22.05.2024

Seminar 11

- Clasificarea transf. ortog. pt $n = 2$ și $n = 3$
- Forme pătratice
- M.V.P. (Met. transf. ortog)
- P.O.G-S
- Aplicații ortog.

P.O.G-S. formule:

I $\{f_1 \to f_n\}$ b. arbitrară

$e'_1 = f_1$

$$e'_i = f_i - \sum_{j=1}^{i-1} \frac{\langle f_i, e'_j \rangle}{\|e'_j\|^2} e'_j \quad \forall\, i = \overline{2,n}$$

$B' = \{e'_1, \dots e'_n\}$ b. ortog. $\xrightarrow{\text{normare}}$ $B = \{e_1, \dots e_n\}$ b. orton.

$$e_i = \frac{e'_i}{\|e'_i\|} \quad \forall i = \overline{1,n}$$

II $\{f_1 \to f_n\}$ b. arbitrară

$e'_1 = f_1$ unde $e_1 = \frac{f_1}{\|f_1\|}$

$$e_i = \frac{e'_i}{\|e'_i\|} \text{ unde } e'_i = f_i - \sum_{j=1}^{i-1} \langle f_i, e_j \rangle e_j \quad \forall\, i = \overline{2,n}$$

$B = \{e_1, \dots e_n\}$ b. orton.

Metoda val. proprii (Met transf. ortog)

Fie $Q: V \to \mathbb{R}$ formă pătratică asociată formei biliniare $g: V \times V \to \mathbb{R}$, $Q(x) = g(x,x)$ $\forall\, x \in V$

$G \to$ mat. asoc. f.p. în rap. cu baza B

Aducerea f.p. Q la o formă canonică
adică $Q(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_m x_m^2$ $\quad r = rg\, G$

$$G \to \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & & \ddots & \\ 0 & 0 & 0 & \lambda_r \end{pmatrix}$$

1) Met. Gauss

2) Met. Jacobi

3) Met val. proprii / Met transf. ortogonale

Det val. proprii pt. mat G simetrică (mereu)

În rap. cu o bază B' formată din vectori proprii, ortonormați prin $P.O. G = S$

La val. proprii distincte corespund vectori proprii ortogonali

$\{\lambda_1, \dots \lambda_p\}$ $\quad V_{\lambda_1}, \dots V_{\lambda_p}$ ssp. pr.

$m_1, \dots m_p \to$ ~~m. alg~~ multiplicități algebrice $\quad B_1, \dots B_p$

$\boxed{P}$ $\lambda_i = \lambda_j$

V_{λ_1} $\quad V_{\lambda_j}$ $\quad v \perp w$

$Q(x) = \lambda_1 x_1^2 + \dots + \lambda_p x_p^2$, unde $x = (x_1', \dots x_m')$

① Fie forma pătratică $Q: \mathbb{R}^3 \to \mathbb{R}$,

$Q(x) = x_1^2 + 2x_2^2 + 3x_3^2 - 4x_1x_2 - 4x_2x_3$ $\quad \forall x = (x_1, x_2, x_3)$

Să se aducă Q la o formă canonică utilizând

a) Met. Gauss

$Q(x) = (x_1 - 2x_2)^2 - 4x_2^2 + 2x_2^2 + 3x_3^2 - 4x_2x_3 =$

$= (x_1 - 2x_2)^2 - 2\{(x_2 + x_3)^2 - x_3^2\} + 3x_3^2 =$

$= (x_1 - 2x_2)^2 - 2(x_2 + x_3)^2 + 5x_3^2 = y_1^2 - 2y_2^2 + 5y_3^2$

$y_1 = x_1 - 2x_2 \qquad y_2 = x_2 + x_3 \qquad y_3 = x_3$

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \det A = 1 \neq 0$$

pt $\begin{cases} y_1 = \\ y_2 = \\ y_3 = \end{cases}$

$z_1 = y_1$; $z_2 = y_2\sqrt{2}$

$z_3 = \sqrt{5}\, y_3$

$$\begin{cases} x_1 = y_1 + 2y_2 - 2y_3 \\ x_2 = y_2 - y_3 \\ x_3 = y_3 \end{cases} \Rightarrow A' = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow B' = \{v_1, v_2, v_3\}$$

$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$

B' baza în rap cu coord vect $\Rightarrow$ se obține forma canonică

b) Metoda Jacobi

$$G = \begin{pmatrix} \boxed{1} & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix} \Rightarrow \begin{aligned} \Delta_1 &= 1 \neq 0 \\ \Delta_2 &= 2 - 4 = -2 \neq 0 \\ \Delta_3 &= -10 \neq 0 \end{aligned}$$

$$Q(x) = \frac{1}{\Delta_1} y_1^2 + \frac{\Delta_1}{\Delta_2} y_2^2 + \frac{\Delta_2}{\Delta_3} y_3^2$$

unde $x = (y_1, y_2, y_3)$ coord. vect. pt f. can Jacobi

~~$z_1 = y_1$~~ $Q(x) = y_1^2 - \frac{1}{2} y_2^2 + \frac{1}{5} y_3^2 \quad x = (y_1, y_2, y_3)$

c) M.V.P.

$$G = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix} \to$$ m. asoc. f p Q în rap cu

baza $B_0 = \{e_1, e_2, e_3\} \subset \mathbb{R}^3$

Det. v.p. corесp. mat G.

Rezolv. ec. c. în $K = \mathbb{R}$

$$P(\lambda) = 0 \Leftrightarrow \det(G - \lambda I_3) = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 2-\lambda & -2 \\ 0 & -2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(3-\lambda) - 4(3-\lambda) - 4(1-\lambda) = 0$$

$$-\lambda^3 + 6\lambda^2 - 3\lambda - 10 = 0$$

	λ^3	λ^2	λ^1	λ^0
	-1	6	-3	-10
-1	-1	7	-10	0

$\Rightarrow \lambda_1 = -1 \Rightarrow \lambda_2 = 2$ și $\lambda_3 = 5$

$Q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

$Q(x) = -y_1^2 + 2y_2^2 + 5y_3^2$ unde $x = (y_1, y_2, y_3)$ coord. vect x în rap. cu baza B' în care se realiz. f. can. constantă

$V_{\lambda_1} = \langle \underbrace{(2,2,1)}_{v_1} \rangle$

$V_{\lambda_2} = \langle \overbrace{(-2,1,2)}^{v_2} \rangle$ ~~(e gresit un minus undeva)~~

$V_{\lambda_3} = \langle \underbrace{(1,-2,2)}_{v_3} \rangle$

$v_1 \perp v_2,\ v_2 \perp v_3,\ v_1 \perp v_3 \Rightarrow$

$\Rightarrow B'' = \{v_1, v_2, v_3\}$ bază ortogonală

$e_1 = \frac{v_1}{\|v_1\|},\ e_2 = \frac{v_2}{\|v_2\|},\ e_3 = \frac{v_3}{\|v_3\|}$

$e_1 = \frac{1}{3} v_1,\ e_2 = \frac{1}{3} v_2,\ e_3 = \frac{1}{3} v_3$

$B' = \left\{ \frac{1}{3}(2,2,1), \frac{1}{3}(-2,1,2), \frac{1}{3}(1,-2,2) \right\}$ b. ortom.

Produsul vectorial în $\mathbb{R}^3/\mathbb{R}$

Fie $B_0 = \{e_1, e_2, e_3\} \subset \mathbb{R}^3$ b. canonică

$\langle , \rangle \rightarrow$ produsul scalar canonic pe $\mathbb{R}^3$

notez cu $\det(x, y, z) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

unde $x = \sum_{i=1}^{3} x_i e_i$

$y = \sum_{i=1}^{3} y_i e_i$

$z = \sum_{i=1}^{3} z_i e_i$

Def: Fie $x, y \in \mathbb{R}^3$ unic vector $\in \mathbb{R}^3$ not $x \times y$ și

def prin $\langle x \times y, z \rangle = \det(x, y, z)$ $\forall z \in \mathbb{R}^3$ PA

$z = e_1$, $z = e_2$, $z = e_3$

$$x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} e_1 - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} e_2 +$$

$$+ \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} e_3 = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$$

Proprietăți:

1) $x \times y = -y \times x$ $\forall x, y \in \mathbb{R}^3$ anticomutativitate

2) $\otimes$ $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ apl. bilin. ($\mathbb{R}$-bilin)

$(a_1 x_1 + a_2 x_2) \times y = a_1(x_1 \times y) + a_2(x_2 \times y)$

$x \times (b_1 y_1 + b_2 y_2) = b_1(x \times y_1) + b_2(x \times y_2)$

3) $x \times y = 0 \iff \{x, y\}$ S.L.D

4) $\begin{cases} \langle x \times y, x \rangle = 0 \\ \langle x \times y, y \rangle = 0 \end{cases}$ i.e. $x \times y \perp x$, $\perp y$

$R = \{x, y, x \times y\}$ reper. vect. (orientat) pozitiv

$\det(x, y, x \times y) = \langle x \times y, x \times y \rangle = \|x \times y\|^2 > 0$

$\langle x \times y, z \rangle = \det(x, y, z)$

Lemă: $\langle u \times v, x \times y \rangle = \begin{vmatrix} \langle u, x \rangle & \langle v, x \rangle \\ \langle u, y \rangle & \langle v, y \rangle \end{vmatrix}$

Caz particular: $u = x$, $v = y$

$\|x \times y\|^2 = \langle x \times y, x \times y \rangle = \|x\|^2 \|y\|^2 - \langle x, y \rangle^2$

$= \|x\|^2 \|y\|^2 - \langle x, y \rangle^2$

$\cos \theta = \dfrac{\langle x, y \rangle}{\|x\| \|y\|} \Rightarrow \langle x, y \rangle = \|x\| \|y\| \cos \theta$

$$\Rightarrow \|x \times y\|^2 = \|x\|^2 \|y\|^2 \underbrace{(1-\cos^2\theta)}_{\sin^2\theta}$$

$$\|x \times y\| = \|x\| \|y\| \sin\theta$$