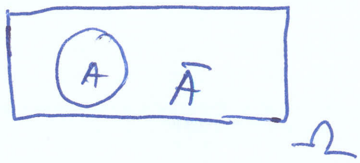
Formule probabilități – demonstrații
(∃ m. m. dem. posibile la o formulă ...)

1) $P(\emptyset) = 0$

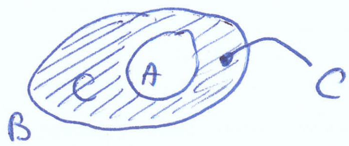
$$\left.\begin{aligned} \Omega &= \Omega \cup \emptyset \\ \Omega \cap \emptyset &= \emptyset \end{aligned}\right\} \xRightarrow[\text{num.}]{\text{adit.}} \cancel{P(\Omega)} = \cancel{P(\Omega)} + P(\emptyset) \Rightarrow P(\emptyset) = 0$$

2) $P(\bar{A}) = 1 - P(A)$



$$\left.\begin{aligned} \Omega &= \bar{A} \cup A \\ \bar{A} \cap A &= \emptyset \end{aligned}\right\} \xRightarrow[\text{num.}]{\text{adit.}} \underbrace{P(\Omega)}_{1} = P(\bar{A}) + P(A) \Rightarrow P(\bar{A}) = 1 - P(A)$$

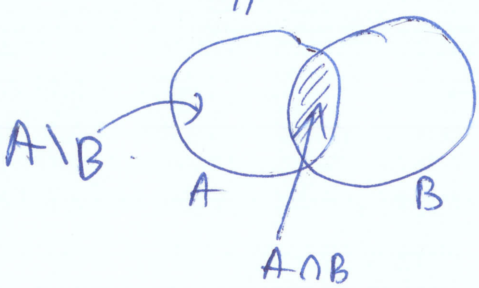
3) $A \subseteq B \Rightarrow P(A) \leq P(B)$



$A \subseteq B \Rightarrow \exists\, C,\ A \cap C = \emptyset$, a.î. $B = A \cup C$

$$\left.\begin{aligned} B &= A \cup C \\ A \cap C &= \emptyset \end{aligned}\right\} \xRightarrow[\text{num.}]{\text{adit.}} P(B) = P(A) + \underbrace{P(C)}_{\in[0,1]} \geq P(A) \Rightarrow P(B) \geq P(A)$$

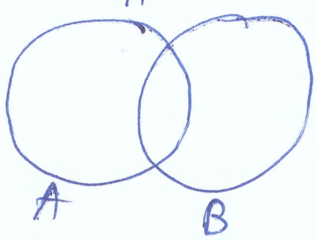
4) $P(A \setminus B) = P(A) - P(A \cap B)$



$$\left.\begin{aligned} A &= (A \cap B) \cup (A \setminus B) \\ (A \cap B) &\cap (A \setminus B) = \emptyset \end{aligned}\right\} \xRightarrow[\text{num.}]{\text{adit.}} P(A) = P(A \cap B) + P(A \setminus B) \Rightarrow$$

$$\Rightarrow P(A \setminus B) = P(A) - P(A \cap B)$$

5) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$$\left.\begin{aligned} &A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A) \\ &A \setminus B,\ A \cap B,\ B \setminus A \text{ – disj. 2-2} \end{aligned}\right\} \xRightarrow[\text{num.}]{\text{adit.}}$$

$$\Rightarrow P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A)$$

$$\overset{4)}{=} P(A) - \cancel{P(A \cap B)} + \cancel{P(A \cap B)} + P(B) - P(B \cap A)$$

$$= P(A) + P(B) - P(A \cap B)$$

$$\left.\begin{aligned} A\cup B &= A\cup (B\setminus A) \\ A\cap (B\setminus A) &= \emptyset \end{aligned}\right\} \xRightarrow[\text{num.}]{\text{adit.}} P(A\cup B) = P(A) + P(B\setminus A)$$

$$\overset{4)}{=} P(A) + P(B) - P(B\cap A)$$

$$= P(A) + P(B) - P(A\cap B)$$

6) $\underline{P(A\cap B) = P(A|B)P(B)}$

$$P(A|B) \xlongequal[\text{cond.}]{\text{def. de}} \frac{P(A\cap B)}{P(B)} \Rightarrow P(A\cap B) = P(A|B)P(B)$$

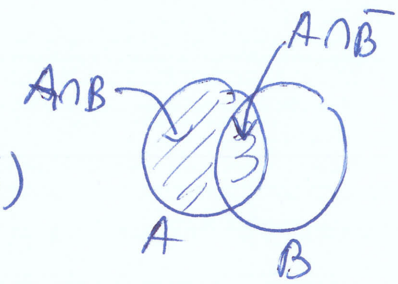
7) $P(A_1, A_2, \dots, A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2, A_1)\cdot$

$$\dots\cdot P(A_n|A_{n-1}A_{n-2}\dots, A_1)$$

$$P(A_1)P(A_2|A_1)P(A_3|A_2, A_1)\cdot\dots\cdot P(A_n|A_{n-1}, A_{n-2}, \dots, A_1) =$$

$$\overset{\text{def. de cond.}}{=} P(\cancel{A_1})\cdot\frac{P(\cancel{A_2, A_1})}{\cancel{P(A_1)}}\cdot\frac{P(\cancel{A_3, A_2, A_1})}{\cancel{P(A_2, A_1)}}\cdot\dots\cdot\frac{P(A_n, A_{n-1}, \dots, A_1)}{P(\cancel{A_{n-1}, \dots, A_1})}$$

$$= P(A_n, A_{n-1}, \dots, A_1) = P(A_1, \dots, A_n)$$



8) $\underline{P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$

$$P(A|B)P(B) + P(A|\bar{B})P(\bar{B}) = P(A\cap B) + P(A\cap\bar{B})$$

$$(A\cap B)\cap(A\cap\bar{B}) = \emptyset \xRightarrow[\text{num.}]{\text{aditiv.}} P(\underline{(A\cap B)\cup(A\cap\bar{B})}) =$$

$$= P(A\cap B) + P(A\cap\bar{B})$$

$$(A\cap B)\cup(A\cap\bar{B}) = A$$

(Dacă nu e așa de evident, putem folosi:
$(A\cap B)\cup(A\cap\bar{B}) = A\cap(\underbrace{B\cup\bar{B}}_{\Omega}) = A\cap\Omega = A$)

9) $A \subseteq B_1 \cup \dots \cup B_n$

B_i, B_j – disj. 2-2

$$P(A) = P(A|B_1)P(B_1) + \dots + P(A|B_n)P(B_n)$$

$$P(A|B_1)P(B_1) + \dots + P(A|B_n)P(B_n) =$$

$$= P(A\cap B_1) + \dots + P(A \cap B_n)$$

B_i, B_j – disj. 2-2 $\Rightarrow (A\cap B_1), \dots, (A\cap B_n)$ – disj. 2-2 $\xrightarrow[\text{num.}]{\text{aditiv.}}$

$$\Rightarrow P\big(\underbrace{(A\cap B_1)\cup \dots \cup (A\cap B_n)}\big) = P(A\cap B_1) + \dots + P(A\cap B$$

$$(A\cap B_1)\cup \dots \cup (A\cap B_n) = A\cap (B_1 \cup \dots \cup B_n) \xlongequal{A\subseteq B_1\cup\dots\cup B_n} A$$

$$\Rightarrow P(A) = P(A\cap B_1) + \dots + P(A\cap B_n)$$

Reluând, avem:

$$\underline{P(A|B_1)P(B_1) + \dots + P(A|B_n)P(B_n)} =$$

$$= P(A\cap B_1) + \dots + P(A\cap B_n)$$

$$= \underline{\underline{P(A)}}$$

Observații

(1) Formulele sunt valabile și ~~pentru~~ în variantele lor condiționale.

demonstrație cerută la parțial 1 2016-2017

ex.:

$$P(\bar{A}) = 1 - P(A) \quad \longrightarrow \quad P(\bar{A}|B) = 1 - P(A|B)$$

$(P(B) \neq 0)$

$$P(A \cap B) = P(A|B)\,P(B) \quad \longrightarrow \quad P(A \cap B|C) = P(A|B, C)\,P(B|C)$$

$P(C) \neq 0$

$P(B, C) \neq 0$

etc.

(2) Ca regulă generală, demonstrațiile trebuie gândite cu „$\cup$" (nu cu „$\setminus$", spre exemplu).

ex.:

la 4), intenția ar fi să scriem: $A \setminus B = A \setminus (A \cap B)$, dar nu putem aplica propr. de aditiv. numărabilă. Așa că vom regândi astfel: $A = (A \setminus B) \cup (A \cap B) \Rightarrow \ldots$