03.04.2024

Seminar 6

① Altă variantă a ex. din seminarul trecut la subpct d)

$Im f = \{(x', y', z') \in \mathbb{R}^3 \mid x' - y' + z' = 0\} \subset \mathbb{R}^3 \Rightarrow$

$Im f \ni (x', y', z')$, $x' - y' + z' = 0$

$$y' = x' + z'$$

$\cancel{(x', x'+z', z') = (x', x', 0) + (0, z}$

$(x', x'+z', z') = (x', x', 0) + (0, z', z') =$

$= x'(1,1,0) + z'(0,1,1) =$

$= x' v_1 + y' v_2$

$S = \{v_1, v_2\} \subset Im f \Rightarrow \dim_{\mathbb{R}} Im f = 2$

② $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (x+y-2z, 2x+y+z, 3x+2y-z)$

a) apl. lin.? (endomorf. liniar chiar) - TEMĂ

b) $Ker f$, $Im f$, B_1 pt. $Ker f$ și B_2 pt $Im f$

c) f. inj, surj., bij?

d) Th. rang. defect

b) $Ker f = \{v \in \mathbb{R}^3 \mid f(v) = 0_{\mathbb{R}^3}\} \subseteq \mathbb{R}^3$ subsp. vect.

$v = (x, y, z)$

$$f(x,y,z) = (0,0,0) \Rightarrow \begin{cases} x+y+2z = 0 \\ 2x+y+z = 0 \\ 3x+2y-z = 0 \end{cases}$$

Fie $A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$$\begin{array}{ccc} \| & \| & \| \\ f(e_1) & f(e_2) & f(e_3) \end{array}$$

$$\det A = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \end{vmatrix} = 0 \Rightarrow rg\, A \le 2 \quad \Big| \Rightarrow rg\, A = 2$$

$$\Delta_p = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \neq 0$$

x, y nec. principale

$z = \alpha \in \mathbb{R}$ nec. sec.

$$\begin{cases} x + y = 2\alpha \\ 2x + y = -\alpha \\ z = \alpha \in \mathbb{R} \end{cases} \Leftrightarrow \begin{cases} x = -3\alpha \\ y = 5\alpha \\ z = \alpha \in \mathbb{R} \end{cases} \Bigg| \Rightarrow S = \{(-3\alpha, 5\alpha, \alpha) \mid \alpha \in \mathbb{R}\}$$

$$\mathrm{Ker}\, f = \{ \alpha \underbrace{(-3, 5, 1)}_{u} \mid \alpha \in \mathbb{R} \} = \langle u \rangle$$

$B_1 = \{u\} \subset \mathrm{Ker}\, f$

$\mathrm{Im} f = \{ (x', y', z') \in \mathbb{R}^3 \mid \exists\, f(x, y, z) = (x', y', z') \} \subseteq \mathbb{R}^3$

$\mathrm{Im} f$ subsp. vect.

$$\begin{cases} x + y - 2z = x' \\ 2x + y + z = y' \\ 3x + 2y - z = z' \end{cases} \Bigg| \xrightarrow[\text{elim}]{x, y, z} x' + y' = z' \quad \text{(var. observabilă)}$$

$\alpha x' + \beta y' + \gamma z' = 0$ (var 2)

var 3 : Fie $f : V^n \to W^m$ apl. lin (Th.)

a) f inj $\Leftrightarrow \forall\, S = \{e_1, \ldots e_k\} \subset V \Rightarrow \{f(e_1), \ldots f(e_k)\} \subset W$
$\quad n \le m \qquad k \le n \quad$ S.L.i $\qquad k \le m \quad$ S.L.i

b) f surj $\Leftrightarrow \forall\, \tilde{S} = \{f_1, \ldots f_k\} \subset V \Rightarrow \{f(f_1), \ldots f(f_k)\} \subset W$
$\quad m \le n \qquad$ S.G. $\qquad$ S.G.

c) f bij $\Leftrightarrow \forall\, S' \subset V \Rightarrow f(S') \subset W \Rightarrow n = m$
$\qquad$ bază $\qquad$ bază

$$\mathrm{Im} f \ni (x', y', z') = (x', y', x' + y') = (x', 0, x') + (0, y', y') =$$
$$= x'(1, 0, 1) + y'(0, 1, 1) = x' v_1 + y' v_2$$

$B_2 = \{v_1, v_2\} \subset \mathrm{Im}\, f$ S.G. + SLi și $\mathrm{Im} f \subset \mathbb{R}^3$

$\dim_{\mathbb{R}} \mathrm{Im} f = 2$ (și rg. A o poate afla)

c) $\mathrm{Ker} f \neq \{0_{\mathbb{R}}\} \Rightarrow f$ nu e inj.
$\mathrm{Im} f \neq \mathbb{R}^3 \Rightarrow f$ nu e surj. $\Big| \Rightarrow f$ nu e bij.

d) Conform T. rg. dif $\Rightarrow \dim_{\mathbb{R}} \mathrm{Ker} f + \dim_{\mathbb{R}} \mathrm{Im} f = \dim_{\mathbb{R}} \mathbb{R}^3$

~~$\Rightarrow$ dr~~

$$\left.\begin{array}{l} \dim_{\mathbb{R}} \mathrm{Ker} f = 1 \\ \dim_{\mathbb{R}} \mathrm{Im} f = 2 \\ \dim_{\mathbb{R}} \mathbb{R}^3 = 3 \end{array}\right| \Rightarrow 1+2=3 \rightarrow \text{Th. adev.}$$

(3) TEMĂ (+ POZĂ), multe poze

Fie $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) \;\middle|\; \begin{cases} a-b+c-d=0 \\ a+2b-c+3d=0 \end{cases} \right\}$

a) $V \subset \mathcal{M}_2(\mathbb{R})$ subsp. vect.

b) ~~det~~. o bază & $\dim_{\mathbb{R}} V = ?$

Teorema dimensiunii (Grassman)

Fie V/K sp. vect., $V_1, V_2 \subset V/K$ subsp. vect.

Atunci $\dim_K (V_1 + V_2) = \dim_K V_1 + \dim_K V_2 - \dim_K (V_1 \cap V_2)$

$V_1 + V_2 = \{ v \in V \mid (\exists)\, v_1 \in V_1$ a.î. $v = v_1 + v_2,\ v_2 \in V_2 \}$

$V_1 + V_2 = \langle V_1 \cup V_2 \rangle$ (reuniunea)

(4) Fie $V_1 = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \subset \mathbb{R}^3$

$V_2 = \{(t, 0, t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^3$

a) $V_1, V_2 \subset \mathbb{R}^3$ subsp. vect.

$V_1 \ni (x, y, 0) = (x, 0, 0) + (0, y, 0) = x e_1 + y e_2$

unde $e_1 = (1, 0, 0)$ și $e_2 = (0, 1, 0)$

$S_1 = \{e_1, e_2\} \subset V_1$ SG, SLI $\Rightarrow S_1$ și bază

$\dim_{\mathbb{R}} V_1 = 2$ (plan vect)

analog $V_2 = \{ t(1, 0, 1) \mid t \in \mathbb{R} \} = \langle u \rangle \Rightarrow S_2 = \{u\} \subset V_2 \Rightarrow$

$\Rightarrow S_2$ și bază $\rightarrow \dim_K V_2 = 1$ (dr.)

b) $V_1 + V_2 = ?$

Aplicăm Th. Grassman $\Rightarrow$

$\Rightarrow \dim_{\mathbb{R}}(V_1+V_2) = \dim_{\mathbb{R}} V_1 + \dim_{\mathbb{R}} V_2 - \dim_{\mathbb{R}}(V_1 \cap V_2)$

$V_1 \cap V_2 \ni v = \begin{cases} (x, y, 0) \\ (t, 0, t) \end{cases} \Rightarrow t = x = y = 0 \rightarrow V_1 \cap V_2 = \{0_{\mathbb{R}}\}$

$\dim_{\mathbb{R}}(V_1 \cap V_2) = 0$

$\left.\begin{array}{l} \dim_{\mathbb{R}}(V_1+V_2) = 1+2-0 = 3 \\ V_1+V_2 \subseteq \mathbb{R}^3 \text{ subsp. vect.} \end{array}\right| \Rightarrow V_1+V_2 = \mathbb{R}^3$

Pt. un ex din teme (ultim.):

$\mathcal{M}_n(K) = S \oplus A \qquad S: {}^tA = A$ și $A: {}^tB = -B$

$K = \mathbb{R}$

$\mathcal{M}_n(\mathbb{R}) = V_1 \oplus V_2 \Leftrightarrow \forall M \in \mathcal{M}_n(\mathbb{R}), \exists A \in V_1$

$V_1, V_2 \subset \mathcal{M}_n(K)$ subsp. vect.

$B \in V_2, B = \lambda I_n$