Algorithms and Data Structures

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Course 0: Heapsort, Lower bounds.

Slide credit (partial): David Luebke (Virginia)

Sorting Revisited

So far: two algorithms to sort an array of numbers

What is the advantage of merge sort?

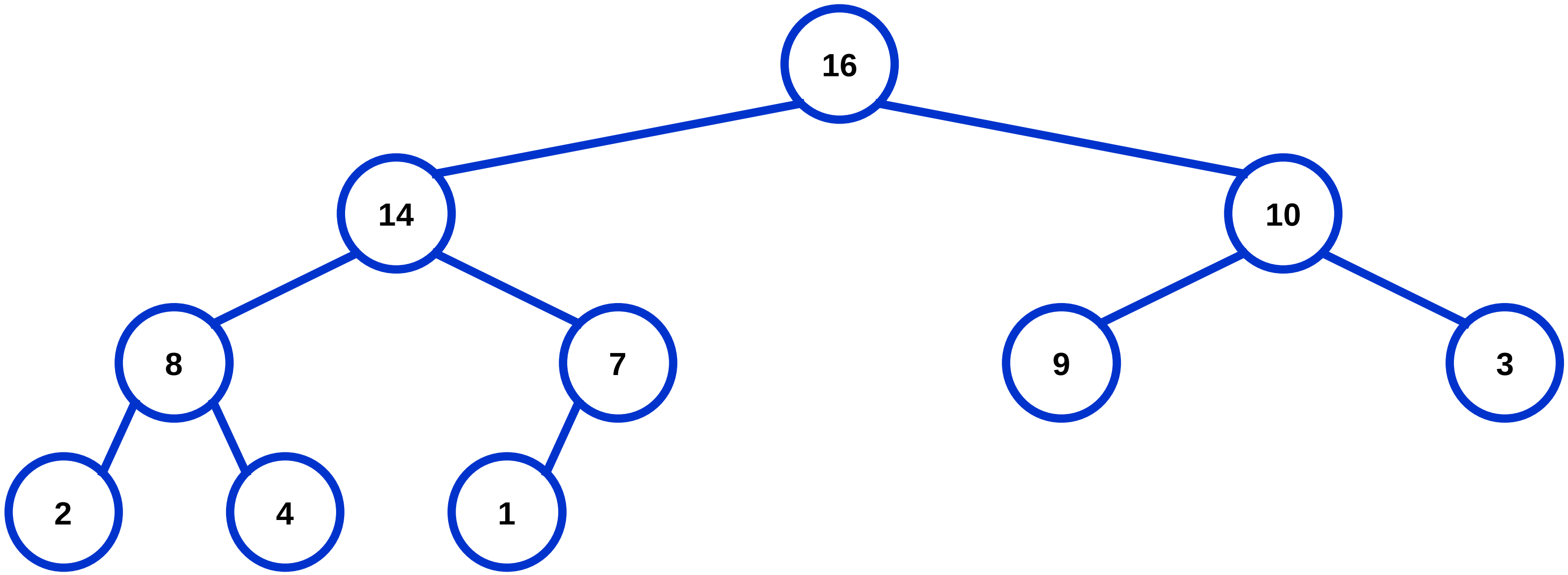
What is the advantage of insertion sort?

Next on the agenda: *Heapsort*

Combines advantages of both previous algorithms

Heaps

A *heap* can be seen as a complete binary tree:

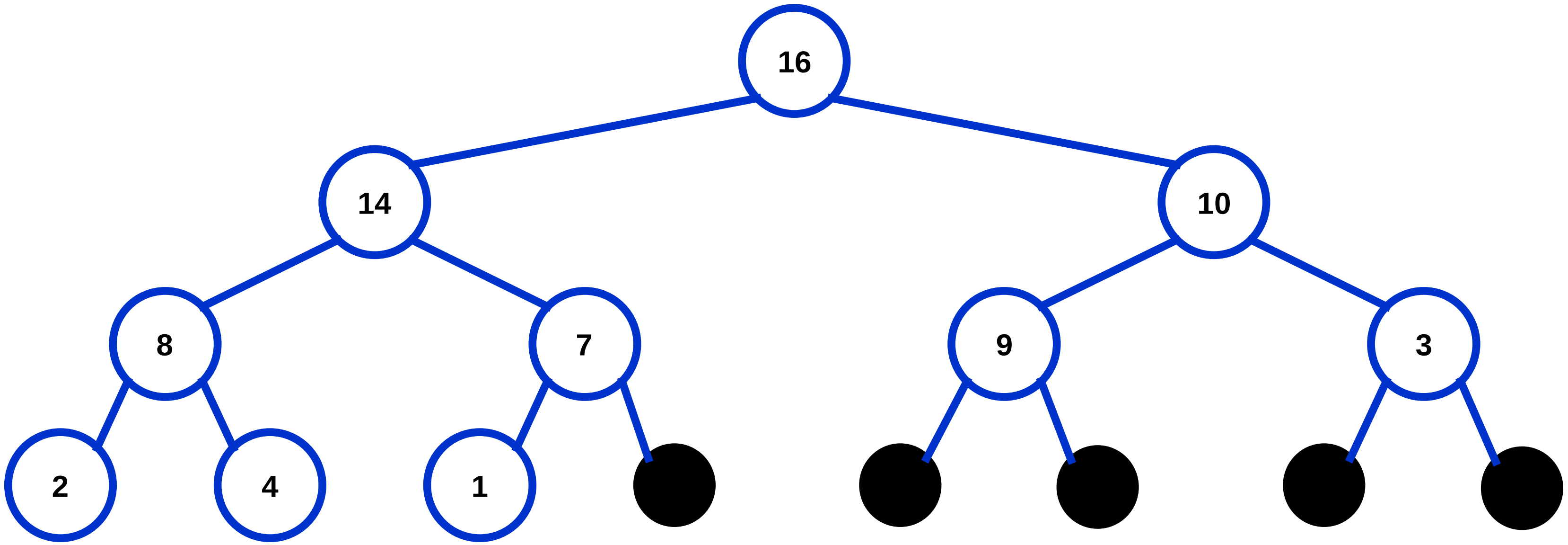


What makes a binary tree complete?

Is the example above complete?

Heaps

A *heap* can be seen as a complete binary tree:



The book calls them “nearly complete” binary trees; can think of unfilled slots as null pointers

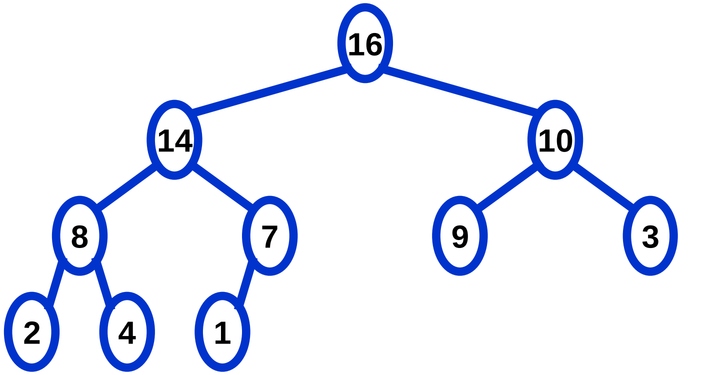
Heaps

In practice, heaps are usually implemented as arrays:

$A =$

16	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

=



Heaps

To represent a complete binary tree as an array:

The root node is A[1]

Node *i* is A[*i*]

The parent of node *i* is A[*i*/2] (note: integer divide)

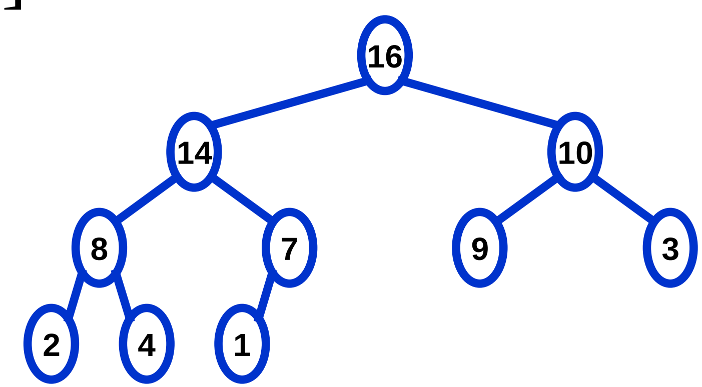
The left child of node *i* is A[2*i*]

The right child of node *i* is A[2*i* + 1]

A =

16	14	10	8	7	9	3	2	4	1

=



Referencing Heap Elements

So…

```
Parent(i) { return ⌊i/2⌋; }
Left(i) { return 2*i; }
right(i) { return 2*i + 1; }
```

An aside: *How would you implement this most efficiently?*

Another aside: *Really?*

The Heap Property

Heaps also satisfy the *heap property*:

$A[\mathit{Parent}(i)] \geq A[i]$ for all nodes $i > 1$

In other words, the value of a node is at most the value of its parent

Where is the largest element in a heap stored?

Definitions:

The *height* of a node in the tree = the number of edges on the longest downward path to a leaf

The height of a tree = the height of its root

Heap Height

What is the height of an n-element heap? Why?

This is nice: basic heap operations take at most time proportional to the height of the heap

log2(n) - inaltimea unui arbore binar

Heap Operations: Heapify()

`Heapify()`: maintain the heap property

Given: a node *i* in the heap with children *l* and *r*

Given: two subtrees rooted at *l* and *r*, assumed to be heaps

Problem: The subtree rooted at *i* may violate the heap property (*How?*)

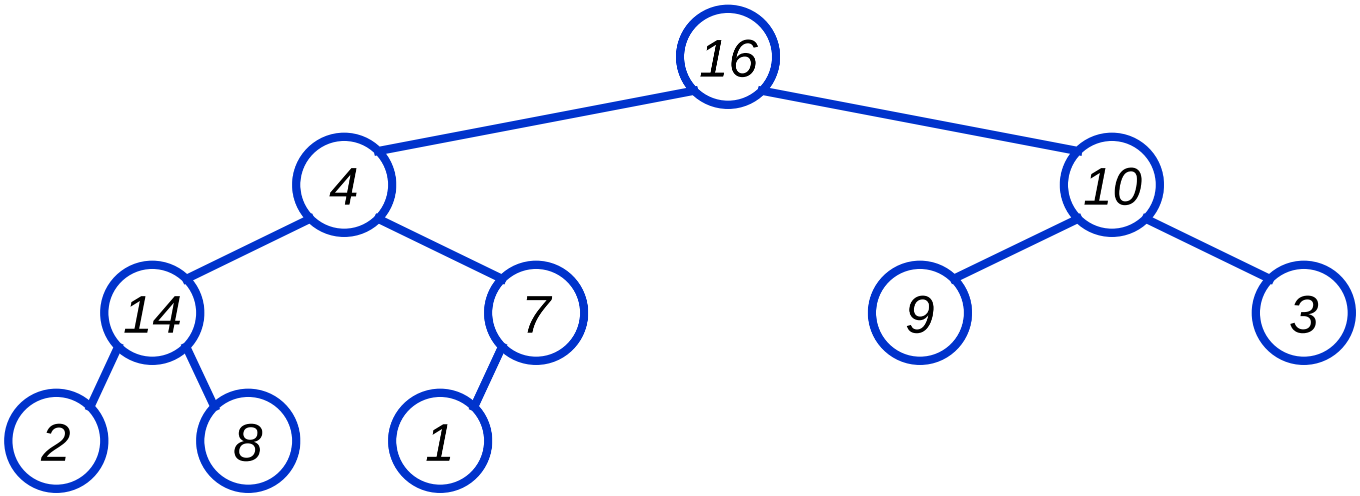
Action: let the value of the parent node "float down" so subtree at *i* satisfies the heap property

- *What do you suppose will be the basic operation between i, l, and r?*

Heap Operations: Heapify()

```
Heapify(A, i)
{
  l = Left(i); r = Right(i);
  if (l <= heap_size(A) && A[l] > A[i])
   largest = l;
  else
   largest = i;
  if (r <= heap_size(A) && A[r] > A[largest])
   largest = r;
  if (largest != i)
   Swap(A, i, largest);
   Heapify(A, largest);
}
```

Heapify() Example



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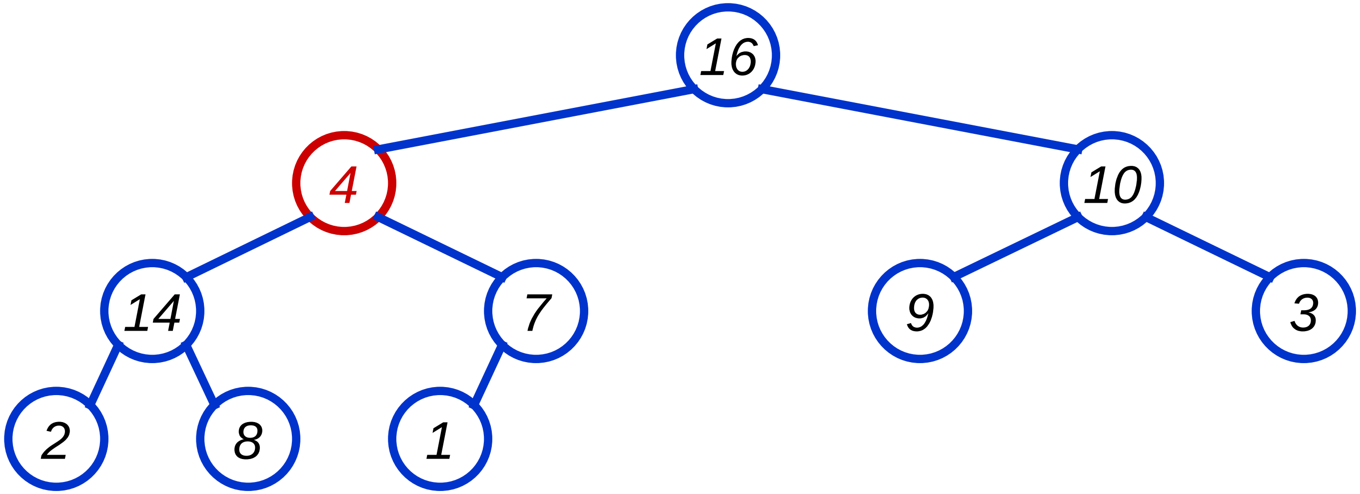
16	4	10	14	7	9	3	2	8	1

Heapify() Example



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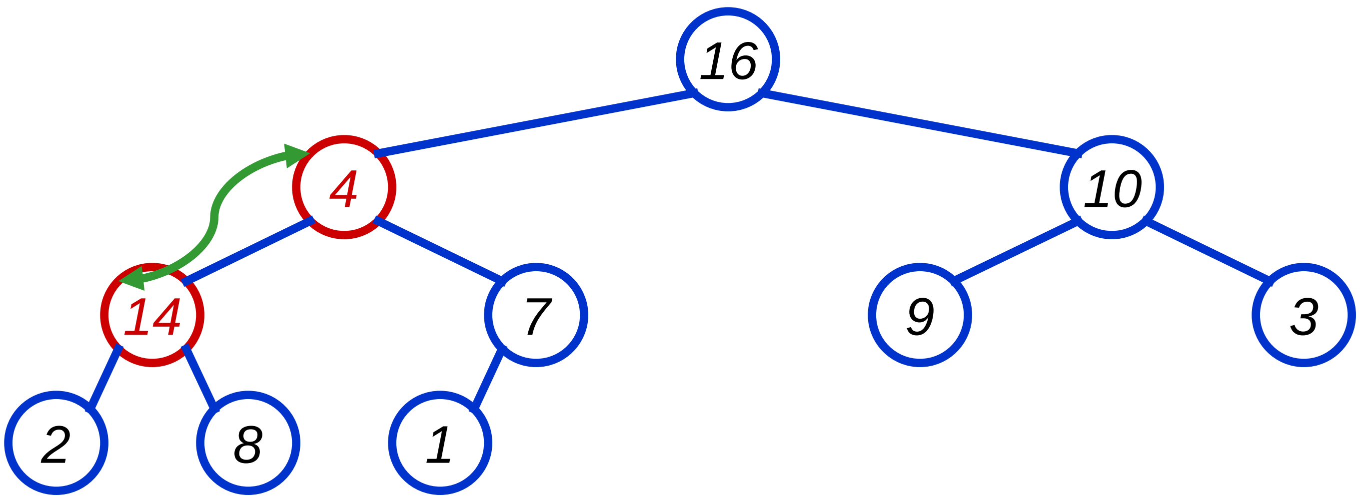

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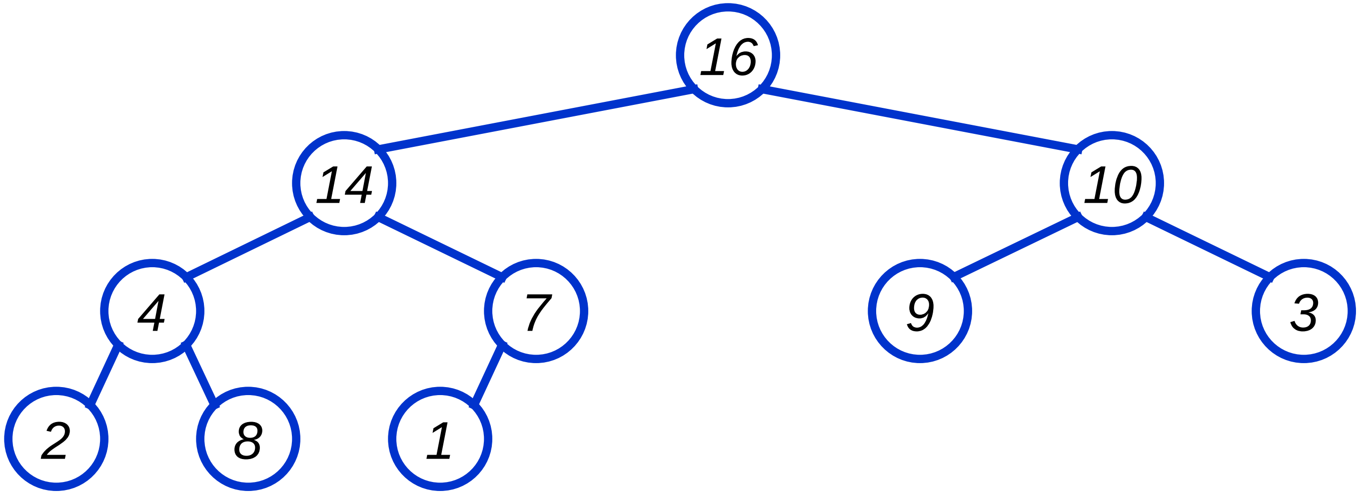

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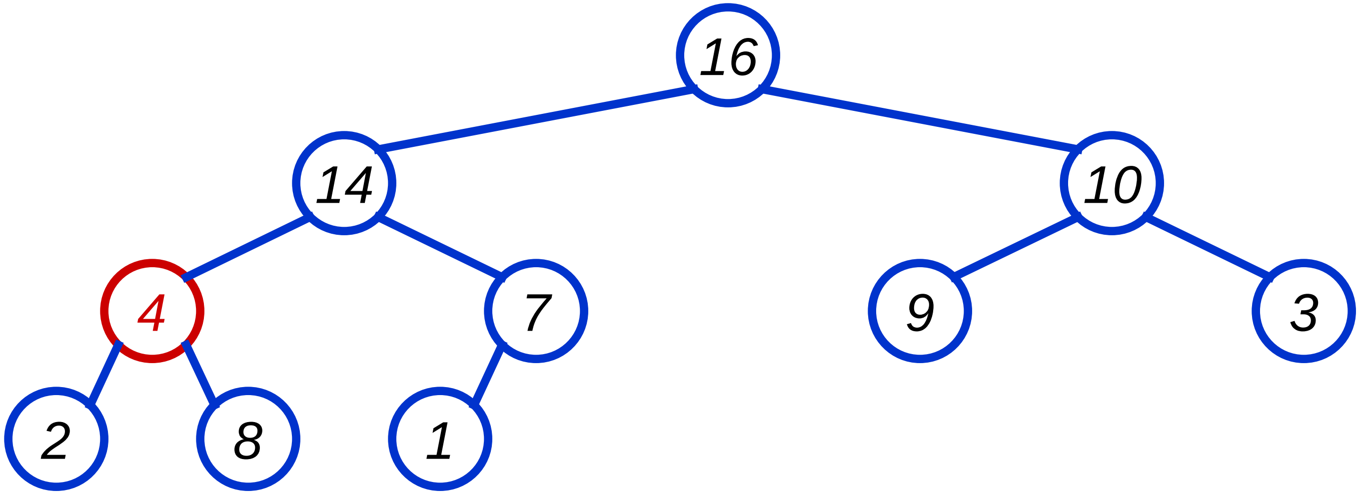

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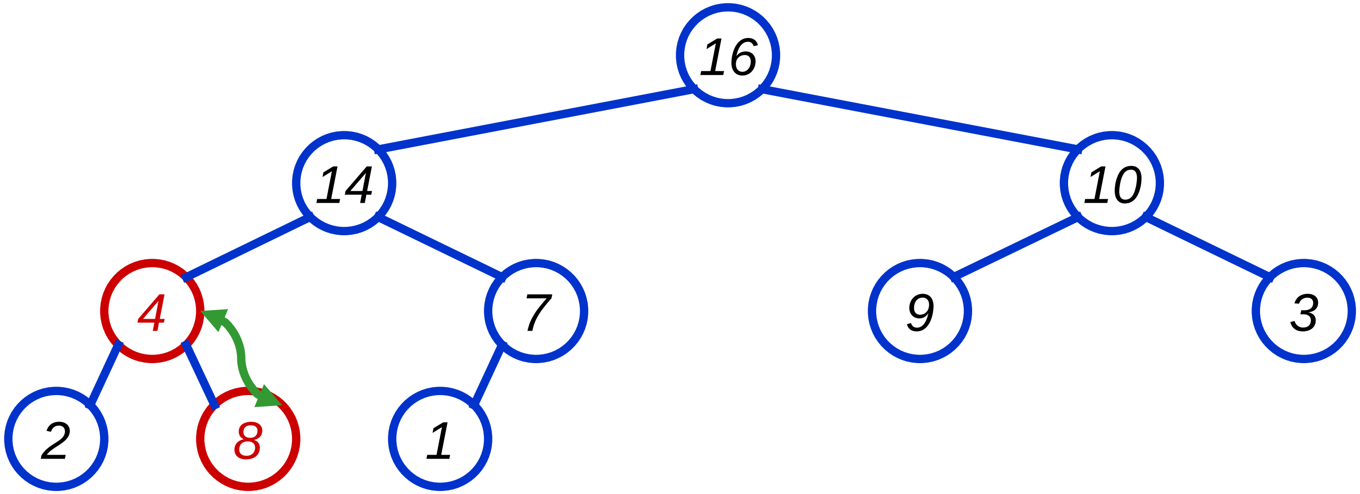

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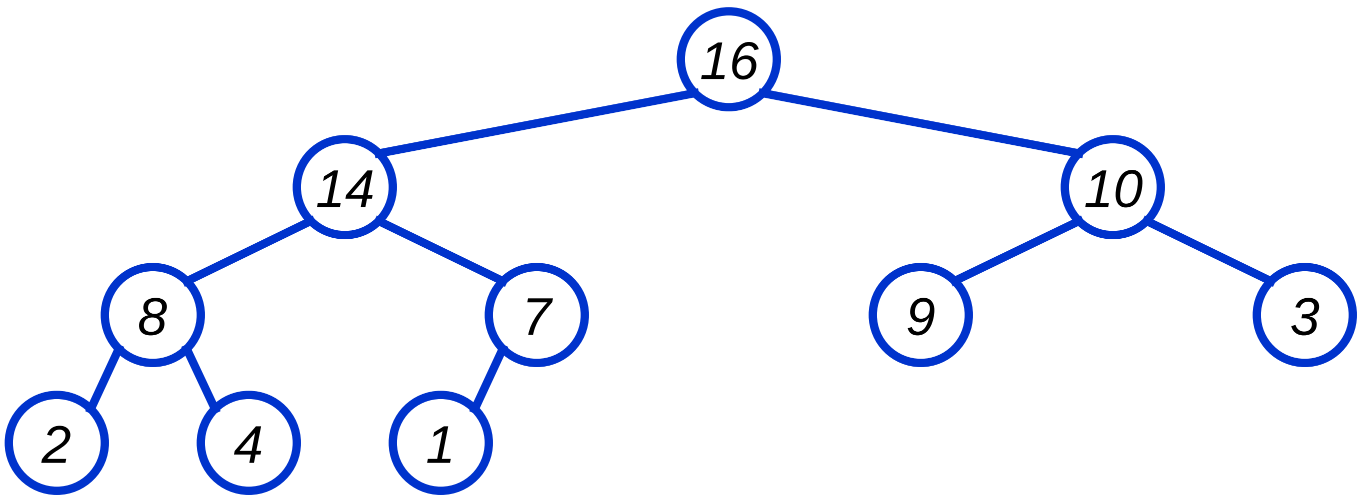

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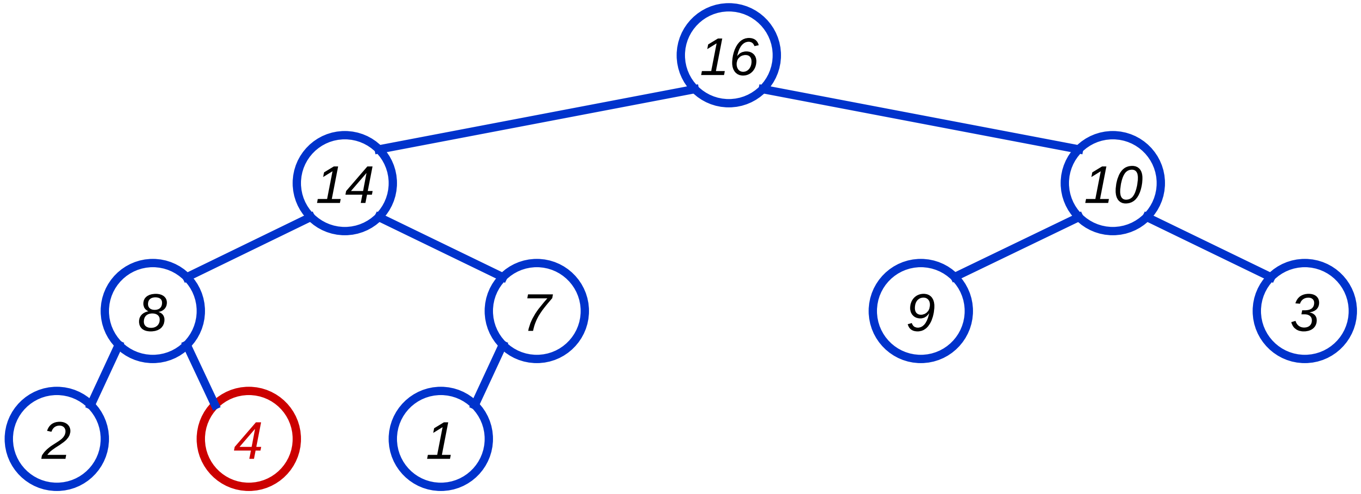

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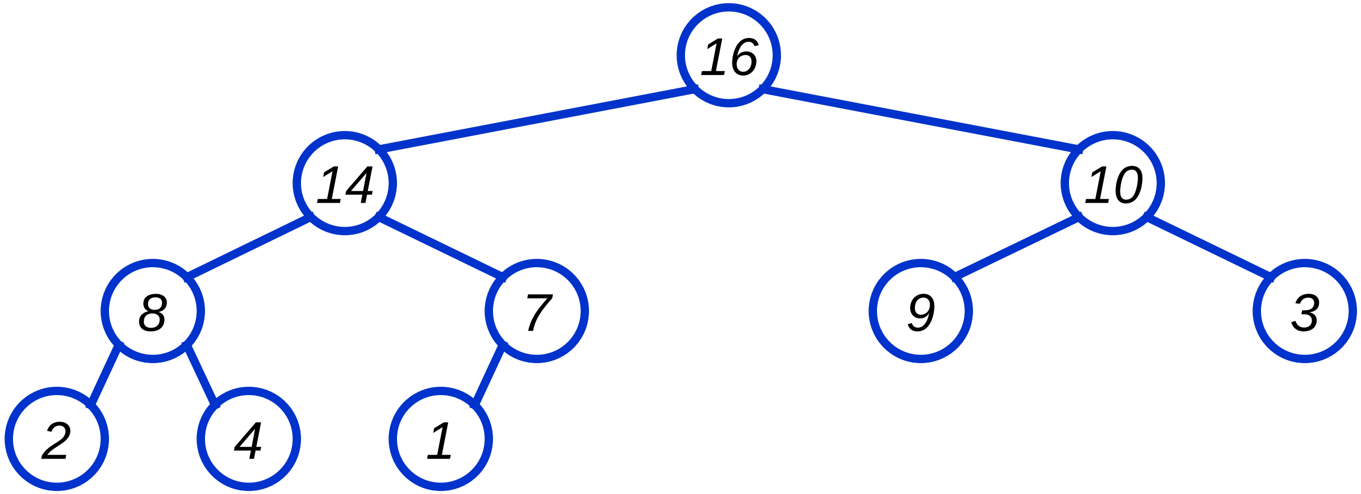

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Analyzing Heapify(): Informal

- *Aside from the recursive call, what is the running time of* **`Heapify()`***?*
- *How many times can* **`Heapify()`** *recursively call itself?*
- *What is the worst-case running time of* **`Heapify()`** *on a heap of size n?*

Analyzing Heapify(): Formal

- Fixing up relationships between i, l, and r takes $\Theta(1)$ time
- *If the heap at i has n elements, how many elements can the subtrees at l or r have?*
- Draw it
- Answer: $2n/3$ (worst case: bottom row 1/2 full)
- So time taken by **`Heapify()`** is given by

$$T(n) \leq T(2n/3) + \Theta(1)$$

Analyzing Heapify(): Formal

So we have

$$T(n) \le T(2n/3) + \Theta(1)$$

By case 2 of the Master Theorem,

$$T(n) = O(\lg n)$$

Thus, **`Heapify()`** takes linear time

Heap Operations: BuildHeap()

We can build a heap in a bottom-up manner by running **`Heapify()`** on successive subarrays

Fact: for array of length *n*, all elements in range A[$\lfloor n/2 \rfloor$ + 1 .. n] are heaps (*Why?*)

So:

- Walk backwards through the array from n/2 to 1, calling **`Heapify()`** on each node.
- Order of processing guarantees that the children of node *i* are heaps when *i* is processed

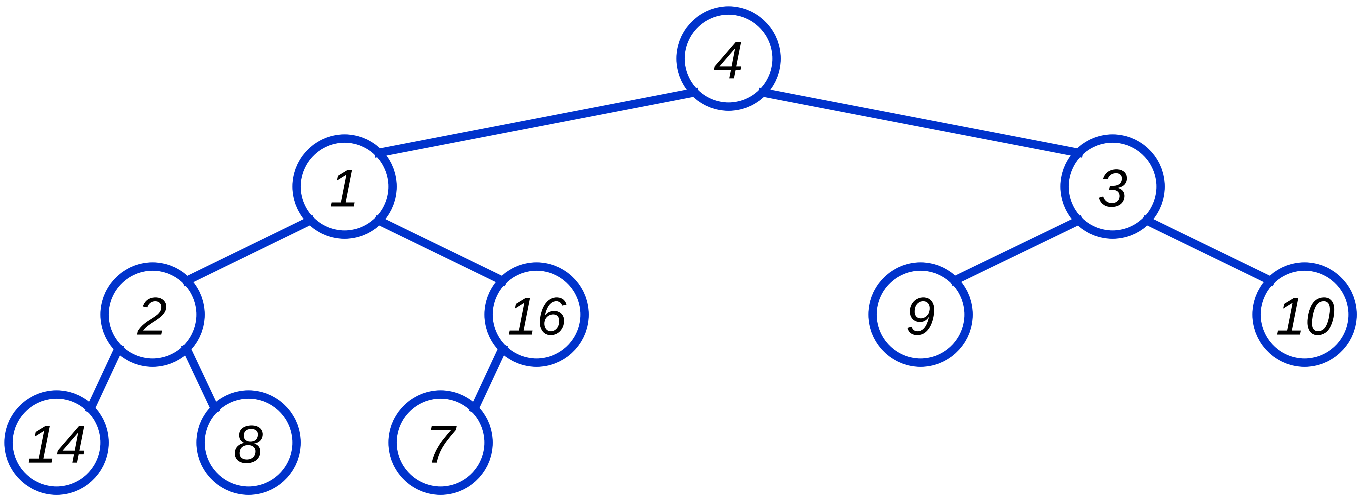
BuildHeap()

```
// given an unsorted array A, make A a heap
BuildHeap(A)
{
  heap_size(A) = length(A);
  for (i = ⌊length[A]/2⌋ downto 1)
   Heapify(A, i);
}
```

BuildHeap() Example

Work through example

A = {4, 1, 3, 2, 16, 9, 10, 14, 8, 7}



Analyzing BuildHeap()

Each call to **`Heapify()`** takes $O(\lg n)$ time

There are $O(n)$ such calls (specifically, $\lfloor n/2 \rfloor$)

Thus the running time is $O(n \lg n)$

Is this a correct asymptotic upper bound?

Is this an asymptotically tight bound?

A tighter bound is $O(n)$

How can this be? Is there a flaw in the above reasoning?

Analyzing BuildHeap(): Tight

- To **`Heapify()`** a subtree takes O(h) time where h is the height of the subtree
  - $h = O(\lg m)$, m = # nodes in subtree
  - The height of most subtrees is small
- Fact: an n-element heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height h
- CLR 7.3 uses this fact to prove that **`BuildHeap()`** takes O(n) time

Heapsort

Given **`BuildHeap()`**, an in-place sorting algorithm is easily constructed:

Maximum element is at A[1]

Discard by swapping with element at A[n]

- Decrement heap_size[A]
- A[n] now contains correct value

Restore heap property at A[1] by calling **`Heapify()`**

Repeat, always swapping A[1] for A[heap_size(A)]

Heapsort

```
Heapsort(A)
{
   BuildHeap(A);
   for (i = length(A) downto 2)
   {
      Swap(A[1], A[i]);
      heap_size(A) -= 1;
      Heapify(A, 1);
   }
}
```

Analyzing Heapsort

- The call to **`BuildHeap()`** takes $O(n)$ time
- Each of the $n - 1$ calls to **`Heapify()`** takes $O(\lg n)$ time
- Thus the total time taken by **`HeapSort()`**

$$= O(n) + (n - 1)\, O(\lg n)$$
$$= O(n) + O(n \lg n)$$
$$= O(n \lg n)$$

Priority Queues

- Heapsort is a nice algorithm, but in practice Quicksort (coming up) usually wins
- But the heap data structure is incredibly useful for implementing *priority queues*
  - A data structure for maintaining a set *S* of elements, each with an associated value or *key*
  - Supports the operations **`Insert()`**, **`Maximum()`**, and **`ExtractMax()`**
  - ***What might a priority queue be useful for?***

Priority Queue Operations

Insert(S, x) inserts the element x into set S

Maximum(S) returns the element of S with the maximum key

ExtractMax(S) removes and returns the element of S with the maximum key

How could we implement these operations using a heap?

Sorting So Far

- Insertion sort:
  - Easy to code
  - Fast on small inputs (less than ~50 elements)
  - Fast on nearly-sorted inputs
  - $O(n^2)$ worst case
  - $O(n^2)$ average (equally-likely inputs) case
  - $O(n^2)$ reverse-sorted case

Sorting So Far

- Merge sort:
  - Divide-and-conquer:
    - Split array in half
    - Recursively sort subarrays
    - Linear-time merge step
  - O(n lg n) worst case
  - Doesn’t sort in place

Sorting So Far

- Heap sort:
  - Uses the very useful heap data structure
    - Complete binary tree
    - Heap property: parent key > children's keys
  - O(n lg n) worst case
  - Sorts in place
  - Fair amount of shuffling memory around

Sorting So Far

- Quick sort:
  - Divide-and-conquer:
    - Partition array into two subarrays, recursively sort
    - All of first subarray < all of second subarray
    - No merge step needed!
  - $O(n \lg n)$ average case
  - Fast in practice
  - $O(n^2)$ worst case
    - Naïve implementation: worst case on sorted input
    - Address this with randomized quicksort

How Fast Can We Sort?

- We will provide a lower bound, then beat it
  - *How do you suppose we'll beat it?*
- First, an observation: all of the sorting algorithms so far are *comparison sorts*
  - The only operation used to gain ordering information about a sequence is the pairwise comparison of two elements
  - Theorem: all comparison sorts are Ω(n lg n)
    - A comparison sort must do O(n) comparisons (*why?*)
    - What about the gap between O(n) and O(n lg n)

Decision Trees

- *Decision trees* provide an abstraction of comparison sorts
  - A decision tree represents the comparisons made by a comparison sort. Every thing else ignored
  - (Draw examples on board)
- *What do the leaves represent?*
- *How many leaves must there be?*

Decision Trees

- Decision trees can model comparison sorts. For a given algorithm:
  - One tree for each n
  - Tree paths are all possible execution traces
  - *What's the longest path in a decision tree for insertion sort? For merge sort?*
- *What is the asymptotic height of any decision tree for sorting n elements?*
- Answer: $\Omega(n \lg n)$ (now let's prove it…)

Lower Bound For Comparison Sorting

- Thm: Any decision tree that sorts n elements has height $\Omega(n \lg n)$
- *What's the minimum # of leaves?*
- *What's the maximum # of leaves of a binary tree of height h?*
- Clearly the minimum # of leaves is less than or equal to the maximum # of leaves

Lower Bound For Comparison Sorting

- So we have…
  $n! \le 2^h$
- Taking logarithms:
  $\lg(n!) \le h$
- Stirling's approximation tells us:
  $$n! > \left(\frac{n}{e}\right)^n$$
- Thus: $h \ge \lg\left(\frac{n}{e}\right)^n$

Lower Bound For Comparison Sorting

- So we have
$$h \geq \lg\left(\frac{n}{e}\right)^n$$
$$= n\lg n - n\lg e$$
$$= \Omega(n\lg n)$$

- Thus the minimum height of a decision tree is $\Omega(n \lg n)$

Lower Bound For Comparison Sorts

- Thus the time to comparison sort n elements is $\Omega(n \lg n)$
- Corollary: Heapsort and Mergesort are asymptotically optimal comparison sorts
  - *How can we do better than* $\Omega(n \lg n)$*?*