27.03.2024

Seminar 5

Coord și schimbări de coord.

Fie V/K sp. vect, $\dim_K V = n < \infty$

$B, B' \subset V$ baze

$[v]_B \Rightarrow$ coord vect. v în B (spațiu)

$[v]_{B'} \Rightarrow$ —————— " —— B'

$B \rightsquigarrow B'$

$B \overset{S}{\rightsquigarrow} B'$

$[v]_B$

$\Rightarrow [v]_B = S[v]_{B'} \; \Big| \; S^{-1}$ la stg, $\det S \neq 0$

↳ mat. de trecere

$V/K \to$ sp. vect.

$B, B', B'' \subset V$

1) $B \xrightarrow{S} B' \Rightarrow B' \xrightarrow{S^{-1}} B$

2) $B \underset{S_1^{-1}}{\overset{S_1}{\rightleftarrows}} B' \underset{S_2^{-1}}{\overset{S_2}{\rightleftarrows}} B'' \Rightarrow B \underset{S_2^{-1} S_1^{-1}}{\xrightarrow{S_1 S_2}} B''$

$(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^t = B^t A^t$

K^n/K sp. vect

$B_0 = \{e_1 = (1, 0, \ldots 0), \ldots, e_n = (0, 0, \ldots 0, 1)\}$ bază arbitrară

$B = \{f_1, \ldots f_n\}$ bază arbitrară $\Bigg| \Rightarrow$

$\Rightarrow S = \begin{pmatrix} | & | & \ldots & | \\ f_1 & f_2 & & f_n \end{pmatrix} \in \mathcal{M}_n(K)$

$B_0 \overset{S}{\rightsquigarrow} B$ de trecere

① $\mathbb{R}^3/\mathbb{R}$ sp. vect

$B_1 = \{ f_1 = (0,1,2),\ f_2 = (1,0,2),\ f_3 = (2,2,2) \}$

$B_2 = \{ g_1 = (1,1,-1),\ g_2 = (1,-1,1),\ g_3 = (-1,1,1) \}$

a) Dem. că $B_1, B_2 \subset \mathbb{R}^3$ baze

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$$

$\det A_1 = \cancel{} 4 + 4 - 2 + 0 + 0 = 6 \neq 0 \Rightarrow B_1 \subset \mathbb{R}^3$

$\det A_2 = -4 \neq 0 \Rightarrow B_2 \subset \mathbb{R}^3$ bază

b) Det. coord vect f_1, f_2, f_3 în rap. cu B_2

$[f_1]_{B_2} = ?$ $\quad [f_2]_{B_2} = ?$ $\quad [f_3]_{B_2} = ?$

Varianta 1:

$B_2 \subset \mathbb{R}^3$ bază $\Rightarrow \forall v \in V^3\ \exists!\ \alpha, \beta, \gamma \in \mathbb{R}$ a.î.

$v = \alpha g_1 + \beta g_2 + \gamma g_3$

$[v]_{B_2} = (\alpha, \beta, \gamma)$

Luăm $v = f_1 \Rightarrow f_1 = \alpha_1 g_1 + \beta_1 g_2 + \gamma_1 g_3 \Rightarrow [f_1]_{B_2} = (\alpha_1, \beta_1, \gamma_1)$

$(0,1,2) = \alpha_1 (1,1,-1) + \beta_1 (1,-1,1) + \gamma (1,1,-1)$

$$\Leftrightarrow \begin{cases} \alpha_1 + \beta_1 - \gamma_1 = 0 \\ \alpha_1 - \beta_1 + \gamma_1 = 1 \\ -\alpha_2 + \beta_1 + \gamma_1 = 2 \end{cases} \Rightarrow \alpha_1 = ?\ \beta_1 = ?\ \gamma_1 = ?$$

$\det A_2 \neq 0 \Rightarrow$ sol. unică (Sist. Cramer)

Analog f_2, f_3

Varianta 2:

$[f_1]_{B_2}$, $B_0 \overset{S_2}{\rightsquigarrow} B_2$, $B_0 = \{ e_1 = (0,1,2); e_2 = (1,0,2); e_3 = (2,2,2) \}$ bază canonică

$$S_2 = \begin{pmatrix} | & | & | \\ g_1 & g_2 & g_3 \end{pmatrix} = A_2$$

$[v]_{B_2} = A_2 [v]_{B_0}$

Pt $v = f_1 \Rightarrow [f_1]_{B_2} = A_2 [f_1]_{B} \;\Big|\; \cdot A_2^{-1}$ la stg $\Rightarrow$

$\Rightarrow [f_1]_{B_2} = A_2^{-1} [f_1]_{B_0}$

$[f_2]_{B_2} = A_2^{-1} [f_2]_{B_0}$

$[f_3]_{B_2} = A_2^{-1} [f_3]_{B_0}$

$$B_1 \xrightarrow{(A_2)^{-1}} B_2$$

$$A_{12} = \begin{pmatrix} | & | & | \\ [f_1]_{B_1} & [f_2]_{B_1} & [f_3]_{B_1} \end{pmatrix}$$

c) $B_0 \xrightarrow[A_1]{} B_1 \xrightarrow[A_{12}]{} B_2 \quad ; \quad B_0 \xrightarrow{A_2} B_2$

$A_{12} = A_1^{-1} A_2$

$B_2 \xrightarrow{A_{21}} B_1 \iff A_{21} = (A_{12})^{-1} = (A_1^{-1} A_2)^{-1} = A_2^{-1} A_1$

Aplicații liniare (morf. de sp. vect.)

Fie $V, W / K$ sp. vect.

O apl. $f : V \to W$ s.n. apl. lin. dacă

1) $\forall\, v_1, v_2 \in V \Rightarrow f(v_1 + v_2) = f(v_1) + f(v_2)$

2) $\forall\, v \in V$ și $\alpha \in K \Rightarrow f(\alpha v) = \alpha f(v)$ $\iff$

$\iff \forall\, v_1, v_2 \in V,\ \alpha_1, \alpha_2 \in K \Rightarrow f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2)$

Ex: 1) $1_V : V \to V,\ 1_V(v) = v \quad \forall\, v \in V \to$ morf. id

2) $O : V \to W,\ O(v) = 0_W$ morf. nul

3) $Tr : \mathcal{M}_n(K) \to K$

$A \in \mathcal{M}_n(K) \mapsto$

$A \to Tr(A) = \sum_{i=1}^{n} a_{ii} \to$

$Tr(A+B) = Tr\,A + Tr\,B$

$Tr(\alpha A) = \alpha\, Tr\,A \quad ; \ \alpha \in K$

5) $\det : \mathcal{M}_n(K) \to K$ NU e apl. lin.

Avem $\det(A+B) \neq \det A + \det B$ în general

$\det(\alpha A) \neq \alpha \det A$, $\alpha \in K$

Dar $\det(AB) = \det A \det B$, $\forall A, B \in \mathcal{M}_m(K)$

Fie $f: V \to W$ apl. lin., $B_V \subset V$, $B_W \subset W$

$B_V = \{e_1, \dots, e_m\}$ și $B_W = \{f_1, \dots, f_m\}$

$[f]_{B_V, B_W} \to$ m. asoc. apl. lin f în rap. cu bela B_V și B_W

$B_V \xrightarrow{S} B'_V \subset V$ și $B_W \xrightarrow{T} B'_W \subset W$

$[f]_{B'_V, B'_W} = T^{-1} [f]_{B_V, B_W} S$

Dacă era endomorf. $f: V \to V \Rightarrow$

$\Rightarrow B \xrightarrow{S} B \Rightarrow [f]_{B'} = S^{-1} [f]_B S$

$f: V \to W$ apl. lin.

$\operatorname{Ker} f = \{v \in V \mid f(v) = 0_W\} \subseteq V$

$\quad\; = f^{-1}(0_W)$

$\operatorname{Im} f = \{w \in W \mid \exists v \in V \text{ a.î. } f(v) = w\} \subseteq W$

Teorema rang defect

Fie $f: V \to W$ apl. lin., $\dim_K V < \infty$

Atunci $\dim_K \operatorname{Ker} f + \dim_K \operatorname{Im} f = \dim_K V$

(2) $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (x+y, x-y, y)$

a) f apl. lin?

$f(X) = AX$ unde $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_{3,2}(\mathbb{R})$

Fie $\left.\begin{matrix} X_1, X_2 \in \mathbb{R}^2 \\ \alpha_1, \alpha_2 \in \mathbb{R} \end{matrix}\right| \Rightarrow f(\alpha_1 X_1 + \alpha_2 X_2) = A(\alpha_1 X_1 + \alpha_2 X_2) =$

$= \alpha_1 (AX_1) + \alpha_2 (AX_2) = \alpha_1 f(X_1) + \alpha_2 f(X_2) \Rightarrow f$ apl. lin.

b) $[f]_{B_0, B'_0} = A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$ $f(e_1) = f(1,0)$

$B_0 = [e_1, e_2] \subset \mathbb{R}^2$ $f(e_2) = f(0,1)$

$\underbrace{}_{f(e_1)} \underbrace{}_{f(e_2)}$

c) $B_1 = \{ f_1 = (1,2),\ f_2 = (2,1) \}$

$B_1' = \{ f_1' = (1,1,0),\ f_2' = (1,0,1),\ f_3' = (0,1,1) \} \subset \mathbb{R}^3$

Th $\Rightarrow$ $\underbrace{[f]_{B_1,B_1'}}_{A} = \underset{(3,3)}{T^{-1}} \cdot \underset{(3,2)}{A} \cdot \underset{(2,2)}{S}$

$B_0 \xrightarrow{S} B_1$ unde $S = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$B_0' \xrightarrow{T} B_1'$ unde $T = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ \quad f_1' \quad f_2' \quad f_3' \end{array}$

(3)

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$f(x,y) = (x+y,\ x,\ -y)$

a) apl. lin. T.A.

b) $\operatorname{Ker} f = ?$ $\operatorname{Im} f = ?$

$\operatorname{Ker} f = \{ v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid f(x) = 0_{\mathbb{R}^2} \} \subset \mathbb{R}^2$ subsp. vect.

$f(x,y) = (0,0,0) \Leftrightarrow \begin{cases} x+y = 0 \\ x = 0 \\ -y = 0 \end{cases} \Rightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{M}_{3,2}\,\mathbb{R}$

$rg\ A = 2 \Rightarrow x = y = 0$ sol. unică

$\operatorname{Ker} f = \{ 0_{\mathbb{R}^2} \} \subset \mathbb{R}^2$

$\operatorname{Im} f = \{ w \in \mathbb{R}^3 \mid \exists!\ v \in \mathbb{R}^2$ a.î. $f(v) = w \}$

$\operatorname{Im} f = \{ (x', y', z') \in \mathbb{R}^3 \mid x' - y' + z' = 0 \} \subset \mathbb{R}^3$

c) f inj. și surj., bij.

1) f inj. $\Leftrightarrow \operatorname{Ker} f = \{ 0_V \}$ $\Rightarrow f$ inj.

2) f surj. $\Leftrightarrow \operatorname{Im} f = W$ $\Rightarrow f$ nu e surj. $\Big| \Rightarrow$

3) f bij. $\Leftrightarrow \begin{cases} \operatorname{Ker} f = \{ 0_V \} \\ \operatorname{Im} f = W \end{cases}$

$\Rightarrow f$ nu e bij.

T.A $\Rightarrow$ d) T rg. def? $\Rightarrow \underbrace{\dim \operatorname{Ker} f}_{0} + \underbrace{\dim_{\mathbb{R}} \operatorname{Im} f}_{2} = \underbrace{\dim_{\mathbb{R}} \mathbb{R}^2}_{2}$