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Seminar 7

Endomorfism vectorial

Def: $f: V \to V$ apl. lin.

V/K ~~sp~~ vect. s.m. end. vect.

Def: V/K sp vect $f: V \to V$

a) $\lambda \in K$ s.m. val. proprii dacă

$\exists\, v \in V^* = V \setminus \{0_V\}$ a.î. $f(v) = \lambda v$

b) $v \in V \setminus \{0_V\}$ s.m. vector propriu

$Spec(f) = \{\lambda \in K \mid \lambda$ ~~-~~ val. proprie coresp. lui $f\}$

$V_\lambda = \{v \in V \mid f(v) = \lambda v\} \subset V$

{subsp. propriu corep. λ}

$f: V \to V$ endom
$n < \infty$

$B \subset V$ bază, $f \to A_f$ mat. asoc. lui f

$\lambda I_n - A_f \to$ mat carac. coresp. lui lui f

$P(\lambda) = \det(\lambda I_n - A_f)$ pol. caracteristic lui f

$P(\lambda) = 0 \iff \det(\lambda I_n - A_f) = 0$

$\lambda\, v$ coresp. lui lui $f \iff \lambda$ răd. pt. ec. caract.

$\det(\lambda I_n - A_f) = 0$

Det val. proprii se reduce la det răd ec. caract. în corpul scalarilor

$V_\lambda \Rightarrow f(v) = \lambda v$

$A_f v = \lambda v \quad (\lambda I_n - A_f) v = 0 \quad SLO$

Diagonalizare

Fie $f: V \to V$ endm, V/K sp. vect $n < \infty$

f s.n. endm diagonalizabil $\Rightarrow \exists B \subset V$ a.î. A_f

să aibă f. diag. ~~Δ~~ ~~diag~~ : $\Delta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & \lambda_n \end{pmatrix}$

$\exists B \subset V$ bază formată numai din vect.

(T) Toate ec. ~~cara~~ răd. ec. carac. sunt în K

Dim. fiecărui subsp. propriu coincide cu ord. de multiplicitate (alg.) p.t. fiecare val. proprie

$f: V \to V$

endm diag $\Leftrightarrow \begin{cases} 1)\ m_a(\lambda_1) + \ldots + m_a(\lambda_p) = n \\ 2)\ m_a(\lambda_i) = m_g(\lambda_i) \quad \forall i = \overline{1,p} \end{cases}$

* Propr.:

$m_g(\lambda) \le m_a(\lambda)$ multiplicitatea geom. proprie $\le$ multip. alg. proprie

(T) End. sim. (mat. sim.) reale sunt diagonalizabile

$\forall$ mat. sim. este diag.

Orice end (real) care are n valori proprii dist. este diag.; unde n = dim. sp. vect. al cărui end. rulează.

(1) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 4y \\ 5x + 2y \end{pmatrix}$ apl. lin. (end.)

a) A_f în rap. cu baza

$B_0 = \{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \} \subset \mathbb{R}^2$

b) $B = \{ f_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, f_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \} \subset \mathbb{R}^3$

$A_f = ?$ mat în rap. cu B

c) $\lambda \to$ v. p. ?

$V_\lambda \to$ ssp. proprii

d) Stabiliţi dc. endomorf. diag. în cor. afir., scrieţi f. diag. coresp.

e) Verif. rez. obţinut

f) Calc $A_f^n = ?$ $n \in \mathbb{N}^*$

a) $A_f = \begin{array}{c} \begin{pmatrix} \boxed{\begin{matrix}3\\5\end{matrix}} & \boxed{\begin{matrix}4\\2\end{matrix}} \end{pmatrix} \\ \begin{matrix} \shortparallel & \shortparallel \\ f(e_1) & f(e_2) \end{matrix} \end{array} \in \mathcal{M}_2(\mathbb{R})$ mat. asoc. endm. rep. B_0

b) $f: V \to V$

$$\begin{tikzcd} B \arrow[r, "S"] \arrow[d] \arrow[dr, phantom, ""] & B' \arrow[d] \\ A_f & A'_f \end{tikzcd}$$

$S \to$ mat. de trecere. $A'_f = S^{-1} A_f S$

$B_0 \xrightarrow{S} B \implies S = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

(P) Fie $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(K)$

$\det A = ad - bc \neq 0 \implies A$ inversab. $\implies A^{-1} = \frac{1}{\det A} A^*$

$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

c) $\det(\lambda I_2 - A_f) = 0$

$\begin{vmatrix} \lambda - 3 & -4 \\ -5 & \lambda - 2 \end{vmatrix} = 0 \iff (\lambda - 3)(\lambda - 2) - 20 = 0 \implies$

$\iff \lambda^2 - 5\lambda - 14 = 0 \implies \lambda \in \{-2, 7\}$

$\begin{cases} \lambda_1 = -2 \quad m_a(\lambda_1) = 1 \\ \lambda_2 = 7 \quad m_a(\lambda_2) = 1 \end{cases}$

$V_{\lambda_1} = ?$ şi $V_{\lambda_2} = ?$

$$V_{\lambda_1} = -2 = \{ \underset{\substack{\| \\ \begin{pmatrix} x \\ y \end{pmatrix}}}{\lambda} \in \mathbb{R}^2 \mid f(\lambda) = \underset{\substack{\| \\ -2}}{\lambda} v \} \subset \mathbb{R}^2$$

$$(\lambda I_2 - A_f) v = 0_{21}$$

$$(-2 I_2 - A_f) v = 0_{21} \iff \begin{pmatrix} -5 & -4 \\ -5 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -5x - 4y = 0 \\ -5y - 4y = 0 \end{cases} \Bigg| \Rightarrow \begin{matrix} x = -\frac{4}{5}\alpha \in \mathbb{R} \\ y = \alpha \in \mathbb{R} \end{matrix}$$

$$V_{\lambda_1} = -2 = \{ \alpha \underbrace{\left(-\tfrac{4}{5}, 1\right)}_{v_1} \mid \alpha \in \mathbb{R} \} =$$

$$B_1 = \{ v_{\lambda_1} \} \subset V_{\lambda_1}$$

analog $V_{\lambda_2} = 7 = \{ \lambda \in \mathbb{R}^2 \mid f(\lambda_2) = \lambda_2 v \} \subset \mathbb{R}^2$

$$(\lambda I_2 - A_f) v = 0_{21} \iff \begin{pmatrix} 4 & -4 \\ -5 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} 4x - 4y = 0 \\ -5x + 5y = 0 \end{cases} \Bigg| \Rightarrow x = y = \alpha$$

$$V_{\lambda_2} = 7 = \{ \underbrace{\alpha(1, 1)}_{v_2} \mid \alpha \in \mathbb{R} \} \Rightarrow B_2 = \{ v_2 \} \subset V_{\lambda_2}$$

d) $A_f = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$

1) $\underset{\substack{\| \\ 1}}{m_a(\lambda_1)} + \underset{\substack{\| \\ 1}}{m_a(\lambda_2)} = \dim_{\mathbb{R}} \mathbb{R}^2 = 2$

2) $m_a(\lambda_i) = m_g(\lambda_i) \quad i = \overline{1,2}$

$m_g(\lambda_1 = -2) = \dim_{\mathbb{R}} V_{\lambda_1} = 1$

$m_g(\lambda_2 = 7) = \dim_{\mathbb{R}} V_{\lambda_2} = 1$

$\Rightarrow$

$\Rightarrow f$ end. diag.

$D = \begin{pmatrix} -2 & 0 \\ 0 & 7 \end{pmatrix}$ forma diag.

e) $f: \mathbb{R}^2 \to \mathbb{R}^2$ end. liniar

$B_0 \xrightarrow{\tilde{S}} \tilde{B}$ unde $\begin{cases} B_0 \text{ în rap. cu } A_f \\ \tilde{B} \text{ în rap. cu } \Delta = \tilde{S}^{-1} A_f \tilde{S} \end{cases}$

$\tilde{B} = B_1 \cup B_2 = \{v_1, v_2\}$

$$\tilde{S} = \begin{pmatrix} -4 & 1 \\ 5 & 1 \end{pmatrix}$$

f) $A_f^m = ?$ $m \in \mathbb{N}^*$

$\Delta = \tilde{S}^{-1} A_f \tilde{S}$ | $\tilde{S}$ la stg și $\tilde{S}^{-1}$ la dr

$A_f = \tilde{S} \Delta \tilde{S}^{-1}$

$$A_f^m = \tilde{S} \Delta \underbrace{\tilde{S}^{-1} \tilde{S}}_{I_2} \Delta \tilde{S}^{-1} \dots \tilde{S} \Delta \tilde{S}^{-1} = \tilde{S} \Delta^m \tilde{S}^{-1}$$

$$\Delta^m = \begin{pmatrix} (-2)^m & 0^m \\ 0^m & 7^m \end{pmatrix} = \begin{pmatrix} (-2)^m & 0 \\ 0 & 7^m \end{pmatrix}$$

$$\tilde{S} = \begin{pmatrix} -4 & 1 \\ 5 & 1 \end{pmatrix} \qquad \tilde{S}^{-1} = \begin{pmatrix} -1/9 & 1/9 \\ +5/9 & +4/9 \end{pmatrix}$$

$\boxed{\text{Temă acasă}}$

(2) $f: \mathbb{R}^3 \to \mathbb{R}^3$ endom. φ

$f(x, y, z) = (3x + y - z,\ 2y,\ x + y + z)$

a) $B_0 = \{e_1, e_2, e_3\} \subset \mathbb{R}^3$

$e_1 = (1, 0, 0)$ $e_2 = (0, 1, 0)$ $e_3 = (0, 0, 1)$

$$A_f = \begin{array}{c} \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \\ \begin{array}{ccc} f(e_1) & f(e_2) & f(e_3) \end{array} \end{array}$$

b) Val. proprii ?

$\det(\lambda I_3 - A_f) = 0$

$$\begin{vmatrix} \lambda-3 & -1 & +1 \\ 0 & \lambda-2 & 0 \\ -1 & -1 & \lambda-1 \end{vmatrix} = 0 \iff$$

$(\lambda-2)(-1)^4 [(\lambda-1)(\lambda-3)+1] = 0 \iff (\lambda-2)^3 = 0 \Rightarrow$

$\Rightarrow \lambda \in \{2\} \quad m_a(\lambda_1) - 3$

c) Stab. ~~desp.~~ proprii

$V_{\lambda_1} - 2 = \{ v \in \mathbb{R}^3 \mid f(v) = \lambda_1 v \}$

$(\lambda_1 I_3 - A_f) v = 0_{31}$

$(2 I_3 - A_f) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{cases} -x - y + z = 0 \\ 0 = 0 \quad \text{adev.} \\ -x - y + z = 0 \end{cases}$ $\Rightarrow -x - y + z = 0$ $\Rightarrow x = \beta - \alpha$

$rg(2I_3 - A_f) = 1$ $\quad y = \alpha$ și $z = \beta$

$V_{\lambda_1} = 2 = \{ (-\alpha + \beta, \alpha, \beta) \in \mathbb{R}^3 \mid \alpha, \beta \in \mathbb{R} \}$

d) f end. diag?

P_1 ✓

P_2 $m_g(\lambda_1) = \dim_{\mathbb{R}} V_{\lambda_1}$

$(-\alpha + \beta, \alpha, \beta) \in V_{\lambda} \iff (\alpha, \alpha, 0) + (\beta, 0, \beta) \in V_{\lambda_1}$

$\iff \alpha(-1, 1, 0) + \beta(1, 0, 1) \in V_{\lambda_1}$

$\iff \alpha v_1 + \beta v_2 \in V_{\lambda_1}$

$B_1 = \{ v_1, v_2 \} \subset V_{\lambda_2} \Rightarrow \dim_{\mathbb{R}} V_{\lambda_1} = 2 \Rightarrow$ ~~$m_g(\lambda_1) = 2 \neq 3 = m_a(\lambda_1)$~~

$m_g(\lambda_1) = 2 \neq 3 = m_a(\lambda_1)$

f end. nu e diag.