SAI

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Seminar 8

① Presupunem că $(\mathbb{Q},+)$ e ciclic

Atunci $\exists\ a,b \in \mathbb{Z}^*$ cu $(a,b)=1$, astfel ca $\mathbb{Q} = \langle a/b \rangle$

Atunci $\exists\ \alpha \in \mathbb{Z}$, $\mathbb{Q} \ni \frac{a}{2b} = \alpha \cdot \frac{a}{b}$, deci $a = 2\alpha a \Leftrightarrow (2\alpha - 1)a = 0 \Leftrightarrow a = 0$

dar $2\alpha - 1 = 0 \notin 2\mathbb{Z} \Rightarrow$ ↯

Rămâne deci că $(\mathbb{Q},+)$ nu e ciclic

Curs: $\mathbb{Z}_{mn} \xrightarrow[\text{grup}]{\sim} \mathbb{Z}_m \times \mathbb{Z}_n \Leftrightarrow (m,n)=1$ (teoremă)

$\hat{a} \xrightarrow{\varphi} (\bar{a}, \mathring{a})$; unde $\hat{a} \in \mathbb{Z}_{mn}$, $\bar{a} \in \mathbb{Z}_m$ și $\mathring{a} \in \mathbb{Z}_n$

$\hat{a} = \hat{b} \Leftrightarrow m \cdot n \mid (a-b) \Rightarrow \begin{cases} m \mid (a-b) \Leftrightarrow \bar{a} = \bar{b} \\ n \mid (a-b) \Leftrightarrow \mathring{a} = \mathring{b} \end{cases} \Rightarrow$

$\Rightarrow (\bar{a}, \mathring{a}) = (\bar{b}, \mathring{b}) \Rightarrow \varphi$ e corect definită

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② $(8,9)=1$, $\Rightarrow \mathbb{Z}_8 \times \mathbb{Z}_9 \xrightarrow{\sim} \mathbb{Z}_{72}$ care e ciclic (că e generat de $\{\hat{1}\}$)

Deci $\mathbb{Z}_8 \times \mathbb{Z}_9$ e ciclic

Presupunem că $\mathbb{Z}_8 \times \mathbb{Z}_{10}$ e ciclic $\Rightarrow \exists\ m,n \in \mathbb{Z}$ a.î. $\mathbb{Z}_8 \times \mathbb{Z}_{10} = \langle \{(\hat{m}, \bar{n})\} \rangle$

Atunci $\exists\ \alpha, \beta \in \mathbb{Z}$ a.î. $\alpha \cdot (\hat{m}, \bar{n}) = (\hat{1}, \bar{0}) \Leftrightarrow \begin{cases} \alpha \hat{m} = \hat{1} \\ \alpha \bar{n} = \bar{0} \end{cases}$ $\Rightarrow \beta\alpha\hat{m} = \hat{\beta} \Rightarrow$ $\hat{0}$

$\beta \cdot (\hat{m}, \bar{n}) = (\hat{0}, \bar{1}) \Leftrightarrow \begin{cases} \beta \hat{m} = \hat{0} \\ \beta \bar{n} = \bar{1} \end{cases}$

$\Rightarrow \beta \vdots 8 \Rightarrow \exists\ \gamma \in \mathbb{Z}, \beta = 8\gamma$ $\Big|$ $\Rightarrow \exists\ \gamma \in \mathbb{Z}$, $\overline{8\gamma} \neq \hat{1} \Rightarrow \exists\ \gamma, \delta \in \mathbb{Z}$ $8\gamma n - 1 = 10\delta$

$\beta \bar{n} = \bar{1} \Rightarrow \beta \bar{n} = \bar{1}$

Curs: Grup finit generat - grup care admite sistem de gen. finit

(3) $\mathbb{Q}$ nu e finit generat?

Presupunem că e. Fie $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots \frac{a_m}{b_m}$ sistem finit de generatori pt el. ($m \in \mathbb{N}^*$; $a_1, \dots a_m, b_m \in \mathbb{Z}^*$)

Le aducem la același numitor; ele devin: $\frac{a_1}{b}; \frac{a_2}{b}; \dots \frac{a_m}{b}$

Atunci $\exists\, \alpha_1, \alpha_2, \dots \alpha_m \in \mathbb{Z}$ a.î. $\frac{1}{2b} = \alpha_1 \frac{a_1}{b} + \alpha_2 \frac{a_2}{b} + \dots + \alpha_m \frac{a_m}{b} \Rightarrow$

$\Rightarrow 1 = 2(\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_m a_m) \Rightarrow 2 \mid 1 \Rightarrow$ abs

Deci $(\mathbb{Q}, +)$ nu e finit generat

$\boxed{\text{O temă}}$: Arătați că $(\mathbb{R}, +)$ nu e finit generat

- Proprietăți ale ordinelor elementelor din grupuri:

1) G = grup; $a \in G$; $k \in \mathbb{Z}$ $\Rightarrow$ $ord_G(x^k) = \frac{ord\, x}{(k, ord\, x)}$

2) $ord_{G_1 \times G_2}((x_1, x_2)) = [ord_{G_1} x_1, ord_{G_2} x_2]$

(4) Determinați $ord_{\mathbb{Z}_{330}} \widehat{120}$

$ord_{\mathbb{Z}_{330}} \widehat{120} = ord_{\mathbb{Z}_{330}}(120 \cdot \hat{1}) = \frac{ord_{\mathbb{Z}_{330}}(\hat{1})}{(120, ord_{\mathbb{Z}_{330}} \hat{1})} = \frac{330}{(120, 330)} = \frac{330}{30} = 11$

(5) Determinați elem. de ord 40 din $\mathbb{Z}_{300}$ și pe cele din $\mathbb{Z}_{360}$

~~Nu~~ $\nexists$ $40 \mid 300 \Rightarrow \nexists$ elem de ord 40 (consecința t. Lagrange)

Fie $a \in \{0, 1, \dots 359\}$

$ord_{\mathbb{Z}_{360}}(\hat{a}) = 40 \Leftrightarrow ord(a \cdot \hat{1}) = 40 \Leftrightarrow \frac{ord\, \hat{1}}{(a, ord\, \hat{1})} = 40 \Leftrightarrow \frac{360}{(a, 360)} = 40 \Rightarrow$

$\Rightarrow (a, 360) = 9 \Leftrightarrow a \in \{9\alpha \mid \alpha \in \{0, 1, \dots 39\} \wedge \underbrace{(\alpha, 40) = 1}_{\text{prime între ele}}\} \Rightarrow$

$\Rightarrow a \in \{\hat{9}, \widehat{27}, \widehat{63}, \widehat{81}, \widehat{99}, \widehat{117}, \widehat{153}, \widehat{171}, \widehat{189}, \widehat{207}, \widehat{243}, \widehat{279}, \widehat{297}, \widehat{333}, \widehat{351}\}$

(6) Determinați $ord_{\mathbb{Z}_{10} \times \mathbb{Z}_{16}}((\hat{6}, \overline{10}))$

$ord_{\mathbb{Z}_{10} \times \mathbb{Z}_{16}}((\hat{6}, \overline{10})) = [ord_{\mathbb{Z}_{10}} \hat{6}, ord_{\mathbb{Z}_{16}} \overline{10}] = [5, 8] = 40$

x a b

$(a \cdot b) \vdots x$

(7) Determinați elementele de ord 12 din $\mathbb{Z}_4 \times \mathbb{Z}_6$

Fie $(\hat{a}, \bar{b}) \in \mathbb{Z}_4 \times \mathbb{Z}_6$

$$\operatorname{ord}_{\mathbb{Z}_4 \times \mathbb{Z}_6}((\hat{a}, \bar{b})) = 12 \iff [\operatorname{ord}_{\mathbb{Z}_4} \hat{a}, \operatorname{ord}_{\mathbb{Z}_6} \bar{b}] = 12 \Rightarrow$$

$$\Rightarrow \begin{cases} \operatorname{ord}_{\mathbb{Z}_4} \hat{a} \mid 4 \\ \operatorname{ord}_{\mathbb{Z}_6} \bar{b} \mid 6 \end{cases} \Rightarrow \operatorname{ord}_{\mathbb{Z}_4} \hat{a} = 4 \text{ și } \operatorname{ord}_{\mathbb{Z}_6} \bar{b} \in \{3, 6\} \Rightarrow$$

$$\Rightarrow \hat{a} \in \{\hat{1}, \hat{3}\} \text{ și } \bar{b} \in \{\bar{1}, \bar{2}, \bar{4}, \bar{5}\}$$

Ca urmare, elem. de ord 12 sunt $(\hat{1}, \bar{1}), (\hat{1}, \bar{2}), (\hat{1}, \bar{4}), (\hat{1}, \bar{5})$; $(\hat{3}, \bar{1}), (\hat{3}, \bar{2}), (\hat{3}, \bar{4}), (\hat{3}, \bar{5})$.

(8) Reluăm determinarea ciclicității lui $\mathbb{Z}_8 \times \mathbb{Z}_{10}$

Presupunem că e ciclic. Fie $(\hat{m}, \bar{n})$ un generator al său.

Atunci $80 = |\mathbb{Z}_8 \times \mathbb{Z}_{10}| = \operatorname{ord}_{\mathbb{Z}_8 \times \mathbb{Z}_{10}}(\hat{m}, \bar{n}) = [\operatorname{ord}_{\mathbb{Z}_8} \hat{m}, \operatorname{ord}_{\mathbb{Z}_{10}} \bar{n}] = x$

Dar $x \mid [8, 10] = 40 \Rightarrow \mathbb{Z}_8 \times \mathbb{Z}_{10}$ nu e ciclic

(9) Fie $m, n \in \mathbb{Z}^*$. Presupunem că $\mathbb{Z}_m \times \mathbb{Z}_n \xrightarrow{\sim} \mathbb{Z}_{mn}$ (grup)

$\boxed{\text{Temă:}}$ 1) $\mathbb{Z} \times \mathbb{Z}$ nu e ciclic

2) elem. de ord a) 40, b) 50 din $\mathbb{Z}_{600}$

3) elem. de ord 18 din $\mathbb{Z}_6 \times \mathbb{Z}_9$

Atunci, întrucât $\mathbb{Z}_{mn}$ e ciclic $\Rightarrow \mathbb{Z}_m \times \mathbb{Z}_n$ e ciclic

Fie $(\hat{a}, \bar{b})$ un generator al lui $\mathbb{Z}_{mn}$

Atunci $mn = |\mathbb{Z}_m \times \mathbb{Z}_n| = \operatorname{ord}_{\mathbb{Z}_m \times \mathbb{Z}_n}(\hat{a}, \bar{b}) = [\operatorname{ord}_{\mathbb{Z}_m} \hat{a}, \operatorname{ord}_{\mathbb{Z}_n} \bar{b}] = x$

dar $x \mid [m, n] \Rightarrow [m, n] \cdot (m, n) = mn \mid [m, n]$

deci $(m, n) = 1$