Tema

① $f: \mathbb{R}^3 \to \mathbb{R}^3$, ~~$f(x,y,z) = (2x+y+z,\ x-y+z,\ x-y-z)$~~

$f(x,y,z) = (2x-y+2z,\ -x+2y-z,\ x+y+z)$

a) $A_f = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

b) $\det(A_f - \lambda \cdot I_3) = 0 \Rightarrow \left| \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| =$

$$= \begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$[(2-\lambda)^2(1-\lambda)] + (-1)\cdot 1\cdot 2 + (-1)(-1)\cdot 1 - 2(2-\lambda) - 1(-1)(2-\lambda) - (-1)(-1)(1-\lambda) =$

$= (2-\lambda)^2(1-\lambda) - 2 + 1 - 2(2-\lambda) + (2-\lambda) - (1-\lambda) =$

$= (2-\lambda)^2(1-\lambda) - 1 - (2-\lambda) - (1-\lambda) =$

$= (2-\lambda)^2(1-\lambda) - 1 - 2 + \lambda - 1 + \lambda =$

$= (1-\lambda)(2-\lambda)^2 - 4 + 2\lambda = (1-\lambda)(2-\lambda)^2 - 2(2-\lambda) =$

$= (2-\lambda)[(1-\lambda)(2-\lambda) - 2] =$

$= (2-\lambda)\{\cancel{2} - \lambda - 2\lambda + \lambda^2 - \cancel{2}\} = \underbrace{(2-\lambda)}_{\lambda_1 = 2}(\underbrace{\lambda^2 - 3\lambda}_{\lambda^2 - 3\lambda = 0}) = 0$

$\lambda^2 - 3\lambda = 0 \Leftrightarrow \lambda(\lambda - 3) = 0$, $\lambda = 0$, $\lambda = 3$

Deci val. proprii: $\lambda_1 = 2$, $\lambda_2 = 0$, $\lambda_3 = 3$

$V_{\lambda_1} = \left\{ v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; f(v) = \lambda_1 v \right\} \Rightarrow (A_f - \lambda_1 I_3)v = 0_{\mathbb{R}^3} \Rightarrow$

$\Leftrightarrow \underbrace{\begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}}_{rg = 2} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -y + 2z = 0 \Leftrightarrow 2z = y \\ -x - z = 0 \\ x + y - z = 0 \end{cases} \Rightarrow y = 2\alpha$ și $x = -\alpha$

$rg = 2 \Rightarrow z =$ nec. sec. $= \alpha$

$x, y =$ nec. principale

$V_{\lambda_1} = \{ \alpha \underbrace{(-1, 2, 1)}_{v_1} \mid \alpha \in \mathbb{R} \} = \langle v_1 \rangle \Rightarrow \dim_{\mathbb{R}} V_{\lambda_1} = 1$

$V_{\lambda_2} = \left\{ v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; f(v) = \lambda_2 v \right\} \Rightarrow (A_f - \lambda_1 I_3)v = 0_{\mathbb{R}^3}$

$$\underbrace{\begin{pmatrix}2&-1&2\\-1&2&-1\\1&1&1\end{pmatrix}}_{rg=2}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{cases}2x-y+2z=0\\-x+2y-z=0\\x+y+z=0\end{cases}\Rightarrow\begin{cases}x+y=-\alpha\\-x+2y=\alpha\end{cases}\Rightarrow$$

$rg = 2 \Rightarrow z =$ nec. secundară $= \alpha$

$x, y =$ nec. princip.

$\Rightarrow -x-2\alpha-2x=\alpha \Leftrightarrow x=-\alpha \Rightarrow y=-\alpha-\alpha=0$

$V_{\lambda_2}=\{\alpha\underbrace{(-1,0,1)}_{v_2} \mid \alpha\in\mathbb{R}\} = \langle v_2\rangle \Rightarrow \dim_{\mathbb{R}} V_{\lambda_2}=1$

$V_{\lambda_3}=\{v=\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \mid f(v)=\lambda_3 v\}$

$$\underbrace{\begin{pmatrix}-1&-1&2\\-1&-1&-1\\1&1&-2\end{pmatrix}}_{rg=2}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{cases}-x-y+2z=0\\-x-y-z=0\\x+y-2z=0\end{cases}\Rightarrow\begin{cases}y+z=-\alpha\\y-2z=-\alpha\end{cases}\Rightarrow$$

$rg=2 \Rightarrow x =$ nec. sec. $=\alpha$

$y, z =$ nec. princip.

$\Rightarrow z=0$ și $y=-\alpha$

$V_{\lambda_3}=\{\alpha\underbrace{(1,-1,0)}_{v_3} \mid \alpha\in\mathbb{R}\}=\langle v_3\rangle \Rightarrow \dim_{\mathbb{R}} V_{\lambda_3}=1$

c) diag?

$m_a(\lambda_1)+m_a(\lambda_2)+m_a(\lambda_3)=\dim_{\mathbb{R}}\mathbb{R}^3 \Leftrightarrow 1+1+1=3$ (adev)

$m_a(\lambda_1)=m_g(\lambda_1)=1$ și $m_a(\lambda_2)=m_g(\lambda_2)=1$ și $m_a(\lambda_3)=m_g(\lambda_3)=1$ $\Rightarrow$

$\Rightarrow f$ diagonalizabilă

d) Forma diag $D=\begin{pmatrix}2&0&0\\0&0&0\\0&0&3\end{pmatrix}=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}$

② $f:\mathbb{R}^3\to\mathbb{R}^3,\ f(x,y,z)=(x+4y,\ 2y+3z,\ y)$

Fie $v_1=(x_1,y_1,z_1)$, $v_2=(x_2,y_2,z_2)$ $\Rightarrow f(v_1+v_2)=f(x_1+x_2+4y_1-4y_2,\ 2y_1+2y_2+3z_1+3z_2,\ y_1+y_2)=$
$=f(v_1)+f(v_2)$ (1)

Fie $v=(x,y,z)$ și $\alpha\in\mathbb{R} \Rightarrow f(\alpha v)=(\alpha x+\alpha 4y,\ \alpha 2y+\alpha 3z,\ \alpha y)=\alpha(x+4y,\ 2y+3z,\ y)=$
$=\alpha f(v)$ (2)

(1) + (2) $\Rightarrow$ apl. lin.