15.05.2024

Seminar 10

(*) Fie $V/\mathbb{R}$ sp. vect. real

$g: V \times V \to \mathbb{R}$, g bilin, sim + poz. def

$\to Q: V \to \mathbb{R}$ g poz. def dacă $\begin{cases} Q(x) > 0, \forall x \in V \setminus \{0_V\} \\ Q(x) = 0 \iff x = 0_V \end{cases}$

g s.n. produs scalar (real) $\to \langle , \rangle$

$(V/\mathbb{R}, +, g)$ spațiu vect. euclidian

Exemplu: $(\mathbb{R}^n/\mathbb{R}, +, \cdot)$; $\langle , \rangle \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$\langle x, y \rangle \overset{def}{=} \sum_{i=1}^{n} x_i y_i \quad \forall x = (x_1, \dots x_n), y = (y_1 \dots y_n)$$

produs scalar canonic

$x \in V \Rightarrow \|x\| = \sqrt{\langle x, x \rangle}$ spațiu normat

↳ normă euclidiană

$d: V \times V \to \mathbb{R}_+$, $d(x, y) = \|y - x\| =$

$= \sqrt{\langle y - x, y - x \rangle}$ spațiu metric

Inegalitatea C-B-S:

În orice sp. vect. euclidian $(E/\mathbb{R}, \langle , \rangle)$ are loc urm. inegalitate $|\langle x, y \rangle| \leq \|x\| \, \|y\|$

În plus avem „=" $\iff \{x, y\}$ s.v. lin. dep.

CP. $E = \mathbb{R}^n/\mathbb{R}$

$\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$

$\forall x = (x_1, \dots x_n)$

$y = (y_1, \dots y_n)$

$$G = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = I_n$$

Cr. S.: $|\langle x, y\rangle| \leq \|x\| \, \|y\|$

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i=1}^{m} x_i^2}$$

$$\left| \sum_{i=1}^{m} x_i y_i \right| \leq \sqrt{\sum_{i=1}^{m} x_i^2} \sqrt{\sum_{j=1}^{m} y_j^2}$$

$$(x_1 y_1 + \dots + x_m y_m)^2 \leq (x_1^2 + \dots + x_m^2)(y_1^2 + \dots y_m^2)$$

"=" $\Leftrightarrow \{x, y\}$ lin dep. i.e. x, y coliniari

$\exists \lambda \in \mathbb{R}^*$ a.î. $y = \lambda x$

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \dots = \frac{x_m}{y_m} \left(\frac{1}{\lambda}\right)$$

Fie $x, y \in V$ $\measuredangle(x, y) = \theta \Rightarrow \cos\theta = \dfrac{\langle x, y\rangle}{\|x\| \, \|y\|}$ $\theta \in [0, \pi]$

Baze ortogonale și baze ortonormate

Fie $(E/\mathbb{R}, \langle , \rangle)$ sp. vect. euclidian, $\dim_{\mathbb{R}} E = m$

Def: a) $B' = \{e_1', e_2', \dots e_m'\} \subset E/\mathbb{R}$ bază ortog.

dacă: $e_i' \perp e_j' \Leftrightarrow \langle e_i', e_j'\rangle = 0$ $\forall i \neq j \leq m$

(mutual ortog.)

b) $B'' = \{e_1'', e_2'', \dots e_m''\} \subset E/\mathbb{R}$ bază ortom.

dacă: bază ortom = bază ortog. + $\|e_i''\| = 1 \Leftrightarrow$

$\Leftrightarrow \langle e_i'', e_j''\rangle = \delta_{ij}$ $\forall i, j = \overline{1, m}$

$\delta_{ij} \Rightarrow$ simbolul lui Kronecker

$$\delta_{ij} = \begin{cases} 1, & \text{dacă } i = j \\ 0, & i \neq j \end{cases}$$

Procedeul de ortonormalizare Gram-Schmidt

Fie $(E/\mathbb{R}, \langle , \rangle)$ sp. vect. eucl, m-dim

$\forall \{f_1, \dots f_m\} \subset E \Rightarrow \exists \{e_1, \dots e_m\} \subset E$ bază ortom.

bază arbitrară

a $\uparrow$. $\langle \{f_1, \dots f_i\} \rangle = \langle \{e_1, \dots e_i\} \rangle$, $\forall\, i = \overline{1,m}$

① În sp. vect. euc $(\mathbb{R}^3/\mathbb{R}, \langle , \rangle)$ să se construească o bază ortom. pornind de la baza arbitrară:

$B = \{f_1 = (-1,1,1), f_2 = (1,-1,1), f_3 = (1,1,-1)\} \subset E^3$

P.O.G.S $\longrightarrow$ B'' b. ortom.

V_1

$e_1' = f_1$, $e_i' = f_i = \sum_{j=1}^{i-1} \frac{\langle f_i, e_j' \rangle}{\|e_j\|^2} e_j$ $\forall i = \overline{2,m}$

$B' = \{e_1', e_2' \dots e_m'\}$ b. orog.

~~$\langle e_i', e_j' \rangle = \delta_{ij}$ $\forall i,j = \overline{1,m}$~~

$B' \xrightarrow{\text{normare}} B'' = \{e_1, \dots e_m\}$ b. ortom.

$\langle e_i', e_j' \rangle = \delta_{ij}$ $\forall i,j = \overline{1,m}$

$e_i = \frac{e_i}{\|e_i\|}$ $\forall i = \overline{1,m}$

$e_1' = f_1 = (-1,1,1)$

$e_2' = f_2 - \frac{\langle f_2, e_1' \rangle}{\|e_1'\|^2} e_1' = (1,-1,1) + \frac{+1}{3}(-1,1,1) =$

$= \left(\frac{2}{3}, -\frac{2}{3}, \frac{4}{3}\right) = \frac{2}{3}(1,-1,2)$

$\|e_1'\|^2 = \|f_1\|^2 = \langle f_1, f_1 \rangle = 3$

$e_3' = f_3 - \frac{\langle f_3, e_2' \rangle}{\|e_2'\|^2} e_2' - \frac{\langle f_3, e_1' \rangle}{\|e_1'\|^2} e_1' =$

$= (1,1,-1) - \frac{-1}{3}(-1,1,1) - \frac{-4/3}{8/3} \cdot \frac{2}{3}(1,-1,2) =$

$= (1,1,0)$

$\langle f_3, e_2' \rangle = 2/3\,(-2)$

$\|e_2'\|^2 = 4/9 \cdot 6 = 8/3$

$e_1' = (-1,1,1)$; $e_2' = 2/3\,(1,-1,2)$; $e_3' = (1,1,0)$

$B' = \{e_1', e_2', e_3'\}$ b ortog.

$\langle e_1', e_2' \rangle = \langle e_1', e_3' \rangle = \langle e_2', e_3' \rangle = 0$

$\| e_1' \| = \sqrt{3} \Rightarrow e_1 = e_1'/\|e_1'\| = 1/\sqrt{3}\,(-1,1,1)$

$\| e_2' \| = 2/3 \cdot \sqrt{6} \Rightarrow e_2 = e_2'/\|e_2'\| = \frac{3}{2\sqrt{6}} \cdot \frac{2}{3}(1,-1,2) = \frac{1}{\sqrt{6}}(1,-1,2)$

$\| e_3' \| = \sqrt{2} \Rightarrow e_3 = \frac{e_3'}{\|e_3'\|} = \frac{1}{\sqrt{2}}(1,1,0)$

$B'' = \{ e_1, e_2, e_3 \} = \{ \frac{1}{\sqrt{3}}(-1,1,1), \frac{1}{\sqrt{6}}(1,-1,2), \frac{1}{\sqrt{2}}(1,1,0) \}$

bază orton.

$\langle e_i, e_j \rangle = \delta_{ij}$ pt $i,j = \overline{1,3}$ (verif)

$$\begin{cases} \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1 \\ \langle e_1, e_2 \rangle = \langle e_1, e_3 \rangle = \langle e_2, e_3 \rangle = 0 \end{cases}$$

V_2

$e_1 = \frac{f_1}{\|f_1\|}$

$e_i = \frac{e_i'}{\|e_i'\|}$, $e_i' = f_i - \sum_{j=1}^{i-1} \langle f_i, e_j \rangle e_j \quad \forall i = \overline{2,m}$

Avem $\|f_1\| = \sqrt{3} \Rightarrow e_1 = \cancel{1/\sqrt{3}}\,(-1,1,1)$

$e_2 = \frac{e_2'}{\|e_2\|}$ unde $e_2' = f_2 - \langle f_2, e_1 \rangle e_1 = \cancel{(1,-1,1)}$

$= (1,-1,1) - \left(\frac{-1}{\sqrt{3}}\right) \frac{1}{\sqrt{3}}(-1,1,1) =$

$= (1,-1,1) + \frac{1}{3}(-1,1,1) = (\frac{2}{3}, -\frac{2}{3}, \frac{4}{3}) =$

$= \frac{2}{3}(4,-1,2) \cancel{= \frac{3}{2\sqrt{6}} \cdot \frac{2}{3}}$

$\langle f_2, e_1 \rangle = \frac{1}{3}(-1) = -\frac{1}{3}$

$e_2 = \frac{3}{2\sqrt{6}} \cdot \frac{2}{3}(1,-1,2) = \frac{1}{\sqrt{6}}(1,-1,2)$

$\| e_2' \| = \frac{2}{3}\sqrt{6}$

$e_3 = \frac{e_3'}{\|e_3'\|}$ unde $e_3' = f_3 - \langle f_3, e_1\rangle e_1 - \langle f_3, e_2\rangle e_2 =$

$= (1,1,-1) + \frac{1}{3}\cdot\frac{1}{\sqrt{3}}(-1,1,1) + \frac{2}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}(1,-1,2) =$

$= (1,1,-1) + \frac{1}{3}(-1,1,1) + \frac{1}{3}(1,-1,2) = (1,1,0)$

$\langle f_3, e_2\rangle = 1/\sqrt{6}\,(-2)$

$\boxed{\text{TEMĂ}}$: Aceleași enunț, Încă 2 ex pentru ~~F~~

② $(E^3/\mathbb{R}, \langle , \rangle)$ sp. vect. eucl.

$B' = \{e_1 = 1/\sqrt{2}\,(0,1,1),\ e_2 = 1/\sqrt{6}\,(2,-1,1),\ e_3 = 1/\sqrt{3}\,(1,1,-1)\}$

bază orton.

a) $v = (1,2,3)$ b) Temă $u = (-1,4,2)$

a)

$[v]_{B'} = (\alpha, \beta, \gamma)$

$\exists!\ \alpha, \beta, \gamma \in \mathbb{R}$ $v = \alpha e_1 + \beta e_2 + \gamma e_3$ $\Big|\ \langle\ , e_1\rangle,\ \langle\ , e_2\rangle,\ \langle\ , e_3\rangle$

$\langle v, e_1\rangle = \langle \alpha e_1 + \beta e_2 + \gamma e_3, e_1\rangle =$

$= \alpha \underbrace{\langle e_1, e_1\rangle}_{=1} + \beta \underbrace{\langle e_2, e_1\rangle}_{=0} + \gamma \underbrace{\langle e_3, e_1\rangle}_{=0} \Rightarrow$

$\Rightarrow \alpha = \langle v, e_1\rangle = \frac{5}{\sqrt{2}}$

Analog: $\beta = \langle v, e_2\rangle = \frac{3}{\sqrt{6}}$

$\gamma = \langle v, e_3\rangle = 0$

Deci $[v]_{B'} = \left(\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{6}}, 0\right)$

③ $E_3 = (\mathbb{R}^3/\mathbb{R}, \langle , \rangle)$ sp. vect. eucl.

$f_1 = (2, 2, 1)$; $f_2 = (-2, -1, 2)$

a) Calculați $\|f_1\|$, $\|f_2\|$ și unghiul dintre f_1 și f_2

$\|e_1\| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

$\|e_2\| = 3$

$\cos\theta = \dfrac{\langle f_1, f_2 \rangle}{\|e_1\| \|e_2\|}$ unde $\theta = \widehat{f_1 f_2}$

$\cos\theta = \frac{-4}{9} \Leftrightarrow \theta = \arccos\left(-\frac{4}{9}\right) = \pi - \arccos\left(\frac{4}{9}\right)$

b) $f_3 = ? \in E_3$ dif. de 0_E

a.î. $\begin{cases} f_3 \perp f_1 \\ f_3 \perp f_2 \end{cases} \Leftrightarrow \begin{cases} \langle f_3, f_1 \rangle = 0 \\ \langle f_3, f_2 \rangle = 0 \end{cases}$

$f_3 = (x, y, z)$

SLO $\begin{cases} 2x + 2y + z = 0 \\ -2x - y + 2z = 0 \end{cases}$ $A = \left(\begin{array}{cc|c} 2 & 2 & 1 \\ -2 & -1 & 2 \end{array}\right)$

x, y nec. princip

$z = \alpha \in \mathbb{R}$ nec. sec.

$\Delta_p = \begin{vmatrix} 2 & 2 \\ -2 & -1 \end{vmatrix} \neq 0$ $\begin{cases} 2x + 2y = -\alpha \\ -2x - y = -2\alpha \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x = \frac{-5}{2}\alpha \\ y = -3\alpha \\ z = \alpha \in \mathbb{R} \end{cases}$

b arb. $\begin{cases} f_1 = (2, 2, 1) \\ f_2 = (-2, -1, 2) \\ f_3 = (5, -6, 2) \end{cases}$

1x pt. 1 temă

c) P.O.G-S B orton. ?

$f_3 = \alpha(\frac{5}{2}, -3, 1)$

$f_3 = \beta(5, -6, 2)$

$e'_1 = f_1 = (2, 2, 1)$

$e'_2 = f_2 - \frac{\langle f_2, e'_1 \rangle}{\|e'_1\|^2} e'_1 = (-2, -1, 2) + \frac{5}{9}(2, 2, 1) =$

$= \left(-\frac{10}{9}, -\frac{1}{9}, \frac{22}{9}\right)$

$e'_3 = f_3 - \frac{\langle f_3, e'_1 \rangle}{\|e'_1\|^2} e_1 - \frac{\langle f_3, e'_2 \rangle}{\|e'_2\|^2} e'_2 =$

$= (5, -6, 2) - 0 - 0 = (5, -6, 2)$

B' orthog $= \{e'_1, e'_2, e'_3\}$

$e_1 = \frac{e'_1}{\|e'_1\|} = \frac{1}{3}(2, 2, 1)$

$e_2 = \frac{e'_2}{\|e'_2\|} = \frac{3}{\sqrt{65}}(-10, -1, 22)$

$e_3 = \frac{e'_3}{\|e'_3\|} = \frac{1}{\sqrt{65}}(5, -6, 2)$

$\langle f_3, e'_2 \rangle \Rightarrow$ $f_3 = (5, -6, 2)$

$e'_2 = \frac{1}{9}(-10, -1, 22)$

$\rightarrow \langle f_3, e'_2 \rangle = \frac{1}{9} \cdot (5 \cdot (-10) + (-6)(-1) + (2)(22)) =$

$=$

$\|e\| = \sqrt{\sum e_i^2} = \sqrt{\langle e, e \rangle}$