Tema

① $f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x,y,z) = (x+y+z,\ x-y+z,\ x-y-z)$

a) apl. liniară?

b) Ker f, Im f, B_1 pt Ker f și B_2 pt ~~Ker~~ f

c) f inj, surj, bij?

d) Th. rang defect

a) $f(X) = AX$ unde $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ și $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

Fie $\left.\begin{array}{l} X_1, X_2 \in \mathbb{R}^3 \\ \alpha_1, \alpha_2 \in \mathbb{R} \end{array}\right| \Rightarrow f(\alpha_1 X_1 + \alpha_2 X_2) = A(\alpha_1 X_1 + \alpha_2 X_2) =$

$= \alpha_1 (AX_1) + \alpha_2 (AX_2)$

b) $\mathrm{Ker}\, f = \left\{ v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3} \right\} \subset \mathbb{R}^3$ subsp. vect.

$f(x,y,z) = (0,0,0) \Leftrightarrow \begin{cases} x+y+z=0 \\ x-y+z=0 \\ x-y-z=0 \end{cases} \Rightarrow A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$

$\left.\begin{array}{l} \mathrm{rg}\, A \le 3 \\ \det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 4 \neq 0 \end{array}\right| \Rightarrow$ ∃ o singură sol. unică $\Rightarrow$

$\Rightarrow x = y = z = 0 \Rightarrow \mathrm{Ker}\, f = \{0_{\mathbb{R}^3}\} \subset \mathbb{R}^3$

Fie B_1 bază a lui $\mathrm{Ker} f$
dar $\mathrm{Ker} f = \{0_{\mathbb{R}^3}\}$ $\Big| \Rightarrow B_1 = \mathrm{Ker} f$

$\mathrm{Im} f = \{(x', y', z') \in \mathbb{R}^3 \mid f(x, y, z) = (x', y', z')\} \subseteq \mathbb{R}^3$

$(x', y', z') \in \mathrm{Im} f \Rightarrow (x', y', z') = (x', y', x'-y') =$

$= (x', 0, x') + (0, y', -y') = x'(1, 0, 1) + y'(0, 1, -1) =$

$= x' \cdot v_1 + y' v_2$

$B_2 = \{v_1, v_2\} \subset \mathrm{Im} f$

c) $\mathrm{Ker} f = \{0_{\mathbb{R}^3}\} \Rightarrow f$ inj
$\mathrm{Im} f \neq \mathbb{R}^3 \Rightarrow f$ nu e surj $\Big| \Rightarrow f$ nu e bij

d) Th. rg. def.: $\dim_{\mathbb{R}} \mathrm{Ker} f + \dim_{\mathbb{R}} \mathrm{Im} f = \dim_{\mathbb{R}} \mathbb{R}^3$

$\dim_{\mathbb{R}} \mathrm{Ker} f = 0$
$\dim_{\mathbb{R}} \mathrm{Im} f = 2$
$\dim_{\mathbb{R}} \mathbb{R}^3 = 3$ $\Big| \Rightarrow 0 + 2 \neq 3 \Rightarrow$ fals.

(2) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (x + y + 2z, 2x + y + z, 3x + 2y - z)$

f apl. lin ?

Fie $v_1, v_2 \in \mathbb{R}^3$
$\alpha_1, \alpha_2 \in \mathbb{R}$ $\Big| \Rightarrow f(\alpha_1 v_1 + \alpha_2 v_2) \overset{?}{=} \alpha_1 f(v_1) + \alpha_2 f(v_2)$

$f(X) = AX$ unde $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ şi $A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$f(\alpha_1 X_1 + \alpha_2 X_2) = A(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 (AX_1) + \alpha_2 (AX_2)$

este apl. liniară

(3) Fie $V_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$
$V_2 = \{(u, 0, v) \mid u, v \in \mathbb{R}\}$

a) Ar. că $V_2 \subset \mathbb{R}^3$ ssp. vect. şi $\dim_{\mathbb{R}} V_2 = ?$

b) Dem $V_1 + V_2 = \mathbb{R}^3$ şi este adv. rel $V_1 \oplus V_2 = \mathbb{R}^3$?

a) $V_2 \ni (u, 0, v) = (u, 0, 0) + (0, 0, v) =$

$= u(1, 0, 0) + v(0, 0, 1) =$

$= u e_1 + v e_2$

$V_2 = \{ u e_1 + v e_2 \mid u, v \in \mathbb{R} \} \subset \mathbb{R}^3$ subsp vect.

$\dim_{\mathbb{R}} V_2 = 2$

b) analog subpct a) pt $V_1 \subset \mathbb{R}^3 \Rightarrow$

$\Rightarrow V_1 = \{ x(1,0,0) + y(0,1,0) \mid x, y \in \mathbb{R} \} \subset \mathbb{R}^3$

$\dim_{\mathbb{R}} V_1 = 2$

Aplicăm Th. Grassmann $\Rightarrow$

$\Rightarrow \dim_{\mathbb{R}} V_1 + \dim_{\mathbb{R}} V_2 - \dim_{\mathbb{R}} (V_1 \cap V_2) = \dim_{\mathbb{R}} (V_1 + V_2)$

$\left.\begin{array}{l} V_1 = \{ x(1,0,0) + y(0,1,0) \mid x, y \in \mathbb{R} \} \\ V_2 = \{ u(1,0,0) + v(0,0,1) \mid u, v \in \mathbb{R} \} \end{array}\right| \Rightarrow V_1 \cap V_2 = \{ (1,0,0) \}$

Deci $\dim_{\mathbb{R}} (V_1 \cap V_2) = 1$

$\left.\begin{array}{l} \dim_{\mathbb{R}} (V_1 + V_2) = 2 + 2 - 1 = 3 \\ V_1, V_2 \subset \mathbb{R}^3 \text{ subsp. vect} \Rightarrow V_1 + V_2 \subset \mathbb{R}^3 \text{ subsp vect} \end{array}\right| \Rightarrow$

$\Rightarrow V_1 + V_2 = \mathbb{R}^3$

$V_1 \oplus V_2 = \mathbb{R}^3 \Rightarrow \begin{cases} \mathbb{R}^3 = V_1 + V_2 & \text{(adevărat)} \\ V_1 \cap V_2 = \{ 0_{\mathbb{R}^3} \} \end{cases}$

dar $V_1 \cap V_2 = \{ <(1,0,0)> \neq 0_{\mathbb{R}^3}$ $\Rightarrow$ abs $\Rightarrow$

$\Rightarrow$ relația $V_1 \oplus V_2 = \mathbb{R}^3$ nu e adevărată

(4) Fie $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) \;\middle|\; \begin{cases} a - b + c - d = 0 \\ a + 2b - c + 3d = 0 \end{cases} \right\}$

a) $V \subset \mathcal{M}_2(\mathbb{R})$ subsp. vect?

b) det o bază și $\dim_{\mathbb{R}} V = ?$

Fie $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ unde $A = O_2 \Rightarrow$

$\Rightarrow \begin{cases} a - b + c - d = 0 \\ a + 2b - c + 3d = 0 \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ 0 = 0 \end{cases}$ (1)

Fie $A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$ și $A_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in V$

$$A_1 + A_2 = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \Rightarrow \begin{cases} a+e-b-f+c+g-d-h=0 \\ a+e+2b+2f-c-g+3d+3h=0 \end{cases}$$

$$\Rightarrow \begin{cases} (a-b+c-d)+(e-f+g-h)=0 \Rightarrow 0=0 \\ (a+2b-c+3d)+(e+2f-g+3h)=0 \Leftrightarrow 0=0 \end{cases} \Rightarrow A_1+A_2 \in V \quad (2)$$

Fie $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$ și $\alpha \in \mathbb{R}$

$$\alpha A = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} \alpha a - \alpha b + \alpha c - \alpha d = 0 \\ \alpha a + 2\alpha b - \alpha c + 3\alpha d = 0 \end{cases} \Rightarrow \begin{cases} \alpha(\underbrace{a-b+c-d}_{0}) = 0 \\ \alpha(\underbrace{a+2b-c+3d}_{0}) = 0 \end{cases} \quad (3)$$

Din (1), (2), (3) $\Rightarrow V \subset \mathcal{M}_2(\mathbb{R})$ subsp. vect.

b) Fie $A \in V \Rightarrow \begin{cases} a+c = b+d & (E_1) \\ a+3d = c-2b & (E_2) \end{cases}$

$E_1 \Rightarrow \begin{cases} a = b+d-c \\ c = b+d-a \end{cases}$

$E_2 \Rightarrow a+3d = c-2b$

$\Rightarrow b+d-c+3d = b+d-a-2b \Leftrightarrow b-c+4d = -b+d-(b+d-c)$

$\Leftrightarrow b-c+4d = -2b + c \Leftrightarrow 3b+4d = 2c \Leftrightarrow$

$\Leftrightarrow c = \frac{3}{2}b + 2d$

$$A = \begin{pmatrix} -1/2\, b - d & b \\ 3/2\, b + 2d & d \end{pmatrix} = b \underbrace{\begin{pmatrix} -1/2 & 1 \\ 3/2 & 0 \end{pmatrix}}_{v_1} + d \underbrace{\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}}_{v_2}$$

Deci $B = \{v_1, v_2\}$ bază a lui V

$\dim_{\mathbb{R}} V = 2$

5) Fie $V = \{ P(x) \in \mathbb{R}_2[X] \mid a_1 + a_2 + a_0 = 0 \}$ unde

$P(x) = a_0 + a_1 x + a_2 x^2$

a) $V \subset \mathbb{R}_2[X]$ subsp. vect.

b) ~~Det~~ Det. o bază și $\dim_{\mathbb{R}} V = ?$

a) $P(x) = 0 \iff a_0 = a_1 = a_2 = 0$ (1)

Fie $P(x)$ și $Q(x) \in V$ unde $P(x) = a_0 + a_1 x + a_2 x^2$ și

$Q(x) = b_0 + b_1 x + b_2 x^2$

$P(x) + Q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$

$P(x) + Q(x) = 0$ $\Rightarrow$

$\Rightarrow a_0 + b_0 = a_1 + b_1 = a_2 + b_2 = 0 \Rightarrow P(x) + Q(x) \in V$ (2)

Fie $P(x) = a_0 + a_1 x + a_2 x^2$ și $\alpha \in \mathbb{R}$

~~$P(\alpha x) = a_0$~~ $\alpha P(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2$

$\alpha P(x) = 0$ $\Rightarrow$

$\Rightarrow \alpha \underbrace{(a_0 + a_1 x + a_2 x^2)}_{0 \in V} = 0 \Rightarrow$ (3)

Din (1), (2), (3) $\Rightarrow V \subset \mathbb{R}_2[X]$ subsp. vect.

b) Fie $p = p_0 + p_1 X + p_2 x^2$ $p \in V$

$p = 0 \iff p_2 = -p_0 - p_1$

$V = \{ p_0 + p_1 X + (-p_0 - p_1)x^2 \mid p_0, p_1 \in \mathbb{R} \}$

$V = \{ p_0 \underbrace{(1, 0, -x^2)}_{v_1} + p_1 \underbrace{(0, x, -x^2)}_{v_2} \mid p_0, p_1 \in \mathbb{R} \}$

$B = \{ v_1, v_2 \}$ s.g.

$\dim_{\mathbb{R}} V = 2$