20FALL Introduction to Deep Learning Recitation 10

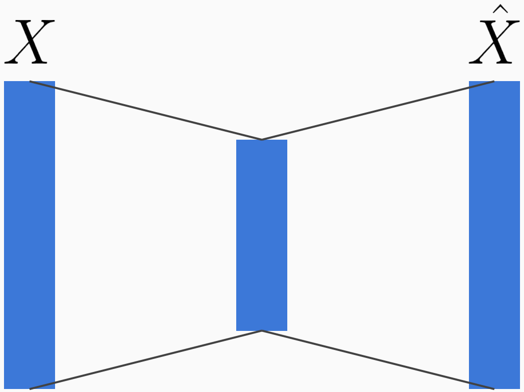
Variational Autoencoders

By Akshat Gupta, Jiachen Lian 11/13/2020

Auto-Encoder

- Output is the input itself.

$$min\ E(||\hat{X} - X||^2)$$



Auto-Encoder

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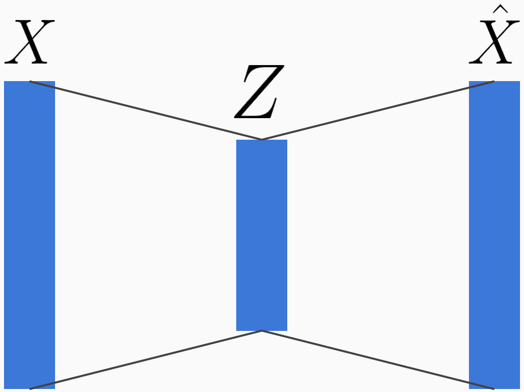
- Compressed Latent Representation.



Auto-Encoder

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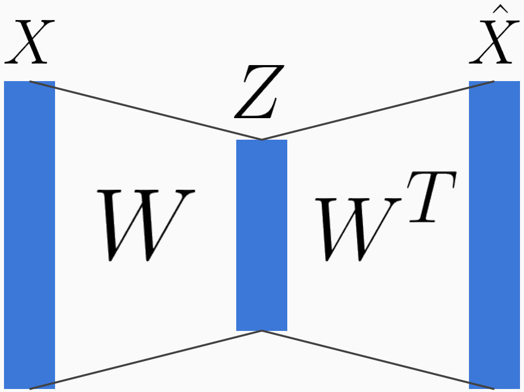
- Compressed Latent Representation.
- Linear single-layer AE performs PCA

$$\hat{X} = WW^TX$$



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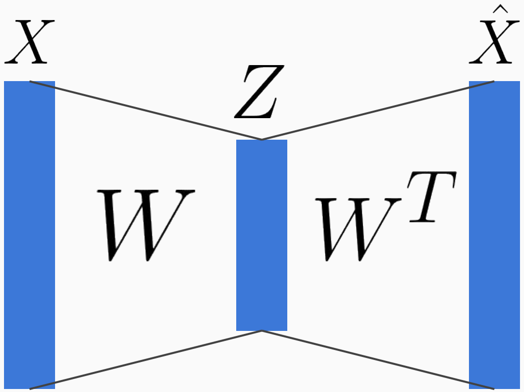
  - It is not PCA.

$$WW^T \neq I$$



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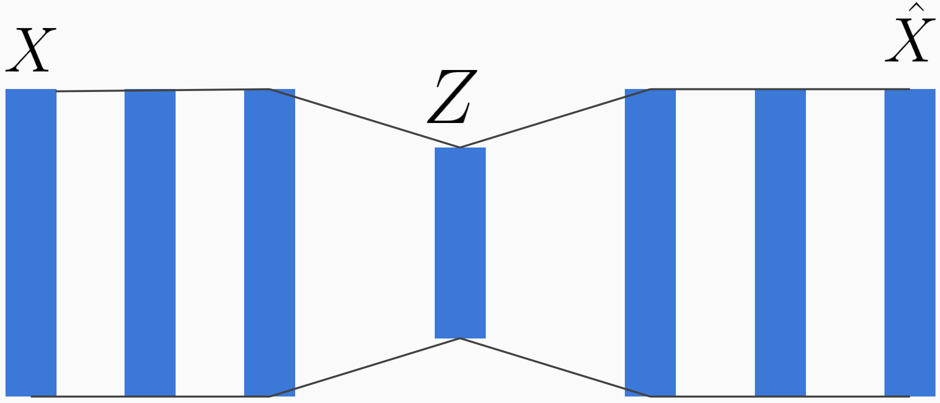
- Deep non-linear AE generates powerful representation



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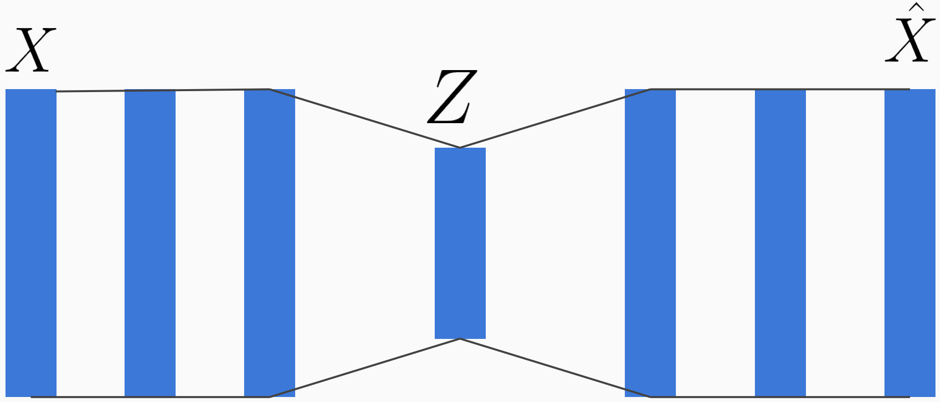

Auto-Encoder

- Why AE?

  Compressed Latent Representation.

  Promising Applications. E.g. Image denoising(Super Resolution), Neural Machine Translation

- Why not AE?

  Zero Reconstruction loss on a limited dataset

  It is difficult to generate a new datapoint without using the data itself

Why Variational Auto-Encoder?

Generate an image without the image itself

$$P(X) \rightarrow X$$

Why Variational Auto-Encoder?

Generate an image without the image itself

$$P(X) \rightarrow X$$

X_1

X_2

X_3

X_4

Why Variational Auto-Encoder?

Generate an image without the image itself

$$P(X) \to X$$

X_1

X_2

X_3

X_4

$$X_i = [X_{i1}, X_{i2}, ..., X_{ik}]^T$$

$$X \sim N(\mu, \Sigma)$$

$$\mu = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

$$\Sigma = \Sigma_{i=1}^{4} \frac{(X_i - \mu)(X_i - \mu)^T}{4}$$

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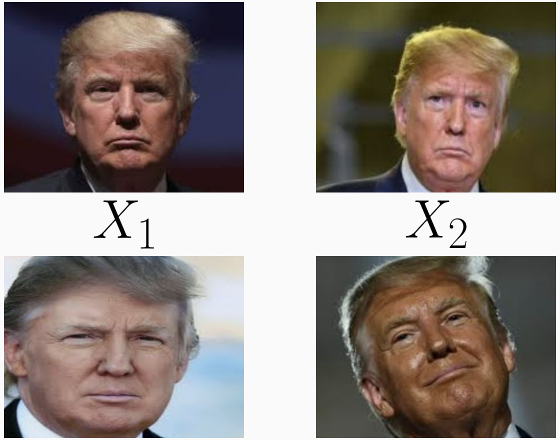

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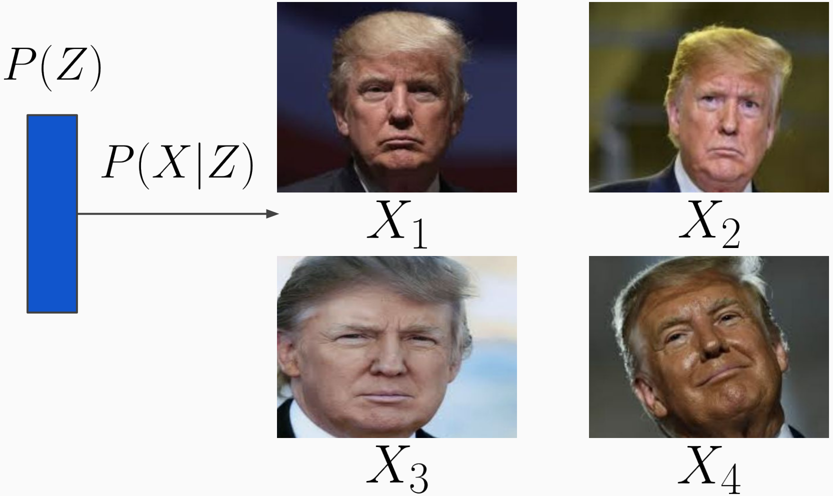
This is incorrect

Why Variational Auto-Encoder?

Generate an image without the image itself

$$\cancel{P(X) \to X}$$

$$P(X) = \int P(Z)P(X|Z)\,dZ$$



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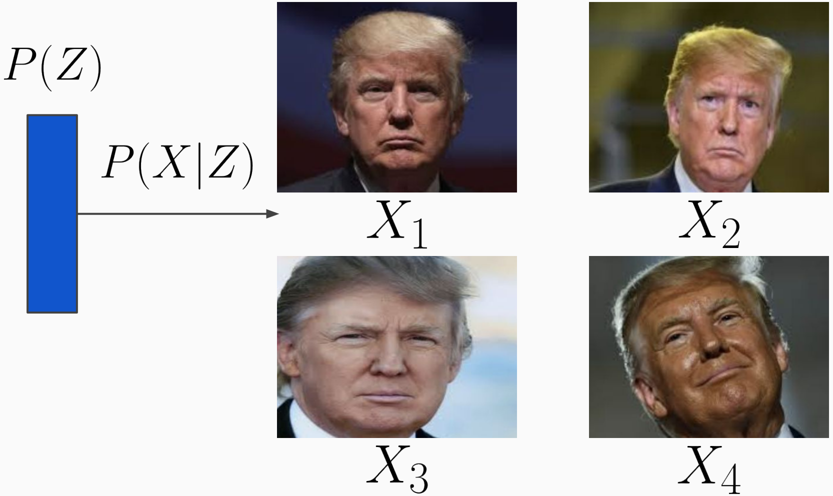
↓

Variational Inference



Why Variational Auto-Encoder?

Generate an image without the image itself

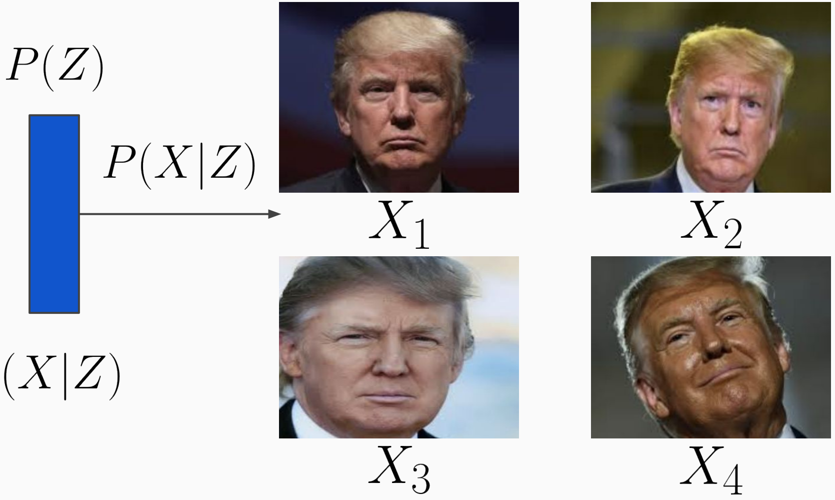
$$\cancel{P(X) \rightarrow X}$$

$$\cancel{P(X) = \int P(Z)P(X|Z)\, dZ}$$

$\downarrow$

Variational Inference

$$P(X, Z) = P(X)P(Z|X) = P(Z)P(X|Z)$$



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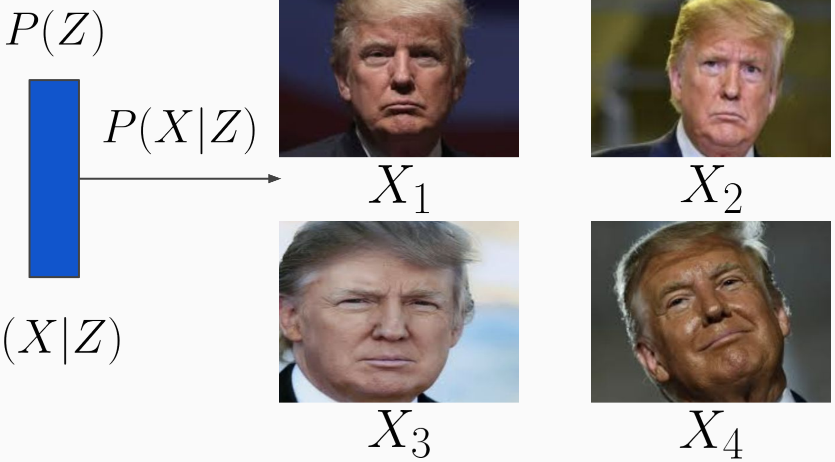
$$\downarrow$$

Variational Inference

$$P(X,Z) = P(X)\boxed{P(Z|X)} = \boxed{P(Z)}P(X|Z)$$



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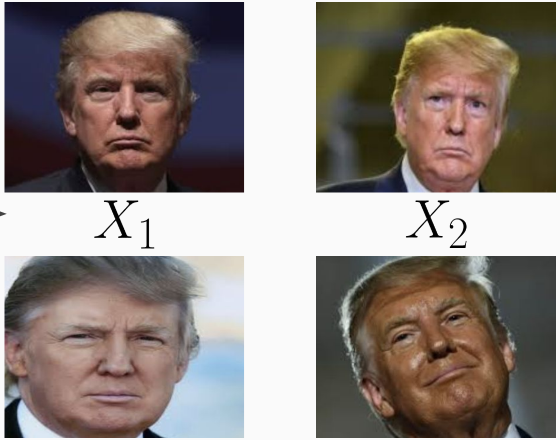
$P(Z)$

$P(X|Z)$



X_1 X_2

X_3 X_4

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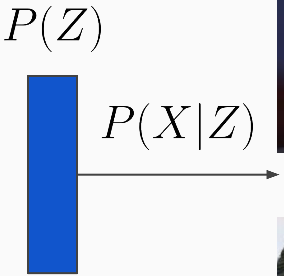

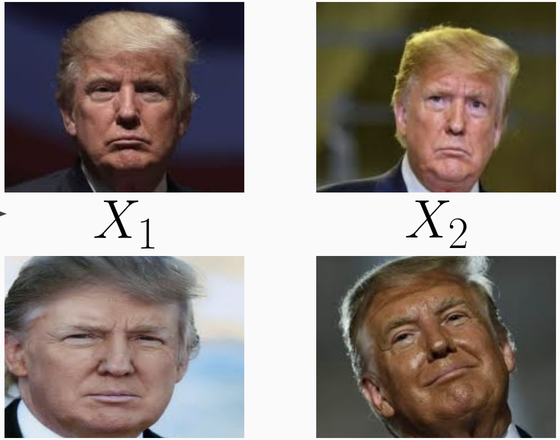

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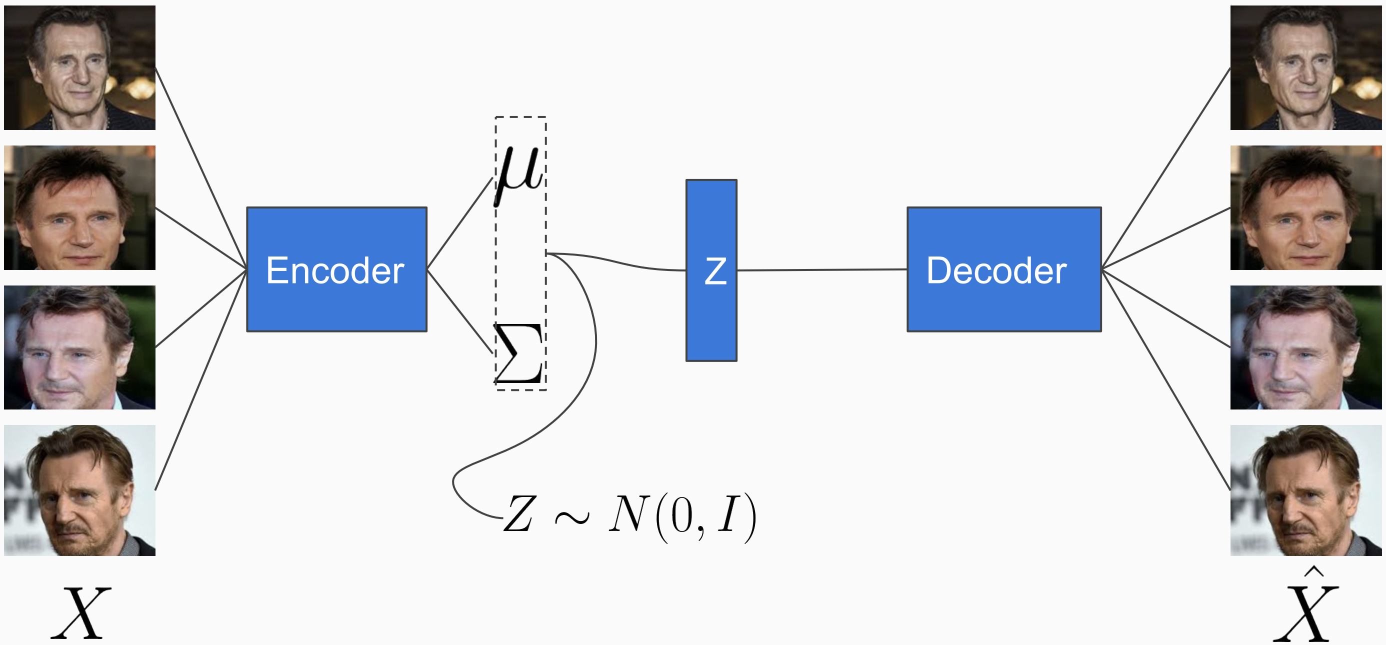
Variational Auto-Encoder



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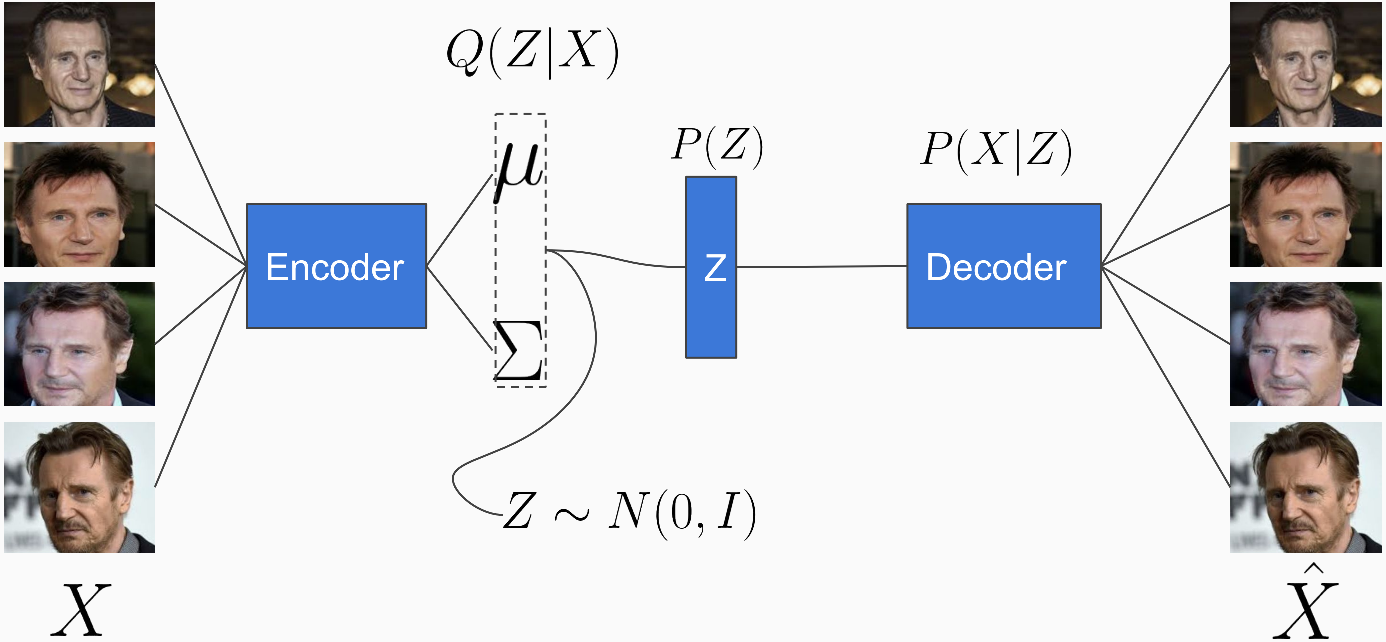

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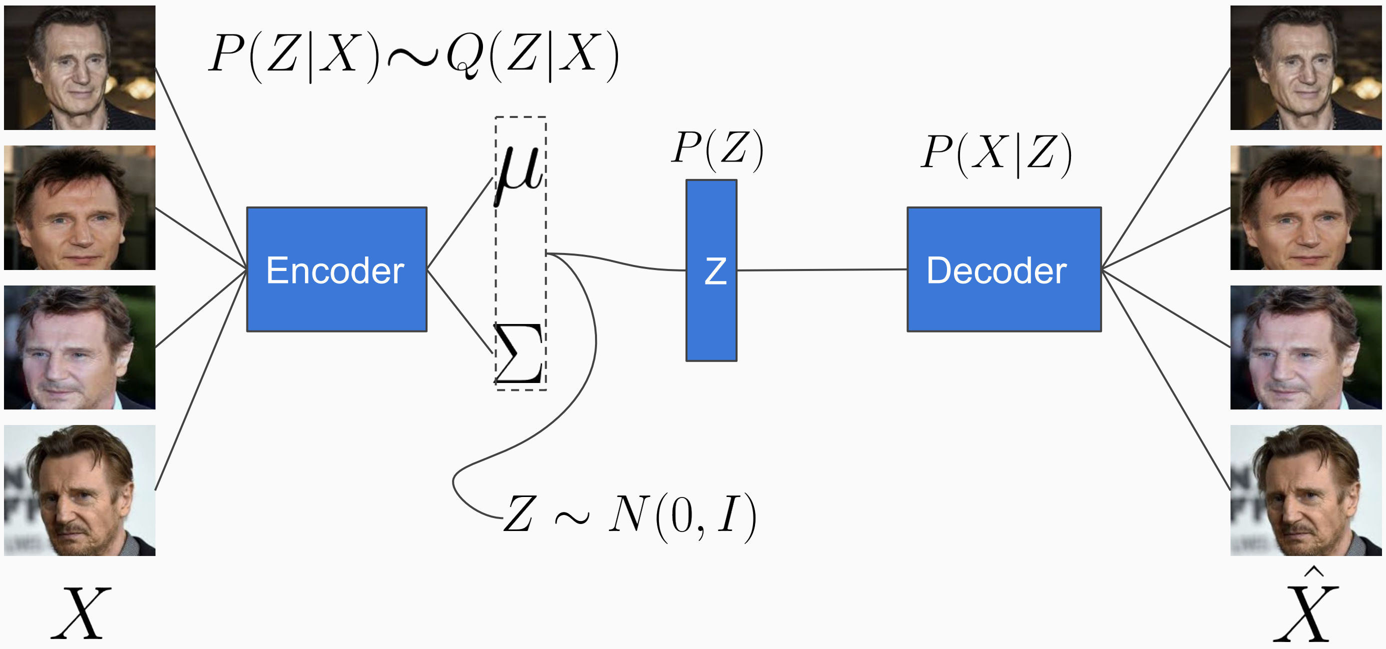

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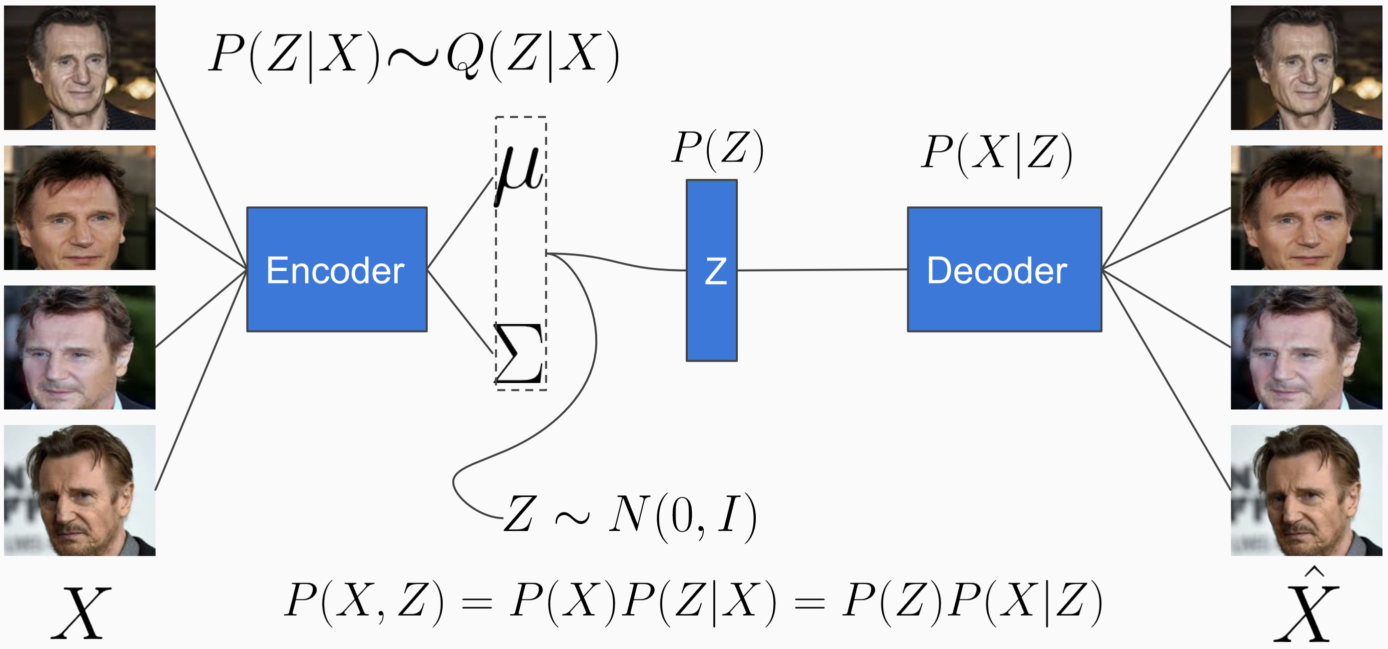

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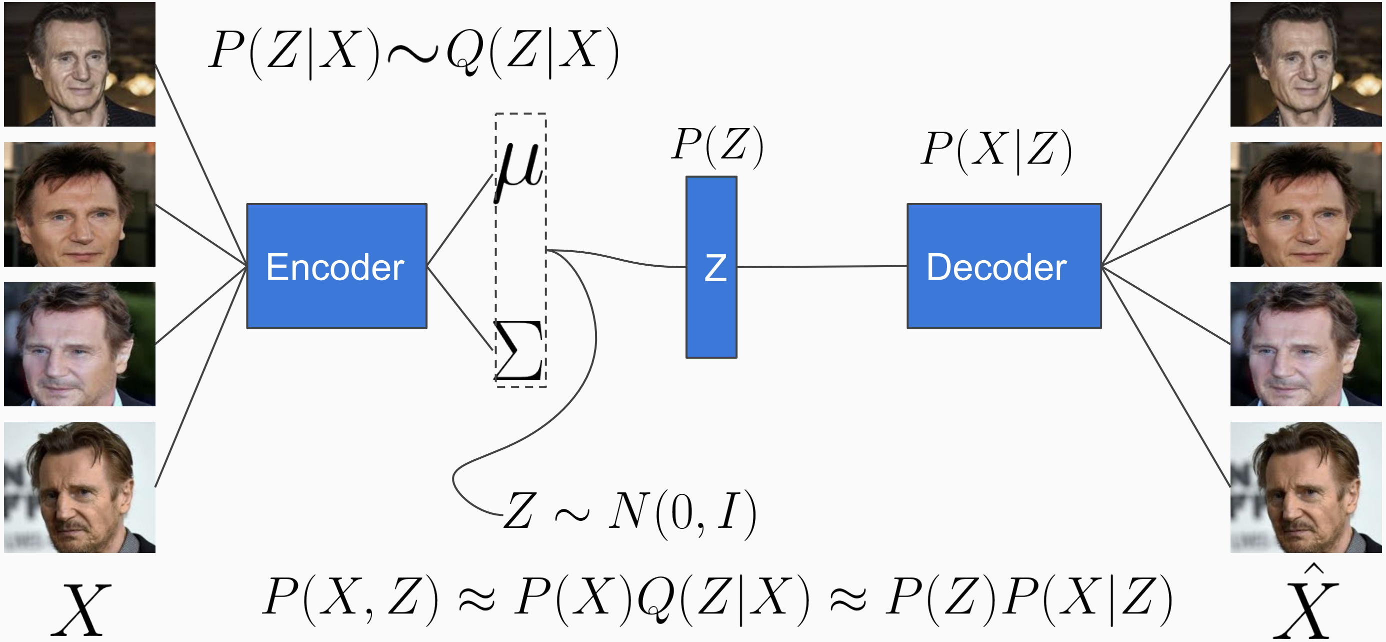

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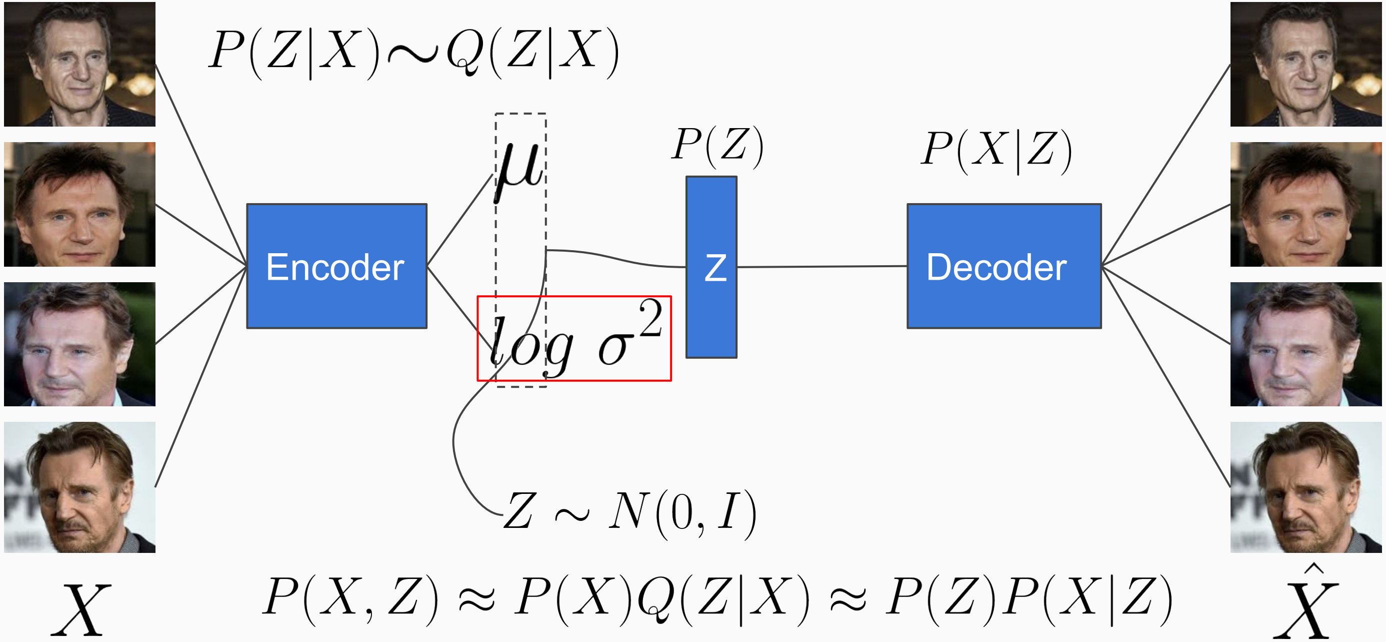

Variational Auto-Encoder

Statistical Distance between Distributions $P(X)$ and $Q(X)$

Variational Auto-Encoder

Statistical Distance between Distributions P(X) and Q(X)

KL-Divergence

Variational Auto-Encoder

Statistical Distance between Distributions P(X) and Q(X)

KL-Divergence

Information: $I(p(X)) = -log\ p(X)$

Variational Auto-Encoder

Statistical Distance between Distributions P(X) and Q(X)

KL-Divergence

Information: $I(p(X)) = -log\ p(X)$

Entropy: $H = \Sigma_i p(X_i) I(p(X_i)) = \Sigma_i -p(X_i) log\, p(X_i)$

Variational Auto-Encoder

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$$KL(P||Q) = -\Sigma_i P(X_i) log \frac{Q(X_i)}{P(X_i)} \qquad - \int p\, log \frac{q}{p}\, dx$$

Variational Auto-Encoder

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$$KL(Q||P) = -\Sigma_i Q(X_i) log \frac{P(X_i)}{Q(X_i)} \qquad - \int q\ log \frac{p}{q}\ dx$$

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$$\boxed{KL(Q||P) \neq KL(P||Q)}$$

You can pick either!

Variational Auto-Encoder

Statistical Distance between Distributions P(X) and Q(X)

KL-Divergence

$$lemma\ 1 : KL(P||Q) \geq 0$$

Variational Auto-Encoder

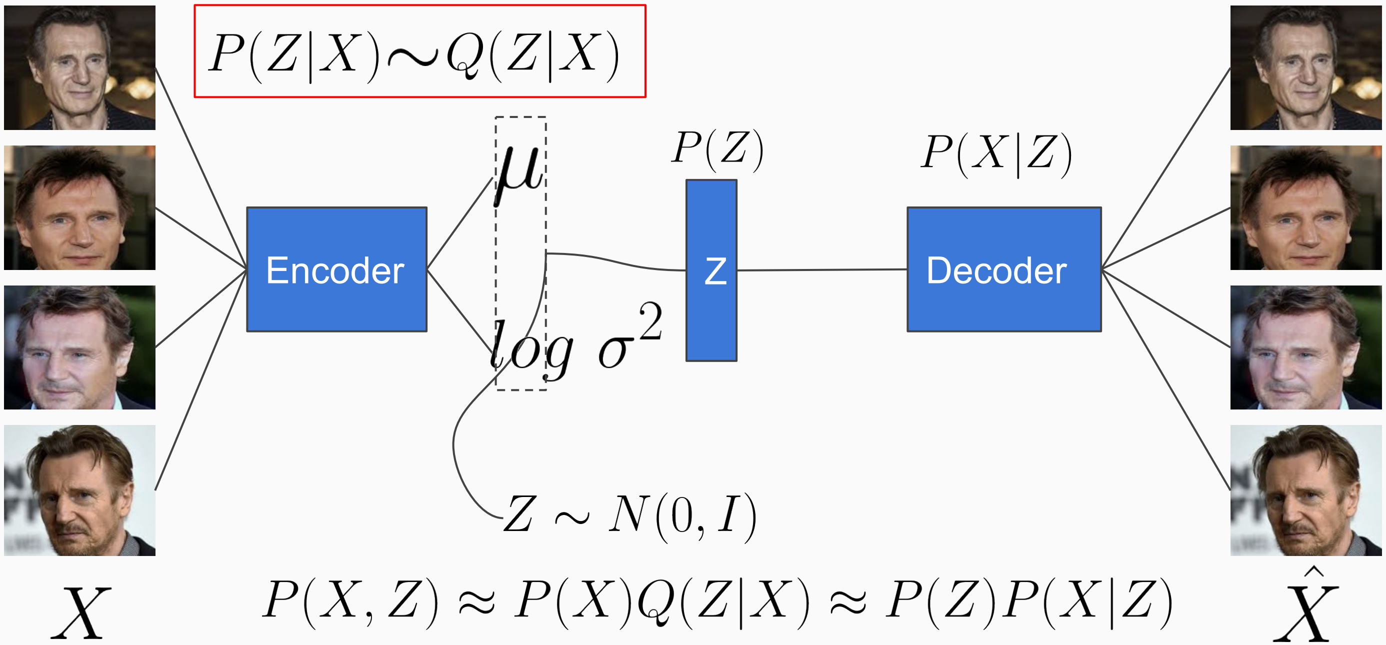
Statistical Distance between Distributions P(X) and Q(X)

KL-Divergence

$$lemma\ 1 : KL(P||Q) \geq 0$$

Proof: $$KL(P||Q) = -\Sigma P log\frac{Q}{P} \geq log(\Sigma P\frac{Q}{P}) = 0$$

Variational Auto-Encoder



Variational Auto-Encoder

$$\boxed{P(Z|X) \sim Q(Z|X)}$$

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Variational Auto-Encoder

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Variational Auto-Encoder

$$logP(X) = KL(Q(Z|X), P(Z|X)) + \Sigma Q(Z|X) log\frac{P(X,Z)}{Q(Z|X)}$$

Constant

Unknown

Evidence Lower Bound(ELBO)

Variational Auto-Encoder

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Unknown

Evidence Lower Bound(ELBO)

$$Max\ L = \Sigma Q(Z|X) log\frac{P(X,Z)}{Q(Z|X)}$$

Variational Auto-Encoder

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$$= E_{Q(Z|X)}logP(X|Z) - KL(Q(Z|X)||P(Z))$$

Variational Auto-Encoder

$$L = E_{Q(Z|X)} log P(X|Z) - KL(Q(Z|X)||P(Z))$$

Reconstruction Loss

Tractable

$$Loss = -E_{Q(Z|X)} log P(X|Z) + KL(Q(Z|X)||P(Z))$$

Variational Auto-Encoder

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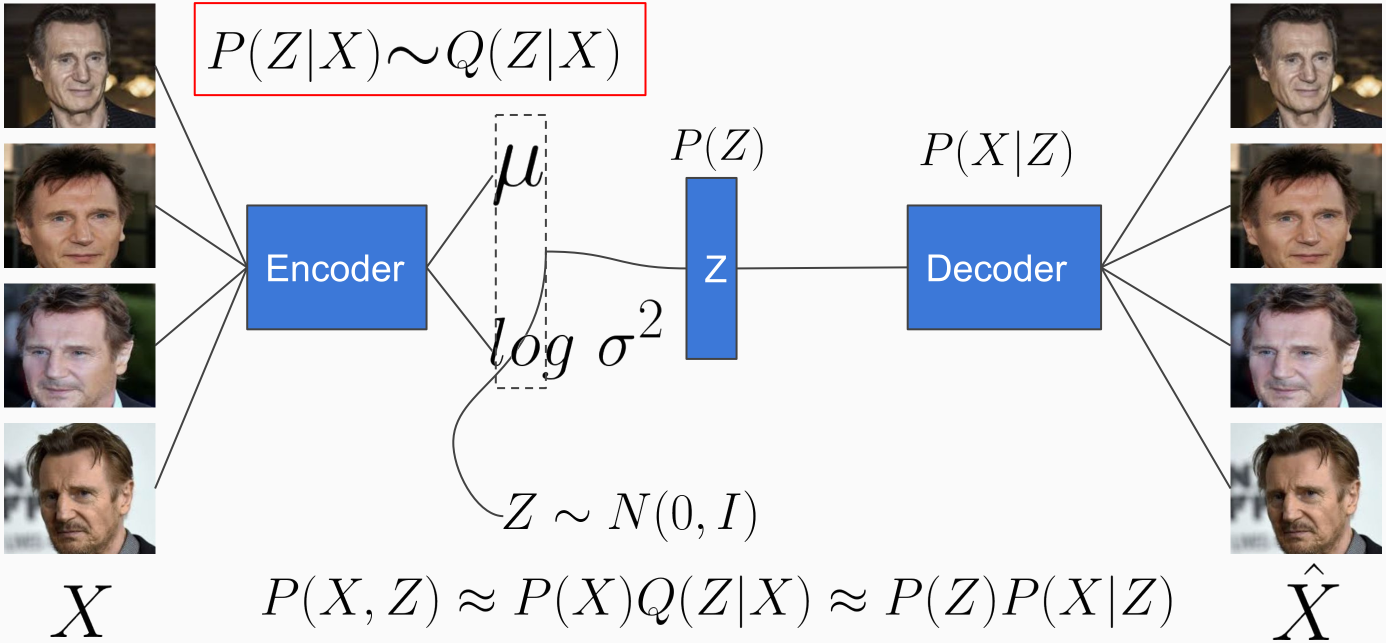
$$\boxed{Loss = -E_{Q(Z|X)} logP(X|Z) + KL(Q(Z|X)||P(Z))}$$

Variational Auto-Encoder



Variational Auto-Encoder

- Discussion 1

$$Q(Z|X) \sim P(Z) \sim P(Z|X)$$

Variational Auto-Encoder

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$$Q(Z|X) \sim P(Z) \sim P(Z|X)$$

$$KL(N(\mu, \sigma^2)||N(0,1)) = \frac{1}{2}(-log\sigma^2 + \mu^2 + \sigma^2 - 1)$$

$$\boxed{KL(p||q) = -\int p\, log\frac{q}{p}\, dx}$$

Variational Auto-Encoder

- Discussion 1

$$Q(Z|X) \sim P(Z) \sim P(Z|X)$$

$$KL(N(\mu, \sigma^2)||N(0,1)) = \frac{1}{2}(-log\sigma^2 + \mu^2 + \sigma^2 - 1)$$

$$\boxed{KL(p||q) = -\int p\, log\frac{q}{p}\, dx}$$

$$Loss = E(||X - \hat{X}||^2) + \frac{1}{2}(-log\sigma^2 + \mu^2 + \sigma^2 - 1)$$

Variational Auto-Encoder

- Discussion 2

$$Q(Z|X) \sim P(Z) \sim P(Z|X)$$

$$Loss = -E_{Q(Z|X)} log P(X|Z) + KL(Q(Z|X)||P(Z))$$

Variational Auto-Encoder

- Discussion 2

$$Q(Z|X) \sim P(Z) \sim P(Z|X)$$

$$Loss = -E_{Q(Z|X)} logP(X|Z) + KL(Q(Z|X)||P(Z))$$

$$P(Z) = P(Z|X) \to P(Z, X) = P(Z)P(X)$$

Variational Auto-Encoder

- Discussion 2

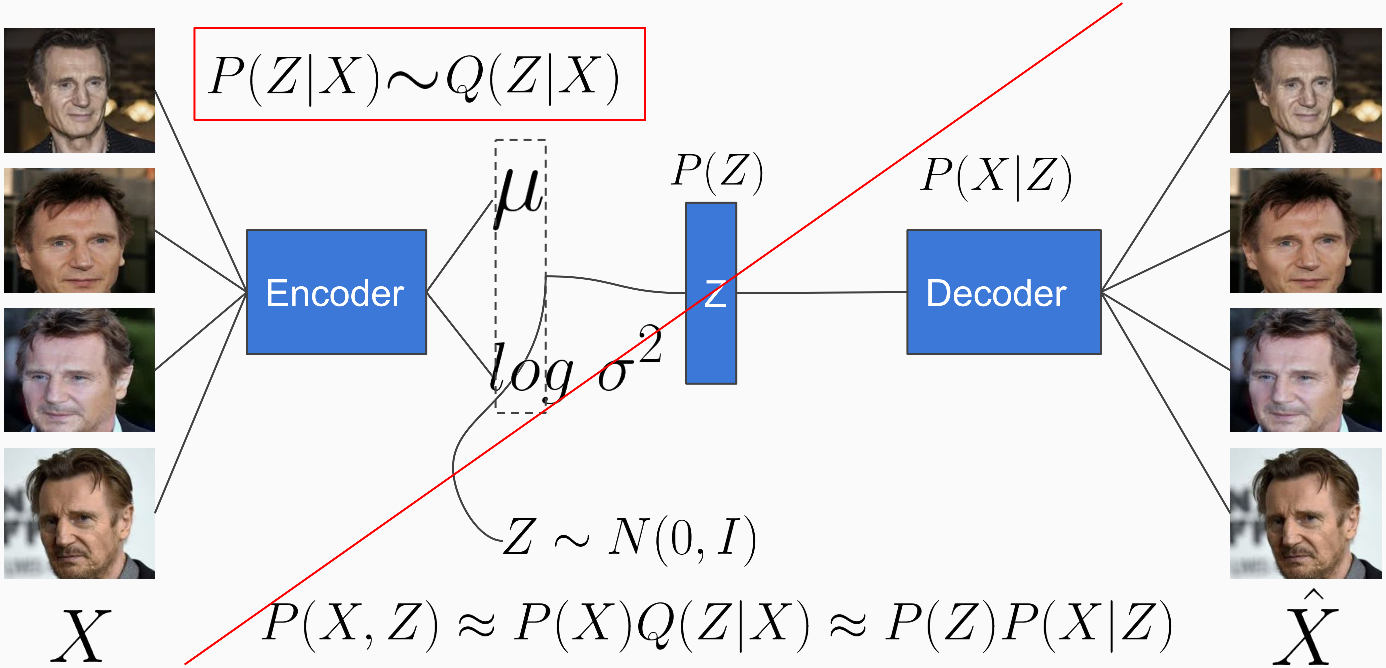
$$Q(Z|X) \sim P(Z) \sim P(Z|X)$$

$$Loss = -E_{Q(Z|X)} logP(X|Z) + KL(Q(Z|X)||P(Z))$$

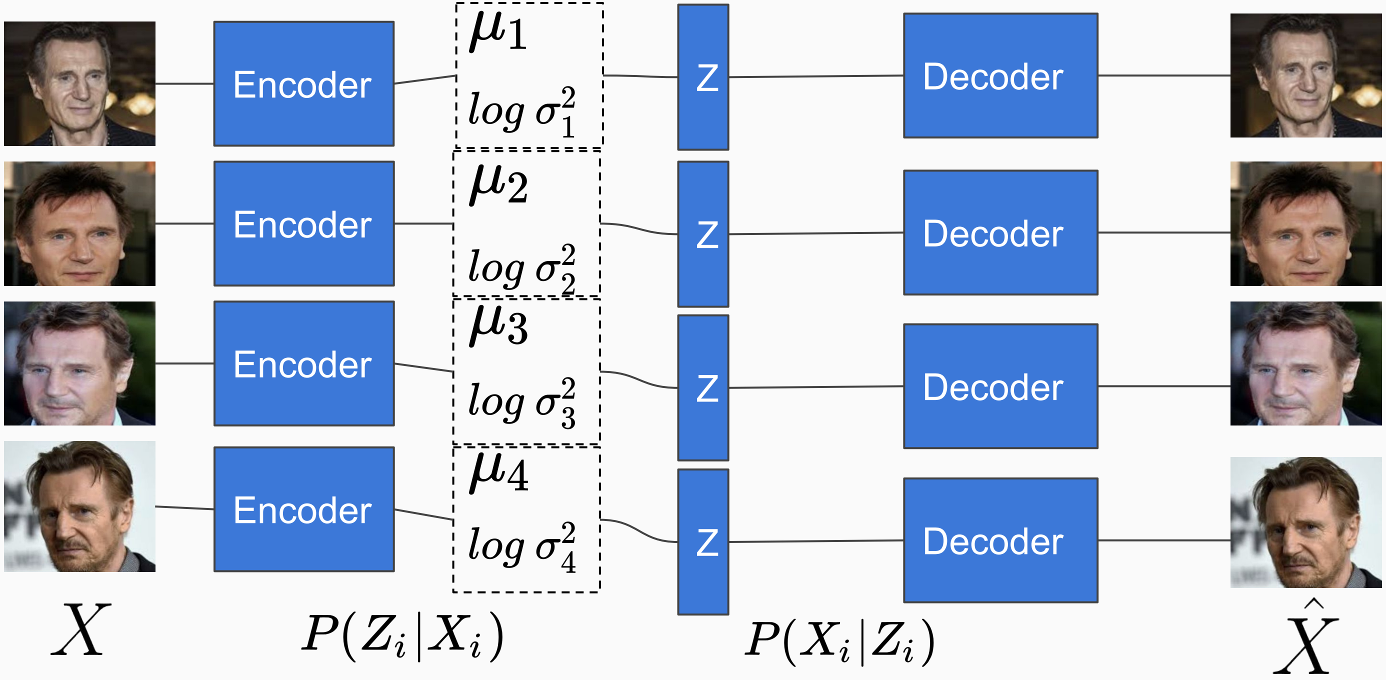
$$P(Z) = P(Z|X) \rightarrow P(Z,X) = P(Z)P(X)$$

Undesired!

Variational Auto-Encoder



Variational Auto-Encoder



Variational Auto-Encoder

- Discussion 3

  Reparameterization

$$Z = \mu + \epsilon\sigma \;\sim N(\mu, \sigma^2)$$

$$\epsilon \sim N(0, 1)$$

Variational Auto-Encoder