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Seminar 2

• Determinanți și regula lui Laplace

⊛ Fie $A \in \mathcal{M}_n(K)$, K corp. comutativ

$\det : \mathcal{M}_n(K) \to K$

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma)\, a_{1\sigma(1)}\, a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

$\varepsilon(\sigma) = (-1)^{m(\sigma)}$ signatura lui σ

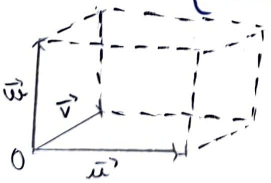
$\begin{cases} m(\sigma) = \text{nr inversiunilor} \\ (i,j) \Rightarrow i<j \text{ și } \sigma(i) > \sigma(j) \end{cases}$

Obs: 1) $\det \to \mathcal{M}_n(K)$

$\to$ scalar ($\in K$)

2) $A, B \in \mathcal{M}_n(K) \Rightarrow \begin{cases} \det(A+B) \neq \det A + \det B \\ \det(AB) = \det A \cdot \det B \\ \det(\alpha A) = \alpha^n \det A \end{cases}$

În geometrie:



în $\mathbb{R}^3$; $\vec{w}, \vec{v}, \vec{u}$ vectori necoplanari cu aceeași origine

$V_{\text{paralelipiped}} = |\Delta|$ $\underset{(1,0,0)}{\bar{i}}, \underset{(0,1,0)}{\bar{j}}, \underset{(0,0,1)}{\bar{k}}$

$u = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$

$v = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

$w = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$

$$\Delta = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$V_{\text{tetraedru}} = \frac{1}{6}|\Delta|$

$A_1 = (x_1, x_2, x_3)$ $\tilde{\Delta}$ = det de ord 4

$B_1 = (y_1, y_2, y_3)$

$C_1 = (z_1, z_2, z_3)$

$D_1 = (t_1, t_2, t_3)$

$$\tilde{\Delta} = \begin{vmatrix} x_1 & x_2 & x_3 & 1 \\ y_1 & y_2 & y_3 & 1 \\ z_1 & z_2 & z_3 & 1 \\ t_1 & t_2 & t_3 & 1 \end{vmatrix}$$

$$V_{A_1B_1C_1D_1} = \frac{1}{6}|\tilde{\Delta}|$$

(*) Regula lui Laplace

Fie $A \in \mathcal{M}_m(K)$, $1 \le p \le m; p \in \mathbb{N}$

$$\det A = \sum M \cdot M' = \sum_{J} \det A_{IJ} \cdot (-1)^{i_1 + \dots i_p + j_1 + \dots j_p} \cdot \det A_{\bar{I}\bar{J}}$$

M = minor de ord p format la „n" a „p" linii fixate cu „p" col var.

M' = complement alg. al lui M

$I = \{1 \le i_1 \le i_2 \le \dots \le i_p \le m\}$ fixat $\Rightarrow \bar{I} = \{1, \dots m\} \setminus I$

$J = \{1 \le j_1 \le j_2 \le \dots \le j_p \le m\}$ variabil $\Rightarrow \bar{J} = \{1, \dots m\} \setminus J$

<u>Obs:</u> card $J = C_m^p$

<u>Caz particular:</u> $p = 1 \Rightarrow \det A = a_{i1}C_1 + a_{i2}C_2 + \dots + a_{im}C_m$ — complementarul

$(-1)^{i+j} \cdot A_{ij}$

dezvoltare după linia i

analog pt. dezv. după col

$\det A = a_{1j}C_{1j} + a_{2j} + C_{2j} + \dots + a_{mj}C_{mj}$ $\forall j = \overline{1,m}$

L dezv. după col j

(1) $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix}$ dezvoltare după prima linie

$$\det A = 1 \cdot (-1)^2 \begin{vmatrix} 1 & 3 & 4 \\ 5 & 1 & -1 \\ -2 & 2 & 4 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & -1 \\ -1 & 2 & 4 \end{vmatrix} + 2(-1)^4 \begin{vmatrix} 1 & 1 & 4 \\ 2 & 5 & -1 \\ -1 & -2 & 4 \end{vmatrix}$$

$$+ 3 \cdot (-1)^5 \begin{vmatrix} 1 & 1 & 3 \\ 2 & 5 & 1 \\ -1 & -2 & 2 \end{vmatrix} = 4 + 40 + 9 + (-8) + (-2) + 60 - (4 + 16 + 3 - 4$$

$-2 + 24) + 20 + 1(\cancel{+}16) \dots = -5$

② dezvoltare după primele 2 linii cu regula lui Laplace.

$$\det A = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{vmatrix} = (-1)^{\overset{\text{linii}}{\widehat{1+2}} + \overset{\text{col}}{\widehat{1+2}}} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} +$$

$$+ \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} (-1)^{1+2+2+3} \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} +$$

$$+ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} (-1)^{1+2+2+4} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} +$$

$$+ \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} (-1)^{1+2+3+4} \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix} = -5$$

• Spații vectoriale

$V \neq \emptyset$, K corp comutativ

$+ : V \times V \to V$ adunarea vectorilor

$(v_1, v_2) \to v_1 + v_2$ operație unitară

$\cdot : V \times V \to V$ înmulțirea vectorilor

$(k, v) \to k \times v$ operație externă

I $(V, +)$ grup comutativ

II

1) $(k_1 + k_2) v = k_1 v + k_2 v$

2) $k(v_1 + v_2) = k v_1 + k v_2$

3) $(k_1 k_2) v = k_1 (k_2 v)$

4) $1 \cdot v = v$ $\quad \forall k_1, k_2, k \ldots$

$(V/k, +, \cdot)$ s.n. spațiu vectorial V peste k (V/k)

$v \in V$ vectori

$k \in K$ scalari

$K = \mathbb{R}$ sp. vect. real

$K = \mathbb{C}$ sp. vect. complex

Exemplu: 1) K = corp. comutativ

$(K/_K, +, \cdot)$ sp. vect. pe k

$H \subseteq K$ subcorp

~~$(K/_H)$~~ $(K/_H, +, \cdot) \to$ sp. vect peste H

Caz particular: $\mathbb{Q}/\mathbb{Q}$, $\mathbb{R}/\mathbb{R}$, $\mathbb{Z}_p/\mathbb{Z}_p$ p-prim, $\mathbb{C}/\mathbb{C}$, $\mathbb{C}/\mathbb{R}$, $\mathbb{R}/\mathbb{Q}$

$V_1, V_2/K \to$ 2 sp. vect $/K$

$V_1 \times V_2/K \to$ 2 sp. vect $/K$

$V \overset{not}{=} V_1 \times V_2/K$

$+ : V \times V \to V$

$(v_1, v_2) + (w_1, w_2) \overset{def}{=} (v_1 + w_1, v_2 + w_2)$

$\cdot : K \times V \to V$

$k(v_1, v_2) \overset{def}{=} (kv_1, kv_2)$

$(V/K, +, \cdot)$ ~~$(V/K, +, \cdot) \to$ sp.~~ spațiu vect. peste K

Caz general: $V_1, \dots V_{m}/K$ mult. numărabilă

$V_i = K$, $\forall\ i = \overline{1,m}$

$(K^m/K, +, \cdot)$ spațiu vect. peste K

$K^m = \{ (x_1, \dots x_m) \mid x_i \in K, \forall\ i = \overline{1,m} \}$

$+ : K^m \times K^m \to K^m$ operație internă

$(x_1, \dots x_m) + (y_1, \dots y_m) = (x_1 + y_1, \dots x_m + y_m)$

$\cdot : K \times K^m \to K^m$

$k(x_1, \dots x_m) = (kx_1, \dots kx_m)$

Caz particular: $K = \mathbb{C}$ avem $(\mathbb{C}^m/_\mathbb{C}, +, \cdot)$

$\mathbb{R} = K$ avem $(\mathbb{R}^m/_\mathbb{R}, +, \cdot)$

$\mathbb{C}/\mathbb{R}$ $\dim_\mathbb{R} \mathbb{C} = 2$ dim

1 Demonstrăm că $(\mathbb{R}^3/\mathbb{R}, +, \cdot)$ sp. vect. real

$+ : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$\cdot : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$

$k(x_1, x_2, x_3) = (kx_1, kx_2, kx_3)$

Dem: I $(\mathbb{R}^3, +)$ grup comut

II 1) $k(x+y) = kx + ky$

2) $(k_1 + k_2)x = k_1 x + k_2 x$

3) $(k_1 k_2)x = k_1(k_2 x)$

4) $1 \cdot x = x$

$\to (\mathbb{R}, +, \cdot)$ corp. comutativ