Calc. dif. și int.

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Seminar 2

Limita inf. și limita sup. - șiruri

$(x_n)_{n\in\mathbb{N}}$ șir din $\mathbb{R}$

$\overline{\lim\limits_{n\to\infty}}\, x_n \in \overline{\mathbb{R}}$ (sup) $\quad \underline{\lim\limits_{n\to\infty}}\, x_n \in \overline{\mathbb{R}}$ (inf)

$\underline{\lim\limits_{n\to\infty}}\, x_n \le \overline{\lim\limits_{n\to\infty}}\, x_n$

$\exists \lim\limits_{n\to\infty} x_n \in \overline{\mathbb{R}} \iff \underline{\lim\limits_{n\to\infty}}\, x_n = \overline{\lim\limits_{n\to\infty}}\, x_n$

$\nexists \lim\limits_{n\to\infty} x_n \iff \underline{\lim\limits_{n\to\infty}}\, x_n \ne \overline{\lim\limits_{n\to\infty}}\, x_n$

① Să se calculeze $\underline{\lim\limits_{n\to\infty}}\, x_n$ și $\overline{\lim\limits_{n\to\infty}}\, x_n$ pt $x_n = \dfrac{3\cdot(-1)^n}{2+n(-1)^{n+1}} + \sin\dfrac{n\pi}{2}$

unde $n \ne 2$

$(-1)^n = \begin{cases} -1, & n = 2k+1 \\ 1, & n = 2k \end{cases}$

$(-1)^{n+1} = \begin{cases} -1, & n = 2k \\ 1, & n = 2k+1 \end{cases}$

$\sin\dfrac{n\pi}{2} = \begin{cases} -1, & n = 4k+3 \\ 0, & n = 2k \\ 1, & n = 4k+1 \end{cases}$

$n=0 \to \sin 0 = 0$

$n=1 \Rightarrow \sin\frac{\pi}{2} = 1$

$n=2 \Rightarrow \sin\pi = 0$

$n=3 \Rightarrow \sin\frac{3\pi}{2} = -1$

$n=4 \Rightarrow \sin 2\pi = 0$

Se aleg subșirurile: $(x_{2k})_{k\in\mathbb{N}}$, $(x_{4k+3})_{k\in\mathbb{N}}$, $(x_{4k+1})_{k\in\mathbb{N}}$

Pt $(x_{2k})_{k\in\mathbb{N}}$: $\lim\limits_{k\to\infty} x_{2k} = \dfrac{3\cdot 1}{2+2k\cdot(-1)} + 0 = \lim\limits_{k\to\infty} \dfrac{3}{2-2k} = 0$

Pt $(x_{4k+3})_{k\in\mathbb{N}}$: $\lim\limits_{k\to\infty} x_{4k+3} = \dfrac{3\cdot(-1)}{2+(4k+3)\cdot 1} + (-1) =$

$= \lim\limits_{k\to\infty} \dfrac{-3}{4k+5} - 1 = -1$

Pt $(x_{4k+1}) \Rightarrow \lim_{k\to\infty} x_{4k+1} = \lim_{k\to\infty} \frac{3\cdot(-1)}{2+(4k+1)(-1)} + 1 = 1$

$A = \{-1, 0, 1\} \Rightarrow \begin{cases} \underline{\lim}_{m\to\infty} x_m = \inf A = -1 \\ \overline{\lim}_{m\to\infty} x_m = \sup A = 1 \end{cases} \Rightarrow \underline{\lim} \neq \overline{\lim} \Rightarrow$

$\Rightarrow \nexists \lim_{m\to\infty} x_m$

- Şirul este şi mărginit inf şi sup., deoarece:

$\begin{cases} \underline{\lim} = -1 \text{ e finită} = m \\ \overline{\lim} = 1 \text{ e finită} = M \end{cases}$

(2) $\underline{\lim} = ?$ şi $\overline{\lim} = ?$ pt $x_m = \left[m \cdot \ln\left(1+\frac{1}{m}\right)\right]\left[3+(-1)^m\right] + \cos\frac{m\pi}{2}$

$(-1)^m = \begin{cases} -1, & m = 2k+1 \\ 1, & m = 2k \end{cases}$

$\cos\frac{m\pi}{2} = \begin{cases} 1, & m = 4k \\ 0, & m = 2k+1 \\ -1, & m = 4k+2 \end{cases}$

Subşirurile sunt: (x_{2k+1}); (x_{4k}); (x_{4k+2})

$\lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} \left[(2k+1)\cdot\ln\left(1+\frac{1}{2k+1}\right)\right]\left[3-1\right] + 0 =$

$= \lim_{k\to\infty} 2(2k+1)\cdot\ln\left(1+\frac{1}{2k+1}\right) =$

$= 2\cdot\lim_{k\to\infty} \ln\left(1+\frac{1}{2k+1}\right)^{2k+1} = 2\cdot\ln e = 2\cdot 1 = 2$

$\lim_{k\to\infty} x_{4k} = \lim_{k\to\infty} \left[(4k)\cdot\ln\left(1+\frac{1}{4k}\right)\right]\left[3+1\right] + 1 =$

$= 4\cdot\lim_{k\to\infty} k\cdot\ln\left(1+\frac{1}{4k}\right)\cdot 4 + 1 =$

$= 4\cdot\lim_{k\to\infty} \ln\left(1+\frac{1}{4k}\right)^{4k} + 1 = 4\cdot\ln e + 1 = 4+1 = 5$

$$\lim_{k\to\infty} x_{4k+2} = \lim_{k\to\infty} \left[(4k+2)\cdot \ln\left(\frac{1}{4k+2}+1\right)\right] [3+1] - 1 =$$

$$= 4\lim_{k\to\infty} \ln\left(1+\frac{1}{4k+2}\right)^{4k+2} - 1 =$$

$$= 4\cdot \ln e - 1 = 4-1 = 3$$

$$A = \{2;3;5\} \Rightarrow \begin{cases} M = \overline{\lim_{n\to\infty}}\, x_n = 5 \\ m = \underline{\lim_{n\to\infty}}\, x_n = 2 \end{cases} \Bigg| \Rightarrow M \neq m \Rightarrow \nexists \lim_{n\to\infty} x_n$$

③ $\underline{\lim}$ și $\overline{\lim} = ?$ pt $x_n = (-1)^{n+1} \cdot \frac{n}{n+1} + tg\, \frac{n\pi}{3}$

$$(-1)^{n+1} = \begin{cases} 1, & n = 2k+1 \\ -1, & n = 2k \end{cases}$$

$$tg\, \frac{n\pi}{3} = \begin{cases} -\sqrt{3}, & n = 3k+2 \\ 0, & n = 3k \\ \sqrt{3}, & n = 3k+1 \end{cases}$$

$n = 0 \Rightarrow tg\, 0 = 0$

$n = 1 \Rightarrow tg\, \frac{\pi}{3} = \sqrt{3}$

$n = 2 \Rightarrow tg\, \frac{2\pi}{3} = -\sqrt{3}$

$n = 3 \rightarrow tg\, \pi = 0$

$n = 4 \Rightarrow tg\, \frac{4\pi}{3} = \sqrt{3}$

Nu avem indici comuni $\Rightarrow$ luăm cmmmc $= 6$ pt subșiruri

Se aleg subșirurile: (x_{6k}); (x_{6k+1}); (x_{6k+2}); (x_{6k+3}); (x_{6k+4}); (x_{6k+5})

$$\lim_{k\to\infty} x_{6k} = \lim_{k\to\infty} (-1)\cdot \frac{6k}{6k+1} + 0 = -1$$

$$\lim_{k\to\infty} x_{6k+1} = \lim_{k\to\infty} 1 \cdot \frac{6k+1}{6k+2} + \sqrt{3} = 1+\sqrt{3}$$

$$\lim_{k\to\infty} x_{6k+2} = \lim_{k\to\infty} (-1)\cdot \frac{6k+2}{6k+3} - \sqrt{3} = -1-\sqrt{3} = -(1+\sqrt{3})$$

$$\lim_{k\to\infty} x_{6k+3} = \lim_{k\to\infty} 1 \cdot \frac{6k+3}{6k+4} + 0 = 1$$

$$\lim_{k\to\infty} x_{6k+4} = \lim_{k\to\infty} (-1)\cdot \frac{6k+4}{6k+5} + \sqrt{3} = -1+\sqrt{3}$$

$6k+4 = 3(2k+1)+1$

$$\lim_{k\to\infty} x_{6k+5} = \lim_{k\to\infty} 1 \cdot \frac{6k+5}{6k+6} - \sqrt{3} = 1-\sqrt{3}$$

$A = \{-1;\ 1;\ 1-\sqrt{3};\ \sqrt{3}-1;\ -\sqrt{3}-1;\ 1+\sqrt{3}\}$

$$\left.\begin{array}{l} M = \overline{\lim\limits_{n\to\infty}}\, x_n = 1+\sqrt{3} \\ m = \underline{\lim\limits_{n\to\infty}}\, x_n = -1-\sqrt{3} \end{array}\right| \Rightarrow m \neq M \Rightarrow \nexists \lim_{n\to\infty} x_n$$

Serii de nr. reale

$\sum\limits_{n\geq 0} x_n \to$ absolut convergentă (modul = absolut $\Rightarrow |x_n| = x_n$, poz)

$\Downarrow$

$\to$ convergentă

$\to$ divergentă

- Dacă $x_n \geq 0\ \forall n \geq p$ sau $x_n \leq 0\ \forall n \geq p$, noțiunile de serii absolut conv. $\Leftrightarrow$ conv.

$\sum\limits_{n\geq 0} x_n \longrightarrow (S_n)_{n\in\mathbb{N}}$ unde $S_n = x_0 + x_1 + \dots + x_n$ (suma parțială)

$\sum\limits_{n\geq 0} x_n \to$ convergentă dacă S_n = șir conv.

$\to$ divergentă dacă S_n = șir divergent

- Cerința: „Studiați natura seriei" = aflăm term. gen. x_n

① Studiați natura seriei de nr. reale $\sum\limits_{n\geq 1} \frac{3}{n(n+2)}$

Fie term. gen. $x_n = \frac{3}{n(n+2)}$, $n \geq 1$

Studiem semnul: $x_n \geq 0\ \forall n \geq 1$

Prin natura seriei: studierea conv. sau div.

$S_n = x_1 + x_2 + \dots + x_n \overset{?}{=} 1 + \frac{3}{8} + \dots + \frac{3}{n(n+2)} = \frac{3}{1\cdot 3} + \frac{3}{2\cdot 4} + \frac{3}{3\cdot 5} + \dots + \frac{3}{n(n+2)}$

$S_n = 3\left(\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \dots + \frac{1}{n(n+2)}\right) = \cancel{\frac{3}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \dots\right.}$

$S_n = \frac{3}{2}\left(\frac{2}{1\cdot 3} + \frac{2}{2\cdot 4} + \frac{2}{3\cdot 5} + \dots + \frac{2}{n(n+2)}\right) = \frac{3}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \dots + \frac{1}{n} - \frac{1}{n+2}\right)$

$S_n = \frac{3}{2}\left(1 + \frac{1}{2} \overset{+\frac{1}{n+1}}{-} \frac{1}{n+2}\right) = \frac{3}{2}\left(\frac{3}{2} - \frac{1}{n+2}\right) = \frac{3}{2} \cdot \frac{3n+4}{2n+4} \cdot \frac{3}{2}$

$\lim\limits_{n\to\infty} S_n = \frac{3}{2} \cdot \frac{3}{2} = \frac{9}{4} \Rightarrow \sum\limits_{n\geq 1} x_n$ este serie conv.