CDI

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Mulțimi măsurabile Jordan

- Fie $K \subseteq \mathbb{R}^m$ mulțime măsurabilă Jordan
  $f: K \to \mathbb{R}$ fct. continuă
  Atunci: $G_f = \{(x, f(x)) \mid x \in K\} \subseteq \mathbb{R}^{m+1}$ mult. măsurabilă Jordan
  și $\lambda(G_f) = 0$
- Fie $K \subseteq \mathbb{R}^m$ mult. măsurabilă Jordan
  $g, h: K \to \mathbb{R}$ funcții continue
  Atunci mulțimea $D = \{(x, y) \in \mathbb{R}^{m+1} \mid g(x) \le y \le h(x)\}$ mult. m. Jordan
- $D \subseteq \mathbb{R}^m$ mult. măsurabilă Jordan $\Rightarrow \bar{D}, \mathring{D}$ m. Jordan
  Și $\lambda(D) = \lambda(\bar{D}) = \lambda(\mathring{D})$
- $D_1, D_2 \subseteq \mathbb{R}^m$ m. m. Jordan $\Rightarrow D_1 \cup D_2, D_1 \cap D_2, D_1 \setminus D_2, D_2 \setminus D_1$
  sunt m. m. J.

(1) Să se demonstreze că urm. mulțimi sunt m. m. J

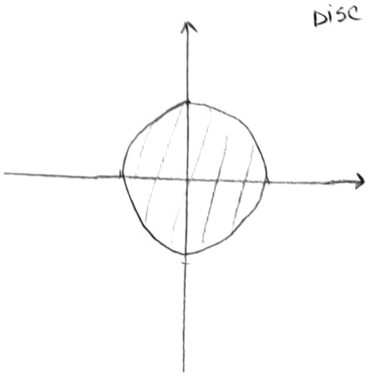
a) $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le r^2\}, r > 0$

b) $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < r^2\}, r > 0$

c) $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 \le y, y < 2x + 3\}$

a) $D \subseteq \mathbb{R}^2$ mult. închisă

Disc



$x^2 + y^2 \le r^2 \mid -y^2$

$x^2 \le r^2 - y^2$

$x^2 \le (r - y)(r + y)$

c.e. $r^2 - y^2 \ge 0 \Leftrightarrow r^2 \ge y^2 \Rightarrow y \in [-r, r)$

$$\left.\begin{array}{l}|x| \leq \sqrt{(r-y)(r+y)} \\ y \in [-r, r]\end{array}\right| \Leftrightarrow -\sqrt{r^2-y^2} \leq x \leq \sqrt{r^2-y^2}$$

$$D = \{ (x,y) \in \mathbb{R}^2 \mid y \in [-r, r] , \underbrace{-\sqrt{r^2-y^2}}_{g(y)} \leq x \leq \underbrace{\sqrt{r^2-y^2}}_{h(y)} \}$$

$g, h : [-r, r] \to \mathbb{R}$ interval 1 dimensional $\Rightarrow$ m.m.J. în $\mathbb{R}$

$g(y) = -\sqrt{r^2-y^2}$ și $h(y) = \sqrt{r^2-y^2}$ funcții continue

Deci D e m.m.J. în $\mathbb{R}^2$

$$\lambda(D) = \iint_D 1 \, dx\, dy = \int_{-r}^{r} \left(\int_{g(y)}^{h(y)} 1 \, dx \right) dy$$

$$\int_{g(y)}^{h(y)} 1\, dx = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} 1 \, dx = x \Big|_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} = 2\sqrt{r^2-y^2}$$

$$\int_{-r}^{r} 2\sqrt{r^2-y^2}\, dy = 2\int_{-r}^{r} \sqrt{r^2-y^2}\, dy = 4 \int_0^r \sqrt{r^2-y^2}\, dy =$$

Not $y = r \sin t$

$dy = y\, dt = r \cos t \, dt$

$y = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0$

$y = r \Rightarrow \sin t = 1 \Rightarrow t = \pi/2$

$$= 4\int_0^{\pi/2} \sqrt{r^2 - r^2 \sin t}\; r \cos t \, dt = 4 \int_0^{\pi/2} |r \cos t|\, r \cdot \cos t \, dt =$$

$$= 4 \int_0^{\pi/2} r^2 \cos^2 t \, dt = 4 \int_0^{\pi/2} \frac{\cos 2t + 1}{2} \, dt =$$

$$= 2r^2 \int_0^{\pi/2} \cos 2t + 2r^2\int_0^{\pi/2} 1\, dt = 2r^2 \left[(\tfrac{1}{2} \underset{0}{\underbrace{\sin 2t}}) + t \right] \Big|_0^{\pi/2} =$$

$$= 2r^2 \cdot \frac{\pi}{2} = \pi r^2$$

b) $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < r^2 \}$, $r > 0$ mulț. deschisă

O descriem ca dif de 2 mulț. m. J.

$$D = \underbrace{\{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2 \}}_{D_1} \setminus \underbrace{\{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 = r^2 \}}_{D_2}$$

$D_1 \subseteq \mathbb{R}^2$ m. m. J. cu $\lambda(D_1) = \pi r^2$ (1)

$\left.\begin{array}{l} x^2 + y^2 = r^2 \Leftrightarrow y^2 = r^2 - x^2 \\ \text{c.e. } r^2 - x^2 \geq 0 \Leftrightarrow x \in [-r, r] \end{array}\right| \Rightarrow y = \sqrt{r^2 - x^2}$ sau $y = -\sqrt{r^2 - x^2}$

$D_2 = \{(x,y) \mid x \in [-r,r], \underbrace{y = \sqrt{r^2 - x^2}}_{g(x)}\} \cup \{(x,y) \mid x \in [-r,r], \underbrace{y = -\sqrt{r^2-x^2}}_{h(x)}\}$

$g, h : [-r, r] \to \mathbb{R}$, $g(x) = \sqrt{r^2 - x^2}$ și $h(x) = -\sqrt{r^2 - x^2}$ f. cont

$[-r, r]$ m. m. J., interval închis 1 dimensional

$D_2 = G_g \cup G_h \Rightarrow$ m. m. J.

$0 \leq \lambda(D_2) = \lambda(G_g \cup G_h) \leq \lambda(G_g) + \lambda(G_h) = 0 \Rightarrow \lambda(D_2) = 0 \Rightarrow (2)$

$\left.\begin{array}{l} \text{Din } (1), (2) \\ D = D_1 \setminus D_2 \end{array}\right| \Rightarrow D$ este m. m. J $\Rightarrow \lambda(D) = \lambda(D_1 \setminus D_2) =$

$= \lambda(D_1) - \lambda(D_2) = \pi r^2$

c) D nu e mulțime închisă și nici deschisă

$D = \{(x,y) \mid x^2 \leq y, y < 2x + 3\}$

$D = \underbrace{\{(x,y) \mid x^2 \leq y, y \leq 2x+3\}}_{D_1} \setminus \underbrace{\{(x,y) \mid x^2 \leq y, y = 2x+3\}}_{D_2}$

$\left.\begin{array}{l} x^2 = y \\ y = 2x + 3 \end{array}\right| \Rightarrow x^2 \leq y = 2x + 3 \Leftrightarrow x^2 \leq 2x + 3 \Leftrightarrow -x^2 + 2x + 3 \geq 0 \Rightarrow$

$\Rightarrow \Delta = 16 \Rightarrow x = -1$ sau $x = 3 \Rightarrow x \in [-1, 3]$

$\left.\begin{array}{l} D_1 = \{(x,y) \mid x \in [-1,3], \underbrace{x^2}_{g(x)} \leq y \leq \underbrace{2x+3}_{h(x)}\} \\ g, h : [-1, 3] \to \mathbb{R} \\ g(x) = x^2 \text{ și } h(x) = 2x + 3 \\ [-1,3] \text{ mult. m. J.} \end{array}\right| \Rightarrow D_1$ m. m. J.

$\lambda(D_1) = \iint_{D_1} 1 \, dy \, dx = \int_{-1}^{3} \left(\int_{g(x)}^{h(x)} 1 \, dy \right) dx = \int_{-1}^{3} \left(\int_{x^2}^{2x+3} 1 \, dx \right) dy =$

$= \int_{-1}^{3} \left(1 \Big|_{x^2}^{2x+3} \right) dx = \int_{-1}^{3} (-x^2 + 2x + 3) \, dx = -\frac{x^3}{3} + \frac{2x^2}{2} \Big|_{-1}^{3} = \frac{34}{3}$

$D_2 = \{(x, y) \in \mathbb{R}^2 \mid \cancel{x^2 \leq 2x+3} \; x^2 \leq y, y = 2x + 3\} =$

$= \{(x,y) \in \mathbb{R}^2 \mid x \in [-1, 3], y = \underbrace{2x+3}_{h(x)}\} = \cancel{G_h} \; G_h$

$h: [-1,3] \to \mathbb{R}$, $h(x) = 2x+3$ f. cont, m. m. $J = [-1,3]$

D_2 e m. m. J. în $\mathbb{R}^2$ $\Rightarrow \lambda(D_2) = 0$

$D = D_1 \setminus D_2 = \lambda(D_1) - \lambda(D_2) = 34/3 - 0 = 34/3$

$D =$ m. m. J.

Observații:

1) $D_1 \subseteq D_2 \Rightarrow \lambda(D_2 \setminus D_1) = \lambda(D_2) - \lambda(D_1)$

2) $D_1 \cap D_2 = \emptyset \Rightarrow \lambda(D_1 \cup D_2) = \lambda(D_1) + \lambda(D_2)$

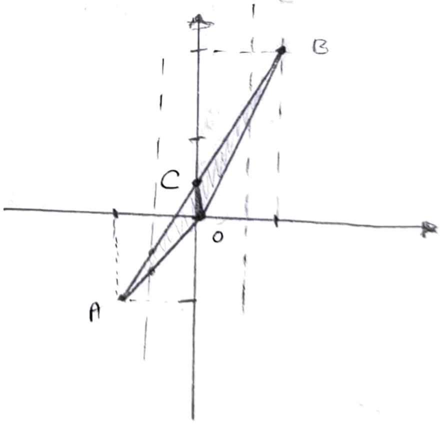
3) $\lambda(D_1 \cup D_2) = \lambda(D_1) + \lambda(D_2) - \lambda(D_1 \cap D_2)$

4) $\lambda(D_1 \cup D_2) \leq \lambda(D_1) + \lambda(D_2)$

(2) Se consideră punctele: $O(0,0)$, $A(-1,-1)$, $B(1,2)$

Să se arate că mulțimea delimitată de ΔOAB este măs. m. J. $\subset \mathbb{R}^2$

$OA = \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \end{vmatrix} = 0 \Leftrightarrow -(-x+y) = 0 \Leftrightarrow y = x$



$OB = \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0 \Leftrightarrow (2x-y) = 0 \Leftrightarrow y = 2x$

$AB = \begin{vmatrix} x & y & 1 \\ -1 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0 \Leftrightarrow -3x + 2y - 1 = 0 \Leftrightarrow y = \frac{3x+1}{2}$

Fie $C(0, 1/2)$

$D = D_1 \cup D_2$, $x \in [-1,1]$, $y \in [x, \frac{3x+1}{2}]$

$D_1 = \{(x,y) \in \mathbb{R}^2 \mid x \in [-1,0], x \leq y \leq \frac{3x+1}{2}\}$ m. m. J.

$x = [0,1]$, $y \in [2x, \frac{3x+1}{2}]$

$D_2 = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1], 2x \leq y \leq \frac{3x+1}{2}\}$ m. m. J

$\Rightarrow D$ m. m. J.

meep ♡