Temă

(1) Fie $(\mathbb{R}^3/\mathbb{R}, +, \cdot)$ ~~sub~~sp. vect real

$V_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y + z = 0\} \subset \mathbb{R}^3$

$V_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + z = 0\} \subset \mathbb{R}^3$

a) Dem. că V_1, V_2 subsp. vect.

Fie $\left.\begin{matrix} u, v \in V_1 \\ \alpha, \beta \in \mathbb{R} \end{matrix}\right| \Rightarrow \alpha u + \beta v \in V_1$

$u = (x_1, y_1, z_1)$

$v = (x_2, y_2, z_2)$

$$\alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) =$$

$$= (\alpha x_1, \alpha y_1, \alpha z_1) + (\beta x_2, \beta y_2, \beta z_2) =$$

$$= \underbrace{\alpha x_1 + \beta x_2}_{x} + \underbrace{\alpha y_1 + \beta y_2}_{y} + \underbrace{\alpha z_1 + \beta z_2}_{z} =$$

$$= x + y + z$$

$$2x - y + z = 2\alpha x_1 + 2\beta x_2 - \alpha y_1 - \beta y_2 + \alpha z_1 + \beta z_2 =$$

$$= \alpha(\underbrace{2x_1 - y_1 + z_1}_{0}) + \beta(\underbrace{2x_2 - y_2 + z_2}_{0}) =$$

$= 0 \cdot \alpha + \beta \cdot 0 = 0 \in V_1 \Rightarrow V_1 \leq \mathbb{R}^3$ subsp.

Fie $\left.\begin{array}{l} u, v \in V_2 \\ \alpha, \beta \in \mathbb{R} \end{array}\right| \Rightarrow \alpha u + \beta v \in V_2$

$u = (x_1, y_1, z_1)$

$v = (x_2, y_2, z_2)$

$\alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) =$

$= (\alpha x_1, \alpha y_1, \alpha z_1) + (\beta x_2, \beta y_2, \beta z_2) =$

$= \underbrace{\alpha x_1 + \beta x_2}_{x} + \underbrace{\alpha y_1 + \beta y_2}_{y} + \underbrace{\alpha z_1 + \beta z_2}_{z} =$

$= x + y + z$

$x + 2y + z = \alpha x_1 + \beta x_2 + 2\alpha y_1 + 2\beta y_2 + \alpha z_1 + \beta z_2 =$

$= \alpha(x_1 + 2y_1 + z_1) + \beta(x_2 + 2y_2 + z_2) =$

$= \alpha \cdot 0 + 0\beta = 0 \in V_2 \Rightarrow V_2 \leq \mathbb{R}^3$ subsp.

c) Continuare din seminar

$$\left\{\begin{array}{l} 2x - y + z = 0 \\ x + 2y + z = 0 \\ x = x_1 + x_2 \\ y = y_1 + y_2 \\ z = z_1 + z_2 \end{array}\right. \Rightarrow \left\{\begin{array}{l} 2(x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2) = 0 \\ (x_1 + x_2) + 2(y_1 + y_2) + (z_1 + z_2) = 0 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} 2x_1 + 2x_2 - y_1 - y_2 + z_1 + z_2 = 0 \\ x_1 + x_2 + 2y_1 + 2y_2 + z_1 + z_2 = 0 \end{array}\right. \Leftrightarrow$$

$$\Leftrightarrow \left\{\begin{array}{l} 2x_1 - y_1 + z_1 + 2x_2 - y_2 + z_2 = 0 \\ x_1 + 2y_1 + z_1 + x_2 + 2y_2 + z_2 = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} v_1 + v_2 = 0 \\ v_1 + v_2 = 0 \end{array}\right. \Rightarrow$$

$\Rightarrow v_1 + v_2 = 0 \in \mathbb{R}^3$

(2) $A \in \mathcal{M}_n(K)$ $\Rightarrow$ mat sim $\Leftrightarrow$ ${}^tA = A$

$\Rightarrow$ mat. antisim $\Leftrightarrow$ ${}^tA = -A$

$(\mathcal{M}_2(\mathbb{R}) / \mathbb{R}, +, \cdot)$ sp. vect. real

$\mathcal{S} = \{A \in \mathcal{M}_2(\mathbb{R}) \mid A$ mat sim$\}$

$\mathcal{A} = \{A \in \mathcal{M}_2(\mathbb{R}) \mid A$ mat. antisim$\}$

a) Dem. că $\mathcal{S}$ și $\mathcal{A}$ subsp. vect.

Fie $X_1, X_2 \in \mathcal{S}$, $\alpha_1, \alpha_2 \in K$ $\Big| \Rightarrow \alpha_1 X_1 + \alpha_2 X_2 \in \mathcal{S}$

$\alpha_1 X_1 + \alpha_2 X_2 = \alpha_1 {}^tX_1 + \alpha_2 {}^tX_2 =$

$= {}^t(\alpha_1 X_1) + {}^t(\alpha_2 X_2) \in \mathcal{S} \Rightarrow$

$\Rightarrow \mathcal{S} \leq \mathcal{M}_n(K)$

Fie $X_1, X_2 \in \mathcal{A}$, $\alpha_1, \alpha_2 \in K$ $\Big| \Rightarrow \alpha_1 X_1 + \alpha_2 X_2 \in \mathcal{A}$

$\alpha_1 X_1 + \alpha_2 X_2 = \alpha_1 {}^t(-X_1) + \alpha_2 {}^t(-X_2) =$

$= {}^t(-\alpha_1 X_1) + {}^t(-\alpha_2 X_2) \in \mathcal{S} \Rightarrow$

$\cancel{{}^t(-\alpha_1 X_1) + {}^t(-\alpha_2 X_2) = {}^t(-\alpha(-X_1)) + {}^t(-\alpha_2(-X_2))}$

$\cancel{= {}^t({}^t(\alpha_1 X_1)) + {}^t({}^t(\alpha_2 X_2)) =}$

$\cancel{= \alpha_1 X_1 + \alpha_2 X_2 \in \mathcal{A} \Rightarrow}$

$\Rightarrow \mathcal{A} \leq \mathcal{M}_n(K)$

b) Arătați că $\forall C \in \mathcal{M}_2(\mathbb{R})$ $\exists A \in \mathcal{S}$ și $B \in \mathcal{A}$ a.î.

$C = A + B$ (unice)

$A + B = \cancel{{}^tA + {}^tB} = A - (-B) = {}^tA - {}^tB =$

$= {}^t(A - B)$