28.02.2025

Seminar 1

- sisteme de ecuații liniare -

- Metoda Gauss - Jordan

(1) $(S): \sum_{j=1}^{m} a_{ij} x_j = b_i \ ; \ \forall i = \overline{1,m} \ ; \ a_{ij}, b \in K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_{prim}\}$

(m ecuații liniare, m necunoscute)

$A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,m}}} \in \mathcal{M}_{m,m}(\mathbb{R})$

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \quad \Rightarrow AX = B$

$\bar{A} = (A \mid B)$ matricea extinsă

$S_1 \sim S_2 \Leftrightarrow \mathcal{S}_1 = \mathcal{S}_2$ (sist echiv dacă au ac. soluții)

$A_1 \sim A_2$ mat. echiv

Transf. elem $\longrightarrow$ 1) $L_i \leftrightarrow L_j \quad i \neq j$

2) $L_i \leftarrow a L_j \ , \ a \in K^*$

3) $L_i \leftarrow L_i + \beta L_j \ , \ \beta \in K$

Orice matrice poate fi adusă printr-un nr. finit de transf. elementare la forma eșalon standard.

- Algoritm

$AX = B$

$\bar{A} = (A \mid B) \rightarrow$ aducem la forma eșalon $= \bar{E}$

Interpretări

Nr total al pivoților = rangul mat. extinse $\bar{A}$

① a) $\begin{cases} x+y-z=1 \\ x-y+z=2 \\ 2x+y+2z=4 \end{cases}$ b) $\begin{cases} 2x+y-z+t=1 \\ x-y+2z+t=3 \\ x+2y-3z-2t=6 \end{cases}$

a) $A=\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$ $B=\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$

$(A|B)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & 2 & 4 \end{array}\right)=\overline{A}$ mat extinsă a sist

Aducem $\overline{A}$ la forma eșalon redusă $\overline{E}$

$$\overline{A}=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & 2 & 4 \end{array}\right) \xrightarrow[L_3 \leftarrow L_3-2L_1]{L_2 \leftarrow L_2-L_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 1 \\ 0 & -1 & 4 & 2 \end{array}\right)$$

$$\xrightarrow{L_2 \leftarrow \left(-\frac{1}{2}\right)L_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1/2 \\ 0 & -1 & 4 & 2 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3+L_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 3 & 3/2 \end{array}\right)$$

$$\xrightarrow[\text{forma eș. standard}]{L_3 \leftarrow \left(\frac{1}{3}\right)L_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right) \xrightarrow{L_1 \leftarrow L_2-L_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

$$\xrightarrow{L_2 \leftarrow L_2+L_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{array}\right)=\overline{E}$$ sist comp. det $\Rightarrow$

$\Rightarrow x=3/2$ $y=0$ și $z=1/2$ $\Rightarrow S=\{(3/2, 0, 1/2)\}$

b) $$\overline{A}=(A|B)=\left(\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 1 \\ 1 & -1 & 2 & 1 & 3 \\ 1 & 2 & -3 & -2 & 6 \end{array}\right) \xrightarrow{L_1 \leftrightarrow L_2}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 3 \\ 2 & 1 & -1 & 1 & 1 \\ 1 & 2 & -3 & -2 & 6 \end{array}\right) \xrightarrow{L_2 \leftarrow L_2-2L_1} \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 3 \\ 0 & 3 & -5 & -1 & -5 \\ 0 & 3 & -5 & -3 & 3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 3 \\ 0 & 1 & -\frac{5}{3} & -1/3 & -5/3 \\ 0 & 3 & -5 & -3 & -3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1/3 & 2/3 & 4/3 \\ 0 & 1 & -5/3 & -1/3 & -5/3 \\ 0 & 0 & 0 & -2 & 8 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1/3 & 2/3 & 4/3 \\ 0 & 1 & -5/3 & -1/3 & -5/3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1/3 & 0 & 4 \\ 0 & 1 & -5/3 & 0 & -3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right) = E$$

$\overline{E}$ mat camp redus (simplu)

x, y, t nec principale

$z = \alpha$ nec sec, $\alpha \in \mathbb{R}$

$$\begin{cases} x + \frac{1}{3}\alpha = 4 \\ y - \frac{2}{3}\alpha = -3 \\ t = -4 \end{cases} \iff \begin{cases} x = 4 - \frac{1}{3}\alpha \\ y = \frac{2}{3}\alpha - 3 \\ t = -4 \end{cases}$$

$$S = \left\{ \left(4 - \tfrac{1}{3}\alpha ;\ \tfrac{2}{3}\alpha - 3 ;\ \alpha ;\ -4\right) \mid \alpha \in \mathbb{R} \right\}$$

$$S = \left\{ \underbrace{(4, -3, 0, -4)}_{v_1} + \alpha \underbrace{\left(-\tfrac{1}{3}, \tfrac{5}{3}, 1, 0\right)}_{v_2} \mid \alpha \in \mathbb{R} \right\}$$

$S = \{ v_1 + \alpha v_2 \mid \alpha \in \mathbb{R} \}$ unde $v_1, v_2 \in \mathbb{R}^4$

- Det. inversei unei mat. pătratice (dacă $\exists$)

Fie $A \in \mathcal{M}_m(K)$

$(A \mid I_m) \in \mathcal{M}_{m,2m}(K) \xrightarrow{G-J} E = (B \mid C)$ eș. red.

Forma eșalon redusă pt $\forall$ mat A din $\mathcal{M}_m(K)$ este mat unitate I_m

Temă a) $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 3 & 2 \\ 4 & 3 & -2 \end{pmatrix}$ inv. dacă $\exists$

b) $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$$\overline{A} = (A \mid I_3) = \left(\begin{array}{ccc|ccc} \boxed{1} & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} \boxed{1} & 1 & -1 & 1 & 0 & 0 \\ 0 & \boxed{-2} & 2 & -1 & 1 & 0 \\ 0 & -1 & 4 & -2 & 0 & 1 \end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc} \boxed{1} & 1 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & -1 & 1/2 & -1/2 & 0 \\ 0 & -1 & 4 & -2 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & \boxed{1} & -1 & 1/2 & -1/2 & 0 \\ 0 & 0 & 3 & -3/2 & -1/2 & 1 \end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & \boxed{1} & -1 & 1/2 & -1/2 & 0 \\ 0 & 0 & \boxed{1} & -1/2 & -1/2 & 1/3 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & -2/3 & 1/3 \\ 0 & 0 & 1 & -1/2 & -1/6 & 1/3 \end{array}\right)$$

$\underbrace{}_{I_3}$ $\underbrace{}_{A^{-1}}$

este inversabilă $\Rightarrow A^{-1}$ inversa

Temă

① a) $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 3 & 2 \\ 4 & 3 & -2 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

c) $B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ d) $C = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & -2 \\ 1 & -1 & 4 \end{pmatrix}$

a) $\bar{A} = (A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 3 & 2 & 0 & 1 & 0 \\ 4 & 3 & -2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_2 \leftarrow L_2 + L_1}$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 2 & 0 \\ 4 & 3 & -2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_3 \leftrightarrow L_1} \left(\begin{array}{ccc|ccc} 4 & 3 & -2 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 2 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_1 \leftarrow L_1 - 3L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & -3 \\ 0 & 4 & 4 & 0 & 2 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 - L_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & -3 \\ 0 & 4 & 4 & 0 & 2 & 0 \\ 0 & 1 & -2 & -1 & 0 & 4 \end{array}\right)$$

$$\xrightarrow{L_2 \leftrightarrow L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & -3 \\ 0 & 1 & -2 & -1 & 0 & 4 \\ 0 & 4 & 4 & 0 & 2 & 0 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 - 4L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & -3 \\ 0 & 1 & -2 & -1 & 0 & 4 \\ 0 & 0 & 12 & 4 & 2 & -16 \end{array}\right)$$

$$\xrightarrow{L_3 \leftarrow \frac{1}{12} L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & -3 \\ 0 & 1 & -2 & -1 & 0 & 4 \\ 0 & 0 & 1 & 1/3 & 1/6 & -3/4 \end{array}\right) \xrightarrow[L_2 \leftarrow L_2 + 2L_3]{L_1 \leftarrow L_1 - 4L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & -2/3 & 0 \\ 0 & 1 & 0 & \cancel{-1/3} & \cancel{1/3} & 5/2 \\ 0 & 0 & 1 & 1/3 & 1/6 & -3/4 \end{array}\right)$$

b) $\bar{A} = (A \mid I_2) = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \xrightarrow{L_1 \leftrightarrow L_2} \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right)$

$$\sim\left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -5 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & -1/5 & 2/5 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 3/5 & -1/5 \\ 0 & 1 & -1/5 & 2/5 \end{array}\right)$$

c) $\bar{A} = (A \mid I_3) = \left(\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 0 & 1 & 0 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right) \sim \cancel{\left(\begin{array}{ccc|ccc} -3 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 3 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 3 & 0 & -1 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -2/3 & 1 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

d) $\bar{A} = (A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 1 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \overset{L_1 - L_3 \to L_1}{\sim}$

$$\left(\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & -1 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 1 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & -1 \\ 1 & 2 & 2 & 0 & 1 & -1 \\ 1 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & -1 \\ 1 & 0 & 3 & -1 & 1 & 0 \\ 1 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & -1 \\ 0 & 1 & 7 & -1 & 1 & -1 \\ 1 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & -15 & 3 & -2 & 1 \\ 0 & 1 & 7 & -1 & 1 & -1 \\ 1 & -1 & 4 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & -1/5 & 2/15 & -1/15 \\ 0 & 1 & 7 & -1 & 1 & -1 \\ 1 & -1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & -1/5 & 2/15 & -1/15 \\ 0 & 1 & 7 & -1 & 1 & -1 \\ 1 & 0 & 11 & -1 & 1 & 0 \end{array}\right) \sim \cancel{\left(\begin{array}{ccc|} 0 & 0 & 1 \end{array}\right.}$$

$$\sim \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & -1/5 & 2/15 & -1/15 \\ 0 & 1 & 0 & -2/5 & 1/15 & -8/15 \\ 1 & 0 & 11 & -1 & 1 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 11 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2/5 & 1/15 & -8/15 \\ 0 & 0 & 1 & -1/5 & 2/15 & -1/15 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6/5 & -7/15 & 11/15 \\ 0 & 1 & 0 & -2/5 & 1/15 & -8/15 \\ 0 & 0 & 1 & -1/5 & 2/15 & -1/15 \end{array}\right)$$