Calcul dif. & integr.

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ŞIRURI DE NR. REALE

① Se consideră şirul $x_n = \frac{2n}{n^2+1} \cdot (-1)^n$

a) $\lim\limits_{x \to n} x_n = ?$ b) monotonia şirului

a) $\lim\limits_{x \to n} \frac{2n}{n^2+1} \cdot (-1)^n = ?$ $\Rightarrow$ $-\frac{2n}{n^2+1} \leq x_n \leq \frac{2n}{n^2+1}$ $\forall n \in \mathbb{N}$

$-\frac{2n}{n^2+1} \xrightarrow{n \to 0} 0$, $x_n \to 0$, $\frac{2n}{n^2+1} \xrightarrow{n \to 0} 0$ (criteriul cleştelui)

$\lim\limits_{x \to n} (-1)^n \Rightarrow \lim \nexists$

$\lim\limits_{x \to n} x_n = 0$

b) $x_0 > x_1 \Leftrightarrow 0 > -1$
$x_1 < x_2 \Leftrightarrow -1 < \frac{4}{5}$
$x_2 > x_3$
$x_3 < x_4$
$\Rightarrow (x_n)$ nu e monoton

② a) $\lim\limits_{n \to \infty} \frac{a^n}{n!}$; $a > 0$

b) $\lim\limits_{n \to \infty} \frac{4 \cdot 7 \cdot \ldots \cdot (3n+1)}{6 \cdot 10 \cdot \ldots \cdot (4n+2)}$

c) $\lim\limits_{n \to \infty} \frac{\sqrt[n]{n!}}{n}$

d) $\lim\limits_{n \to \infty} \frac{(n+1)^p}{1^p + 2^p + \ldots + n^p}$; $p > 0$

a) I: $a \in (0,1) \Rightarrow \lim\limits_{n \to \infty} \frac{a^n}{n!} = \frac{0}{+\infty} = 0$

II: $a = 1 \Rightarrow \lim\limits_{n \to \infty} \frac{a^n}{n!} = \frac{1}{+\infty} = 0$

III: $a \in (1; +\infty) \Rightarrow \lim\limits_{n \to \infty} \frac{a^n}{n!}$

Aplicăm „criteriul raportului" pt şiruri cu term. >0

$x_n = \frac{a^n}{n!}$; $n > 1 \Rightarrow x_n > 0$ $\forall n \in (1; +\infty)$

$\lim\limits_{n \to \infty} \frac{x_{n+1}}{x_n}$ comparăm cu 1

! Nu se poate folosi criteriul dacă lim = 1

$$\left\{\begin{array}{l}\lim\limits_{m\to\infty}\dfrac{x_{m+1}}{x_m}<1 \Rightarrow \lim\limits_{m\to\infty} x_m=0\\ \lim\limits_{m\to\infty}\dfrac{x_{m+1}}{x_m}>1 \Rightarrow \lim\limits_{m\to\infty} x_m=+\infty\end{array}\right. \quad \Rightarrow \lim\limits_{m\to\infty} x_m=0$$

$$\lim_{m\to\infty}\frac{a^{m+1}}{(m+1)!}\cdot\frac{m!}{a^m}=\lim_{m\to\infty}\frac{a}{m+1}=0<1$$

b) $$\lim_{m\to\infty} x_m=\lim_{m\to\infty}\frac{4\cdot 7\cdot\ldots\cdot(3m+1)}{6\cdot 10\cdot\ldots\cdot(4m+2)}=\lim_{m\to\infty}\frac{a_m}{b_m}$$

Putem aplica Stolz-Cesaro, mai simplu „criteriul raportului"

$x_m>0 \ \forall m\in\mathbb{N} \Rightarrow \lim\limits_{m\to\infty}\dfrac{4\cdot 7\cdot\ldots\cdot(3m+1)(3m+4)}{6\cdot 10\cdot\ldots\cdot(4m+2)(4m+6)}\cdot\dfrac{6\cdot 10\cdot\ldots\cdot(4m+2)}{4\cdot 7\cdot\ldots\cdot(3m+1)}=$

$$=\lim_{m\to\infty}\frac{x_{m+1}}{x_m}=\lim_{m\to\infty}\frac{3m+4}{4m+6}=\frac{3}{4}<1\Rightarrow\lim_{m\to\infty}x_m=0$$

$$\frac{3m+1}{4m+2}<\frac{3m+2}{4m+2} \quad \frac{4}{5}\ (\text{sc})$$

c) $$\lim_{m\to\infty}a_m=\lim_{m\to\infty}\frac{\sqrt[m]{m!}}{m}$$

Aplicăm criteriul radicalului pt. șiruri cu term. >0

Se introduc termenii sub radical.

$$a_m=\sqrt[m]{\frac{m!}{m^m}}$$

Notăm șirul de sub $\sqrt{\ }$ cu b_m

$$a_m=\sqrt[m]{b_m}$$

Calculăm $\lim\limits_{m\to\infty}\dfrac{b_{m+1}}{b_m}$

$$\lim_{m\to\infty}\frac{b_{m+1}}{b_m}=\lim_{m\to\infty}\frac{(m+1)!}{(m+1)^{m+1}}\cdot\frac{m^m}{m!}=\lim_{m\to\infty}\frac{m^m}{(m+1)^m}=$$

$$=\lim_{m\to\infty}\left(\frac{m}{m+1}\right)^m=1^\infty=\lim_{m\to\infty}\left(1+\frac{m}{m+1}-1\right)^m=$$

$$=\lim_{m\to\infty}\left(1+\frac{-1}{m+1}\right)^m=\lim_{m\to\infty}\left[\left(1+\frac{-1}{m+1}\right)^{-(m+1)}\right]^{\frac{m}{-(m+1)}}=$$

$$=e^{\lim\limits_{m\to\infty}\frac{-m}{m+1}}=e^{-1}=\frac{1}{e}$$

$$\lim_{m\to\infty}\frac{b_{m+1}}{b_m}=\frac{1}{e}\Rightarrow\lim_{m\to\infty}\sqrt[m]{b_m}=\frac{1}{e}=\lim_{m\to\infty}a_m$$

d) $x_n = \dfrac{(n+1)^p}{1^p + 2^p + \dots + n^p}$; $p > 0$

Aplicăm lema Cesaro-Stolz. (cazul $\frac{\infty}{\infty}$). Avem voie?

Alegem 2 șiruri.

$x_n = \dfrac{a_n}{b_n} \Rightarrow \begin{cases} a_n = (n+1)^p \\ b_n = 1^p + 2^p + \dots + n^p \end{cases}$

Se calculează $\lim b_n$

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} 1^p + 2^p + \dots + n^p = +\infty$

- Dacă $\begin{cases} \lim\limits_{n\to\infty} b_n = +\infty \Rightarrow \text{str. cresc (să se dem.)} \\ \lim\limits_{n\to\infty} b_n = -\infty \Rightarrow \text{str. descresc (să se dem.)} \end{cases}$

- Dacă $\lim\limits_{n\to\infty} b_n = 0 \Rightarrow$ dem. că $(b_n)_{n\in\mathbb{N}}$ este str. mon.
  $\lim\limits_{n\to\infty} a_n = 0$

Dem că b_n str. cresc.

$b_{n+1} - b_n = 1^p + 2^p + \dots + (n+1)^p - 1^p - 2^p - \dots - n^p = (n+1)^p > 0 \ \forall n \in \mathbb{N} \Rightarrow$

$\Rightarrow (b_n)$ str. cresc.

Se calculează $\lim\limits_{n\to\infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} \in \overline{\mathbb{R}}$

$\lim\limits_{n\to\infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim\limits_{n\to\infty} \dfrac{(n+2)^p - (n+1)^p}{(n+1)^p} = \lim\limits_{n\to\infty} \left(\dfrac{n+2}{n+1}\right)^p - 1 =$

$= \cancel{\lim\limits_{n\to\infty}} = 1 - 1 = 0$

Aplicăm lema C-S. și obținem $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$

$\lim\limits_{n\to\infty} x_n = 0$

Teoremă: Șirul $x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$, $n \in \mathbb{N}^*$ este convergent

Dem. că este monoton și mărg. (conv.) TEMĂ

$\lim\limits_{n\to\infty} 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n = c \in (0,1)$ CONSTANTA LUI EULER

3) $\lim\limits_{n\to\infty} 1 + \frac{1}{2} + \dots + \frac{1}{n} = \lim\limits_{n\to\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n\right) + \ln n = c + \lim\limits_{n\to\infty} \ln n =$

$= c + \infty = +\infty$