13.03.2024

# Seminar 3

- Subspații vectoriale

Fie $V/K$ sp. vect. și $U \subseteq V$ submulț., $U \neq \emptyset$

$U$ = subsp. vect. al lui $V$ dacă: $\Leftrightarrow$

$$\left.\begin{array}{l} 1)\ v_1, v_2 \in U \Rightarrow v_1 + v_2 \in U \\ 2)\ \cancel{v_1, v_2 \in}\ v \in U,\ \alpha \in K \Rightarrow \alpha v \in U \end{array}\right| \Rightarrow$$

$\Rightarrow \forall v_1, v_2 \in U$ ; $\alpha_1, \alpha_2 \in K \Rightarrow \alpha_1 v_1 + \alpha_2 v_2 \in U$

Exemple: 1) $V/K$ sp. vect.

$\{0_V\}$, $V \to$ subsp. improprii ale lui $V$

2) $(K^n/K, +, \cdot) \to$ sp. vect.

$\tilde{K}^m \subseteq K^n$, $m \leq n$

$\tilde{K}^m \to K^m \times \underbrace{\{0, \dots 0\}}_{\text{de } n-m \text{ ori}} \Rightarrow \{(x_1, x_2, \dots x_m, 0, 0, \dots 0)\}$

① $V_1 = \{(x, y) \in \mathbb{R}^2 \mid x = y\} \overset{?}{\leq} \mathbb{R}^2$

Fie $\left.\begin{array}{l} u, v \in V_1 \\ \alpha, \beta \in \mathbb{R} \end{array}\right| \Rightarrow \alpha u + \beta v \in V_1$

$u = (x_1, y_1) = \cancel{(x_1, x_1)}$

$v = (x_2, y_2) = \cancel{(y_1, y_1)}$

$\alpha u + \beta v = \alpha(x_1, y_1) + \beta(x_2, y_2) =$

$= \alpha x_1 + \alpha y_1 + \beta x_2 + \beta y_2 =$

$= \underbrace{\alpha x_1 + \beta x_2}_{x} + \underbrace{\alpha y_1 + \beta y_2}_{y} =$

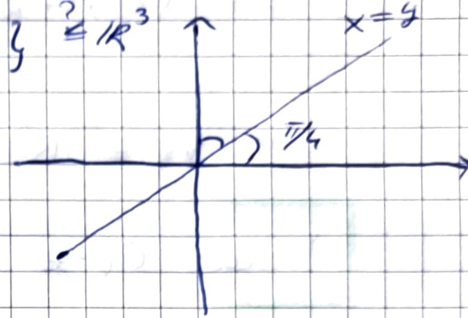
$= x + y$

$x-y = \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2 = \cancel{x-y}$

$= \alpha(\underbrace{x_1-y_1}_{x_1=y_1}) + \beta(\underbrace{x_2-y_2}_{x_2=y_2}) = 0 \in V_1 \Rightarrow$

$\rightarrow$ Deci $V_1 \leq \mathbb{R}^2$ ~~spp~~. vect

(2) $V_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x+2y+3z=0\} \overset{?}{\leq} \mathbb{R}^3$




Fie $u, v \in V_2$

$\alpha, \beta \in \mathbb{R}$

$u = (x_1, y_1, z_1)$

$v = (x_2, y_2, z_2)$ $\Rightarrow$

$\Rightarrow \alpha u + \beta v = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) =$

$\cancel{= \alpha(x_1 + 2y_1 + 3z_1) + \beta(x_2 + 2y_2 + 3z_2) =}$

$\cancel{= \alpha x_1 + 2\alpha y_1 + 3\alpha z_1 + \beta x_2 + 2\beta y_2 + 3\beta z_2 =}$

$\cancel{= \alpha x_1 + \beta x_2 + 2\alpha y_1 + 2\beta y_2 + 3\alpha z_1 + 3\beta z_2 =}$

$= \underbrace{\alpha x_1 + \beta x_2}_{x} + \underbrace{\alpha y_1 + \beta y_2}_{y} + \underbrace{\cancel{3}\alpha z_1 + \beta z_2}_{z}$

$x + 2y + 3z = \alpha x_1 + \beta x_2 + 2\alpha y_1 + 2\beta y_2 + 3\alpha z_1 + 3\beta z_2 =$

$= \alpha(x_1 + 2y_1 + 3z_1) + \beta(x_2 + 2y_2 + 3z_2) =$

$= \alpha \cdot 0 + \beta \cdot 0 = 0 \Rightarrow V_2 \leq \mathbb{R}^3$ subsp. vect.

Subsp. vect. $\Rightarrow$ $\left\{\begin{array}{l} 0 \rightarrow \{0_V\} \\ 1 \rightarrow \text{dr. vect} \\ 2 \rightarrow \text{plane vect} \\ 3 \rightarrow \mathbb{R}^3 \end{array}\right.$ $\Rightarrow$ în $\mathbb{R}^3$

$\Rightarrow \left\{\begin{array}{l} 0 \rightarrow \{0_V\} \\ 1 \rightarrow \text{dr. vect} \\ 2 \rightarrow \mathbb{R}^2 \end{array}\right.$ $\Rightarrow$ în $\mathbb{R}^2$

(3) $V_3 = \{(x,y,z) \in \mathbb{R}^3 \mid x - y + 2z = 1\} \overset{?}{\leq} \mathbb{R}^3$

Fie $u, v \in V_3$ și $\alpha, \beta \in \mathbb{R}$

$u = (x_1, y_1, z_1)$ , $x_1 - y_1 + 2z_1 = 1$

$v = (x_2, y_2, z_2)$ , $x_2 - y_2 + 2z_2 = 1$

$\alpha u + \beta v = (\underbrace{\alpha x_1 + \beta x_2}; \underbrace{\alpha y_1 + \beta y_2}; \underbrace{\alpha z_1 + \beta z_2}) =$

$= (x; y; z) \Rightarrow$

$\Rightarrow x - y + 2z = \alpha(x_1 - y_1 + 2z_1) + \beta(x_2 - y_2 + 2z_2) =$

$= \alpha \cdot 1 + \beta \cdot 1 = \alpha + \beta \not\ni \alpha u + \beta v \Rightarrow$

$\Rightarrow V_3 \not\leq \mathbb{R}^3$ nu e subsp. vect. a lui $\mathbb{R}^3$

Dacă $O_V \notin U \Rightarrow U$ nu este subsp. al lui $V$

Folosind obs. de mai sus $\Rightarrow O_{\mathbb{R}^3} \notin V_3 \Rightarrow$

$\Rightarrow V_3 \not\leq \mathbb{R}^3$ nu e subsp. vect.

(4) Fie $A \in \mathcal{M}_{m,n}(K)$

rang $A = m \leq n$

$S(A) = \{ X \in K^n \mid AX = O \} \subseteq K^n$

$S(A) =$ mult. sol. sist. lin. omogen

$S(A) \subseteq K^n$ subsp. vect

$\dim_K S(A) = \underbrace{n - \text{rang } A}_{\text{corang } A}$ (nr. nec. sec. gr. de mulț.)

Fie $X_1, X_2 \in S(A)$, $\alpha_1, \alpha_2 \in K$ $\Big| \Rightarrow \alpha_1 X_1 + \alpha_2 X_2 \in S(A)$

$A(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 A X_1 + \alpha_2 A X_2 = O$

$\alpha_1 X_1 + \alpha_2 X_2 \in S(A)$

$S(A) \subseteq K^n / K$ subsp. vect.

(5) Fie $(\mathbb{R}^3/\mathbb{R}, +, \cdot)$ sp. vect. real

$V_1 = \{ (x, y, z) \in \mathbb{R}^3 \mid 2x - y + z = 0 \} \subset \mathbb{R}^3$

$V_2 = \{ (x, y, z) \in \mathbb{R}^3 \mid x + 2y + z = 0 \} \subset \mathbb{R}^3$

a) Dem că $V_1, V_2$ subsp. (TA)

b) $V_1 \cap V_2 = ?$

c) Dem. că $\mathbb{R}^3 = V_1 \oplus V_2 \Rightarrow \begin{cases} \rightarrow \mathbb{R}^3 = V_1 + V_2 \\ \Rightarrow V_1 \cap V_2 = \{0_{\mathbb{R}^3}\} \end{cases}$

b) $V_1, V_2 \subset V/K \Rightarrow \begin{cases} V_1 \cap V_2 \subset V/K \\ V_1 \cup V_2 \text{ ~~nu~~ e subsp. vect. în gen.} \end{cases}$

$V_1, V_2 \subset V/K$ subsp. vect $\Rightarrow V_1 \cup V_2 \subset V/K$ dacă $V_1 \subseteq V_2$ sau $V_2 \subseteq V_1$

$$V_1 \cap V_2 = \begin{cases} 2x - y + z = 0 \\ x + 2y + z = 0 \end{cases} \qquad A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$= \begin{cases} 2x - y = -\alpha \\ x + 2y = -\alpha \end{cases} ; z = \alpha \in \mathbb{R} \Rightarrow$$

$\Rightarrow x = -\frac{3}{5}\alpha$ și $y = -\frac{1}{5}\alpha$

$$V_1 \cap V_2 = \{(-3/5\,\alpha; -1/5\,\alpha; \alpha) \mid \alpha \in \mathbb{R}\} =$$

$$= \{\alpha(-3; -1; 5) \mid \alpha = 5\beta, \beta \in \mathbb{R}\} =$$

$$= \langle\{u\}\rangle \text{ dr. vect}$$

c) $\mathbb{R}^3 = V_1 \oplus V_2 \Leftrightarrow \forall v \in \mathbb{R}^3\ \exists\, v_1 \in V_1$ și $v_2 \in V_2 \Rightarrow$

$\Rightarrow$ a.î. $v = v_1 + v_2$

Fie $v = (x, y, z)$

$v_1 = (x_1, y_1, z_1)$

$v_2 = (x_2, y_2, z_2)$

$\Rightarrow (x, y, z) = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \Rightarrow$

$$\Rightarrow \begin{cases} x = x_1 + x_2 \\ y = y_1 + y_2 \\ z = z_1 + z_2 \end{cases}$$

dar $\begin{cases} 2x - y + z = 0 \\ x + 2y + z = 0 \end{cases}$

$$\Rightarrow \begin{cases} 2x - y + z = 0 \\ x + 2y + z = 0 \\ x = x_1 + x_2 \\ y = y_1 + y_2 \\ z = z_1 + z_2 \end{cases}$$