# Sine Approximation for Direct Digital Frequency Synthesizers and Function Generators

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*Abstract:* - There are several compression methods for phase to amplitude converter for sine function. In this paper, the linear and parabolic approximation of equi-section division utilizing the symmetry property and amplitude approximation of a sinusoidal waveform for direct digital frequency synthesizer (DDS) and also for other waveform applications, e.g. function generators and telecommunication devices is proposed. The sinusoidal phase from 0 to π/2 is divided into different numbers of equi-sections. The linear and parabolic approximations are compared. The simulation results show spectral properties of booth approach. The presented sine approximation is usable for FPGA implementation.

*Key-Words:* Direct digital synthesis, frequency spectrum, frequency synthesizers, linear approximation, parabolic approximation, phase to amplitude converter, simulation.

## 1 Introduction

Direct digital frequency synthesis is increasingly welcomed in modern communication systems and precise electronic systems, due to their significant advantages over phase-locked loop (PLL) based synthesizers. Fast frequency switching, excelent frequency resolution, continuous-phase switching and low phase noise are important features easily obtainable in direct digital synthesizers is a well-known technique for the generation of a sinusoidal waveform reconfigurably [1]. DDS techniques have lots of advantages over analog approaches beyond flexibility, including excellent frequency resolution, fast response time, fast frequency switching, and low phase noise.

The DDS architecture, is shown in Fig. 1. The system has two inputs: a clock frequency *fclk* and a frequency control word (*FCW*). The phase accumulator accumulates the value of the *FCW* in every clock cycle, periodically overflowing. Its ramp output is considered as a phase in the interval ⟨0, 2π⟩, while the phase-to-sinusoid amplitude converter provides a digital amplitude value of the sinusoidal waveform of that phase. The output frequency of DDS is given by

$$f_o = f_{clk} \frac{FCW}{2^N} \qquad \text{[Hz]} \qquad (1)$$

The sequences of digital amplitude values pass through a digital-to-analog converter (DAC) and usualy to low-pass filter (LPF) to generate a sinusoidal signal [2].

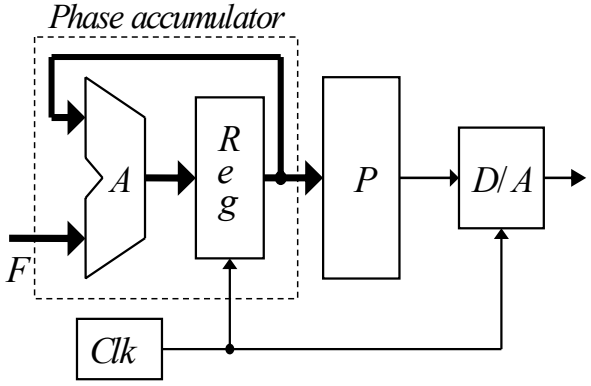


Fig. 1. The DFS architecture block diagram. *F*- frequency control word, *Clk*- clock generator, *Reg* – register, P – phase-to-sine amplitude converter, *D/A* – digital/analog converter.

There are a wide variety of methods for phase to sine-amplitude conversion [3, 4, 5], from full-memory to full-hardware approaches. The methods can be placed into three categories.

a) Based on ROM lookup tables (LUT), containing amplitude samples of the complete, half, or quarter period of sinusoid, and addressed by the digital phase, which is generated by the phase accumulator [1, 2].

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b) Full computational methods, trying to compute the sine amplitude samples from the digital phase, which is generated by the phase accumulator. Sine computation by using Taylor series [1] and CORDIC algorithm [6] are two examples of this category.

c) Initial gusess correction methods, in which an initial guess for the sine function is generated by digital hardware and then is corrected by small ROM lookup table, containing the difference between the initial guess and the accurate value for the sine amplitude. If the initial guess is properly provided, in each memory location just small correcting data will be stored instead of the whole sine amplitude. It is obvious that the closer the approximation is to ideal sine function, the more memory sawing there will be.

There are several ROM compression methods, such as the quarter-wave symmetry method [7], the sine amplitude approximation, eg.

Sine-Phase Difference Algori thm [2].

Modified Sunderland Archite cture [4].

Nicholas' Architecture [2].

Taylor Series Approximation [5].

The another possibility is ROM-less DDS with Sin-weighted DAC.

With the quarter-wave symmetry method, the ROM only needs to store one-quarter (the phase from 0 to $\pi/2$) of the sine wave sequences [8 - 15]. The two most significant bits (MSBs) of the phase value determine to which quadrant an angle belongs, while the residual $M-2$ bits are used to address a one-quadrant sine LUT.

In this paper, the linear and parabolic approximation of equi-section division is proposed as a ROM compression method which utilizes the property of the quarter-wave symmetry method as well as sine amplitude approximation. The length of period in all simulations is 4096 and sine amplitude is ± 4095 ($2^{13}$-1).

The MSE (mean square error) is calculated according (2)

$$MSE = \sum_{i=1}^{N} \left( y_i - sin_i(.) \right)^2 \qquad (2)$$

## 2 Linear and Parabolic Approximation

The sinusoidal phase from 0 to π/2 is divided into $s$ equi-sections, where $s$ is the parameter by which to decide the number of equi-sections. The coefficients for linear and polynomial approximations are derived by least squares sense method.

The example of linear approximation (for 4 sections) is shown in Fig. 2, and example of parabolic approximation (also for 4 sections) is displayed in Fig, 3. The error of approximation is also shown. The sinusoidal phase from 0 to $\pi/2$ is divided into equi-sections.

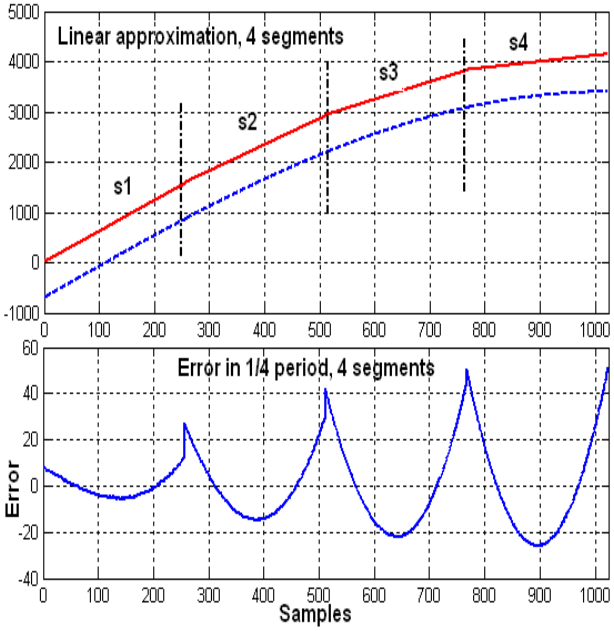



Fig. 2. Example of linear approximation quarter of sine (top) and approximation error (bottom). Approximation – solid line (red), sine – dash line (blue), shifted down.
(4 segments s1, s2, s3, s4 are used).

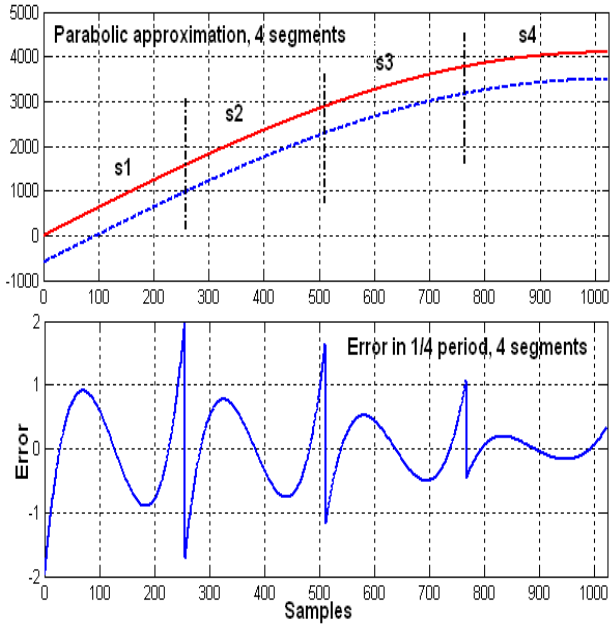



Fig. 3. Example of parabolic approximation quarter of sine (top) and approximation error (bottom). Approximation – solid line (red), sine – dash line (blue), shifted down.
(4 segments s1, s2, s3, s4 are used).

Equation (3) below gives the slopes and shift for linear approximation (example for 4 segments) as

$$4095 \cdot sin\left(2\pi \frac{x}{4096}\right) \approx y = \begin{cases} 6.138x + 8; & 0 \le x < 256 \\ 5.205x + 261; & 256 \le x < 512 \\ 3.48x + 1155; & 512 \le x < 768 \\ 1.225x + 2892; & 768 \le x < 1024 \end{cases} \quad (3)$$

Result is shown in Fig. 2.

Equation (4) shows the equations for parabolic approximation (example for 4 segments)

$$4095 \cdot sin\left(2\pi \frac{x}{4096}\right) \approx y = \begin{cases} -0.0009x^2 + 6.37x - 1.98; & 0 \le x < 256 \\ -0.0027x^2 + 7.25x - 116; & 256 \le x < 512 \\ -0.0040x^2 + 8.59x - 456; & 512 \le x < 768 \\ -0.0047x^2 + 9.66x - 860; & 768 \le x < 1024 \end{cases} \quad (4)$$

Result is shown in Fig. 3.

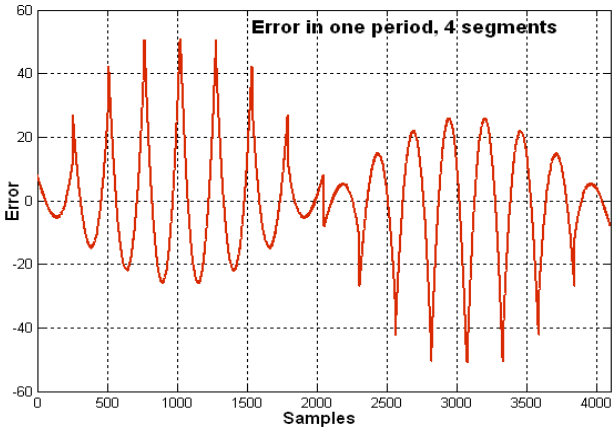


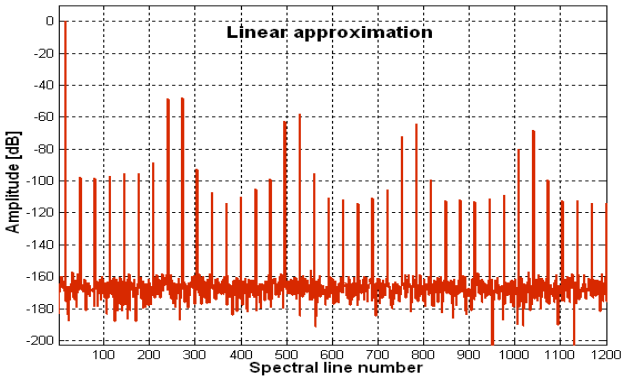

Fig. 4. Example of error calculated for 1 period for linear approximation and 4 segments.

## 3 Linear Approximation for 4, 8, 16, 32 and 64 segments

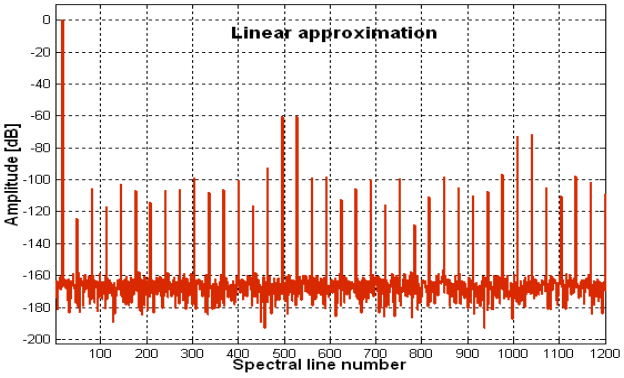
In this chapter the simulation results for segment number of 4, 8, 16, 32 and 64 are shown. The MSE (Mean square error) errors and spectral properties were calculated and are presented in Tab. 1. The example of error for 1 period is shown in Fig. 4, spectral properties in Fig. 5 - 9. From Tab. 1 can be seen, that almost optimal is 32 segments (quarter of period, 0 to π/2 is divided to 32 equidistant segments).

Tab. 1. Linear approximations with different number of segments.

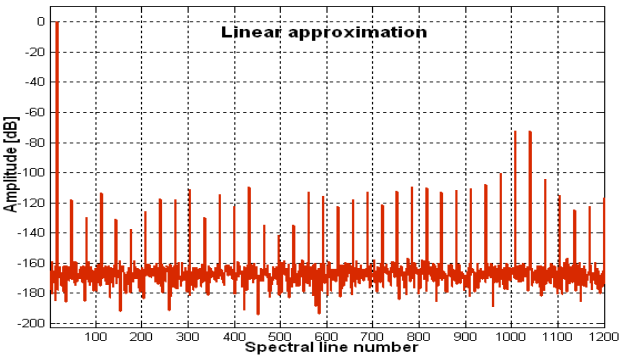
| No. of segment | MSE | dBc |
|---|---|---|
| 4 | 1.13e+6 | -47.9 |
| 8 | 7.10e+4 | -60.1 |
| 16 | 4.76e+3 | -72.1 |
| 32 | 612.2 | -106.0 |
| 64 | 378.7 | -104.9 |




Fig, 5. The output frequency spectrum, calculated for 4 segments and linear approximation.




Fig, 6. The output frequency spectrum, calculated for 8 segments and linear approximation.




Fig, 7. The output frequency spectrum, calculated for 16 segments and linear approximation.

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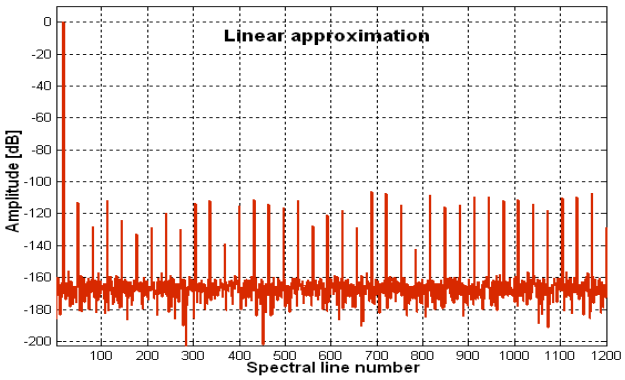
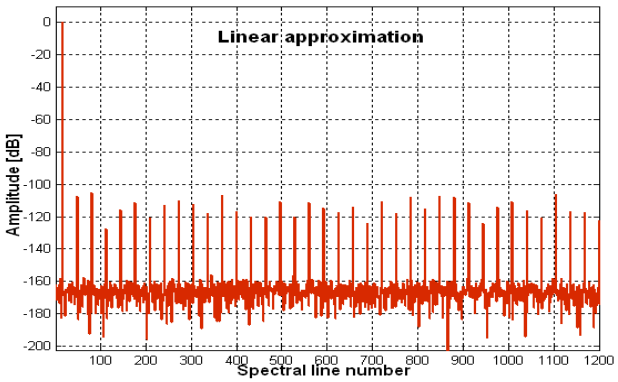
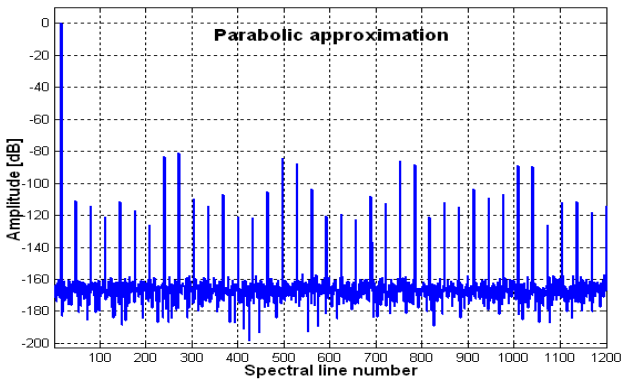



Fig, 8. The output frequency spectrum, calculated for 32 segments and linear approximation.




Fig, 9. The output frequency spectrum, calculated for 32 segments and linear approximation.



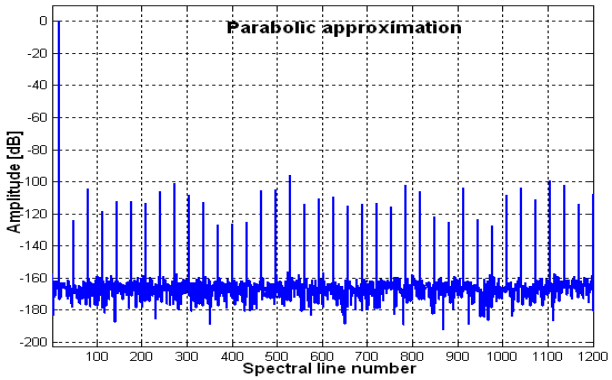

Fig, 10. The output frequency spectrum, calculated for 4 segments and parabolic approximation.

## 4 Parabolic Approximation for 4, 8 and 16 segments

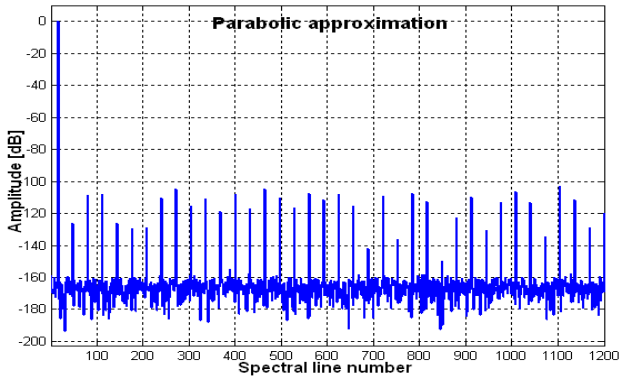
In this chapter the simulation results for segment number of 4, 8, and 16 are shown. The MSE (Mean square error) errors and spectral properties were calculated and are presented in Tab. 2. The spectral properties are also displayed in Fig. 10 - 12. From Tab. 2., can be seen, that 16 segments of parabolic approximation is similar to 64 segments of linear approximation.

Tab. 2. Parabolic approximations with different number of segments.

| No. of segment | MSE | dBc |
|---|---|---|
| 4 | 1.5307e+003 | -81.2 |
| 8 | 374.6 | -95.8 |
| 16 | 356.9 | -102.9 |




Fig, 11. The output frequency spectrum, calculated for 8 segments and parabolic approximation.




Fig, 12. The output frequency spectrum, calculated for 16 segments and parabolic approximation.

## 5 Conclusion

Ultrahigh speed direct digital synthesizers will play important roles in next generation radar and communication systems. Recent developments in radar systems require frequency synthesis with low power consumption, high output frequency, fine frequency resolution, fast channel switching and versatile modulation capabilities. Linear frequency modulation or chirp modulation is widely used in radars to achieve high range resolution, while pulsed phase modulation can provide anti-jamming capability. With fine frequency resolution, fast channel switching and versatile modulation capabilities, the DDS provides frequency synthesis and direct modulation capabilities that cannot be

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easily implemented by other synthesizer tools such as analog-based phase-locked loop synthesizers.

In this paper, the linear and parabolic approximation of equi-section division utilizing the quarter-wave symmetry method and sine amplitude approximation was described, calculated and simulated. The sinusoidal phase from 0 to $\pi/2$ was divided into different numbers of equi-sections, in linear approximation from 4 to 64, in parabolic approximation from 4 to 16 sections. The calculated values were rounded to integer numbers and frequency spectrum of simulated sine function was presented. The proposed compression method can be applied for DDS, software defined radio (SDR) systems, spread spectrum frequency hopping systems, and a variety of signaling applications.

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## Appendix: Spurious modulation of DDS

The DDS output waves at the digital to analog convertors (DAC), with normalized output frequencies close to the ratio of small integers, exhibit either a "quasi-amplitude" modulation in some instances (see Fig. 13) and a "quasi-superposition" (see Fig. 14) of the desired high frequency wave on a low frequency signal, as mentioned in [16, 17], e.g. for expansion:

$$FCW/2^n = 253/1024$$

$$\frac{X}{Y} = \frac{253}{1024} = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3}}} = \frac{1}{4 + \frac{1}{21 + \frac{1}{12}}} \quad \text{(A1)}$$

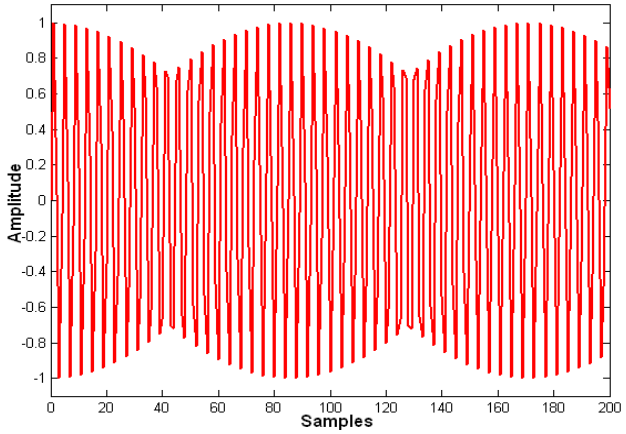


Fig. 13. Example of quasi-amplitude modulation for DDS, $FCW/2^n$=253/1024

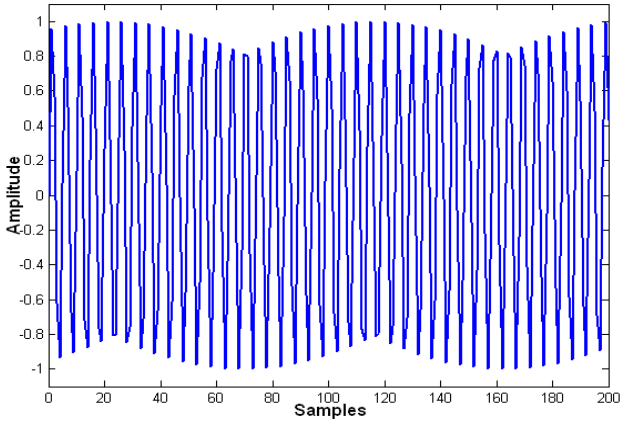


Fig. 14. Example of quasi-superposition for DDS, $FCW/2^n$=207/1024

The continued fraction expansion is given by

$$\frac{X}{Y} = \frac{1}{B_0 B_1} - \frac{1}{B_1 B_2} + \frac{1}{B_2 B_3} \ldots \frac{(-1)^n}{B_n B_{n+1}} \quad \text{(A2)}$$

where

$$\begin{aligned} B_0 &= 1 \\ B_1 &= b_1 \\ B_2 &= b_2 B_1 + B_0 \\ B_3 &= b_3 B_2 + B_1 \\ &\ldots. \\ B_n &= b_n B_{n-1} + B_{n-2} \end{aligned} \quad \text{(A3)}$$

In previous example ($X/Y$ = 253/1024) $b_1$=4, $b_2$=21, $b_3$=12 (function "rat" in MATLAB returns the continued fraction representation, $b_1$, $b_2$, $b_3$) and according (A3):

$B_1$=4; $B_2$=21*4+1=85; $B_3$=12*85+4=1024

and therefore:

$$\frac{X}{Y} = \frac{253}{1024} = \\ = \frac{1}{3} - \frac{1}{4*85} + \frac{1}{85*1024} = 0.2470703125$$

For second example $X/Y$ = 207/1024, the $b_1$=54, $b_2$=-19, $b_3$=5, $b_4$=2 and expansion is:

$$\frac{X}{Y} = \frac{207}{1024} = \frac{1}{5} - \frac{1}{5*(-94)} + \frac{1}{(-94)*(-465)} \\ - \frac{1}{(-465)*(-1024)} = 0.2021484375$$

The output sine wave of DDS can be simplified to:

$$\begin{aligned} s(m) &= sin\left(2\pi m \frac{X}{Y}\right) \approx \\ &\approx sin\left[2\pi m\left(\frac{1}{B_1} \pm \frac{1}{B_1 B_2}\right)\right] \end{aligned} \quad \text{(A4)}$$

The trigonometric expansion gives:

$$\begin{aligned} s(m) &= sin\left(2\pi m \frac{1}{B_1}\right) cos\left(2\pi m \frac{1}{B_1 B_2}\right) \\ &\pm cos\left(2\pi m \frac{1}{B_1}\right) sin\left(2\pi m \frac{1}{B_1 B_2}\right) \\ &= s_1(m) \pm s_2(m) \end{aligned} \quad \text{(A5)}$$

From expansion can be seen that low frequency component cause "quasi-amplitude" modulation or "quasi-superposition" for some FCW numbers.

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