in section 1.2 above, motion in more than one dimension is a straightforward generalisation of what we have learned so far. This is because the set of all points in 3D space forms a **vector space**, called $\mathbb{R}^3$, so we can pick 3 **orthogonal** axes and an origin to use as our coordinate system and define three basis vectors to span all of space. In **cartesian** coordinates, which are the most commonly used system to label points in 3D space, we label the axes x, y and z, and choose the unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ to point along each axis respectively.



A general vector in 3D cartesian coordinates is represented by a sum of components along each direction.

$$\boldsymbol{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}. \tag{1.15}$$

In 2D (where we don't have a z component), there is a simple way to calculate the components using the angle that the vector makes with the x axis.



First, note that the length of the vector is given by Pythagoras' theorem:

$$|\boldsymbol{A}| = \sqrt{A_x^2 + A_y^2}. \tag{1.16}$$

Then, using trigonometry, the components A_x and A_y are given by

$$A_x = |\boldsymbol{A}|\cos\theta \tag{1.17}$$
$$A_y = |\boldsymbol{A}|\sin\theta. \tag{1.18}$$

We also have a relation for the angle:

$$\tan\theta = \frac{A_y}{A_x}. \tag{1.19}$$

In 3D we need two angles to describe the direction of a vector, so the relations become slightly more complicated.

Note that it is important that our basis vectors are **orthonormal**, meaning both orthogonal and of unit length, as this means we have the relations

$$|\hat{\boldsymbol{\imath}}| = |\hat{\boldsymbol{\jmath}}| = |\hat{\boldsymbol{k}}| = 1 \tag{1.20}$$
$$\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}} = 1 \tag{1.21}$$
$$\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\jmath}}\cdot\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\imath}} = 0. \tag{1.22}$$

If these results were not true, then some of the mathematics we will do later on (involving dot and cross products) would become much more complicated than it needs to be.

Definition 1.5. The **trajectory** of an object in three dimensions is a vector-valued function $\boldsymbol{r}(t)$ where $t \in \mathbb{R}$, $\boldsymbol{r}(t) \in \mathbb{R}^3$. We write

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}} \tag{1.23}$$

in cartesian coordinates, where $x(t)$, $y(t)$ and $z(t)$ are the 1D trajectories of the object along each axis. For example, if $\boldsymbol{r}(t_0) = \boldsymbol{r}_0 = x_0\hat{\boldsymbol{\imath}} + y_0\hat{\boldsymbol{\jmath}} + z_0\hat{\boldsymbol{k}}$, then the object is located at position x_0 along the x axis, y_0 along the y axis and z_0 along the z axis. $\boldsymbol{r}(t)$ is known as the **position vector**.

At this point it is worth introducing a new notation which will simplify our expressions going forward. We will represent a time derivative of a quantity by simply writing a dot above the letter, and put two dots for a second derivative.

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \ddot{x} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \tag{1.24}$$

i.e. $\dot{x}$ represents velocity and $\ddot{x}$ represents acceleration. The notation is due to Newton and so it is fitting that we use it a lot in mechanics. We have also stopped notating dependence on time explicitly for brevity and to reduce clutter in the notation.

Definition 1.6. Just as in one dimension. The velocity is defined as the time derivative of position.

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\boldsymbol{\jmath}} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\boldsymbol{k}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}}. \tag{1.25}$$

Sometimes we denote $\dot{x}$, $\dot{y}$ and $\dot{z}$ as v_x, v_y and v_z respectively.

Definition 1.7. Likewise, acceleration is the time derivative of velocity, or the second time derivative of position.

$$\boldsymbol{a} = \ddot{\boldsymbol{r}} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}. \tag{1.26}$$

Sometimes $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$ are called a_x, a_y and a_z.

The key insight is that motion in 3D Cartesian coordinates is simply a superposition of three one-dimensional motions. Because of this, it is possible (and convenient) for simple problems to ignore the vector nature of the problem and just treat motion along each axis as a separate scalar problem.

Suppose we want to take the dot product of two vectors. We can compute it by writing each vector as a sum of components and then multiplying out the brackets. Going through the steps in cartesian coordinates, we get

$$\begin{aligned}
\boldsymbol{A} \cdot \boldsymbol{B} &= (A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}) \cdot (B_x\hat{\boldsymbol{\imath}} + B_y\hat{\boldsymbol{\jmath}} + B_z\hat{\boldsymbol{k}}) && (1.27)\\
&= A_x\hat{\boldsymbol{\imath}} \cdot B_x\hat{\boldsymbol{\imath}} + A_x\hat{\boldsymbol{\imath}} \cdot B_y\hat{\boldsymbol{\jmath}} + A_x\hat{\boldsymbol{\imath}} \cdot B_z\hat{\boldsymbol{k}} + A_y\hat{\boldsymbol{\jmath}} \cdot B_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} \cdot B_y\hat{\boldsymbol{\jmath}} + A_y\hat{\boldsymbol{\jmath}} \cdot B_z\hat{\boldsymbol{k}} \\
&\quad + A_z\hat{\boldsymbol{k}} \cdot B_x\hat{\boldsymbol{\imath}} + A_z\hat{\boldsymbol{k}} \cdot B_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}} \cdot B_z\hat{\boldsymbol{k}} && (1.28)\\
&= A_xB_x\hat{\boldsymbol{\imath}} + A_yB_y\hat{\boldsymbol{\jmath}} + A_zB_z\hat{\boldsymbol{k}}. && (1.29)
\end{aligned}$$

The last step only follows because the basis vectors are **orthonormal**. This is because the terms with two different basis vectors dotted together vanish (equation 1.22) and the terms with two of the same basis vector become just the coordinates multiplied together (equation 1.21). All the coordinate systems we have dealt with so far have an orthonormal basis because of this fact.

1.5 Forces

Now we know how to describe the motion of an object and changes in the motion (kinematics), but we still don't know how to describe *why* the motion of an object changes (dynamics). This is the focus of this section.

A force is some influence on an object that changes the object's motion. In classical mechanics, we describe forces as vectors and denote a generic force with the symbol F. The exact dynamics of how forces affect motion are described in Newton's laws of motion, which we will write now. In the following the symbol F stands for the sum of all forces acting on the object, otherwise known as the **net force**.

Definition 1.8 (Newton's First Law)**.** An object moving with constant velocity v, will stay at the same constant velocity unless acted upon by a force. In other words:

$$v = \text{const.} \iff F = 0. \tag{1.30}$$

Note that this includes an object at rest, which has velocity $v = 0$.

Mass is defined as an objects resistance to acceleration. In order words, a more massive object will accelerate slower relative to a less massive object when under the influence of identical forces. This is quantified by Newton's Second Law.

Definition 1.9 (Newton's Second Law)**.** The net force on the object is equal to the object's mass times the object's acceleration.

$$F = ma. \tag{1.31}$$

Note that force is always parallel to acceleration.

Since acceleration is the second derivative of position, we can write Newton's Second Law as

$$F(t) = m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}, \tag{1.32}$$

which is a differential equation for $x(t)$. This is known as an **equation of motion**, and all classical mechanics problems boil down to solving the equation of motion to obtain the trajectory of the object.

Example 1.4. Suppose an object is acted upon by a constant force F_0, then we align the x axis with the direction of the force and the equation of motion is

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = \frac{F_0}{m}.$$

This is a very easy differential equation to solve. Integrating twice, we get

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \int \frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = v_0 + \frac{F_0}{m}t \\ x(t) &= \int \frac{\mathrm{d}x(t)}{\mathrm{d}t} = x_0 + v_0t + \frac{F_0}{2m}t^2, \end{aligned}$$

where $v_0 = v(0)$ and $x_0 = x(0)$ as before. Note that comparing to equation 1.13, we can identify $a = \frac{F_0}{m}$ i.e. acceleration is constant, which is consistent with what we developed before.

It is important to note that Newton's Laws of Motion are only valid in **inertial reference frames**, which are reference frames travelling at a constant velocity v. If we are in a nonintertial reference frame, i.e. one that is accelerating, and we try to apply Newton's Laws, we will encounter odd things such as phantom forces which have no source. One way to test if we are in an inertial frame or not us by using Newton's First Law. If an object accelerates while under the influence of no forces, then our reference frame must be noninertial.

Definition 1.10 (Newton's Third Law)**.** When two objects interact with each other, the forces on each object due to the other are **equal in magnitude** and **opposite in direction**. In other words, if object A exerts a force $F_{A\to B}$ on object B, then object B exerts a force $F_{B\to A}$ on object A and we may write

$$F_{A\to B} = -F_{B\to A}. \tag{1.33}$$

These two forces are then known as a "Newton (III) pair".

Force is a vector quantity, so in more than dimension it can be decomposed into multiple components which are the forces along each axis. In three dimensions, Newton's Second Law is

$$\begin{aligned}\boldsymbol{F}(t) &= F_x(t)\hat{\boldsymbol{\imath}} + F_y(t)\hat{\boldsymbol{\jmath}} + F_z(t)\hat{\boldsymbol{k}}\\ &= ma_x(t)\hat{\boldsymbol{\imath}} + ma_y(t)\hat{\boldsymbol{\jmath}} + ma_z(t)\hat{\boldsymbol{k}}\\ &= m\boldsymbol{a} = m\ddot{\boldsymbol{r}}.\end{aligned}$$

We can now solve any problem in classical mechanics.

1.6 Projectile Motion

Let's now look at a concrete example of an experiment which will bring together everything we have looked at in this chapter. Suppose we have a cannon situated at the origin which shoots a projectile with a fixed initial velocity $\boldsymbol{v}_0$ which makes an initial angle θ relative to the ground. Can we work out how far the projectile will fly and what its flight time is?

This is going to be a 2D problem as we have two axes of motion. We shall label the horizontal direction that the cannon is shooting along the x axis and the vertical direction the y axis. As stated in the problem, the cannon is located at the origin. We can now write the initial velocity as

$$\boldsymbol{v}_0 = v_0\cos(\theta)\hat{\boldsymbol{\imath}} + v_0\sin(\theta)\hat{\boldsymbol{\jmath}}, \tag{1.34}$$

where $v_0 = |\boldsymbol{v}_0|$. Ignoring air resistance, there are no forces acting on the projectile in the x direction. By Newton's second law this means that $a_x = 0$ and we can immediately write

$$x(t) = v_0\cos(\theta)t, \tag{1.35}$$

using the SUVAT equation 1.13. In the y direction, the only force acting on the projectile is the constant force of gravity, so Newton's second law tells us

$$F_y = -mg = ma_y. \tag{1.36}$$

Hence, we have

$$y(t) = v_0\sin(\theta)t - \frac{1}{2}gt^2, \tag{1.37}$$

again by equation 1.13.

Now, the time of flight t_f will be given when $y(t_f) = 0$. Solving for this, we get

$$y(t_f) = v_0\sin(\theta)t_f - \frac{1}{2}gt_f^2 = 0 \tag{1.38}$$

$$v_0\sin(\theta)t_f = \frac{1}{2}gt_f^2 \tag{1.39}$$

$$t_f = \frac{2v_0\sin(\theta)}{g}. \tag{1.40}$$

Let's consider briefly if this answer makes physical sense. If the initial speed of the projectile v_0 was higher, then the flight time would be longer. Additionally, for a fixed initial speed, a projectile fired at a higher angle would have a longer flight time because more of the initial velocity was aimed along the vertical direction. On the other hand, if gravity was stronger, the projectile would fall to the ground faster and the flight time would be shorter.

Finally, the range of the projectile is given by

$$x_f = x(t_f) = v_0 \cos(\theta) t_f \tag{1.41}$$
$$= \frac{2v_0^2 \sin(2\theta)}{g}. \tag{1.42}$$

So the maximum range is given when $\theta = \frac{\pi}{4}$.

Example 1.5. A motorcyclist is doing a stunt jump between two buildings. If the buildings are separated by a distance d and have a vertical height difference h, what is the minimum velocity the motorcyclist needs to make the jump?

1.7 Friction

Friction is a very complicated process which occurs on a miroscopic scale, so in order to model in on a macroscopic scale we must use simplified empirical laws. In general, friction is a force which opposes change in motion. Hence if a force is applied parallel to a surface, then the frictional force will be antiparallel to this, perpendicular to the normal force.

Satic friction appears when two objects are motionless with respect to one another. If a force is applied between the two objects and they don't move, there must be a frictional force opposing the motion.

$$\boldsymbol{f}_s = -\boldsymbol{F}_{app}. \tag{1.43}$$

As the applied force gets larger, the static friction must get larger to preserve equilibrium, until a maximum limit is reached and the object starts moving.

Definition 1.11. The maximum magnitude of **static friction** is given by

$$f_{s,max} = \mu_s |\boldsymbol{N}|. \tag{1.44}$$

Hence,

$$0 \leq |\boldsymbol{f}_s| \leq f_{s,max}. \tag{1.45}$$

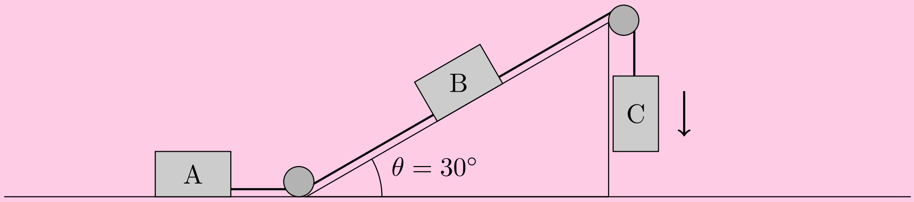
Kinetic friction opposes the motion of two surfaces sliding against each other.

Definition 1.12. **Kinetic friction** is given by

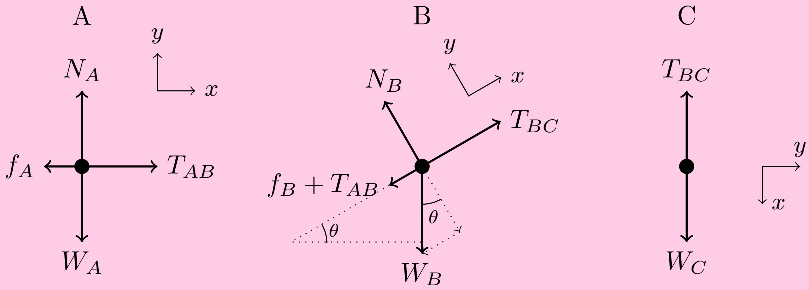
$$\boldsymbol{f}_k = -\mu_k |\boldsymbol{N}| \hat{\boldsymbol{v}}. \tag{1.46}$$

Example 1.6. Find the stopping distance of a block sliding down a slope.

Example 1.7. Three blocks — A, B, and C — are connected over a slope by massless inextensible ropes going through frictionless pulleys. Blocks A and B both have a weight of 25 N a coefficient of kinetic friction of $\mu_k = 0.35$. The angle of the slope is 30° and block C is falling with constant velocity. What is the weight of block C?



We will start to solve this problem by drawing free-body diagrams for all the blocks. In each diagram, we will orient the x-axis along the direction of motion.



Since the ropes are inextensible, the tension throughout them must be uniform. Therefore all the blocks must be moving together at the same speed. This speed is constant, so Newton's second law tells us that the resultant forces along both axes, ΣF_x and ΣF_y, must be zero for all three blocks. Looking at the diagram for block C, this tells us that W_C, the weight of block C, is equal in magnitude to T_{BC}, the tension in the rope connecting B and C.

From the diagram for block B we have

$$\Sigma F_y = N_B - W_B \cos\theta = 0 \tag{1.47}$$
$$\implies N_B = W_B \cos\theta \tag{1.48}$$

and

$$\Sigma F_x = T_{BC} - (f_B + T_{AB}) - W_B \sin\theta = 0 \tag{1.49}$$
$$\implies T_{BC} = f_B + T_{AB} + W_B \sin\theta \tag{1.50}$$
$$= \mu_k W_B \cos\theta + T_{AB} + W_B \sin\theta, \tag{1.51}$$

from block A we get

$$\Sigma F_y = N_A - W_A = 0 \tag{1.52}$$
$$\implies N_A = W_A \tag{1.53}$$

and

$$\Sigma F_x = T_{AB} - f_A = 0 \tag{1.54}$$
$$\implies T_{AB} = f_A = \mu_k N_A \tag{1.55}$$
$$= \mu_k W_A. \tag{1.56}$$

Combining all these results, we get

$$
\begin{aligned}
W_C &= T_{BC} && (1.57)\\
&= \mu_k W_B \cos\theta + W_B \sin\theta + \mu_k W_A && (1.58)\\
&= 7.58\,\mathrm{N} + 12.5\,\mathrm{N} + 8.75\,\mathrm{N} && (1.59)\\
&= 28.83\,\mathrm{N}. && (1.60)
\end{aligned}
$$

Now, imagine if the rope between blocks A and B is cut. What will happen to block C?

With both blocks A and B providing block C with enough friction to balance gravity, it seems that if we remove block A then there will no longer be enough resistance and block C will have to accelerate downwards. We can try calculating the acceleration and find out if our hypothesis is correct. The forces on blocks B and C are all the same except that we no longer have T_{AB}. Thus for block C we get

$$\Sigma F_x = W_C - T_{BC} = \frac{W_C}{g}a, \tag{1.61}$$

$$\tag{1.62}$$

and for block B we get

$$
\begin{aligned}
\Sigma F_x = T_{BC} - f_B - W_B \sin\theta &= \frac{W_B}{g}a && (1.63)\\
\implies T_{BC} = \mu_k W_B \cos\theta + W_B \sin\theta &+ \frac{W_B}{g}a. && (1.64)
\end{aligned}
$$

Combining these, we get

$$
\begin{aligned}
W_C - \frac{W_C}{g}a &= \mu_k W_B \cos\theta + W_B \sin\theta + \frac{W_B}{g}a && (1.65)\\
\frac{(W_C + W_B)}{g}a &= W_C - W_B \sin\theta - \mu_k W_B \cos\theta && (1.66)\\
a &= \frac{W_C - W_B(\sin\theta + \mu_k \cos\theta)}{W_C + W_B}g && (1.67)\\
&= 1.59\,\mathrm{m\,s^{-2}}. && (1.68)
\end{aligned}
$$

This acceleration is nonzero and positive, which indicates that block C does indeed accelerate downwards as we thought.

1.8 Gravitation

Aside from inventing classical mechanics and calculus, Isaac Newton's most well-known contribution to science is his discovery of the universal law of gravitation. If we consider two objects, one of which is located at the origin, then the gravitational force between the two bodies has magnitude

$$F = \frac{Gm_1m_2}{r^2}. \tag{1.69}$$

In vectorial form, these forces form a Newton III pair.

$$\boldsymbol{F}_{1\to 2} = -\frac{Gm_1 m_2}{|\boldsymbol{r}_{1\to 2}|^3}\boldsymbol{r}_{1\to 2} \tag{1.70}$$

$$\boldsymbol{F}_{2\to 1} = -\frac{Gm_1 m_2}{|\boldsymbol{r}_{2\to 1}|^3}\boldsymbol{r}_{2\to 1} = \frac{Gm_1 m_2}{|\boldsymbol{r}_{1\to 2}|^3}\boldsymbol{r}_{1\to 2} = -\boldsymbol{F}_{2\to 1}, \tag{1.71}$$

where $\boldsymbol{r}_{1\to 2} = \boldsymbol{r}_{2\to 1} = \boldsymbol{r}_2 - \boldsymbol{r}_1$.

Consider an object of mass m near the surface on the Earth. The radius of the Earth is very large so we can approximate the distance between the Earth and the object as simply the radius of the Earth. The gravitational force which the Earth exerts on the object is given by

$$F_G \approx \frac{GM_{\text{Earth}}}{R^2_{\text{Earth}}} m = mg, \tag{1.72}$$

where $g = \frac{GM_{\text{Earth}}}{R^2_{\text{Earth}}}$. Thus we have recovered the weight force that we have been using for the force due to gravity. If the height of the object is so large that the approximation no longer holds, then g depends on the height $R_{\text{Earth}} + h = r$. This is the same as the general case, the force only depends on the distance between the centre of the Earth and the object. Now consider the case where the object is *below* the surface of the Earth. In this case, the mass of the Earth depends on the distance between the centre and the object. The mass enclosed within a radius r is given by the density multiplied by the enclosed volume:

$$M(r) = \frac{4}{3}\rho\pi r^3. \tag{1.73}$$

If we assume that the density of the Earth is constant, then it is given by the total mass divided by total volume:

$$\rho = \frac{M_{\text{Earth}}}{\frac{4}{3}\pi R^3_{\text{Earth}}}. \tag{1.74}$$

Thus, $g(r)$ is given by

$$g(r) = \frac{GM(r)}{r^2} = \frac{G}{r^2}\frac{M_{\text{Earth}}}{R^3_{\text{Earth}}} r^3 \tag{1.75}$$

$$= \frac{GM_{\text{Earth}}}{R^3_{\text{Earth}}} r. \tag{1.76}$$

Example 1.8. Calculate g at the surface of Earth.

Chapter 2

Linear Momentum

2.1 Conservation of Momentum

Consider two objects interacting with each other via some forces. They could be two electrons repelling each other because of the electrostatic force or two planets falling together due to gravity. By Newton's third law,

$$\boldsymbol{F}_{A\to B}(t) = -\boldsymbol{F}_{B\to A}(t), \tag{2.1}$$

and so by Newton's second law we can write

$$m_A\boldsymbol{a}_A(t) + m_B\boldsymbol{a}_B(t) = 0. \tag{2.2}$$

We can integrate this equation over some arbitrary time period $t_1 < t_2$ to get

$$\int_{t_1}^{t_2} \left(m_A\boldsymbol{a}_A(t) + m_B\boldsymbol{a}_B(t)\right) \mathrm{d}t = m_A\left(\boldsymbol{v}_A(t_2) - \boldsymbol{v}_A(t_1)\right) + m_B\left(\boldsymbol{v}_B(t_2) - \boldsymbol{v}_B(t_1)\right) \tag{2.3}$$

$$= 0 \tag{2.4}$$

$$\implies m_A\boldsymbol{v}_A(t_1) + m_B\boldsymbol{v}_B(t_1) = m_A\boldsymbol{v}_A(t_2) + m_B\boldsymbol{v}_B(t_2). \tag{2.5}$$

Thus, we have discovered that Newton's third law implies that the quantity $m_A\boldsymbol{v}_A + m_B\boldsymbol{v}_B$ is **conserved**. This means it is constant for all time. We call this quantity the **linear momentum**.

2.2 Centre of Mass

Some times it is useful to consider the **centre of mass** of a system, which is defined as the *average* position of all the objects in the system. For the system of two objects, this is calculated as

$$\boldsymbol{r}_{\mathrm{COM}} = \frac{m_A\boldsymbol{r}_A + m_B\boldsymbol{r}_B}{m_A + m_B}. \tag{2.6}$$

This defines a position vector which points to the centre of mass of the system. By differentiating this vector with respect to time, we can get the velocity of the centre of mass

$$\boldsymbol{v}_{\mathrm{COM}} = \frac{\mathrm{d}\boldsymbol{r}_{\mathrm{COM}}}{\mathrm{d}t} = \frac{m_A\frac{\mathrm{d}\boldsymbol{r}_A}{\mathrm{d}t} + m_B\frac{\mathrm{d}\boldsymbol{r}_B}{\mathrm{d}t}}{m_A + m_B} = \frac{m_A\boldsymbol{v}_A + m_B\boldsymbol{v}_B}{m_A + m_B}. \tag{2.7}$$

This is just the total momentum of the two objects divided by their total mass, and we found before that the total momentum was conserved.

$$(m_A + m_B)\boldsymbol{v}_{\text{COM}} = m_A\boldsymbol{v}_A + m_B\boldsymbol{v}_B = \text{constant}. \tag{2.8}$$

From these definitions we can see that equation 2.2 is the accleration of the centre of mass multiplied by the total mass, which by Newton's second law is the force on the centre of mass. This implies that if the resultant force on the centre of mass is 0, then the centre of mass moves with constant velocity, just like a single object following Newton's first law, and the total linear momentum is conserved.

Example 2.1. For a system of two objects, show that the centre of mass is always located on the line that connects the two objects.

For a general system of N objects, we define the total mass and total momentum as

$$M = m_1 + m_2 + \cdots + m_N \tag{2.9}$$

$$= \sum_{i=1}^{N} m_i \tag{2.10}$$

$$\boldsymbol{P} = \sum_{i=1}^{N} m_i \boldsymbol{v_i}. \tag{2.11}$$

The net force on each individual part of the system can be written as the sum of any external forces and internal forces from all the other parts.

$$\boldsymbol{F}_{i,\text{net}} = \boldsymbol{F}_{i,\text{ext}} + \sum_{j\neq i}^{N} \boldsymbol{F}_{j\to i}. \tag{2.12}$$

Let's now add all the forces together to get the net force on the centre of mass:

$$\boldsymbol{F}_{\text{net}} = \sum_{i=1}^{N} \boldsymbol{F}_{i,\text{net}} = \underbrace{\sum_{i=1}^{N} \boldsymbol{F}_{i,\text{ext}}}_{=\boldsymbol{F}_{\text{ext}}} + \underbrace{\sum_{i=1}^{N}\sum_{j\neq i}^{N} \boldsymbol{F}_{j\to i}}_{=0}. \tag{2.13}$$

The second term on the far right-hand side is 0 by Newton's third law (you can prove this by induction).

Now we define the centre of mass position, velocity, and acceleration as the weighted average of each of the individual particles' quantities.

Definition 2.1. Consider a system of N objects. The **centre of mass** is is a vector function defined as

$$\boldsymbol{r}_{\text{COM}} = \frac{1}{M}\sum_{i=1}^{N} m_i \boldsymbol{r}_i. \tag{2.14}$$

The **centre of mass velocity** is defined as

$$\boldsymbol{v}_{\mathrm{COM}} = \frac{\mathrm{d}\boldsymbol{r}_{\mathrm{COM}}}{\mathrm{d}t} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{1}{M}\sum_{i=1}^{N} m_i \boldsymbol{v}_i = \frac{\boldsymbol{P}}{M}, \tag{2.15}$$

and the **centre of mass acceleration** is defined as

$$\boldsymbol{a}_{\mathrm{COM}} = \frac{\mathrm{d}^2\boldsymbol{r}_{\mathrm{COM}}}{\mathrm{d}t^2} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{\mathrm{d}^2\boldsymbol{r}_i}{\mathrm{d}t^2} = \frac{1}{M}\sum_{i=1}^{N} m_i \boldsymbol{a}_i = \frac{1}{M}\sum_{i=1}^{N} \boldsymbol{F}_{i,\mathrm{net}} = \frac{\boldsymbol{F}_{\mathrm{ext}}}{M}, \tag{2.16}$$

Notice that in the definition of centre of mass acceleration, we have shown that

$$M\boldsymbol{a}_{\mathrm{COM}} = \boldsymbol{F}_{\mathrm{ext}}. \tag{2.17}$$

This result is actually quite profound because it is what allows us to treat systems of particles as point masses themselves while ignoring all of the internal forces between the particles since they all cancel out. Also note that because the centre of mass momentum is equal to the total momentum, systems of many particles act as if all the mass was concentrated in a point at the centre of mass, moving with the the centre of mass velocity. Without these results, we could not apply the laws of mechanics as we have been learning them to macroscopic bodies!

If $\boldsymbol{F}_{\mathrm{ext}} = 0$, i.e. the system is isolated and there are no external forces, then the centre of mass moves in a straight line with a constant velocity and the total linear momentum is conserved.

Since the centre of mass quantites are vectors, we can break them into components just like all the other vector quantites we have seen. For example, in 2D cartesian coordinates we have

$$\boldsymbol{r}_{\mathrm{COM}} = \frac{1}{M}\sum_{i=1}^{N} m_i \boldsymbol{r}_i = \frac{1}{M}\sum_{i=1}^{N} m_i (x_i\hat{\boldsymbol{\imath}} + y_i\hat{\boldsymbol{\jmath}}) = \frac{1}{M}\sum_{i=1}^{N} m_i x_i \hat{\boldsymbol{\imath}} + \frac{1}{M}\sum_{i=1}^{N} m_i y_i \hat{\boldsymbol{\jmath}}, \tag{2.18}$$

so we can define the weighted average along the x and y axes:

$$\boldsymbol{r}_{\mathrm{COM}} = \bar{x}\hat{\boldsymbol{\imath}} + \bar{y}\hat{\boldsymbol{\jmath}}, \quad \bar{x} = \frac{1}{M}\sum_{i=1}^{N} m_i x_i, \quad \bar{y} = \frac{1}{M}\sum_{i=1}^{N} m_i y_i \tag{2.19}$$

2.3 Continuous Extended Objects

In the case where the number of particles in the system becomes so large that the distinction between the individual particles becomes smooth, we stop computing the centre of mass quantities as discrete sums and switch to integrals. The intuition for this is as follows. As the number of particles N tends to infinity, the mass m_i becomes a small mass element $\mathrm{d}m$, which is what we integrate over.

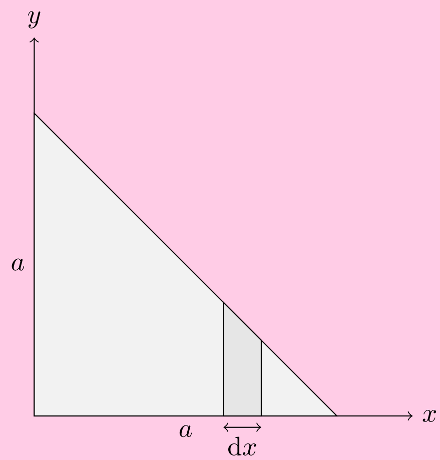
$$M = \int \mathrm{d}m, \quad \boldsymbol{r}_{\mathrm{COM}} = \frac{1}{M}\int \boldsymbol{r}\,\mathrm{d}m. \tag{2.20}$$

To evaluate this integral, we will need to convert the mass element $\mathrm{d}m$ into a spatial element, for example a length, area, or volume element. This is done using a linear, surface, or volume density.

$$\mathrm{d}m = \lambda(x)\,\mathrm{d}x, \quad \mathrm{d}m = \sigma(\boldsymbol{r})\,\mathrm{d}A, \quad \mathrm{d}m = \rho(\boldsymbol{r})\,\mathrm{d}V. \tag{2.21}$$

We will look at some examples here in the form of uniform lamina, which are flat (two-dimensional) extended objects with uniform surface density.

Example 2.2. Find the centre of mass of a right triangle with both small side lengths a.



We will find each component of the centre of mass position separately. In fact, by the symmetry of the shape we only need to find the average x position, because the average y position will be the same. We will break the triangle up into thin vertical slices of area $y\,\mathrm{d}x$. Note that $y = a - x$ (equation of a straight line), so we have

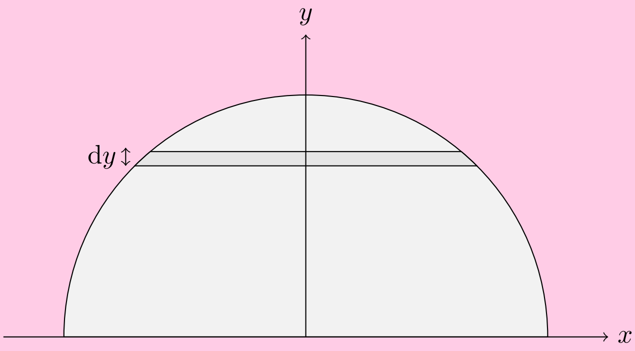
$$\bar{x} = \frac{1}{M}\int x\,\mathrm{d}m = \frac{1}{M}\int_0^a \sigma x y\,\mathrm{d}x = \frac{1}{M}\int_0^a \sigma x(a-x)\,\mathrm{d}x. \tag{2.22}$$

The total mass of the uniform lamina is $M = \frac{1}{2}\sigma a^2$, so the integral becomes

$$\bar{x} = \frac{2}{a^2}\int_0^a (ax - x^2)\,\mathrm{d}x = \frac{2}{a^2}\left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{a}{3}. \tag{2.23}$$

Thus the centre of mass position is $\boldsymbol{r}_{\mathrm{COM}} = \left(\frac{a}{3}, \frac{a}{3}\right)$.

Example 2.3. Find the centre of mass of a hemicircular uniform lamina of radius r.



By symmetry, we can see that $\bar{x}$ must be zero. Then, dividing the lamina into horizontal slices of area $2x\,\mathrm{d}y$ (remember the factor of two because the slice goes from negative x to positive x!), and using the equation of a circle to write $x = \sqrt{r^2 - y^2}$, we get

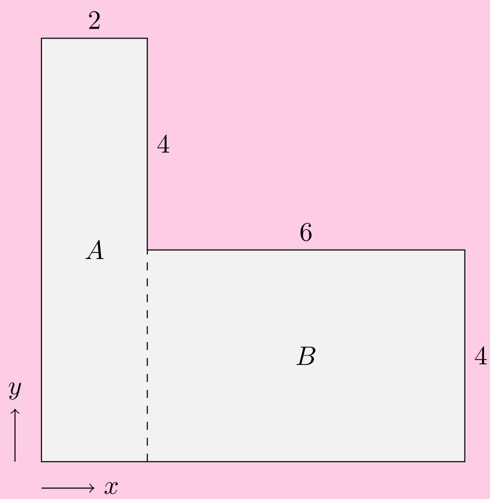
$$\bar{y} = \frac{1}{M}\int y\,\mathrm{d}m = \frac{1}{M}\int_0^r 2\sigma xy\,\mathrm{d}y = \frac{1}{M}\int_0^r 2\sigma y\sqrt{r^2-y^2}\,\mathrm{d}y. \tag{2.24}$$

The total mass of the hemicircle is $M = \frac{1}{2}\sigma\pi r^2$, so

$$\bar{y} = \frac{4}{\pi r^2}\int_0^r y\sqrt{r^2-y^2}\,\mathrm{d}y = \frac{4r}{3\pi}. \tag{2.25}$$

The center of mass is therefore $\boldsymbol{r}_{\mathrm{COM}} = \left(0, \frac{4r}{3\pi}\right)$.

Example 2.4. Sometimes we can solve the problem using symmetries without having to do any integrals at all. Consider the following uniform lamina.



Using the techniques we have built up in this chapter, we can find the centre of mass without doing any integrals.

Firstly, we can consider the whole object in two separate subsections. We will call the left section A and the right section B. Then the centre of mass of the whole lamina will be the average of the centres of mass of each subsection.

$$\boldsymbol{r}_{\mathrm{COM}} = \left(\frac{m_A x_{\mathrm{COM},A} + m_B x_{\mathrm{COM},B}}{m_A + m_B}, \frac{m_A y_{\mathrm{COM},A} + m_B y_{\mathrm{COM},B}}{m_A + m_B} \right) \tag{2.26}$$

Since both subsections are rectangles, by symmetry the centre of mass must be the geometric centre. Thus, the centre of mass of section A is $(1, 4)$ and for section B it is $(5, 2)$. Now, we need to know the mass of each subsection. Note that since the surface density is uniform, the mass is proportional to area. So the mass of section A is proportional to the area of section A, likewise for section B, and the total mass is proportional to the total area.

$$m_A \propto A_A, \quad m_B \propto A_B, \quad m_A + m_B \propto A_A + A_B. \tag{2.27}$$

The area of section A is 16 square units, and the area of section B is 24 square units, so now we can calculate the ratios of mass of subsection to total mass:

$$\frac{m_A}{m_A + m_B} = \frac{A_A}{A_A + A_B} = \frac{16}{40} = \frac{2}{5}, \quad \frac{m_B}{m_A + m_B} = \frac{A_B}{A_A + A_B} = \frac{24}{40} = \frac{3}{5}. \tag{2.28}$$

Therefore, the centre of mass of the whole lamina is

$$\boldsymbol{r}_{\mathrm{COM}} = \left(\frac{2}{5}1 + \frac{3}{5}5, \frac{2}{5}4 + \frac{3}{5}2 \right) = (3.4, 2.8). \tag{2.29}$$

2.4 Impulse

We have seen that an object has a linear momentum given by $\boldsymbol{p} = m\boldsymbol{v}$. How does the momentum change under the action of a force? Notice that

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} m\boldsymbol{v} = m\boldsymbol{a} = \boldsymbol{F}, \tag{2.30}$$

so we have a new way to write Newton's second law. Let's now integrate this equation over an arbitrary time interval:

$$\int_{t_1}^{t_2} \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\, \mathrm{d}t = \boldsymbol{p}(t_2) - \boldsymbol{p}(t_1) \tag{2.31}$$

$$= \Delta\boldsymbol{p} = \int_{t_1}^{t_2} \boldsymbol{F}(t)\, \mathrm{d}t. \tag{2.32}$$

We call the integral of a force over a time interval the **impulse**.

Definition 2.2. The **impulse** of a force $\boldsymbol{F}$ over a time interval $t_1 \le t_2$ is defined as

$$\boldsymbol{J} = \int_{t_1}^{t_2} \boldsymbol{F}(t)\, \mathrm{d}t. \tag{2.33}$$

It has units of Newton second (N·s). As we have seen, the impulse is equal to the change in momentum.

$$\Delta \boldsymbol{p} = \boldsymbol{J}. \tag{2.34}$$

This result is sometimes called the **impulse-momentum theorem**.

If we had a constant force, then the impulse would be $\boldsymbol{J} = \boldsymbol{F}\Delta t$ ($\Delta t = t_2 - t_1$). In most problems we want to solve this will not be the case. However, we can define the **average force** such that

$$\boldsymbol{J} = \int_{t_1}^{t_2} \boldsymbol{F}(t)\,\mathrm{d}t = \boldsymbol{F}_{\mathrm{avg}}\Delta t. \tag{2.35}$$

This is quite useful because if we have a short interaction, we can simply consider the average force over the interval which is a good approximation.

For a general system of N objects, the total impulse on the system over a time interval is

$$\boldsymbol{J} = \int_{t_1}^{t_2} \sum_{i=1}^{N} \boldsymbol{F}_i(t)\,\mathrm{d}tt = \int_{t_1}^{t_2} \boldsymbol{F}_{\mathrm{ext}}\,\mathrm{d}t. \tag{2.36}$$

By equation 2.17 in the previous section,

$$\begin{aligned}
\boldsymbol{J} = \int_{t_1}^{t_2} \boldsymbol{F}_{\mathrm{ext}}\,\mathrm{d}t &= \int_{t_1}^{t_2} M\boldsymbol{a}_{\mathrm{COM}}\,\mathrm{d}t && (2.37)\\
&= M\boldsymbol{v}_{\mathrm{COM}}(t_2) - M\boldsymbol{v}_{\mathrm{COM}}(t_1) && (2.38)\\
&= \sum_{i=1}^{N} (m_i\boldsymbol{v}_i(t_2) - m_i\boldsymbol{v}_i(t_1)) && (2.39)\\
&= \sum_{i=1}^{N} (\boldsymbol{p}_i(t_2) - \boldsymbol{p}_i(t_1)) && (2.40)\\
&= \boldsymbol{P}(t_2) - \boldsymbol{P}(t_1) = \Delta\boldsymbol{P}, && (2.41)
\end{aligned}$$

where $\boldsymbol{P}$ denotes the total momentum of the system. So the impulse-momentum theorem still holds for composite systems.

Example 2.5. Consider a baseball of mass m =0.3kg being thrown at a speed of 15ms^{-1}. If the batter bats the ball at a speed of 25ms^{-1} and the bat is in contact with the ball for 0.005s, what is the impulse imparted to the ball? What is the average force exerted on the ball? What is the average acceleration of the ball?

2.5 Transforming Between Reference Frames

To transform between one frame of reference to another, we subtract the constant velocity between the frames from the position vector.

$$\boldsymbol{r}'(t) = \boldsymbol{r}(t) - Vt. \tag{2.42}$$

We can then transform the velocity and acceleration as

$$\boldsymbol{v}'(t) = \frac{\mathrm{d}\boldsymbol{r}'(t)}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}(t)}{\mathrm{d}t} - V = \boldsymbol{v}(t) - V \tag{2.43}$$

$$\boldsymbol{a}'(t) = \frac{\mathrm{d}^2\boldsymbol{r}'(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\boldsymbol{r}(t)}{\mathrm{d}t^2} = \boldsymbol{a}(t). \tag{2.44}$$

One of the most useful reference frames to transform into is the centre of mass frame.

Chapter 3

Energy & Work

3.1 The Conservation of Energy

Lets calculate the rate of change of velocity squared.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}v^2 &= \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &(3.1)\\ &= 2\boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &(3.2)\\ &= 2\boldsymbol{v}\cdot\frac{\boldsymbol{F}}{m}, &(3.3)\end{aligned}$$

where in the last line we have used Newton II. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right)=\boldsymbol{F}\cdot\boldsymbol{v}. \tag{3.4}$$

We define the quantity in parentheses as the kinetic energy.

Definition 3.1. **Kinetic energy** is given by

$$K=\frac{1}{2}mv^2. \tag{3.5}$$

Let's see how the kinetic energy changes for a given force. We can find the change in kinetic energy by integrating:

$$\begin{aligned}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right)\mathrm{d}t &= \frac{1}{2}m(v(t_2)^2-v(t_1)^2) &(3.6)\\ &= \Delta K=\int_{t_1}^{t_2}\boldsymbol{F}\cdot\boldsymbol{v}\,\mathrm{d}t. &(3.7)\end{aligned}$$

Consider the gravitational force $\boldsymbol{F} = -mg\hat{\boldsymbol{k}}$. Then

$$\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right)\mathrm{d}t = \frac{1}{2}m(v(t_2)^2 - v(t_1)^2) \tag{3.8}$$

$$= \int_{t_1}^{t_2} \boldsymbol{F}\cdot\boldsymbol{v}\,\mathrm{d}t \tag{3.9}$$

$$= -\int_{t_1}^{t_2} mgv_z\,\mathrm{d}t \tag{3.10}$$

$$= -mg(z(t_2) - z(t_1)), \tag{3.11}$$

and hence,

$$\frac{1}{2}mv_1^2 + mgz_1 = \frac{1}{2}mv_2^2 + mgz_2. \tag{3.12}$$

So this quantity is **constant** over the path of the object (since t_1 and t_2 were arbitrary). If we write

$$K = \frac{1}{2}mv^2, \quad U_g = mgz, \tag{3.13}$$

then we have

$$E = K + U_g. \tag{3.14}$$

Example 3.1. Consider a pendulum on the end of a string. What is the maximum speed that the pendulum attains as it swings?

Example 3.2 (The Epitaph of Stevinus)**.** Consider a chain of uniform density draped over a trianglular block. If the chain is free to move with no friction, will it fall to the left, right, or will it balance in place.

The answer is that the chain will not move. We will prove this three different ways. The first way by analysing the forces as we have learned in chapter 1, the second way using energy conservation, and the third way using a clever 16^{th} century thought experiment.

To prove that there is no movement using forces, we will look at the components of the weight parallel to the incline for the left and right halves of the chain separately and show that they are equal in magnitude — therefore the total force must be zero. First, note that if the total mass of the chain is m, then the magnitude of the weight force on the left side is $\frac{3}{7}mg$ and on the right it is $\frac{4}{7}mg$. On the left, the x-component of weight is $\frac{3}{7}mg\cos\theta$ and on the right it is $\frac{4}{7}mg\sin\theta$. Using trigonometry, we have that

$$\cos\theta = \frac{4}{5}, \quad \sin\theta = \frac{3}{5}, \tag{3.15}$$

and therefore

$$F_{\text{left}} - F_{\text{right}} = \frac{3}{7}mg(\frac{4}{5}) - \frac{4}{7}mg(\frac{3}{5}) = 0. \tag{3.16}$$

To use energy conservation, we will first restate the problem slightly. Instead of a chain draped over the whole wedge, suppose that left and right sides of the chain are replaced with blocks of equivalent mass located at the centre of masses of each part. The blocks are connected with a light inextensible string which runs over the top of the wedge via a frictionless pulley. Now, we can calculate the total energy before the system starts moving, given by the sum of gravitational potential energy of each part (we set the zero point to be the bottom of the wedge):

$$E_{\text{before}} = U_L + U_R = \frac{3}{7}mgh_L + \frac{4}{7}mgh_R \tag{3.17}$$

$$= \frac{3}{7}mg(\frac{3}{2}\cos\theta) + \frac{4}{7}mg(2\sin\theta) \tag{3.18}$$

$$= \frac{18}{35}mg + \frac{24}{35}mg = \frac{6}{5}mg. \tag{3.19}$$

Suppose the system moves. Then at a later instant the blocks will have a collective velocity v, the left block will have moved a distance l along the incline, and the right block will have moved $-l$. The total energy at this time is

$$E_{\text{after}} = U_L' + U_R' + K = \frac{3}{7}mgh_L' + \frac{4}{7}mgh_R' + \frac{1}{2}mv^2 \tag{3.20}$$

$$= \frac{3}{7}mg\left(\frac{3}{2} + l\right)\cos\theta + \frac{4}{7}mg(2-l)\sin\theta + \frac{1}{2}mv^2 \tag{3.21}$$

$$= \frac{18}{35}mg + \frac{12}{35}mgl + \frac{24}{35}mg - \frac{12}{35}mgl + \frac{1}{2}mv^2 \tag{3.22}$$

$$= \frac{6}{5}mg + \frac{1}{2}mv^2. \tag{3.23}$$

Since energy is conserved, we should have $E_{\text{before}} = E_{\text{after}}$, which implies that

$$\frac{1}{2}mv^2 = 0, \quad \implies v = 0. \tag{3.24}$$

Finally, we will prove this using a thought experiment by Flemish scientist Simon Stevin. Suppose we attach another length of chain to both ends such that it forms a closed loop, with the new section hanging freely below the wedge. Since it hangs symmetrically, the forces on both sides (the tension at the lower vertices of the wedge) must be equal. If the forces on the upper part of the wedge were unbalanced, the whole loop would begin to rotate around the wedge in perpetual motion. This cannot happen, thus the forces on the upper part must be balanced. This proof is known as the Epitaph of Stevinus.

3.2 Work

As force is applied to an object, energy will be added or taken away. We call this energy **work**. Work is defined by the force applied multiplied by the distance the force was applied over. For a constant force, this can be written mathematically as $W = \boldsymbol{F}\cdot\boldsymbol{s} = |\boldsymbol{F}||\boldsymbol{s}|\cos\theta$ (check that this has units of energy!). We can think of work as the mechanism for transferring energy.

If the force is changing continuously over time, we need to be more careful about how we define the work. We break up the path of the object into small sections over which the force is approximately constant and integrate over all these small sections to get the total work.

Definition 3.2. The work done by a force over an infinitesimal distance $\mathrm{d}\boldsymbol{r}$ is

$$\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}. \tag{3.25}$$

The total work is then defined as

$$W = \int \mathrm{d}W = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}, \tag{3.26}$$

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the start and end points of the path being integrated over.

This is a line integral, so evaluating it will involve choosing a coordinate system and splitting the path up into components along each axis. Let's do a couple of examples now.

In cartesian coordinates, the force vector is given by $\boldsymbol{F} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}}$ and the infinitesimal displacement is $\mathrm{d}\boldsymbol{r} = \mathrm{d}x\hat{\boldsymbol{\imath}} + \mathrm{d}y\hat{\boldsymbol{\jmath}} + \mathrm{d}z\hat{\boldsymbol{k}}$. The dot product $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ is $F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z$. Thus, the integral in equation 3.26 becomes

$$W = \int_{x_1}^{x_2} F_x\,\mathrm{d}x + \int_{y_1}^{y_2} F_y\,\mathrm{d}y + \int_{z_1}^{z_2} F_z\,\mathrm{d}z, \tag{3.27}$$

where the limits of the integrals are the components of the start and end points of the path.

Example 3.3. Consider a force $\boldsymbol{F} = xy\hat{\boldsymbol{\imath}} + x^2y\hat{\boldsymbol{\jmath}}$. What is the work done on an object that moves along a straight line from $(0, 0)$ to $(4, 3)$ while under the influence of this force.

Note that the components of this force F_x and F_y both depend on x and y, so to evaluate the integrals we will have to eliminate the other variable by parameterising the path. Luckily this is easy for a straight line, the path can be written as $y = \frac{3}{4}x$. Then the work is

$$\begin{aligned} W &= \int_0^4 xy\,\mathrm{d}x + \int_0^3 x^2y\,\mathrm{d}y && (3.28)\\ &= \int_0^4 x\left(\frac{3}{4}x\right)\mathrm{d}x + \int_0^3 \left(\frac{4}{3}y\right)^2 y\,\mathrm{d}y && (3.29)\\ &= \frac{3}{4}\int_0^4 x^2\,\mathrm{d}x + \frac{16}{9}y^3\,\mathrm{d}y && (3.30)\\ &= \frac{1}{4}x^3\bigg|_0^4 + \frac{4}{9}y^4\bigg|_0^3 && (3.31)\\ &= 52\,\mathrm{J}. && (3.32) \end{aligned}$$

Because work is defined by a line integral, its value may change depending on the path taken, even if the start and end points are the same. This is quite often the case. Let's do the above example again, but have the object follow a different path.

Example 3.4. Calculate the work done on an object under the influence of the force from example 3.3, but this time following a path from $(0, 0)$ to $(4, 0)$, then to $(4, 3)$.

We can break this path up into two separate straight lines to integrate over. The total work done is the sum of the work over both paths. Over the first part, $y = 0$, so $F_x = xy = 0$, so no work is done over this section of the path. For the second part, $x = 4$ and the work done is

$$W = \int_0^3 16y\,\mathrm{d}y = 8y^2\Big|_0^3 = 72\,\mathrm{J}. \tag{3.33}$$

Now let's look at how to calculate work done in polar coordinates, in 2D to begin with. Force in polar coordinates is given by $\boldsymbol{F} = F_r\hat{\boldsymbol{r}} + F_\theta\hat{\boldsymbol{\theta}}$, and the infinitesimal displacement is $\mathrm{d}\boldsymbol{r} = \mathrm{d}r\hat{\boldsymbol{r}} + r\,\mathrm{d}\theta\hat{\boldsymbol{\theta}}$. Thus the work done is given by

$$W = \int (F_r\hat{\boldsymbol{r}} + F_\theta\hat{\boldsymbol{\theta}}) \cdot (\mathrm{d}r\hat{\boldsymbol{r}} + r\,\mathrm{d}\theta\hat{\boldsymbol{\theta}}) \tag{3.34}$$

$$= \int_{r_1}^{r_2} F_r\,\mathrm{d}r + \int_{\theta_1}^{\theta_2} F_\theta r\,\mathrm{d}\theta. \tag{3.35}$$

Example 3.5.

3.3 Work-Energy Theorem

As a force does work on an object, its speed will increase or decrease. This relationship is clarified by the work-energy theorem.

Theorem 3.1 (Work-Energy Theorem) *The net work on an object is equal to the change in its kinetic energy.*

$$W = \int_{\text{path}} \mathrm{d}W \tag{3.36}$$

$$= \int_{r_1}^{r_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \tag{3.37}$$

$$= \int_{t_1}^{t_2} \boldsymbol{F} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\mathrm{d}t \tag{3.38}$$

$$= \int_{t_1}^{t_2} \boldsymbol{F} \cdot \boldsymbol{v}\mathrm{d}t \tag{3.39}$$

$$= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right)\mathrm{d}t = \Delta K. \tag{3.40}$$

The kinetic energy depends on the speed of the object, so if the net work is > 0 then the object must have sped up. Likewise, if the net work is < 0, the object has slowed down. If the net work is 0, the object must be at the same speed that it started at.

Example 3.6. Consider a particle of mass 2 kg. It is being acted on by a force with the form $\boldsymbol{F} = 4x\hat{\boldsymbol{\imath}}$, and at $t = 0$ its position is $x = 4\,\mathrm{m}$ and its velocity is $v_0 = 1\,\mathrm{m\,s^{-1}}$. What is the particle's velocity when it reaches $x = 6\,\mathrm{m}$?

We can solve this using the work energy theorem. First we calculate the work done:

$$W = \int_{x_1}^{x_2} F_x \,\mathrm{d}x = \int_4^6 4x \,\mathrm{d}x = 2x^2\Big|_4^6 = 40\,\mathrm{J}. \tag{3.41}$$

Now we can use the fact that the work done is equal to the change in kinetic energy to get a formula for the final velocity:

$$W = \Delta K = \frac{1}{2}m(v_f^2 - v_0^2) \tag{3.42}$$

$$\implies v_f = \sqrt{\frac{2W}{m} + v_0^2} \tag{3.43}$$

$$= 6.4\,\mathrm{m\,s^{-1}}. \tag{3.44}$$

Now let's look at a mass on a spring. The restoring force on the mass always acts opposite to displacement. Hooke's law says

$$F_s = -k(x - x_0), \tag{3.45}$$

where k is the spring constant, x_0 is the equilibrium position of the spring. Then the work done by the spring force is

$$W_s = \int_{x_0}^{x_f} F_s(x)\,\mathrm{d}x = -\int_{x_0}^{x_f} k(x - x_0)\,\mathrm{d}x \tag{3.46}$$

$$= -\frac{1}{2}k(x_f - x_0)^2. \tag{3.47}$$

Note that the work done is always negative no matter if the displacement is positive or negative because the force points in the opposite direction. Now we define the spring potential energy as

$$U_s = -W_s = \frac{1}{2}k(x_f - x_0)^2, \tag{3.48}$$

Then by the work-energy theorem we have

$$\Delta K = W_s = -\Delta U_s, \tag{3.49}$$

so

$$K + U_s = E \tag{3.50}$$

is constant.

3.4 Conservative Forces

We have seen that in some cases the total energy is conserved and in others it is not. As we have seen above, work done is defined by a line integral, so in general it depends on the path chosen for integration, which corresponds to the path of the object through space. However, we have seen for the case of gravity and the spring force that the work done depends *only* on the initial and final positions; the integral is *path independent.* We also saw that in this case we can define a potential energy function for which

$$W = \int_{r_1}^{r_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = -(U(r_2) - U(r_1)), \tag{3.51}$$

where $\boldsymbol{F} = -\nabla U$ (mathematicians will note that this is just a special case of the **gradient theorem**, a multidimensional generalisation of the fundamental theorem of calculus). We call forces which have this property **conservative forces**. They are called this because by the work-energy theorem:

$$W = -(U(r_2) - U(r_1)) = -\Delta U = \Delta K, \tag{3.52}$$

and hence the total energy $E = K + U$ is conserved.

To fully define a potential energy function, we must explicitly say where potential energy is zero. Suppose we choose the point $\boldsymbol{r}_0$, so $U(\boldsymbol{r}_0) = 0$. Then we can define

$$U(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}. \tag{3.53}$$

We can choose whatever point is most convenient because it won't affect the physics, since force is minus the derivative of the potential energy any constant value we add to it will disappear. Only *differences* in potential (potential difference) have physical significance.

The gradient formula $\boldsymbol{F} = -\nabla U$ is the most general way to define potential energy, but it can be written more explicitly by choosing a coordinate system. For example, in 1D it reduces to

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x}, \tag{3.54}$$

in 3D cartesian coordinates it is

$$\boldsymbol{F} = -\frac{\partial U}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial U}{\partial y}\hat{\boldsymbol{\jmath}} - \frac{\partial U}{\partial z}\hat{\boldsymbol{k}}, \tag{3.55}$$

in 2D polar coordinates it is

$$\boldsymbol{F} = -\frac{\partial U}{\partial r}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\boldsymbol{\theta}}, \tag{3.56}$$

and so on.

These discoveries unlock a very powerful and intuitive way to look at dynamics. Rather than looking at the force and trying to figure out at what points it is pushing in which direction, we can simply look at the graph of the potential energy, specifically the slope. For example, suppose a force has the following potential energy function as a function of radial distance r. At the points B and D, the gradient is flat, so $\frac{\mathrm{d}U}{\mathrm{d}r} = 0$ and $F = 0$, there is no force at these points. At A and E we have a positive gradient, which means force is negative, and at C the gradient is negative which implies a positive force. We can immediately see that an object under the influence of this force will be pushed towards B or to $r = 0$, depending on where it starts and how much kinetic energy it has.

We will now list a few equivalent properties of conservative forces.

Definition 3.3. A force $\boldsymbol{F}$ is **conservative** if it satisfies any of the following conditions, which are all equivalent.

- The work done by the force is path independent.
- The work done by the force on a closed loop is zero.

$$W = \oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0. \tag{3.57}$$

- A potential energy function U can be defined such that $\boldsymbol{F} = -\nabla U$.
- The work done by the force is equal to minus the difference in U between the start and end points.
- $\boldsymbol{F}$ is irrotational, i.e. the curl of $\boldsymbol{F}$ is zero.

$$\nabla \times \boldsymbol{F} = 0. \tag{3.58}$$

For a central force, looking at equation 3.35 we can see that the second integral will be zero since there is no force in the tangential direction. Thus the work done by a central force is always given by

$$W = \int_{r_1}^{r_2} F(r)\,\mathrm{d}r. \tag{3.59}$$

So the work depends only on the initial and final radial distance and is thus path independent, which implies that central forces are *always* conservative.

Another category of conservative forces are those that point along a single axis and where the strength of the force depends only on the distance along that axis, if it depends on anything at all (constant forces are conservative!). These are forces of the form

$$\boldsymbol{F} = F(x)\hat{\boldsymbol{\imath}}. \tag{3.60}$$

Hooke's law and the constant gravitational field fall into this category of force. The proof is basically the same as for central forces, but we will show that the work done over a closed path is zero.

$$W = \oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \oint F(x)\hat{\boldsymbol{\imath}} \cdot (\mathrm{d}x\hat{\boldsymbol{\imath}} + \mathrm{d}y\hat{\boldsymbol{\jmath}} + \mathrm{d}z\hat{\boldsymbol{k}}) \tag{3.61}$$

$$= \int_{x_1}^{x_2} F(x)\,\mathrm{d}x + \int_{x_2}^{x_1} F(x)\,\mathrm{d}x \tag{3.62}$$

$$= \int_{x_1}^{x_2} F(x)\,\mathrm{d}x - \int_{x_1}^{x_2} F(x)\,\mathrm{d}x = 0. \tag{3.63}$$

Another two familiar examples of conservative forces are the universal law of gravitation and Coulomb's law. Being central forces, they are therefore conservative. Let's calculate their potential energy functions. For gravity, we get

$$U(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} -\frac{GMm}{r^2}\hat{\boldsymbol{r}}\cdot(\mathrm{d}r\hat{\boldsymbol{r}} + r\,\mathrm{d}\theta\hat{\boldsymbol{\theta}}) \tag{3.64}$$

$$= GMm\int_{r_0}^{r}\frac{1}{r^2}\,\mathrm{d}r \tag{3.65}$$

$$= -\frac{GMm}{r}\bigg|_{r_0}^{r}. \tag{3.66}$$

We can choose r_0 for our convenience, in this case if we choose $r_0 = \infty$ then the lower limit vanishes and the potential energy has the simple form

$$U(r) = -\frac{GMm}{r}. \tag{3.67}$$

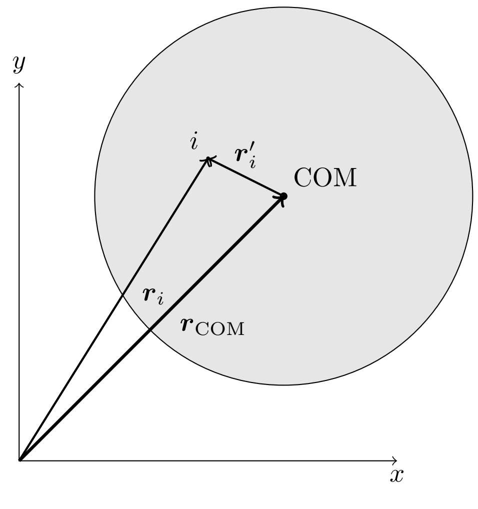
Similarly, for Coulomb's law $\boldsymbol{F} = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$ we get (taking the same zero point $r_0 = \infty$)

$$U(r) = \frac{q_1 q_2}{4\pi\varepsilon_0 r}. \tag{3.68}$$

In the case where q_1 and q_2 have opposite signs, the numerator q_1q_2 is negative and we get a graph that looks just like the one for gravity. The potential energy is negative for all values of r, the gradient is positive and therefore the force is always negative. This means that the Coulomb force is attractive in the case of opposite charges. Where q_1 and q_2 have the same sign, the numerator is positive and we have $U(r) > 0$ for all r. This implies that the gradient is always negative so the force is positive, indicating a repulsive force.

3.5 Energy in a System of Multiple Particles

Let's look at motion relative to a centre of mass. Consider a macroscopic body composed of many particles with a centre of mass at $\boldsymbol{r}_{\text{COM}}$.



We can label the position of particle i inside the body as $\boldsymbol{r}_i$, and its position relative to the centre of mass at $\boldsymbol{r}'_i$. Then

$$\boldsymbol{r}_i = \boldsymbol{r}_{\text{COM}} + \boldsymbol{r}'_i. \tag{3.69}$$

If we differentiate this equation, we can find the velocity of particle i:

$$\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i = \dot{\boldsymbol{r}}_{\text{COM}} + \dot{\boldsymbol{r}}'_i \tag{3.70}$$
$$= \boldsymbol{v}_{\text{COM}} + \boldsymbol{v}'_i, \tag{3.71}$$

where $\boldsymbol{v}'_i$ is the velocity of particle i relative to the centre of mass.

Let's look at the total kinetic energy of the system. It is the sum of the kinetic energy of all the constituent particles.

$$K = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 \tag{3.72}$$
$$= \frac{1}{2}\sum_{i=1}^{N} m_i (\boldsymbol{v}_{\text{COM}} + \boldsymbol{v}'_i)^2 \tag{3.73}$$
$$= \frac{1}{2}\sum_{i=1}^{N} m_i v_{\text{COM}}^2 + \frac{1}{2}\sum_{i=1}^{N} m_i v_i'^2 + \frac{1}{2}\sum_{i=1}^{N} m_i (2\boldsymbol{v}_{\text{COM}} \cdot \boldsymbol{v}'_i). \tag{3.74}$$

In the first term, v_{COM}^2 is constant and can be pulled out of sum, so the sum is simply the total mass M. The second term is the total kinetic energy in the centre of mass frame of reference, we call this the kinetic energy *in* the centre of mass. Looking at the last term, we can once again take $\boldsymbol{v}_{\text{COM}}$ out of the sum, so the term becomes $\boldsymbol{v}_{\text{COM}} \cdot \sum_{i=1}^{N} m_i \boldsymbol{v}'_i$. The sum is the total momentum in the centre of mass frame, which is zero. This can be justified by noting that in the centre of mass frame we have $\sum_{i=1}^{N} m_i \boldsymbol{r}'_i = 0$, so

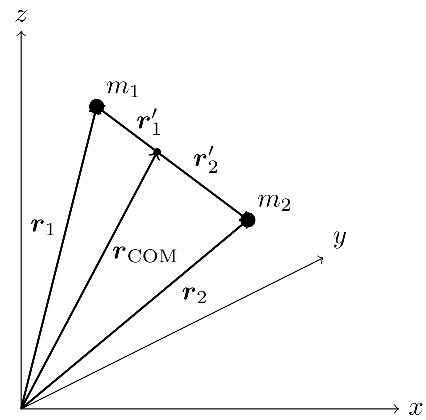
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} m_i \boldsymbol{r}'_i\right) = \sum_{i=1}^{N} m_i \boldsymbol{v}'_i = 0. \tag{3.75}$$

Therefore the total expression for the kinetic energy of the system is

$$K = \frac{1}{2} M v_{\text{COM}}^2 + \frac{1}{2}\sum_{i=1}^{N} m_i v_i'^2. \tag{3.76}$$

This is the kinetic energy associated with the total mass moving with speed v_{COM} and the *internal* kinetic energy from motion of particles relative to the centre of mass (the kinetic energy in the centre of mass). Depending on the problem, we can ignore the second term as it may not be relevant. For example, if we are examining the acceleration of a train, for the most part we don't care about the kinetic energy of objects within the vehicle. However, if we are looking at something that is rotating, like a flywheel for instance, we do care about the second term because there is a lot of motion relative to the centre of mass.

Consider a system of two particles.



In the centre of mass frame we have that $\sum_{i=1}^{N} m_1 \boldsymbol{r}_i = 0$, which in this case implies that

$$m_1 \boldsymbol{r}_1 = -m_2 \boldsymbol{r}_2. \tag{3.77}$$

We can define a vector $\boldsymbol{r}$ that points from m_1 to m_2, given by either

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1, \tag{3.78}$$

or equivalently,

$$\boldsymbol{r} = \boldsymbol{r}'_2 - \boldsymbol{r}'_1. \tag{3.79}$$

Substituting these relations into equation 3.77 above and rearranging, we get

$$\boldsymbol{r}'_1 = \frac{-m_2}{m_1 + m_2}\boldsymbol{r}, \quad \boldsymbol{r}'_2 = \frac{m_1}{m_1 + m_2}\boldsymbol{r}. \tag{3.80}$$

Now let's look at the kinetic energy. In the centre of mass frame, it is

$$K' = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2, \tag{3.81}$$

were $\boldsymbol{v}'_1 = \dot{\boldsymbol{r}}'_1$ and $\boldsymbol{v}'_2 = \dot{\boldsymbol{r}}'_2$. Using the relations for $\boldsymbol{r}'_1$ and $\boldsymbol{r}'_2$ above, we get

$$\boldsymbol{v}'_1 = \frac{-m_2}{m_1 + m_2}\boldsymbol{v}, \quad \boldsymbol{v}'_2 = \frac{m_1}{m_1 + m_2}\boldsymbol{v}, \tag{3.82}$$

where $\boldsymbol{v} = \dot{\boldsymbol{r}}$ is the **relative velocity** between the two particles. Therefore, the kinetic energy in the centre of mass frame becomes

$$K' = \frac{1}{2} m_1 \left(\frac{-m_2}{m_1 + m_2} \boldsymbol{v} \right)^2 + \frac{1}{2} m_2 \left(\frac{m_1}{m_1 + m_2} \boldsymbol{v} \right)^2 \tag{3.83}$$

$$= \frac{1}{2} \frac{m_1 m_2^2 + m_1^2 m_2}{(m_1 + m_2)^2} v^2 \tag{3.84}$$

$$= \frac{1}{2} \frac{m_1 m_2 (m_1 + m_2)}{(m_1 + m_2)^2} v^2 \tag{3.85}$$

$$= \frac{1}{2} \mu v^2, \tag{3.86}$$

where we have defined the **reduced mass** μ as

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{3.87}$$

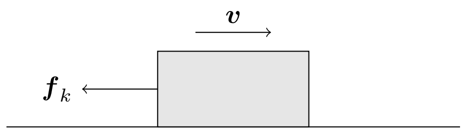
In a general inertial reference frame, the total energy (under the action of a conservative force) is given by

$$E = \frac{1}{2}Mv_{\text{COM}}^2 + \frac{1}{2}\mu v^2 + V(\boldsymbol{r}). \tag{3.88}$$

3.6 Non-conservative Forces

Non-conservative forces are the opposite of conservative forces, that is they don't conserve energy. They can be characterised by failing to meet one or more of the conditions in definition 3.3. I.e. for a non-conservative force, the work done depends on the path taken and there is no potential energy function. Another way of viewing this is that under the action of conservative forces, the work done along a path is equal to the negative of the work done by reversing along the path. So we get back the energy that we put in. But in the case of a non-conservative force like friction, we don't regain energy we put in by moving an object along a path.

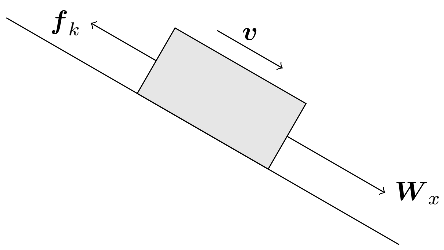
Let's look more closely at kinetic friction as an example. Consider a block sliding along a surface which is being slowed down by friction.



The friction force $\boldsymbol{f}_k$ always points in the opposite direction to $\mathrm{d}\boldsymbol{r}$, which points to the right. So, the work done by friction $W_f = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{f}_k \cdot \mathrm{d}\boldsymbol{r}$ is *always* negative. This is true for all paths, even closed ones, so we have

$$\oint \boldsymbol{f}_k \cdot \mathrm{d}\boldsymbol{r} < 0. \tag{3.89}$$

Now let's add another conservative force and verify that energy is not conserved. Suppose the block is sliding down a slope, let $\boldsymbol{W}_x$ denote the component of weight acting parallel to $\mathrm{d}\boldsymbol{r}$.



By the work-energy theorem, we can write

$$\Delta K = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} (\boldsymbol{f}_k + \boldsymbol{W}_x) \cdot \mathrm{d}\boldsymbol{r} \tag{3.90}$$

$$= \int_{\boldsymbol{r_0}}^{\boldsymbol{r}} \boldsymbol{W}_x \cdot \mathrm{d}\boldsymbol{r} + \int_{\boldsymbol{r_0}}^{\boldsymbol{r}} \boldsymbol{f}_k \cdot \mathrm{d}\boldsymbol{r} \tag{3.91}$$

$$= -U(\boldsymbol{r}) + \int_{\boldsymbol{r_0}}^{\boldsymbol{r}} \boldsymbol{f}_k \cdot \mathrm{d}\boldsymbol{r}. \tag{3.92}$$

We know that the weight force is conservative, so we have replaced the integral with the potential energy function. Now let's rearrange this to get the final total energy on the left:

$$\frac{1}{2}mv^2 + U(\boldsymbol{r}) = \frac{1}{2}mv_0^2 + \int_{\boldsymbol{r_0}}^{\boldsymbol{r}} \boldsymbol{f}_k \cdot \mathrm{d}\boldsymbol{r}. \tag{3.93}$$

Since the integral on the right is negative, the right hand side is always less than $\frac{1}{2}mv_0^2$, which is the initial energy. This implies

$$E_{\text{final}} < E_{\text{initial}}, \tag{3.94}$$

so energy is lost over time.

In a general system, an object may be under the influence of multiple forces which can be conservative or non-conservative. If we split the resultant force on the system into a conservative part and a non-conservative part: $\boldsymbol{F} = \boldsymbol{F}_{\text{conservative}} + \boldsymbol{F}_{\text{non-conservative}}$, then using the work energy theorem again we can write

$$\Delta K = W = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \tag{3.95}$$

$$= \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}_{\text{conservative}} \cdot \mathrm{d}\boldsymbol{r} + \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}_{\text{non-conservative}} \cdot \mathrm{d}\boldsymbol{r} \tag{3.96}$$

$$= -\Delta U + W_{\text{non-conservative}}. \tag{3.97}$$

If we call the sum of kinetic energy and the potential energy from conservative forces $K + U$ the **mechanical energy** E_{mech}, then we get

$$\Delta E_{\text{mech}} = W_{\text{non-conservative}}. \tag{3.98}$$

Consider the case where the non-conservative force is friction, so most of the work done is converted to heat, or **thermal energy**. So $W_{\text{non-conservative}} = -\Delta E_{\text{thermal}}$. This implies that we can write energy conservation as

$$\Delta E_{\text{mech}} + \Delta E_{\text{thermal}} = 0. \tag{3.99}$$

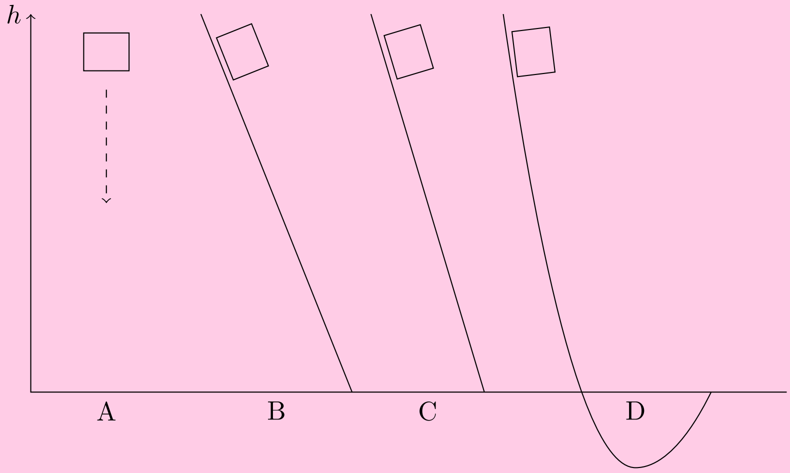
This implies that $\Delta E_{\text{mech}} \leq 0$, so the total mechanical energy in a closed system can only stay the same or decrease. We have basically just discovered the first and second laws of thermodynamics, but that is a topic for another time. If our system is not isolated and is acted on by an external force, we can say

$$\Delta E_{\text{mech}} + \Delta E_{\text{thermal}} = W_{\text{ext}}, \tag{3.100}$$

where W_{ext} is the work done by the external force on the system.

Example 3.7. A 2000kg elevator cable snaps at a height of 20m above a spring with $k = 10,000\text{Nm}^{-1}$. Taking into consideration that the friction of the shaft walls exert a constant force of 15,000N to resist the fall of the elevator, what is the maximum compression of the spring?

Example 3.8. Consider four possible paths of an object falling that start and end at the same height. Order the paths in terms of the final kinetic energy when there is no friction. What changes if there is friction?



With no friction, the final velocity is the same for all paths because the change in gravitional potential energy U_g is the same. With friction, $v_A > v_B > v_C > v_D$.

3.7 Collisions

A collision is an interaction between two objects over a short time interval. To solve these problems, we can use the concept of momentum conservation and energy conservation that we have been studying in the last two chapters. Consider two blocks sliding along a frictionles surface towards each other (1-dimensional problem). The blocks have masses m_1, m_2 and velocities v_1, v_2 respectively. What we want to find is the velocities of the blocks after the collision. To do this, we write the total momentum and kinetic energy before and after as

$$\text{Before:} \quad P = m_1 v_1 + m_2 v_2 \tag{3.101}$$

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \tag{3.102}$$

$$\text{After:} \quad P' = m_1 v_1' + m_2 v_2' \tag{3.103}$$

$$K' = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2. \tag{3.104}$$

Total momentum is always conserved in collisions. On the other hand, depending on the forces involved during the collision, total kinetic energy may or may not be conserved. We call the case where it is conserved “**elastic**” and the case where it is not “**inelastic**”.

In the case of elastic collisions, where total kinetic energy is conserved, we can write

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \tag{3.105}$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \tag{3.106}$$

This is a system of two equations for two unknowns, v_1' and v_2'. We can solving this system using algebra, and the solution is

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2m_2}{m_1 + m_2} v_2 \tag{3.107}$$

$$v_2' = \frac{2m_1}{m_1 + m_2} v_1 + \frac{m_2 - m_1}{m_1 + m_2} v_2. \tag{3.108}$$

Does this make sense? To examine whether this answer makes physical sense we can take some limits and see what happens to the solution. Set $v_2 = 0$ and then consider the limit where $m_1 \gg m_2$. In this case, $v_1' \to v_1$ and $v_2' \to 2v_1$. This is like a bowling ball colliding with a ping-pong ball, the bowling ball keeps on going and the ping ball gets deflected in the same direction with twice the speed. If the two masses are equal, $v_1' = 0$ and $v_2' = v_1$, which is like a perfect billiard ball collision. On the other hand, if $m_1 \ll m_2$, $v_1' \to -v_1$ and $v_2' \to 0$. This corresponds to a ping-pong ball hitting a bowling ball at rest. It bounces off with the same speed in the opposite direction while the bowling ball stays still.

If we transform the velocities into the centre of mass frame we get

$$v_{1,\mathrm{COM}} = \frac{m_2(v_1 - v_2)}{m_1 + m_2} \tag{3.109}$$

$$v_{2,\mathrm{COM}} = \frac{m_1(v_2 - v_1)}{m_1 + m_2} = -\frac{m_1}{m_2} v_{1,\mathrm{COM}} \tag{3.110}$$

$$v_{1,\mathrm{COM}}' = -v_{1,\mathrm{COM}} \tag{3.111}$$

$$v_{2,\mathrm{COM}}' = -v_{2,\mathrm{COM}}. \tag{3.112}$$

So in the centre of mass frame, the two objects approach each other from opposite directions with velocities antiproportional to thei masses. After the collision, the magnitude of the velocities remains the same but they switch sign.

In an inelastic collision, we only have conservation of momentum since some energy is lost to non-conservative forces in the collision. To solve the system, we need another constraint on the velocities after the collision. In the case where the *maximum* kinetic energy is lost, which is when the objects stick together and move as a single body with velocity $v' = v_1' = v_2'$. This reduces the two equations for two unknowns that we had to solve before to one equation for one unknown.

$$m_1 v_1 + m_2 v_2 = m_1 v' + m_2 v' \tag{3.113}$$

$$\implies v' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}. \tag{3.114}$$

Notice that v' is simply the centre of mass velocity. So, if we transform into the centre of mass frame the final velocity is 0

$$v_{\mathrm{COM}}' = v' - v_{\mathrm{COM}} = 0. \tag{3.115}$$

This means that in a **perfectly inelastic** collision seen from the centre of mass frame, the objects approach each other with the same velocities as in the elastic case, but then come together at rest at the origin.

Example 3.9. Golf ball on a basketball.

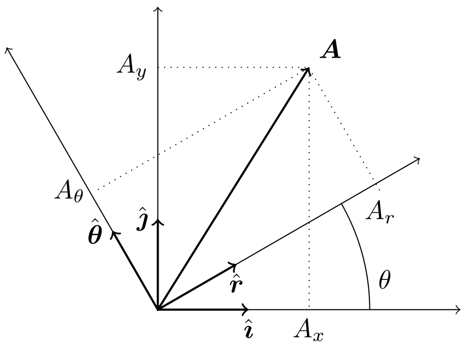
Chapter 4

Angular Motion

4.1 Polar Coordinates in 2D

We have studied in great detail the mechanics of objects travelling in straight lines. Now we want to extend this to more general situations where objects can move along curved paths. We will first study motion in 2D as it is much simpler than 3D. As we saw in section 1.4, vectors in 2D can be equivalently described by two cartesian components or their length together with the angle they make with the x-axis. Depending on the type of motion in the problem, it is much simpler to describe the trajectory of an object the latter way. These are known as **polar coordinates**.

To set up the coordinate system, we define two new coordinate axes relative to the position vector $\boldsymbol{r}$. The first axis points along the direction of the position vector, and the second points one quarter turn anticlockwise from the first. These axes get two orthonormal vectors to form a basis which we label $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ respectively.



Then a 2D vector $\boldsymbol{A}$ has a cartesian representation $\boldsymbol{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\imath}}$ and a polar representation $\boldsymbol{A} = A_r\hat{\boldsymbol{r}} + A_\theta\hat{\boldsymbol{\theta}}$. To find the components in one coordinate system using the components in the other, we have to know the relations between all the basis vectors. Using trigonometry, we can see that

$$\hat{\boldsymbol{r}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}} \tag{4.1}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}. \tag{4.2}$$

Substituting this into the polar coordinate representation of $\boldsymbol{A}$, we get

$$\begin{aligned}\boldsymbol{A} &= A_r(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) + A_\theta(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) && (4.3)\\ &= (A_r\cos\theta - A_\theta\sin\theta)\hat{\boldsymbol{\imath}} + (A_r\sin\theta + A_\theta\cos\theta)\hat{\boldsymbol{\jmath}}. && (4.4)\end{aligned}$$

Comparing coefficients with the cartesian representation, we see that

$$\begin{aligned}A_x &= A_r\cos\theta - A_\theta\sin\theta && (4.5)\\ A_y &= A_r\sin\theta + A_\theta\cos\theta. && (4.6)\end{aligned}$$

By doing the same procedure the other way around, we can deduce that

$$\begin{aligned}A_r &= A_x\cos\theta + A_y\sin\theta && (4.7)\\ A_\theta &= -A_x\sin\theta + A_y\cos\theta. && (4.8)\end{aligned}$$

The definition of the axes in polar coordinates may seem confusing at first as the position vector moves around over time. This means that the axes and unit vectors also change over time, and we will see how this affects calculation of the motion of an object in the next section.

Once thing that doesn't change is the representation of the position vector itself, which is always given by $\boldsymbol{r} = r\hat{\boldsymbol{r}}$. The conversion between representations of the trajectory in cartesian and polar coordinates is then given by equations 1.16, 1.17, 1.18, and 1.19.

$$\begin{aligned}x &= r\cos\theta && (4.9)\\ y &= r\sin\theta && (4.10)\\ r &= \sqrt{x^2+y^2} && (4.11)\\ \tan\theta &= \frac{y}{x}. && (4.12)\end{aligned}$$

4.2 Angular Kinematics

Similar to how we analysed motion along each axis seperately in cartesian coordinates in chapter 1, we can do the same thing in polar coordinates, except with some differences. In the special case of circular motion, there is no motion in the radial direction and 2D motion is reduced to a 1D problem. Displacement has units of length, but angles are unitless, so we will now examine how the angle of the position vector changes over time.

Definition 4.1. The **angular displacement** of an object is the difference in angle to the x axis between two times t_1 and $t_2 > t_1$.

$$\Delta\theta = \theta(t_2) - \theta(t_1) = \theta_2 - \theta_1. \quad (4.13)$$

Now consider the velocity of the object, analogously to the linear case, we can define the angular velocity as the rate of change of angular displacement.

Definition 4.2. The **instananeous angular velocity** of an object is defined as the time derivative of angular displacement.

$$\omega(t) = \lim_{\Delta t \to 0} \frac{\theta(t+\Delta t) - \theta(t)}{\Delta t} = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \dot{\theta}. \tag{4.14}$$

Likewise, the angular acceleration is given by the rate of change of angular velocity.

Definition 4.3. The **instaneous angular acceleration** of an object is defined as the time derivative of angular velocity.

$$\alpha(t) = \lim_{\Delta t \to 0} \frac{\omega(t+\Delta t) - \omega(t)}{\Delta t} = \frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = \dot{\omega} = \ddot{\theta}. \tag{4.15}$$

Again analogously to the linear case, we define the average angular velocity and angular acceleration as integrals.

$$\bar{\omega}(t) = \frac{\Delta\theta}{\Delta t} = \frac{1}{\Delta t}\int_{t_1}^{t_2} \omega(t)\,\mathrm{d}t \tag{4.16}$$

$$\bar{\alpha}(t) = \frac{\Delta\omega}{\Delta t} = \frac{1}{\Delta t}\int_{t_1}^{t_2} \alpha(t)\,\mathrm{d}t. \tag{4.17}$$

Now, let us go back to the full 2D vectors and see how they change with time. The velocity vector is the time derivative of the position vector:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{\boldsymbol{r}}). \tag{4.18}$$

We need to use the product rule for this since in general both r (the length of the position vector) and $\hat{\boldsymbol{r}}$ (the direction that the position vector points) change over time. What is the time derivative of $\hat{\boldsymbol{r}}$? We can work it out using equation 4.1:

$$\begin{aligned}
\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\cos\theta\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}}{\mathrm{d}t}\sin\theta\hat{\boldsymbol{\jmath}} && (4.19)\\
&= -\dot{\theta}\sin\theta\hat{\boldsymbol{\imath}} + \dot{\theta}\cos\theta\hat{\boldsymbol{\jmath}} && (4.20)\\
&= \omega(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) && (4.21)\\
&= \omega\hat{\boldsymbol{\theta}}, && (4.22)
\end{aligned}$$

where we have used equation 4.2 to substitute $\hat{\boldsymbol{\theta}}$ in the last line. We can work out the time derivative of $\hat{\boldsymbol{\theta}}$ now as well, it is

$$\begin{aligned}
\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} &= -\frac{\mathrm{d}}{\mathrm{d}t}\sin\theta\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}}{\mathrm{d}t}\cos\theta\hat{\boldsymbol{\jmath}} && (4.23)\\
&= -\dot{\theta}\cos\theta\hat{\boldsymbol{\imath}} - \dot{\theta}\sin\theta\hat{\boldsymbol{\jmath}} && (4.24)\\
&= -\omega\hat{\boldsymbol{r}}. && (4.25)
\end{aligned}$$

As an aside, think about why these two results make physical sense. Why should the rate of change of $\hat{\boldsymbol{r}}$ be in the $\hat{\boldsymbol{\theta}}$ direction?

We can now finish off calculating the velocity vector:

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} \tag{4.26}$$
$$= \dot{r}\hat{\boldsymbol{r}} + r\omega\hat{\boldsymbol{\theta}}. \tag{4.27}$$

We label the component pointing along the radial direction $v_r = \dot{r}$ and call it the **radial velocity**, and the component pointing along $\hat{\boldsymbol{\theta}}$, $v_\theta = r\omega$, is the **tangential velocity**.

The benefit of using the $\hat{\boldsymbol{r}} - \hat{\boldsymbol{\theta}}$ axes with polar coordinates is immediately apparent. Using the cartesian coordinate axes, the position vector is given by $\boldsymbol{r} = r\cos\theta\hat{\boldsymbol{\imath}} + r\sin\theta\hat{\boldsymbol{\jmath}}$, so the time derivative is

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = (\dot{r}\cos\theta - r\omega\sin\theta)\hat{\boldsymbol{\imath}} + (\dot{r}\sin\theta + r\omega\cos\theta)\hat{\boldsymbol{\jmath}} \tag{4.28}$$
$$= \dot{r}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) + r\omega(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) \tag{4.29}$$
$$= \dot{r}\hat{\boldsymbol{r}} + r\omega\hat{\boldsymbol{\theta}}. \tag{4.30}$$

In the last two lines we have regrouped the terms to write $\boldsymbol{v}$ in the $\hat{\boldsymbol{r}} - \hat{\boldsymbol{\theta}}$ basis and recovered our previous result. In the cartesian coordinate system, we get $v_x = \dot{r}\cos\theta - r\omega\sin\theta$ and $v_y = \dot{r}\sin\theta + r\omega\cos\theta$ which is rather opaque to interpretation.

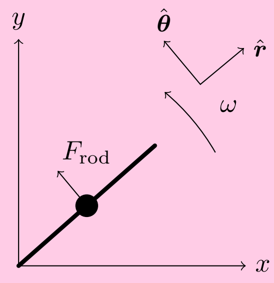
Now, let's calculate the acceleration vector in polar coordinates.

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}\hat{\boldsymbol{r}} + r\omega\hat{\boldsymbol{\theta}}) \tag{4.31}$$
$$= \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}t} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}t} \tag{4.32}$$
$$= \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\boldsymbol{\theta}} \tag{4.33}$$
$$= (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}. \tag{4.34}$$

Therefore, the radial acceleration is $a_r = \ddot{r} - r\dot{\theta}^2$, and the tangential acceleration is $a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$.

Here is an important thing to note about using polar coordinates. Although the unit vectors change over time, the reference frame is inertial. Our perspective is fixed, it does not move or co-rotate with the coordinate system. This an lead to some, at first counterintuitive, results compared to cartesian coordinates. One may assume that if there is no force in the radial direction, then there is no acceleration in the radial direction and $\ddot{r} = 0$. However, as we see above, the radial component of the acceleration vector, which takes into account the change of the unit vectors over time, is $a_r = \ddot{r} - r\dot{\theta}^2$. The radial component of acceleration is *not* the derivative of radial velocity, $\dot{r}$. This can be confusing because it implies that if $F_r = 0$ and so $a_r = 0$ by Newton's second law, then there can still be motion in the radial direction $\ddot{r} \neq 0$. We will now look at an example to illustrate this.

Example 4.1. Consider a bead which is free to move along a frictionless rod. The rod is anchored at one end and is rotating at constant angular velocity ω. As the rod rotates, the bead will swing out to the end of the rod. Before it reaches the end, what does its motion look like?



Since the total force on the bead is $\boldsymbol{F} = F_{\text{rod}}\hat{\boldsymbol{\theta}}$, the instantaneous radial acceleration $a_r = \ddot{r} - r\dot{\theta}^2$ must be zero. This is a differential equation that we can solve for the radial motion.

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \omega^2 r = 0. \tag{4.35}$$

Using an ansatz of the form $r = Ae^{Bt}$, with two unknown constants A and B, we get

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}(Ae^{Bt}) - \omega^2 Ae^{Bt} &= 0 && (4.36)\\ AB^2 e^{Bt} - \omega^2 Ae^{Bt} &= 0 && (4.37)\\ B^2 - \omega^2 &= 0. && (4.38) \end{aligned}$$

This gives $B = \pm\omega$, so we get the general solution

$$r = A_+ e^{\omega t} + A_- e^{-\omega t}. \tag{4.39}$$

When $t = 0$, r is some value r_0 and $\dot{r} = 0$, so

$$\begin{aligned} \dot{r}(0) &= A_+\omega e^{\omega t} - A_-\omega e^{-\omega t}\big|_{t=0} && (4.40)\\ &= \omega(A_+ - A_-) = 0, && (4.41) \end{aligned}$$

therefore $A_+ = A_- =$, so we will relabel them both A. Finally, $r(0) = 2A = r_0$, so we have

$$\begin{aligned} r &= \frac{1}{2} r_0 (e^{\omega t} + e^{-\omega t}) && (4.42)\\ &= r_0 \cosh(\omega t). && (4.43) \end{aligned}$$

4.3 Constant Angular Acceleration

In the case where we have a constant angular acceleration, we can derive a set of equations analogous to the SUVAT equations for linear motion derived in section 1.3. From the equation for average acceleration above, we get

$$\begin{aligned} \Delta\omega = \omega(t) - \omega_0 &= \alpha t && (4.44)\\ \omega(t) &= \omega_0 + \alpha t, && (4.45) \end{aligned}$$

which is analogous to equation 1.12. Then from the definition of angular displacement,

$$\Delta\theta = \int_{t_1}^{t_2} (\omega_0 + \alpha t)\,\mathrm{d}t \tag{4.46}$$

$$= \omega_0 t + \frac{1}{2}\alpha t^2 \tag{4.47}$$

$$\implies \theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2. \tag{4.48}$$

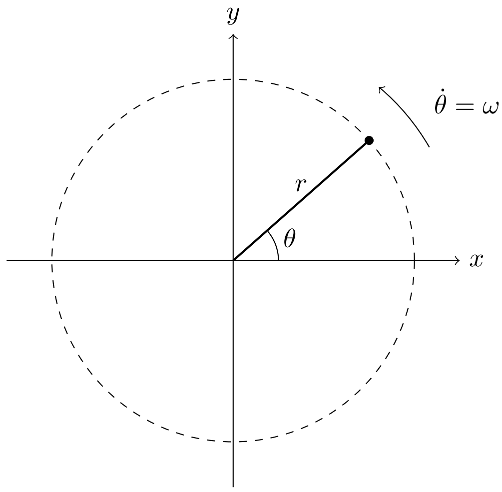
This is like equation 1.13 Finally, squaring equation 4.45 and substituting it into equation 4.48 gives the last equation:

$$\omega^2(t) = \omega_0^2 + 2\alpha\theta(t), \tag{4.49}$$

which is an angular version of equation 1.14. These equations are useful in solving problems where we don't have to worry (or don't care) about motion in the radial direction.

4.4 Uniform Circular Motion

The simplest case of angular motion is where an object is moving in a circle, so there is no motion in the radial direction, and going at a constant speed. This is called **uniform circular motion**, and objects moving this way can be analysed using 1D techniques we studied in section 1.2.



First, note that the displacement is the arc length of the circular path, given by $s = r\theta$, where θ is the angular displacement in radians and r is the radius of the circle. Thus, the velocity along the path is given as

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = r\omega. \tag{4.50}$$

Since the speed is constant, ω is constant as well. Therefore, by equation 4.48, we have $\theta = \omega t$.

Does this mean acceleration is zero? No. Since ω is constant, $\alpha = \dot{\omega} = 0$, but this does not mean that the linear acceleration is zero. Recall Newton's first law of motion, definition 1.8: if the *velocity* is constant, the force (acceleration) is zero. Since the path of motion is a circle, the velocity vector is constantly changing direction and is therefore *not* constant.

We will calculate the velocity and accelearation in cartesian coordinates first. Taking equation 4.28 and dropping the terms with $\dot{r}$, we get

$$\boldsymbol{v} = -r\omega\sin(\omega t)\hat{\boldsymbol{\imath}} + r\omega\cos(\omega t)\hat{\boldsymbol{\jmath}} \tag{4.51}$$

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \tag{4.52}$$

$$= -r\omega^2\cos(\omega t)\hat{\boldsymbol{\imath}} - r\omega^2\sin(\omega t)\hat{\boldsymbol{\jmath}}. \tag{4.53}$$

As a sanity check, we can see that the speed is given by

$$|\boldsymbol{v}| = \sqrt{(-r\omega\sin(\omega t))^2 + (r\omega\cos(\omega t))^2} \tag{4.54}$$

$$= \sqrt{r^2\omega^2(\sin^2(\omega t) + \cos^2(\omega t))} \tag{4.55}$$

$$= r\omega, \tag{4.56}$$

which is what we found before. It is not immediately clear in this coordinate system which direction the velocity points in, let's calculate the dot product with the position vector to find out.

$$\boldsymbol{v}\cdot\boldsymbol{r} = (-r\omega\sin(\omega t)\hat{\boldsymbol{\imath}} + r\omega\cos(\omega t)\hat{\boldsymbol{\jmath}}) \cdot (r\cos(\omega t)\hat{\boldsymbol{\imath}} + r\sin(\omega t)\hat{\boldsymbol{\jmath}}) \tag{4.57}$$

$$= -r^2\omega\sin(\omega t)\cos(\omega t) + r^2\omega\sin(\omega t)\cos(\omega t) \tag{4.58}$$

$$= 0. \tag{4.59}$$

Hence the velocity and position vectors are perpendicular. This makes sense because as we know from section 1.4, the velocity is always tangent to the trajectory, which corresponds to being perpendicular to the position vector in the case of a circle.

Finally, notice that since $\boldsymbol{r} = r\sin(\omega t)\hat{\boldsymbol{\imath}} + r\cos(\omega t)\hat{\boldsymbol{\jmath}}$,

$$\boldsymbol{a} = -\omega^2\boldsymbol{r}. \tag{4.60}$$

The acceleration is antiparallel to the position vector, pointing in towards the centre of the circle. The magnitude of the acceleration is

$$|\boldsymbol{a}| = |-\omega^2||\boldsymbol{r}| \tag{4.61}$$

$$= r\omega^2 \tag{4.62}$$

$$= \frac{v^2}{r}, \tag{4.63}$$

where we have substituted equation 4.50 in the last line Thus the magnitude of the acceleration is constant, but the direction changes as the object moves on its circular path. By Newton's second law, a nonzero acceleration implies an unbalanced force. This force is what keeps the object moving on its circular path and is known as the **centripetal force**. It is given by

$$\boldsymbol{F}_{\text{centripetal}} = m\boldsymbol{a} = -mr\omega^2\hat{\boldsymbol{r}} = -\frac{mv^2}{r}\hat{\boldsymbol{r}}. \tag{4.64}$$

Let us see if we can reproduce these results using the 2D polar coordinate vectors. Using equation 4.27 and noting that r is constant, we immediately recover

$$\boldsymbol{v} = r\omega\hat{\boldsymbol{\theta}}. \tag{4.65}$$

Here it is immediately apparent that the velocity always points perpendicular to the position vector. In equation 4.34, any time derivatives of r drop out, as well as $\ddot{\theta}$, leaving us with

$$\boldsymbol{a} = -r\dot{\theta}^2\hat{\boldsymbol{r}}, \tag{4.66}$$

which is the same as above.

Example 4.2. Consider a child on a merry-go-round. If the platform is rotating at 60rpm and the child is holding on, what is the force on the child's arm?

Example 4.3. Consider a conical pendulum. If the bob of mass of 200g on a string of length 50cm is swinging around at a frequency of 1 rotation per second, what is the angle that the pendulum makes with the vertical?

Example 4.4. Consider a car going round a circular bend. Find the maximum velocity that the car can take the bend at without skidding.

Example 4.5. Consider a ball rolling on a circular banked curve. What is the speed required to maintain a constant height on the curve as a function on the banking angle?

This next example uses the concept of energy conservation in combination with the traditional method of comparing forces to solve the problem.

Example 4.6. Consider a mass on a string. The mass starts hanging vertically downwards, then it gets projected sideways at a speed v_0. When the angle between the string and the vertical is 120°, the string becomes slack and the mass falls. Find the initial speed v_0 in terms of the length of the string. First use energy conservation to relate the change in kinetic energy to the change in gravitational potential energy. Then evaluate the forces at the top of the path to get a formula for the final velocity. Finally use the conservation of energy to solve for v_0.

How fast would the mass haves to be projected to get to the top of the loop?

4.5 Rigid Body Rotation

A rigid body is a system of particles which are fixed together such that they move together, no matter the forces applied. We have seen in previous chapters that it is possible to treat rigid bodies as point masses when considering linear motion. However, when considering rotational effects as well, we need to consider the macroscopic extent of an object. Consider a rigid body in motion and look at the total kinetic energy, which is given by the sum of kinetic energy of each constituent particle:

$$K = \sum_i \frac{1}{2} m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i. \tag{4.67}$$

We can split the velocity of each particle into the sum of velocity around the centre of rotation and velocity along the line of motion.

$$K = \sum_i \frac{1}{2} m_i (\boldsymbol{v}_{i,\mathrm{rot}} + \boldsymbol{v}_{i,\mathrm{lin}})^2 \tag{4.68}$$

$$= \sum_i \frac{1}{2} m_i (v_{i,\mathrm{rot}}^2 + v_{i,\mathrm{lin}}^2 + 2\boldsymbol{v}_{i,\mathrm{rot}} \cdot \boldsymbol{v}_{i,\mathrm{lin}}) \tag{4.69}$$

$$= \underbrace{\sum_i \frac{1}{2} m_i v_{i,\mathrm{rot}}^2}_{K_{\mathrm{rotational}}} + \underbrace{\sum_i \frac{1}{2} m_i v_{i,\mathrm{lin}}^2}_{K_{\mathrm{linear}}}. \tag{4.70}$$

The cross-term in the square cancels out (why?). Assuming the rigid body is rotating about a *fixed* axis, all particles rotate in circles with the same angular velocity ω because their distance from the axis of rotation is constant. This means the rotational kinetic energy can be written as

$$K_{\mathrm{rot}} = \sum_i \frac{1}{2} m_i v_{i,\mathrm{rot}}^2 = \sum_i \frac{1}{2} m_i (r_i \omega)^2 \tag{4.71}$$

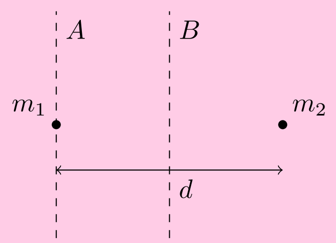
$$= \frac{1}{2} \omega^2 \sum_i m_i r_i^2 \tag{4.72}$$

$$= \frac{1}{2} I \omega^2. \tag{4.73}$$

I is called the **moment of inertia**, and it is kind of an angular equivalent of mass. Notice how it is calculated in a similar way to the centre of mass except with the square of r. Also notice how in the formula for K_{rot}, I and ω play the role of m and v respectively, which shows how they are analogous to the linear quantities. However I, just like the centre of mass, depends on how the mass is distributed in an object. For example, consider a solid disc and a hoop of the same mass. From the formula for I, we can see that since all of the mass in the hoop is concentrated further out, the moment of inertia will be larger than the disc. This means that for the same angular velocity, the rotational kinetic energy of the hoop will be larger than the disc. I also depends on the axis of rotation, since a different axis will have a different mass distribution around it.

Let's do a few examples of calculating moments of inertia.

Example 4.7. Consider a system of two particles with masses m_1 and m_2 separated by a distance d.



What is the moment of inertia about the axis which passes through m_1 (A) and about the axis which passes halfway between m_1 and m_2 (B)?

Lets to the axis which runs through the midpoint first. Using the formula for moment of inertia, we get

$$\begin{aligned} I_B &= m_1 r_1^2 + m_2 r_2^2 && (4.74) \\ &= m_1 \left(\frac{d}{2}\right)^2 + m_2 \left(\frac{d}{2}\right)^2 && (4.75) \\ &= \frac{m_1 + m_2}{4} d^2. && (4.76) \end{aligned}$$

For the moment of inertia about axis A, the distance of m_1 from the axis is zero, so the moment of inertia is

$$I_A = m_2 d^2. \tag{4.77}$$

The key takeaway here is that any mass located at the axis of rotation contributes *nothing* to the moment of inertia, no matter how much it is.

Just like calculating the centre of mass, when we have a continuous body we have to calculate moments of inertia using integrals. The formula is then

$$I = \int r^2 \, \mathrm{d}m, \tag{4.78}$$

where we have to find a way to express the mass element $\mathrm{d}m$ in terms of length, area, or volume elements. We will now find the moments of inertia of common shapes.

Example 4.8. Find the moment of inertia of a thin rod of uniform density rotating about its midpoint.

Let L be the length of the rod and λ be its linear mass density. Then from the diagram, we can see that $\mathrm{d}m = \lambda \, \mathrm{d}x$. For the integral, the limits are $-L/2$ and $L/2$ and $r^2 = x^2$, so we have

$$I = \int_{\frac{L}{2}}^{\frac{L}{2}} x^2 \lambda \, \mathrm{d}x = \left. \frac{\lambda x^3}{3} \right|_{\frac{L}{2}}^{\frac{L}{2}} = \frac{\lambda L^3}{12}. \tag{4.79}$$

If we let M be the total mass then $M = \lambda L$, and we have

$$I = \frac{1}{12} M L^2. \tag{4.80}$$

Example 4.9. Find the moment of inertia of a uniform density thin hoop of radius r in the x-y plane rotating around the z-axis about its centre.

We can use a symmetry argument to solve this. The distance from the axis is constant, so it doesn't change for any mass element and can be taken out of the integral. Thus the moment of inertia just the total mass multiplied by the radius squared.

$$I = M r^2. \tag{4.81}$$

Note that this is the same result as a point mass of mass M revolving around the z-axis at the same radius r.

Example 4.10. Find the moment of inertia of a uniform density thin disc of radius R in the x-y plane rotating about the z-axis about its centre.
Here, we can use the result from the thin hoop. If we split the disc up into thin hoops, then we have $\mathrm{d}I = r^2\,\mathrm{d}m$ and we can integrate this from 0 to R. But first, we need to write the mass element in terms of the radius of the thin hoop. We can write $\mathrm{d}m = \mu\,\mathrm{d}A$ where μ is the surface density of the disc, and the area of a thin hoop is $\mathrm{d}A = 2\pi r\,\mathrm{d}r$. Putting this all together, we get

$$I = \int_0^R 2\pi\mu r^3\,\mathrm{d}r = \left.\frac{2\pi\mu r^4}{4}\right|_0^R = \frac{1}{2}\pi\mu R^4. \tag{4.82}$$

The total mass of the disc is $M = \mu A = \mu\pi R^2$, so this simplifies to

$$I = \frac{1}{2}MR^2. \tag{4.83}$$

So the moment of inertia of a thin disc is half that of a thin hoop with the same mass and radius.

Example 4.11. Find the moment of inertia of a uniform density sphere of radius R rotating about its centre.
Here we will again build upon previous results, this time using the moment of inertia of a thin disc. By aligning the axis of rotation with the z-axis, we can split the sphere into thin discs of different radii and at different heights on the z-axis. Then the total moment of inertia is found by integrating over $\mathrm{d}I = \frac{1}{2}r^2\,\mathrm{d}m$, where r is the radius of the thin disc. $\mathrm{d}m$ is given by $\rho\,\mathrm{d}V$ where ρ is the volume density, and the volume element $\mathrm{d}V$ is the area of the disc multiplied by its height, $\mathrm{d}V = \pi r^2\,\mathrm{d}z$. So far, we have

$$\mathrm{d}I = \frac{1}{2}\rho\pi r^4\,\mathrm{d}z. \tag{4.84}$$

In order to integrate this we need a relation between r, the radius of each disc, and z, its height. This is given by Pythagoras' theorem (see the diagram above) as $r^2 = R^2 - z^2$ so we get

$$\mathrm{d}I = \frac{1}{2}\rho\pi(R^2 - z^2)^2\,\mathrm{d}z. \tag{4.85}$$

We can now tackle the integral:

$$\begin{align}
I &= \int_{-R}^{R} \frac{1}{2}\rho\pi(R^2 - z^2)^2\,\mathrm{d}z \tag{4.86}\\
&= \int_0^R \rho\pi(R^4 + z^4 - 2R^2z^2)\,\mathrm{d}z \tag{4.87}\\
&= \rho\pi\left[R^4 z + \frac{z^5}{z} - \frac{2}{3}R^2z^3\right]_0^R \tag{4.88}\\
&= \frac{8}{15}\rho\pi R^5. \tag{4.89}
\end{align}$$

Note that in the second line we have used symmetry to ignore half of the integral and double the result. The mass of a sphere is $M = \frac{4}{3}\rho\pi R^3$, so we have

$$I = \frac{2}{5}MR^2. \tag{4.90}$$

We will now look at two useful results that help with calculating more moments of inertia.

Theorem 4.1 (Parallel Axis Theorem) *For a system of total mass M and moment of inertia about an axis passing through the centre of mass I_{COM}, the moment of inertia about a parallel axis a distance d away is*

$$I_{\parallel} = I_{COM} + Md^2. \tag{4.91}$$

Proof. Let's put ourselves in the centre of mass frame, align the z-axis with the axis of rotation and the x-axis with the along the perpendicular distance between the two axes. The moment of inertia relative to the z-axis is

$$I_{\mathrm{COM}} = \int (x^2 + y^2)\,\mathrm{d}m, \tag{4.92}$$

and the moment of inertia relative to the other axis is

$$I_{\parallel} = \int [(x-d)^2 + y^2]\,\mathrm{d}m \tag{4.93}$$

$$= \int (x^2 + y^2)\,\mathrm{d}m + d^2 \int \mathrm{d}m + 2d \int x\,\mathrm{d}m \tag{4.94}$$

$$= I_{\mathrm{COM}} + Md^2 + 2d \int x\,\mathrm{d}m. \tag{4.95}$$

The integral in the last term is x coordinate of the centre of mass, which is zero in the centre of mass frame. So the last term disappears, leaving us with the result. ■

We can use this result to calculate moments of inertia of shapes we have seen before about different axes that might be difficult to find via a straightforward approach.

Example 4.12. Consider a uniform density thin disc of radius R in the x-y plane, but instead of rotating around the z-axis about its centre like before, it is rotating about a point on the circumference.

We know that the moment of inertia of the thin disc about the centre is $I_{\mathrm{COM}} = \frac{1}{2}MR^2$, and the distance to the parallel axis is simply the radius R, so applying the parallel axis theorem we get

$$I_{\parallel} = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2. \tag{4.96}$$

Theorem 4.2 (Perpendicular Axis Theorem) *Suppose we have a planar lamina lying in the x-y plane. The moment of inertia about the z axis, I_z, is related to the moments of inertia about the x and y-axes, I_x and I_y, by*

$$I_z = I_x + I_y. \tag{4.97}$$

Proof. The proof is straightforward: manipulating the integral for moment of inertia about the z-axis we get

$$I_z = \int (x^2 + y^2)\,\mathrm{d}m = \int x^2\,\mathrm{d}m + \int y^2\,\mathrm{d}m = I_y + I_x. \tag{4.98}$$

Note that $\int x^2\,\mathrm{d}m = I_y$, not I_x since x^2 in this case denotes the perpendicular distance from the axis of rotation, which must be around the y-axis. ■

Example 4.13. Consider the uniform density thin disc again. What is its moment of inertia about the x and y axes?

We know that $I_z = \frac{1}{2}MR^2$ for the thin disc, and by the perpendicular axis theorem we have $I_z = I_x + I_y$. In this case we can invoke symmetry of the disc to argue that $I_x = I_y$, and therefore $I_z = 2I_x = 2I_y$. Therefore we must have

$$I_x = I_y = \frac{I_z}{2} = \frac{1}{4}MR^2. \tag{4.99}$$

Theorem 4.3 (Stretch Rule) *The moment of inertia of a rigid object is unchanged when the object is stretched parallel to the axis of rotation and the distribution of mass is kept the same (except along the axis of rotation). This allows us to take a planar object and extrude it into 3D along the z-axis while keeping the same mass and moment of inertia.*

Proof. Have an object of cross-sectional area in the x-y plane A and height L. Then the moment of inertia around the z-axis can be expressed as

$$I_z = \int (x^2 + y^2)\rho(x, y, z)\,\mathrm{d}V = \int_0^L \mathrm{d}z \int_A (x^2 + y^2)\rho(x, y, z)\,\mathrm{d}A. \tag{4.100}$$

If we stretch this object along z by a factor of a, then we have to divide the mass density by a and change the upper limit on the z integral to the new height aL. This means the total mass remains unchanged. Then the new moment of inertia is

$$I_z' = \int_0^{aL} \mathrm{d}z \int_A (x^2 + y^2)\frac{\rho(x, y, z/a)}{a}\,\mathrm{d}A. \tag{4.101}$$

Making the substitution $z' = z/a$, we get

$$I'_z = \int_0^L a\,\mathrm{d}z' \int_A (x^2+y^2)\frac{\rho(x,y,z')}{a}\,\mathrm{d}A \tag{4.102}$$

$$= \int_0^L \mathrm{d}z' \int_A (x^2+y^2)\rho(x,y,z')\,\mathrm{d}A \tag{4.103}$$

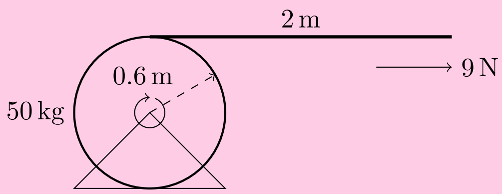
$$= I_z. \tag{4.104}$$

■

Using the stretch rule, we can say that the moment of inertia of a cylinder rotating about its main axis is the same as a thin disc.

Let's now look at some problems which combine all the ideas of this chapter.

Example 4.14. A light inextensible cable is being pulled out from a reel with a force of 9 N. The reel weighs 50 kg and has a radius of 0.6 m. After 2 m of the cable have been pulled out, what it its velocity?



To solve this, we will look at the rotational kinetic energy of the pulley before and after and then invoke the work-energy theorem. Its initial kinetic energy is zero, and its final kinetic energy is its rotational kinetic energy $K_{\text{rot}} = \frac{1}{2}I\omega^2$. By the work-energy theorem, the final kinetic energy is equal to the work done, which is the length of rope pulled out multiplied by the tension $W = Ts$. Putting these two equations together and rearranging for ω, we get

$$\omega = \sqrt{\frac{2Ts}{I}}. \tag{4.105}$$

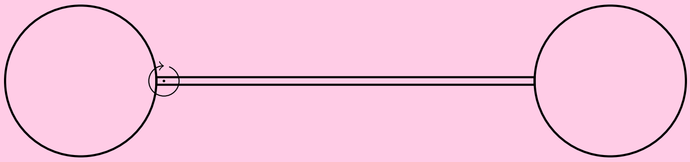
By the stretch rule, the moment of inertia of the reel is the same as a thin disc, namely $I = \frac{1}{2}MR^2$, so we get

$$\omega = \sqrt{\frac{4Ts}{MR^2}} = \sqrt{\frac{4\cdot 9\,\mathrm{N}\cdot 2\,\mathrm{m}}{50\,\mathrm{kg}\cdot(0.6\,\mathrm{m})^2}} = 2\,\mathrm{rad\,s^{-1}}. \tag{4.106}$$

Now that we have the final angular velocity of the reel, the speed of the cable is just the tangential speed of the reel at its edge, which is

$$v_{\text{cable}} = v_{\text{reel}} = R\omega = 0.6\,\mathrm{m}\cdot 2\,\mathrm{rad\,s^{-1}} = 1.2\,\mathrm{m\,s^{-1}}. \tag{4.107}$$

Example 4.15. Suppose a dumbbell-shaped pendulum is allowed to fall from horizontal. The weights at each end are spherical with radius $r = 10\,\mathrm{cm}$ and they weigh $m_{\mathrm{weight}} = 10\,\mathrm{kg}$. The bar is $l = 60\,\mathrm{cm}$ long and weighs $m_{\mathrm{bar}} = 1\,\mathrm{kg}$. The pivot point is at the left end of the bar. What is angular velocity of the dumbbell when it is vertical?



The first step to solving this problem is working out the moment of inertia of the dumbbell about the pivot point. If we split the dumbbell into the two weights (A for the one above the pivot and B for the one below) and the bar, the total moment of inertia can be written as

$$I = I_A + I_{\mathrm{bar}} + I_B. \tag{4.108}$$

Using the parallel axis theorem, the moment of inertia of the bar is $I_{\mathrm{bar}} = \frac{1}{12}m_{\mathrm{bar}}l^2 + m_{\mathrm{bar}}\left(\frac{1}{2}l\right)^2 = \frac{1}{3}m_{\mathrm{bar}}l^2$. For the weights, we can again use the parallel axis theorem since we know that the moment of inertia of a sphere is $\frac{2}{5}mr^2$. Hence we get

$$\begin{aligned} I &= \left(\frac{2}{5}m_{\mathrm{weight}}r^2 + m_{\mathrm{weight}}r^2\right) + \frac{1}{3}m_{\mathrm{bar}}l^2 + \left(\frac{2}{5}m_{\mathrm{weight}}r^2 + m_{\mathrm{weight}}(l+r)^2\right) && (4.109)\\ &= \frac{9}{5}\cdot 10\,\mathrm{kg}\cdot(0.1\,\mathrm{m})^2 + \frac{1}{3}\cdot 1\,\mathrm{kg}\cdot(0.6\,\mathrm{m})^2 + 10\,\mathrm{kg}\cdot(0.7\,\mathrm{m})^2 && (4.110)\\ &= 5.2\,\mathrm{kg\,m^2}. && (4.111)\end{aligned}$$

Now, to use energy conservation we also need to know the potential energy difference. To do this we need to know where the centre of mass is, since the potential energy difference is just $-Mgh$, where M is the total mass and h is the distance that the centre of mass dropped. By symmetry, the centre of mass is clearly just the centre of the dumbbell which is a distance of $30\,\mathrm{cm}$ from the pivot. Thus the height by which the centre of mass has dropped when it reaches the bottom is $30\,\mathrm{cm}$. Setting the zero point of the potential energy to be the initial height, the total energy before is zero since the dumbbell is not moving. The final energy is

$$E_f = \frac{1}{2}I\omega^2 - Mgh = 0. \tag{4.112}$$

Rearranging for ω, we get

$$\begin{aligned} \omega &= \sqrt{\frac{2Mgh}{I}} && (4.113)\\ &= \sqrt{\frac{2\cdot 21\,\mathrm{kg}\cdot 9.8\,\mathrm{m\,s^{-2}}\cdot 0.3\,\mathrm{m}}{5.2\,\mathrm{kg\,m^2}}} && (4.114)\\ &= 4.87\,\mathrm{rad\,s^{-1}}. && (4.115)\end{aligned}$$

Chapter 5

Angular Momentum

5.1 Moments

When dealing with problems involving rotation, we have mechanical quantities which are analogous to ones used to solve linear problems. We have seen a few of these before: angular velocity $\omega = v_\theta/r$, angular acceleration $\alpha = a_\theta/r$ (for circular motion), and moment of inertia $I = mr^2$. As we can see, the definitions of these quantities all involve the analogous linear quantity and the distance from the origin r. Quantities that involve the *product* of a linear quantity with the radius are called **moments**. Moments we have seen already are the center of mass, which is the first moment of mass (sum of mr) normalised by the total mass, and the moment of inertia, which is the second moment of mass (mass multiplied by r^2) In this chapter we will introduce two more moments: torque — the moment of force — and angular momentum — the moment of momentum. When vector quantities like force and momentum are involved, moments are defined using the cross product.

Definition 5.1. **Angular Momentum** is defined as the first moment of momentum.

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times \boldsymbol{v}. \tag{5.1}$$

Note that because the cross product is antisymmetric, the order of $\boldsymbol{r}$ and $\boldsymbol{p}$ matters! The motivation for introducing angular momentum will become clear in the next section.

5.2 Central Forces

We will now examine a very special type of force. These forces have many useful properties which make solving problems involving them a lot simpler.

Definition 5.2. A **central force** is a force which acts along the radial direction and only depends on the radial distance r. Central forces thus have the form

$$\boldsymbol{F} = F(r)\hat{\boldsymbol{r}}. \tag{5.2}$$

If $F(r) < 0$, the force is attractive, and if $F(r) > 0$, the force is repulsive.

We have seen one example of a central force already, the universal law of gravitation. Another example is Coulomb's law.

Motion under a central force obeys the following rules:

- The motion is confined to a plane.
- The angular momentum is conserved ($\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = 0$).
- The position vector sweeps out equal area in equal time.

Let's prove that angular momentum is conserved under the influence of a central force. Looking at the derivative, we get

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times \boldsymbol{p}) && (5.3)\\
&= \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} && (5.4)\\
&= \boldsymbol{v} \times m\boldsymbol{v} + \boldsymbol{r} \times F(R)\hat{\boldsymbol{R}} && (5.5)\\
&= 0. && (5.6)
\end{aligned}$$

Therefore, $\boldsymbol{L}$ is constant over time, it is conserved.

By definition, $\boldsymbol{L}$ is perpendicular to $\boldsymbol{r}$. We can also see this by taking the dot product of $\boldsymbol{L}$ with the position vector:

$$\boldsymbol{r} \cdot \boldsymbol{L} = \boldsymbol{r} \cdot (\boldsymbol{r} \times \boldsymbol{p}) = \boldsymbol{p} \cdot (\boldsymbol{r} \times \boldsymbol{r}) = 0. \tag{5.7}$$

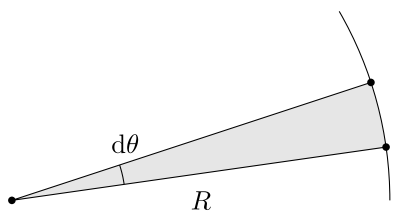
Since $\boldsymbol{L}$ is constant, this implies that $\boldsymbol{r}$ is confined to the plane perpendicular to $\boldsymbol{L}$.

Because of this, it is convenient to use cylindrical coordinates with the plane $z = 0$ as the plane of motion. Using equation **??** (note that $\dot{z} = 0$) and the cross products between unit vectors, we get

$$\begin{aligned}
\boldsymbol{L} &= m\boldsymbol{r} \times \boldsymbol{v} && (5.8)\\
&= mR\hat{\boldsymbol{R}} \times (\dot{R}\hat{\boldsymbol{R}} + R\dot{\theta}\hat{\boldsymbol{\theta}}) && (5.9)\\
&= mR^2\dot{\theta}\hat{\boldsymbol{k}} && (5.10)\\
&= mh\hat{\boldsymbol{k}}. && (5.11)
\end{aligned}$$

In the last line we have defined $h = R^2\dot{\theta}$. This is the **specific angular momentum** (angular momentum normalised by mass), and is useful in some problems.

Finally, we will prove the last fact about motion under central forces, the law of equal areas, which is a more general version of Kepler's second law of planetary motion. Consider an object on a curved path at two points separated by a small time interval $\mathrm{d}t$.



The shaded arc is approximately a triangle, so the area get closer and closer to $A = \frac{1}{2}R^2\,\mathrm{d}\theta$ as $\mathrm{d}t \to 0$. Therefore, we have

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}R^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}h = \text{const.} \tag{5.12}$$

So the rate of area swept out over time is constant.

What does motion under a central force look like? By Newton's second law, we know that

$$a_R = \ddot{R} - R\dot{\theta}^2 = \frac{F(R)}{m}. \tag{5.13}$$

Substituting in the specific angular momentum, we get

$$\ddot{R} - \frac{h^2}{R^3} = \frac{F(R)}{m}. \tag{5.14}$$

This is the differential equation we need to solve for any central force problem.

Example 5.1. Prove Kepler's third law: the square of a planet's orbital period is proportional to the cube of the semi-major axis of its orbit.

Planets move according to the universal law of gravitation, which is

$$\boldsymbol{F} = -\frac{GMm}{R^2}\hat{\boldsymbol{R}}. \tag{5.15}$$

We will make an approximation and assume that the sun is fixed at the origin and the planets orbit in circular orbits. In reality, the a given planet and the sun orbit their common centre of mass in elliptical paths, but the difference is very small making this approximation makes the problem much simpler. Since $R = \text{const}$, then $\dot{R} = \ddot{R} = 0$. So $a_R = -R\dot{\theta}^2$. Since angular momentum $L = mR^2\dot{\theta}$ is conserved, $\dot{\theta} = \omega$ is a constant. Thus the equation of motion 5.13 becomes

$$-\frac{GMm}{R^2} = -mR\omega^2 \tag{5.16}$$

$$GM = R^3\omega^2. \tag{5.17}$$

The period of the orbit is equal to the displacement divided by the speed, in this case $T = \frac{2\pi}{\omega}$, so rearranging for R^3 we get

$$R^3 = \frac{GM}{4\pi^2}T^2. \tag{5.18}$$

Example 5.2. Consider a central force of the form $\boldsymbol{F} = -\frac{mk^2}{r^3}\hat{\boldsymbol{r}}$. If a particle under the influence of this force starts at a distance $r = r_0$ from the origin with no radial speed ($v_{r,0} = 0$), what will its motion look like for different values of k?

Substituting in the force into equation 5.13, we get

$$\ddot{r} - \frac{h}{r^3} = -\frac{k^2}{r^3} \tag{5.19}$$

$$\ddot{r} + \frac{k^2 - h^2}{r^3} = 0. \tag{5.20}$$

For $k^2 = h^2$, we get $\ddot{r} = 0$, which can be integrated twice to give $r = r_0 + v_{r,0}t$. Since $v_{r,0} = 0$, we have $r = r_0$ which is circular motion. The angular speed is given by equation 5.10, $\omega = \frac{L}{mr_0^2} = \frac{mvr_0}{mr_0^2} = \frac{v}{r_0}$. This is just our familiar result from uniform circular motion, which makes sense because if we plug $k^2 = h^2$ into the form of the force above, we get

$$\boldsymbol{F} = -\frac{mr^4\omega^2}{r^3}\hat{\boldsymbol{r}} = -mr\omega^2\hat{\boldsymbol{r}}, \tag{5.21}$$

which is just the formula for centripetal force.

What about the cases where $k^2 \neq h^2$? Unfortunately, the differential equation is not solvable analytically for any other case. Luckily, there is another way to solve it! What we have to do is parameterise the orbit in terms of the angular displacement θ instead of t by making use of the relation $h = r^2\dot{\theta}$. We will also make the substition $u = 1/r$, which is a very useful substition when solving problems with central forces of the form $F(r) = \frac{\alpha}{r^n}$. Therefore, the aim is to replace all time derivatives of r with derivatives of u with respect to θ. Using the chain rule, we get

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}u}\dot{u} = \frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\theta}\dot{\theta} \tag{5.22}$$

$$= -r^2\dot{\theta}\frac{\mathrm{d}u}{\mathrm{d}\theta} \tag{5.23}$$

$$= -h\frac{\mathrm{d}u}{\mathrm{d}\theta}. \tag{5.24}$$

For $\ddot{r}$, we get

$$\ddot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-h\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = -h\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\dot{\theta} \tag{5.25}$$

$$= -\frac{h^2}{r^2}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} \tag{5.26}$$

$$= -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}. \tag{5.27}$$

Now, the equation of motion becomes

$$-h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + (k^2 - h^2)u^3 = 0 \tag{5.28}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - \frac{k^2 - h^2}{h^2}u = 0. \tag{5.29}$$

Letting $\beta^2 = |\frac{k^2 - h^2}{h^2}|$, if $k^2 > h^2$ then the differential equation becomes

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - \beta^2u = 0. \tag{5.30}$$

The general solution is $u = Ae^{\beta\theta} + Be^{-\beta\theta}$.

For the opposite case $k^2 < h^2$, the negative sign gets cancelled out and we get

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + \beta^2 u = 0. \tag{5.31}$$

This has the general solution $u = A\cos(\beta\theta) + B\sin(\beta\theta)$.

5.3 Torque

Torque is defined as the **moment** of force, that is, the product of the distance from a reference point and the force. In terms of vectors, this is given by the cross product

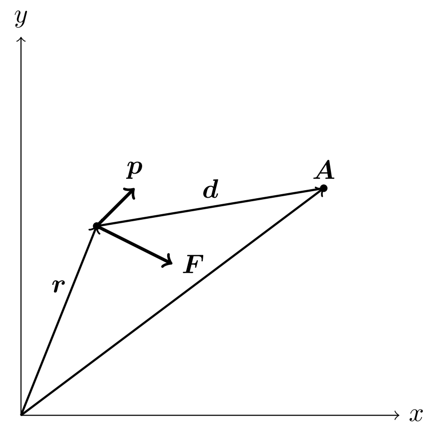
$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}. \tag{5.32}$$

The magnitude of $\boldsymbol{\tau}$ is given by

$$\begin{aligned} \tau &= |\boldsymbol{r}||\boldsymbol{F}|\sin\theta && (5.33) \\ &= rF_{\mathrm{tan}}. && (5.34) \end{aligned}$$

This formula tells us that for a fixed radius, the torque has maximum magnitude when $\theta = \pm\frac{\pi}{2}$ i.e. the force acts *perpendicular* to the radius. It also tells us that if the force acts along the same line as the radius ($\theta = 0$ or $\theta = \pi$), then the torque is equal to 0. Since torque is defined as a moment like angular momentum, its value is relative to an arbitrary reference point. Changing the reference point will change the value of the torque. For a fixed reference point, if the same force is applied further away from the centre, the torque will be greater.

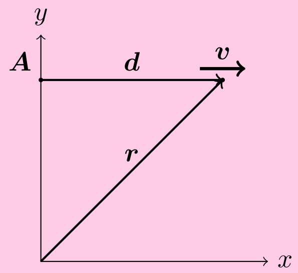
Let's look at how changing the reference point changes the angular momentum and the torque. Consider the following diagram.



The angular momentum and torque relative to the origin are $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ and $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$ like before, but what are the angular momentum and torque relative to the point $\boldsymbol{A}$? The radius vector or lever arm is now the vector $\boldsymbol{d}$ between the points $\boldsymbol{A}$ and $\boldsymbol{r}$, i.e.

$$\boldsymbol{L_A} = \boldsymbol{d} \times \boldsymbol{p} \tag{5.35}$$
$$= (\boldsymbol{r} - \boldsymbol{A}) \times \boldsymbol{p} \tag{5.36}$$
$$\boldsymbol{\tau_A} = \boldsymbol{d} \times \boldsymbol{F} \tag{5.37}$$
$$= (\boldsymbol{r} - \boldsymbol{A}) \times \boldsymbol{F}. \tag{5.38}$$

Example 5.3. Consider a 2 kg particle at $y = 3\,\mathrm{m}$ moving to the right with velocity $\boldsymbol{v} = 5\,\mathrm{m\,s^{-1}}\hat{\boldsymbol{\imath}}$. What is its angular momentum relative to the origin and relative to the point $\boldsymbol{A} = 3\,\mathrm{m}\hat{\boldsymbol{\jmath}}$.



The position vector is $\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + 3\,\mathrm{m}\hat{\boldsymbol{\jmath}}$ and the radius vector relative to $\boldsymbol{A}$ is $\boldsymbol{d} = x\hat{\boldsymbol{\imath}}$. The momentum vector is $\boldsymbol{p} = mv_x\hat{\boldsymbol{\imath}} = 10\,\mathrm{kg\,m\,s^{-1}}$, so the angular momentum relative to the origin is

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ x & 3 & 0 \\ 10 & 0 & 0 \end{vmatrix} \tag{5.39}$$
$$= -30\,\mathrm{kg\,m^2\,s^{-1}}\hat{\boldsymbol{k}}, \tag{5.40}$$

at the angular momentum relative to $\boldsymbol{A}$ is

$$\boldsymbol{L_A} = \boldsymbol{d} \times \boldsymbol{p} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ x & 0 & 0 \\ 10 & 0 & 0 \end{vmatrix} \tag{5.41}$$
$$= 0. \tag{5.42}$$

This makes sense because relative to $\boldsymbol{A}$ the motion is radial so there is no lever arm.

Example 5.4. Let's now look at a familiar example, uniform circular motion. First, what is the angular momentum relative to the origin for a particle travelling with angular speed ω?

Working in polar coordinates, the radius vector is $\boldsymbol{r} = r\hat{\boldsymbol{r}}$ and the velocity vector is $\boldsymbol{v} = r\omega\hat{\boldsymbol{\theta}}$. Then the angular momentum is $\boldsymbol{L} = r\hat{\boldsymbol{r}} \times mr\omega\hat{\boldsymbol{\theta}} = mr^2\omega\hat{\boldsymbol{k}}$. This is a constant, which we know to be the case since the centripetal force is a central force.

Now, suppose the particle is moving around a circle at $y = h$. What is the angular momentum relative to the origin?

The position vector is now given by $\boldsymbol{r} = r\hat{\boldsymbol{r}} + h\hat{\boldsymbol{k}}$, so the angular momentum becomes

$$\boldsymbol{L} = m\begin{vmatrix}\hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{k}} \\ r & 0 & h \\ 0 & r\omega & 0\end{vmatrix} \tag{5.43}$$

$$= -mhr\omega\hat{\boldsymbol{r}} + mr^2\omega\hat{\boldsymbol{k}}. \tag{5.44}$$

The unit vector $\hat{\boldsymbol{r}}$ changes over time, so $\boldsymbol{L}$ is not constant (its magnitude is constant but its direction changes). This is because the centripetal force is no longer central relative to the origin.

Torque is measured in N m. This is dimensionally equivalent to Joules, but this does not mean that torque is a kind of energy. Energy is a scalar quantity, whereas torque is a vector, so they are really different things.

In linear dynamics, we have Newton's second law which states $\boldsymbol{F} = \dot{\boldsymbol{p}}$. We will now show that there is a rotational equivalent of Newton II that relates torque to angular momentum. Using the product rule to differentiate $\boldsymbol{L}$, we find

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times \boldsymbol{p}) \tag{5.45}$$

$$= (\dot{\boldsymbol{r}} \times \boldsymbol{p}) + (\boldsymbol{r} \times \dot{\boldsymbol{p}}) \tag{5.46}$$

$$= \left(\frac{1}{m}\boldsymbol{p} \times \boldsymbol{p}\right) + (\boldsymbol{r} \times \boldsymbol{F}) \tag{5.47}$$

$$= \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{\tau}. \tag{5.48}$$

This is the same calculation we did earlier for central forces. We now have a shortcut to prove that angular momentum is conserved under the influence of a central force. Since central forces are always parallel to the radius vector, the torque is zero.

$$\boldsymbol{\tau} = \boldsymbol{r} \times F(r)\hat{\boldsymbol{r}} = 0. \tag{5.49}$$

Then using the angular equivalent of Newton II, $\dot{\boldsymbol{L}} = \boldsymbol{\tau} = 0$. So central forces exert no torque and therefore angular momentum is conserved.

5.4 Torque and Angular Momentum for Systems of Particles

In a system of many particles, each particle with a position $\boldsymbol{r}_i$ and momentum $\boldsymbol{p}_i$, the total angular momentum is the sum of each particle's angular momentum.

$$\boldsymbol{L} = \sum_i \boldsymbol{L}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i. \tag{5.50}$$

If the particles are subject to some torques, their angular momenta will change. The total angular momentum changes as

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}\boldsymbol{L}_i}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_i \times \boldsymbol{p}_i) = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = \sum_i \boldsymbol{\tau}_i. \tag{5.51}$$

Let's look at the individual torques $\boldsymbol{\tau}_i$ more closely. The forces on each particle can be split into internal and external forces, $\boldsymbol{F}_i = \boldsymbol{F}_{i,\text{int}} + \boldsymbol{F}_{i,\text{ext}}$. Internal forces are those from other particles, so $\boldsymbol{F}_{i,\text{int}}$ can be written as

$$\boldsymbol{F}_{i,\text{int}} = \sum_{j\neq i} \boldsymbol{F}_{j\to i}. \tag{5.52}$$

So the torques become

$$\sum_i \boldsymbol{\tau}_i = \sum_i \boldsymbol{r}_i \times \left(\sum_{j\neq i} \boldsymbol{F}_{j\to i} + \boldsymbol{F}_{i,\text{ext}} \right) \tag{5.53}$$

$$= \sum_i \sum_{j\neq i} \boldsymbol{r}_i \times \boldsymbol{F}_{j\to i} + \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_{i,\text{ext}}. \tag{5.54}$$

Looking at the first term more closely, we can use a similar argument to back in chapter 2 to show that it is zero. The sum will pairs of terms of the form $\boldsymbol{r}_i \times \boldsymbol{F}_{j\to i} + \boldsymbol{r}_j \times \boldsymbol{F}_{i\to j}$. By Newton III, these two forces are equal and opposite, $\boldsymbol{F}_{j\to i} = -\boldsymbol{F}_{i\to j}$, so the pair of terms becomes

$$\boldsymbol{r}_i \times \boldsymbol{F}_{j\to i} + \boldsymbol{r}_j \times \boldsymbol{F}_{i\to j} = \boldsymbol{r}_i \times \boldsymbol{F}_{j\to i} - \boldsymbol{r}_j \times \boldsymbol{F}_{j\to i} \tag{5.55}$$

$$= (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{j\to i}. \tag{5.56}$$

The vector $\boldsymbol{r}_i - \boldsymbol{r}_j$ is a vector which points from particle j to particle i, which is parallel to the force $\boldsymbol{F}_{j\to i}$, therefore the whole sum is zero. Therefore the contribution to the torques comes *only* from the external forces. Internal forces do *not* contribute to the change in angular momentum.

$$\dot{\boldsymbol{L}} = \sum_i \boldsymbol{\tau}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_{i,\text{ext}}. \tag{5.57}$$

Is $\dot{\boldsymbol{L}} = \tau_{\text{ext}}$? Only in two situations: when we consider the torque about the origin of an inertial frame of reference, and when we consider the torque about the centre of mass (even if the centre of mass frame is *not* inertial). This is very important because it allows us to use the angular equivalent of Newton II in many situations. For example, if we are analysing a rigid body rolling down a hill, we can measure the angular momentum and torque from the centre of mass then we can still use $\dot{\boldsymbol{L}} = \boldsymbol{\tau}$.

5.5 Dynamics of Rigid Bodies

As stated in chapter 4, rigid bodies are distinguished by the fact that all mass elements have the same angular velocity ω. Each mass element is therefore moving in circles with speed $v_i = R_i\omega$, where R_i is the distance from the axis of rotation. The smart choice for a coordinate system would be to align the z-axis with the axis of rotation and use cylindrical coordinates (R, ϕ, z), but let's look at the angular momentum in cartesian coordinates. The velocity of a mass element (assuming that the rigid body has no linear motion) is $\boldsymbol{v}_i = v_{x,i}\hat{\boldsymbol{\imath}} + v_{y,i}\hat{\boldsymbol{\jmath}} = -\omega y_i\hat{\boldsymbol{\imath}} + \omega x_i\hat{\boldsymbol{\jmath}}$. Then the total angular momentum is

$$\boldsymbol{L} = \sum_i \boldsymbol{L}_i = \sum_i \boldsymbol{r}_i \times m_i\boldsymbol{v_i} \tag{5.58}$$

$$= \sum_i m_i \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ x_i & y_i & z_i \\ -\omega y_i & \omega x_i & 0 \end{vmatrix} \tag{5.59}$$

$$= \sum_i m_i x_i z_i \omega\hat{\boldsymbol{\imath}} + \sum_i m_i y_i z_i \omega\hat{\boldsymbol{\jmath}} + \sum_i m_i(x_i^2 + y_i^2)\omega\hat{\boldsymbol{k}}. \tag{5.60}$$

The last term can be simplified to $\sum_i m_i R_i^2 \omega\hat{\boldsymbol{k}} = I_z\omega\hat{\boldsymbol{k}}$ where I_z is the moment of inertia about the z axis. The first two terms are the **products of inertia** multiplied by ω. In the case where the products of inertia are zero, we have

$$\boldsymbol{L} = I_z\omega\hat{\boldsymbol{k}}, \tag{5.61}$$

and the object is rotating around a **principle axis**.

Assuming this angular momentum is measured from the centre of mass of the rigid body, we can use $\boldsymbol{\tau} = \dot{\boldsymbol{L}}$ to get

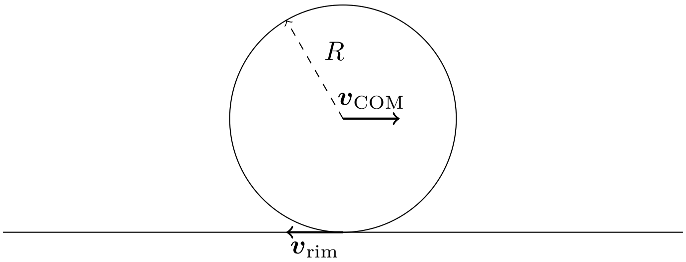
$$\boldsymbol{\tau} = \dot{\boldsymbol{L}} = I\dot{\omega}\hat{\boldsymbol{k}} = I\alpha\hat{\boldsymbol{k}}. \tag{5.62}$$

This is an angular analogue of Newton II where moment of inertia takes the place of mass. This derivation has also implicitly assumed that the torque is being applied around the axis of rotation and hence the direction of $\boldsymbol{L}$ does not change. A torque in another direction would change the direction of angular momentum, which leads to *precession* In the case of constant torque around the axis of rotation, the equation above implies that α is constant, which means we can use the angular SUVAT equations from chapter 4 to describe the motion.

Example 5.5. Consider two connected masses on a massless pulley with $m_1 > m_2$. Suppose the system starts from rest and assume the string is massless, inextensible, and lies vertically. Find an expression for the magnitude of acceleration of the masses. To do this, we have to analyse the forces acting on the blocks and also the torques acting on the pulley.

5.6 Rolling Motion

One of the most familiar examples of torques in everyday life are rolling objects. Rolling motion is a superposition of a rotation about the centre of mass and a translation. In this section we will study **rolling without slipping**, which is the easiest type of rolling motion to analyse. We will also only look at circular rigid objects for now (wheels, balls, cylinders).



In this case, we have that the contact point with the ground has *zero* instantaneous velocity. For a circular rolling object, we can write

$$v_{\text{COM}} + v_{\text{rim}} = v_{\text{COM}} - R\omega = 0. \tag{5.63}$$

This implies that

$$v_{\text{COM}} = R\omega, \tag{5.64}$$

which makes sense when we realise that for a circle that is rolling without slipping, its total distance travelled is $s = R\theta$ so $v_{\text{COM}} = \dot{s} = R\omega$.

For a disc (or a cylinder), its moment of inertia while rolling is $\frac{1}{2}mR^2$. Then its total kinetic energy is

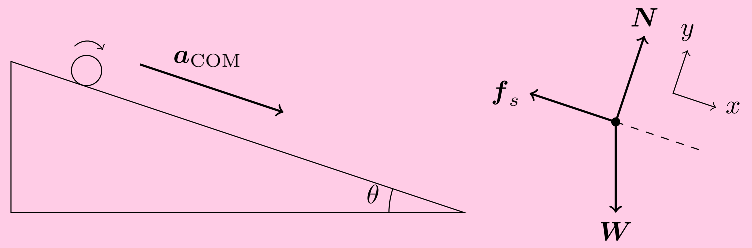
$$\begin{aligned} K &= \frac{1}{2}mv_{\text{COM}}^2 + \frac{1}{2}I\omega^2 && (5.65)\\ &= \frac{1}{2}mv_{\text{COM}}^2 + \frac{1}{4}(mR^2)\left(\frac{v_{\text{COM}}}{R}\right)^2 && (5.66)\\ &= \frac{1}{2}mv_{\text{COM}}^2 + \frac{1}{4}mv_{\text{COM}}^2 = \frac{3}{4}mv_{\text{COM}}^2. && (5.67) \end{aligned}$$

So the rotational energy is half of its translational energy, or to put it another way a third of its kinetic energy is stored in the rotational motion.

Let's look at the forces and torques involved in rolling. On a flat surface, if we have $a_{\text{COM}} = 0$ then $\alpha = a/R = 0$. This implies that the torque about the centre of mass is zero which means no force is acting on the rim of the object. All this is to say that there is no tendency to slide. If $a_{\text{COM}} \neq 0$, then there is a torque about the centre of mass and therefore there is a force. This must be a friction force, specifically static friction in the case of rolling without slipping. Thus we cannot have rolling without friction!

Which way does the static friction point? We are used to having static friction point to the left if the object is moving to the right, but there is a subtle gotcha here. In the case of rolling motion, the contact point wants to slip to the left, which means the static friction force must point to the right. On an incline, the weight does not exert a torque because it acts on the centre of mass, and the normal force does not exert a torque because it acts parallel to the radius vector. The only force which exerts a torque is the static friction, so $\tau_{f_s} = I\alpha$.

Example 5.6. A disc is rolling down an inclined plane. Find an expression for its acceleration.



Looking at the forces in the x direction, we get

$$\sum F_x = ma_{\text{COM}} = |\boldsymbol{W}|\sin\theta - |\boldsymbol{f}_s| \tag{5.68}$$

$$= mg\sin\theta - f_s. \tag{5.69}$$

Now we need to find an expression for the magnitude of static friction f_s, which we can do by analysing the torques. Since the only force which exerts a torque is static friction, we have $\tau_{f_s} = Rf_s = I\alpha$. So using the moment of inertia for a disc $\frac{1}{2}mR^2$ and substituting $\alpha = a_{\text{COM}}/R$, we get

$$f_s = \frac{Ia_{\text{COM}}}{R^2} = \frac{1}{2}ma_{\text{COM}}. \tag{5.70}$$

Substituting this into the equation above, we get

$$ma_{\text{COM}} = mg\sin\theta - \frac{1}{2}ma_{\text{COM}} \tag{5.71}$$

$$\implies a_{\text{COM}} = \frac{2}{3}g\sin\theta. \tag{5.72}$$

This result is somewhat interesting, the acceleration does not depend on the mass or radius of the disc, only the angle of the plane. However, the acceleration does depend on the distribution of mass within the rolling object, i.e. the moment of inertia. If we had left the moment of inertia unspecified, we would get

$$a_{\text{COM}} = \frac{mg\sin\theta}{\frac{I}{R^2} + m}. \tag{5.73}$$

This is valid for all circular rolling objects. As we have seen, the moment of inertia for most circular objects is proportional to mR^2, leading to a simplification with a different constant multiplying $g\sin\theta$.

5.7 Static Equilibrium

In the net force and the net torque on a rigid body are both 0, then it is in **static equilibrium**.

$$\sum_i \boldsymbol{F}_i = 0 \quad \text{and} \quad \sum_i \boldsymbol{\tau}_i = 0. \tag{5.74}$$

Note that we are free to choose any point as the origin to make finding the net torqe easier.

Example 5.7. Consider a beam of mass 10kg and length 4m. It sits on a fulcrum placed 1m from one end of the beam, and is supported from the other end by a string. Find the tension in the string and the force of the beam on the fulcrum.

Example 5.8. A ladder weighing 10kg rests on a smooth wall. Find the the static friction force between the floor and the ladder.

Example 5.9. Consider a sign hanging from a bar attached to a wall supported by a string. Find the force between the bar and the wall.

Chapter 6

Oscillations

6.1 Simple Harmonic Motion

Oscillations are periodic variations of a quantity about some equilibrium. This could be an object moving back and forth, a voltage going up and down, or any other physical variable. Let's look at a system we have seen before, a mass on a horizontal spring. We are going to study the motion of the mass in much more detail than earlier. The only force acting on the mass is the spring restoring force, which is given by Hooke's law:

$$\boldsymbol{F} = -kx\hat{\boldsymbol{\imath}}. \tag{6.1}$$

By Newton II, we can write

$$a_x = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{k}{m}x. \tag{6.2}$$

The general solution to this differential equation is

$$x(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t), \tag{6.3}$$

where

$$\omega = \sqrt{\frac{k}{m}}. \tag{6.4}$$

Systems where displacement is a sinusoidal function of time are said to exhibit **simple harmonic motion**. This motion is characteristic of any system where the force is oppositely proportional to displacement. We will see more examples of this later.

We can simplify the equation for $x(t)$ using a trigonometric addition formula. If we let $A_1 = x_m \cos\phi$, $A_2 = x_m \sin\phi$, where ϕ is some angle. Then we have

$$\begin{aligned} x(t) &= x_m \cos\phi \cos(\omega t) + x_m \sin\phi \sin(\omega t) \qquad (6.5)\\ &= x_m \cos(\omega t + \phi). \qquad (6.6) \end{aligned}$$

We still have two constants x_m and ϕ rather than A_1 and A_2 to specify the particular solution. Since cosine has a maximum value of 1, x_m must represent the maximum amplitude of the oscillation. ϕ is the **initial phase** of the oscillation, i.e. the point on the cosine curve where the mass is at $t = 0$. Calculating the derivatives of $x(t)$, we find

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = -x_m\omega\sin(\omega t + \phi) \tag{6.7}$$

$$a(t) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x_m\omega^2\cos(\omega t + \phi) = -\omega^2 x(t). \tag{6.8}$$

Thankfully we have recovered the equation of motion (**??**) for the acceleration. The two constants can be found using two initial conditions, usually $x(0)$ and $v(0)$. For example, let's suppose $x(0) = 0$ and $v(0) = v_0$, then we have

$$x(0) = x_m\cos\phi = 0 \implies \phi = \pm\frac{\pi}{2} \tag{6.9}$$

$$v(0) = -x_m\omega\sin\phi = v_0 \implies \phi < 0, \tag{6.10}$$

so $x_m = \frac{v_0}{\omega}$, $\phi = \frac{\pi}{2}$.

ω has units of $\mathrm{rad\,s^{-1}}$ (the same as angular velocity!), and we call it the **natural** or **resonant (angular) frequency**. Let's work out the period of oscillation. It should be equal to the time taken for the phase to change by 2π.

$$\omega t + \phi = \omega(t + T) + \phi \tag{6.11}$$

$$\implies \omega T = 2\pi \tag{6.12}$$

$$T = \frac{2\pi}{\omega}. \tag{6.13}$$

Linear frequency f is a more familiar unit, being measured in cycles per second or Hz. It is related to the period by $f = 1/T$, and therefore to angular frequency by

$$\omega = 2\pi f. \tag{6.14}$$

Let's look at a pendulum on a string of length L now. We are assuming that the string is massless and ignoring the effects of air resistance. Given that the arc length of the pendulum is related to the angular displacement by $s = L\theta$ and that the restoring force is $F = -mg\sin(\theta)$, we have the equation of motion:

$$F = m\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = -mg\sin(\theta). \tag{6.15}$$

This is a nonlinear differential equation. We will simplify this by assuming that the angle θ is small so $\sin(\theta) \approx \theta$ (small-angle approximation). Then the equation of motion becomes

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = -g\theta = -\frac{g}{L}s. \tag{6.16}$$

Note that this has the same form as equation **??**, so the trajectory will have the same form! What we have effectively done is assume that the displacement s and the restoring force F are pointing in a straight line, not along the arc of a circle, so that the force becomes identical to Hooke's law.

$$s(t) = s_m\cos(\omega t + \phi), \tag{6.17}$$

where ω in this case is given by

$$\omega = \sqrt{\frac{g}{L}}. \tag{6.18}$$

Example 6.1. As an exercise, let's rederive this result by looking at the torques on the pendulum rather than the forces.

The pendulum, being a point mass, has a moment of inertia $I = mL^2$. Then the torque on the pendulum is

$$\tau = rF \sin\theta = -mgL\sin\theta \approx -mgL\theta. \tag{6.19}$$

Using $\tau = I\alpha$, we get

$$\alpha = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{mgL}{I}\theta \tag{6.20}$$
$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{mgL}{mL^2}\theta \tag{6.21}$$
$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{L}\theta. \tag{6.22}$$

Solving this, we get

$$\theta(t) = \theta_m \cos(\omega t + \phi), \tag{6.23}$$

which is the same solution as before where $s_m = L\theta_m$ and $\omega = \sqrt{\frac{g}{L}}$. Note that we have to be quite careful here, as the angular frequency ω is *not* the angular velocity $\dot{\theta}$, since the latter changes with time.

What happens if we don't use the small-angle approximation? Unfortunately, equation 6.15 cannot be solved analytically. Fortunately, it is easy to solve numerically! Analysing the solution is outside the scope of these notes, but to put it simply, as the maximum amplitude of the oscillation increase, the resonant frequency decreases. I.e. ω becomes a function of s_m (or θ_m).

There is another example of simple harmonic motion that we have seen already. Think back to uniform circular motion. An object moving with UCM has a trajectory of the form

$$\theta(t) = \omega t + \phi, \tag{6.24}$$

where ω is the angular velocity and ϕ is the angular displacement at $t = 0$. If we describe this motion in cartesian coordinates, the trajectory takes the form

$$\boldsymbol{r}(t) = r\cos(\omega t + \phi)\hat{\boldsymbol{\imath}} + r\sin(\omega t + \phi)\hat{\boldsymbol{\jmath}}, \tag{6.25}$$

so we see that UCM is really SHM in two dimensions! If we look at the object dead-on from the y or x direction, we would see identical motion to that of a mass on a spring. For UCM, the motion in the two perpendicular directions are perfectly out of phase (out of phase by one quarter turn, or $\pi/2$).

Example 6.2. Consider a pendulum of mass m and length L which is swinging around in uniform circular motion. What is the frequency of the oscillation?

The radial acceleration is $a_r = \dot{\theta}^2 r = \omega^2 L \sin\theta$. Therefore, we have

$$mg\sin\theta = m\omega^2 L\sin\theta \tag{6.26}$$

$$\implies \omega = \sqrt{\frac{g}{L}}, \tag{6.27}$$

which is the same frequency as a normal pendulum.

6.2 Physical Pendula

Let's now analyse the motion of a pendulum that is not a point mass. In this case, the we will look at the torques to find the motion. The weight force acts on the centre of mass, and the lever arm for the torque is the distance from the pivot point to the centre of mass. We call this h, then the torque is

$$\tau = I\alpha = mgh\sin\theta \approx mgh\theta. \tag{6.28}$$

If we write this as a differential equation using the small-angle approximation, we will get equation 6.20 for the pendulum above just without the moment of inertia specified. A physical pendulum therefore also obeys SHM (under the small-angle approximation) with frequency and period

$$\omega = \sqrt{\frac{mgh}{I}}, \quad T = 2\pi\sqrt{\frac{I}{mgh}}. \tag{6.29}$$

Here we use the small-angle approximation because it allows us to solve for the motion analytically for lots of physical pendula. Of course, if we had some object with a moment of inertia that cannot be expressed analytically, we might as well do away with the small-angle approximation since we are going to have to solve the equation of motion numerically anyway!

Let's do some examples of physical pendula.

Example 6.3. Consider a long thin rod swinging about a pivot at one end. If m is the mass of the rod and L is its length, what is its resonant frequency?

As we have seen, the moment of inertia of a thin rod rotating about its end is $\frac{1}{3}mL^2$. The distance between the pivot and the centre of mass is $\frac{L}{2}$, so by equation 6.29 the resonant frequency is

$$\omega = \sqrt{\frac{mg\frac{L}{2}}{\frac{1}{3}mL^2}} = \sqrt{\frac{3g}{2L}}. \tag{6.30}$$

Example 6.4. Consider a thin hoop pendulum of mass m and radius R. What is its resonant frequency?

The moment of inertia of a thin ring rotating about its centre is mR^2, so by the parallel axis theorem the moment of inertia about the pivot is $2mR^2$. Therefore, the resonant frequency is

$$\omega = \sqrt{\frac{mgR}{2mR^2}} = \sqrt{\frac{g}{2R}}. \tag{6.31}$$

6.3 Energy in Simple Harmonic Motion

As we have seen back in chapter3, Hooke's law is a conservative force, so the total energy is conserved. Let's show this explicitly for the mass on a spring. From before, we know the total energy is given by

$$\begin{aligned} E &= K + U_s & (6.32) \\ &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2. & (6.33) \end{aligned}$$

Substituting equations 6.6 and 6.7, we get

$$\begin{aligned} E &= \frac{1}{2}mx_m^2\omega^2\sin^2(\omega t+\phi) + \frac{1}{2}kx_m^2\cos^2(\omega t+\phi) & (6.34) \\ &= \frac{1}{2}mx_m^2\left(\frac{k}{m}\right)\sin^2(\omega t+\phi) + \frac{1}{2}kx_m^2\cos^2(\omega t+\phi) & (6.35) \\ &= \frac{1}{2}kx_m^2\left(\sin^2(\omega t+\phi)+\cos^2(\omega t+\phi)\right) & (6.36) \\ &= \frac{1}{2}kx_m^2. & (6.37) \end{aligned}$$

Where in the last line we have used the identity $\sin^2 A + \cos^2 A = 1$. This is independent of time, so the total energy is conserved as we found before. By using trigonometric identities for $\sin^2$ and $\cos^2$, we have

$$\begin{aligned} K &= \frac{1}{2}kx_m^2\left(\frac{1}{2} - \frac{1}{2}\cos(2(\omega t+\phi))\right) & (6.38) \\ U &= \frac{1}{2}kx_m^2\left(\frac{1}{2} + \frac{1}{2}\cos(2(\omega t+\phi))\right). & (6.39) \end{aligned}$$

This shows that the energies oscillate sinusoidally as well, but with a frequency that is twice the resonant frequency.

In general, for an system exhibiting SHM in the quantity A, the equation of motion will be

$$\frac{\mathrm{d}^2A}{\mathrm{d}t^2} = -\omega^2 A, \tag{6.40}$$

and the potential and kinetic energies will have the form

$$U = \frac{1}{2}\alpha A^2, \quad K = \frac{1}{2}\beta\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)^2, \tag{6.41}$$

where

$$\omega = \sqrt{\frac{\alpha}{\beta}}. \tag{6.42}$$

Example 6.5. Let's solve the pendulum using the energy approach.

The rotational kinetic energy of a pendulum is given by

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}mL^2\omega^2, \tag{6.43}$$

and the gravitational potential energy is

$$U = mgL(1 - \cos\theta) \approx mgL\frac{\theta^2}{2}, \tag{6.44}$$

where we have used the small-angle approximation $1-\cos\theta \approx \frac{\theta^2}{2}$. Then using the formulae above (6.41 and 6.42), we get

$$\omega = \sqrt{\frac{mgL}{mL^2}} = \sqrt{\frac{g}{L}}, \tag{6.45}$$

which is what we found before.

The following example is more complicated than anything we have seen so far.

Example 6.6. Consider a small marble of mass m and radius r rolling without slipping in a large spherical dish of radius R. Does the marble move with simple harmonic motion? If so, find the resonant frequency.

We will analyse the motion of the marble using energies, which is much simpler than solving this problem using forces. The height that the centre of mass changes by when it moves around in the dish is $h = (R-r)(1-\cos\theta)$, so the potential energy is

$$U = mgh = mg(R-r)(1-\cos\theta) \approx mg(R-r)\frac{\theta^2}{2}. \tag{6.46}$$

The fact that $R \gg r$ justifies the use of the small-angle approximation. The marble has two different kinetic energies, its translation (which is actually rotation of its centre of mass around the pivot) and rotation around its own centre of mass. The marble is a solid sphere, so its moment of inertia about its centre is $\frac{2}{5}mr^2$ and the latter kinetic energy is given by

$$K_{\text{rot}} = \frac{1}{2}I\dot{\phi}^2 = \frac{1}{2}\left(\frac{2}{5}mr^2\right)\dot{\phi}^2, \tag{6.47}$$

where $\dot{\phi}$ is the angular velocity of the marble around its centre of mass. The former kinetic energy is given by

$$K_{\text{tran}} = \frac{1}{2}I_{\text{pivot}}\dot{\theta}^2 = \frac{1}{2}\left(m(R-r)^2 + \frac{2}{5}mr^2\right)\dot{\theta}^2, \tag{6.48}$$

where we have used the parallel axis theorem to get the moment of inertia about the pivot point. We would like to find a relation between $\dot{\phi}$ and $\dot{\theta}$ to simplify the total kinetic energy. Note that when the marble rolls a distance $s = R\theta$ in the dish, which is equal to $r(\phi+\theta)$.

Then we have

$$\phi = \frac{R-r}{r}\theta \tag{6.49}$$

$$\implies \dot{\phi} = \frac{R-r}{r}\dot{\theta}. \tag{6.50}$$

Thus the total kinetic energy is

$$K = K_{\text{tran}} + K_{\text{rot}} = \frac{1}{2}\left[m(R-r)^2\dot{\theta}^2 + \frac{2}{5}mr^2\dot{\theta}^2 + \frac{2}{5}mr^2\left(\frac{R-r}{r}\dot{\theta}\right)^2\right] \tag{6.51}$$

$$= \frac{1}{2}\left[\frac{7}{5}m(R-r)^2 + \frac{2}{5}mr^2\right]\dot{\theta}^2. \tag{6.52}$$

These energies satisfy the forms in equation 6.41, so the marble does exhibit simple harmonic motion. The resonant frequency is given by equation 6.42:

$$\omega = \sqrt{\frac{mg(R-r)}{\frac{7}{5}m(R-r)^2 + \frac{2}{5}mr^2}}. \tag{6.53}$$

6.4 Damped Oscillations

An object moving with simple harmonic motion will continue to do so indefinitely. A more realistic scenario is to include the effects of other forces, such as friction and air resistance, that slow down the oscillations until they eventally decay away to nothing. These effects are collectively known as **damping**.

Depending on the strength of the damping forces, they can result one of of three scenarios:

- **Underdamping** — The damping effects aren't that strong, there are still oscillations but they decay away over time.
- **Overdamping** — Damping is very strong, there are no oscillations and the system smoothly decays to equilibrium.
- **Critial damping** — The system returns to equilibrium in the fastest possible way without overshooting.

Damping forces can be proportional to velocity (as with viscous drag), proportional to the square of velocity (quadratic drag e.g. air resistance), or something else (friction etc.). In this chapter we will focus on linear drag as it provides an instructive example of damping effects which we can solve analytically. We write the damping force as $F = -bv$, where b is some constant with units of $\mathrm{kg\,s^{-1}}$. The minus sign indicates that the damping force acts in the opposite direction of the velocity. Using Newton II with the two forces we have now, the equation of motion becomes

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0. \tag{6.54}$$

To solve this we use an ansatz of the form $x(t) = Ae^{\xi t}$. Differentiating twice and substituting into the above, we get an equation for ξ:

$$\xi^2 + \frac{b}{m}\xi + \omega_0^2 = 0, \tag{6.55}$$

where we have written the resonant frequency as $\omega_0^2 = \frac{k}{m}$. The subscript 0 is important to distinguish the resonant frequency from other frequencies we will define later. If we also define the **damping coefficient** γ as

$$\gamma = \frac{b}{2m}, \tag{6.56}$$

then we can write ξ as

$$\xi = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}, \tag{6.57}$$

and the general solution is

$$x(t) = A_1 e^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + A_2 e^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t}. \tag{6.58}$$

Now, we have four different cases that can arise depending on the strength of the damping force i.e. the size of b compared to ω_0.

Case 1: $b = 0$ (no damping). Then $\gamma = 0$ and the general solution becomes

$$\begin{aligned} x(t) &= A_1 e^{i\omega_0^2 t} + A_2 e^{-i\omega_0^2 t} && (6.59)\\ &= A\cos(\omega_0 t + \phi), && (6.60) \end{aligned}$$

which is exactly the SHM that we found before. ξ is purely imaginary.

Case 2: $\gamma^2 - \omega_0^2 < 0$. In this case we define the **damped frequency** $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$, so the solution becomes

$$\begin{aligned} x(t) &= A_1 e^{(-\gamma + i\omega_d)t} + A_2 e^{(-\gamma - i\omega_d)t} && (6.61)\\ &= Ae^{-\gamma t}\cos(\omega_d t + \phi). && (6.62) \end{aligned}$$

So there are still oscillations, but the amplitude decays over time ($x_m(t) = Ae^{-\gamma t}$). This is an *underdamped* system. ξ is a complex number.

Case 3: $\gamma^2 - \omega_0^2 > 0$. Then ξ is a real number and the solution just becomes the sum of two decaying exponentials.

$$x(t) = A_1 e^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + A_2 e^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t}. \tag{6.63}$$

There are two slightly different decay rates, the second term decays faster since its decay constant is larger. This is an *overdamped* system.

Case 4: $\gamma^2 = \omega_0^2$. Here, ξ simplifies to just $-\gamma$ (a real number) and the general solution becomes

$$x(t) = (A_1 + A_2 t)e^{-\gamma t}. \tag{6.64}$$

This is a *critically damped* system.

In the case of an underdamped system, notice that the total mechanical energy, given by equation 6.37, will decay exponentially.

$$E_{\text{mech}}(t) = E_{\text{mech}}(0)e^{-\frac{b}{m}t}. \tag{6.65}$$

To quantify the level of underdamping, it is common to define the **Q-factor** as the ratio of the initial energy in the oscillator to the energy dissipated in one radian of the oscillation, or

$$Q = 2\pi \frac{\text{initial energy}}{\text{energy dissipated in one cycle}}. \tag{6.66}$$

It can be shown that for our linear damping the Q-factor is equal to

$$Q = \frac{m}{b}\omega_d = \tau\omega_d, \tag{6.67}$$

where $\tau = \frac{m}{b}$ is the **time constant** for the decay.

6.5 Forced Oscillations

A forced oscillator is an oscillator subject to a periodic external force.

$$F_{\text{net}} = -kx - bv + F_{\text{drive}}. \tag{6.68}$$

Consider a sinusoidal driving force $F_{\text{drive}} = F_0 \cos(\omega_{\text{dr}}t)$. Then the equation of motion becomes

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0\cos(\omega_{\text{dr}}t). \tag{6.69}$$

When we solve this equation, what we find is that, when damping is small, there is a huge build-up of energy when the system is driven at its resonant frequency ω_0. This phenomenon is called **resonance**. Mathematically, the condition for resonance is

$$\omega_{\text{drive}} = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}} = \sqrt{\omega_0^2 - 2\gamma^2}. \tag{6.70}$$

We call this ω_{res}. Note that this is not the damped frequency ω_d from above. We see that when damping is weak, the condition for resonance becomes $\omega_{\text{drive}} \approx \omega_0$.

To find the resonant frequency of a system, there are two general methods we can use. There is the "impulse method", where we excite all frequencies in the system at once. The system will then resonate at the resonant frequency and the vibrations at other frequencies will decay away (we will see why shortly). An example of this is striking a bell. The other method is the "frequency-scan method", where we use a low-ampltidue signal and scan through frequencies until we hit resonance, when energy will rapidly build up.

For the case of underdamping ($\gamma < \omega_0$), the general solution is a sum of a transient (decaying) oscillation at the damped frequency and a steady state (not decaying) oscillation at the driven frequency.

$$x(t) = A_{\text{decay}}e^{-\gamma t}\cos(\omega_d t + \phi_{\text{decay}}) + A_{\text{steady}}\cos(\omega_{\text{drive}}t - \phi_{\text{steady}}). \tag{6.71}$$

The ampltiude and phase of the steady state oscillation depend on the driven frequency and the resonant frequency. These dependencies have the form

$$A_{\text{steady}} = \frac{F_0/m}{\sqrt{(\omega_{\text{dr}}^2 - \omega_0^2)^2 + \left(\frac{b}{m}\right)^2 \omega_{\text{dr}}^2}}, \quad \phi_{\text{steady}} = \tan^{-1}\left(\frac{\left(\frac{b}{m}\right)\omega_{\text{drive}}}{\omega_0^2 - \omega_{\text{drive}}^2}\right) + \phi_0 \tag{6.72}$$

A_{steady} has a maximum at ω_{res}, which is the resonance phenomenon we have been discussing. Note that we now have three different "resonance" frequencies for oscillators. ω_0 for when we have an undamped, undriven oscillator (SHM), ω_d for a damped oscillator, and ω_{res} for a driven oscillator. We see that at low driving frequencies, the ampltide of the steady state is small (but nonzero) but the oscillation is in-phase with the driving force. For high driving frequencies, the amplitude goes to zero, and the oscillation is in anti-phase with the driving force. At resonance, the phase of the steady state is $\pi/2$, which means the driving force is in-phase with the velocity. This makes sense because then we have a *constant* power input to the system ($P = \boldsymbol{F} \cdot \boldsymbol{v}$).

For a driven oscillator, we find that the Q-factor is

$$Q = \frac{\omega_{\text{res}}}{\text{FWHM of energy curve}} \approx \omega_{\text{res}} \frac{m}{b}. \tag{6.73}$$

If damping is low, we can say $Q \approx \omega_0 \frac{m}{b}$ and write

$$A_{\text{steady}} \approx \frac{F_0(\omega_0/\omega_{\text{drive}})}{k\sqrt{\left(\frac{\omega_0}{\omega_{\text{drive}}} - \frac{\omega_{\text{drive}}}{\omega_0}\right)^2 + \frac{1}{Q^2}}}. \tag{6.74}$$

So for $Q \gg 1$, we get the following cases. For $\omega_{\text{drive}} \ll \omega_{\text{res}}$, $A_{\text{steady}} = F_0/k$. For $\omega_{\text{drive}} = \omega_{\text{res}}$, we get $A_{\text{steady}} = QF_0/k$. For $\omega_{\text{drive}} \gg \omega_{\text{res}}$, we have $A_{\text{steady}} = -\frac{\omega_0}{\omega_{\text{drive}}} F_0/k$.

6.6 Coupled Oscillations and Normal Modes

A coupled oscillator is a system of more than one oscillators that have some way of transferring energy to one another. Generally what we see is that the kinetic energy is passed between each oscillator. A **normal mode** is a collective excitation of the whole system where all parts move with the same frequency. Normal modes are the generalisation of the resonant frequency for a single oscillator, and once the system is in a normal mode it does not decay or change its motion into another mode (unless there is damping). If we take a system of two pendula coupled by a spring as an example, the system has two normal modes. One where the pendula are swinging in-phase, which has a frequency $\omega_1 = \sqrt{\frac{g}{L}}$, and one where they are swinging out-of-phase, which has a frequency $\omega_2 = \sqrt{\frac{g}{L} + 2\frac{k}{m}}$.

The **principle of superposition** states that any motion of the system can be expressed as a superposition (sum) of motion due to normal modes. Since the frequencies of the normal modes are unequal, this means the motion will evolve over time, which leads to the kinetic energy being passed around as stated above. This exchange of energy will happen at a **beat frequency** $\omega_2 - \omega_1$. For weakly coupled oscillators (in this case small k), they are almost independent and so all the normal modes will be very similar in frequency. This leads to the beat frequency being very small. Note that *no* energy is passed between the normal modes themselves. The designation "normal" means they are independent from each other.

Example 6.7. Consider the example of two blocks, both of mass m, coupled to three springs, all with spring constant k, from above. Solve for the general motion of each block.

We will solve this with an elementary approach, by looking at the forces on each block. If we set $x_1 = 0$, and $x_2 = 0$ at the equilibrium positions of each block respectively, then the force from each spring is

$$F_1 = -kx_1 \tag{6.75}$$
$$F_2 = -k(x_1 - x_2) \tag{6.76}$$
$$F_3 = kx_2. \tag{6.77}$$

so by Newton II, the equations of motion for each block are

$$m\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} + 2kx_1 - kx_2 = 0 \tag{6.78}$$
$$m\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} - kx_1 + 2kx_2 = 0. \tag{6.79}$$

These are two coupled second-order differential equations, which would be quite hard to solve. Fortunately, by writing the positions of the blocks x_1 and x_2 in terms of the normal modes, we can uncouple them and solve them easily! First, we will rearrange slightly and substitute in the resonant frequency of a mass on a spring $\omega_0 = \sqrt{k/m}$,

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} + \omega_0^2x_1 + \omega_0^2(x_1 - x_2) = 0 \tag{6.80}$$
$$\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} + \omega_0^2x_2 - \omega_0^2(x_1 - x_2) = 0, \tag{6.81}$$

and then we let $X_1 = x_1 + x_2$ (the normal mode where the masses are swinging together) and $X_2 = x_1 - x_2$ (the normal mode where the masses are swinging apart). Then we can get differential equations for X_1 and X_2 by adding and subtracting the two equations we have for x_1 and x_2.

$$\frac{\mathrm{d}^2X_1}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} + \frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} = -\omega_0^2x_1 - \omega_0^2x_1 = -\omega_0^2X_1 \tag{6.82}$$
$$\frac{\mathrm{d}^2X_2}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} - \frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} = -3\omega_0^2(x_1 - x_2) = -3\omega_0^2X_2. \tag{6.83}$$

These are just the equations of motion for SHM, so we can solve for the amplitude of the normal modes.

$$X_1(t) = A_1\cos(\omega_0t + \phi_1) \tag{6.84}$$
$$X_2(t) = A_2\cos(\sqrt{3}\omega_0t + \phi_2). \tag{6.85}$$

So for the first mode $X_1 = x_1 + x_2$ we have $x_1 = x_2$, and for the second mode $X_2 = x_1 - x_2$ we have $x_1 = -x_2$. The general solution is a superposition of motion due to normal modes, giving

$$x_1(t) = A_1'\cos(\omega_1t + \phi_1) + A_2'\cos(\omega_2t + \phi_2) \tag{6.86}$$
$$x_2(t) = A_1'\cos(\omega_1t + \phi_1) + A_2'\cos(\omega_2t + \phi_2), \tag{6.87}$$

where $\omega_1 = \omega_0 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3}\omega_0 = \sqrt{3k/m}$. This system requires four initial conditions to specificy a particular solution i.e. A_1', A_2', ϕ_1, and ϕ_2. These could be $x_1(0)$, $x_2(0)$, $v_1(0)$, and $v_2(0)$.

In general, for a system of N coupled simple harmonic oscillators, there will be N normal modes (N ways they can move collectively). Therefore, the general solution will be a superposition of N normal modes and $2N$ initial conditions will be required to specify a particular solution. In the previous example, we defined the variables $X_1 = x_1 + x_2$ and $X_2 = x_1 + x_2$ to represent the normal modes. These quantities along with their derivatives $\dot{X}_1$ and $\dot{X}_2$ are known as the **normal coordinates** of the system. Each way a system can store energy is known as a **degree of freedom**, and each degree of freedom has a normal coordinate. In simple harmonic oscillators, the degrees of freedom are called **quadratic** because the energies are proportional to the squares of the normal coordinates $\alpha_1 X_1^2$, $\alpha_2 X_2^2$, $\beta_1 \dot{X}_1^2$, and $\beta_2 \dot{X}_2^2$ for some constants α_1, α_2, β_1, and β_2. Recall that the total energy in each normal mode is constant (if there is no damping) as they do not exchange energy with each other. For a given normal mode, all of the oscillators will pass through their equilibrium points at the same time and they will have a fixed phase and amplitude relationship with each other.

Chapter 7

Waves

7.1 The Wave Equation

A wave is a periodic variation or disturbance which travels at a well defined speed through space. How do we describe waves mathematically? Suppose $f(\chi)$ is some periodic function which takes a phase χ measured in radians (fractions of 2π). Then we can describe the variation in space at a specific point in time as a snapshot:

$$y(x) = Af\left(\frac{2\pi}{\lambda}x + \delta\right), \tag{7.1}$$

where λ is the **wavelength** of the wave (the spatial period). We can also describe the variation in amplitude at a single point in space over time:

$$y(t) = Af\left(\frac{2\pi}{T}t + \theta\right), \tag{7.2}$$

where T is the temporal period.

To put these pictures together, we can consider the snapshot picture with a shift $x - vt$ where v is the speed of the wave. Then we have

$$\begin{aligned}
y(x,t) &= Af\left(\frac{2\pi}{\lambda}(x - vt) + \phi\right) && (7.3)\\
&= Af\left(\frac{2\pi}{\lambda}x - \frac{2\pi v}{\lambda}t + \phi\right) && (7.4)\\
&= Af\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t + \phi\right) && (7.5)\\
&= Af(kx - \omega t + \phi). && (7.6)
\end{aligned}$$

Where we have defined the **wavenumber** (spatial frequency measured in radians/m) $k = 2\pi/\lambda$ and recalled $v = f\lambda$, $f = 1/T$ and $\omega = 2\pi f$.

Example 7.1. Suppose we have a sinusoidal wave given by $y(x,t) = A\cos(kx - \omega t + \phi)$. What is the particle velocity at fixed position x?

To solve this, we take the *partial derivative* with respect to time (to keep x constant)

$$\frac{\partial y(x,t)}{\partial t} = A\omega \sin(kx - \omega t + \phi). \tag{7.7}$$

Note that this is different to the propagation speed of the wave itself, which is constant.

For simplicity, let's consider a general right-travelling wave $f(x - vt)$. We can make a substitution $u = x - vt$. Then we get

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} && (7.8)\\
&= \frac{\partial f}{\partial u} && (7.9)\\
\implies \frac{\partial^2 f}{\partial x^2} &= \frac{\partial^2 f}{\partial u^2}, && (7.10)\\
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} && (7.11)\\
&= -v\frac{\partial f}{\partial u} && (7.12)\\
\implies \frac{\partial^2 f}{\partial t^2} &= v^2\frac{\partial^2 f}{\partial u^2}. && (7.13)
\end{aligned}$$

If we do this same calculation with a left-travelling wave $f(x + vt)$, we get the same relation. Thus, by construction, the general solution to the differential equation

$$\frac{\partial^2 f}{\partial t^2} = v^2\frac{\partial^2 f}{\partial x^2}, \tag{7.14}$$

is $f(x,t) = f_l(x + vt) + f_r(x - vt)$. This linear differential equation is known as the **wave equation**. All functions which satisfy our definition of a wave solve this equation.

7.2 Superposition of Waves

Since the wave equation is linear, the solutions follow the principle of **linear superposition**. This means that when multiple waves come together, the amplitude at every point in space and time is determined by the sum of all the waves at that point.

Example 7.2. Consider two sinusoidal waves with the travelling with the same frequency and direction. Then the superposition is given by

$$\begin{aligned}
y(x,t) &= A\cos(kx - \omega t) + A\cos(kx - \omega t) && (7.15)\\
&= 2A\cos(kx - \omega t). && (7.16)
\end{aligned}$$

So, the resultant wave has the same frequency and direction but double the amplitude.

Now consider what happens if one of the waves has a phase shift of π radians. The resultant wave is

$$
\begin{aligned}
y(x,t) &= A\cos(kx-\omega t) + A\cos(kx-\omega t+\pi) &(7.17)\\
&= A\cos(kx-\omega t) - A\cos(kx-\omega t) &(7.18)\\
&= 0. &(7.19)
\end{aligned}
$$

The two waves cancel each other out completely.

In the general case with a phase shift Ω, we get

$$
\begin{aligned}
y(x,t) &= A\cos(kx-\omega t) + A\cos(kx-\omega t+\Omega) &(7.20)\\
&= 2A\cos\left(kx-\omega t+\frac{\Omega}{2}\right)\cos\left(-\frac{\Omega}{2}\right) &(7.21)\\
&= \underbrace{2A\cos\left(\frac{\Omega}{2}\right)}_{\text{Amplitude } \leq 2A}\underbrace{\cos\left(kx-\omega t+\frac{\Omega}{2}\right)}_{\text{Time-dependent part}}. &(7.22)
\end{aligned}
$$

Note that we have used the identity $\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$.

Most superpositions of two sinusoidal functions do not have a nice simplification which is easily interpreted like this, but from these basic examples we can build an intuition for what happens when two waves meet.

Now consider what happens if the waves still have the same frequency but are moving in opposite directions. In this case the superposition is

$$
\begin{aligned}
y(x,t) &= A\cos(kx-\omega t) + A\cos(kx+\omega t) &(7.23)\\
&= 2A\cos\left(\frac{2kx}{2}\right)\cos\left(-\frac{2\omega t}{2}\right) &(7.24)\\
&= \underbrace{2A\cos(kx)}_{A(x)}\cos(\omega t). &(7.25)
\end{aligned}
$$

So we have a spatially varying amplitude $A(x)$ multiplied by a time-dependent variation. This is known as a **standing wave**.

Example 7.3. In the case where one of the waves has a phase shift Ω. The relation above becomes

$$
\begin{aligned}
y(x,t) &= A\cos(kx-\omega t) + A\cos(kx+\omega t+\Omega) &(7.26)\\
&= 2A\cos\left(\frac{2kx+\Omega}{2}\right)\cos\left(-\frac{w\omega t+\Omega}{2}\right) &(7.27)\\
&= 2A\cos\left(kx+\frac{\Omega}{2}\right)\cos\left(\omega t+\frac{\Omega}{2}\right). &(7.28)
\end{aligned}
$$

The points on the standing wave which don't move at all are known as **nodes**, and points which move up the twice the ampltide of the waves are called **antinodes**. Nodes are separated by half a wavelength. In a standing wave, the whole string oscillates in simple harmonic motion.

The standing waves allowed in a one-dimensional region of length L are given by

$$\lambda = \frac{2L}{p}, \quad f_p = p\frac{v}{2L} = pf_1, \tag{7.29}$$

where $p \in \mathbb{Z}$.

Note that sometimes it is more convenient to express waves in terms of complex exponential functions according to Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{7.30}$$

so a general sinusoidal wave like above would be written as

$$y(x,t) = Ae^{i(kx-\omega t+\phi)}. \tag{7.31}$$

We recover the trigonometric form (cosine in this case) by taking the real part of this function. For the standing wave example above, we have

$$y(x,t) = Ae^{i(kx-\omega t)} + Ae^{i(kx+\omega t)} \tag{7.32}$$

$$= Ae^{ikx}\left[e^{-i\omega t} + e^{i\omega t}\right] \tag{7.33}$$

$$= 2Ae^{ikx}\cos(\omega t), \tag{7.34}$$

where in the last line we have used Euler's identity to get $e^{-i\theta} + e^{i\theta} = 2\cos\theta$. Taking the real part of this, we recover $y(x,t) = 2A\cos(kx)\cos(\omega t)$ which we found above.

Example 7.4. Given a periodic wave, what is the phase difference between two points on the wave separated by a distance Δx?

$$\Delta\phi = 2\pi\frac{\Delta x}{\lambda} = k\Delta k. \tag{7.35}$$

What is the phase difference between a single point over a interval of time Δt?

$$\Delta\phi = 2\pi\frac{\Delta t}{T} = \omega\Delta t. \tag{7.36}$$

7.3 Phase Velocity & Group Velocity

As we have seen, the speed you need to keep up with a point of constant phase along the wave is given by

$$v_\phi = f\lambda = \frac{\omega}{k}. \tag{7.37}$$

This is known as the **phase velocity**. The dependence of ω on k (or vice-versa) is called the **dispersion relation**. If the relationship is linear, i.e. if v_ϕ is constant, the wave is said to be dispersionless. Otherwise, the wave will undergo dispersion as different frequencies will travel at difference speeds.

The **group velocity** is defined as

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{7.38}$$

So if a wave is dispersionless, the phase velocity and group velocity will be the same. In the case where $v_g \neq v_\phi$, the group velocity is the speed that the wave envelope propagates.

7.4 Transverse Waves on a String

Consider an infinite string under constant tension T. We will now show that the equation of motion of the string is the wave equation and derive the wave speed. Consider a short section of the string of length Δx. We are assuming that the string has linear density μ, zero stiffness, and we are ignoring the effects of gravity. Then assuming that there are only small displacements on the string, then $\frac{\partial y}{\partial x}$ is small, so the angles θ_1 and θ_2 are also small. Hence we use the small angle approximation and say that $\cos\theta_1 \approx \cos\theta_2 \approx 1$.

$$\sum F_x = -|\boldsymbol{T}_1|\cos\theta_1 + |\boldsymbol{T}_2|\cos\theta_2 = 0 \tag{7.39}$$

$$\implies |T_{1,x}| \approx |T_{2,x}| \approx T. \tag{7.40}$$

From these we get that $T_{1,x} \approx -T$ and $T_{2,x} \approx T$.

Now, using some trigonometry, notice that

$$\left.\frac{\partial y}{\partial x}\right|_x = \frac{T_{1,y}}{T_{1,x}} \approx -\frac{T_{1,y}}{T} \tag{7.41}$$

$$\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} = \frac{T_{2,y}}{T_{2,y}} \approx \frac{T_{2,y}}{T}. \tag{7.42}$$

Thus the net force in the y-direction is given by

$$F_y = T_{1,y} + T_{2,y} \tag{7.43}$$

$$= T\left(-\left.\frac{\partial y}{\partial x}\right|_x + \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}\right). \tag{7.44}$$

Using Newton's second law, we get

$$F_y = ma_y \tag{7.45}$$

$$= \mu\Delta x a_y \tag{7.46}$$

$$= \mu\Delta x \left.\frac{\partial^2 y}{\partial t^2}\right|_{x+\frac{\Delta x}{2}} = \left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_x\right)T. \tag{7.47}$$

Finally we divide by Δx on both sides and take the limit as $\Delta x \to 0$ to get

$$\lim_{\Delta x\to 0}\left(\mu \left.\frac{\partial^2 y}{\partial t^2}\right|_{x+\frac{\Delta x}{2}}\right) = \lim_{\Delta x\to 0}\left[\frac{\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_x\right)T}{\Delta x}\right] \tag{7.48}$$

$$\mu\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2} \tag{7.49}$$

$$\implies \frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2}. \tag{7.50}$$

This is the wave equations and we can see that for waves on a string, $v = \sqrt{\frac{T}{\mu}}$.

What is the mechanical energy stored in a wave on a string? It will have two contributions, potential energy which depends on the displacement of every point and kinetic energy which depends on the velocity of every point. Consider a segment of the string of length $\mathrm{d}x$, mass $\mathrm{d}m = \mu \mathrm{d}x$. The infinitesimal contribution to the kinetic energy of the wave is given by

$$\mathrm{d}K = \frac{1}{2}\mathrm{d}m v_y^2 = \frac{1}{2}\mu \mathrm{d}x \left(\frac{\partial y}{\partial t}\right)^2. \tag{7.51}$$

To get a value for this, we integrate it over some length L.

$$K = \frac{1}{2}\mu \int_0^L \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x. \tag{7.52}$$

The potential energy is due to the stretching of the string. A segment of length $\mathrm{d}x$ stretches to a length $\mathrm{d}s$, and we can calculate the relationship between the two as follows:

$$\begin{aligned}
\mathrm{d}s &= \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} && (7.53)\\
&= \sqrt{\mathrm{d}x^2 + \mathrm{d}x^2\left(\frac{\partial y}{\partial x}\right)^2} && (7.54)\\
&= \mathrm{d}x\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} && (7.55)\\
&\approx \mathrm{d}x\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right), && (7.56)
\end{aligned}$$

where in the last line we have used the Taylor expansion $\sqrt{1+u^2} \approx 1 + \frac{1}{2}u^2$ when u is small. This means we can calculate the potential energy as

$$\begin{aligned}
\mathrm{d}U &= T(\mathrm{d}s - \mathrm{d}x) && (7.57)\\
&\approx \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x && (7.58)\\
\implies U &= \frac{1}{2}T\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x. && (7.59)
\end{aligned}$$

Example 7.5. Consider a sinusoidal wave $y = A\cos(kx - \omega t)$. What is the energy per unit wavelength? The partial derivatives are given by

$$\begin{aligned}
\frac{\partial y}{\partial t} &= A\omega \sin(kx - \omega t) && (7.60)\\
\frac{\partial y}{\partial x} &= -Ak\sin(kx - \omega t), && (7.61)
\end{aligned}$$

so the infinitesimal contribution to the total energy is

$$\begin{aligned} \mathrm{d}E &= \mathrm{d}K + \mathrm{d}U && (7.62)\\ &= \frac{1}{2}\left[\mu\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right]\mathrm{d}x && (7.63)\\ &= \frac{1}{2}A^2\sin^2(kx-\omega t)(\mu\omega^2 + Tk^2)\mathrm{d}x. && (7.64)\end{aligned}$$

Note that $v = \frac{\omega}{k} = \sqrt{T}\mu$, so $Tk^2 = \mu\omega^2$. Hence for a sinudoidal wave, the kinetic and potential energies are the same. The energy per unit wavelength is then

$$\begin{aligned} E_\lambda &= \mu A^2\omega^2 \int_0^\lambda \sin^2(kx)\mathrm{d}x && (7.65)\\ &= \frac{1}{2}\lambda\mu A^2\omega^2. && (7.66)\end{aligned}$$

Note that we choose to write the energy in terms of μ rather than T because linear density is an easily measurable property whereas the tension is not. One important thing to mention is that the dependence on A^2 is actually general to all forms of waves, not just sinusoidal. We can calculate the power transmitted through a single point by the wave as

$$\begin{aligned} P = E_\lambda f &= \frac{1}{2}\lambda f\mu A^2\omega^2 && (7.67)\\ &= \frac{1}{2}v\mu A^2\omega^2 && (7.68)\\ &= \frac{1}{2}\sqrt{\mu T}A^2\omega^2. && (7.69)\end{aligned}$$

We can calculate the power transmitted through the wave from first principles as well. Each string segment exerts a force and does work on the adjoining segments. For a point x_0 on the string, work is done on the string by to the right of x_0 by a tension force T_y applied by the string to the left of x_0. Assuming motion in the x-direction is negligible, we have that the work done by the tension is $\mathrm{d}W = T_y\,\mathrm{d}y$. Since we are assuming $\frac{\partial y}{\partial x}$ is small, $T_y = -T\frac{\partial y}{\partial x}$. Then we have that the instantaneous power is

$$\begin{aligned} P_{\mathrm{inst}}(x,t) = \frac{\mathrm{d}W}{\mathrm{d}t} &= T_y\frac{\partial y}{\partial t} && (7.70)\\ &= -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} && (7.71)\\ &= \frac{T}{V}\left(\frac{\partial y}{\partial t}\right)^2 = \sqrt{\mu T}\left(\frac{\partial y}{\partial t}\right)^2, && (7.72)\end{aligned}$$

where in the last line we have used the wave equation. For a sinusoidal pattern of motion, we have

$$P_{\mathrm{inst}}(x,t) = \sqrt{\mu T}A^2\omega^2\sin^2(kx-\omega t). \tag{7.73}$$

The average value of $\sin^2$ is $\frac{1}{2}$, so the average power transmitted is

$$P_{\mathrm{avg}} = \sqrt{\mu T}A^2\omega^2, \tag{7.74}$$

which is what we derived before.

7.5 Boundaries, Transmission, and Reflection

A boundary for a wave is a change in medium. When a wave encounters a boundary, some energy is transmitted across the boundary and some is reflected. Consider a pulse wave travelling along a string:

- At a *fixed* end, the string exerts an upward force on the fixed end which pulls back down on the string according to Newton III. This generates an upside-down wave pulse travelling in the opposite direction (what we actually see while this is happening is a superposition of both pulses).
- At a *free* end, the same thing happens except that the reflected pulse is the same way up. Because of the superposition, the free end reaches twice the peak amplitude of the pulse.
- When we have a mix between these two cases, for example a change in linear density of the string, we will see partial transmission and reflection.

To figure out how much gets reflected and transmitted, we must make some assumptions about what happens to the wave at the boundary. At the boundary x_b we have:

- $y(x_b, t)$ must be continuous (no gaps in the string).
- $\frac{\partial y(x_b,t)}{\partial x}$ must be continuous (no sharp kinks).

Therefore, we must have the following two conditions at the boundary (they do not have to hold anywhere else!):

$$y_i(x_b, t) + y_r(x_b, t) = y_t(x_b, t) \tag{7.75}$$

$$\frac{\partial y_i(x_b, t)}{\partial t} + \frac{\partial y_r(x_b, t)}{\partial t} = \frac{\partial y_t(x_b, t)}{\partial t}, \tag{7.76}$$

where subscript i, r, and t represent the incident, reflected, and transmitted waves respectively. Solving for y_r and y_t using the wave equation, we get

$$y_r(x_b, t) = \frac{v_2 - v_1}{v_2 + v_1} y_i(x_b, t) = r y_i(x_b, t) \tag{7.77}$$

$$y_t(x_b, t) = \frac{2v_2}{v_1 + v_2} y_i(x_b, t) = \tau y_i(x_b, t), \tag{7.78}$$

where we have defined the **reflection coefficient** r and the **transmission coefficient** τ as

$$r = \frac{v_2 - v_1}{v_1 + v_2}, \quad \tau = \frac{2v_2}{v_1 + v_2}, \tag{7.79}$$

where v_1 is the wave velocity in the left medium and v_2 is the velocity on the right. Note that $-1 \leq r \leq 1$, and $0 \leq \tau \leq 2$. These coefficients can also be defined in terms of the wave impedance $Z = \sqrt{\mu T}$:

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad \tau = \frac{2Z_1}{Z_1 + Z_2}. \tag{7.80}$$

Because r and τ are defined as the ratios of the amplitudes of the reflected and transmitted waves to the incident wave, we have a constraint $|r| + |\tau| = 1$. Note that this has nothing to do with energy conservation, it holds even when energy is not conserved!

Consider a boundary where the density increases. The wave speed is slower on the other side of the boundary, $v_2 < v_1$, so $r < 0$ and $\tau < 1$. The pulse also becomes narrower because the wave speed is slower. What about where the density decreases? The new wave speed is faster, $v_2 > v_1$, so $r > 0$ and $\tau > 1$. The reflected pulse has the same width but with smaller amplitude and the transmitted pulse is broader because its speed is higher. The maths for these scenarios is general to *all* shapes of waves. For periodic waves, the left-hand side of the boundary consists of the superposition of the incident and reflected waves. For the transmitted wave, since the frequency is fixed by the source the wavelength *must* be the quantity to change. If energy is conserved, it is conserved at the boundary, so we also have the constraint

$$|P_{\text{inst}_i}(x_b, t)| = |P_{\text{inst}_r}(x_b, t)| + |P_{\text{inst}_t}(x_b, t)|. \tag{7.81}$$

Note that this is different to the equations above where incident and reflected were on the same side, here reflected and transmitted are on the same side.

7.6 Normal Modes and Fourier Series

Consider a standing wave $y(x,t) = Ae^{i(kx-\omega t)} + Be^{i(kx+\omega t)}$ on a string clamped at both ends. Suppose the string runs from $x = 0$ to $x = L$, then at the boundaries we have $y(0,t) = y(L,t) = 0$. Applying the left boundary condition to the standing wave, we get

$$\begin{aligned} y(0,t) = \mathfrak{Re}(Ae^{-i\omega t} + Be^{i\omega t}) = 0 && (7.82)\\ A\cos(\omega t) + B\cos(\omega t) = 0 && (7.83)\\ \implies A = -B. && (7.84) \end{aligned}$$

This makes physical sense because as we have seen above, wave pulses invert at fixed boundaries. For the right boundary condition, we get

$$\begin{aligned} y(L,t) &= \mathfrak{Re}(A[e^{i(kL-\omega t) - e^{i(kL+\omega t)}}]) && (7.85)\\ &= \mathfrak{Re}(Ae^{ikL}[\underbrace{e^{-i\omega t} - e^{i\omega t}}_{-2i\sin(\omega t)}]) && (7.86)\\ &= 2A\sin(kL)\sin(\omega t) = 0 && (7.87)\\ &\implies \sin(kL) = 0 && (7.88)\\ & && (7.89) \end{aligned}$$

$\sin(kL) = 0$ implies that $kL = n\pi$, so we can write the standing wave in the form

$$y(x,t) = A\sin(k_n x)\cos(\omega_n t + \phi_0), \tag{7.90}$$

where

$$k_n = \frac{n\pi}{L}, \quad \omega_n = \frac{n\pi v}{L}. \tag{7.91}$$

Note that wavelength is related to angular wavenumber by $\lambda = \frac{2\pi}{k}$, so $\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}$. These allowed wavenumbers/frequencies are the **normal modes** of vibration for the string clamped at both ends.

Recall in chapter 6 we discussed exciting all the frequencies of a system at once using the impulse method. If we pluck the string, multiple normal modes are excited. The resulting motion of the string is a superposition of the normal modes, given by

$$y(x,t) = \sum_n A_n \sin(k_n x)\cos(\omega_n t). \tag{7.92}$$

We have chosen cos instead of sin here for the temporal part because the displacement is maximal at $t = 0$. It makes no difference to the physics which one we choose. At $t = 0$, we have

$$y(x,0) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right). \tag{7.93}$$

If we can calculate all the A_n's, we can determine the subsequent motion of the string. But there may be infinitely many A_n's! Luckily, we can use a mathematical tool called **Fourier series** which makes the calculation of all of them straightforward.

If we have a periodic function (for simplicity we can assume the period is 2π, since we can stretch or shrink it otherwise), it is (almost) always possible to represent it as an infinite series of sines and cosines.

$$f(x) = \frac{1}{2}a_0 + \sum_{j=1}^{\infty} a_j \cos(jx) + \sum_{j=1}^{\infty} b_j \sin(jx). \tag{7.94}$$

Using some convenient properties of integrals of sines and cosines, there is a simple method to calculate what the coefficients a_j and b_j should be for any function $f(x)$. We can see how this will help us solve our problem of representing a wave pulse as a sum of normal modes.

Specifically, we will make use of the following three integrals:

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \begin{cases} 0 \text{ if } m \neq n \\ \pi \text{ if } m = n \end{cases} \tag{7.95}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \begin{cases} 0 \text{ if } m \neq n \\ \pi \text{ if } m = n \end{cases} \tag{7.96}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = 0 \quad \forall m, n. \tag{7.97}$$

Now consider multiplying equation 7.94 by $\cos(nx)$ and integrating from $-\pi$ to π. We get

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nx) f(x)\,\mathrm{d}x &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x + \sum_{j=1}^{\infty} a_j \int_{-\pi}^{\pi} \cos(nx)\cos(jx)\,\mathrm{d}x \\ &\quad + \underbrace{\sum_{j=1}^{\infty} b_j \int_{-\pi}^{\pi} \sin(nx)\cos(jx)\,\mathrm{d}x}_{=0} \end{aligned} \tag{7.98}$$

$$= \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x + \sum_{j=1}^{\infty} a_j \int_{-\pi}^{\pi} \cos(nx)\cos(jx)\,\mathrm{d}x. \tag{7.99}$$

If $n = 0$ we get

$$\int_{-\pi}^{\pi} \cos(0) f(x)\, \mathrm{d}x = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos(0)\, \mathrm{d}x = \pi a_0. \tag{7.100}$$

Whereas if $n > 0$ we find

$$\int_{-pi}^{\pi} \cos(nx) f(x)\, \mathrm{d}x = \sum_{j=1}^{\infty} a_j \int_{-\pi}^{\pi} \cos(nx) \cos(jx)\, \mathrm{d}x. \tag{7.101}$$

All the terms on the right-hand side are zero except the one where $j = n$, where the result will be πa_n. So we have found formula for a_0 and a_n, they are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, \mathrm{d}x \tag{7.102}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) f(x)\, \mathrm{d}x. \tag{7.103}$$

Similarly, if we multiply by $\sin(nx)$ and integrate from $-\pi$ to π we find

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x)\, \mathrm{d}x. \tag{7.104}$$

More generally, if a function has a period P, then we can adjust the periods of sine and cosine and the formulae for the coefficients as follows:

$$f(x) = \frac{1}{2} a_0 + \sum_{j=1}^{\infty} a_j \cos\left(\frac{2\pi n}{P} x\right) + \sum_{j=1}^{\infty} b_j \sin\left(\frac{2\pi n}{P} x\right) \tag{7.105}$$

$$a_0 = \frac{2}{P} \int_{-\frac{P}{2}}^{\frac{P}{2}} f(x)\, \mathrm{d}x \tag{7.106}$$

$$a_n = \frac{2}{P} \int_{-\frac{P}{2}}^{\frac{P}{2}} \cos\left(\frac{2\pi n}{P} x\right) f(x)\, \mathrm{d}x \tag{7.107}$$

$$b_n = \frac{2}{P} \int_{-\frac{P}{2}}^{\frac{P}{2}} \sin\left(\frac{2\pi n}{P} x\right) f(x)\, \mathrm{d}x. \tag{7.108}$$

In fact, the integrals don't necessarily have to be from $-\frac{P}{2}$ to $\frac{P}{2}$, as long as they go over one full period. We can choose the most convenient range to integrate over.

Note that if a function is odd ($f(-x) = -f(x)$), the Fourier series will *only* contain sine terms. Likewise, if a function is even ($f(-x) = f(x)$) its Fourier series will *only* contain cosine terms. This is because sine is an odd function and cosine is even.

Example 7.6. Find the Fourier series of a square wave of amplitude d and period P, which has the form

$$f(x) = \begin{cases} -d, & -\frac{P}{2} < x < 0 \\ d, & 0 < x < \frac{P}{2} \end{cases} \tag{7.109}$$

on the domain $\left[-\frac{P}{2}, \frac{P}{2}\right]$. Outside of this domain it repeats periodically.

We have defined the square wave above as an odd function, so there should be no cosine terms. We can show this explicitly by calculating the a_n coefficients:

$$a_n = \frac{2}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} \cos\left(\frac{2\pi n}{P}x\right) f(x)\,\mathrm{d}x \tag{7.110}$$

$$= \frac{2}{P}\left[\int_{-\frac{P}{2}}^{0} -d\cos\left(\frac{2\pi n}{P}x\right)\mathrm{d}x + \int_{0}^{\frac{P}{2}} d\cos\left(\frac{2\pi n}{P}x\right)\mathrm{d}x\right] \tag{7.111}$$

$$= \frac{2d}{P}\underbrace{\left[\int_{0}^{\frac{P}{2}} \cos\left(\frac{2\pi n}{P}x\right)\mathrm{d}x - \int_{-\frac{P}{2}}^{0} \cos\left(\frac{2\pi n}{P}x\right)\mathrm{d}x\right]}_{=0 \text{ since } \cos(-x)=\cos(x)}. \tag{7.112}$$

Now calculating the b_n coefficients, we find

$$b_n = \frac{2}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} \sin\left(\frac{2\pi n}{P}x\right) f(x)\,\mathrm{d}x \tag{7.113}$$

$$= \frac{2}{P}\left[\int_{-\frac{P}{2}}^{0} -d\sin\left(\frac{2\pi n}{P}x\right)\mathrm{d}x + \int_{0}^{\frac{P}{2}} d\sin\left(\frac{2\pi n}{P}x\right)\mathrm{d}x\right] \tag{7.114}$$

$$= \frac{2d}{P}\left[\int_{0}^{\frac{P}{2}} \sin\left(\frac{2\pi n}{P}x\right)\mathrm{d}x - \int_{-\frac{P}{2}}^{0} \sin\left(\frac{2\pi n}{P}x\right)\mathrm{d}x\right] \tag{7.115}$$

$$= \frac{2d}{P}\left(\left[\frac{P}{2\pi n}\cos\left(\frac{2\pi n}{P}x\right)\right]_{-\frac{P}{2}}^{0} - \left[\frac{P}{2\pi n}\cos\left(\frac{2\pi n}{P}x\right)\right]_{0}^{\frac{P}{2}}\right) \tag{7.116}$$

$$= \frac{d}{\pi n}([1-\cos(-\pi n)] - [\cos(\pi n) - 1]) \tag{7.117}$$

$$= \frac{2d}{\pi n}[1-\cos(\pi n)]. \tag{7.118}$$

If n is even then $\cos(\pi n) = 1$ and if n is odd then $\cos(\pi n) = -1$, so we get

$$b_n = \begin{cases} 0 \text{ if } n \text{ is even} \\ \frac{4d}{\pi n} \text{ if } n \text{ is odd.} \end{cases} \tag{7.119}$$

Therefore the Fourier series for the square wave consists only of sine terms with odd n. The first few terms are

$$f(x) = \frac{4d}{\pi}\sin\left(\frac{2\pi}{P}x\right) + \frac{4d}{3\pi}\sin\left(\frac{6\pi}{P}x\right) + \frac{4d}{5\pi}\sin\left(\frac{10\pi}{P}x\right) + \ldots. \tag{7.120}$$

Let's now apply this amazing mathematical technique to the motion of a plucked string clamped at both ends. We know that the initial displacement of the string is given by equation 7.93, and we want to find the A_n's. However, the displacement of the string is not periodic in x because the string is finite in length! How we can find a Fourier series? We can just find a Fourier series for a periodic function that matches our string in the range 0 to L. Really, what we are asking is "what normal modes are excited when we pluck the string", so we actually want our Fourier series to contain only the normal modes. Since $\lambda_1 = 2L$, our periodic representation of the string *must* have period $2L$ otherwise the Fourier series won't contain the fundamental mode. Note that it must also be an odd function, because the initial conditions for the string only contains sines. With these constraints, we end up with the following function:

$$y(x,0) = \begin{cases} 4d\frac{x}{L}, & 0 < x < \frac{L}{4} \\ \frac{4d}{3}\left(1 - \frac{x}{L}\right), & \frac{L}{4} < x < L, \end{cases} \tag{7.121}$$

where d is the maximum initial displacement of the string. Using the formula for the Fourier coefficients with $y'(x)$ as the periodic extension of $y(x,0)$, we get

$$\begin{aligned} A_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{\sin\left(\frac{2\pi}{\lambda_n}x\right)}_{\text{odd}} \underbrace{y'(x)}_{\text{odd}}\,\mathrm{d}x && (7.122)\\ &= \frac{2}{L}\int_0^L \sin\left(\frac{2\pi}{\lambda_n}x\right) y'(x)\,\mathrm{d}x && (7.123)\\ &= \frac{2}{L}\int_0^L \sin\left(\frac{2\pi}{\lambda_n}x\right) y(x,0)\,\mathrm{d}x. && (7.124) \end{aligned}$$

The second line follows because two odd functions multiplied together make an even function, so we can cut the range of the integral in half and double the result. The last line follows because $y'(x) = y(x,0)$ on the domain $[0, L]$, so actually only the initial displacement between 0 and L matters! Now we can find the A_n's:

$$A_n = \frac{8d}{L^2}\int_0^{\frac{L}{4}} x\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x + \frac{8d}{3L}\int_{\frac{L}{4}}^{L} \sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x - \frac{8d}{3L^2}\int_{\frac{L}{4}}^{L} x\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x. \tag{7.125}$$

We can solve these integrals using the following results

$$\int \sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x = -\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right) + c \tag{7.126}$$

$$\int x\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x = \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{L}x\right) - \frac{Lx}{n\pi}\cos\left(\frac{n\pi}{L}x\right). \tag{7.127}$$

So we find

$$\begin{aligned} A_n = 8d&\left[\frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right) - \frac{1}{4\pi n}\cos\left(\frac{n\pi}{4}\right)\right] \\ &+ \frac{8d}{3}\left[\frac{1}{n\pi}\cos\left(\frac{n\pi}{4}\right) - \frac{1}{n\pi}\cos(n\pi)\right] \\ &+ \frac{8d}{3}\left[\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right) - \frac{1}{4\pi n}\cos\left(\frac{n\pi}{4}\right)\right]. \end{aligned} \tag{7.128}$$

The cosine terms all cancel out, and we are left with

$$A_n = \frac{32d}{3n^2\pi^2}\sin\left(\frac{n\pi}{4}\right). \tag{7.129}$$

$\sin\left(\frac{n\pi}{4}\right)$ follows the repeating sequence $\frac{1}{\sqrt{2}}$, 1, $\frac{1}{\sqrt{2}}$, 0, $-\frac{1}{\sqrt{2}}$, -1, $-\frac{1}{\sqrt{2}}$, 0..., so the first few terms of the Fourier series are

$$\begin{aligned} y(x,0) = \frac{32d}{3\pi^2}\Bigg[& \frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{L}x\right) + \frac{1}{4}\sin\left(\frac{2\pi}{L}x\right) + \frac{1}{9\sqrt{2}} + \sin\left(\frac{3\pi}{L}x\right) \\ & - \frac{1}{25\sqrt{2}}\sin\left(\frac{5\pi}{L}x\right) - \frac{1}{36}\sin\left(\frac{6\pi}{L}x\right) - \frac{1}{49\sqrt{2}}\sin\left(\frac{7\pi}{L}x\right) + \dots\Bigg] \end{aligned} \tag{7.130}$$

We can plot the **frequency spectrum** of $y(x,0)$, which is a plot showing A_n against n. We see that the lowest n are the largest components, and they can smaller as n increases. The gaps in the frequency spectrum are where the modes have a node at the point where the string was plucked. This means that because of where we chose to pluck the string, the resulting vibration will not contain any of those frequencies at all.

With damping effects, we find that all of the normal modes decay away, with the higher frequency modes decaying faster.

7.7 Sound Waves

So far we have been looking at transverse waves, where the displacement is perpendicular to the direction of wave travel. Examples of transverse waves in real life are waves on a string and light waves. Waves can also be **longitudinal**, where the displacement is along the direction of travel. Sound waves are longitudinal, with regions of high density along the direction of travel (compression) and low density (rarefaction).

Let's look at some snapshots of longitudinal waves. We denote the displacement of a particle from its equilibrium position by $s(x,t)$. Then the particles oscillate back and forth along the direction of motion with SHM. Where s is positive, the particles are displaced to the right, and where s is negative they are displaced to the left. The points of zero displacement represent pressure maxima and minima.

It is a general rule for all mechanical waves that the wave speed is related to the ratio of restoring force to inertia. For liquids and gases, this is

$$v = \sqrt{B}\rho, \tag{7.131}$$

where ρ is the density and B is the **bulk modulus**, which is a measure of how easy it is to compress the medium, given by

$$B = -V\frac{\mathrm{d}P}{\mathrm{d}V}. \tag{7.132}$$

In everday situations, we can approximate $\frac{\mathrm{d}P}{\mathrm{d}V}$ as $\frac{\Delta P}{\Delta V}$ i.e. the pressure change that accompanies a small volume change. We find that for air $B_{\mathrm{air}} = 1.42\times10^5\,\mathrm{N\,m^{-2}}$, and for water $B_{\mathrm{water}} = 2.2\times10^9\,\mathrm{N\,m^{-2}}$. The density of air will change in everyday situations depending on the temperature. At room temperature (293 K), we have $v = 343\,\mathrm{m\,s^{-1}}$. More generally, we have $v \propto T$.

The equations of pressure waves have the same form as transverse waves:

$$s(x,t) = A\cos(kx - \omega t + \phi_0). \tag{7.133}$$

Can we get a formula for the variation of pressure across a sound wave? Along the direction of propogation, volume elements oscillate in SHM and pressure variations cause the volumes to change slightly. This is because the left and right-hand sides of undergo slightly different displacements. Thus by the definition of the bulk modulus, we get

$$P(x,t) = -B\frac{\partial s(x,t)}{\partial x} \tag{7.134}$$
$$= BAk\sin(kx - \omega t + \phi_0) \tag{7.135}$$
$$= P_{\text{max}}\sin(kx - \omega t + \phi_0). \tag{7.136}$$

Note that this pressure is the excess pressure deviation from equilibrium, *not* the absolute pressure in the fluid. From this, we can see that on the plot of $s(x,t)$, points of zero displacement with *positive* derivative are the pressure *maxima*, and those with *negative* derivative are the pressure *minima*.

When we calculated the power transmitted by a transverse wave on a string, we could neglect the other directions and consider the problem in 1D. For a sound wave this is not possible, so we have to consider plane waves (waves that have constant value on a plane perpendicular to the direction of motion). Instead of power, we look at *intensity* which is defined as power per unit area, and we measure it across surfaces perpendicular to the propagation of the wave. We have a force F which does work along a distance $\mathrm{d}s$, so $\mathrm{d}W = F\,\mathrm{d}s$, then the intensity is the power $\frac{\mathrm{d}W}{\mathrm{d}t}$ divided by the area which the force acts on S:

$$I_{\text{inst}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F}{S}\,\mathrm{d}s\right) \tag{7.137}$$
$$= \frac{F}{S}\frac{\partial s}{\partial t} \tag{7.138}$$
$$= P(x,t)\frac{\partial s}{\partial t} \tag{7.139}$$
$$= -B\frac{\partial s}{\partial x}\frac{\partial s}{\partial t} \tag{7.140}$$
$$= BA^2k\omega\sin^2(kx - \omega t + \phi_0). \tag{7.141}$$

Note how similar the second last line looks to the equation for instantaneous power of a transverse wave on a string. Taking the time average, we get

$$I_{\text{avg}} = \frac{1}{2}BA^2k\omega \tag{7.142}$$
$$= \frac{1}{2}\frac{P_{\text{max}}^2}{\sqrt{\rho B}}, \tag{7.143}$$

where in the second line we have substituted more physical quantities. The denominator $\sqrt{\rho B}$ is known as the **specific acoustic impedance** Z, and this can be used to define transmission and reflection coefficients as we did before.

7.8 Longitudinal Standing Waves

We now have two ways of describing longitudinal waves, by the displacement of the particles and by the pressure along the direction of travel.

$$s(x,t) = A\cos(kx - \omega t + \phi_0) \tag{7.144}$$
$$P(x,t) = P_{\max}\sin(kx - \omega t + \phi_0). \tag{7.145}$$

We can see that if we have a longitudinal standing wave, displacement nodes will be pressure antinodes, and pressure nodes will be displacement antinodes.

Let's look at the allowed wavenumbers for longitudinal waves in a pipe with closed and open ends. In the case with both ends closed, we will have displacement nodes at each end of the pipe (the medium cannot move through the end). Thus the normal modes will have the form

$$s_n(x,t) = A\sin(k_n x)\cos(\omega_n t + \phi_0), \tag{7.146}$$

where $k_n = \frac{n\pi}{L}$, $\lambda_n = \frac{2L}{n}$, and $f_n = \frac{nv}{2L} = nf_1$. In the case where both ends are open, we have displacement antinodes at each end. This leads to normal modes of the form

$$s_n(x,t) = A\cos(k_n x)\cos(\omega_n t + \phi_0), \tag{7.147}$$

with the same allowed wavenumbers. In the case of pipe with one and closed and one end open, there will be a displacement node at the closed end and a displacement antinode at the open end. Depending on which end of the pipe is open, we would use sine or cosine for the spatial part of the normal modes. The allowed wavenumbers are then $k_n = \frac{(2n-1)\pi}{2L}$, $\lambda_n = \frac{4L}{2n-1}$, $f_n = \frac{(2n-1)v}{4L} = (2n-1)f_1$. Note that in this case the fundamental mode has a wavelength of $4L$.

7.9 The Doppler Effect

Consider a source at rest emitting spherical waves with frequency f_s and speed v. What does a detector towards the source with speed v_d see? In this situation, the wavefronts approach the detector with speed $v + v_d$, so the frequency as measured by the detector is

$$f_d = \frac{v + v_d}{\lambda} = f_s\frac{v + v_d}{v}. \tag{7.148}$$

If the detector is moving away from the source, then the observed frequency will be

$$f_d = f_s\frac{v - v_d}{v}. \tag{7.149}$$

This change in frequency due to a difference in velocity is known as the **Doppler effect**.

What about a moving source and a stationary detector? During one period T_s of the source emitting, the waves will propagate a distance vT_s. The source itself moves a distance v_sT_s. If the source is moving towards the detector, then the observed wavelength is $\lambda_d = vT_s - v_sT_s = T_s(v - v_s)$, which gives an observed frequency of

$$f_d = \frac{v}{T_s(v - v_s)} = f_s\frac{v}{v - v_s}. \tag{7.150}$$

If the source is moving away from the detector, then we get

$$f_d = f_s\frac{v}{v + v_s}. \tag{7.151}$$

In the general case of a moving source and detector, we have

$$f_d = f_s \frac{v \pm v_d}{v \mp v_s}, \tag{7.152}$$

where we have the $\frac{+}{-}$ case when the source and detector are moving closer and the $\frac{-}{+}$ case when they are moving apart.

7.10 Wave Interference

When two waves come together the result is the sum of both waves at every point. A special case of superposition is created when two waves of the same frequency (coherent waves) meet, called **interference**. When the crests of two coherent waves meet, they add together and we get a crest which is larger than either of the initial ones, which is called **constructive interference**. When a crest of one wave meets a trough of another, they cancel out and we get a much smaller amplitude, or even nothing if the initial amplitudes match. This is **destructive interference**. Let's look at this mathematically. Consider two waves of the same amplitude and frequency travelling in the same direction in 1D. Lets look at the amplitude at a point x_i which is a distance x_1 from the source of the first wave and a distance x_2 from the source of the second. The resulting amplitude is

$$y(x_i, t) = A\cos(kx_1 - \omega t + \phi_{0,1}) + A\cos(kx_2 - \omega t + \phi_{0,2}). \tag{7.153}$$

Now, we will make use of the addition formula

$$A\cos(\theta_1) + A\cos(\theta_2) = 2A\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_1 - \theta_2}{2}\right). \tag{7.154}$$

Then the amplitude at x_i becomes

$$y(x_i, t) = 2A\cos\left(\frac{k(x_1 + x_2) - 2\omega t + \phi_{0,1} + \phi_{0,2}}{2}\right)\cos\left(\frac{k(x_1 - x_2) + \phi_{0,1} - \phi_{0,2}}{2}\right) \tag{7.155}$$

$$= 2A\cos\left(\frac{\Delta\phi}{2}\right)\cos(k\bar{x} - \omega t + \bar{\phi}_0), \tag{7.156}$$

where we have defined the *total* phase difference $\Delta\phi = k(x_2 - x_1) + \phi_{0,2} - \phi_{0,1}$, the average distance from x_i to each source $\bar{x} = \frac{x_1 + x_2}{2}$, and the average initial phase $\bar{\phi}_0 = \frac{\phi_{0,1}+\phi_{0,2}}{2}$. The maximum amplitude of the superposition is $2A\cos\left(\frac{\Delta\phi}{2}\right)$, so we see that this is maximised — i.e. we get destructive interference — for $\Delta\phi = 2m\pi$ for integer m. Likewise the maximum amplitude goes to zero and we get destructive interference when $\Delta\phi = (2m + 1)\pi$.

To extend this to more than one dimension, we must keep in mind that wavefronts are circular in 2D and spherical in 3D. If we are very far away from the source, the curvature of the waves becomes negligible and we get **plane waves**. This is known as the **far-field limit**.

Consider an colinear array of n wave sources all separated by a distance d. Let's put ourselves in the far-field limit, so that all the incoming wavefronts from the array will be parallel. The **path difference** (the extra distance one wave has travelled relative to another) is given by $h = d\sin\theta$. Then the **phase difference** is just the path difference multiplied by the wavelength

$\Delta\phi = kh = \frac{2\pi}{\lambda} d \sin\theta$. If this phase difference is a multiple of 2π, we get constructive interference. This is given by the condition

$$d \sin\theta_m = m\lambda, \tag{7.157}$$

where θ_m labels the angles where we get maxima in amplitude. The value of $m = 0, 1, 2, ...$ is called the **order** of the maxima. Inbetween the maxima, we get secondary maxima with amplitude given by

$$s(\theta) = \varepsilon \frac{\sin\left(n\frac{\Delta\phi}{2}\right)}{\sin\left(\frac{\Delta\phi}{2}\right)} = \varepsilon \frac{\sin\left(n\frac{kd\sin\theta}{2}\right)}{\sin\left(\frac{kd\sin\theta}{2}\right)}, \tag{7.158}$$

where n is the number of sources and ε is the amplitude of a single source. The intensity will be proportional to the square of the amplitude.

$$I(\theta) = I_1 \frac{\sin^2\left(n\frac{kd\sin\theta}{2}\right)}{\sin^2\left(\frac{kd\sin\theta}{2}\right)}, \tag{7.159}$$

where I_1 is the intensity from a single source.

Constructive interference is quite often used to examine structures which are too small to observe with visible light, as was historically the case with **Bragg scattering**. Lawrence and William Henry Bragg used constructive interference of X-rays to measure the distance between atoms in crystals in the early 1910s. Consider a crystal structure with layers of atoms separated by a distance d. When X-rays enter the crystal, they scatter off the layers. Consider two parallel incident rays which reflect off the top two layers. The path difference between the two rays will be $2d\sin\theta$, so therefore we get constructive interference when the path difference is an integer multiple of the wavelength:

$$2d\sin\theta = n\lambda. \tag{7.160}$$

Interference also gives rise to the phenomenon known as **diffraction**, which is where waves are observed to curve around apertures and obstructions in their path. Each part of the wavefront in the gap or around the barrier becomes a secondary source of spherical waves, and these waves interfere to give a curved interference pattern. This method of analysis is known as the **Huygens-Fresnel principle**.

Consider a plane wave moving through a gap of height d. Let's look at a thin slice of the gap of length $\mathrm{d}y$ at a distance y from the middle as a point source. The amplitude due to this small slice at a far away point P is $\mathrm{d}s = \varepsilon_R \, \mathrm{d}y \cos(kr - \omega t)$, where r is the distance from the thin slice to P and R is the distance from the middle of the gap to P. If $R \gg d$, then $r \approx R$ so ε_R only depends on R. It can be shown that $r \approx R - y\sin\theta$, which we will need to use because small differences in the distance *will* make a big difference for the phase of the wave at P. Integrating over the whole gap, we get

$$s(\theta) = \int \mathrm{d}s = \varepsilon_R \int_{-\frac{d}{2}}^{\frac{d}{2}} \cos(k(R - y\sin\theta) - \omega t)\,\mathrm{d}y \tag{7.161}$$

$$= 2\varepsilon_R d \frac{sin\left(\frac{kd\sin\theta}{2}\right)}{kd\sin\theta} \cos(kR - \omega t). \tag{7.162}$$

The intensity is proportional to the time average of the square of the amplitude.

$$I(\theta) = 4I_0 \frac{\sin^2\left(\frac{kd\sin\theta}{2}\right)}{k^2 d^2 \sin^2\theta}, \tag{7.163}$$

where I_0 is the intensity at $\theta = 0$. Note that in this situation minima occur at integer multiples of the wavelength, not maxima.

If we superpose two sounds with a very similar frequency, we will hear a periodic variation in sound intensity. Consider two longitudinal sound waves:

$$s_1(x,t) = A\cos(k_1 x - \omega_1 t) \tag{7.164}$$
$$s_2(x,t) = A\cos(k_2 x - \omega_2 t), \tag{7.165}$$

where $k_1 \approx k_2$ and $\omega_1 \approx \omega_2$. Then using the addition formula, the superposition is

$$s_1 + s_2 = 2A\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right) \tag{7.166}$$
$$= 2A\cos(\bar{k}x - \bar{\omega}t)\cos(k_{\text{mod}}x - \omega_{\text{mod}}t), \tag{7.167}$$

where we have defined the average wavenumber and frequency $\bar{k}$ and $\bar{\omega}$, and a **modulation wavenumber** and **modulation frequency** k_{mod} and ω_{mod}. This superposition is a product of a wave with the average frequency of the two original ones and a low frequency modulation (since the two original frequencies are close, the modulation frequency is very low). The intensity rises and falls twice per period, so the **beat frequency** is twice the modulation frequency, $f_{\text{beat}} = 2f_{\text{mod}} = f_1 - f_2$.

Now consider the case where we allow the phase velocities of the waves to be different ($v_1 = \frac{\omega_1}{k_1} \neq 2 = \frac{\omega_2}{k_2}$). Then we can write

$$s_1(x,t) = A\cos((k_0 + \Delta k)x - (\omega_0 + \Delta\omega)t) \tag{7.168}$$
$$s_2(x,t) = A\cos((k_0 - \Delta k)x - (\omega_0 - \Delta\omega)t), \tag{7.169}$$

and we get

$$s_1 + s_2 = 2A\cos\left(\frac{2k_0}{2}x - \frac{2\omega_0}{2}t\right)\cos\left(\frac{2\Delta k}{2}x - \frac{2\Delta\omega}{2}t\right) \tag{7.170}$$
$$= 2A\cos(k_0 x - \omega_0 t)\cos(\Delta k x - \Delta\omega t). \tag{7.171}$$

This is the product of a higher frequency wave with a lower frequency envelope. The higher frequency wave has phase speed $v_{\text{crest}} = \frac{\omega_0}{k_0}$ and the envelope has speed $v_{\text{env}} = \frac{\Delta\omega}{\Delta k}$. If $\omega \propto k$, then these speeds are always the same. However, if this is not the case then the wave crests travel at a different speed to the envelope. If the range of wavenumbers in a superposition is small, then the group velocity is the speed of the envelope (the largest amplitude) and we have $v_{\text{gr}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}\Big|_{k_0}$.

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