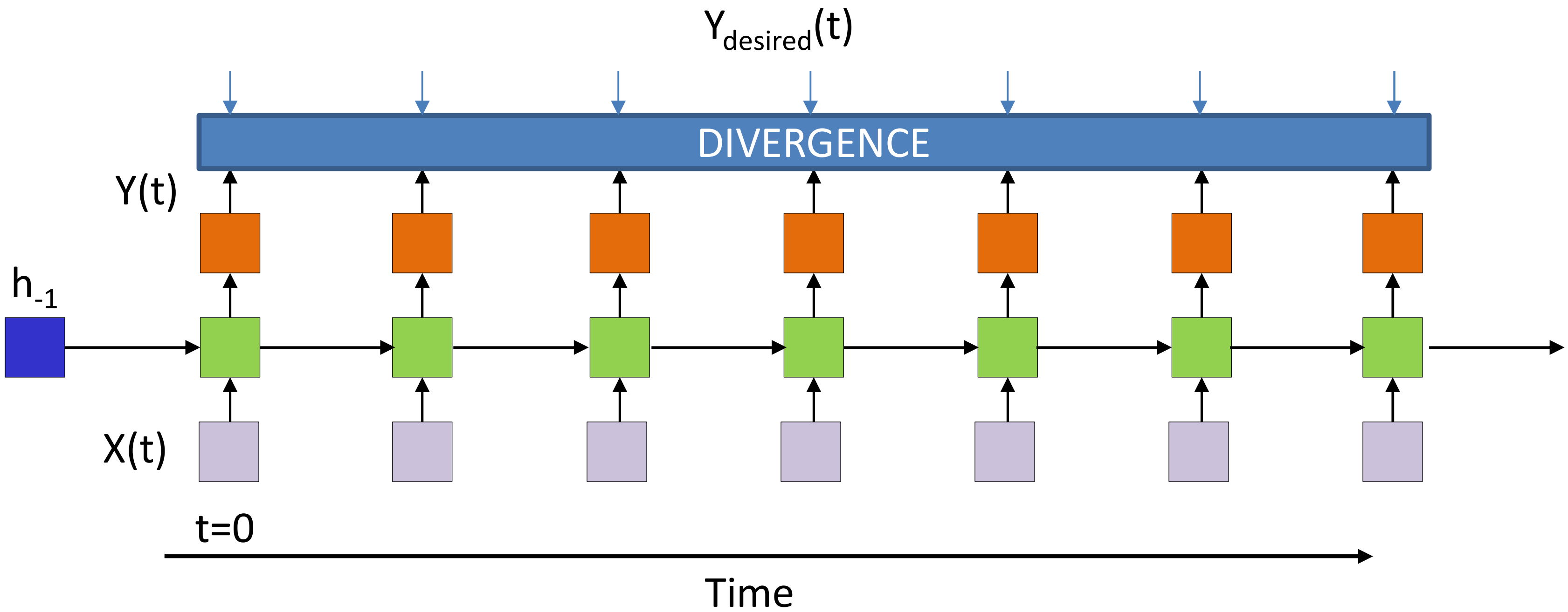
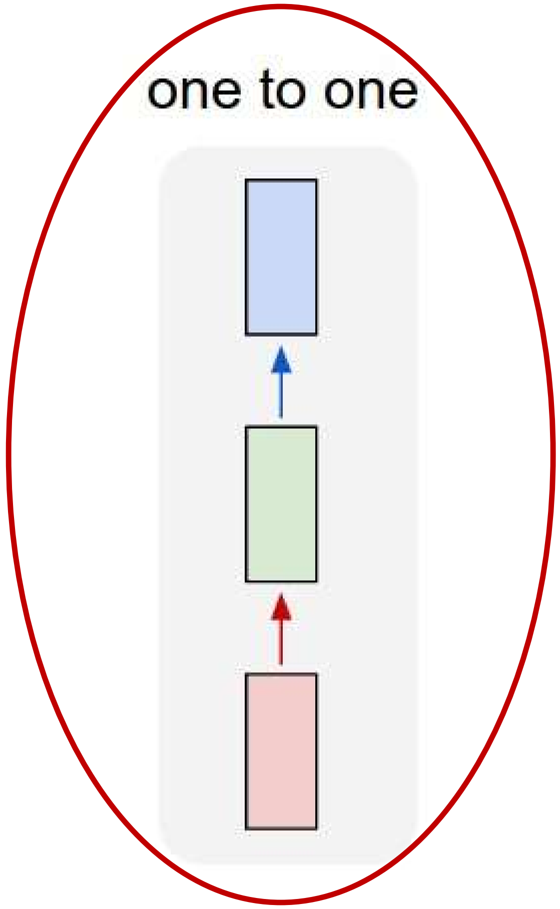
Deep Learning
Recurrent Networks: Part 4
Spring 2021

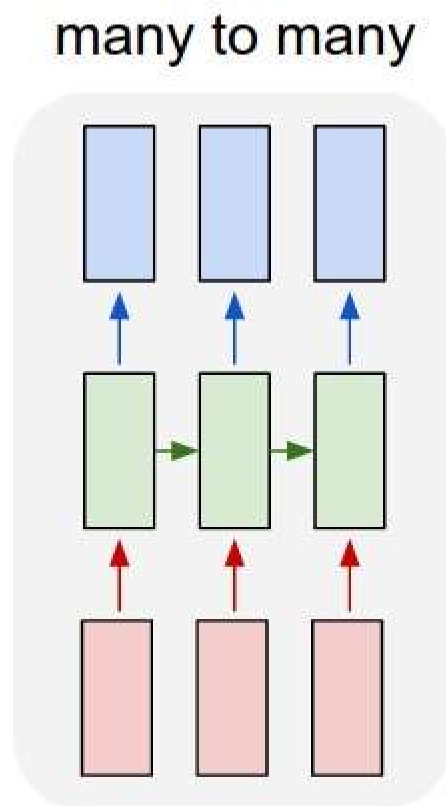
Story so far



- Recurrent structures can be trained by minimizing the divergence between the ***sequence*** of outputs and the ***sequence*** of desired outputs
  – Through gradient descent and backpropagation
- The challenge: Defining this divergence
  – Inputs and outputs may not be time aligned or even synchronous

Variants of recurrent nets

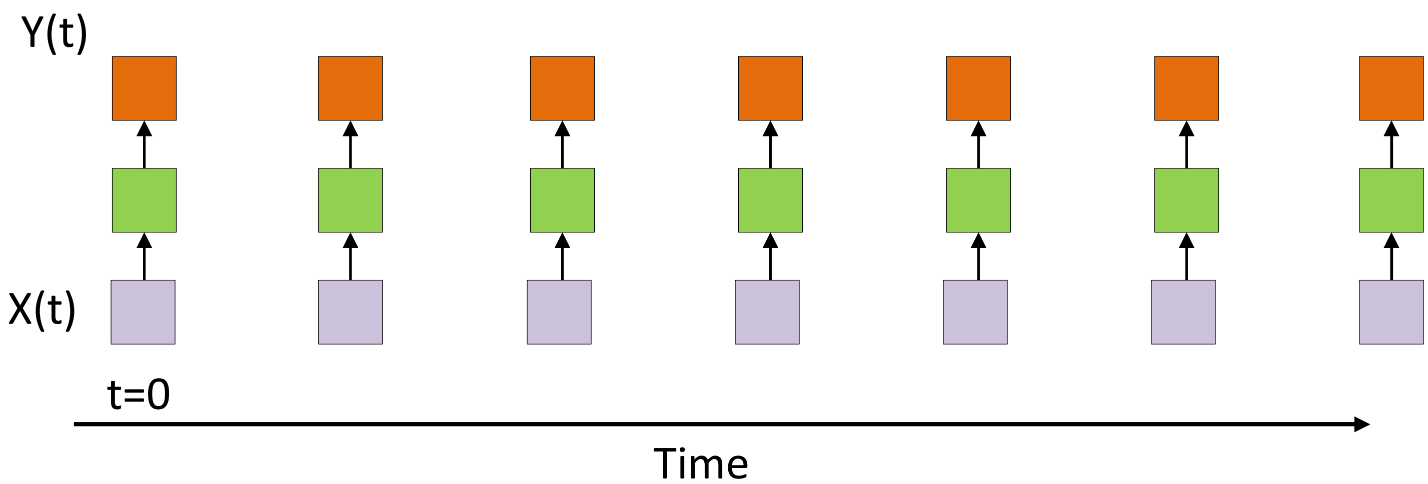




Images from Karpathy

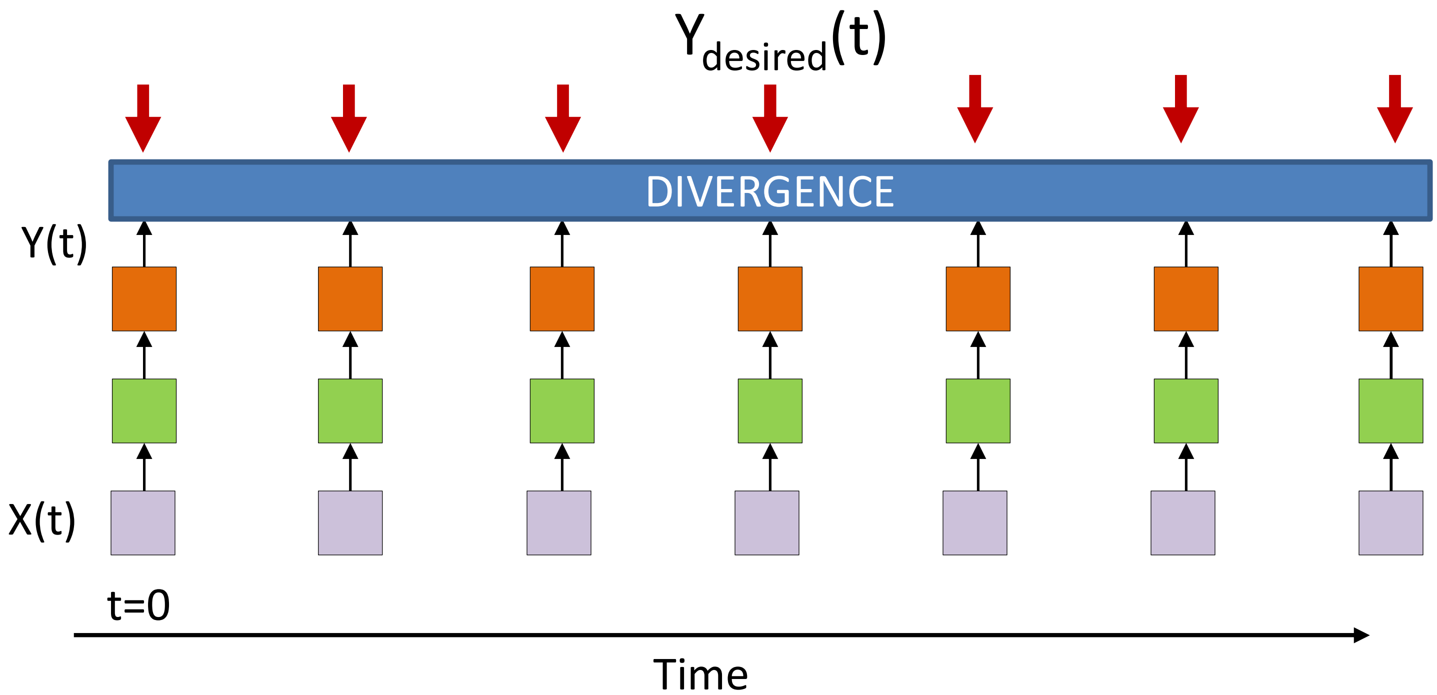
- Conventional MLP
- Time-synchronous outputs
  - E.g. part of speech tagging

This is a regular MLP



- No recurrence
  - Exactly as many outputs as inputs
  - The output at time t is unrelated to the output at $t' \neq t$.

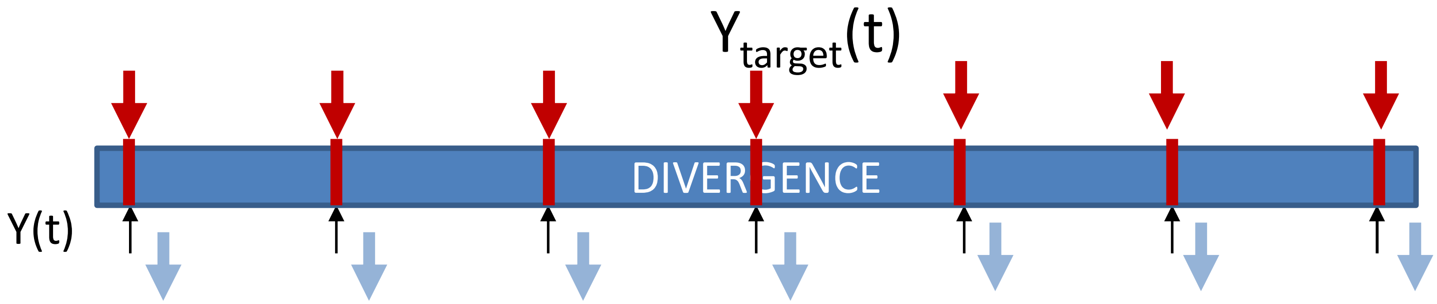
Learning in a regular MLP for series



- In the context of analyzing time series, the divergence to minimize is still the divergence between two series
  - Must be differentiable w.r.t every Y(t)

- In this setting: One-to-one correspondence between actual and target outputs
- Common assumption: Total divergence is the sum of *local* divergences at individual times
  - Simplifies model and maths

"Series MLP" as a regular MLP



- Gradient backpropagated at each time

$$\nabla_{Y(t)} Div\big(Y_{target}(1 \dots T), Y(1 \dots T)\big)$$

- Common assumption: One-to-one corresponde

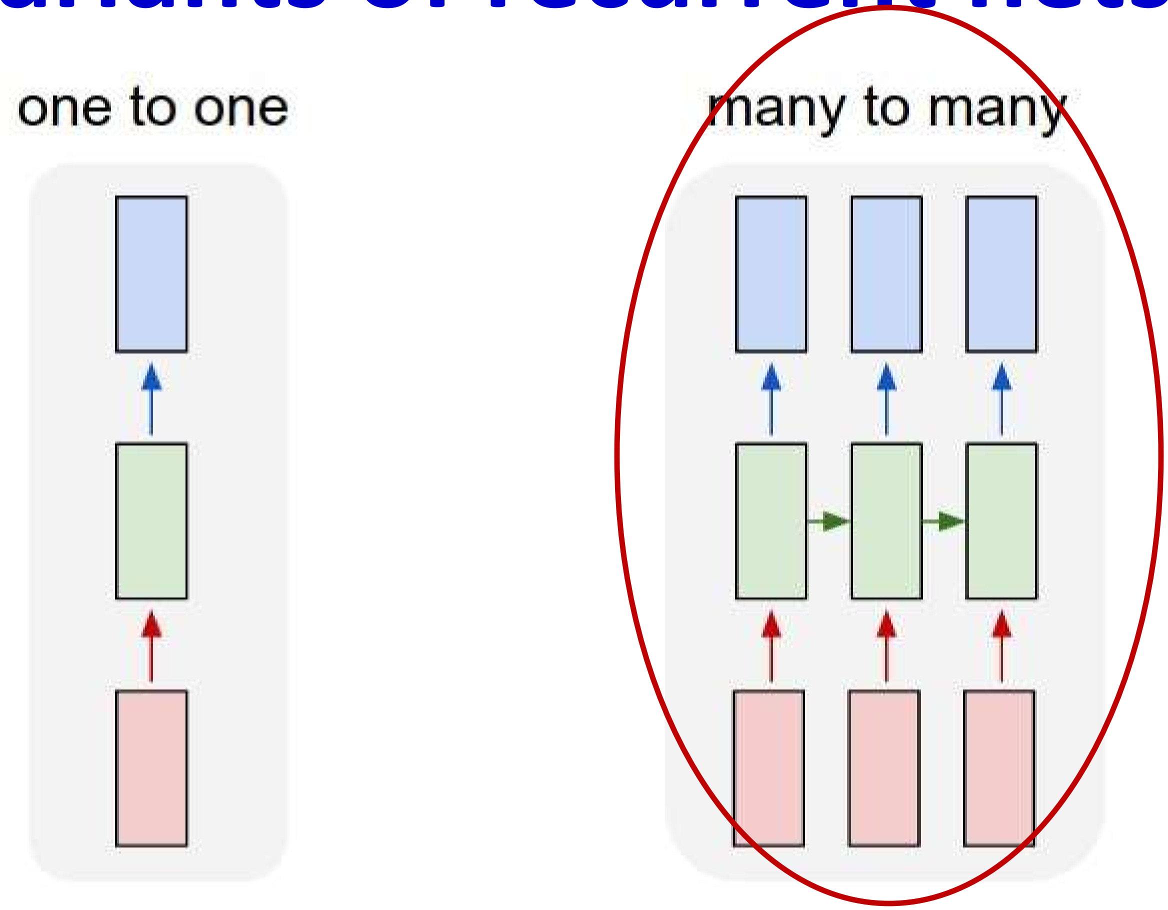
$$Div\big(Y_{target}(1 \dots T), Y(1 \dots T)\big) = \sum_t Div\big(Y_{target}(t), Y(t)\big)$$

$$\nabla_{Y(t)} Div\big(Y_{target}(1 \dots T), Y(1 \dots T)\big) = \nabla_{Y(t)} Div\big(Y_{target}(t), Y(t)\big)$$

  – This is further backpropagated to update weights etc

Typical Divergence for classification: $Div\big(Y_{target}(t), Y(t)\big) = KL(Y_{target}(t), Y(t))$

Variants of recurrent nets



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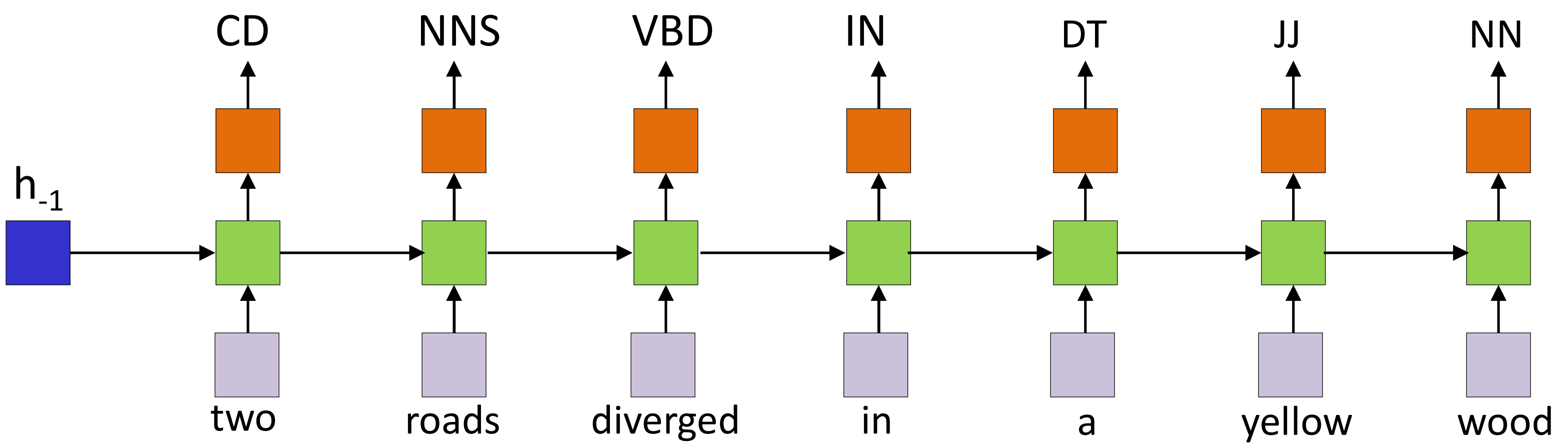
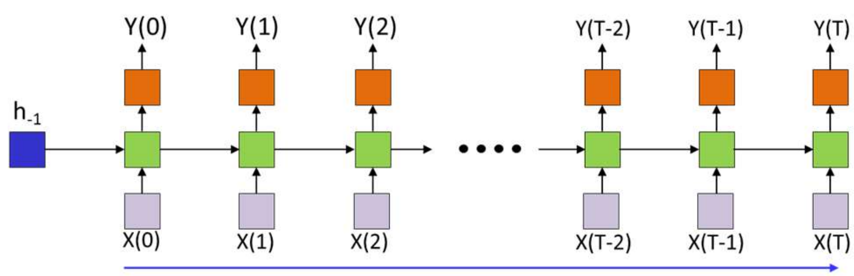
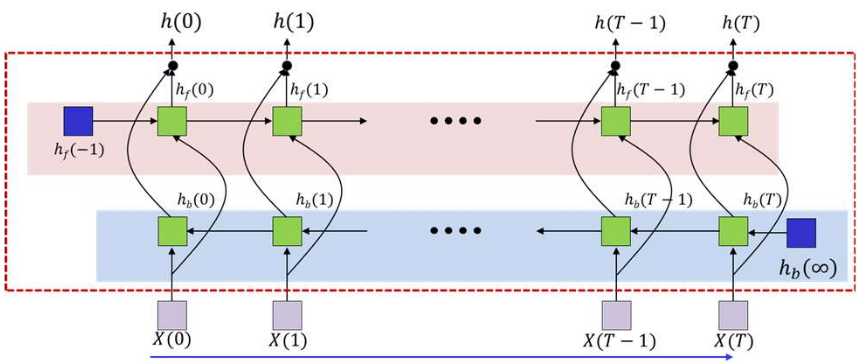
Time synchronous network



- Network produces one output for each input
  - With one-to-one correspondence
  - E.g. Assigning grammar tags to words
    - May require a bidirectional network to consider both past and future words in the sentence

Time-synchronous networks: Inference



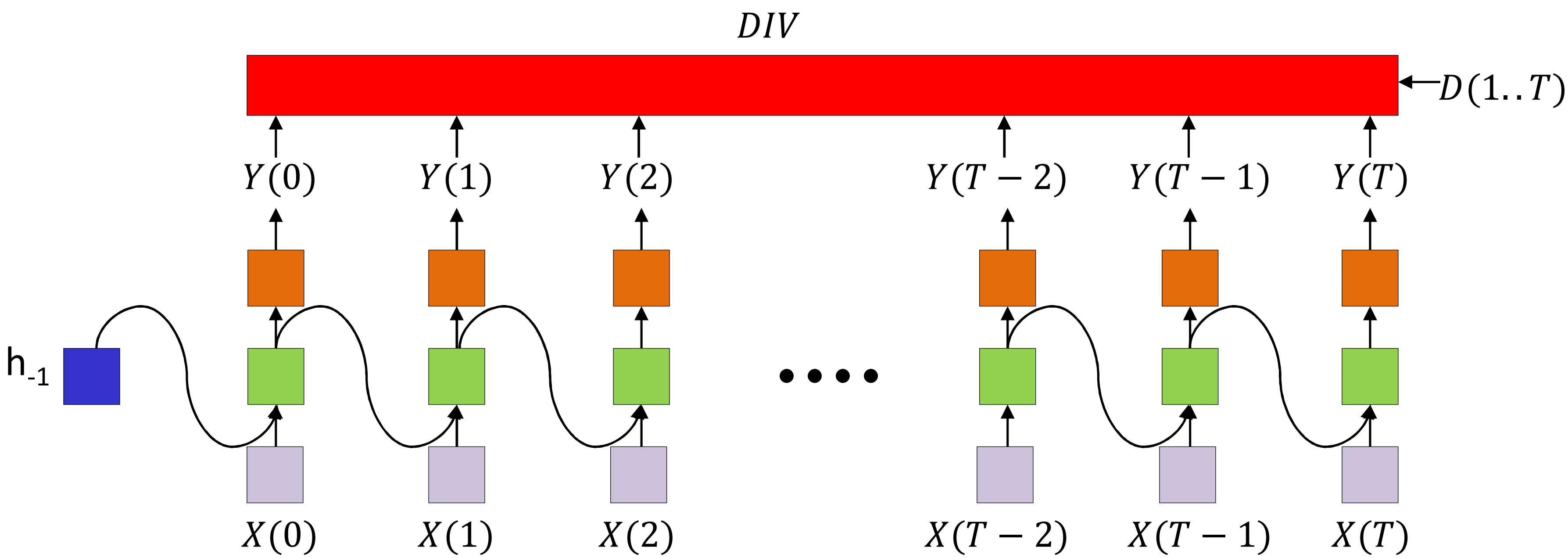


- One sided network: Process input left to right and produce output after each input

- Bi-directional network: Process input in both directions

- In all cases, there is an output for every input with exact one-to-one time-synchronous correspondence
  - Will continue to assume unidirectional models for explanations

Back Propagation Through Time

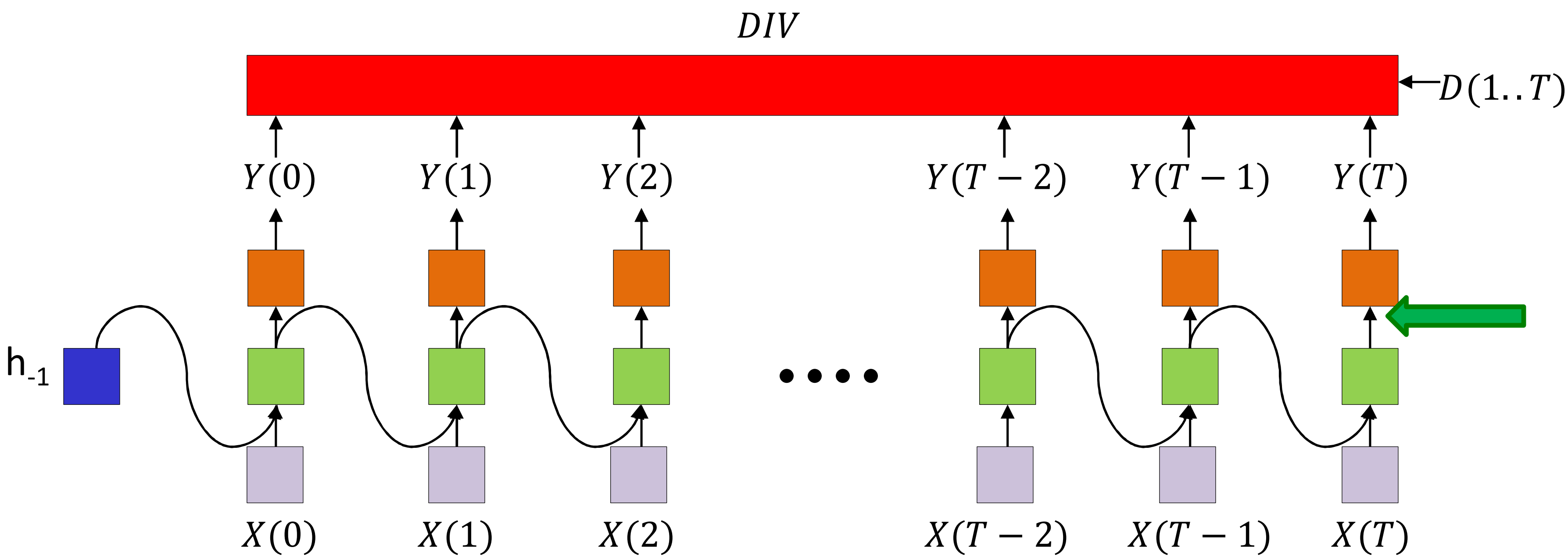


- Train given a set of input-target output pairs that are time synchronous
  - $(\mathbf{X}_i, \mathbf{D}_i)$, where $\mathbf{X}_i = X_{i,0}, \ldots, X_{i,T}$, $\mathbf{D}_i = D_{i,0}, \ldots, D_{i,T}$

- The divergence computed is between the *sequence of outputs* by the network and the *desired sequence of outputs*

$$Div\left(Y_{target}(1 \ldots T), Y(1 \ldots T)\right)$$

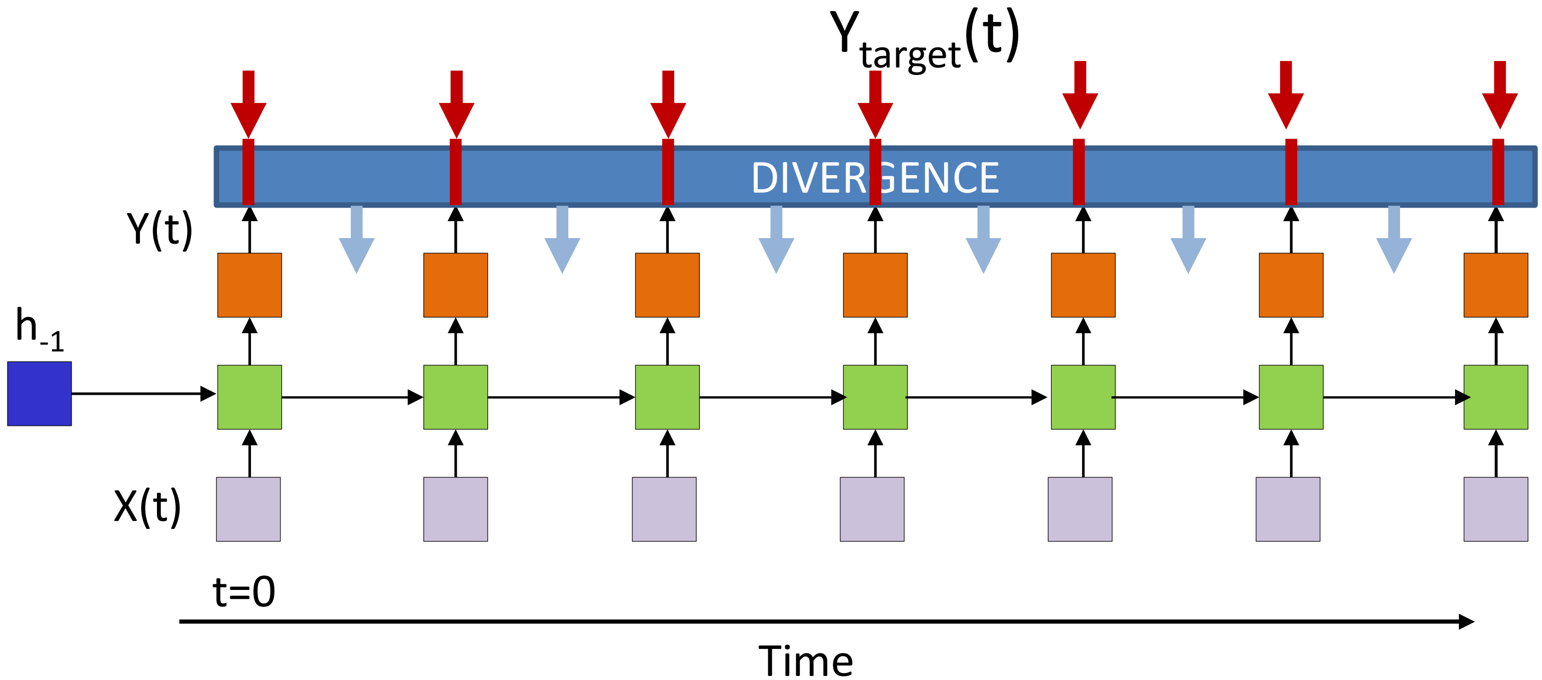
Back Propagation Through Time



First step of backprop: Compute $\nabla_{Y(t)} DIV$ for all t

- The key component is the computation of this derivative!!
- This depends on the definition of "DIV"

BPTT: Time-synchronous recurrence



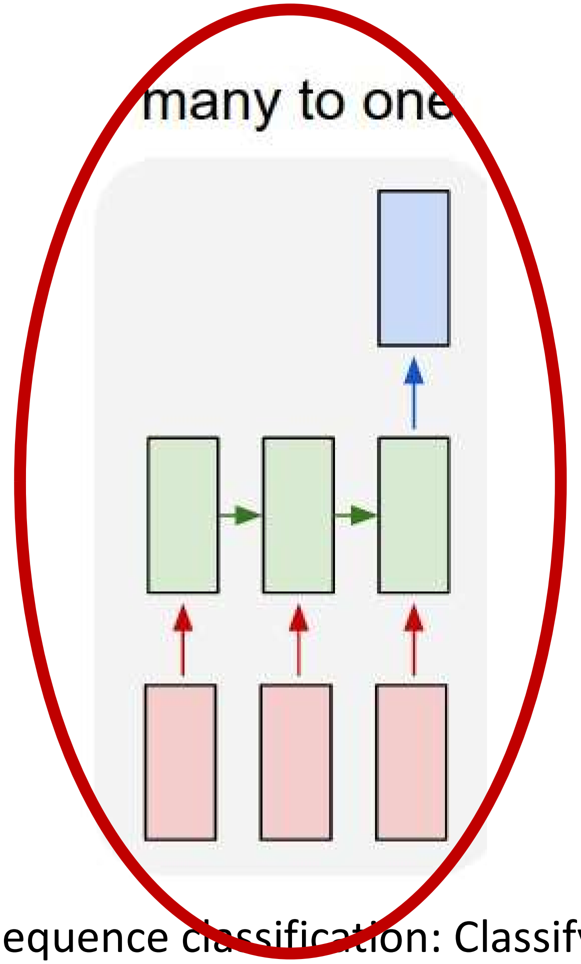
- Usual assumption: ***Sequence divergence is the sum of the divergence at individual instants***

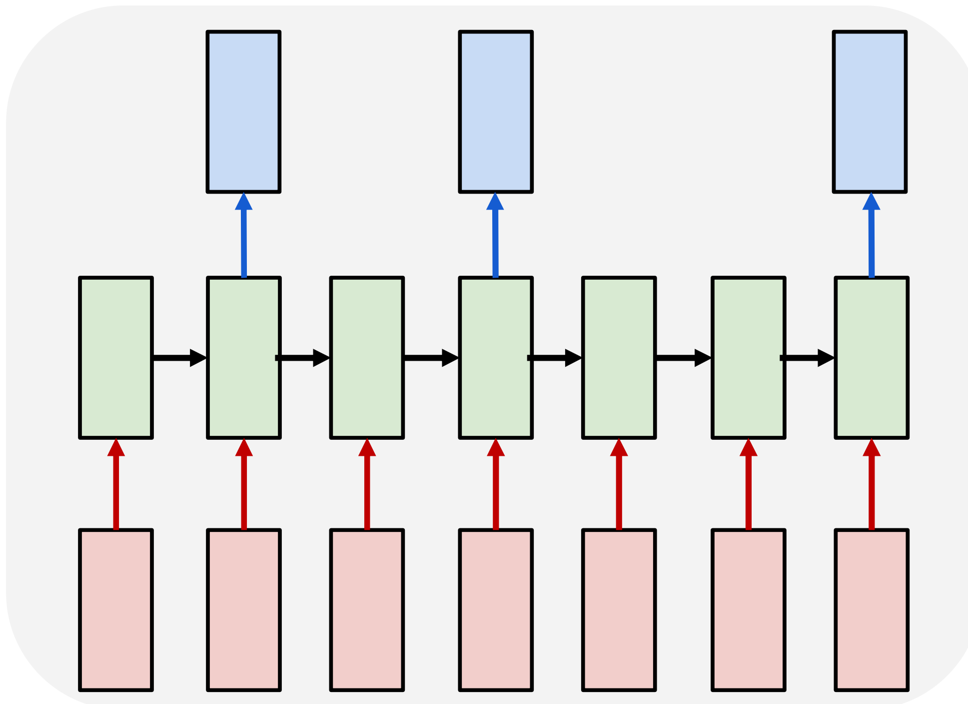
$$Div\left(Y_{target}(1 \dots T), Y(1 \dots T)\right) = \sum_t Div\left(Y_{target}(t), Y(t)\right)$$

$$\nabla_{Y(t)} Div\left(Y_{target}(1 \dots T), Y(1 \dots T)\right) = \nabla_{Y(t)} Div\left(Y_{target}(t), Y(t)\right)$$

Typical Divergence for classification: $Div\left(Y_{target}(t), Y(t)\right) = KL\left(Y_{target}(t), Y(t)\right)$

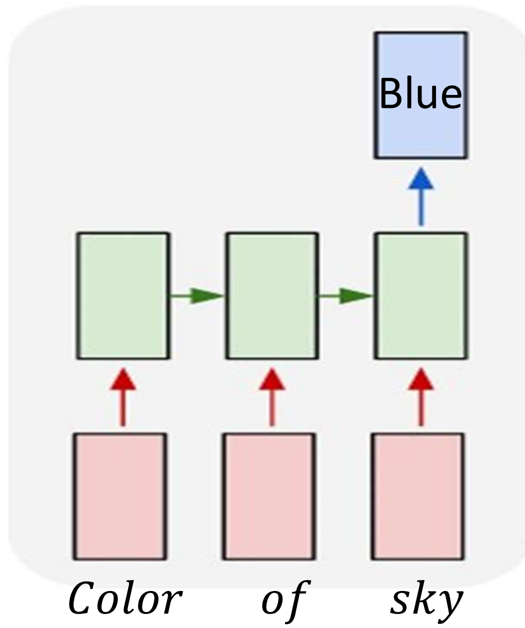
Variants of recurrent nets





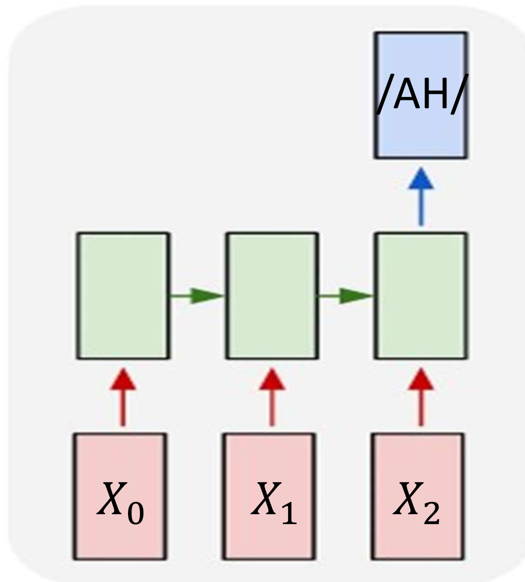
- Sequence classification: Classifying a full input sequence
  - E.g phoneme recognition
- Order synchronous , time asynchronous sequence-to-sequence generation
  - E.g. speech recognition
  - Exact location of output is unknown a priori

Example..



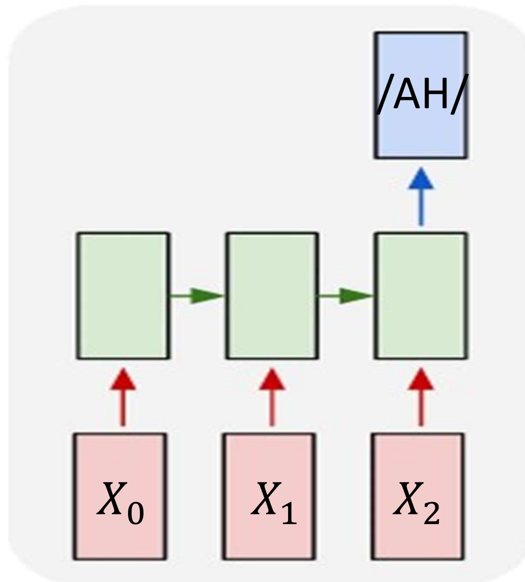
- Question answering
- Input : Sequence of words
- Output: Answer at the end of the question

Example..



- Speech recognition
- Input : Sequence of feature vectors (e.g. Mel spectra)
- Output: Phoneme ID at the end of the sequence
  - Represented as an N-dimensional output probability vector, where N is the number of phonemes

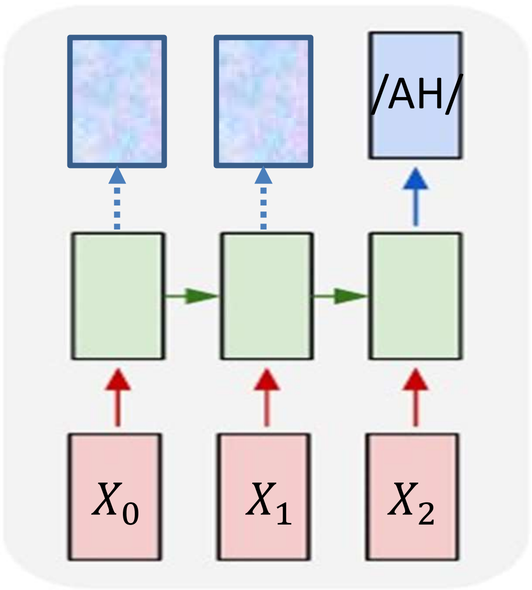
Inference: Forward pass



- Exact input sequence provided
  - Output generated when the last vector is processed
    - Output is a probability distribution over phonemes

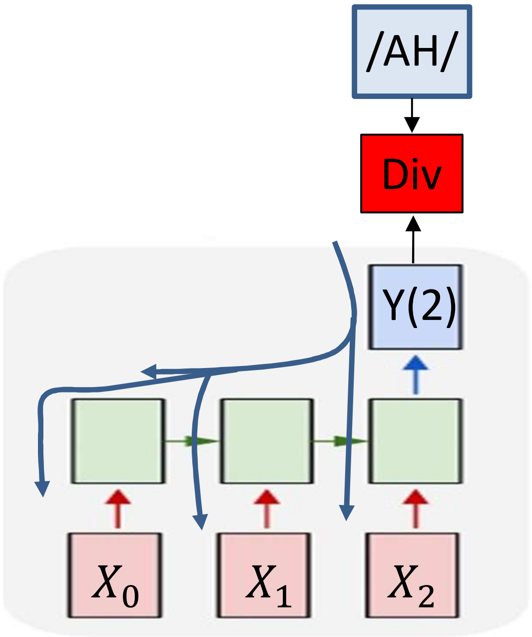
- But what about at *intermediate stages?*

Forward pass



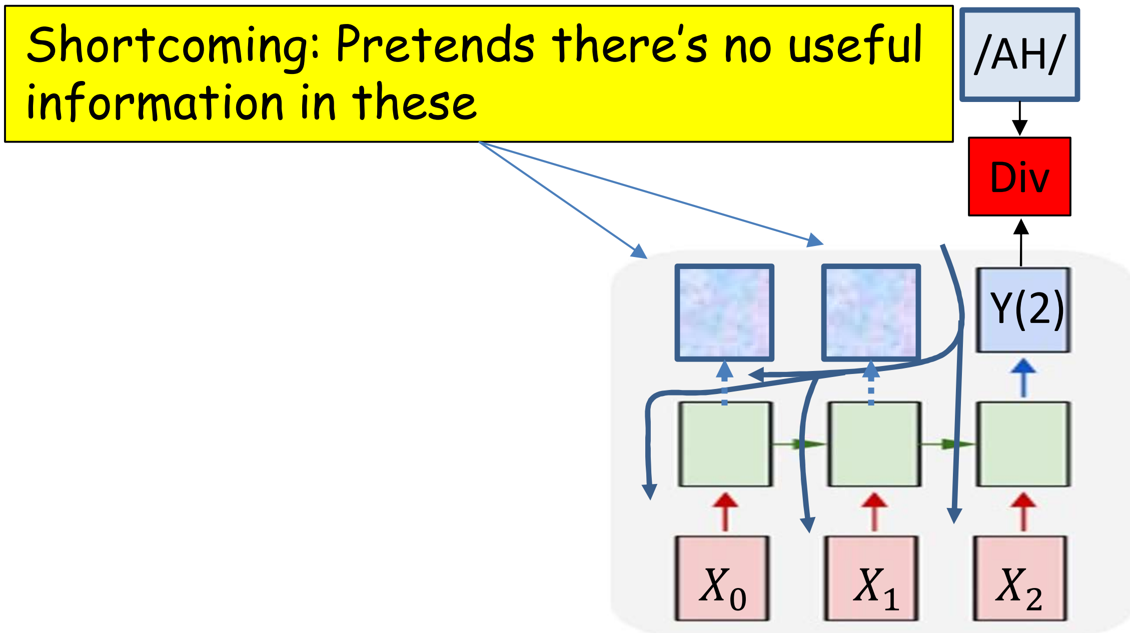
- Exact input sequence provided
  - Output generated when the last vector is processed
    - Output is a probability distribution over phonemes
- Outputs are actually produced for *every* input
  - We only *read* it at the end of the sequence

Training



- The Divergence is only defined at the final input
  - $DIV\left(Y_{target}, Y\right) = KL(Y(T), Phoneme)$
- This divergence must propagate through the net to update all parameters

Training

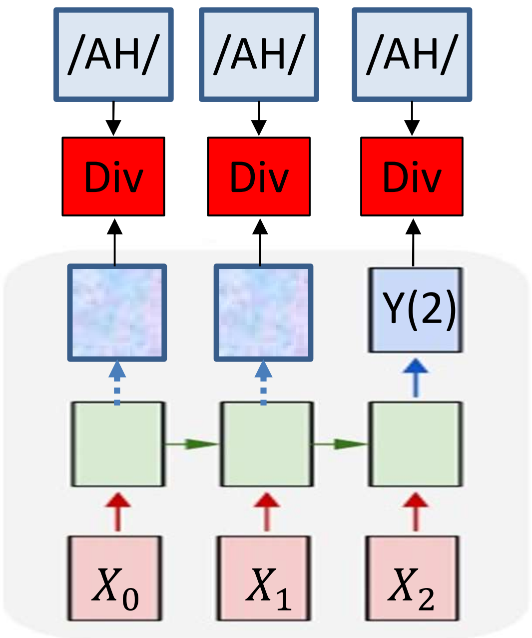


- The Divergence is only defined at the final input
  - $DIV\left(Y_{target}, Y\right) = Xent(Y(T), Phoneme)$
- This divergence must propagate through the net to update all parameters

Training

Fix: Use these outputs too.

These too must ideally point to the correct phoneme



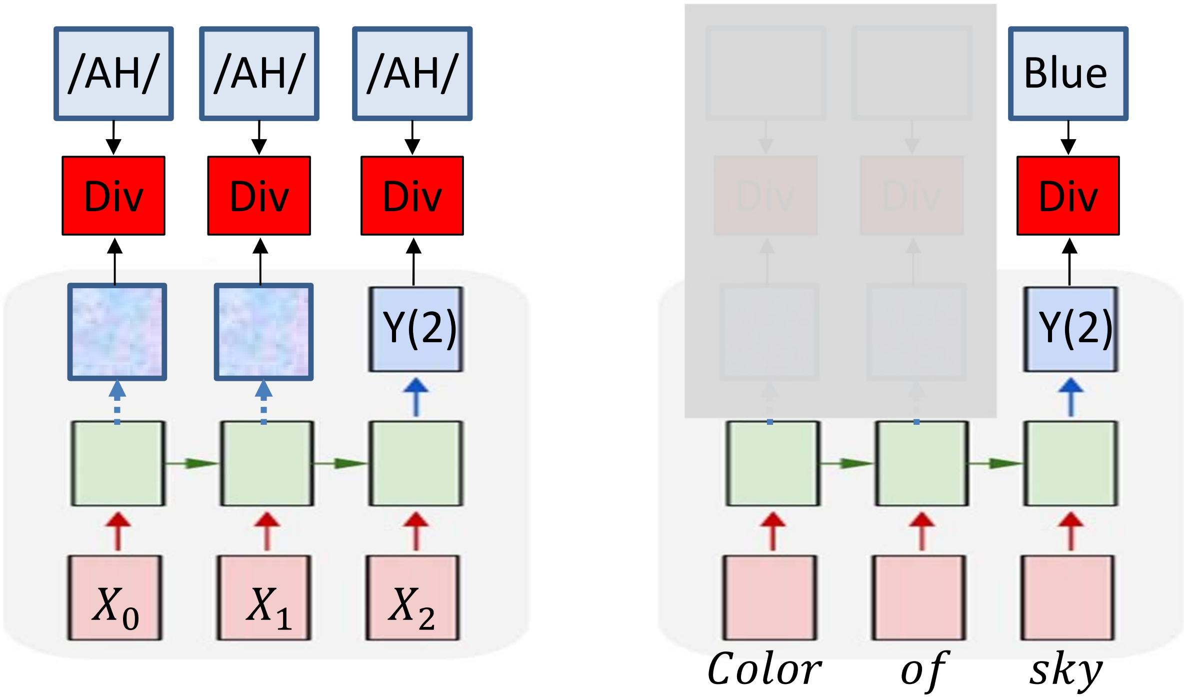
- Exploiting the untagged inputs: assume the same output for the entire input
- Define the divergence everywhere

$$DIV(Y_{target}, Y) = \sum_t w_t Xent(Y(t), Phoneme)$$

Training

Fix: Use these outputs too.

These too must ideally point to the correct phoneme

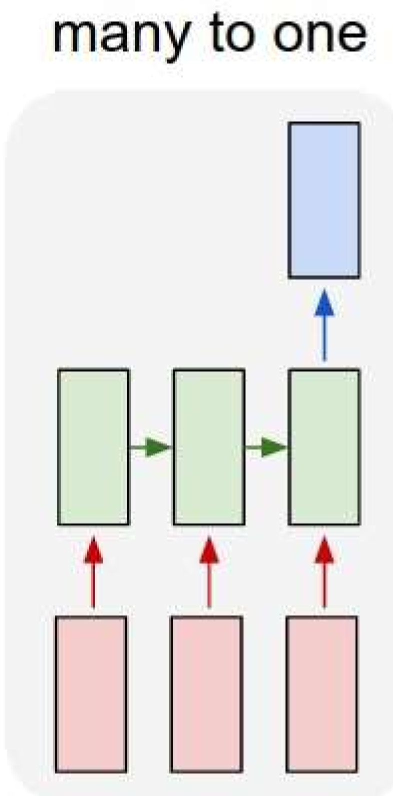


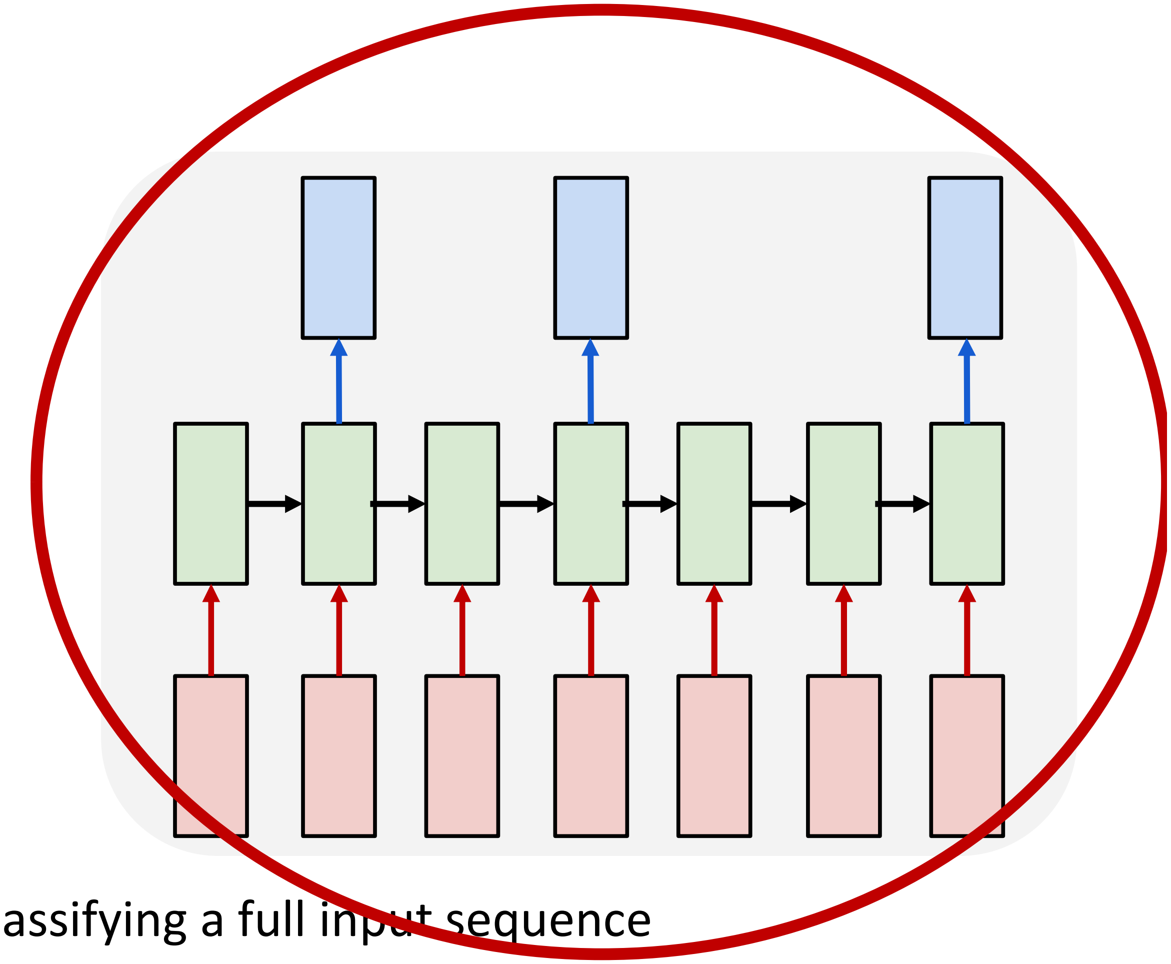
- Define the divergence everywhere

$$DIV\left(Y_{target}, Y\right) = \sum_t w_t Xent(Y(t), Phoneme)$$

- Typical weighting scheme for speech: all are equally important
- Problem like question answering: answer only expected after the question ends
  - Only w_T is high, other weights are 0 or low

Variants on recurrent nets





- Sequence classification: Classifying a full input sequence
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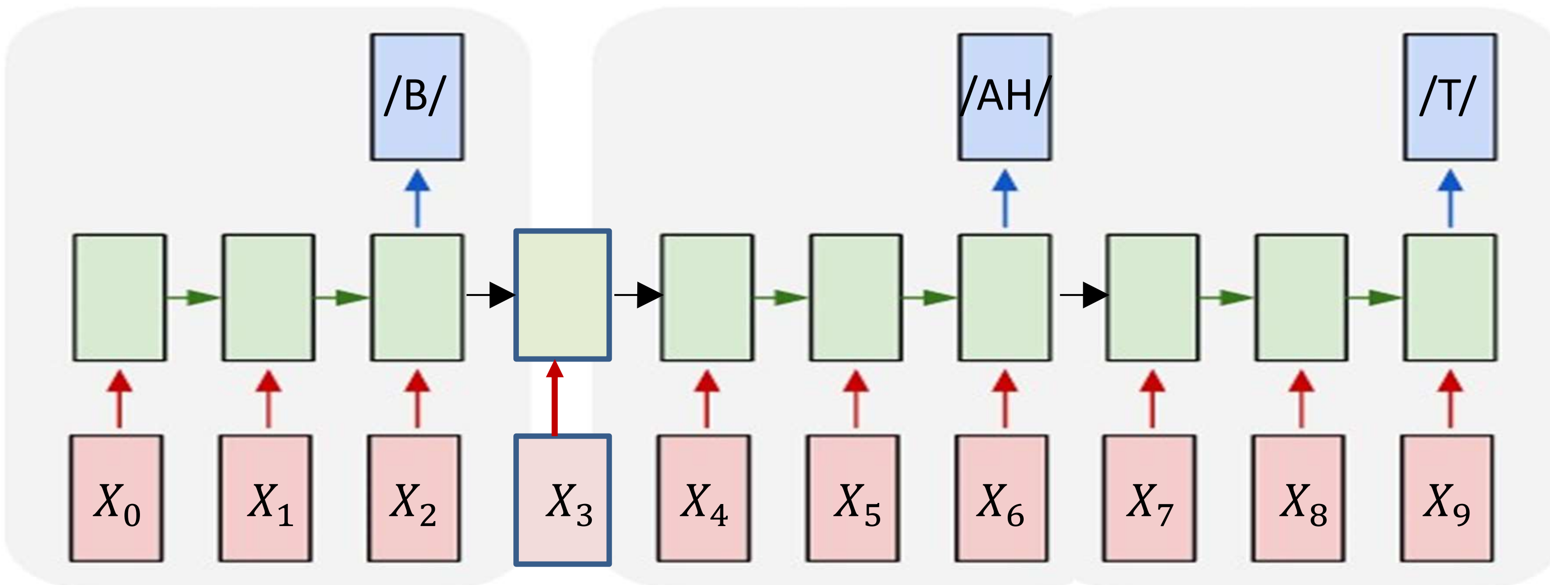
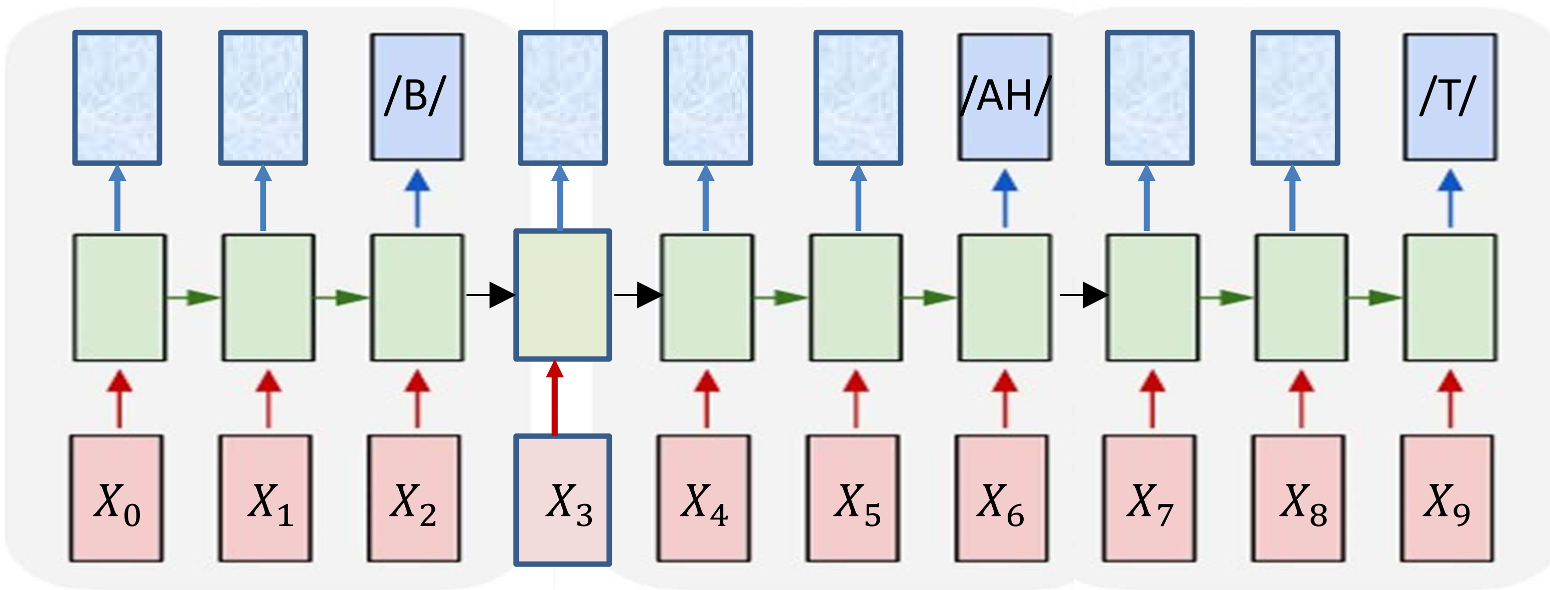
A more complex problem



- Objective: Given a sequence of inputs, asynchronously output a sequence of symbols
  – This is just a simple concatenation of many copies of the simple "output at the end of the input sequence" model we just saw

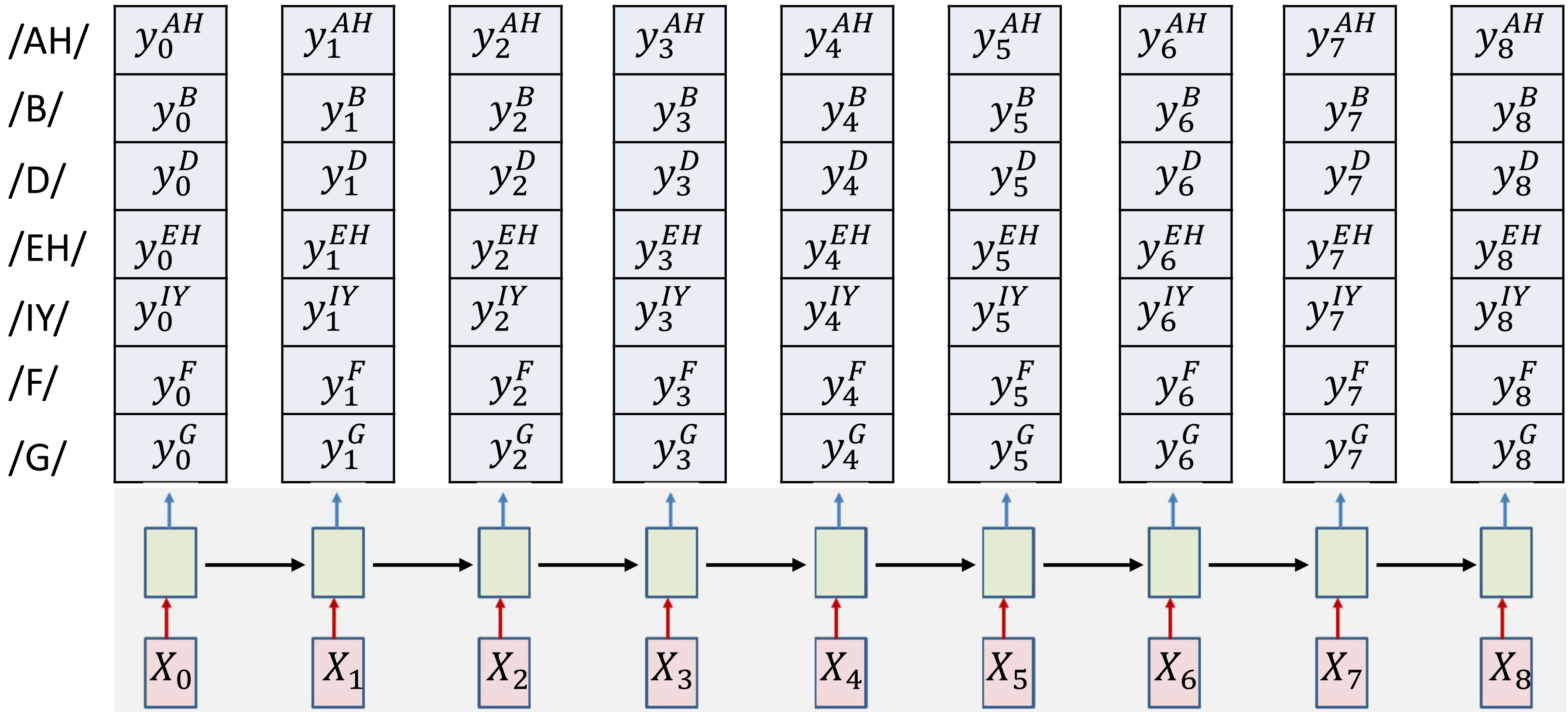
- But this simple extension complicates matters..

The *sequence-to-sequence* problem



- How do we know *when* to output symbols
  - In fact, the network produces outputs at *every* time
  - *Which* of these are the *real* outputs
    - Outputs that represent the definitive occurrence of a symbol

The actual output of the network



- At each time the network outputs a probability for *each* output symbol given all inputs until that time
  - E.g. $y_4^D = prob(s_4 = D|X_0 \dots X_4)$

Recap: The output of a network

- Any neural network with a softmax (or logistic) output is actually outputting an estimate of the *a posteriori* probability of the classes given the output

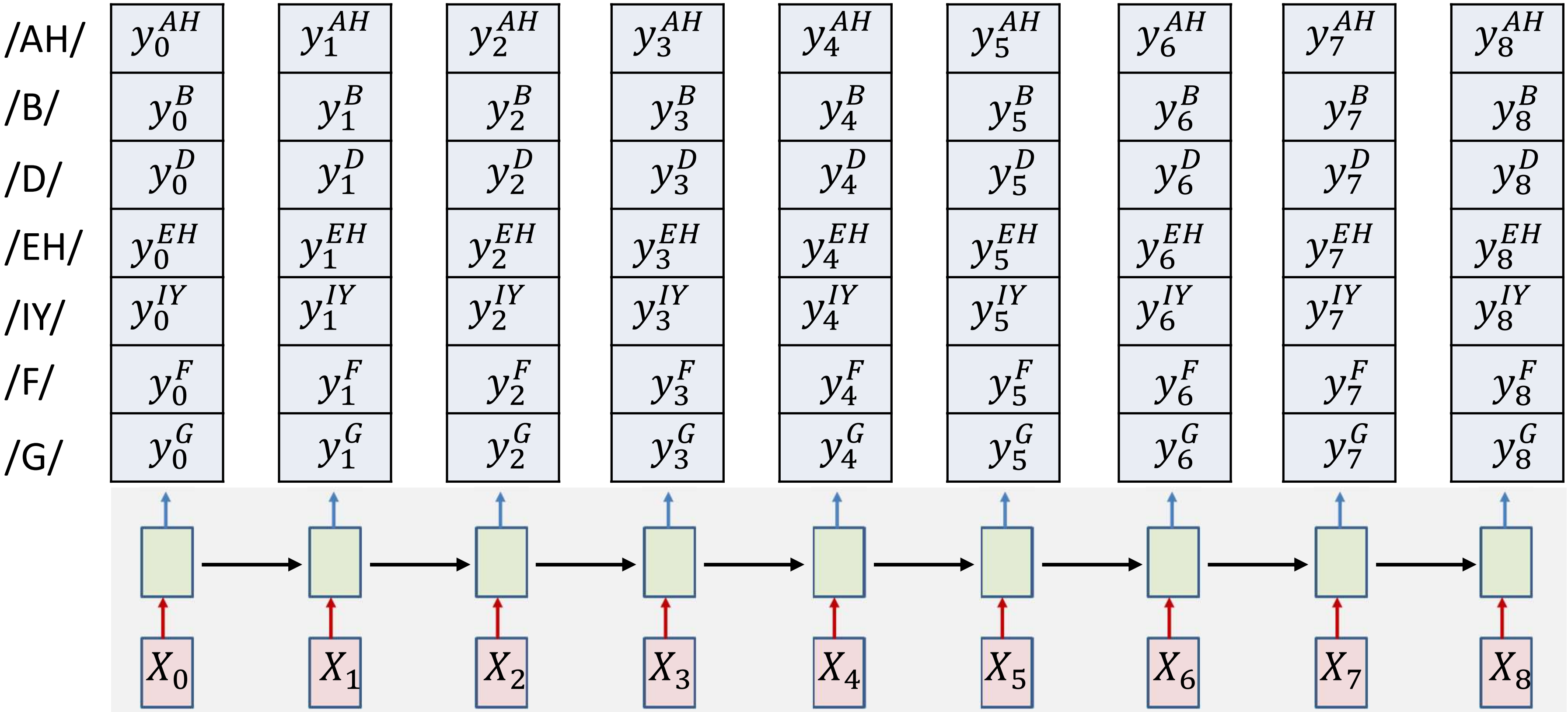
$$[P(c_1|X), P(c_2|X), \dots, P(c_K|X)]$$

- Selecting the class with the highest probability results in *maximum a posteriori probability* classification

$$Class = \underset{i}{\operatorname{argmax}} P(Y_i|X)$$

- We use the same principle here

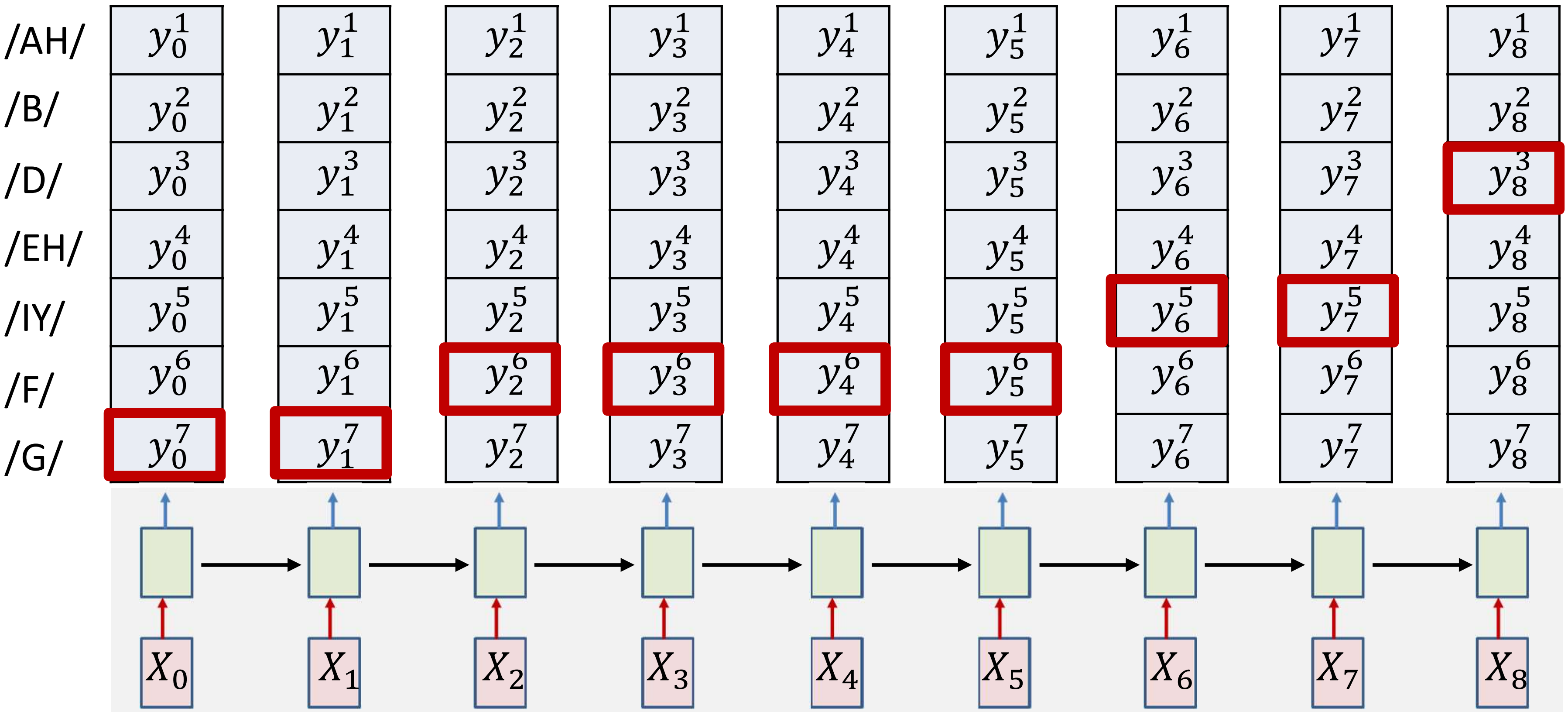
Overall objective



- Find most likely symbol sequence given inputs

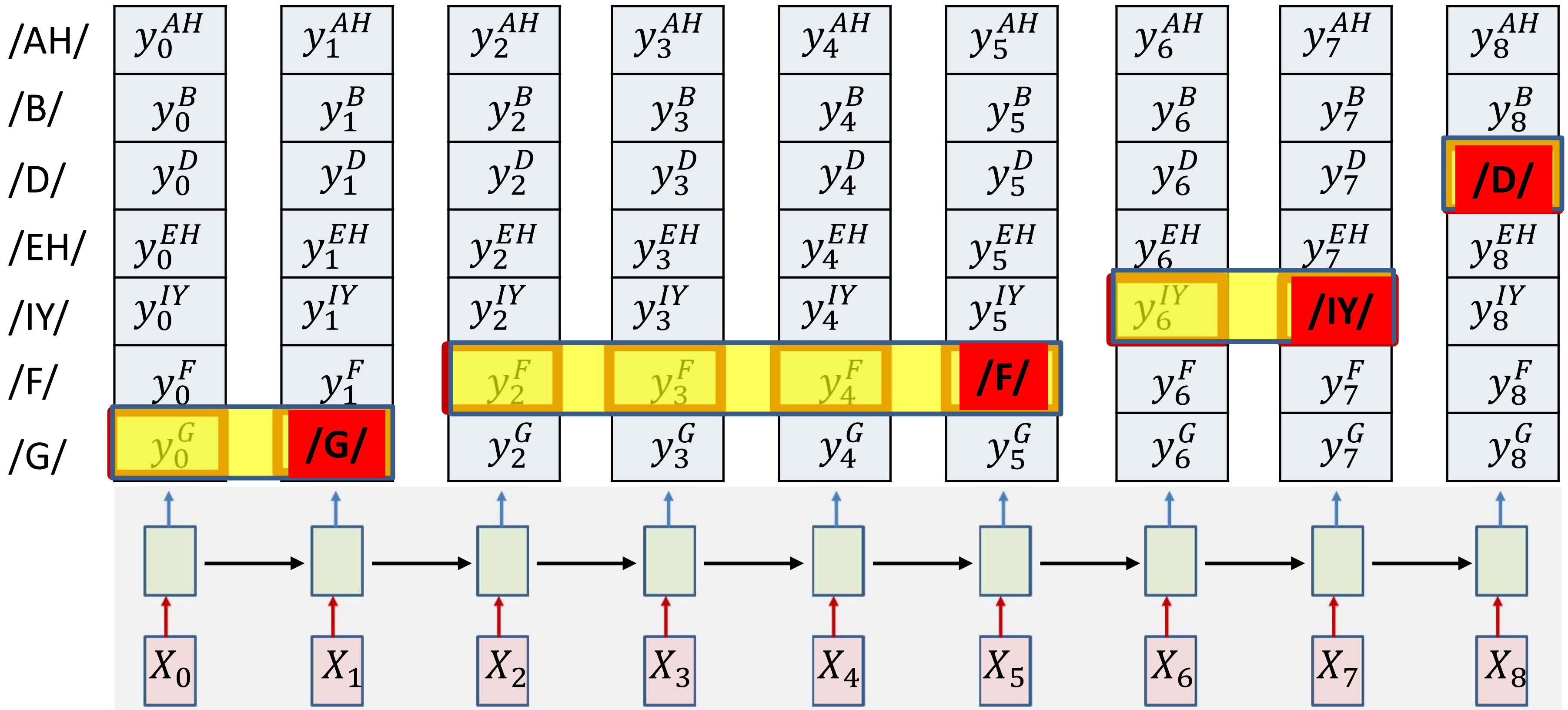
$$S_0 \ldots S_{K-1} = \underset{S'_0 \ldots S'_{K-1}}{\operatorname{argmax}} \; prob(S'_0 \ldots S'_{K-1} | X_0 \ldots X_{N-1})$$

Finding the best output



- Option 1: Simply select the most probable symbol at each time

Finding the best output



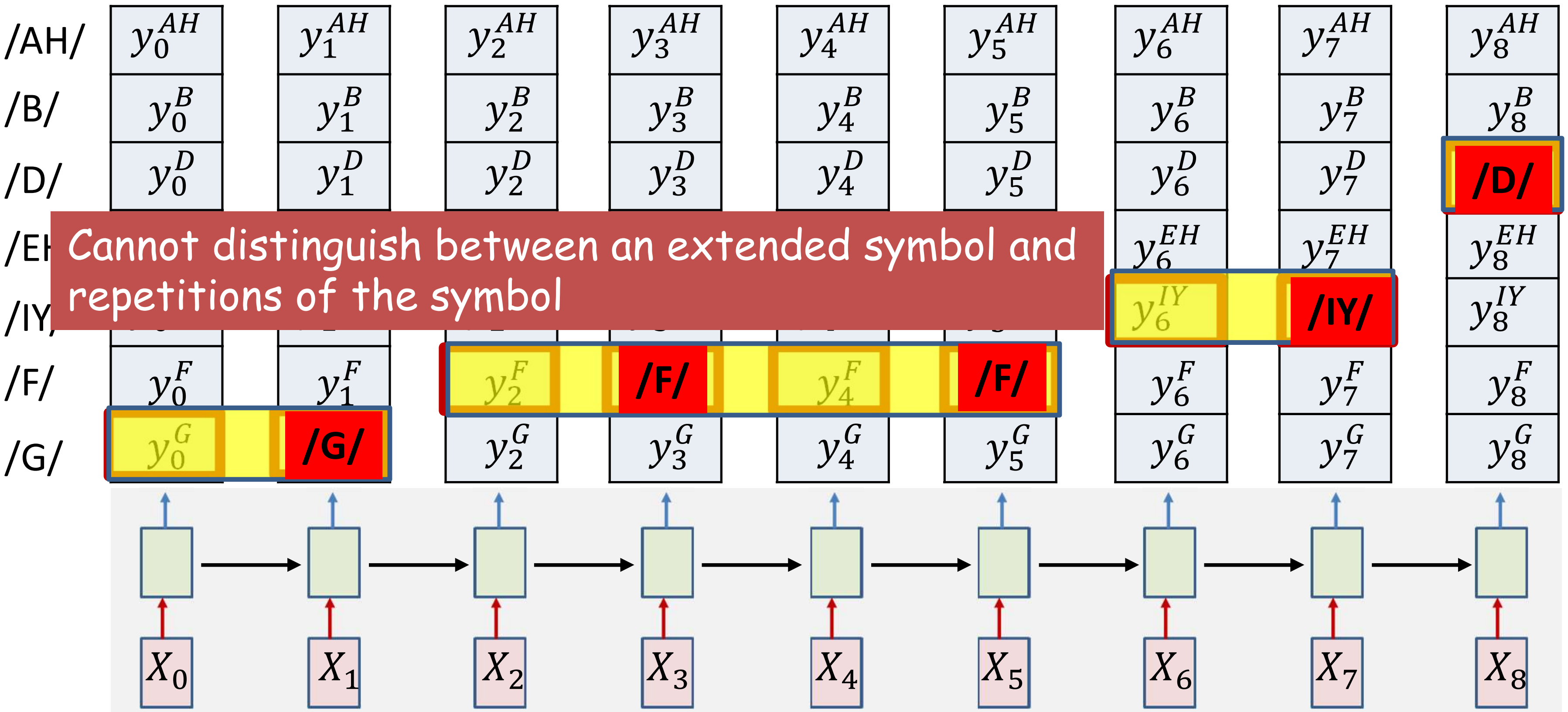
- Option 1: Simply select the most probable symbol at each time
  - *Merge* adjacent repeated symbols, and place the actual emission of the symbol in the final instant

Simple pseudocode

- Assuming $y(t, i), t = 1 \ldots T, i = 1 \ldots N$ is already computed using the underlying RNN

```
n = 1
best(1)= argmax_i(y(1,i))
for t = 1:T
    best(t)= argmax_i(y(t,i))
    if (best(t) != best(t-1))
        out(n) = best(t-1)
        time(n) = t-1
        n = n+1
```

Finding the best output



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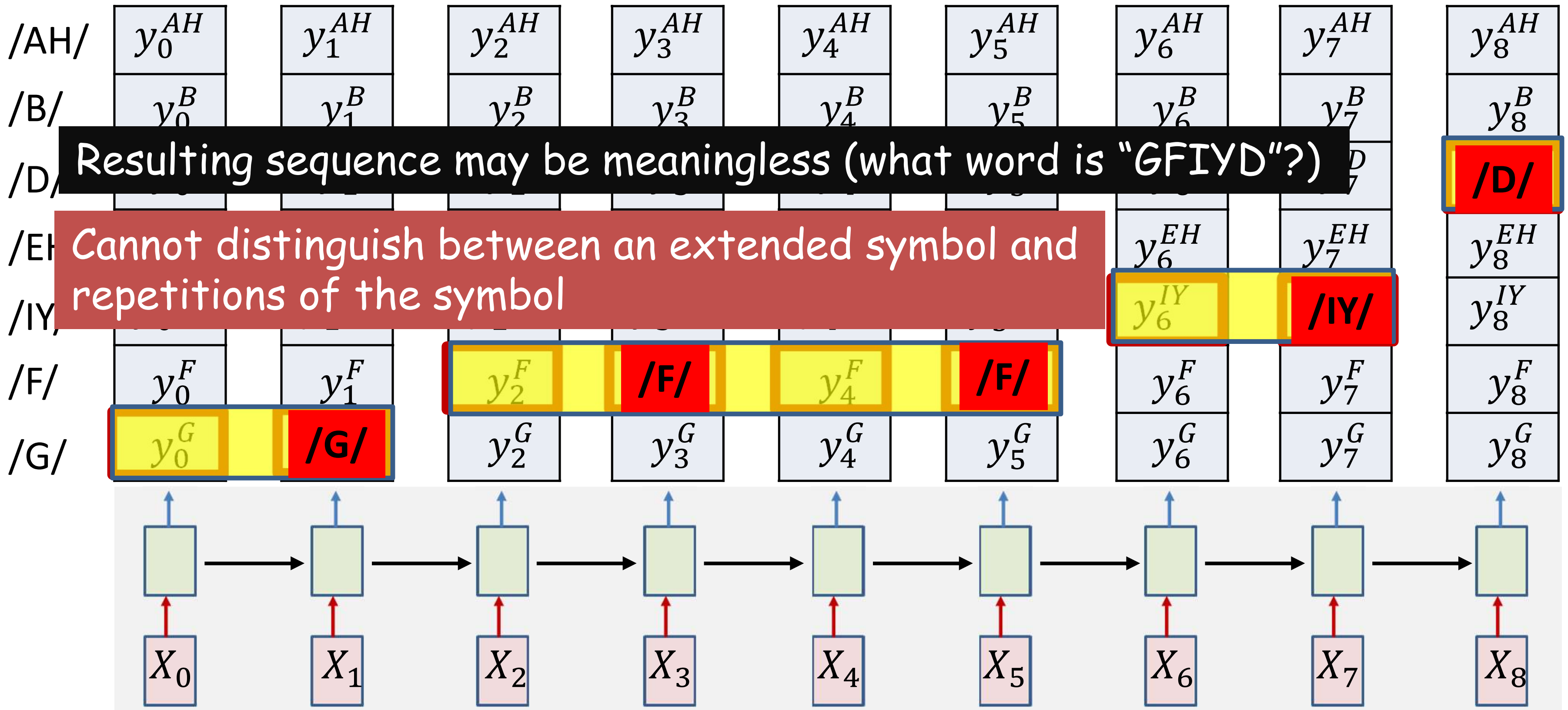

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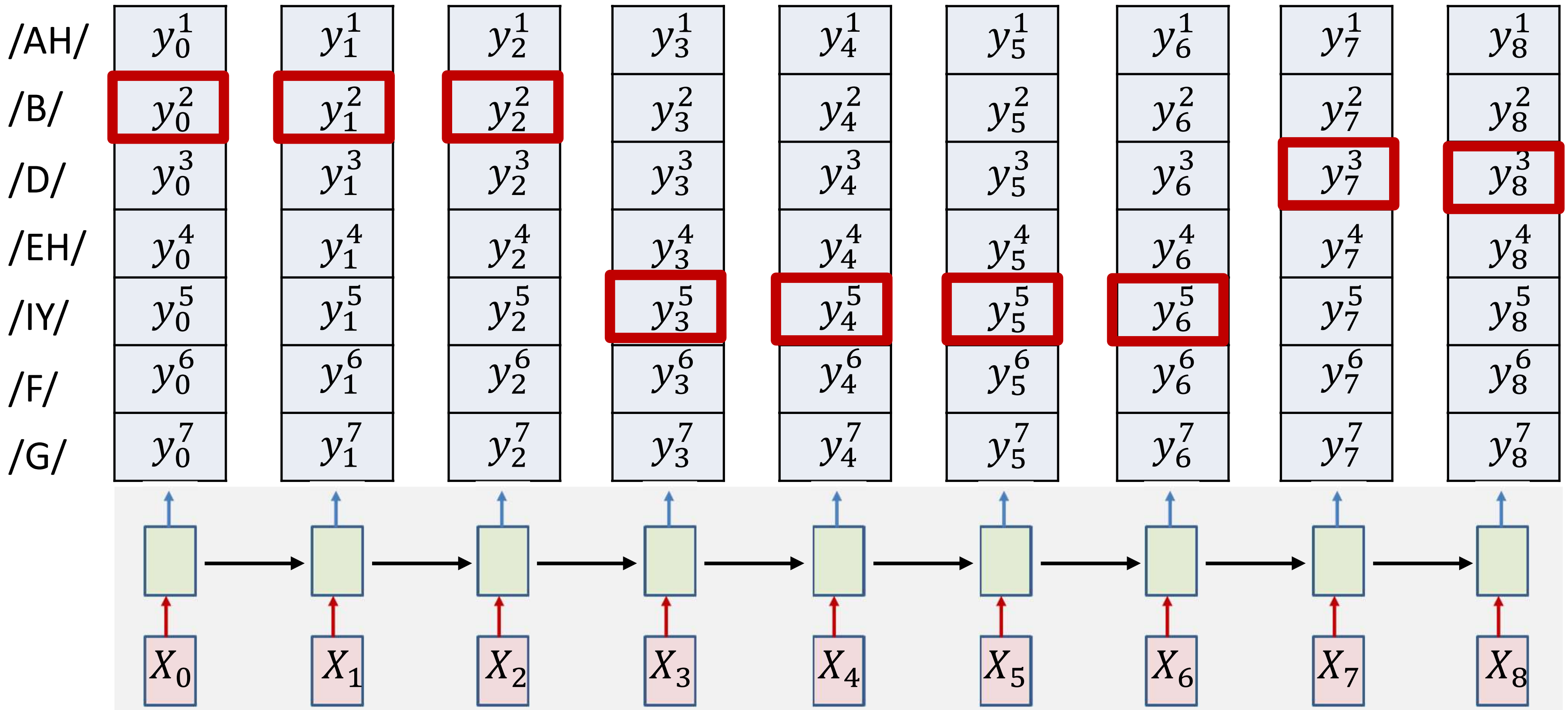
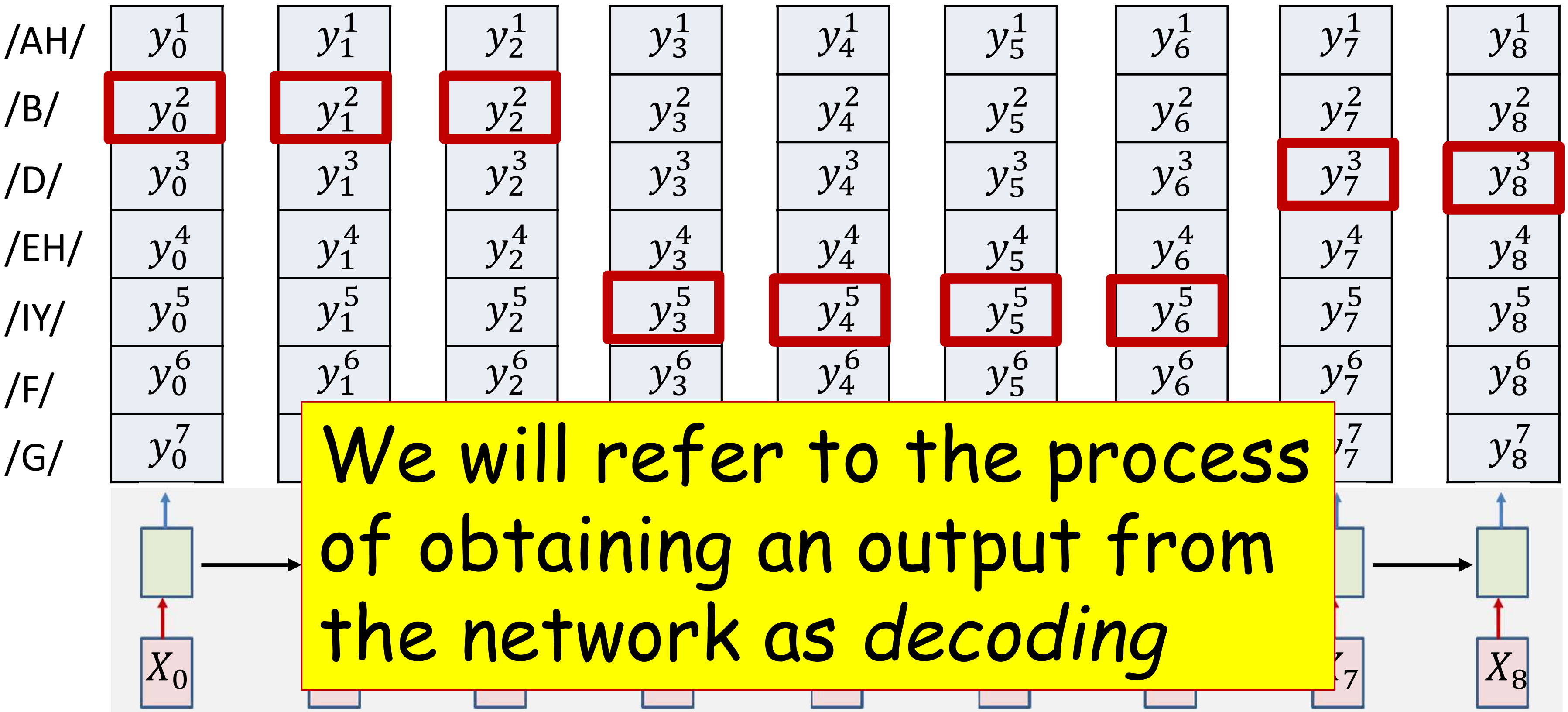
Finding the best output



- Option 2: Impose external constraints on what sequences are allowed
  - *E.g.* only allow sequences corresponding to dictionary words
  - *E.g. Sub-symbol* units (like in HW1 – what were they?)
  - *E.g.* using special “separating” symbols to separate repetitions

Finding the best output



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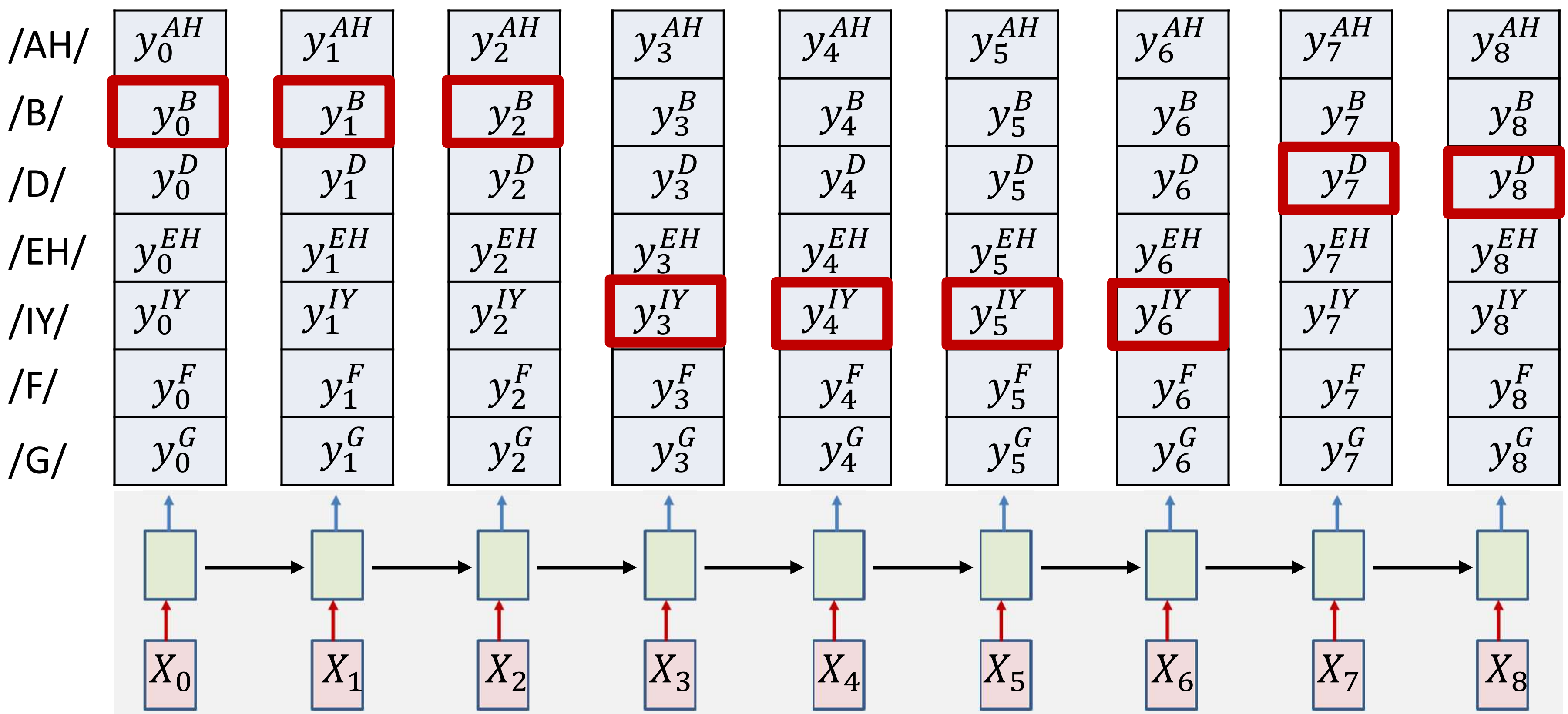
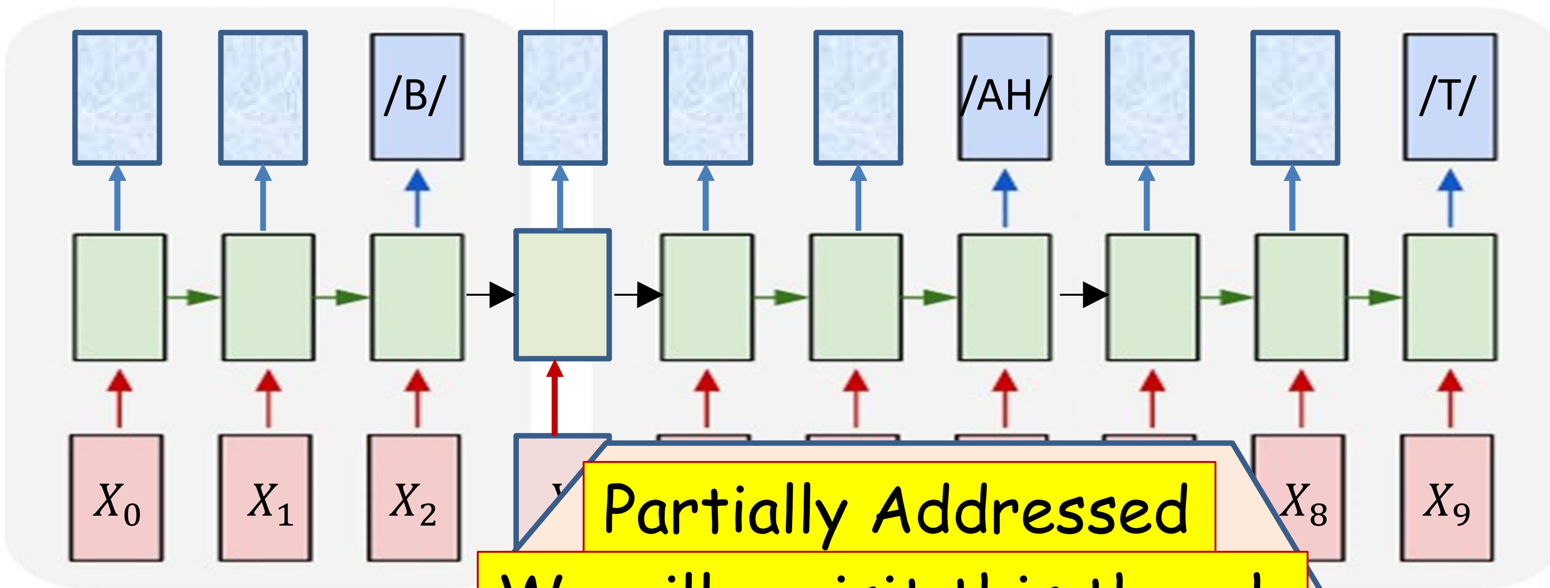
Decoding



- This is in fact a *suboptimal* decode that actually finds the most likely *time-synchronous* output sequence
  - Which is not necessarily the most likely *order-synchronous* sequence
    - The “merging” heuristics do not guarantee optimal order-synchronous sequences
  - We will return to this topic later

The *sequence-to-sequence* problem



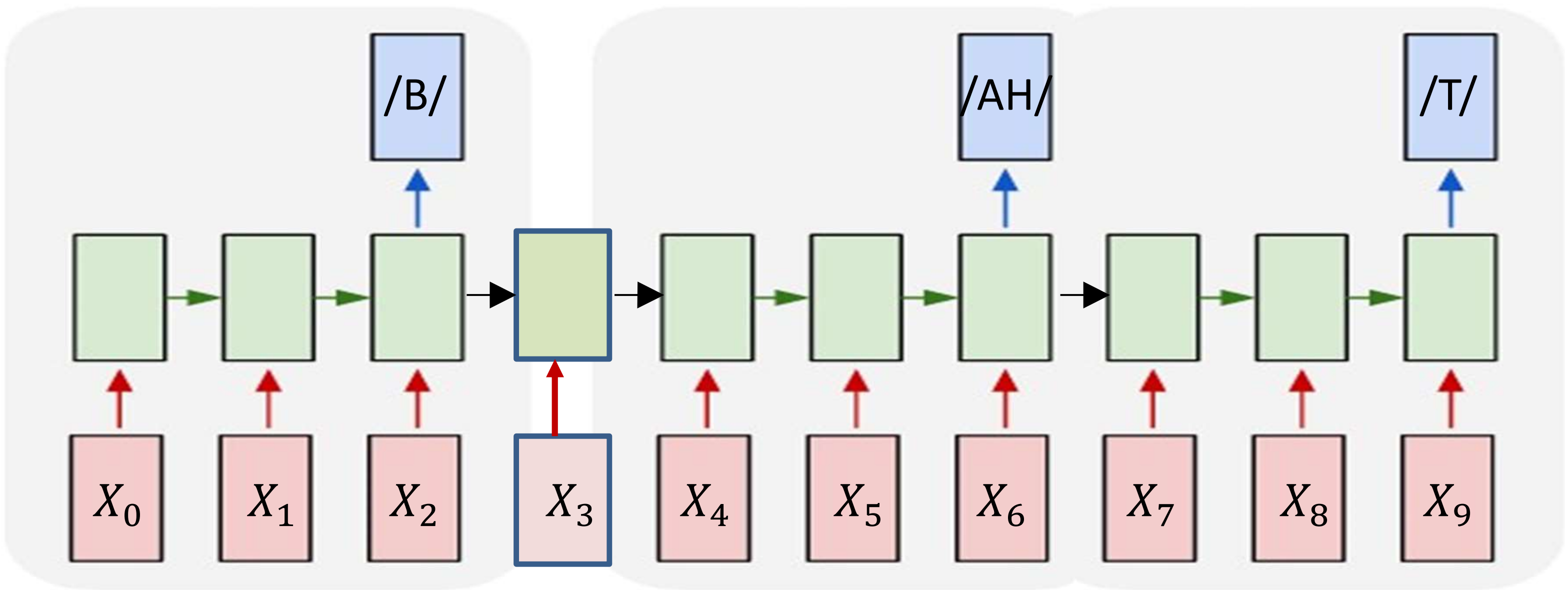
Partially Addressed

We will revisit this though

- How do we know *when* to output symbols
  - In fact, the network produces outputs at *every* time
  - *Which* of these are the *real* outputs

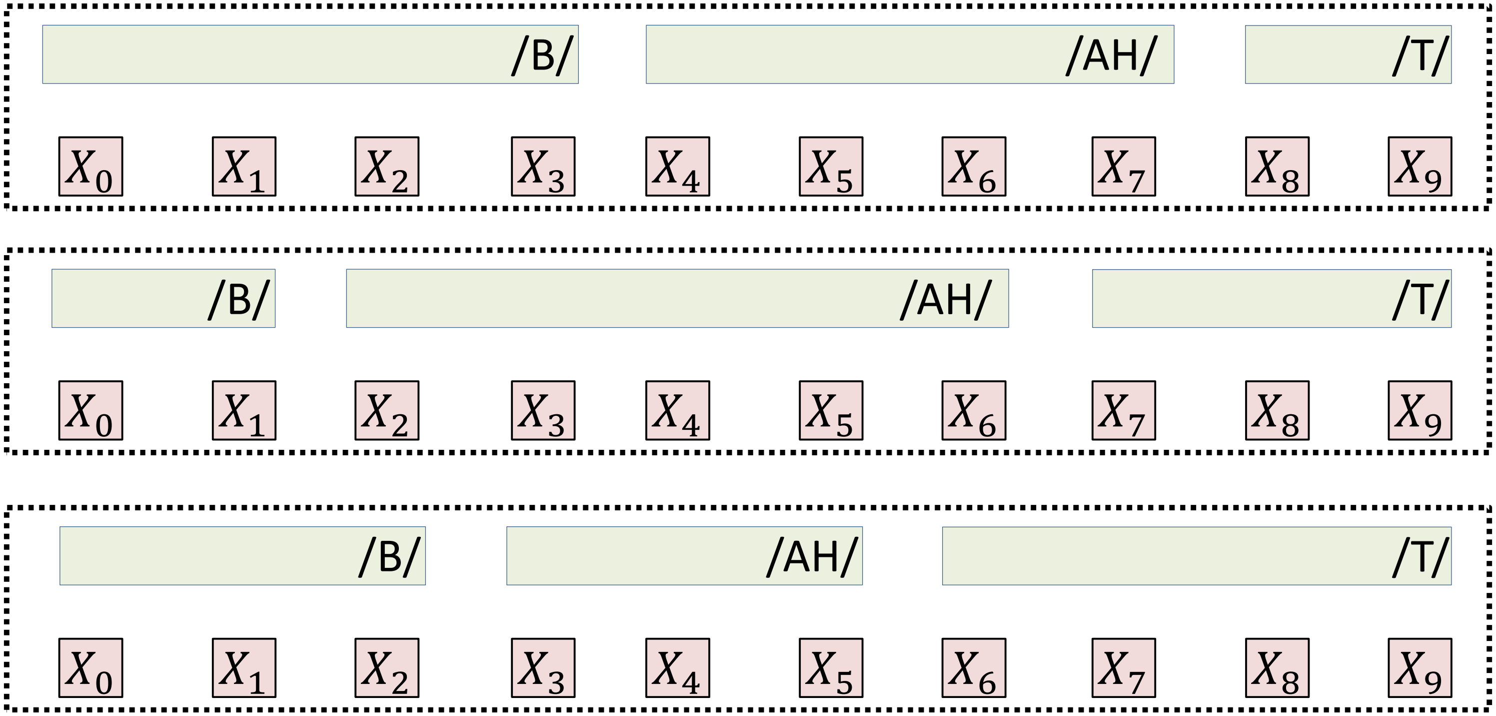
- How do we *train* these models?

Training



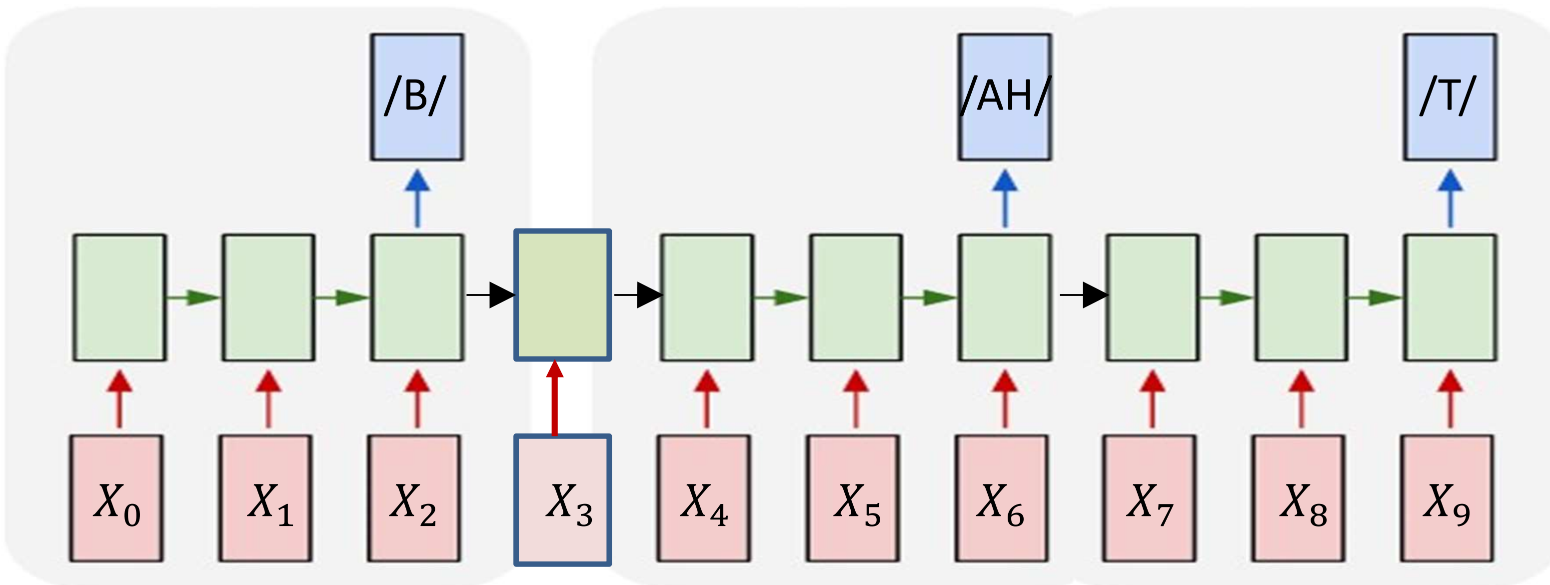
- Training data: input sequence + output sequence
  - Output sequence length <= input sequence length
- Given output symbols *at the right locations*
  - The phoneme /B/ ends at X_2, /AH/ at X_6, /T/ at X_9

The "alignment" of labels



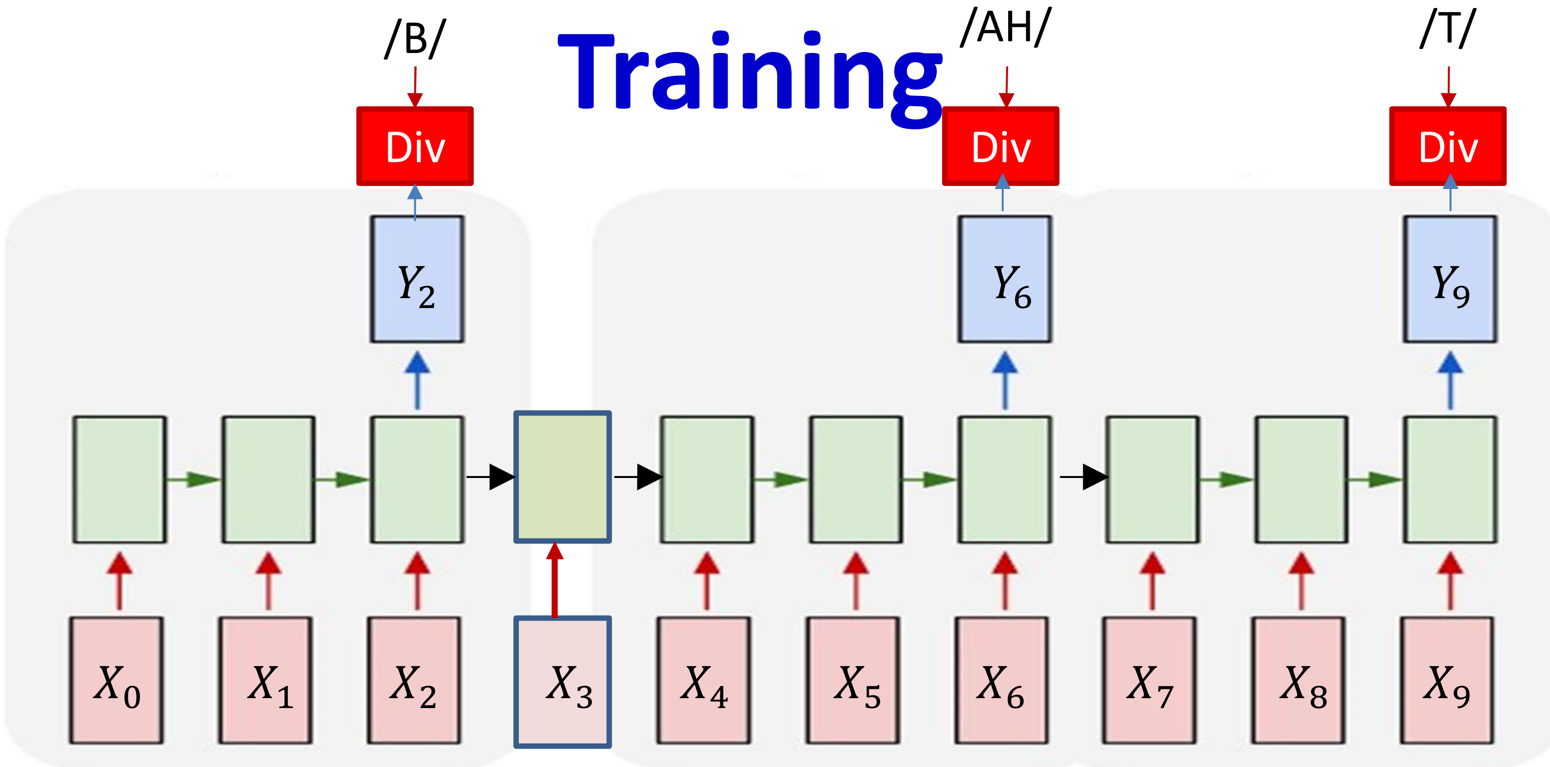
- The time-stamps of the output symbols give us the "alignment" of the output sequence to the input sequence
  - Which portion of the input aligns to what symbol
- Simply knowing the output sequence does not provide us the alignment
  - This is extra information

Training with alignment



- Training data: input sequence + output sequence
  - Output sequence length <= input sequence length
- Given the *alignment* of the output to the input
  - The phoneme /B/ ends at X_2, /AH/ at X_6, /T/ at X_9

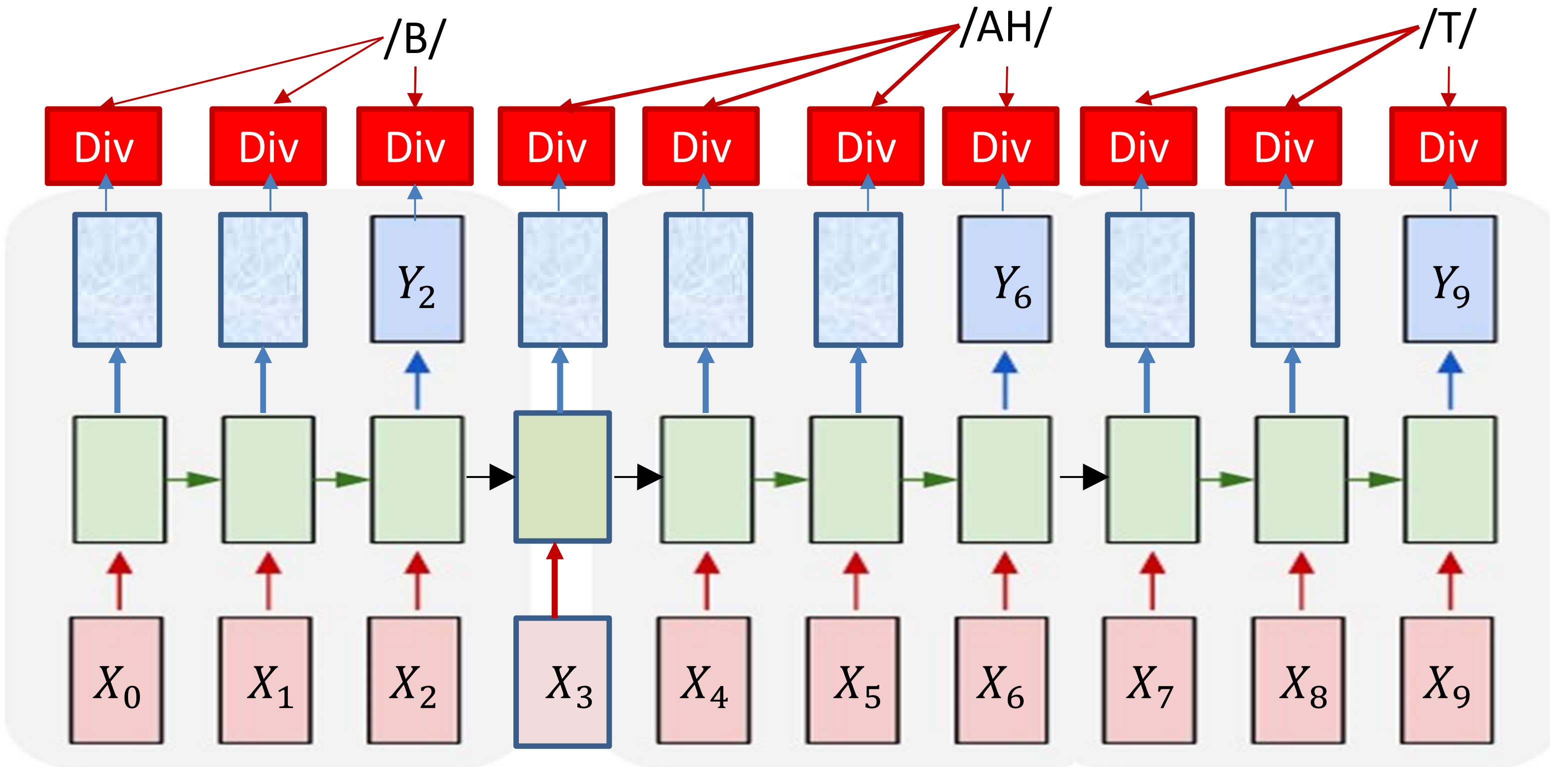
Training



- Either just define Divergence as:

$$DIV = KL(Y_2, B) + KL(Y_6, AH) + KL(Y_9, T)$$

- Or..



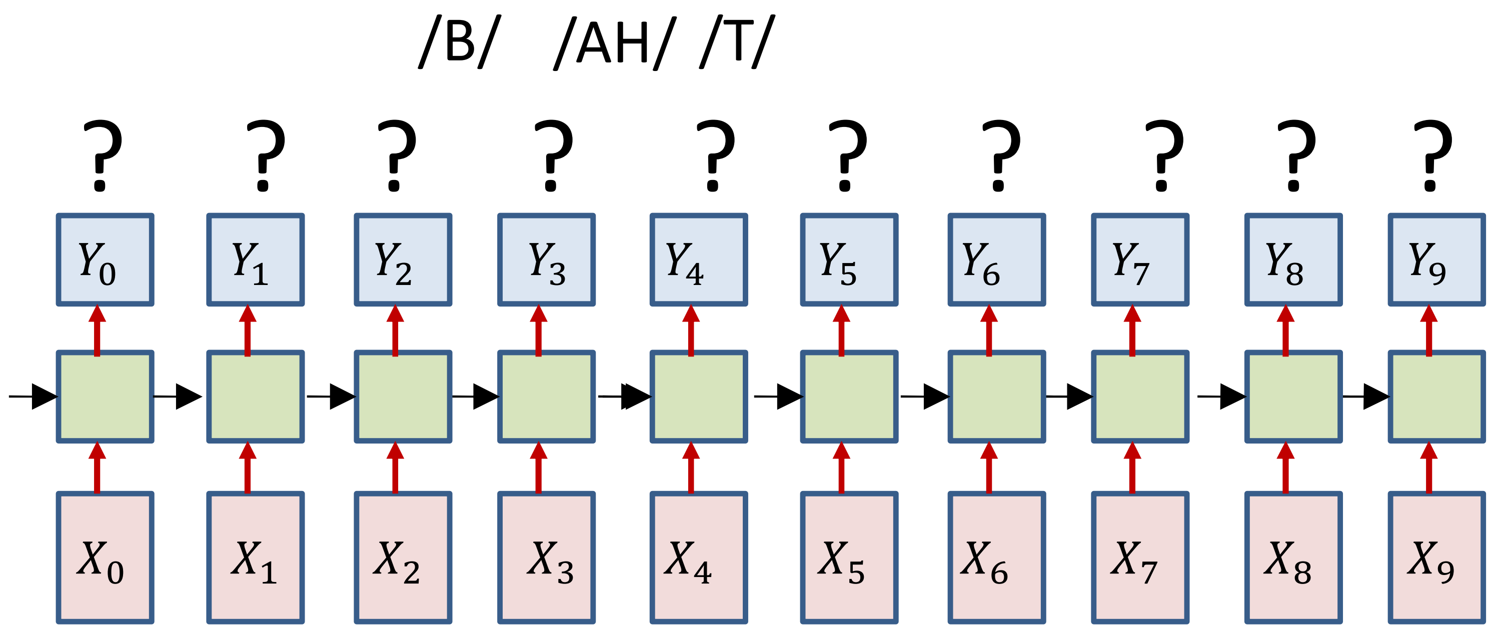
- Either just define Divergence as:

  $DIV = Xent(Y_2, B) + Xent(Y_6, AH) + Xent(Y_9, T)$

- Or *repeat the symbols over their duration*

$$DIV = \sum_t KL(Y_t, symbol_t) = -\sum_t \log Y(t, symbol_t)$$

Problem: No timing information provided



- Only the sequence of output symbols is provided for the training data
  - But no indication of which one occurs where

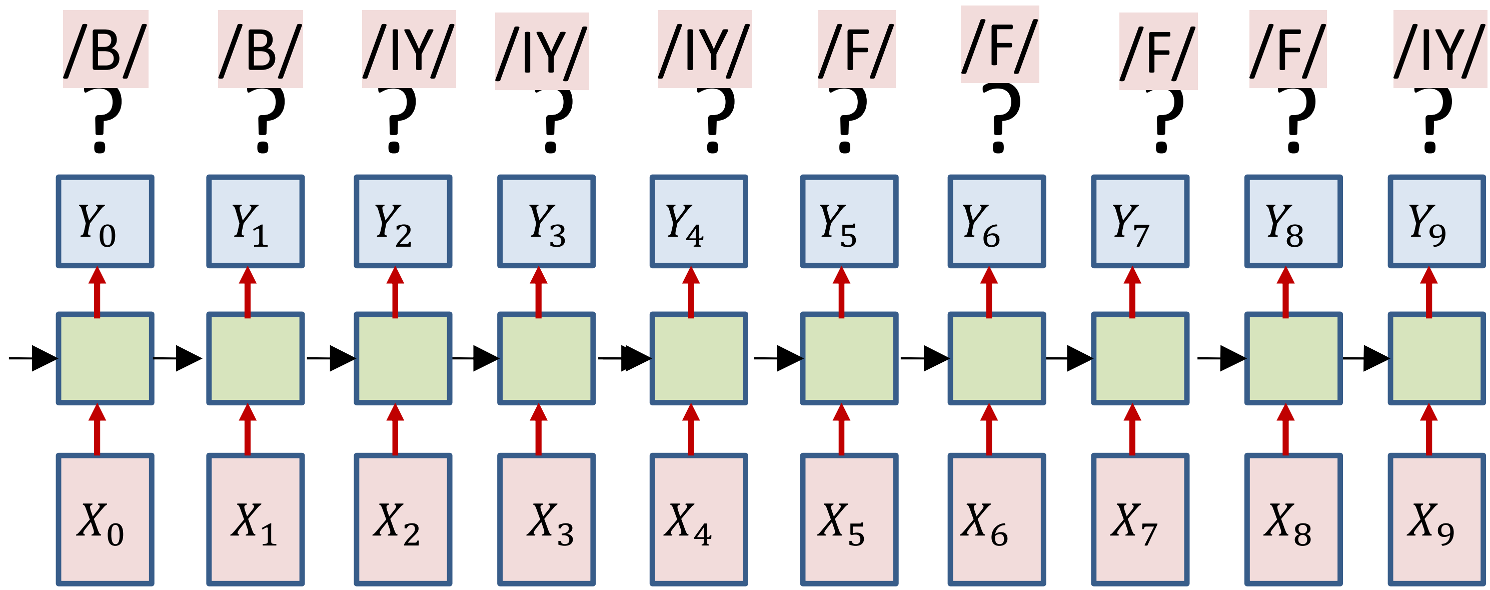
- How do we compute the divergence?
  - And how do we compute its gradient w.r.t. Y_t

Training *without* alignment

- We know how to train if the alignment is provided
- Problem: Alignment is *not* provided

- Solution:
  1. *Guess* the alignment
  2. Consider *all possible* alignments

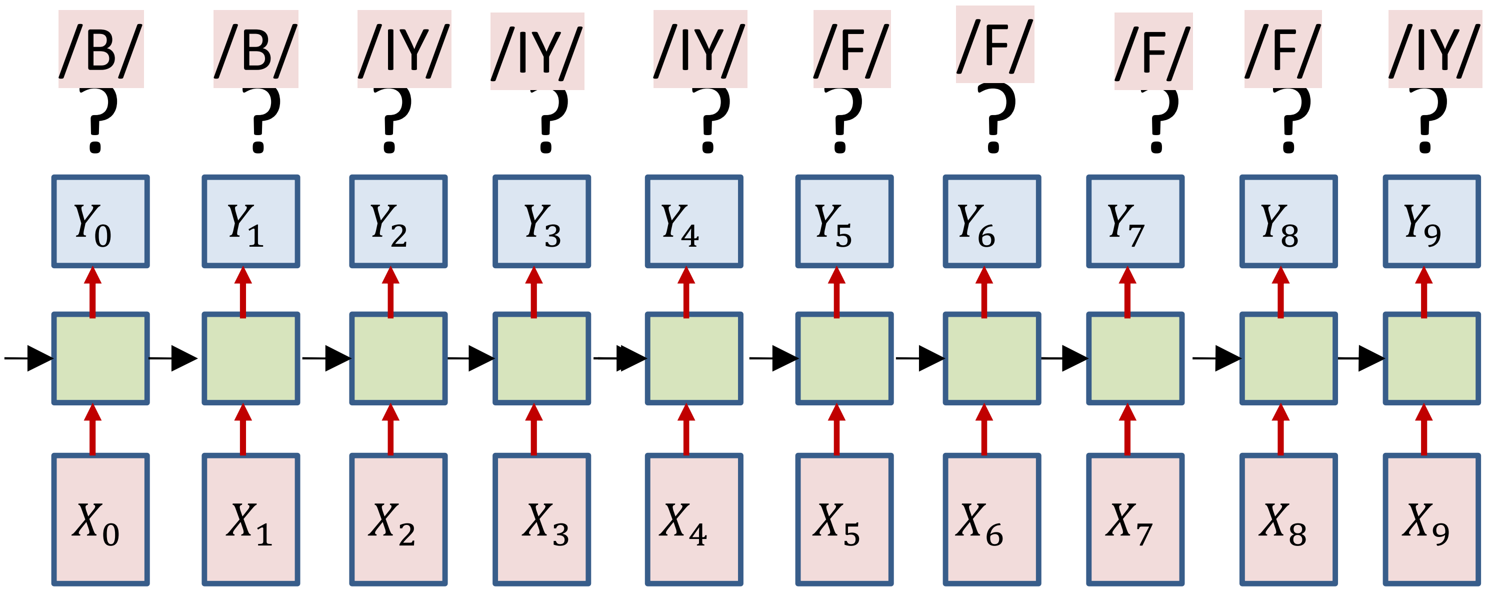
Solution 1: *Guess the alignment*



- Guess an initial alignment and iteratively refine it as the model improves

- Initialize: Assign an initial alignment
  - Either randomly, based on some heuristic, or any other rationale
- Iterate:
  - Train the network using the current alignment
  - *Reestimate* the alignment for each training instance

Solution 1: *Guess the alignment*



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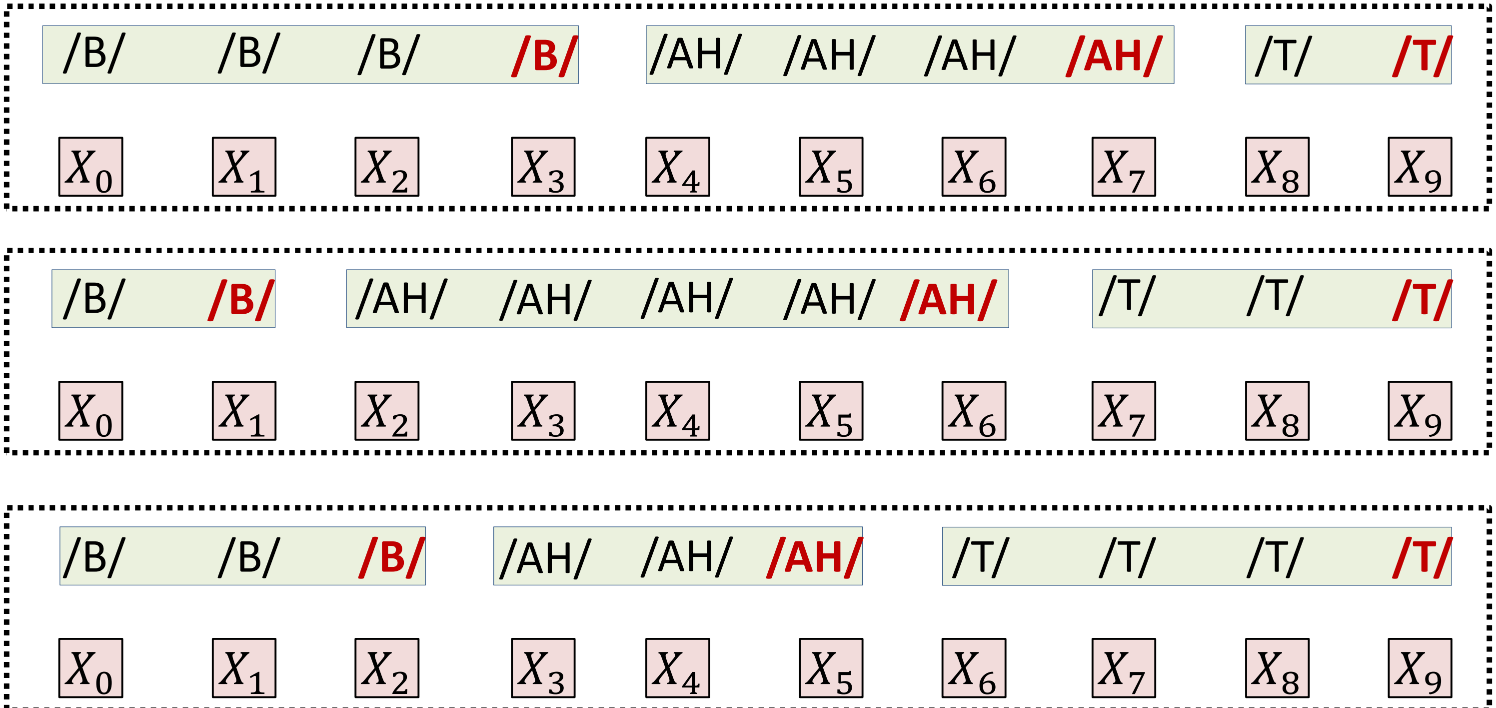
  - *Reestimate* the alignment for each training instance

Characterizing the alignment



- An alignment can be represented as a repetition of symbols
  - Examples show different alignments of /B/ /AH/ /T/ to $X_0 \ldots X_9$

Estimating an alignment

- Given:
  - The unaligned K-length symbol sequence $S = S_0 \dots S_{K-1}$ (e.g. /B/ /IY/ /F/ /IY/)
  - An N-length input ($N \geq K$)
  - And a (trained) recurrent network

- Find:
  - An N-length expansion $s_0 \dots s_{N-1}$ comprising the symbols in S in strict order
    - e.g. $S_0 S_0 S_1 S_1 S_1 S_2 \dots S_{K-1}$
      - i.e.$s_0 = S_0, s_1 = S_0, s_2 = S_1, s_3 = S_1, s_4 = S_1, \dots s_{N-1} = S_{K-1}$
    - E.g. /B/ /B/ /IY/ /IY/ /IY/ /F/ /F/ /F/ /F/ /IY/ ..

- Outcome: an *alignment* of the target symbol sequence $S_0 \dots S_{K-1}$ to the input $X_0 \dots X_{N-1}$

Estimating an alignment

- Alignment problem:

- Find

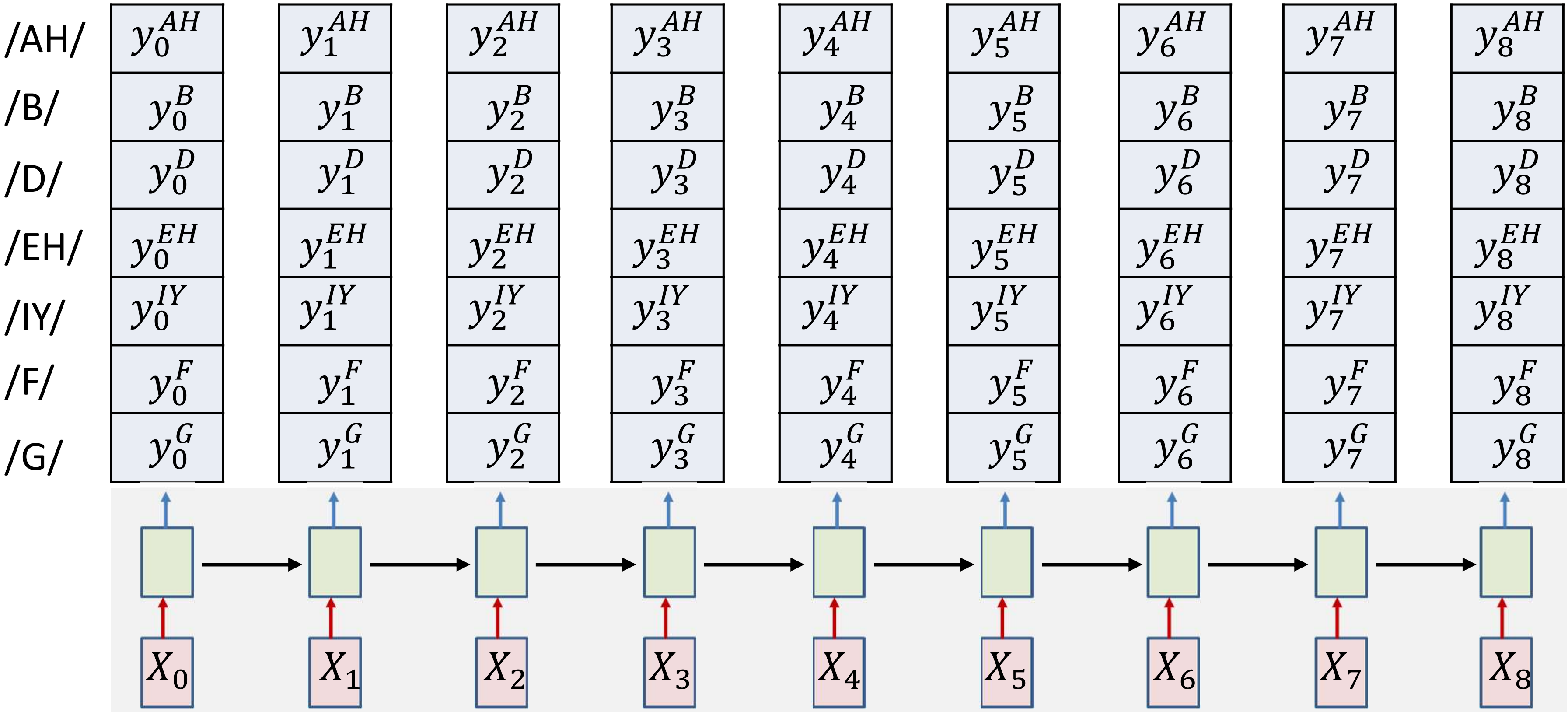
$$argmax\ P(s_0, s_1, \dots, s_{N-1} | S_0, S_1, \dots, S_K, X_0, X_1, \dots, X_{N-1})$$

  - Such that

$$compress(s_0, s_1, \dots, s_{N-1}) \equiv S_0, S_1, \dots, S_K$$

- $compress()$ is the operation of compressing repetitions into one

Recall: The actual output of the network



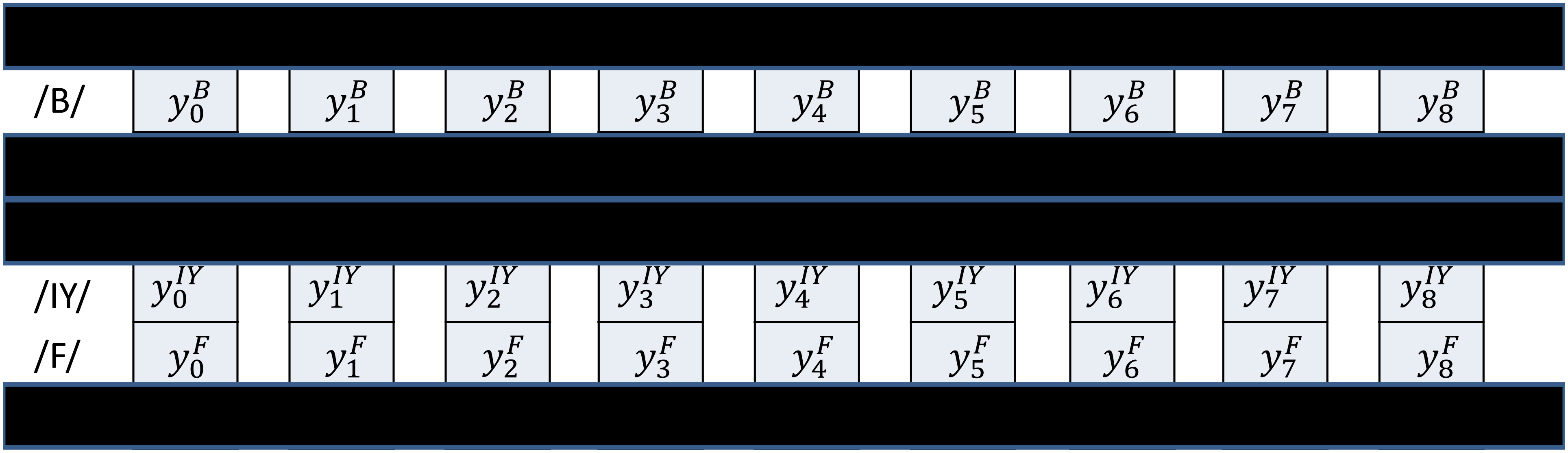
- At each time the network outputs a probability for *each* output symbol

Recall: unconstrained decoding

/AH/	$\boxed{y_0^{AH}}$	$\boxed{y_1^{AH}}$	$\boxed{y_2^{AH}}$	y_3^{AH}	y_4^{AH}	$\boxed{y_5^{AH}}$	y_6^{AH}	y_7^{AH}	y_8^{AH}
/B/	y_0^{B}	y_1^{B}	y_2^{B}	y_3^{B}	y_4^{B}	y_5^{B}	y_6^{B}	y_7^{B}	y_8^{B}
/D/	y_0^{D}	y_1^{D}	y_2^{D}	$\boxed{y_3^{D}}$	$\boxed{y_4^{D}}$	y_5^{D}	y_6^{D}	y_7^{D}	y_8^{D}
/EH/	y_0^{EH}	y_1^{EH}	y_2^{EH}	y_3^{EH}	y_4^{EH}	y_5^{EH}	y_6^{EH}	y_7^{EH}	y_8^{EH}
/IY/	y_0^{IY}	y_1^{IY}	y_2^{IY}	y_3^{IY}	y_4^{IY}	y_5^{IY}	y_6^{IY}	$\boxed{y_7^{IY}}$	$\boxed{y_8^{IY}}$
/F/	y_0^{F}	y_1^{F}	y_2^{F}	y_3^{F}	y_4^{F}	y_5^{F}	$\boxed{y_6^{F}}$	y_7^{F}	y_8^{F}
/G/	y_0^{G}	y_1^{G}	y_2^{G}	y_3^{G}	y_4^{G}	y_5^{G}	y_6^{G}	y_7^{G}	y_8^{G}

- We find the most likely sequence of symbols
  - (Conditioned on input $X_0 \ldots X_{N-1}$)
- This may not correspond to an expansion of the desired symbol sequence
  - E.g. the unconstrained decode may be /AH//AH//AH//D//D//AH//F//IY//IY/
    - Contracts to /AH/ /D/ /AH/ /F/ /IY/
  - Whereas we want an expansion of /B//IY//F//IY/

Constraining the alignment: Try 1



- Block out all rows that do not include symbols from the target sequence
  - E.g. Block out rows that are not /B/ /IY/ or /F/

Blocking out unnecessary outputs

Compute the entire output (for all symbols)

Copy the output values for the target symbols into the secondary reduced structure

Constraining the alignment: Try 1

/B/	y_0^B	y_1^B	y_2^B	y_3^B	y_4^B	y_5^B	y_6^B	y_7^B	y_8^B
/IY/	y_0^{IY}	y_1^{IY}	y_2^{IY}	y_3^{IY}	y_4^{IY}	y_5^{IY}	y_6^{IY}	y_7^{IY}	y_8^{IY}
/F/	y_0^F	y_1^F	y_2^F	y_3^F	y_4^F	y_5^F	y_6^F	y_7^F	y_8^F

- Only decode on reduced grid
  - We are now assured that only the appropriate symbols will be hypothesized

Constraining the alignment: Try 1

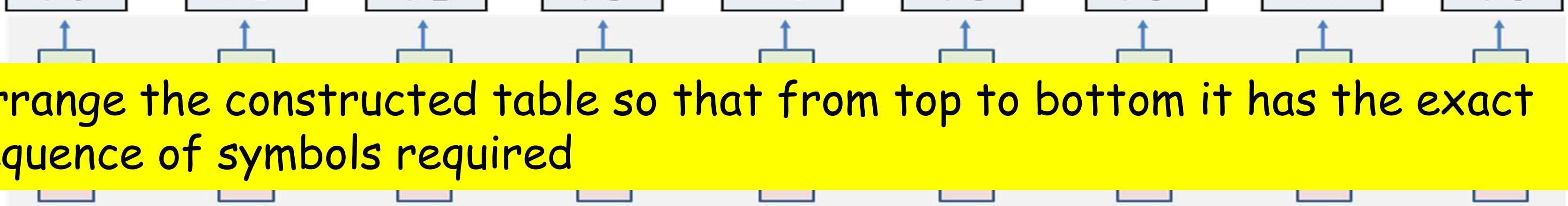
/B/	$\boxed{y_0^B}$	$\boxed{y_1^B}$	y_2^B	y_3^B	$\boxed{y_4^B}$	$\boxed{y_5^B}$	y_6^B	y_7^B	y_8^B
/IY/	y_0^{IY}	y_1^{IY}	$\boxed{y_2^{IY}}$	y_3^{IY}	y_4^{IY}	y_5^{IY}	y_6^{IY}	y_7^{IY}	$\boxed{y_8^{IY}}$
/F/	y_0^F	y_1^F	y_2^F	$\boxed{y_3^F}$	y_4^F	y_5^F	$\boxed{y_6^F}$	$\boxed{y_7^F}$	y_8^F

- Only decode on reduced grid
  - We are now assured that only the appropriate symbols will be hypothesized
- Problem: This still doesn't assure that the decode sequence correctly expands the target symbol sequence
  - E.g. the above decode is not an expansion of /B//IY//F//IY/
- Still needs additional constraints

Try 2: Explicitly arrange the constructed table

/B/	y_0^B	y_1^B	y_2^B	y_3^B	y_4^B	y_5^B	y_6^B	y_7^B	y_8^B
/IY/	y_0^{IY}	y_1^{IY}	y_2^{IY}	y_3^{IY}	y_4^{IY}	y_5^{IY}	y_6^{IY}	y_7^{IY}	y_8^{IY}
/F/	y_0^F	y_1^F	y_2^F	y_3^F	y_4^F	y_5^F	y_6^F	y_7^F	y_8^F
/IY/	y_0^{IY}	y_1^{IY}	y_2^{IY}	y_3^{IY}	y_4^{IY}	y_5^{IY}	y_6^{IY}	y_7^{IY}	y_8^{IY}

/AH/	y_0^{AH}	y_1^{AH}	y_2^{AH}	y_3^{AH}	y_4^{AH}	y_5^{AH}	y_6^{AH}	y_7^{AH}	y_8^{AH}
/B/	y_0^B	y_1^B	y_2^B	y_3^B	y_4^B	y_5^B	y_6^B	y_7^B	y_8^B
/D/	y_0^D	y_1^D	y_2^D	y_3^D	y_4^D	y_5^D	y_6^D	y_7^D	y_8^D
/EH/	y_0^{EH}	y_1^{EH}	y_2^{EH}	y_3^{EH}	y_4^{EH}	y_5^{EH}	y_6^{EH}	y_7^{EH}	y_8^{EH}
/IY/	y_0^{IY}	y_1^{IY}	y_2^{IY}	y_3^{IY}	y_4^{IY}	y_5^{IY}	y_6^{IY}	y_7^{IY}	y_8^{IY}
/F/	y_0^F	y_1^F	y_2^F	y_3^F	y_4^F	y_5^F	y_6^F	y_7^F	y_8^F
/G/	y_0^G	y_1^G	y_2^G	y_3^G	y_4^G	y_5^G	y_6^G	y_7^G	y_8^G



Arrange the constructed table so that from top to bottom it has the exact sequence of symbols required

Try 2: Explicitly arrange the constructed table

/B/	y_0^B	y_1^B	y_2^B	y_3^B	y_4^B	y_5^B	y_6^B	y_7^B	y_8^B
/IY/	y_0^{IY}	y_1^{IY}	y_2^{IY}	y_3^{IY}	y_4^{IY}	y_5^{IY}	y_6^{IY}	y_7^{IY}	y_8^{IY}
/F/	y_0^F	y_1^F	y_2^F	y_3^F	y_4^F	y_5^F	y_6^F	y_7^F	y_8^F
/IY/	y_0^{IY}	y_1^{IY}	y_2^{IY}	y_3^{IY}	y_4^{IY}	y_5^{IY}	y_6^{IY}	y_7^{IY}	y_8^{IY}

Note: If a symbol occurs multiple times, we repeat the row in the appropriate location.
E.g. the row for /IY/ occurs twice, in the 2nd and 4th positions

/B/	y_0^B	y_1^B	y_2^B	y_3^B	y_4^B	y_5^B	y_6^B	y_7^B	y_8^B
/D/	y_0^D	y_1^D	y_2^D	y_3^D	y_4^D	y_5^D	y_6^D	y_7^D	y_8^D
/EH/	y_0^{EH}	y_1^{EH}	y_2^{EH}	y_3^{EH}	y_4^{EH}	y_5^{EH}	y_6^{EH}	y_7^{EH}	y_8^{EH}
/IY/	y_0^{IY}	y_1^{IY}	y_2^{IY}	y_3^{IY}	y_4^{IY}	y_5^{IY}	y_6^{IY}	y_7^{IY}	y_8^{IY}
/F/	y_0^F	y_1^F	y_2^F	y_3^F	y_4^F	y_5^F	y_6^F	y_7^F	y_8^F
/G/	y_0^G	y_1^G	y_2^G	y_3^G	y_4^G	y_5^G	y_6^G	y_7^G	y_8^G

Arrange the constructed table so that from top to bottom it has the exact sequence of symbols required

Composing the graph

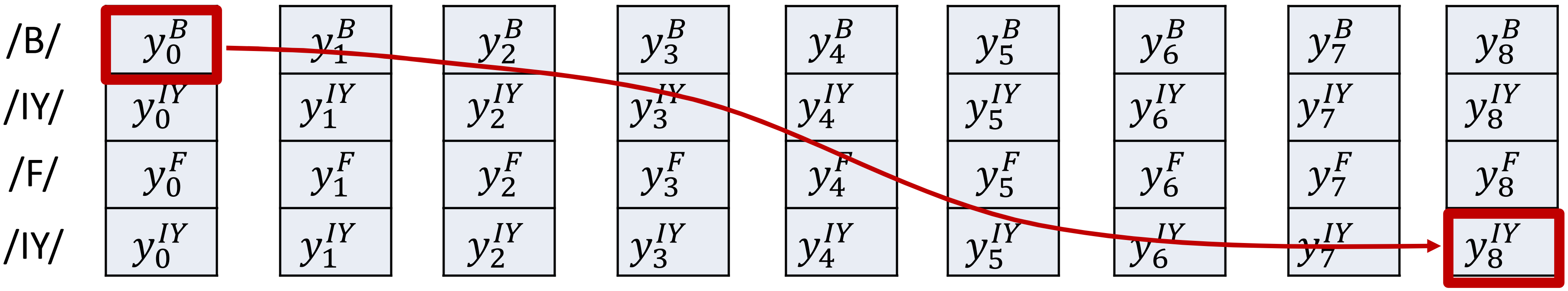
```
#N is the number of symbols in the target output
#S(i) is the ith symbol in target output
#T = length of input

#First create output table
For i = 1:N
    s(1:T,i) = y(1:T, S(i))
```

Using 1..N and 1..T indexing, instead of 0..N-1, 0..T-1, for convenience of notation

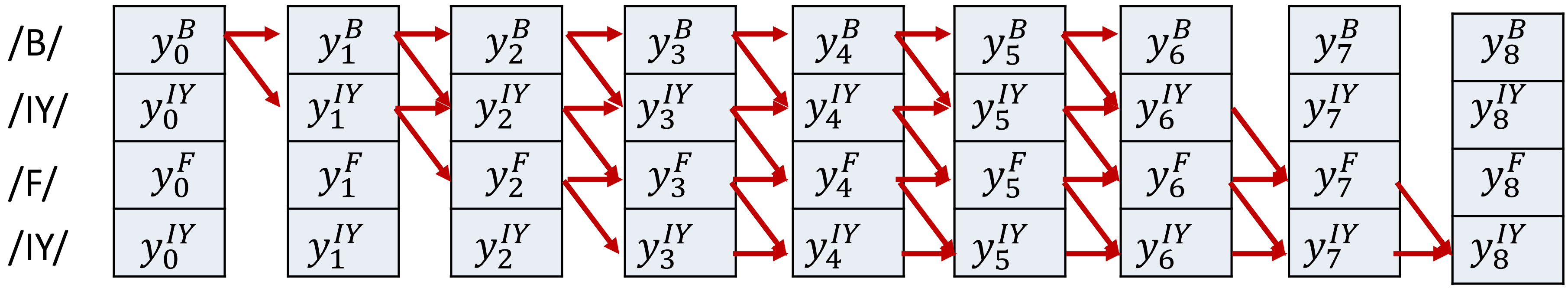
/B/	y_0^B		y_1^B		y_2^B		y_3^B		y_4^B		y_5^B		y_6^B		y_7^B		y_8^B
/IY/	y_0^{IY}		y_1^{IY}		y_2^{IY}		y_3^{IY}		y_4^{IY}		y_5^{IY}		y_6^{IY}		y_7^{IY}		y_8^{IY}
/F/	y_0^F		y_1^F		y_2^F		y_3^F		y_4^F		y_5^F		y_6^F		y_7^F		y_8^F
/IY/	y_0^{IY}		y_1^{IY}		y_2^{IY}		y_3^{IY}		y_4^{IY}		y_5^{IY}		y_6^{IY}		y_7^{IY}		y_8^{IY}

Explicitly constrain alignment



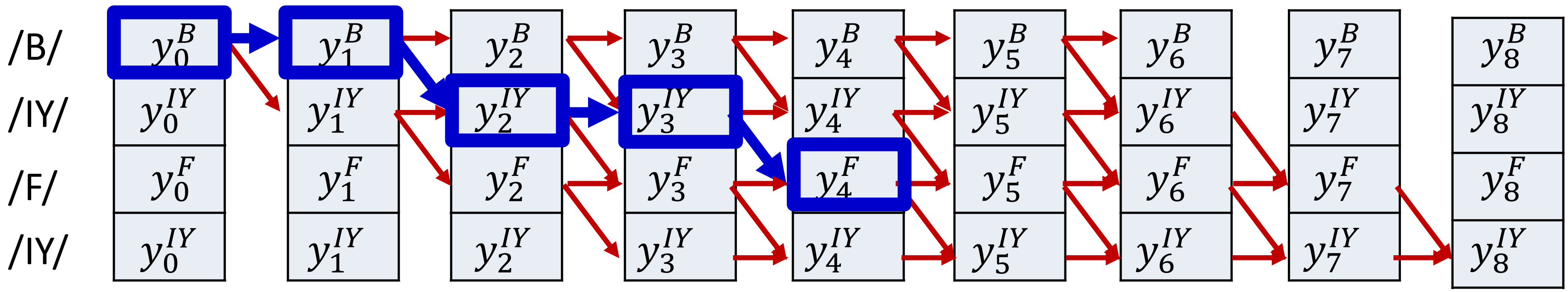
- Constrain that the first symbol in the decode *must* be the top left block
- The last symbol *must* be the bottom right
- The rest of the symbols must follow a sequence that *monotonically* travels down from top left to bottom right
  - I.e. symbol chosen at any time is at the same level or at the next level to the symbol at the previous time
- This guarantees that the sequence *is* an expansion of the target sequence
  - /B/ /IY/ /F/ /IY/ in this case

Explicitly constrain alignment



- Compose a graph such that every path in the graph from source to sink represents a valid alignment
  - Which maps on to the target symbol sequence (/B//IY//F//IY/)
- Edge scores are 1
- Node scores are the probabilities assigned to the symbols by the neural network

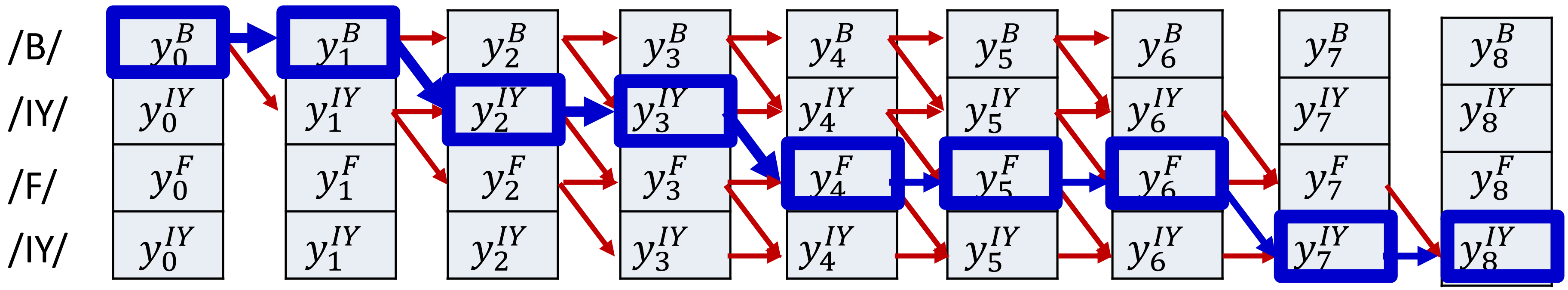
Path Score (probability)



- Compose a graph such that every path in the graph from source to sink represents a valid alignment
  - Which maps on to the target symbol sequence (/B//IY//F//IY/)
- Edge scores are 1
- Node scores are the probabilities assigned to the symbols by the neural network
- **The "score" of a path is the product of the probabilities of all nodes along the path**
- **E.g. the probability of the marked path is**

$$Scr(Path) = y_0^B y_1^B y_2^{IY} y_3^{IY} y_4^F$$

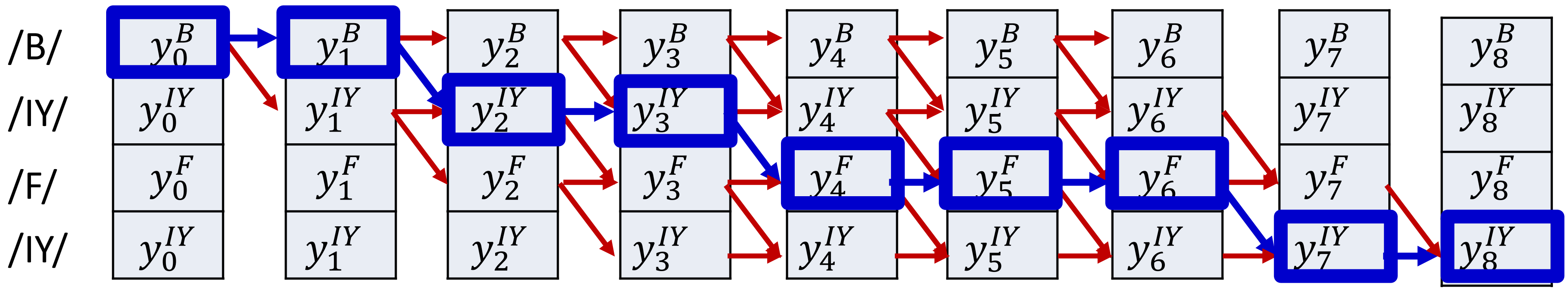
Path Score (probability)



- Compose a graph such that every path in the graph from source to sink represents a valid alignment
  - Which maps on to the target symbol sequence (/B//IY//F//IY/)
- Edge scores are 1
- Node scores are the probabilities assigned to the symbols by the neural network
- **The "score" of a path is the product of the probabilities of all nodes along the path**

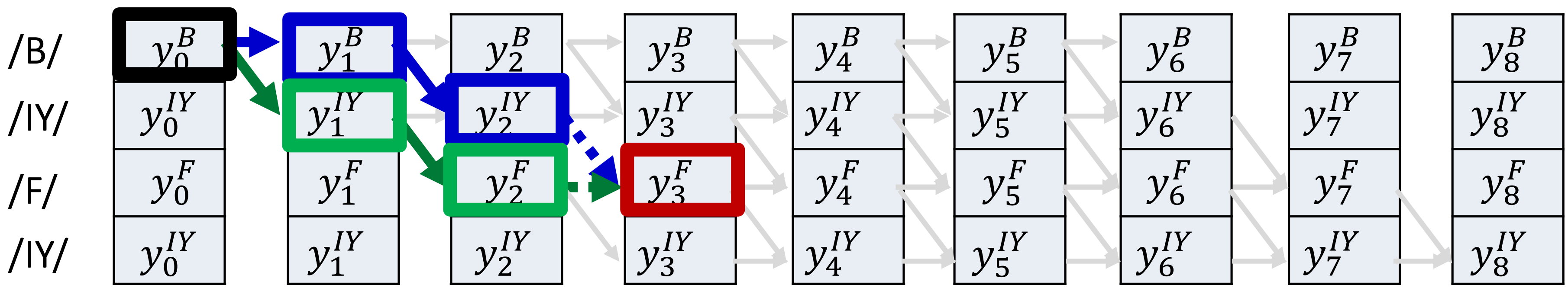
Figure shows a typical end-to-end path. There are an exponential number of such paths. Challenge: Find the path with the highest score (probability)

Explicitly constrain alignment



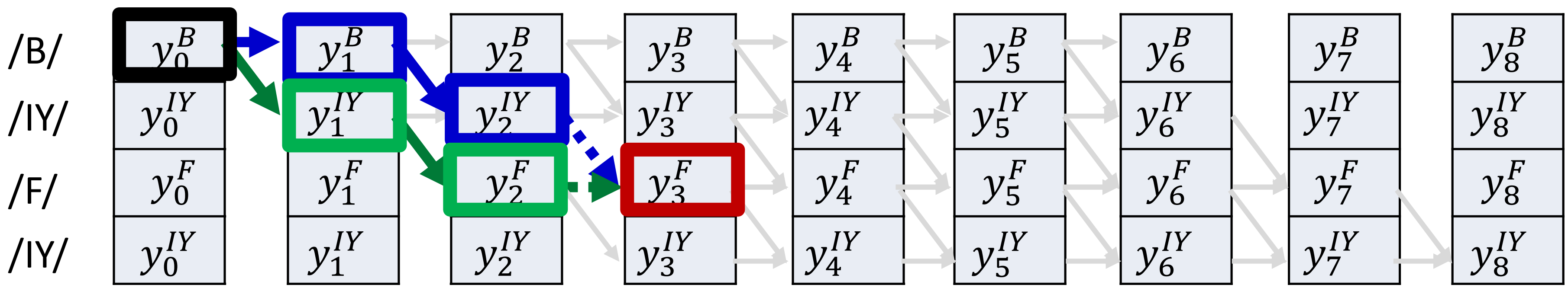
- Find the ***most probable path*** from source to sink using any dynamic programming algorithm
  - E.g. The Viterbi algorithm

Viterbi algorithm: Basic idea



- The best path to any node *must* be an extension of the best path to one of its parent nodes
  - Any other path would necessarily have a lower probability
- The best parent is simply the parent with the best-scoring best path

Viterbi algorithm: Basic idea



$$BestPath(y_0^B \rightarrow y_3^F) = \; BestPath(y_0^B \rightarrow y_2^{IY})y_3^F$$

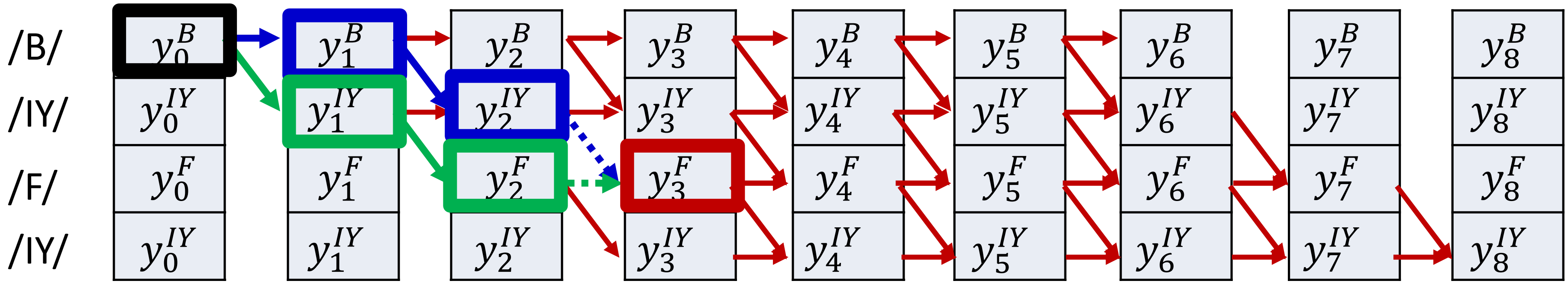
$$or \qquad BestPath(y_0^B \rightarrow y_2^F)y_3^F$$

$$BestPath(y_0^B \rightarrow y_3^F) = BestPath(y_0^B \rightarrow BestParent)y_3^F$$

- The best parent is simply the parent with the best-scoring best path

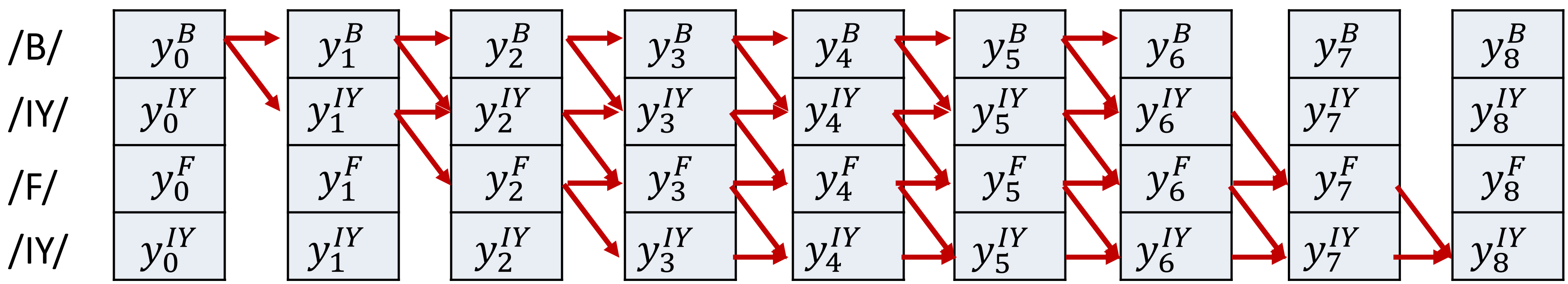
$$BestParent$$
$$= argmax_{Parent \in (y_2^{IY}, y_2^F)}(Score(BestPath(y_0^B \rightarrow Parent)))$$

Viterbi algorithm



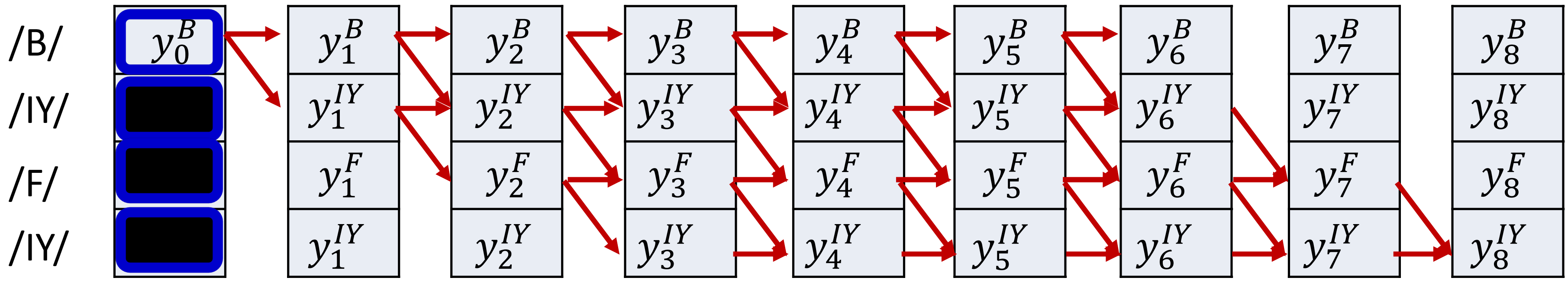
- Dynamically track the best path (and the score of the best path) from the source node to every node in the graph
  - At each node, keep track of
    - The best incoming parent edge
    - The score of the best path from the source to the node through this best parent edge
- Eventually compute the best path from source to sink

Viterbi algorithm



- First, some notation:
- $y_t^{S(r)}$ is the probability of the target symbol assigned to the r-th row in the t-th time (given inputs $X_0 \dots X_t$)
  - E.g., S(0) = /B/
    - The scores in the 0th row have the form y_t^B
  - E.g. S(1) = S(3) = /IY/
    - The scores in the 1st and 3rd rows have the form y_t^{IY}
  - E.g. S(2) = /F/
    - The scores in the 2nd row have the form y_t^F

Viterbi algorithm



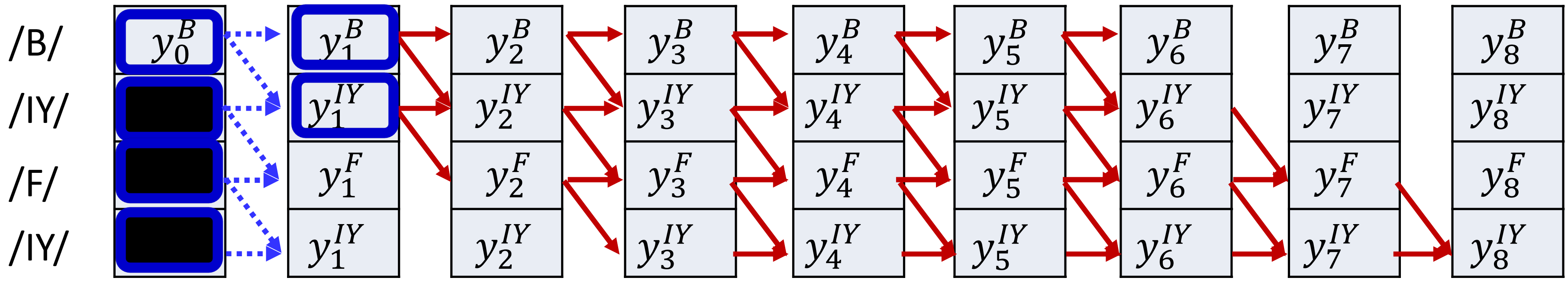
BP := Best Parent
Bscr := Bestpath Score to node

- Initialization:

$$BP(0,i) = null, \; i = 0 \ldots K-1$$

$$Bscr(0,0) = y_0^{S(0)}, \; Bscr(0,i) = 0 \; for \; i = 1 \ldots K-1$$

Viterbi algorithm



- Initialization:

$$BP(0,i) = null,\ i = 0 \ldots K-1$$

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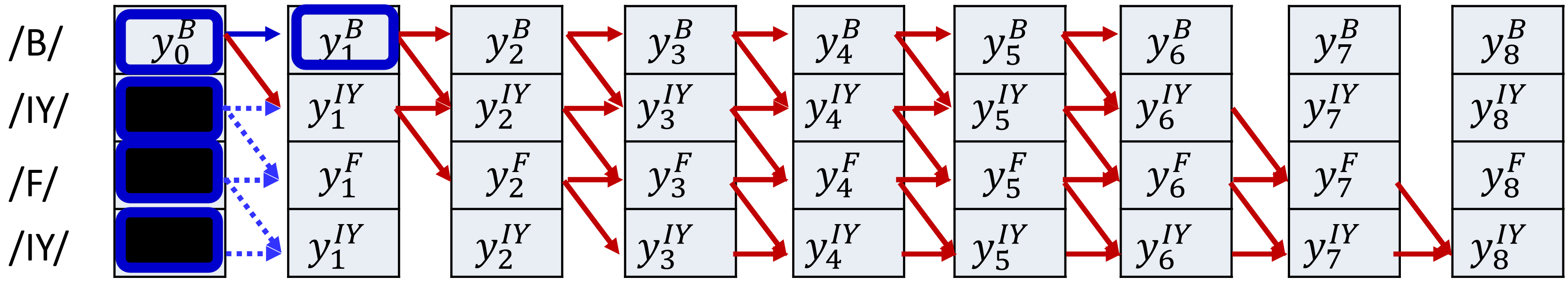
- for $t = 1 \ldots T-1$

  for $l = 0 \ldots K-1$

  - $BP(t,l) = \operatorname{argmax}_{p \in parents(l)} Bscr(t-1,p)$
  - $Bscr(t,l) = Bscr(BP(t,l)) \times y_t^{S(l)}$

Viterbi algorithm



- Initialization:

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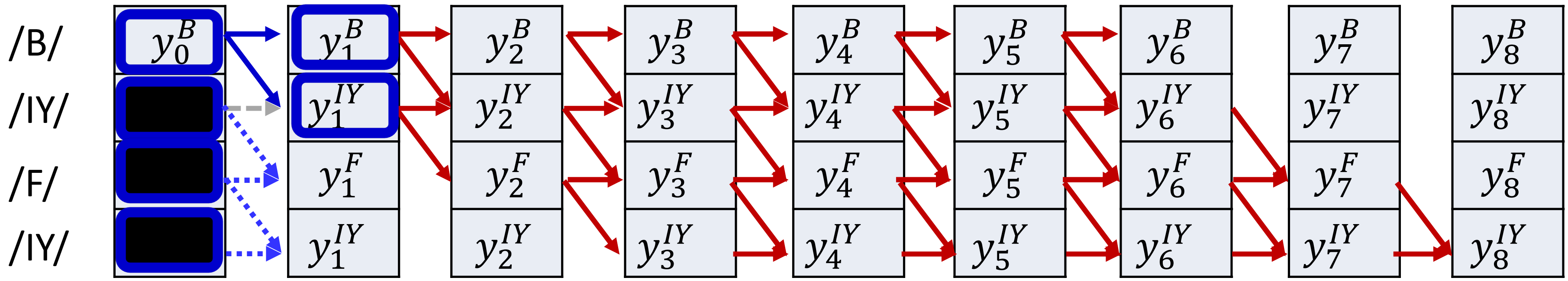
$$Bscr(0,0) = \ y_0^{S(0)}, \ \ Bscr(0,i) = 0 \ for \ \ i \ = \ 1 \ldots K-1$$

- for $t \ = \ 1 \ldots T-1$

$$BP(t,0) = \ 0; \ Bscr(t,0) \ = \ Bscr(t-1,0) \times y_t^{S(0)}$$

Viterbi algorithm



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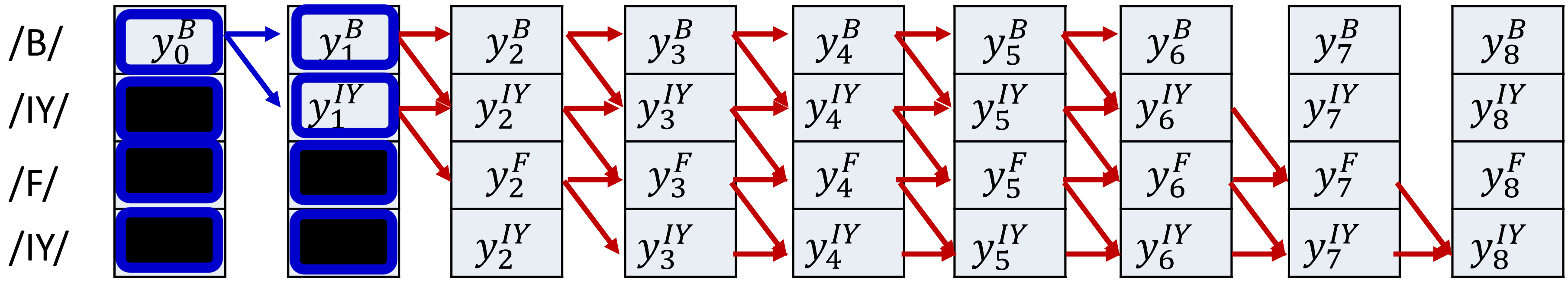
- for $t = 1 \dots T-1$

$$BP(t,0) = 0; \ Bscr(t,0) = Bscr(t-1,0) \times y_t^{S(0)}$$

for $l = 1 \dots K-1$

- $BP(t,l) = \begin{pmatrix} l-1: \ if \ (Bscr(t-1,l-1) > Bscr(t-1,l)) \\ l: else \end{pmatrix}$
- $Bscr(t,l) = Bscr(BP(t,l)) \times y_t^{S(l)}$

Viterbi algorithm



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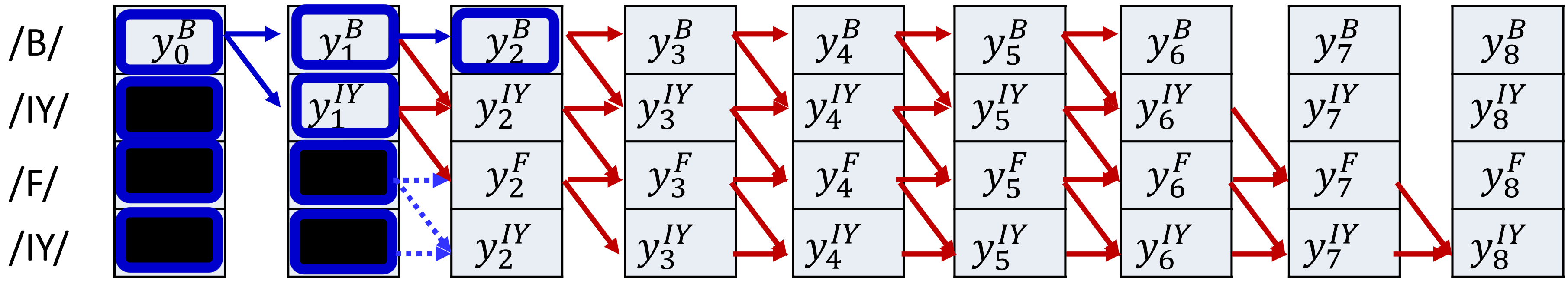
- for $t \; = \; 1 \dots T-1$

$$BP(t,0) = \; 0; \; Bscr(t,0) \; = \; Bscr(t-1,0) \times y_t^{S(0)}$$

for $l \; = 1 \dots K-1$

- $BP(t,l) = \begin{pmatrix} l-1: \;\; if \; (Bscr(t-1,l-1) > \; Bscr(t-1,l)) \;\; l-1; \\ l : else \end{pmatrix}$
- $Bscr(t,l) \; = \; Bscr(BP(t,l)) \; \times y_t^{S(l)}$

Viterbi algorithm



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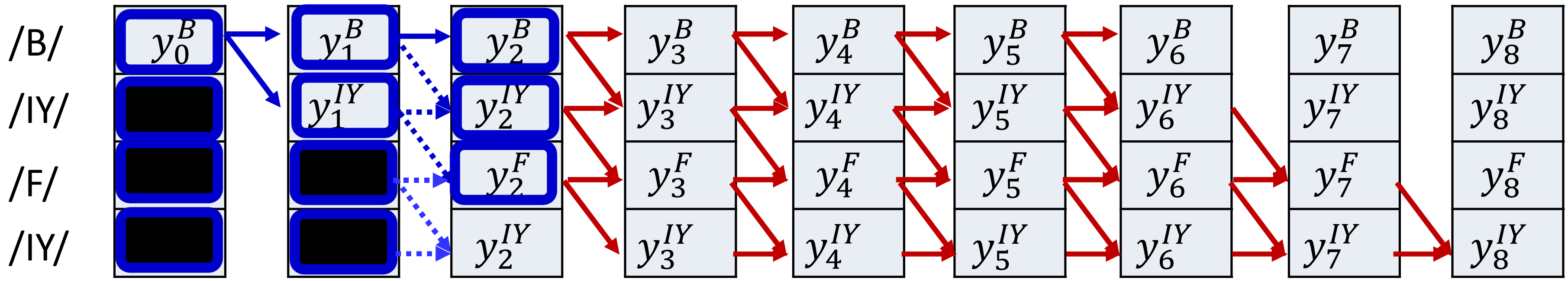
$$BP(t,0) = 0; Bscr(t,0) = Bscr(t-1,0) \times y_t^{S(0)}$$

for $l = 1 \dots K-1$

- $BP(t,l) = \begin{pmatrix} l-1 : \; if \; (Bscr(t-1,l-1) > Bscr(t-1,l)) \; l-1; \\ l : else \end{pmatrix}$
- $Bscr(t,l) = Bscr(BP(t,l)) \times y_t^{S(l)}$

Viterbi algorithm



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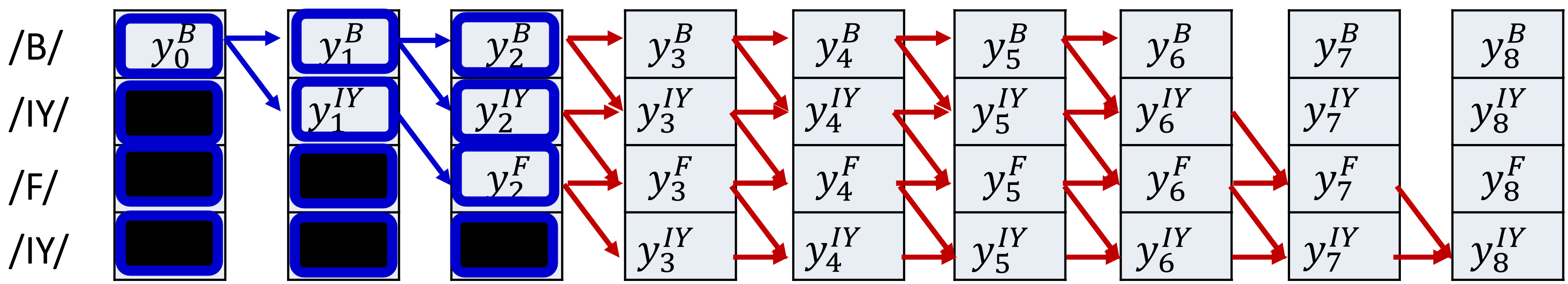
- for $t = 1 \dots T-1$

$$BP(t,0) = 0; \;\; Bscr(t,0) = Bscr(t-1,0) \times y_t^{S(0)}$$

for $l = 1 \dots K-1$

- $BP(t,l) = \left(if \; \left(Bscr(t-1,l-1) > Bscr(t-1,l)\right) \; l-1; \;\; else \; l\right)$
- $Bscr(t,l) = Bscr(BP(t,l)) \times y_t^{S(l)}$

Viterbi algorithm



- Initialization:

$$BP(0,i) = null, \; i = 0 \ldots K-1$$

$$Bscr(0,0) = y_0^{S(0)}, \; Bscr(0,i) = 0 \; for \; i = 1 \ldots K-1$$

- for $t = 1 \ldots T-1$

$$BP(t,0) = 0; \; Bscr(t,0) = Bscr(t-1,0) \times y_t^{S(0)}$$

for $l = 1 \ldots K-1$

- $BP(t,l) = \left(if \left(Bscr(t-1,l-1) > Bscr(t-1,l)\right) \; l-1; \; else \; l\right)$
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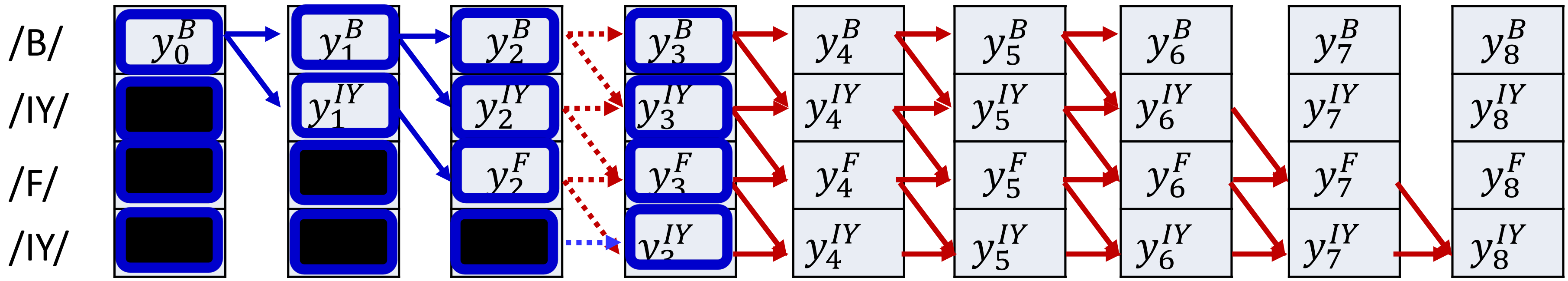
Viterbi algorithm



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- $BP(t,l) = \begin{pmatrix} l-1: \ \ if \ \left(Bscr(t-1,l-1) > \ Bscr(t-1,l)\right) \ l-1; \\ l: else \end{pmatrix}$
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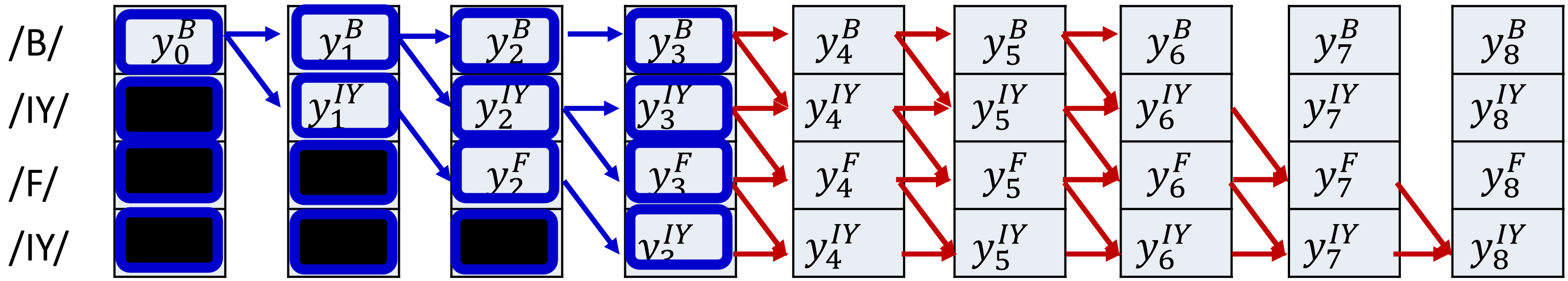
Viterbi algorithm



- Initialization:

$$BP(0, i) = null, \;\; i = 0 \ldots K-1$$

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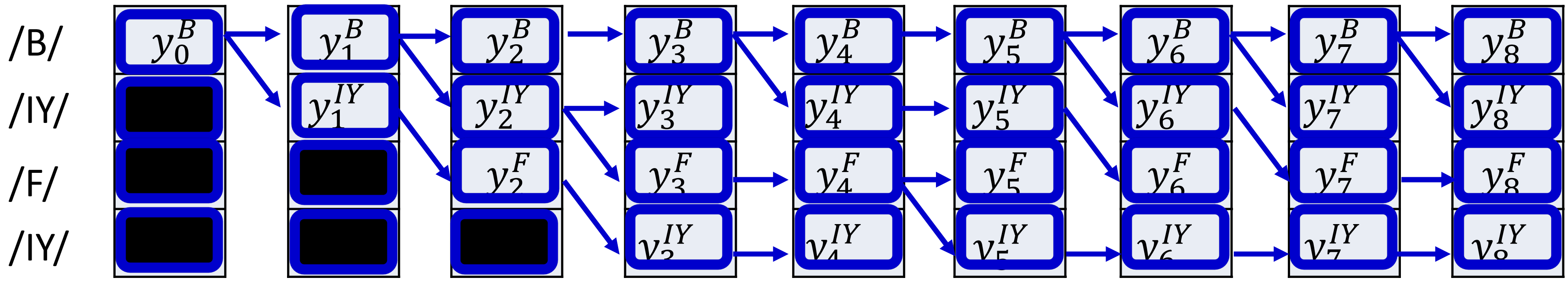
- for $t = 1 \ldots T-1$

$$BP(t, 0) = 0; \;\; Bscr(t, 0) = Bscr(t-1, 0) \times y_t^{S(0)}$$

for $l = 1 \ldots K-1$

  - $BP(t, l) = \begin{pmatrix} l-1 : \;\; if \; \left(Bscr(t-1, l-1) > Bscr(t-1, l)\right) \; l-1; \\ l : else \end{pmatrix}$
  - $Bscr(t, l) = Bscr(BP(t, l)) \times y_t^{S(l)}$

Viterbi algorithm



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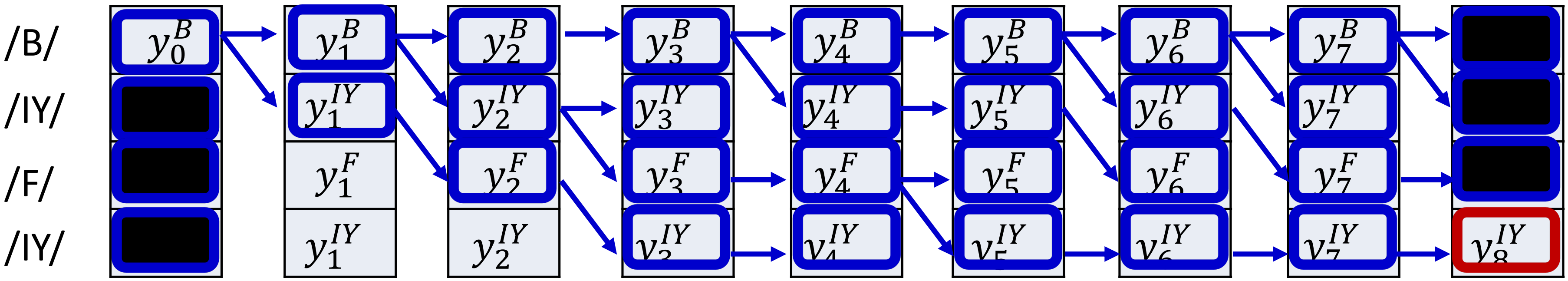
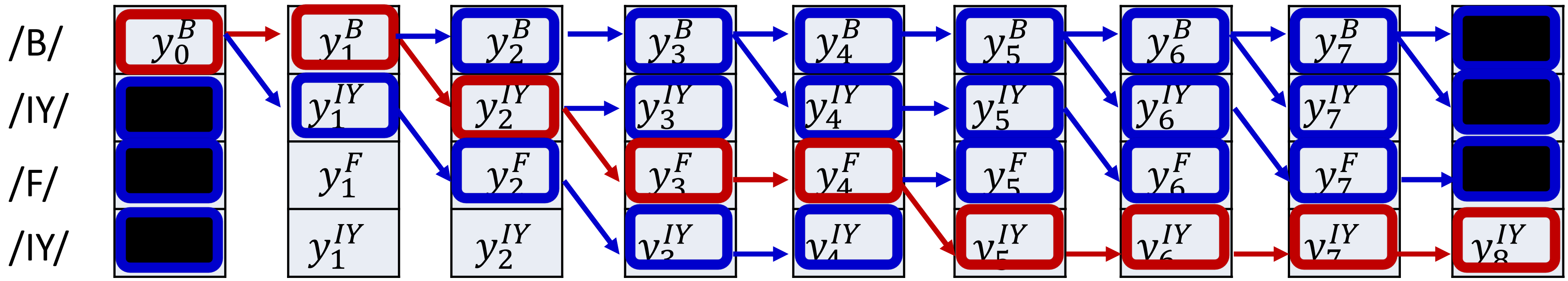
- $Bscr(t,l) \ = \ Bscr(BP(t,l)) \ \times y_t^{S(l)}$

Viterbi algorithm



- $s(T-1) = S(K-1)$

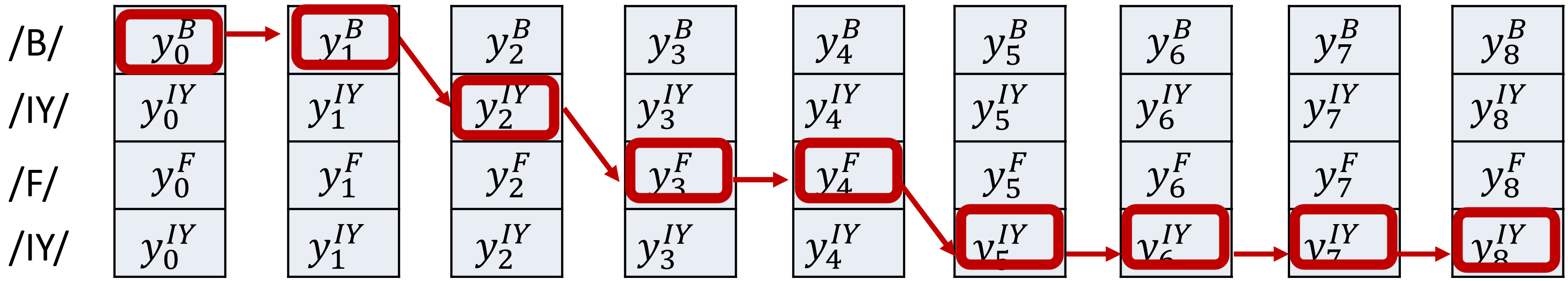
Viterbi algorithm



- $s(T-1) = S(K-1)$
- for $t = T-1\ downto\ 1$

  $s(t-1) = BP(s(t))$

Viterbi algorithm



- $s(T-1) = S(K-1)$
- for $t = T - 1\ downto\ 1$

  $s(t-1) = BP(s(t))$

/B/ /B/ /IY/ /F/ /F/ /IY/ /IY/ /IY/ /IY/

VITERBI

```
#N is the number of symbols in the target output
#S(i) is the ith symbol in target output
#T = length of input

#First create output table
For i = 1:N
    s(1:T,i) = y(1:T, S(i))

#Now run the Viterbi algorithm
# First, at t = 1
BP(1,1) = -1
Bscr(1,1) = s(1,1)
Bscr(1,2:N) = 0
for t = 2:T
    BP(t,1) = 1;
    Bscr(t,1) = Bscr(t-1,1)*s(t,1)
    for i = 1:min(t,N)
          BP(t,i) = Bscr(t-1,i) > Bscr(t-1,i-1) ? i : i-1
          Bscr(t,i) = Bscr(t-1,BP(t,i))*s(t,i)

# Backtrace
AlignedSymbol(T) = N
for t = T downto 2
    AlignedSymbol(t-1) = BP(t,AlignedSymbol(t))
```

Using 1..N and 1..T indexing, instead of 0..N-1, 0..T-1, for convenience of notation

VITERBI

```
#N is the number of symbols in the target output
#S(i) is the ith symbol in target output
#T = length of input

#First create output table
For i = 1:N
    s(1:T,i) = y(1:T, S(i))

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          BP(t,i) = Bscr(t-1,i) > Bscr(t-1,i-1) ? i : i-1
          Bscr(t,i) = Bscr(t-1,BP(t,i))*s(t,i)

# Backtrace
AlignedSymbol(T) = N
for t = T downto 2
    AlignedSymbol(t-1) = BP(t,AlignedSymbol(t))
```

Do not need explicit construction of output table

Information about order already in symbol sequence S(i), so we can use y(t,S(i)) instead of composing s(t,i) = y(t,S(i)) and using s(t,i)

Using 1..N and 1..T indexing, instead of 0..N-1, 0..T-1, for convenience of notation

VITERBI

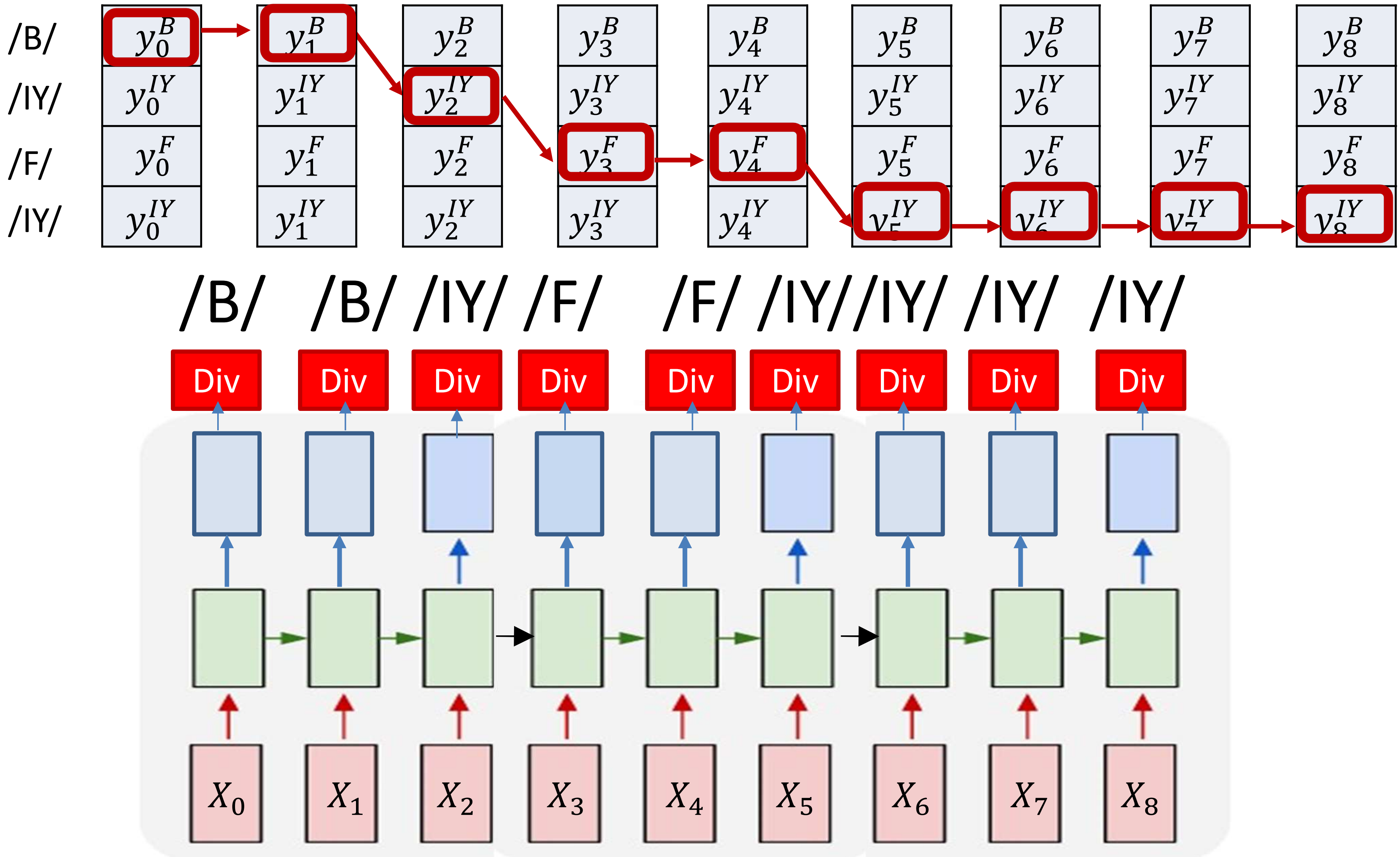
```
#N is the number of symbols in the target output
#S(i) is the ith symbol in target output
#T = length of input
```

Without explicit construction of output table

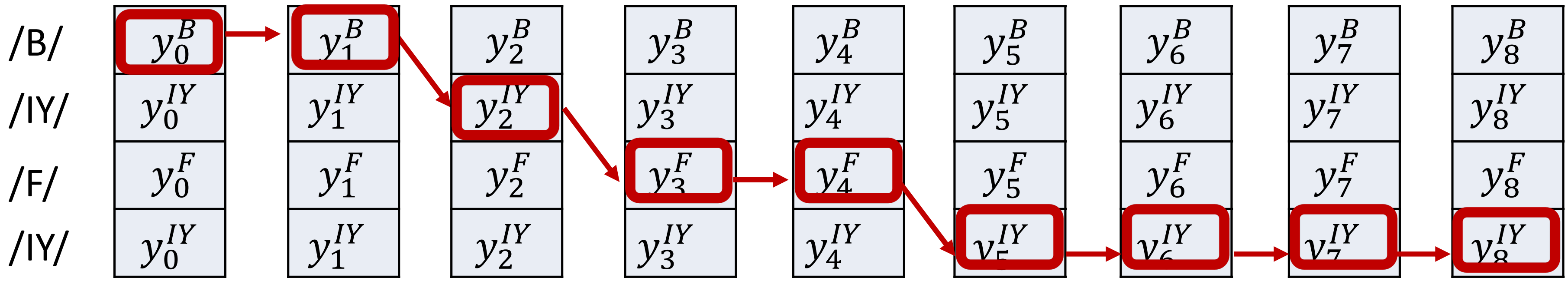
```
# First, at t = 1
BP(1,1) = -1
Bscr(1,1) = y(1,S(1))
Bscr(1,2:N) = 0
for t = 2:T
    BP(t,1) = 1;
    Bscr(t,1) = Bscr(t-1,1)*y(t,S(1))
    for i = 2:min(t,N)
        BP(t,i) = Bscr(t-1,i) > Bscr(t-1,i-1) ? i : i-1
        Bscr(t,i) = Bscr(t-1,BP(t,i))*y(t,S(i))

# Backtrace
AlignedSymbol(T) = N
for t = T downto 2
    AlignedSymbol(t-1) = BP(t,AlignedSymbol(t))
```

Using 1..N and 1..T indexing, instead of 0..N-1, 0..T-1, for convenience of notation

Assumed targets for training with the Viterbi algorithm



Gradients from the alignment



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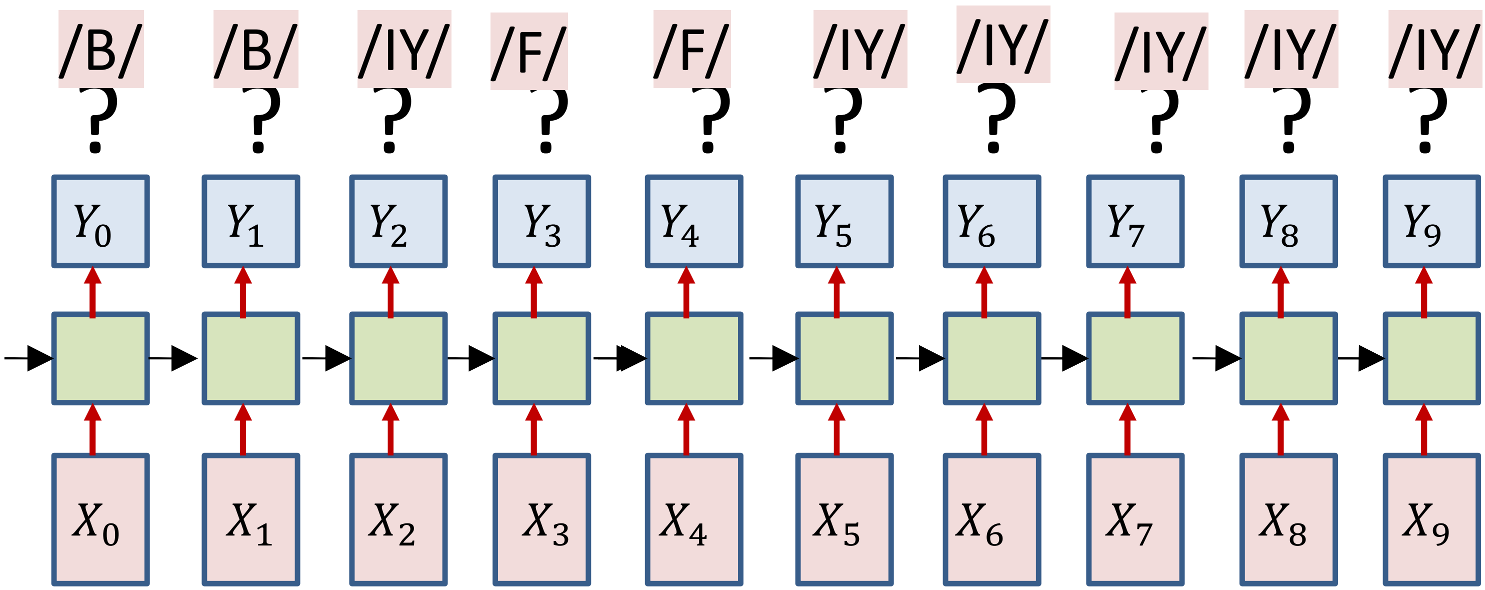
$$DIV = \sum_t KL\left(Y_t, symbol_t^{bestpath}\right) = -\sum_t \log Y\left(t, symbol_t^{bestpath}\right)$$

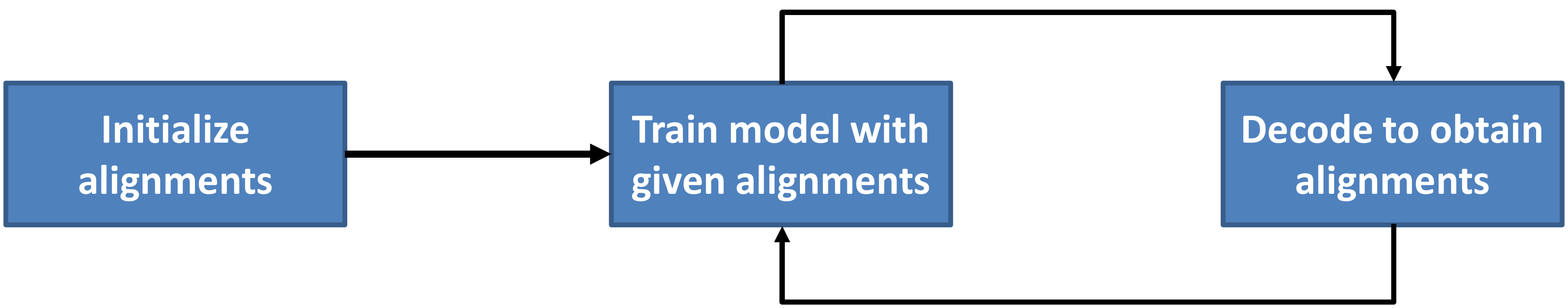
- The gradient w.r.t the t-th output vector Y_t

$$\nabla_{Y_t} DIV = \begin{bmatrix} 0 & 0 & \cdots & \frac{-1}{Y\left(t, symbol_t^{bestpath}\right)} & 0 & \cdots & 0 \end{bmatrix}$$

  – Zeros except at the component corresponding to the target *in the estimated alignment*

Iterative Estimate and Training





The "decode" and "train" steps may be combined into a single "decode, find alignment compute derivatives" step for SGD and mini-batch updates

Iterative update

- Option 1:
  - Determine alignments for every training instance
  - Train model (using SGD or your favorite approach) on the entire training set
  - Iterate

- Option 2:
  - During SGD, for each training instance, find the alignment during the forward pass
  - Use in backward pass

Iterative update: Problem

- Approach heavily dependent on initial alignment

- Prone to poor local optima

- Alternate solution: Do not commit to an alignment during any pass..

Next Class

- Training without explicit alignment..
  - Connectionist Temporal Classification
  - Separating repeated symbols
- The CTC decoder..