Calc. dif & int

16.10.2023

## Seminar 3

### Serii de nr. reale

- Criteriul de divergență:

Dacă $\exists \lim_{n\to\infty} x_n \neq 0$ sau $\nexists \lim_{n\to\infty} x_n \Rightarrow$ seria $\sum_{n\geq 0} x_n$ divergentă

- Criteriul lui Raabe - Duhamel pt serii cu term > 0

Fie $\sum_{n\geq 0} x_n$ cu $x_n \in \mathbb{R}_+, \forall n \geq 0$ pt care $\exists \lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}} - 1\right) = l \in \overline{\mathbb{R}}$

Dacă $l < 1$, seria $\sum_{n\geq 0} x_n$ e divergentă

Dacă $l > 1$, seria $\sum_{n\geq 0} x_n$ e convergentă

### Serii remarcabile de nr $\in \mathbb{R}$

1) Seria armonică: $\sum_{n\geq 1} \frac{1}{n^\alpha}$ → convergentă pt $\alpha > 1$; → divergentă pt $\alpha \leq 1$

2) Seria putere: $\sum_{n\geq 0} a^n$ → absolut convergentă pt $a \in (-1, 1)$; → divergentă pt $a \in (-\infty, -1] \cup [1, +\infty)$

3) Seria exponențială: $\sum_{n\geq 0} \frac{a^n}{n!}$ → absolut conv. pt $a \in \mathbb{R}$

① Să se studieze natura seriilor de nr. reale:

a) $\sum_{n\geq 1} \frac{2^n \cdot n!}{n^n}$ ; $x_n = \frac{2^n\cdot n!}{n^n}$, $n \in \mathbb{N}^*$ ; $x_n > 0$ $\forall n \in \mathbb{N}^*$

$S_n = x_1 + x_2 + \ldots + x_n = \frac{2\cdot 1!}{1^1} + \frac{2^2\cdot 2!}{2^2} + \frac{2^3\cdot 3!}{3^3} + \ldots + \frac{2^n\cdot n!}{n^n}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{2^{n+1}\cdot (n+1)!\cdot n^n}{2^n\cdot n!\cdot (n+1)^{(n+1)}} = \lim_{n\to\infty} \frac{\frac{2}{n+1}}{\frac{1}{n^n}} =$$

$$= 2\cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = 1^\infty = 2\cdot\lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)^n =$$

$$= 2 \cdot \lim_{n\to\infty} \left[ \left(1 + \frac{-1}{n+1}\right)^{n+1} \right]^{\frac{n}{n+1}} = 2 \cdot \cancel{\lim}\, e^{\frac{n}{n+1}} = 2 \cdot e^{-1} = \frac{2}{e} < 1 \Rightarrow$$

$\Rightarrow \sum_{n\geq 1} x_n$ conv.

b) $\displaystyle\sum_{n\geq 0} \frac{n!\, a^n}{(a+1)(2a+1)\cdots(na+1)}$ ; $x_n \geq 0$ ; $x_n = \dfrac{n!\, a^n}{(a+1)(2a+1)\cdots(na+1)}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)!\cdot a^{n+1}\cdot (a+1)(2a+1)\cdots(na+1)}{n!\cdot a^n \cdot (a+1)(2a+1)\cdots(na+1)((n+1)a+1)} =$$

$$= \lim_{n\to\infty} \frac{(n+1)a}{(n+1)a+1} = \lim_{n\to\infty} \frac{an+a}{an+a+1} = 1 \Rightarrow \text{nu se poate aplica} \Rightarrow$$

$\Rightarrow$ nu putem trage o concluzie

analog criteriul radicalului $\Rightarrow$ nu se poate aplica

$$\lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n \left(\frac{an+a+1}{an+a} - 1\right) =$$

$$= \lim_{n\to\infty} n\cdot \frac{1}{an+a} = \frac{1}{a} = l$$

I $a \in (0,1) \Rightarrow l > 1 \Rightarrow$ serie conv.

II $a \in (1, +\infty) \Rightarrow l < 1 \Rightarrow$ serie div

III $a = 1 \Rightarrow l = 1 \Rightarrow$ nu se poate aplica criteriul lui Raad

$$\sum_{n\geq 0} x_n = \sum_{n\geq 0} \frac{1}{n+1} = \sum_{n\geq 0} \frac{1}{n} \Rightarrow \text{serie div.}$$

c) $\displaystyle\sum_{n\geq 1} a^n \left(\frac{n^2+n+1}{n^2}\right)^n$, $a > 0$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} a\cdot \frac{n^2+n+1}{n^2} = a \quad \text{(criteriul radicalului)}$$

I $a \in (0,1) \Rightarrow a < 1 \Rightarrow$ seria conv.

II $a \in (1, \infty) \Rightarrow a > 1 \Rightarrow$ seria div.

III $a = 1 \Rightarrow x_n = \left(\dfrac{n^2+n+1}{n^2}\right)^n$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(\frac{n^2+n+1}{n^2}\right)^n = 1^\infty = \lim_{n\to\infty} \left(1 + \frac{n+1}{n^2}\right)^n =$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{n+1}{n^2}\right)^{\frac{n^2}{n+1}}\right]^{\frac{n+1}{n^2}\cdot n} = \lim_{n\to\infty} e^{\frac{n+1}{n}} = e^1 = e \neq 0 \Rightarrow$$

$\Rightarrow$ criteriul de div se poate aplica $\Rightarrow$ seria e div.