$\boxed{4}$ Bayesian Belief Networks

(also called Bayes Nets)

Interesting because:

- **The Naive Bayes assumption of conditional independence of attributes is too restrictive.**

  (But it's intractable without some such assumptions...)

- **Bayesian Belief networks describe conditional independence among *subsets* of variables.**

- **It allows the combination of prior knowledge about (in)dependencies among variables with observed training data.**

Conditional Independence

Definition: X **is conditionally independent of** Y **given** Z **if the probability distribution governing** X **is independent of the value of** Y **given a value of** Z**:**

$$(\forall x_i, y_j, z_k)\ P(X = x_i | Y = y_j, Z = z_k) = P(X = x_i | Z = z_k)$$

More compactly, we write $P(X|Y, Z) = P(X|Z)$

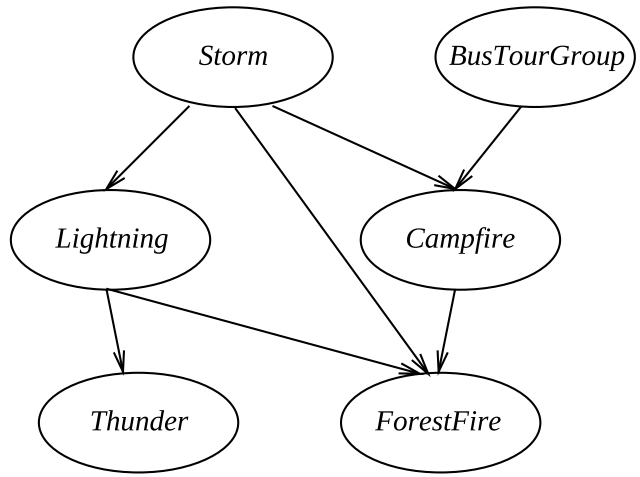
Note: **Naive Bayes uses conditional independence to justify**

$$P(A_1, A_2|V) = P(A_1|A_2, V)P(A_2|V) = P(A_1|V)P(A_2|V)$$

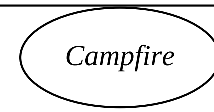
Generalizing the above definition:

$$P(X_1 \ldots X_l|Y_1 \ldots Y_m, Z_1 \ldots Z_n) = P(X_1 \ldots X_l|Z_1 \ldots Z_n)$$

A Bayes Net



	S,B	S,¬B	¬S,B	¬S,¬B
C	0.4	0.1	0.8	0.2
¬C	0.6	0.9	0.2	0.8



The network is defined by

- **A directed acyclic graph, represening a set of conditional independence assertions:**

  Each node — representing a random variable — is asserted to be conditionally independent of its nondescendants, given its immediate predecessors.

  Example: $P(Thunder|ForestFire, Lightning) = P(Thunder|Lightning)$

- **A table of local conditional probabilities for each node/variable.**

A Bayes Net (Cont'd)

represents the joint probability distribution over all variables $Y_1, Y_2, \ldots, Y_n$:

This joint distribution is fully defined by the graph, plus the conditional probabilities:

$$P(y_1, \ldots, y_n) = P(Y_1 = y_1, \ldots, Y_n = y_n) = \prod_{i=1}^{n} P(y_i | Parents(Y_i))$$

where $Parents(Y_i)$ denotes immediate predecessors of Y_i in the graph.

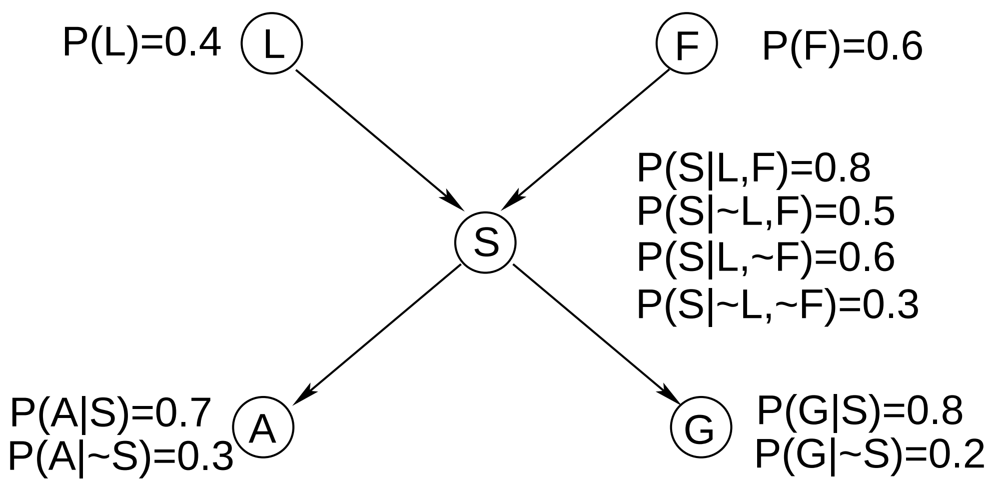
In our example: $P(Storm, BusTourGroup, \ldots, ForestFire)$

Inference in Bayesian Nets

Question: **Given a Bayes net, can one infer the probabilities of values of one or more network variables, given the observed values of (some) others?**

Example:

Given the Bayes net



compute:

(a) $P(S)$

(b) $P(A, S)$

(b) $P(A)$

Inference in Bayesian Nets (Cont'd)

Answer(s):

- **If only one variable is of unknown (probability) value, then it is easy to infer it**
- **In the general case, we can compute the probability distribution for any subset of network variables, given the distribution for any subset of the remaining variables. But...**
- **The exact inference of probabilities for an arbitrary Bayes net is an NP-hard problem!!**

Inference in Bayesian Nets (Cont'd)

In practice, we can succeed in many cases:

- **Exact inference methods work well for some net structures.**
- **Monte Carlo methods "simulate" the network randomly to calculate approximate solutions [Pradham & Dagum, 1996].**

  (In theory even approximate inference of probabilities in Bayes Nets can be NP-hard!! [Dagum & Luby, 1993])

Learning Bayes Nets (I)

There are several variants of this learning task

- The network structure might be either *known* or *unknown* (i.e., it has to be inferred from the training data).
- The training examples might provide values of *all* network variables, or just for *some* of them.

The simplest case:

If the structure is known and we can observe the values of all variables,

then it is easy to estimate the conditional probability table entries. (Analogous to training a Naive Bayes classifier.)

Learning Bayes Nets (II)

When

- **the structure of the Bayes Net is known, and**
- **the variables are only partially observable in the training data**

learning the entries in the conditional probabilities tables is similar to (learning the weights of hidden units in) training a neural network with hidden units:

- **We can learn the net's conditional probability tables using the gradient ascent!**
- **Converge to the network h that (locally) maximizes $P(D|h)$.**

Gradient Ascent for Bayes Nets

Let w_{ijk} denote one entry in the conditional probability table for the variable Y_i in the network

$$w_{ijk} = P(Y_i = y_{ij} | Parents(Y_i) = \text{the list } u_{ik} \text{ of values})$$

It can be shown (see the next two slides) that

$$\frac{\partial ln P_h(D)}{\partial w_{ijk}} = \sum_{d \in D} \frac{P_h(y_{ij}, u_{ik} | d)}{w_{ijk}}$$

therefore perform gradient ascent by repeatedly

1. **update all w_{ijk} using the training data D**

$$w_{ijk} \leftarrow w_{ijk} + \eta \sum_{d \in D} \frac{P_h(y_{ij}, u_{ik} | d)}{w_{ijk}}$$

2. **renormalize the w_{ijk} to assure**

$$\sum_j w_{ijk} = 1 \text{ and } 0 \leq w_{ijk} \leq 1$$

Gradient Ascent for Bayes Nets: Calculus

$$\frac{\partial \; lnP_h(D)}{\partial w_{ijk}} = \frac{\partial}{\partial w_{ijk}} \ln \prod_{d \in D} P_h(d) = \sum_{d \in D} \frac{\partial \ln P_h(d)}{\partial w_{ijk}} = \sum_{d \in D} \frac{1}{P_h(d)} \frac{\partial P_h(d)}{\partial w_{ijk}}$$

Summing over all values $y_{ij'}$ **of** Y_i**, and** $u_{ik'}$ **of** $U_i = Parents(Y_i)$**:**

$$\begin{aligned}
\frac{\partial \; lnP_h(D)}{\partial w_{ijk}} &= \sum_{d \in D} \frac{1}{P_h(d)} \frac{\partial}{\partial w_{ijk}} \sum_{j'k'} P_h(d|y_{ij'}, u_{ik'}) P_h(y_{ij'}, u_{ik'}) \\
&= \sum_{d \in D} \frac{1}{P_h(d)} \frac{\partial}{\partial w_{ijk}} \sum_{j'k'} P_h(d|y_{ij'}, u_{ik'}) P_h(y_{ij'}|u_{ik'}) P_h(u_{ik'})
\end{aligned}$$

Note that $w_{ijk} \equiv P_h(y_{ij}|u_{ik})$**, therefore...**

Gradient Ascent for Bayes Nets: Calculus (Cont'd)

$$
\begin{aligned}
\frac{\partial \; lnP_h(D)}{\partial w_{ijk}} &= \sum_{d \in D} \frac{1}{P_h(d)} \frac{\partial}{\partial w_{ijk}} P_h(d|y_{ij}, u_{ik}) w_{ijk} P_h(u_{ik}) \\
&= \sum_{d \in D} \frac{1}{P_h(d)} P_h(d|y_{ij}, u_{ik}) P_h(u_{ik}) \quad \textbf{(applying Bayes th.)} \\
&= \sum_{d \in D} \frac{1}{P_h(d)} \frac{P_h(y_{ij}, u_{ik}|d) P_h(d) P_h(u_{ik})}{P_h(y_{ij}, u_{ik})} \\
&= \sum_{d \in D} \frac{P_h(y_{ij}, u_{ik}|d) P_h(u_{ik})}{P_h(y_{ij}, u_{ik})} = \sum_{d \in D} \frac{P_h(y_{ij}, u_{ik}|d)}{P_h(y_{ij}|u_{ik})} \\
&= \sum_{d \in D} \frac{P_h(y_{ij}, u_{ik}|d)}{w_{ijk}}
\end{aligned}
$$

Learning Bayes Nets (II, Cont'd)

The EM algorithm (see next sildes) can also be used.

Repeatedly:

1. **Calculate/estimate from data the probabilities of unobserved variables w_{ijk},**
   assuming that the hypothesis h holds
2. **Calculate a new h (i.e. new values of w_{ijk}) so to maximize $E[\ln P(D|h)]$,**
   where D now includes both the observed and the unobserved variables.

Learning Bayes Nets (III)

When the structure is unknown, algorithms usually use greedy search to trade off network complexity (add/substract edges/nodes) against degree of fit to the data.

Example: [**Cooper & Herscovitz, 1992**] **the** $K2$ **algorithm:**
When data is fully observable, use a score metric to choose among alternative networks.
They report an experiment on (re-learning) a network with 37 nodes and 46 arcs describing anesthesia problems in a hospital operating room. Using 3000 examples, the program succeeds almost perfectly: it misses one arc and adds an arc which is not in the original net.

Summary: Bayesian Belief Networks

- **Combine prior knowledge with observed data**
- **The impact of prior knowledge (when correct!) is to lower the sample complexity**
- **Active/Recent research area**
  - **Extend from boolean to real-valued variables**
  - **Parameterized distributions instead of tables**
  - **Extend to first-order instead of propositional systems**
  - **More effective inference methods**
  - **...**