10.04.2024

Seminar 8

Diag. endomorf.

① $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (x+4y, 2y+3z, y)$

a) f apl. lin. (endomorf. lin) TEMĂ

$f(X) = A_f X \iff f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} X$

luăm 2 scalari și arătăm ↑

b) mat. asoc. endomorf. în rap cu baza canonică

$B_0 = \{e_1, e_2, e_3\}$, practic $A_f = ?$

$f(e_1) = (1,0,0)$; $f(e_2) = (4,2,1)$ și $f(e_3) = (0,3,0)$

c) det val. și subsp. proprii coresp.

$K = \mathbb{R}$

Rezolvăm ec. $P(\lambda) = 0$

$\det(A_f - \lambda I_3) = 0 \iff \begin{vmatrix} 1-\lambda & 4 & 0 \\ 0 & 2-\lambda & 3 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \iff (1-\lambda)(\lambda^2 - 2\lambda - 3) = 0$

$\lambda \in \{\pm 1, 3\}$ val. proprii

$m_a(\lambda_1) = m_a(\lambda_2) = m_a(\lambda_3) = 1$

Ssp. proprii

$V_{\lambda_1} = \{v \in \mathbb{R}^3 \mid A_f \cdot v = \lambda_1 v\}$ unde $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$(A_f - I_3) v = 0_{\mathbb{R}^3} \iff \underbrace{\begin{pmatrix} 0 & 4 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & -1 \end{pmatrix}}_{C} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\det C = 0 \Rightarrow \operatorname{rg} C = 2 \Rightarrow \Delta_p = \begin{vmatrix} 4 & 0 \\ 1 & 3 \end{vmatrix} \Rightarrow x$ nec. sec. $\Rightarrow$

$\Rightarrow x = \alpha \to z = y = 0$

$V_{\lambda_1} = \{\alpha \underbrace{(1,0,0)}_{e_1} \mid \alpha \in \mathbb{R}\}$

$V_{\lambda_2} = \{ v \in \mathbb{R}^3 \mid A_f v = \lambda_2 v \}$

$(A_f - \lambda_2 I_3) v = O_{3,1}$

$$\underbrace{\begin{pmatrix} -2 & 4 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{pmatrix}}_{C} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\det C = 0 \Rightarrow \Delta_p = \begin{vmatrix} -2 & 4 \\ 0 & -1 \end{vmatrix} \Rightarrow x, y$ nec. princip. $\Rightarrow$

$\Rightarrow z = \alpha \Rightarrow \begin{cases} -2x + 4y = 0 \\ -y = -3\alpha \\ y = 3\alpha \end{cases} \Rightarrow y = 3\alpha \Rightarrow 2x = 12\alpha \Rightarrow x = 6\alpha$

$V_{\lambda_2} = \{ \alpha \underbrace{(6, 3, 1)}_{v_2} \mid \alpha \in \mathbb{R} \} = \langle v_2 \rangle$

$V_{\lambda_3} = \{ v \in \mathbb{R}^3 \mid A_f v = \lambda_3 v \}$

$$\underbrace{\begin{pmatrix} -2 & 4 & 0 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \end{pmatrix}}_{\det = 0} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow z = \alpha \Rightarrow \begin{cases} 2x + 4y = 0 \\ 3y = -3\alpha \\ y = -\alpha \end{cases} \Rightarrow x = 2\alpha$

$V_{\lambda_3} = \{ \alpha \underbrace{(2, -1, 1)}_{v_3} \mid \alpha \in \mathbb{R} \} = \langle v_3 \rangle$

$\dim_{\mathbb{R}} V_{\lambda_1} = \dim_{\mathbb{R}} V_{\lambda_2} = \dim_{\mathbb{R}} V_{\lambda_3} = 1$

$B_1 = \{ e_1 \} \subset V_{\lambda_1};\ B_2 = \{ v_2 \} \subset V_{\lambda_2},\ B_3 = \{ v_3 \} \subset V_{\lambda_3}$

$m_g(\lambda_1) = m_g(\lambda_2) = m_g(\lambda_3) = 1$

d) Stab. endom. f e diag.? Afirmative $\Rightarrow f$ diag

$\begin{cases} m_a(\lambda_1) + m_a(\lambda_2) + m_a(\lambda_3) = 3 = \dim_{\mathbb{R}} \mathbb{R}^3 \\ m_g(\lambda_i) = m_a(\lambda_i) \quad \forall i = \overline{1,3} \end{cases}$ ✓

Este diagonalizabilă $\Rightarrow D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow f$ diag

e) Verif. rez. obținut

$$\begin{array}{ccl} B_0 & \xrightarrow{S} & B = B_1 \cup B_2 \cup B_3 = \{e_1, v_2, v_3\} \subset \mathbb{R}^3 \\ \downarrow & & \downarrow \\ A_f & & D \quad \text{(în rap. cu care se realiz. f. diag)} \end{array}$$

S = mat. de trecere de la B_0 canonică la B

$D = S^{-1} A_f S$ (*) unde $S = \begin{array}{c} \begin{pmatrix} 1 & 6 & 2 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix} \\ \;\; e_1 \quad v_2 \quad v_3 \end{array}$

f) Calc $A_f^n = ?$ $n \in \mathbb{N}^*$

(*) $A_f = S D S^{-1}$

$$A_f^n = S D \underbrace{S^{-1} S}_{I_3} D \underbrace{S^{-1} \cdot}_{I_3} \cdot \underbrace{\cdot S}_{I_3} D S^{-1} = S D^n S^{-1}$$

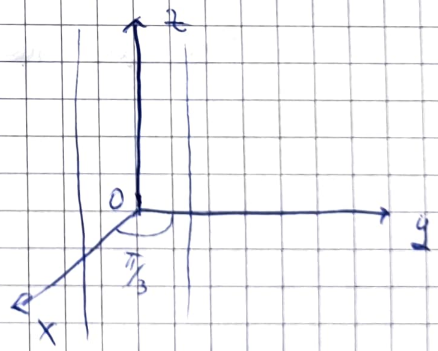
$$D^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}$$

Interpretări geom.

① $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$f = R_{Oz}, \theta = \frac{\pi}{3}$

$f(x) = R_{Oz, \theta}$ (x), $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



$$R_{Oz,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$f(x,y,z) = \left(\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y, z\right) \xleftarrow{\theta = \pi/3}$$

$$\Rightarrow \underbrace{\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A_f = R_{Oz, \theta = \pi/3}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Val. proprii: $\to$ $Spec(g) = \{1\}$

$K = \mathbb{R}$, Ec. car. $\det(A_g - \lambda I_3) = 0 \Rightarrow$

$$\Rightarrow \begin{vmatrix} 1/2 - \lambda & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow (1-\lambda)(-1)\left[(1/2 - \lambda)^2 + \frac{3}{4}\right] = 0$$

$$(1-\lambda)(\lambda^2 - \lambda + 1) = 0 \to \lambda_1 = 1 \in \mathbb{R}$$

$$\Downarrow \quad \lambda_{2,3} \in \mathbb{C}, \quad \lambda_2 = \overline{\lambda_3}$$

Sp. proprii:

$$V_{\lambda_1} = \{ v \in \mathbb{R}^3 \mid f(v) = \underset{1}{\overset{\shortparallel}{\lambda_1}} v \} \Rightarrow f(u) = u$$

$$(A_g - I_3)v = 0$$

$$rg(A_g - I_3) = 2 \Rightarrow \Delta_p = \begin{vmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{vmatrix} = 1 \neq 0$$

x, y nec. princip.

$$\begin{cases} -1/2\, x - \sqrt{3}/2\, y = 0 \\ \sqrt{3}/2\, x - 1/2\, y = 0 \\ z = \alpha \end{cases} \Bigg| \to x = y = 0 \text{ și } z = \alpha$$

$$V_{\lambda_1} = \{\alpha(0,0,1) \mid \alpha \in \mathbb{R}\} = \langle e_3 \rangle \to Oz$$

$\hookrightarrow$ axa de rotație