CA I

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Serii de puteri

$\sum_{n\geq 0} f_n(x) = \sum_{n\geq 0} a_n (x-x_0)^n$ serie de puteri în jurul lui $x_0 \in \mathbb{R}$

$f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = a_n (x-x_0)^n$ $\forall x \in \mathbb{R}$ (monom de gr. n)

$(a_n)_{n\in\mathbb{N}}$ șirul coef. seriei de puteri

$a_0 \Rightarrow$ term. liber al seriei de puteri

$\sum_{n\geq 0} a_n (x-x_0)^n$
- $\to$ raza de convergență $R = \frac{1}{\lim \sqrt[n]{|a_n|}} \in \overline{\mathbb{R}}_+$
- $\to$ intervalul de conv. $(x_0-R, x_0+R) \subseteq \mathbb{R}$, $x_0 \in A$
- $\to$ mulțimea de conv. $A \subseteq \mathbb{R}$; $(x_0-R, x_0+R) \subseteq A \subseteq [x_0-R, x_0+R]$
- $\to$ suma seriei de puteri $f: A \to \mathbb{R}$ continuă pe A

$f|_{(x_0-r, x_0+r)}$ e funcție de clasă C^∞ pe (x_0-r, x_0+r)

$f(x_0) = a_0$

Pe mulț. A converge simplu seria de puteri.

Serii de puteri remarcabile

1. $\sum_{n\geq 0} x^n = \frac{1}{1-x}$ $\forall x \in (-1,1) = A$; $f(x) = \frac{1}{1-x}$ și $r = 1$

2. $\sum_{n\geq 0} (-1)^n x^n = \frac{1}{1+x}$ $\forall x \in (-1,1) = A$; $r = 1$

3. $\sum_{n\geq 0} \frac{x^n}{n!} = e^x$ $\forall x \in \mathbb{R} = A$, $f(x) = e^x$; $r = \infty$

4. $\sum_{n\geq 0} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x$, $\forall x \in \mathbb{R}$, $r = \infty$

5. $\sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x$, $\forall x \in \mathbb{R}$, $r = \infty$

Exerciții

① Să se determine r, A și f pt seria de puteri $\sum_{n\geq 1} (-1)^n \frac{x^n}{n}$

[Dacă $\exists \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \in \overline{\mathbb{R}}_+$, atunci $r = 1 / \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

$x_0 = 0$; $a_n = (-1)^n \cdot \frac{1}{n}$ $\forall n \geq 1$; $a_0 = 0$

$$l = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|(-1)^{n+1} \cdot \frac{1}{n+1}\right|}{\left|(-1)^n \cdot \frac{1}{n}\right|} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \in \overline{\mathbb{R}}_+$$

Deci $r = \frac{1}{l} = 1$

$$\begin{cases} A \subseteq \mathbb{R} \\ (x_0 - r, x_0 + r) \subseteq A \subseteq [x_0 - r, x_0 + r] \end{cases} \Leftrightarrow \begin{cases} A \subseteq \mathbb{R} \\ (-1,1) \subseteq A \subseteq [-1,1] \end{cases}$$

$-1 \in A \Leftrightarrow$ seria de nr. reale $\sum_{n\geq 1} (-1)^n \cdot \frac{(-1)^n}{n}$ este convergentă

$-1 \notin A \Leftrightarrow$ —— " —— $\sum_{n\geq 1} (-1)^n \frac{(-1)^n}{n}$ este divergentă

$1 \in A \Leftrightarrow$ —— " —— $\sum_{n\geq 1} \frac{(-1)^{2n}}{n} = \sum_{n\geq 1} \frac{1}{n}$ cu $\alpha = 1$ e divergentă

$1 \notin A \Leftrightarrow$ —— " —— $\sum_{n\geq 1} \frac{(-1)^n}{n} = \sum_{n\geq 1} (-1)^n \frac{1}{n}$ $\Big|$ $\Rightarrow$ e convergentă

$\lim_{n\to\infty} \frac{1}{n} = 0$ și $\frac{1}{n} > \frac{1}{n+1}$ $\forall n \geq 1$ (crit. Leibniz)

$A = (-1, 1]$

$f : (-1, 1] \to \mathbb{R}$, $f(x) = \sum_{n\geq 1} (-1)^n \frac{x^n}{n}$ cont pe A

$f|_{(-1,1)}$ de clasă C^∞ pe $(-1,1)$

$f(x_0) = a_0 \Leftrightarrow f(0) = 0$

$$f'(x) = \sum_{n\geq 1} \left((-1)^n \frac{x^n}{n}\right)' = \sum_{n\geq 1} (-1)^n \frac{\cancel{n} x^{n-1}}{\cancel{n}} = \sum_{n\geq 1} (-1)^n \cdot x^{n-1} \quad \forall x \in (-1,1)$$

$$f'(x) \overset{n = n-1}{=\!=} \sum_{n\geq 0} (-1)^{n+1} x^n = (-1) \sum_{n\geq 0} (-1)^n x^n = -\frac{1}{1+x}$$

Integrăm nedefinit

$$f(x) = \int \frac{-1}{1+x} dx = -\int \frac{1}{1+x} dx = -\ln|x+1| + \mathcal{C} \quad \forall x \in (-1,1) = -\ln(x+1) + \mathcal{C}$$

$f(x_0)=0 \Rightarrow f(0)=0$

$f(x)=-\ln(x+1)+C$ $\Big| \Rightarrow f(0)=-\ln 1+C=0 \Leftrightarrow C=\ln 1=0 \Rightarrow$

$\Rightarrow f(x)=-\ln(x+1)+\ln 1=-\ln(x+1) \quad \forall x\in(-1,1)$

$f(1)=\lim_{\substack{x\to 1\\ x<1}} f(x)=\lim_{\substack{x\to 1\\ x<1}} -\ln(x+1)=-\ln 2$

(2) Să se det r, A, f pt $\sum_{n\geq 0} n(x+1)^n$

$x_0=-1$, $a_n=n$ şi $a_0=0$

$l=\lim_{x\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{x\to\infty}\frac{n+1}{n}=1\in\mathbb{R} \to r=\frac{1}{l}=1$

$\begin{cases} A\subseteq\mathbb{R} \\ (-2,0)\subseteq A\subseteq[-2,0] \end{cases}$

$x=-2\overset{?}{\in} A \Rightarrow \sum_{n\geq 0} n(-2+1)^n=\sum_{n\geq 0} n(-1)^n=\sum_{n\geq 0} x_n$

$\lim_{n\to\infty} x_n \begin{cases} \lim_{n\to\infty} x_{2k}=+\infty \\ \lim_{n\to\infty} x_{2k+1}=\lim_{n\to\infty}(2k+1)(-1)=-1 \end{cases} \Big| \Rightarrow \nexists \lim_{n\to\infty} x_n$ $\Big| \Rightarrow$

$\Rightarrow$ seria e divergentă $\Rightarrow -2\notin A$ $\Big| \Rightarrow$

$x=0\overset{?}{\in} A \Rightarrow \sum_{n\geq 0} n\cdot 1^n=\sum_{n\geq 0} n$ $\Big| \Rightarrow$ seria e divergentă $\Rightarrow 0\notin A$

$\lim_{n\to\infty} n=\infty\neq 0$

$\Rightarrow A=(-2,0) \Rightarrow f:(-2,0)\to\mathbb{R}$, $f(x)=\sum n(x+1)^n$ cont pe A

f de clasă C^∞ pe $(-2,0)$ şi $f(-1)=0$

$f(x)=\sum_{n\geq 0}(n+1-1)(x+1)^n=\sum_{n\geq 0}(n+1)(x+1)^n-\sum_{n\geq 0}(x+1)^n=g(x)-h(x)$

~~f(x)~~ $\sum_{n\geq 0} x^n=\frac{1}{1-x} \quad \forall x\in(-1,1) \xRightarrow{x\to x+1} \sum_{n\geq 0}(x+1)^n=\frac{1}{-x} \quad \forall x\in(-2,0) \Rightarrow$

$\Rightarrow \sum_{n\geq 0}(x+1)^n=-\frac{1}{x} \quad \forall x\in(-2,0)$

$g(x)\,dx=\sum_{n\geq 0}\int(n+1)(x+1)^n=\sum_{n\geq 0}(n+1)\frac{(x+1)^{n+1}}{n+1}+C=\sum_{n\geq 0}(x+1)^{n+1}+C$

$g(x)\,dx \overset{m=n+1}{=} \left(\sum_{m\geq 1}(x+1)^m\right)-1+C=\frac{1}{x}-1+C \quad \forall x\in(-2,0)$

$g(x) = \left(-\frac{1}{x} - 1 + c\right)' = \frac{1}{x^2} \Rightarrow f(x) = \frac{1}{x^2} - \left(-\frac{1}{x}\right) = \frac{1}{x^2} + \frac{1}{x} \quad \forall x \in (-2, 0)$

(3) Să se determine r și A pt. seria $\sum_{n \geq 0} \frac{2^n}{3^n+1} x^n$

$x_0 = 0, \quad a_n = \frac{2^n}{3^n+1}, \quad a_0 = \frac{1}{2}$

$l = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^{n+1}}{3^{n+1}+1} \cdot \frac{3^n+1}{2^n} = \lim_{n\to\infty} \frac{2 \cdot 3^n\left(1+\frac{1}{3^n}\right)}{3^{n+1}\left(1+\frac{1}{3^{n+1}}\right)} = \frac{2}{3}$

$r = \frac{1}{l} = \frac{3}{2}$

$$\begin{cases} A \subseteq \mathbb{R} \\ \left(-\frac{3}{2}, \frac{3}{2}\right) \subseteq A \subseteq \left[-\frac{3}{2}, 3/2\right] \end{cases}$$

$x = 3/2 \Rightarrow \sum_{n\geq 0} \frac{2^n}{3^n+1}\left(\frac{3}{2}\right)^n = \sum_{n\geq 0} \frac{3^n}{3^n+1} \Rightarrow$ div. $\Rightarrow 3/2 \notin A$

$x = -3/2 \Rightarrow \sum_{n\geq 0} \frac{2^n}{3^n+1}\left(-\frac{3}{2}\right)^n = \sum_{n\geq 0} \frac{(-1)^n \cdot 2^n}{3^n+1}\left(\frac{3}{2}\right)^n = \sum (-1)^n \frac{3^n}{3^n+1} \Rightarrow$

$\Rightarrow$ convergentă $\Rightarrow -3/2 \in A$

$A = [-3/2, 3/2) \Rightarrow f: A \to \mathbb{R}, \; f(x) = \sum_{n\geq 0} \frac{3^n}{3^n+1} x^n$ caut pe A de C^∞ pe $(-3/2, 3/2)$

$f(x_0) = f(0) = 0$

$f'(x) = \sum_{n\geq 0} \left(\frac{3^n \cdot x^n}{3^n+1}\right)' = \sum_{n\geq 0} \frac{3^n}{3^n+1} (n) x^{n-1}$