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Seminar 4

Fie un V/K sp vect.

$S = \{v_1, v_2, \dots v_m\} \subset V$

a) S.m. sist liniar independent (SLI) dacă

$\forall\ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = 0_V \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_m = 0$

Orice combinație liniară nulă se realizează numai prin scalari nuli

b) S.m. sist liniar dependent (SLD) dacă:

$\exists\ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = 0_V$

Există și combinații liniare nule cu scalari nenuli

<u>Caz particular</u>: $V = K^n/K$

$S = \{v_1, v_2, \dots v_m\} \subset K^n$

$A = \begin{pmatrix} | & | & \dots & | \\ v_1 & v_2 & & v_m \end{pmatrix} \in \mathcal{M}_{(n,m)}(K)$

a) SLI dacă $rg\ A = m \leq n$

$\{rg\ A \to max, m \leq n\}$

① SLI sau SLD?

a) $\mathbb{R}^4/\mathbb{R}$

$S_1 = \{v_1 = (1,2,0,3);\ v_2 = (2,-1,3,0);\ v_3 = (-1,3,2,1)\} \subset \mathbb{R}^4$

Fie $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{\mathbb{R}^4}$ unde $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

$\alpha_1(1,2,0,3) + \alpha_2(2,-1,3,0) + \alpha_3(-1,3,2,1) = (0,0,0,0)$

$\begin{cases} \alpha_1 + 2\alpha_2 - \alpha_3 = 0 \\ 2\alpha_1 - \alpha_2 + 3\alpha_3 = 0 \\ 3\alpha_2 + 2\alpha_3 = 0 \\ 3\alpha_1 + \alpha_3 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 0 & 3 & 2 \\ 3 & 6 & 1 \end{pmatrix} \in \mathcal{M}_{4,3}(\mathbb{R})$ $\quad rg\, A = ?$

$\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \neq 0 \Rightarrow rg\, A \geq 2$

$\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 0 & 3 & 2 \end{vmatrix} \neq 0 \Rightarrow rg\, A = 3 \Rightarrow$ SLI pt că $rg\, A = max$

* Altă variantă (Gauss-Jordan)

Aducem matricea la f. eșalon, iar dacă

- avem pivoți pe fiecare col $\Rightarrow$ SLI
- $\exists$ col. fără pivoți $\Rightarrow$ SLD

b) $S_2 = \{ v_1 = (2,3,-1),\ v_2 = (0,1,2),\ v_3 = (2,4,1) \} \subset \mathbb{R}^3$

Fie $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ a.î. $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{\mathbb{R}^3}$

$\alpha_1(2,3,1) + \alpha_2(0,1,2) + \alpha_3(2,4,1) = (0,0,0)$

$$\begin{cases} 2\alpha_1 + 2\alpha_3 = 0 \\ 3\alpha_1 + \alpha_2 + 4\alpha_3 = 0 \\ \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \end{cases}$$

$A = \begin{pmatrix} 2 & 0 & 2 \\ 3 & 1 & 4 \\ 1 & 2 & 1 \end{pmatrix} \Rightarrow \det(A) = 0$

$\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2 \neq 0 \Rightarrow rg\, A = 2 \Rightarrow$ SLD

α_1, α_2 principale, α_3 sec.

$$\begin{cases} 2\alpha_1 = -2\lambda \\ 3\alpha_1 + \alpha_2 = -4\lambda \end{cases} \quad \alpha_3 = \lambda \in \mathbb{R}$$

$\alpha_1 = -\lambda$, $\alpha_2 = -\lambda$ și $\alpha_3 = \lambda$

$-\lambda v_1 - \lambda v_2 + \lambda v_3 = 0_{\mathbb{R}^3}$

Sistem de generatori

Fie V/K sp. vect., $S \subset V$

S = sist. de gen. pt V dacă $\langle S \rangle = V$

Ex: $\forall v \in V$, $\exists$ $\alpha_1 \dots \alpha_m \in K$, $v_1 \dots v_m \in S$ $\Big|$ $\Rightarrow$ a.î. $\alpha_1 v_1 + \dots \alpha_m v_m$

Caz particular: $V/K = K^m/K$, $S = \{v_1, \dots v_m\} \subset V$

$$A = \begin{pmatrix} | & | & | & & | \\ v_1 & v_2 & v_3 & \dots & v_m \end{pmatrix} \in \mathcal{M}_{m,m}(K)$$

a) S = sist. gen. pt $K^m/K \Rightarrow rg\, A = m$

b) $S \neq$ sist gen pt $K^m/K \Rightarrow rg\, A \neq m$

② Fie $L = \langle \{v_1 = (1,2,0,3), v_2 = (2,-1,3,0), v_3 = (-1,3,2,1)\} \rangle$

$L \subset \mathbb{R}^4$ și $v = (1,-1,1,-1)$

Stabiliți dacă $v \in L = \langle S \rangle$

$L = \langle S \rangle = \{\alpha_1 v_1 + \beta v_2 + \gamma v_3,\ \alpha, \beta, \gamma \in \mathbb{R}\}$

$v \in L \Leftrightarrow \exists\ (\alpha, \beta, \gamma) \in \mathbb{R}$ a.î. $v = \alpha v_1 + \beta v_2 + \gamma v_3$

SL. $(1,-1,1,-1) = \alpha(1,2,0,3) + \beta(2,-1,3,0) + \gamma(-1,3,2,1)$

$$\begin{cases} \alpha + 2\beta - \gamma = 1 \\ 2\alpha - \beta + 3\gamma = -1 \\ 3\beta + 2\gamma = 1 \\ 3\alpha + \gamma = -1 \end{cases}$$

$$\bar{A} = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & -1 & 3 & -1 \\ 0 & 3 & 2 & 1 \\ 3 & 0 & 1 & -1 \end{pmatrix} \quad \det \bar{A} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 2 & -1 & 3 & -1 \\ 0 & 3 & 2 & 1 \\ 3 & 0 & 1 & -1 \end{vmatrix}$$

$\det \bar{A} \neq 0 \Rightarrow rg\, \bar{A} = 4 \neq 3 \Rightarrow$ sist. incompatibil $\Rightarrow$

$\Rightarrow \langle S \rangle$ sist de generatori

Baze. Dimensiune - Coord.

Fie V/K sp. vect., $B \subset V$

B = bază pt V/K dacă $\begin{cases} B = \text{s.l.i} \\ B = \text{sist. gen. pt } V, \langle B \rangle = V \end{cases}$

Echivalent: V/K, $B = \{v_1, \dots v_m\} \subset V$

B = bază pt $V/K \iff \forall v \in V \Rightarrow \exists \alpha_1, \alpha_2 \dots \alpha_m \in K$

a.î. $v = \alpha_1 v_1 + \dots \alpha_m v_m$ unice

Prop: 2) $\forall B_1, B_2 \subset V/K \Rightarrow$ coord B_1 = coord B_2

1)

Caz Particular: $V = K^m/K$ sp. vect

P 3) $S = \{v_1, v_2, \dots v_m\} \subset K^m/K$

$S \subset K^m \Rightarrow rg \begin{pmatrix} & A & \\ v_1 & v_2 \dots & v_m \end{pmatrix} = m \iff$

$\iff \det A \neq 0$

Ex: 1) $\mathbb{R}^m/\mathbb{R}$ $\dim_{\mathbb{R}} \mathbb{R}^m = m$ $\in \mathcal{M}_m(K)$

$B_0 = \{e_1 = (1, 0, 0 \dots 0), e_2 = (0, 1, 0 \dots 0), \dots$

$e_m = (0, 0, \dots 0, 1)\} \subset \mathbb{R}^m/\mathbb{R}$ bază canonică

2) $(\mathcal{M}_{m,m}(\mathbb{R})/\mathbb{R}, +, \cdot)$ sp. vect. real

$B_0 = \{E_{ij}, i = \overline{1,m}, j = \overline{1,m}$ unde $e_{ij} = 1$ și 0 în rest

card $B_0 = m \cdot m = \dim_{\mathbb{R}} \mathcal{M}_{m,m}(\mathbb{R})$

3) $\mathbb{C}_m[X]/\mathbb{C} = \{p \in \mathbb{C}[X] \mid \text{grad } p \leq m\}$

$B_0 = \{1, X, X^2, \dots X^m\} \subset \mathbb{C}_m[X]$

↳ bază canonică

$\dim_{\mathbb{C}} \mathbb{C}_m[X] = m+1$

③ $\mathbb{R}^3/\mathbb{R}$, $v_1 = (1, 2, 3)$ și $v_2 = (2, -1, 1)$

a) $v_3 \in \mathbb{R}^3$ a.î. $B = \{v_1, v_2, v_3\} \subset \mathbb{R}^3$, $v_3 = ?$

Fie $v_3 = (x, y, z) \in \mathbb{R}^3$

$$B \subset \mathbb{R}^3 \Leftrightarrow \begin{vmatrix} 1 & 2 & x \\ 2 & -1 & y \\ 3 & 1 & z \\ v_1 & v_2 & v_3 \end{vmatrix} \neq 0 \Leftrightarrow$$

$\Leftrightarrow -z + 6y + 2x + 3x - 4z - y \neq 0$

$5x + 5y - 5z \neq 0$

$x + y - z \neq 0 \Rightarrow$ Deci $v_3 = (x, y, z)$ unde $z \neq x + y$

* Altă variantă (numeric)

$v_3 = (-1, 3, 1 \neq 2)$

b) Det un v_3 particular în rap. cu B coord. vect ~~v~~

$v = (2, 4, 5)$

$B = \{v_1, v_2, v_3\} \subset \mathbb{R}^3$

$v \in \mathbb{R} \Rightarrow \exists\ \alpha, \beta, \gamma \in \mathbb{R}$ unic a.î. $v = \alpha v_1 + \beta v_2 + \gamma v_3$

$$\begin{cases} \alpha + 2\beta - \gamma = 2 \\ 2\alpha - \beta + 3\gamma = 4 \\ 3\alpha + \beta + \gamma = 5 \end{cases}$$

$B_0 \xrightarrow{A} B \Rightarrow A[v]_B = [v]_{B_0} \ \Big|\cdot A^{-1}$ stg

$[v]_B = A^{-1}[v]_{B_0}$

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

$$[v]_{B_0} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$