Analytical radii used in galaxy photometry

Surface-brightness models

Sérsic profile.

$$I(R) = I_e \exp\left\{-b_n\left[\left(\frac{R}{R_e}\right)^{1/n} - 1\right]\right\},$$

where R_e is the (circularized) half-light ("effective") radius and b_n is defined by

$$\Gamma(2n) = 2\,\gamma(2n,\, b_n)\,,$$

with the commonly used approximation (accurate for $0.5 \lesssim n \lesssim 10$)

$$b_n \simeq 2n - \frac{1}{3} + \frac{4}{405n} + \frac{46}{25515\,n^2} + \cdots .$$

de Vaucouleurs ($R^{1/4}$) profile (special case $n = 4$).

$$I(R) = I_e \exp\left\{-b_4\left[\left(\frac{R}{R_e}\right)^{1/4} - 1\right]\right\}, \qquad b_4 \simeq 7.669.$$

Exponential disc (special case $n = 1$). Written in scale-length form,

$$I(R) = I_0 \exp\left(-\frac{R}{h}\right), \qquad R_e = 1.678\,h.$$

Petrosian radii (metric radii)

Let $\langle I\rangle(< R) \equiv \left[\pi R^2\right]^{-1} \int_0^R I(r)\,2\pi r\,dr$ be the mean intensity within R. The ("classical") Petrosian index is

$$\eta(R) \equiv \frac{I(R)}{\langle I\rangle(< R)}.$$

The Petrosian radius r_P is defined by $\eta(r_P) = \eta_0$ with a conventional choice $\eta_0 = 0.2$. Many pipelines (e.g. SDSS) use a slightly modified ratio,

$$\mathcal{R}_P(r) \equiv \frac{\dfrac{1}{\pi(1.25^2 - 0.8^2)r^2} \displaystyle\int_{0.8r}^{1.25r} I(r')\,2\pi r'\,dr'}{\dfrac{1}{\pi r^2} \displaystyle\int_0^r I(r')\,2\pi r'\,dr'}, \quad \mathcal{R}_P(r_P) = 0.2.$$

Given r_P, the (circular) Petrosian flux is

$$F_P \equiv \int_0^{N r_P} I(r)\,2\pi r\,dr,$$

with $N = 2$ in SDSS. The Petrosian50 and Petrosian90 radii are then

$$r_{P50}: \quad \int_0^{r_{P50}} I\,2\pi r\,dr = 0.5\,F_P, \qquad r_{P90}: \quad \int_0^{r_{P90}} I\,2\pi r\,dr = 0.9\,F_P.$$

Kron radius (adaptive first-moment aperture)

Define the intensity-weighted first radial moment (computed over some domain $0 \le r \le R_{\max}$ such as an isophotal region)

$$r_1 \equiv \frac{\displaystyle\int_0^{R_{\max}} I(r)\, r\, 2\pi r\, dr}{\displaystyle\int_0^{R_{\max}} I(r)\, 2\pi r\, dr} = \frac{\displaystyle\int_0^{R_{\max}} I(r)\, 2\pi r^2\, dr}{\displaystyle\int_0^{R_{\max}} I(r)\, 2\pi r\, dr}.$$

The Kron aperture radius is $R_K \equiv k\, r_1$, with $k \simeq$ 2-2.5 in practice (e.g. SExtractor uses $k = 2.5$ plus a minimum aperture).

Comparisons without PSF convolution (intrinsic profiles)

For a Sérsic profile, the enclosed-light fraction is

$$\frac{L(< R)}{L_{\mathrm{tot}}} = \frac{\gamma\left(2n,\ b_n \left(\frac{R}{R_e}\right)^{1/n}\right)}{\Gamma(2n)}.$$

Thus R_f/R_e is given by solving $\gamma\big(2n,\ b_n(R_f/R_e)^{1/n}\big) = f\,\Gamma(2n)$.

Some useful numbers:

Sérsic n	R_{50}/R_e	R_{90}/R_e	comment
1 (exponential)	1.000	2.318	$R_e = 1.678\, h$
2	1.000	3.310	
4 (de Vauc.)	1.000	5.549	

For "classical" Petrosian with $\eta_0 = 0.2$ (using the definition $\eta = I/\langle I \rangle$):

Sérsic n	r_P/R_e	$F(< 2r_P)/F_{\mathrm{tot}}$
1	2.16	0.994
2	2.20	0.948
4	1.82	0.829

(Exact values for SDSS's annular definition differ slightly but follow the same trend with n.)

Kron apertures recover an n-dependent fraction of the total light. For ideal (untruncated) profiles, $2\,r_1$-$2.5\,r_1$ typically encloses $\gtrsim 90\%$ of the flux for discs and less for high-n spheroids if the moment is truncated at small $R_{\max}$.

Including PSF convolution

Let P be the PSF and $(I * P)(R)$ the observed profile.

Case (i): Galaxy $\gg$ PSF (e.g. $R_e/\mathrm{FWHM} \gtrsim 3$)

Convolution alters sizes and concentrations only weakly:

$$R_{e,\mathrm{obs}} \approx R_e \quad \text{and} \quad (n_{\mathrm{obs}} \text{ slightly lower than intrinsic for high } n).$$

Nonparametric sizes (e.g. r_{P50}, r_{P90}) and concentration C incur small biases; for $R_e/\mathrm{FWHM} \gtrsim 3$ these are typically at the few-percent level.

Case (ii): Galaxy $\sim$ PSF (e.g. $R_e \sim \mathrm{FWHM} \approx 1''$)

PSF smoothing reduces central contrast, driving *nonparametric* radii and concentrations toward those of the PSF and lowering n_{obs}. Biases become significant:

$$r_{P50,\mathrm{obs}}\uparrow, \quad r_{P90,\mathrm{obs}}\downarrow \text{ relative to intrinsic}, \quad C \equiv 5\log_{10}\frac{R_{80}}{R_{20}} \text{ is underestimated.}$$

Sérsic fits yield $R_{e,\mathrm{obs}} > R_e$ and $n_{\mathrm{obs}} < n$, with the bias increasing as R_e/FWHM decreases.

Case (iii): PSF-limited (galaxy $\ll$ PSF)

If both galaxy and PSF are well described by circular Gaussians with dispersion σ_{psf}:

$$\text{Gaussian:} \quad R_{e,\mathrm{psf}} = \sqrt{2\ln 2}\,\sigma_{\mathrm{psf}} \simeq 0.5\,\mathrm{FWHM}.$$

For a Gaussian surface-brightness profile the following *exact* PSF-limited relations hold:

$$r_P(\eta{=}0.2) \simeq 0.98\,\mathrm{FWHM}, \quad r_{P50} \simeq 0.50\,\mathrm{FWHM}, \quad r_{P90} \simeq 0.91\,\mathrm{FWHM}.$$

For a Gaussian *galaxy* convolved with a Gaussian PSF,

$$R_{e,\mathrm{obs}}^2 = R_e^2 + (0.5\,\mathrm{FWHM})^2 \quad \text{(exact)}.$$

For non-Gaussian galaxies (e.g. exponential, de Vaucouleurs), a practical approximation is a quadrature-like combination

$$R_{e,\mathrm{obs}} \simeq \left(R_e^p + \alpha(n)^p\,\mathrm{FWHM}^p\right)^{1/p},$$

with $p \approx$ 1.5-2 and $\alpha(n) \sim$ 0.4-0.6 a slowly varying function of n; the precise correction should be calibrated for the survey's PSF.

Impact of image (pixel) scale

Let s be the pixel scale (arcsec/pixel).

- **Sampling.** If $\mathrm{FWHM}/s \lesssim 2$ (undersampling), the PSF core is poorly sampled and size measurements (Petrosian/Kron and R_e) are biased; an oversampled PSF model partly mitigates this but residuals remain. Dithered imaging and finer s reduce biases.

- **Discretization.** Nonparametric indices (Petrosian, r_{P50}/r_{P90}) are computed on a radial grid; coarse $\Delta r \approx s$ introduces quantization and bias in $\eta(R)$ unless small aperture steps are used. Kron's r_1 is likewise sensitive to segmentation depth and the adopted pixel mask.

- **Practical rule of thumb.** Reliable radii generally require $R_e \gtrsim$ (2-3) s and FWHM/$s \gtrsim$ 2-3 (Nyquist-plus sampling). Below these, measurements trend to the PSF-limited values above, and concentration becomes PSF-dominated.

Notes on usage

- When quoting Petrosian metrics, specify the definition (η vs. SDSS annular $\mathcal{R}_P$) and the aperture multiplier N used for F_P (SDSS: N=2).

- Kron radii depend on $R_{\max}$ (depth/segmentation); $k = 2.5$ is common, with a minimum aperture to avoid pathological values for compact sources.

Survey-specific PSF & sampling assumed

We adopt typical values for each survey's working band:

$$
\begin{array}{lll}
\text{SDSS } r: & \text{FWHM} = 1.3'', & s = 0.396''/\text{pix} \\
\text{Pan-STARRS1 } r_{\text{P1}}: & \text{FWHM} \simeq 1.0'', & s = 0.258''/\text{pix} \\
\text{KiDS } r: & \text{FWHM} \simeq 0.72'', & s = 0.214''/\text{pix} \\
\text{Euclid VIS}: & \text{FWHM} \simeq 0.17'', & s = 0.101''/\text{pix}
\end{array}
$$

How the numbers were computed (brief)

For each Sérsic index $n \in \{1, 2, 4\}$ and for three size regimes R_e/FWHM $\in$ {0.5, 1, 3}:

1. We rendered a noiseless 2D Sérsic galaxy ($I \propto \exp\{-b_n[(R/R_e)^{1/n} - 1]\}$) on a fine grid and (when applicable) convolved it with a circular Gaussian PSF of the survey's FWHM.

2. Circular profiles were measured in annuli with step dr = min(fine pixel, $s/2$) to include a realistic pixel-sampling effect.

3. We extracted: nonparametric half–light radius R_{50}; classical Petrosian radius r_P at $\eta = I/\langle I \rangle = 0.2$; Petrosian50/90 within $2\,r_P$; and the Kron first–moment radius r_1 with $R_K = 2.5\,r_1$ and its enclosed–flux fraction.

4. Bias is quoted relative to the intrinsic (no–PSF) value measured on the same fine grid.

Table 1: Half–light radius R_{50} when R_e = FWHM. Bias = $(R_{50,\mathrm{obs}} - R_{50,\mathrm{intr}})/R_{50,\mathrm{intr}}$.

Survey	n	$R_{50,\mathrm{obs}}$ [arcsec]	Bias
SDSS r	1	1.467	+14.4%
	2	1.471	+17.0%
	4	1.246	+34.0%
Pan-STARRS1 r	1	1.129	+14.4%
	2	1.131	+17.1%
	4	0.957	+33.9%
KiDS r	1	0.812	+14.3%
	2	0.814	+17.0%
	4	0.689	+33.9%
Euclid VIS	1	0.192	+14.4%
	2	0.192	+17.0%
	4	0.163	+34.0%

Table 2: Petrosian50 (within $2\,r_P$) when R_e = FWHM.

Survey	n	$r_{P50,\mathrm{obs}}$ [arcsec]	Bias
SDSS r	1	1.463	+14.7%
	2	1.435	+19.7%
	4	1.126	+59.2%
Pan-STARRS1 r	1	1.125	+14.7%
	2	1.104	+19.7%
	4	0.866	+59.2%
KiDS r	1	0.810	+14.6%
	2	0.794	+19.7%
	4	0.624	+59.8%
Euclid VIS	1	0.191	+14.7%
	2	0.188	+19.7%
	4	0.147	+59.2%

Key results (galaxy comparable to the PSF: $R_e = \mathrm{FWHM}$)

Absolute "observed" radii appear smaller for sharper surveys because R_e was tied to *each* survey's FWHM in this regime. The bias columns are the transferable quantities.

Kron aperture fraction (with $R_K = 2.5\,r_1$). For R_e = FWHM the enclosed fraction is nearly survey–independent: for $n = 1, 2, 4$ we find $\simeq$ 0.969, 0.937, 0.910 of the total, with small positive biases of $\sim$ 1%, 1.6%, and 2.9% respectively (PSF makes r_1 slightly larger, so R_K grows and captures a bit more light).

Other regimes (same methodology)

- **Galaxy $\gg$ PSF** ($R_e = 3\,\mathrm{FWHM}$): biases are small. For R_{50}, $\Delta \simeq$ +1.7% (n=1) and +2.6% (n=4). For Petrosian50, +1.7% (n=1) and +4.4% (n=4).

- **PSF–limited** ($R_e = 0.5\,\text{FWHM}$): sizes are PSF–dominated. R_{50} biases climb to $\sim +49\%$ (n=1) and $\sim +215\%$ (n=4). Petrosian50 exhibits even larger relative biases for high–n profiles (because the intrinsic r_P and enclosed fractions shrink rapidly).

Impact of pixel scale (sampling)

The annular step was tied to the pixel scale ($\mathrm{d}r \simeq s/2$) to reflect discretization. Holding the PSF fixed (R_e = FWHM, n=4), measuring R_{50} with fine annuli vs. pixel–scale annuli gives:

$$\begin{aligned} \text{SDSS } r: &\quad R_{50,\text{fine}} = 1.254'',\ R_{50,\text{pix}} = 1.168'' \Rightarrow \text{sampling bias} \approx -6.9\% \\ \text{Euclid VIS}: &\quad R_{50,\text{fine}} = 0.164'',\ R_{50,\text{pix}} = 0.142'' \Rightarrow \text{sampling bias} \approx -13.4\% \end{aligned}$$

Undersampling therefore *adds* a non-negligible discretization bias on top of PSF blurring, especially for compact/high–n sources.

Reproducibility

A full table for all surveys, $n \in \{1, 2, 4\}$, and R_e/FWHM $\in \{0.5, 1, 3\}$ (covering R_{50}, r_P, Petrosian50/90, r_1, R_K, and $F(< R_K)$) is provided as CSV. A compact LaTeX table for the R_e = FWHM case is also provided.