Kinematic Constraint Equations

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Chapter 7
Kinematic Constraint Equations

Abstract This chapter presents a general methodology for the formulation of the kinematic constraint equations at position, velocity and acceleration levels. Also a brief characterization of the different type of constraints is offered, namely the holonomic and nonholonomic constraints. The kinematic constraints described here are formulated using generalized coordinates. The chapter ends with a general approach to deal with the kinematic analysis of multibody systems.

Keywords Kinematic constraints · Positions · Velocities · Accelerations

A constraint condition implies a restriction in the kinematical degrees of freedom of one or more bodies. The classical constraint is usually an algebraic equation that defines the relative translation or rotation between two bodies. There are furthermore possibilities to constrain the relative velocity between two bodies or a body and the ground. This is for example the case of a rolling disc, where the point of the disc that contacts the ground has always zero relative velocity with respect to the ground. In the case that the velocity constraint condition cannot be integrated in time in order to form a position constraint, it is called nonholonomic. This is the case for the general rolling constraint. In addition to that there are non-classical constraints that might even introduce a new unknown coordinate, such as a sliding joint, where a point of a body is allowed to move along the surface of another body. In the case of contact, the constraint condition is based on inequalities and, therefore, such a constraint does not permanently restrict the degrees of freedom of bodies (Huston 1990).

As it was presented previously, the configuration of a multibody system is described by a set of variables called generalized coordinates that completely define the location and orientation of each body in the system. Hereafter, the set of generalized coordinates of a multibody system will be denoted by vector $\mathbf{q} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots, \mathbf{q}_n\}^T$, where n is the number of coordinates. Mostly, in multibody systems formulation the generalized coordinates can be divided into independent and dependent variables, consequently, several algebraic equations are needed to be introduced to relate them. In other words, the constraint equations represent the kinematic relation between independent and dependent coordinates. In a simple

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manner, the constraint equations can arise from the description of the system topology and from the characterization of the driving and guiding constraints that are used to guide the system through the analysis. In this work, the set of constraint equations is denoted by symbol $\mathbf{\Phi}$. In order to distinguish among the different constraint equations, each elementary set of constraints is identified by a superscript containing two parameters. The first parameter denotes the type of constraint, while the second one defines the number of independent equations that it involves. For example, $\mathbf{\Phi}^{(s,3)}$ refers to a spherical (s) joint constraint, which contains three (3) equations.

Kinematic constraints can be classified as holonomic or nonholonomic. Holonomic constraints arise from geometric constraints and are integrable into a form involving only coordinates (*holo* comes from Greek that means whole, integer). Nonholonomic constraints are not integrable. The relation specified by a constraint can be an explicit function of time designated as rheonomic constraints (*rheo* comes from Greek that means hard, inflexible, independent) or not, being designated by scleronomic constraints (*scleros* comes from Greek that means flexible, changing). Figure 7.1 shows a typical spherical joint and a simple human body model placed on a spherical surface, which represents a holonomic and a nonholonomic constraint, respectively. Thus, for instance, in the motion of the human model on the spherical surface, the following mathematical relation has to be satisfied during the analysis (Flores 2006)

$$\mathbf{r}^T\mathbf{r} - R^2 \geq 0 \tag{7.1}$$

where R is the radius of the spherical surface and vector $\mathbf{r}$ represents the position of the model measured from the center of the spherical surface.

The kinematic constraints considered here are assumed to be holonomic, arising from geometrical constraints on the generalized coordinates. Holonomic constraints, also called geometric restrictions, are algebraic equations imposed to the system that are expressed as functions of the displacement and, possibly, time. If the time t does not appear explicitly in the constraint equation, then the system is said to be scleronomic. A simple example of scleronomic constraint equation is the revolute joint between two bodies. Otherwise, when the constraint is holonomic and t appears explicitly, the system is said to be rheonomic (Shabana 1989).

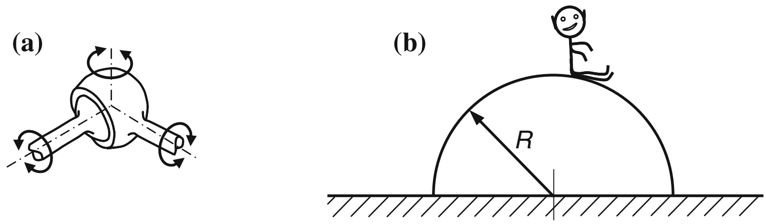

Fig. 7.1 **a** Holonomic constraint; **b** nonholonomic constraint

In general manner, the relations that describe the constraints imposed by kinematic pairs, such as mechanical joints, are formulated using algebraic equations. The kinematic constraint equations can be written as

$$\mathbf{\Phi} \equiv \mathbf{\Phi}(\mathbf{q}) = \mathbf{0} \tag{7.2}$$

where $\mathbf{q}$ denotes the vector of body-coordinates defined by Eq. (6.1) and $\mathbf{\Phi}$ represents a function describing the kinematic constraints. In general, constraint equations may also be functions of time, however, algebraic equality constraints as expressed by Eq. (7.2) are referred to as holonomic constraints.

The first time derivative of Eq. (7.2) yields the velocity constraints that provide relations between the velocity variables of a system. The velocity constraints can be expressed as

$$\dot{\mathbf{\Phi}} \equiv \mathbf{D}\mathbf{v} = \mathbf{0} \tag{7.3}$$

where $\mathbf{D}$ denotes the Jacobian matrix and $\mathbf{v}$ contains the velocity terms defined by Eq. (6.4). For driving elements, the corresponding velocity constraint equations can be written in the form

$$\dot{\mathbf{\Phi}} \equiv \mathbf{D}\mathbf{v} = \boldsymbol{\upsilon} \tag{7.4}$$

in which the right-hand side contains the partial derivates of $\mathbf{\Phi}$ with respect to time, $\partial\mathbf{\Phi}/\partial t$. The constraints at the velocity level are represented by linear algebraic equations.

The second time derivative of Eq. (7.2) results in

$$\ddot{\mathbf{\Phi}} \equiv \mathbf{D}\dot{\mathbf{v}} + \dot{\mathbf{D}}\mathbf{v} = \mathbf{0} \tag{7.5}$$

where $\dot{\mathbf{v}}$ denotes the acceleration terms defined by Eq. (6.6) and the term $-\dot{\mathbf{D}}\mathbf{v}$ is referred to as the right-hand side of the kinematic acceleration equations. By introducing $\boldsymbol{\gamma} = -\dot{\mathbf{D}}\mathbf{v}$, Eq. (7.4) can be rewritten as

$$\mathbf{D}\dot{\mathbf{v}} = \boldsymbol{\gamma} \tag{7.6}$$

In should be highlighted that the terms involved in Eqs. (7.2) through (7.6) appear in a general form, that is, they do not reflect the type of coordinates considered. In addition, the constraint equations represented by Eq. (7.2) are non–linear in terms of $\mathbf{q}$ and are, usually, solved by employing the Newton-Raphson method. Equations (7.3) and (7.6) are linear in terms of $\mathbf{v}$ and $\dot{\mathbf{v}}$, respectively, and can be solved by any usual method adopted for the solution of systems of linear equations. It should be noted that the issues related to the treatment of redundant constraints are not presented in this work. The interested reader in the details on this particular topic is referred to the work by Wehage and Haug (1982).

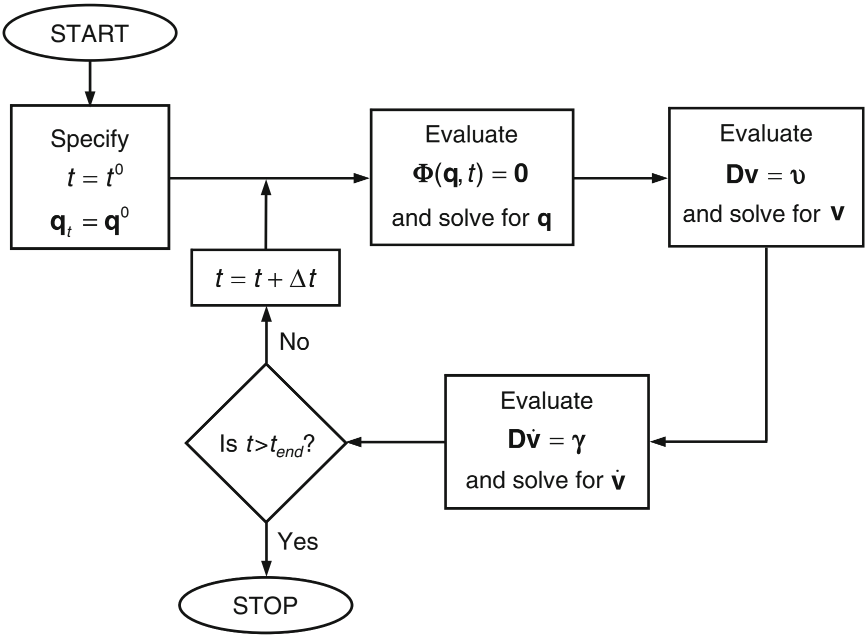

Fig. 7.2 Flowchart of computational procedure for kinematic analysis of a MBS

The kinematic analysis is the study of the motion of a multibody system, independently of the causes that produce it. Since in the kinematic analysis the forces are not considered, the motion of the system is specified by driving or guiding elements that govern the motion of specific degrees of freedom of the system during the analysis. The position, velocity and acceleration of the remaining elements of the system are defined by kinematic constraint equations that describe the system topology. It is clear that in the kinematic analysis, the number of driving and guiding constraints must be equal to the number of degrees of freedom of the multibody system. In short, the kinematic analysis is performed by solving a set of equations that result from the kinematic, driving and guiding constraints (Jalón and Bayo 1994).

The kinematic analysis of a multibody system can be carried out by solving the set of Eqs. (7.2)–(7.6) together with the necessary driver constraints corresponding to the free degrees of freedom. Therefore, the necessary steps to perform this type of analysis, sketched in Fig. 7.2, are summarized as:

1. Specify initial conditions for positions $\mathbf{q}^0$ and initialize the time t^0.
2. Evaluate the position constraint Eq. (7.2) and solve them for positions, $\mathbf{q}$.
3. Evaluate the velocity constraint Eq. (7.4) and solve them for velocities, $\mathbf{v}$.
4. Evaluate the acceleration constraint Eq. (7.6) and solve them for accelerations, $\dot{\mathbf{v}}$.
5. Increment the time. If the time is smaller than final time, go to step (2), otherwise stop the kinematic analysis.

A close observation of the Eqs. (7.4) and (7.6) shows that both expressions represent systems of linear equations, with the same leading matrix and different right-hand side vectors. Moreover, since both expressions share the same leading matrix, Jacobian matrix of the constraints, evaluated with the latest calculated configuration of the system, then this matrix only needs to be factorized once during each step (Nikravesh 1988).

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