CDI

15.01.2024

Integrale multiple

- Integrale multiple
  - pe intervale închise n-dimensionale,
  - mult. simple în rap. cu una dintre axe $\Rightarrow$
  - schimbare de var.

$\Rightarrow$ aplicăm teorema lui Fubini

① Să se calculeze urm. integrale:

a) $\iint\limits_D \frac{x^2}{y^2}\,dx\,dy$, unde $D = [0,1]\times[1,2]$

b) $\iint\limits_D e^{x/y}\,dx\,dy$, unde $D = \{(x,y)\in\mathbb{R}^2 \mid 0\le x\le y^2,\ 0\le y\le 1\}$

a) $D=[0,1]\times[1,2]$ este interval închis 2-dimensional

Integrala dublă $\iint\limits_D \frac{x^2}{y^2}\,dx\,dy$ se calc. cu teorema lui Fubini și alegând ord. de integrare cum dorim.

$$\iint\limits_D \frac{x^2}{y^2}\,dx\,dy = \iint\limits_D \frac{x^2}{y^2}\,dx\,dy \underset{=}{\overset{dx,\,dy}{=}} \int_1^2\left(\int_0^1 \frac{x^2}{y^2}\,dx\right)dy = \star$$

$$\underset{=}{\overset{dy,\,dx}{=}} \int_0^1\left(\int_1^2 \frac{x^2}{y^2}\,dy\right)dx$$

$$\star = \int_1^2\left(\frac{1}{y^2}\int_0^1 x^2\,dx\right)dy = \int_1^2\left(\frac{1}{y^2}\cdot\frac{x^3}{3}\Big|_0^1\right)dy =$$

$$= \int_1^2 \frac{1}{3y^2}\,dy = \frac{1}{3}\int_1^2 y^{-2}\,dy = \frac{1}{3}\cdot\frac{y^{-1}}{-1}\Big|_1^2 =$$

$$= \frac{-1}{3y}\Big|_1^2 = -\frac{1}{6}+\frac{1}{3} = \frac{1}{6}$$

b) $\iint\limits_D e^{\frac{x}{y}}\,dx\,dy$

$$\begin{cases} 0 \le x \le y^2 \\ 0 \le y \le 1 \Rightarrow y \in [0,1] \end{cases} \Rightarrow \underbrace{0}_{g(y)} \le x \le \underbrace{y^2}_{h(y)} \Rightarrow$$

$g, h : [0,1] \to \mathbb{R}$ cont.

$g(y) = 0$

$h(y) = y^2$

$D = \{ (x,y) \in \mathbb{R}^2 \mid y \in [0,1],\ g(y) \le x \le h(y) \} = \Gamma_{g,h}$

x depinde de y $\Rightarrow$ D e simplă în rep cu Ox

Integrala se calc. cu teorema lui Fubini

$$\iint_D f(x,y)\,dx\,dy \overset{dx,\,dy}{=\!=} \int_0^1 \left(\int_{g(y)}^{h(y)} e^{\frac{x}{y}}\,dx \right) dy = I$$

$$\int_{g(y)}^{h(y)} e^{\frac{x}{y}}\,dx = \int_0^{y^2} e^{\frac{x}{y}}\,dx = \frac{e^{\frac{x}{y}}}{\frac{1}{y}} \Big|_0^{y^2} =$$

$$= y\, e^{x/y} \Big|_0^{y^2} = y \cdot e^y - y$$

$$I = \int_0^1 y \cdot e^y - y\,dy = \int_0^1 y(e^y - 1)\,dy = \int_0^1 y (e^y - y)'\,dy$$

$$I = y \cdot (e^y - y)\Big|_0^1 - \int_0^1 y' \cdot (e^y - y)\,dy$$

$$I = (e-1) - \int_0^1 e^y - y\,dy = (e-1) - \left(e^y - \frac{y^2}{2}\right)\Big|_0^1 =$$

$$= (e-1) - \left(e - \frac{1}{2} - 1\right) = \frac{1}{2}$$

(2) Să se calculeze: $\iint_D xy\,dx\,dy$; unde

$D = \{ (x,y) \in \mathbb{R}^2 \mid y^2 \le x ;\ y \ge 0 ;\ y \ge x \}$

$$\begin{cases} y^2 \le x \Rightarrow x \ge 0 \\ y \ge 0 \\ y \ge x \end{cases} \quad \begin{matrix} x \ge 0 \\ y \ge x \end{matrix} \Bigg| \Rightarrow 0 \le x \le y \Rightarrow \begin{matrix} y^2 \ge x^2 \\ y^2 \le x \end{matrix} \Bigg| \Rightarrow x^2 \le y^2 \le x \Rightarrow x^2 \le x \Rightarrow$$

$$\Rightarrow \begin{matrix} x^2 - x \le 0 \Rightarrow x \in [0,1] \\ x^2 \le y^2 \le x \end{matrix} \Bigg| \Rightarrow \underbrace{x}_{g(x)} \le y \le \underbrace{\sqrt{x}}_{h(x)}$$

$g, h : [0,1] \to \mathbb{R} \quad g(x) = x,\ h(x) = \sqrt{x}$

$D = \{ (x,y) \in \mathbb{R} \mid x \in [0,1],\ g(x) \le y \le h(x) \} = \Gamma_{g,h}$

y depinde x

D - mulț. simplă în rap. cu Oy (ord. de int. va fi dy dx)

$$\iint_D f(x,y)\,dx\,dy \underline{\underline{dy,\,dx}} \int_0^1 \left(\int_{g(x)}^{h(x)} xy\,dy \right) dx = I$$

$$\int_{g(x)}^{h(x)} xy\,dy = \int_x^{\sqrt{x}} xy\,dy = x \cdot \frac{y^2}{2} \Big|_x^{\sqrt{x}} = \frac{x^2}{2} - \frac{x^3}{2}$$

$$I = \int_0^1 \frac{x^2 - x^3}{2}\,dx = \frac{1}{2} \int_0^1 x^2 - x^3\,dx = \frac{1}{2}\frac{x^3}{3}\Big|_0^1 - \frac{1}{2}\frac{x^4}{4}\Big|_0^1 =$$

$$= \frac{4-3}{24} = \frac{1}{24}$$

schimbare de var în $\mathbb{R}^2$

- $\varphi : D_1 \to D_2$; $D_1, D_2 \subseteq \mathbb{R}^2$ mulț. deschise
- $\varphi(x_1,x_2) = (\varphi_1(x_1,x_2), \varphi_2(x_1,x_2))$
- φ funcție de clasă C^1
- φ funcție bij
- φ^{-1} funcție de cls. C^1
- $$\frac{D(\varphi_1,\varphi_2)}{D(x_1,x_2)}(x_1,x_2) = \det \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1}(x_1,x_2) & \frac{\partial \varphi_1}{\partial x_2}(x_1,x_2) \\ \frac{\partial \varphi_2}{\partial x_1}(x_1,x_2) & \frac{\partial \varphi_2}{\partial x_2}(x_1,x_2) \end{pmatrix} \neq 0$$

  $\forall (x_1,x_2) \in D_1$

Exemple:

1) Trecerea la coord. polare

$\varphi : (0,\infty) \times (0, 2\pi) \to \mathbb{R}^2 \setminus \{(x,0) \mid x \geq 0\}$

$\varphi : (0,+\infty) \times (-\pi,\pi) \to \mathbb{R}^2 \setminus \{(x,0) \mid x \leq 0\}$

$$\varphi(R,\alpha) = (R\cos\alpha, R\sin\alpha) \Rightarrow \begin{cases} \varphi_1(R,\alpha) = R\cos\alpha \\ \varphi_2(R,\alpha) = R\sin\alpha \end{cases}$$

$$\frac{D(\varphi_1,\varphi_2)}{D(R,\alpha)}(R,\alpha) = \det \begin{pmatrix} (R\cos\alpha)'_R & (R\cos\alpha)'_\alpha \\ (R\sin\alpha)'_R & (R\sin\alpha)'_\alpha \end{pmatrix} =$$

$$= \det \begin{pmatrix} \cos\alpha & -R\sin\alpha \\ \sin\alpha & R\cos\alpha \end{pmatrix} = R\cos^2\alpha + R\sin^2\alpha = R \neq 0$$

$$\boxed{\frac{D(\varphi_1, \varphi_2)}{D(R,\alpha)}(R,\alpha) = R}$$

2) Trec. la coord. pol. generalizate.

$\varphi : (0,\infty)\times(0,2\pi) \to \mathbb{R}^2 \setminus \{(x,0) \mid x \geq 0\}$

$\varphi : (0,\infty)\times(-\pi,\pi) \to \mathbb{R}^2 \setminus \{(x,0) \mid x \geq 0\}$

$\varphi(R,\alpha) = (\underbrace{aR\cos\alpha}_{\varphi_1(R,\alpha)} \quad \underbrace{bR\sin\alpha}_{\varphi_2(R,\alpha)})$; $a, b \in \mathbb{R}^*$

$$\frac{D(\varphi_1,\varphi_2)}{D(R,\alpha)}(R,\alpha) = \det \begin{pmatrix} (aR\cos\alpha)'_R & (aR\cos\alpha)'_\alpha \\ (bR\sin\alpha)'_R & (bR\sin\alpha)'_\alpha \end{pmatrix} =$$

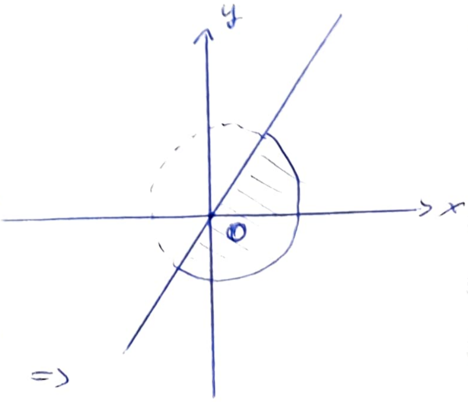
$$= \det \begin{pmatrix} a\cos\alpha & -aR\sin\alpha \\ b\sin\alpha & bR\cos\alpha \end{pmatrix} =$$

$$= abR\cos^2\alpha + abR\sin^2\alpha = abR \neq 0$$

$$\boxed{\frac{D(\varphi_1,\varphi_2)}{D(R,\alpha)}(R,\alpha) = abR}$$

3) Să se calculeze: $\iint_D x\sqrt{x^2+y^2}\,dx\,dy$; $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 6;\ y \leq x\}$

$\left.\begin{array}{l} y = x \\ x^2+y^2 = 6 \end{array}\right| \Rightarrow$ graficul din dr.



$\varphi(R,\alpha) = (R\cos\alpha, R\sin\alpha) \Rightarrow \begin{cases} \varphi_1(R,\alpha) = R\cos\alpha \\ \varphi_2(R,\alpha) = R\sin\alpha \end{cases}$

$D = \varphi(A)$

$D : \begin{cases} x^2+y^2 \leq 6 \\ y \leq x \end{cases} \xrightarrow[y = R\sin\alpha]{x = R\cos\alpha} \begin{cases} R^2\cos^2\alpha + R^2\sin^2\alpha \leq 6 \\ R\sin\alpha \leq R\cos\alpha \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} R^2 \le 6 \\ R\sin\alpha \le R\cos\alpha \end{cases} \qquad \Bigg| \Rightarrow \begin{cases} R \in [0, \sqrt{6}] \\ \sin\alpha \le \cos\alpha \end{cases} \Rightarrow$$

$R \in [0, +\infty)$, $\alpha \in [0, 2\pi]$ sau $\alpha \in [-\pi, \pi]$

$$\Rightarrow \begin{cases} R \in [0, \sqrt{6}] \\ \alpha \in \left[-\frac{3\pi}{4}; \frac{\pi}{4}\right] \end{cases} \Bigg| \Rightarrow A = [0, \sqrt{6}] \times \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$$

$$dx\,dy = \left| \frac{\Delta(\varphi_1, \varphi_2)}{\Delta(R, \alpha)}(R, \alpha) \right| dR\,d\alpha = |R|\,dr\,d\alpha$$

$$\iint_A x\sqrt{x^2+y^2}\,dx\,dy = \iint_A R\cos\alpha\sqrt{R^2\cos^2\alpha + R^2\sin^2\alpha}\,|R|\,dR\,d\alpha =$$

$$= \iint_A R^3\cos\alpha\,dR\,d\alpha \quad \underline{\underline{dR, d\alpha}}$$