CDI

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Integrale improprii

$f:[a,b]\to\mathbb{R}$ funcție integrabilă Riemann $\Rightarrow \exists \int_a^b f(x)\,dx \in \mathbb{R}$

$I \subseteq \mathbb{R}$ interval necompact ($(a,b]$; $[a,b)$; (a,b); $(a,+\infty)$; $[a,+\infty)$; $(-\infty,a)$, $(-\infty,a]$, $\mathbb{R}$)

$f:I\to\mathbb{R}$ integrabilă Riemann pe orice interval $[c,d]\subseteq I \Rightarrow$

$\Rightarrow \exists$ integrală improprie

$f:(a,b]\to\mathbb{R} \Rightarrow \int_{a+0}^{b} f(x)\,dx \Rightarrow$ conv/abs. conv / div

$f:(a,+\infty)\to\mathbb{R} \Rightarrow$

$\int_{a+0}^{+\infty} f(x)\,dx$

$f: \mathbb{R}\to\mathbb{R} \Rightarrow \int_{-\infty}^{+\infty} f(x)\,dx$

① Să se studieze natura următoarelor integrale improprii:

a) $\int_0^{+\infty} e^{-x}\,dx$ și $\int_0^{+\infty} e^{-x^2}\,dx$

b) $\int_{0+0}^{1} \frac{1}{\sqrt[3]{x}+2\sqrt[5]{x}}\,dx$

c) $\int_{0+0}^{\pi/2} \frac{1}{\sqrt{\sin x}}\,dx$

d) $\int_{0+0}^{1} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

a) $f,g:[0,+\infty)\to\mathbb{R}$; $f(x)=e^{-x}$ și $g(x)=e^{-x^2}$

f și g cont pe $[0,+\infty) \Rightarrow f,g$ integ. Riemann pe $[c,d]\subseteq[0,\infty)$

$f(x)>0 \quad \forall x\in[0,+\infty)$

$g(x)>0 \quad \forall x\in[0,+\infty)$

$\int_0^{+\infty} f(x)\,dx = \int_0^{+\infty} e^{-x}\,dx = -e^{-x}\Big|_0^{+\infty} = \lim\limits_{x\to\infty} -e^{-x} - (-e^0) =$

$= \lim_{x\to\infty} -e^{-x} + 1 = 0+1 = 1 \in \mathbb{R} \Rightarrow \int_0^{+\infty} f(x)dx$ este convergentă

$\int_0^{+\infty} e^{-x^2} dx$ nu se poate calcula

$f(x) \geq g(x) \Leftrightarrow e^{-x} \geq e^{-x^2} \Leftrightarrow x \geq x^2 \ \forall\ x \in [1, +\infty)$ (1)

$f(x) \geq g(x) \geq 0$

~~$\Leftrightarrow -x \geq -x^2 \ \forall\ x \in [1, +\infty)$~~

Dacă luam $[0,1]$, atunci era interval compact = nu ne interesează

$\int_1^{+\infty} f(x)\, dx = \int_0^{\infty} f(x)\, dx - \int_0^1 f(x)\, dx \in \mathbb{R} \Rightarrow \int_0^{+\infty} f(x)\, dx$ conv. (2)

din (1) și (2) $\Rightarrow \int_1^{\infty} g(x)\, dx$ conv

$\int_0^1 g(x) + \int_1^{\infty} g(x)\, dx = \int_0^{\infty} g(x)\, dx \in \mathbb{R}$ $\Bigg| \Rightarrow \int_0^{\infty} g(x)\, dx$ conv.

b) $\int_{0+0}^1 \frac{1}{\sqrt[3]{x} + 2\sqrt[5]{x}} dx \Rightarrow f(x) = \frac{1}{\sqrt[3]{x} + 2\sqrt[5]{5}}$, $f: (0,1] \to \mathbb{R} \Rightarrow$

$\Rightarrow f$ cont pe $(0,1]$

$f(x) > 0 \ \forall x \in (0,1]$

$f(x) = \frac{1}{x^{\frac{1}{3}} + 2x^{\frac{1}{5}}} = \frac{1}{x^{\frac{1}{5}}(2 + x^{2/15})}$

Se alege $g: (0,1] \to \mathbb{R}$, $g(x) = \frac{1}{\cancel{2+}x^{\cancel{2/15}}}$ $\forall x \in (0,1]$, g cont. $\Bigg| \Rightarrow$

$\Rightarrow f(x) = g(x) \cdot \frac{1}{2+x^{2/15}} \ \forall\ x \in (0,1]$

$\lim_{\substack{x\to 0\\ x>0}} \frac{f(x)}{g(x)} = \lim_{\substack{x\to 0\\ x>0}} \frac{1}{2+x^{2/15}} = \frac{1}{2} \in (0, +\infty) \Rightarrow \int_{0+0}^1 f(x)$ și $\int_{0+0}^1 g(x)$ au aceeași natură

$\int_{0+0}^1 g(x)\, dx = \int_{0+0}^1 x^{-\frac{4}{5}} dx = \frac{5}{4} x^{\frac{4}{5}} \Big|_{0+0}^1 =$

$= \frac{5}{4} 1^{4/5} - \lim_{\substack{x\to 0\\ x>0}} \frac{5}{4} x^{4/5} = 1 - 0 = 1 \in \mathbb{R} \Rightarrow$

$\Rightarrow \int_{0+0}^1 g(x)\, dx$ conv $\Rightarrow \int_{0+0}^1 f(x)\, dx$ este conv.

c) $\int_{0+0}^{\pi/2} \frac{1}{\sqrt{\sin x}} dx$

$f:[0,\pi/2]\to\mathbb{R}$, $f(x)=\frac{1}{\sqrt{\sin x}}$, f cont și $f(x)>0$ $\forall x\in(0,\pi/2]$

$\lim_{x\to 0}\frac{\sin x}{x}=1 \Rightarrow \lim_{\substack{x\to 0\\ x>0}}\sqrt{\frac{\sin x}{x}}=1 \Rightarrow \lim_{\substack{x\to 0\\ x>0}}\frac{\sqrt{\sin x}}{\sqrt{x}}=1 \Rightarrow$

$\Rightarrow \lim_{\substack{x\to 0\\ x>0}}\frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{\sin x}}}=1$ (ca să putem alege g)

Alegem $g:(0,\pi/2]\to\mathbb{R}$, $g(x)=\frac{1}{\sqrt{x}}$, g cont

$\lim_{\substack{x\to 0\\ x>0}}\frac{g(x)}{f(x)}=1\in(0,+\infty) \Rightarrow \int_{0+0}^{\pi/2}g(x)\,dx$ și $\int_{0+0}^{\pi/2}f(x)\,dx$ au aceeași natură

$\int_{0+0}^{\pi/2}g(x)\,dx=\int_{0+0}^{\pi/2}\frac{1}{\sqrt{x}}\,dx=2\sqrt{x}\Big|_{0+0}^{\pi/2}=2\sqrt{\frac{\pi}{2}}-\lim_{\substack{x\to 0\\ x>0}}2\sqrt{x}=2\sqrt{\frac{\pi}{2}}\in\mathbb{R}$

$\int_{0+0}^{\pi/2}g(x)\,dx$ este convergentă $\Rightarrow \int_{0+0}^{\pi/2}f(x)\,dx$ convergentă

d) $f:(0,1]\to\mathbb{R}$, $f(x)=\frac{\cos\sqrt{x}}{\sqrt{x}}$, f cont, $f(x)>0$

$\int_{0+0}^{1}f(x)\,dx=\int_{0+0}^{1}\frac{\cos\sqrt{x}}{\sqrt{x}}\,dx=2\int_{0+0}^{1}\frac{\cos\sqrt{x}}{2\sqrt{x}}\,dx=2\int_{0+0}^{1}\cos t\,dt=$

$t=\sqrt{x} \Rightarrow dt=\frac{1}{2\sqrt{x}}\,dx$

$x\to 0$, $x>0 \Rightarrow t\to 0$

$x=1 \Rightarrow t=1$

$=2(\sin t)\Big|_{0+0}^{1}=2\sin 1-\lim_{\substack{t\to 0\\ t>0}}2\sin t=2\sin 1.$

② Proprietăți din curs (mai întâi):

- $\Gamma:(0,+\infty)\to\mathbb{R}$, $\Gamma(p)=\int_{0+0}^{+\infty}x^{p-1}e^{-x}\,dx$

  $\Gamma(1)=1$; $\Gamma(1/2)=\sqrt{\pi}$; $\Gamma(p+1)=p\cdot\Gamma(p)$ $\forall p>0$

  $\Gamma(p)\cdot\Gamma(1-p)=\frac{\pi}{\sin(p\pi)}$ $\forall p\in(0,1)$

- $B:(0,\infty)\times(0,\infty)\to\mathbb{R}$; $B(p,q)=\int_{0+0}^{1-0}x^{p-1}\cdot(1-x)^{q-1}\,dx=$

  $=2\int_{0+0}^{\pi/2-0}\sin^{2p-1}x\,\cos^{2q-1}x\,dx=\int_{0+0}^{+\infty}\frac{x^{p-1}}{(1+x)^{p+q}}\,dx$

$$B(p,q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)} \quad \forall\, p,q>0$$

Să se calculeze urm. integrale improprii:

a) $\int_0^{+\infty} e^{-x^2}\,dx$

b) $\int_{0+0}^{1-0} \frac{1}{\sqrt{x(1-x)}}\,dx$

c) $\int_0^{\pi/2} \sin^2 x \cos^3 x\,dx$

a) $\Gamma(p) = \int_0^{+\infty} x^{p-1} e^{-x}\,dx =$

$\int_0^{+\infty} e^{-x^2}\,dx \Rightarrow \int_0^{+\infty} e^{-t}\cdot\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^{+\infty} t^{-1/2} e^{-t}\,dt \Rightarrow p-1=-\frac{1}{2} \Rightarrow$

$x^2=t \Leftrightarrow 2x\,dx = dt$ (nu ne ajută)

$x^2=t \Leftrightarrow x=\sqrt{t} \Leftrightarrow dx = \frac{1}{2\sqrt{t}}\,dt$

$x=0 \Rightarrow t=0$

$x\to\infty \Rightarrow t\to\infty$

$\Rightarrow p=\frac{1}{2} \Rightarrow \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\cdot\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$

b) $\int_{0+0}^{1-0} \frac{1}{\sqrt{x(1-x)}}\,dx = \int_{0+0}^{1-0} x^{-1/2}(1-x)^{-1/2}\,dx$

Asociem B varianta 1 $\Rightarrow B(p,q) = \int_{0+0}^{1-0} x^{p-1}(1-x)^{q-1}\,dx$

$p-1=-\frac{1}{2} \Rightarrow p=q=\frac{1}{2}$ că şi $q-1=-\frac{1}{2} \Rightarrow$

$\Rightarrow B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(1/2)\cdot\Gamma(1/2)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}\cdot\sqrt{\pi}}{1} = \pi$