Baysian Classification
Some exercises

Exemplifying

- **the notion of MAP (Maximum A posteriori Probability) hypotheses**
- **the computation of *expected values* for random variables**

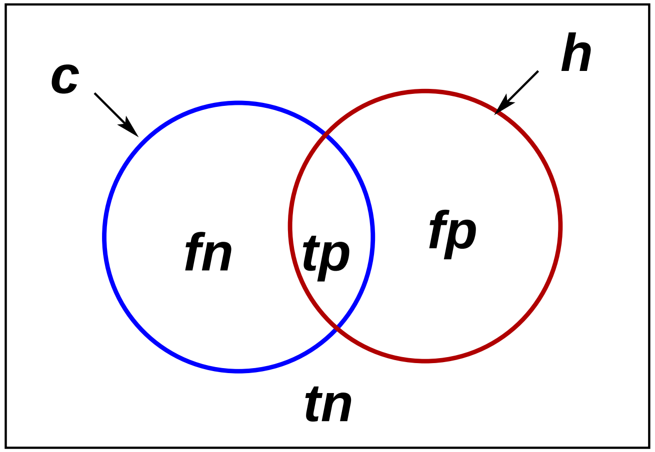
and

- **the [use of] *sensitivity* and *specificity* of a test in a real-world application**

CMU, 2009 fall, Geoff Gordon, HW1, pr. 2

There is a disease which affects 1 in 500 people. A 100.00 dollar blood test can help reveal whether a person has the disease. A positive outcome indicates that the person may have the disease.

The test has perfect sensitivity (*true positive rate*), i.e., a person who has the disease tests positive 100% of the time. However, the test has 99% specificity (*true negative rate*), i.e., a healthy person tests positive 1% of the time.



sensitivity (or: recall): $\frac{tp}{tp + fn}$

specificity: $\frac{tn}{tn + fp}$

a. A randomly selected individual is tested and the result is positive.

What is the *probability* of the individual having the disease?

b. There is a second more expensive test which costs 10, 000.00 dollars but is exact with 100% *sensitivity* and *specificity*.

If we require all people who test positive with the less expensive test to be tested with the more expensive test, what is the *expected cost* to check whether an individual has the disease?

c. A pharmaceutical company is attempting to decrease the cost of the second (perfect) test.

How much would it have to make the second test cost, so that the first test is no longer needed? That is, at what cost is it cheaper simply to use the perfect test alone, instead of screening with the cheaper test as described in part b?

Answer:

Let's define the following *random variables*:

B: $\begin{cases} \textbf{1/true} & \textbf{for persons affected by that disease,} \\ \textbf{0/false} & \textbf{otherwise;} \end{cases}$

T_1: **the result of the first test: + (in case of disease) or − (otherwise);**

T_2: **the result of the second test: again + or −.**

***Known facts*:**

$$P(B) = \frac{1}{500}$$

$$P(T_1 = + \mid B) = 1,\ P(T_1 = + \mid \bar{B}) = \frac{1}{100},$$

$$P(T_2 = + \mid B) = 1,\ P(T_2 = + \mid \bar{B}) = 0$$

a.

$$P(B \mid T_1 = +) \stackrel{TBayes}{=} \frac{P(T_1 = + \mid B) \cdot P(B)}{P(T_1 = + \mid B) \cdot P(B) + P(T_1 = + \mid \bar{B}) \cdot P(\bar{B})}$$

$$= \frac{1 \cdot \dfrac{1}{500}}{1 \cdot \dfrac{1}{500} + \dfrac{1}{100} \cdot \dfrac{499}{500}} = \frac{100}{599} \approx 0.1669$$

b.

Let's consider a new random variable:

$$C = \begin{cases} c_1 & \textbf{if the person only takes the first test} \\ c_1 + c_2 & \textbf{if the person takes the two tests} \end{cases}$$

$$\Rightarrow P(C = c_1) = P(T_1 = -) \textbf{ and } P(C = c_1 + c_2) = P(T_1 = +)$$

$$\begin{aligned} \Rightarrow E[C] &= c_1 \cdot (1 - P(T_1 = +)) + (c_1 + c_2) \cdot P(T_1 = +) \\ &= c_1 - c_1 \cdot P(T_1 = +) + c_1 \cdot P(T_1 = +) + c_2 \cdot P(T_1 = +) \\ &= c_1 + c_2 \cdot P(T_1 = +) \\ &= 100 + 10000 \cdot \frac{599}{50000} = 219.8 \approx 220\$ \end{aligned}$$

Note: Here above we used

$$\begin{aligned} P(T_1 = +) &\overset{\textit{total probability form.}}{=} P(T_1 = + \mid B) \cdot P(B) + P(T_1 = + \mid \bar{B}) \cdot P(\bar{B}) \\ &= 1 \cdot \frac{1}{500} + \frac{1}{100} \cdot \frac{499}{500} = \frac{599}{50000} = 0.01198 \end{aligned}$$

c.

$c_n \stackrel{not.}{=}$ **the new price for the second test** (T_2')

$$\begin{aligned} c_n \leq E[C'] &= c_1 \cdot P(C = c_1) + (c_1 + c_n) \cdot P(C = c_1 + c_n) \\ &= c_1 + c_n \cdot P(T_1 = +) = 100 + c_n \cdot \frac{599}{50000} \end{aligned}$$

$c_n = 100 + c_n \cdot 0.01198 \Rightarrow c_n \approx 101.2125$**.**

Exemplifying

Text classification using the Naive Bayes algorithm

CMU, 2009 spring, Ziv Bar-Joseph, midterm, pr. 2

About 2/3 of your email is spam, so you downloaded an open source spam filter based on word occurrences that uses the Naive Bayes classifier.
Assume you collected the following regular and spam mails to train the classifier, and only three words are informative for this classification, i.e., each email is represented as a 3-dimensional binary vector whose components indicate whether the respective word is contained in the email.

'study'	**'free'**	**'money'**	**Category**	*count*
1	0	0	**Regular**	1
0	0	1	**Regular**	1
1	0	0	**Regular**	1
1	1	0	**Regular**	1
0	1	0	**Spam**	4
0	1	1	**Spam**	4

a. You find that the spam filter uses a prior $P(\text{spam}) = 0.1$. Explain (in one sentence) why this might be sensible.

b. Compute the Naive Bayes parameters, using Maximum Likelihood Estimation (MLE) and applying Laplace's rule ("add-one").

c. Based on the prior and conditional probabilities above, give the model probability $P(\text{spam}|\ s)$ that the sentence

$$s = \text{"money for psychology study"}$$

is spam.

Answer:

a. It is worse for regular emails to be classified as spam than it is for spam email to be classified as regular email.

b. Estimating the Naive Bayes parameters, by MLE and applying Laplace's rule ("add-one"):

$$P(\textbf{study}|\textbf{spam}) = \frac{0+1}{8+2} = \frac{1}{10} \qquad P(\textbf{study}|\textbf{regular}) = \frac{3+1}{4+2} = \frac{2}{3}$$

$$P(\textbf{free}|\textbf{spam}) = \frac{8+1}{8+2} = \frac{9}{10} \qquad P(\textbf{free}|\textbf{regular}) = \frac{1+1}{4+2} = \frac{1}{3}$$

$$P(\textbf{money}|\textbf{spam}) = \frac{4+1}{8+2} = \frac{1}{2} \qquad P(\textbf{money}|\textbf{regular}) = \frac{1+1}{4+2} = \frac{1}{3}$$

c. Classification of the message

$$s = \text{“}\mathbf{money\ for\ psychology\ study}\text{”},$$

using the a priori probability $P(\mathbf{spam}) = 0.1$:

$$P(\mathbf{spam} \mid s) = P(\mathbf{spam} \mid \mathbf{study}, \neg\mathbf{free}, \mathbf{money})$$

$$\stackrel{F.\ Bayes}{=} \frac{P(\mathbf{study},\ \neg\mathbf{free},\ \mathbf{money} \mid \mathbf{spam}) \cdot P(\mathbf{spam})}{P(\mathbf{study},\neg\mathbf{free},\mathbf{money} \mid \mathbf{spam})P(\mathbf{spam}) + P(\mathbf{study},\neg\mathbf{free},\mathbf{money} \mid \mathbf{reg})P(\mathbf{reg})}$$

$$P(\mathbf{study},\ \neg\mathbf{free},\ \mathbf{money}|\mathbf{spam}) \stackrel{\textit{indep. cdt.}}{=} P(\mathbf{study}|\mathbf{spam})\cdot P(\neg\mathbf{free}|\mathbf{spam})\cdot P(\mathbf{money}|\mathbf{spam})$$

$$= \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{2} = \frac{1}{200}$$

$$P(\mathbf{study},\ \neg\mathbf{free},\ \mathbf{money}|\mathbf{reg}) \stackrel{\textit{indep. cdt.}}{=} P(\mathbf{study}|\mathbf{reg})\cdot P(\neg\mathbf{free}|\mathbf{reg})\cdot P(\mathbf{money}|\mathbf{reg})$$

$$= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{27}$$

Therefore,

$$P(\mathbf{spam}|\ s) = \frac{\dfrac{1}{200} \cdot \dfrac{1}{10}}{\dfrac{1}{200} \cdot \dfrac{1}{10} + \dfrac{4}{27} \cdot \dfrac{9}{10}} \approx 0.0037$$

Exemplifying

The computation of the *error rate* for the Naive Bayes algorithm

CMU, 2010 fall, Aarti Singh, HW1, pr. 4.2

Consider a simple learning problem of determining whether Alice and Bob from CA will go to hiking or not $Y : Hike \in \{T, F\}$ given the weather conditions $X_1 : Sunny \in \{T, F\}$ and $X_2 : Windy \in \{T, F\}$ by a Naive Bayes classifier.

Using training data, we estimated the parameters

$$\begin{array}{ll} P(Hike) = 0.5 & \\ P(Sunny \mid Hike) = 0.8, & P(Sunny \mid \neg Hike) = 0.7 \\ P(Windy \mid Hike) = 0.4, & P(Windy \mid \neg Hike) = 0.5 \end{array}$$

Assume that the true distribution of X_1, X_2, and Y satisfies the Naive Bayes assumption of conditional independence with the above parameters.

a. What is the joint probability that Alice and Bob go to hiking and the weather is sunny and windy, that is $P(Sunny, Windy, Hike)$?

Solution:

$$P(Sunny, Windy, Hike) \stackrel{cdt.\ indep.}{=} P(Sunny|Hike) \cdot P(Windy|Hike) \cdot P(Hike) = 0.8 \cdot 0.4 \cdot 0.5 = 0.16.$$

b. What is the expected error rate of the Naive Bayes classifier?
(Informally, the expected error rate is the probability that an "observation"/instance randomly generated according to the *true* probabilistic distribution of data is incorrectly classified by the Naive Bayes algorithm.)

Solution:

X_1	X_2	Y	$P(X_1, X_2, Y) = P(X_1\|Y) \cdot P(X_2\|Y) \cdot P(Y)$	$Y_{NB}(X_1, X_2)$	$P_{NB}(Y\|X_1, X_2)$
F	F	F	$0.3 \cdot 0.5 \cdot 0.5 = 0.075$	F	0.555556
F	F	T	$0.2 \cdot 0.6 \cdot 0.5 = \mathbf{0.060}$	F	0.444444
F	T	F	$0.3 \cdot 0.5 \cdot 0.5 = 0.075$	F	0.652174
F	T	T	$0.2 \cdot 0.4 \cdot 0.5 = \mathbf{0.040}$	F	0.347826
T	F	F	$0.7 \cdot 0.5 \cdot 0.5 = \mathbf{0.175}$	T	0.421686
T	F	T	$0.8 \cdot 0.6 \cdot 0.5 = 0.240$	T	0.578314
T	T	F	$0.7 \cdot 0.5 \cdot 0.5 = 0.175$	F	0.522388
T	T	T	$0.8 \cdot 0.4 \cdot 0.5 = \mathbf{0.160}$	F	0.477612

***Note*:** **Joint probabilities corresponding to incorrect predictions are shown in bold.**

$$
\begin{aligned}
error &\stackrel{def.}{=} E_P[I_{Y_{NB}(X_1,X_2)\neq Y}] \\
&= \sum_{X_1,X_2,Y} I[Y_{NB}(X_1, X_2) \neq Y] \cdot P(X_1, X_2, Y) \\
&= \mathbf{0.060} + \mathbf{0.040} + \mathbf{0.175} + \mathbf{0.160} = \mathbf{0.435}
\end{aligned}
$$

***Note*:**
I **is the *indicator* function; its value is** 1 **whenever the associated condition (in our case,** $Y_{NB}(X_1, X_2) \neq Y$**) is true, and** 0 **otherwise.**

Next, suppose that we gather more information about weather conditions and introduce a new feature denoting whether the weather is X_3 : *Rainy* or not. *Assume* that each day the weather in CA can be either *Rainy* or *Sunny*. That is, it can not be both *Sunny* and *Rainy*. (Similarly, it can not be $\neg$ *Sunny* and $\neg$*Rainy*).

c. In the above new case, are any of the Naive Bayes assumptions violated? Why (not)? What is the joint probability that Alice and Bob go to hiking and the weather is sunny, windy and not rainy, that is $P(\mathit{Sunny}, \mathit{Windy}, \neg \mathit{Rainy}, \mathit{Hike})$?

Solution:

The conditional independence of variables given the class label assumption of Naive Bayes is violated. Indeed, knowing if the weather is *Sunny* completely determines whether it is *Rainy* or not. Therefore, *Sunny* and *Rainy* are clearly NOT conditionally independent given *Hike*.

$$
\begin{aligned}
& P(\mathit{Sunny}, \mathit{Windy}, \neg \mathit{Rainy}, \mathit{Hike}) \\
&\quad = \underbrace{P(\neg \mathit{Rainy} | \mathit{Hike}, \mathit{Sunny}, \mathit{Windy})}_{1} \cdot P(\mathit{Sunny}, \mathit{Windy} | \mathit{Hike}) \cdot P(\mathit{Hike}) \\
&\quad \stackrel{\mathit{cond.\ indep.}}{=} P(\mathit{Sunny} | \mathit{Hike}) \cdot P(\mathit{Windy} | \mathit{Hike}) \cdot P(\mathit{Hike}) \\
&\quad = 0.8 \cdot 0.4 \cdot 0.5 = 0.16.
\end{aligned}
$$

d. What is the expected error rate when the Naive Bayes classifier uses all three attributes? Does the performance of Naive Bayes improve by observing the new attribute Rainy? Explain why.

Solution:

X_1	X_2	X_3	Y	$P(X_1, X_2, Y)$	$P_{NB}(X_1, X_2, X_3, Y) = P(X_3 \vert Y) \cdot P(X_1 \vert Y) \cdot P(X_2 \vert Y) \cdot P(Y)$	$Y_{NB}(X_1, X_2, X_3)$	$P_{NB}(Y \vert X_1, X_2, X_3)$
F	F	F	F	0	$0.075 \cdot 0.7 = 0.0525$	F	0.522388
F	F	F	T	**0**	$0.060 \cdot 0.8 = 0.0480$	F	0.477612
F	F	T	F	0.075	$0.075 \cdot 0.3 = 0.0225$	F	0.652174
F	F	T	T	**0.060**	$0.060 \cdot 0.2 = 0.0120$	F	0.347826
F	T	F	F	0	$0.075 \cdot 0.7 = 0.0525$	F	0.621302
F	T	F	T	**0**	$0.040 \cdot 0.8 = 0.0320$	F	0.378698
F	T	T	F	0.075	$0.075 \cdot 0.3 = 0.0225$	F	0.737705
F	T	T	T	**0.040**	$0.040 \cdot 0.2 = 0.0080$	F	0.262295
T	F	F	F	**0.175**	$0.175 \cdot 0.7 = 0.0525$	T	0.389507
T	F	F	T	0.240	$0.240 \cdot 0.8 = 0.1920$	T	0.610493
T	F	T	F	0	$0.175 \cdot 0.3 = 0.0525$	F	0.522388
T	F	T	T	**0**	$0.240 \cdot 0.2 = 0.0480$	F	0.477612
T	T	F	F	**0.175**	$0.175 \cdot 0.7 = 0.0525$	T	0.489022
T	T	F	T	0.160	$0.160 \cdot 0.8 = 0.1280$	T	0.510978
T	T	T	F	0	$0.175 \cdot 0.3 = 0.0225$	F	0.621302
T	T	T	T	**0**	$0.060 \cdot 0.2 = 0.0120$	F	0.378698

The new error rate is:
$0.060 + 0.040 + 0.175 + 0.175 = 0.45 > 0.435$ (see question b).

The Naive Bayes classifier performance drops because the conditional independence assumptions do not hold for the correlated features.

Computing

The *sample complexity* of the Naive Bayes and Joint Bayes Clssifiers

CMU, 2010 spring, Eric Xing, Tom Mitchell, Aarti Singh, HW2, pr. 1.1

A big reason we use the Naive Bayes classifier is that it requires less training data than the Joint Bayes Classifier. This exercise should give you a "feeling" for how great the disparity really is.

Imagine that each *instance* is an independent "*observation*" of the multi-variate random variable $\bar{X} = X_1, ..., X_d$, where the X_i are i.i.d. and Bernoulli of parameter $p = 0.5$.

To train the Joint Bayes classifier, we need to see every value of $\bar{X}$ "enough" times; training the Naive Bayes classifier only requires seeing both values of X_i "enough" times.

Main Question: How many "observations"/instances are needed until, with probability $1 - \varepsilon$, we have seen every variable we need to see at least once?

Note: To train the classifiers well would require more than this, but for this problem we only require one observation.

Hint: You may want to use the following *inequalities*:

- **For any $k \geq 1$, $(1 - 1/k)^k \leq e^{-1}$**
- **For *any* events $E_1, ..., E_k$, $Pr(E_1 \cup \ldots \cup E_k) \leq \sum_{i=1}^{k} Pr(E_i)$.**
  (This is called the "union bounds" property.)

Consider the Naive Bayes classifier.

a. Show that if N observations have been made, the probability that a given value of X_i (either 0 or 1) has *not* been seen is $\leq \frac{1}{2^{N-1}}$.

b. Show that if more than $N_{NB} = 1 + \log_2 \left(\frac{d}{\varepsilon}\right)$ observations have been made, then the probability that *any* X_i has not been observed in both states is $\leq \varepsilon$.

Solution:

a. $P(\textbf{component } X_i \textbf{ not seen in both states}) = \left(\frac{1}{2}\right)^N + \left(\frac{1}{2}\right)^N = \frac{2}{2^N} = \frac{1}{2^{N-1}}$

b. $P(\textbf{any component not seen in both states})$

$\leq \sum_{i=1}^{d} P(\textbf{component } X_i \textbf{ not seen in both states})$

$= \sum_{i=1}^{d} \frac{1}{2^{N_{NB}-1}} = d \cdot \frac{1}{2^{N_{NB}-1}} = d \cdot \frac{1}{2^{1+\log_2 \frac{d}{\varepsilon}-1}} = d \cdot \frac{1}{2^{\log_2 \frac{d}{\varepsilon}}} = d \cdot \frac{1}{\frac{d}{\varepsilon}} = d \cdot \frac{\varepsilon}{d} = \varepsilon$

Consider the Joint Bayes classifier.

c. **Let $\bar{x}$ be a particular value of $\bar{X}$. Show that after N observations, the probability that we have never seen $\bar{x}$ is $\leq e^{-N/2^d}$.**

d. **Using the "union bounds" property, show that if more than $N_{JB} = 2^d \ln\left(\frac{2^d}{\varepsilon}\right)$ observations have been made, then the probability that an arbitrarily chosen (but fixed) value of $\bar{X}$ has not been seen is $\leq \varepsilon$.**

Solution:

c. $P(\bar{x}$ **not seen in** N **observations**$)$

$$= \left(1 - \frac{1}{2^d}\right)^N = \left[\left(1 - \frac{1}{2^d}\right)^{2^d}\right]^{N/2^d} \leq \left(\frac{1}{e}\right)^{N/2^d} = e^{-N/2^d}$$

d. $P($**any** $\bar{x}$ **not seen in** N_{JB} **observations**$)$

$\leq \sum_{\bar{x}} P(\bar{x}$ **not seen in** N_{JB} **observations**$)$

$$= \sum_{\bar{x}} e^{-N_{JB}/2^d} = 2^d \cdot e^{-N_{JB}/2^d} = 2^d \cdot e^{-\ln\frac{2^d}{\varepsilon}} = 2^d \cdot \frac{1}{e^{\ln\frac{2^d}{\varepsilon}}} = \frac{2^d}{\frac{2^d}{\varepsilon}} = \varepsilon$$

e. Let $d = 2$ and $\varepsilon = 0.1$. What are the values of N_{NB} and N_{JB}?
What about $d = 5$?
And $d = 10$?

Solution:

$$\varepsilon = 0.1,\ d = 2 \quad \Rightarrow \quad \begin{cases} N_{NB} = 1 + \log_2 \dfrac{2}{0.1} = 1 + \log_2 20 \approx 5.32 \\ N_{JB} = 2^2 \cdot \ln \dfrac{2^2}{0.1} = 4 \cdot \ln 40 \approx 14.75 \end{cases}$$

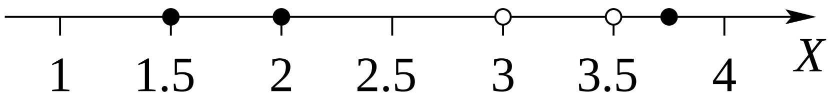
$$\varepsilon = 0.1,\ d = 5 \quad \Rightarrow \quad \begin{cases} N_{NB} = 1 + \log_2 \dfrac{5}{0.1} = 1 + \log_2 50 \approx 6.64 \\ N_{JB} = 2^5 \cdot \ln \dfrac{2^5}{0.1} = 32 \cdot \ln 320 \approx 184.58 \end{cases}$$

$$\varepsilon = 0.1,\ d = 10 \quad \Rightarrow \quad \begin{cases} N_{NB} = 1 + \log_2 \dfrac{10}{0.1} = 1 + \log_2 100 \approx 7.64 \\ N_{JB} = 2^{10} \cdot \ln \dfrac{2^{10}}{0.1} = 1024 \cdot \ln 10240 \approx 9455.67 \end{cases}$$

Exemplifying

ML hypotheses* and *MAP hypotheses

CMU, 2009 spring, Tom Mitchell, midterm, pr. 2.3-4



Let's consider the 1-dimensional data set shown above, based on the single real-valued attribute X. Notice there are two classes (values of Y), and five data points.

Consider a special type of *decision trees* where leaves have *probabilistic labels*. Each leaf node gives the probability of each possible label, where the probability is the fraction of points at that leaf node with that label.

For *example*, a decision tree learned from the data set above with zero splits would say $P(Y=1)=3/5$ and $P(Y=0)=2/5$. A decision tree with one split (at $X=2.5$) would say $P(Y=1)=1$ if $X<2.5$, and $P(Y=1)=1/3$ if $X \geq 2.5$.

a. For the above data set, draw a tree that maximizes the *likelihood* of the data.

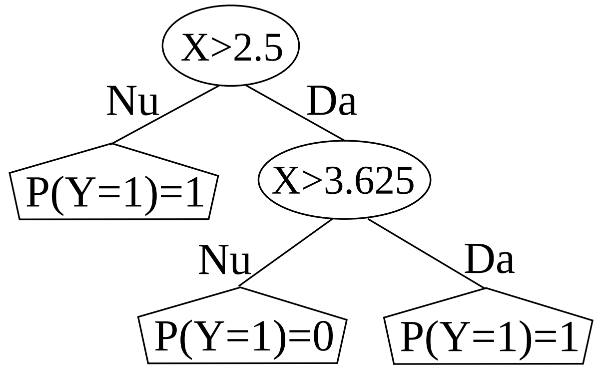
$T_{ML} = \text{argmax}_T \, P_T(D)$, **where**

$P_T(D) \stackrel{def.}{=} P(D|T) \stackrel{i.i.d.}{=} \prod_{i=1}^{5} P(Y=y_i|X=x_i,T)$,

where y_i **is the label/classs of the instance** x_i

$(x_1=1.5,\, x_2=2,\, x_3=3,\, x_4=3.5,\, x_5=3.75.)$

Solution:



b. Consider a prior probability distribution $P(T)$ over trees that penalizes the number of splits in the tree.

$$P(T) \propto \left(\frac{1}{4}\right)^{splits(T)^2}$$

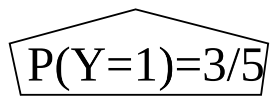
where T is a tree, $splits(T)$ is the number of splits in T, and $\propto$ means "is proportional to".

For the same data set, give the MAP tree when using this prior, $P(T)$, over trees.

Solution:

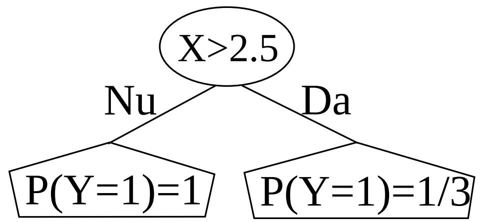
0 nodes:

$$P(T_0 \mid D) \propto \left(\frac{3}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^2 \cdot \left(\frac{1}{4}\right)^0 = \frac{3^3 \cdot 2^2}{5^5} = \frac{108}{3125} = 0.0336$$



1 node:

$$P(T_1 \mid D) \propto 1^2 \cdot \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} \cdot \left(\frac{1}{4}\right)^1 = \frac{1}{27} = 0.037$$



2 nodes:

$$P(T_2) \propto \left(\frac{1}{4}\right)^4 \Rightarrow P(T_2 \mid D) \propto 1 \cdot \left(\frac{1}{4}\right)^4 = \frac{1}{256} = 0.0039 \Rightarrow \textbf{the MAP tree is } T_1.$$

The relationship between [the decision rules of] *Naive Bayes* and *Logistic Regression*; the case of Boolean input variables

CMU, 2005 fall, Tom Mitchell, HW2, pr. 2

CMU, 2009 fall, Carlos Guestrin, HW1, pr. 4.1.2

CMU, 2009 fall, Geoff Gordon, HW4, pr. 1.2

CMU, 2012 fall, Tom Mitchell, Ziv Bar-Joseph, HW2, pr. 3.a

a. [Equivalence of NB and LR]

In Tom's draft chapter (*Generative and discriminative classifiers: Naive Bayes and logistic regression*) it has been proved that when Y follows a Bernoulli distribution and $X = (X_1, \ldots, X_n)$ is a vector of Gaussian variables, then under certain assumptions the Gaussian Naive Bayes classifier implies that $P(Y|X)$ is given by the logistic function with appropriate parameters W. So,

$$P(Y = 1|X) = \frac{1}{1 + \exp(w_0 + \sum_{i=1}^n w_i X_i)}.$$

and therefore,

$$P(Y = 0|X) = \frac{\exp(w_0 + \sum_{i=1}^n w_i X_i)}{1 + \exp(w_0 + \sum_{i=1}^n w_i X_i)}$$

Consider instead the case where Y is Boolean (more generally, Bernoulli) and $X = (X_1, \ldots, X_n)$ is a vector of Boolean variables. Prove for this case also that $P(Y|X)$ follows this same form and hence that Logistic Regression is also the discriminative counterpart to a Naive Bayes generative classifier over Boolean features.

***Note*:**

***Discriminative classifiers* learn the parameters of $P(Y|X)$ directly, whereas *generative classifiers* instead learn the parameters of $P(X|Y)$ and $P(Y)$.**

***Hints*:**

1. Simple notation will help. Since the X_i's are Boolean variables, you need only one parameter to define, $P(X_i|Y = y_k)$, for each $i = 1, \ldots, d$.
Define $\theta_{i1} = P(X_i = 1|Y = 1)$, in which case $P(X_i = 0|Y = 1) = 1 - \theta_{i1}$.
Similarly, use θ_{i0} to denote $P(X_i = 1|Y = 0)$.

2. Notice that with the above notation you can represent $P(X_i|Y = 1)$ as follows:

$$P(X_i = 1|Y = 1) = \theta_{i1}^{X_i}(1 - \theta_{i1})^{(1-X_i)}$$

Note that when $X_i = 1$ the second term is equal to 1 because its exponent is zero. Similarly, when $X_i = 0$ the first term is equal to 1 because its exponent is zero.

Solution

$$
\begin{aligned}
P(Y=1|X=x) &\overset{B.F.}{=} \frac{P(X=x|Y=1)\,P(Y=1)}{\sum_{y'\in\{0,1\}} P(X=x|Y=y')\,P(Y=y')} \\
&= \frac{1}{1+\dfrac{P(X=x|Y=0)P(Y=0)}{P(X=x|Y=1)P(Y=1)}} \\
&= \frac{1}{1+\exp\left(\ln \dfrac{P(X=x|Y=0)P(Y=0)}{P(X=x|Y=1)P(Y=1)}\right)} \\
&= \frac{1}{1+\exp\left(\ln \dfrac{P(X_1=x_1,\ldots,X_d=x_d|Y=0)P(Y=0)}{P(X_1=x_1,\ldots,X_d=x_d|Y=1)P(Y=1)}\right)} \\
&\overset{cond.\ indep.}{=} \frac{1}{1+\exp\left(\ln \frac{P(Y=0)}{P(Y=1)} + \sum_{i=1}^{d} \ln \frac{P(X_i=x_i|Y=0)}{P(X_i=x_i|Y=1)}\right)}
\end{aligned}
$$

Prior probabilities are: $P(Y=1)=\pi$ **and** $P(Y=0)=1-\pi$**.**

Also, each X_i **follows a Bernoulli distribution:**
$P(X_i|Y=1)=\theta_{i1}^{X_i}(1-\theta_{i1})^{(1-X_i)}$**, and** $P(X_i|Y=0)=\theta_{i0}^{X_i}(1-\theta_{i0})^{(1-X_i)}$**.**

So,

$$
\begin{aligned}
P(Y=1|X=x) &= \frac{1}{1+\exp\left(\ln\dfrac{1-\pi}{\pi}+\sum_{i=1}^{d}\ln\dfrac{\theta_{i0}^{X_i}(1-\theta_{i0})^{(1-X_i)}}{\theta_{i1}^{X_i}(1-\theta_{i1})^{(1-X_i)}}\right)} \\
&= \frac{1}{1+\exp\left(\ln\dfrac{1-\pi}{\pi}+\sum_{i=1}^{d}\left(X_i\ln\dfrac{\theta_{i0}}{\theta_{i1}}+(1-X_i)\ln\dfrac{1-\theta_{i0}}{1-\theta_{i1}}\right)\right)} \\
&= \frac{1}{1+\exp\left(\ln\dfrac{1-\pi}{\pi}+\sum_{i=1}^{d}\ln\dfrac{1-\theta_{i0}}{1-\theta_{i1}}+\sum_{i=1}^{d}X_i\left(\ln\dfrac{\theta_{i0}}{\theta_{i1}}+\ln\dfrac{1-\theta_{i0}}{1-\theta_{i1}}\right)\right)}
\end{aligned}
$$

Therefore, in order to reach $P(Y=1|X=x)=1/(1+\exp(w_0+\sum_{i=1}^{d}w_iX_i)$**, we can set**

$$
w_0=\ln\frac{1-\pi}{\pi}+\sum_{i=1}^{d}\ln\frac{1-\theta_{i0}}{1-\theta_{i1}} \quad \textbf{and} \quad w_i=\ln\frac{\theta_{i0}}{\theta_{i1}}+\ln\frac{1-\theta_{i0}}{1-\theta_{i1}} \textbf{ for } i=1,\ldots,d.
$$

b. [Relaxing the conditional independence assumption]

To capture interactions between features, the Logistic Regression model can be supplemented with extra terms. For example, a term can be added to capture a dependency between X_1 and X_2:

$$P(Y=1|X) = \frac{1}{1+\exp(w_0 + w_{1,2}X_1X_2 + \sum_{i=1}^{n} w_i X_i)}$$

Similarly, the conditional independence assumptions made by Naive Bayes can be relaxed so that X_1 and X_2 are not assumed to be conditionally independent. In this case, we can write:

$$P(Y|X) = \frac{P(Y)\, P(X_1, X_2|Y)\, \prod_{i=3}^{n} P(X_i|Y)}{P(X)}$$

Prove that for this case, that $P(Y|X)$ follows the same form as the logistic regression model supplemented with the extra term that captures the dependency between X_1 and X_2 (and hence that the supplemented Logistic Regression model is the discriminative counterpart to this generative classifier).

***Hints*:**

1. Using simple notation will help here as well. You need more parameters than before to define $P(X_1, X_2, Y)$. So let's define $\beta_{ijk} = P(X_1 = i, X_2 = j, Y = k)$, for each i, j and k.

2. The above notation can be used to represent $P(X_1, X_2|Y = k)$ as follows:

$$P(X_1, X_2|Y = k) = (\beta_{11k})^{X_1 X_2} (\beta_{10k})^{X_1(1-X_2)} (\beta_{01k})^{(1-X_1)X_2} (\beta_{00k})^{(1-X_1)(1-X_2)}$$

Solution

$$
\begin{aligned}
P(Y=1|X) &\overset{B.\ F.}{=} \frac{P(X|Y=1)P(Y=1)}{P(X|Y=1)P(Y=1)+P(X|Y=0)P(Y=0)} \\
&= \frac{1}{1+\dfrac{P(X|Y=0)P(Y=0)}{P(X|Y=1)P(Y=1)}} \\
&= \frac{1}{1+\exp\left(\ln \dfrac{P(X|Y=0)P(Y=0)}{P(X|Y=1)P(Y=1)}\right)} \\
&\overset{cdtl.\ indep.}{=} \frac{1}{1+\exp\left(\ln \dfrac{P(X_1,X_2|Y=0)\prod_{i=3}^{d} P(X_i|Y=0)P(Y=0)}{P(X_1,X_2|Y=1)\prod_{i=3}^{d} P(X_i|Y=1)P(Y=1)}\right)} \\
&= \frac{1}{1+\exp\left(\ln \dfrac{1-\pi}{\pi} + \sum_{i=3}^{d} \ln \dfrac{P(X_i|Y=0)}{P(X_i|Y=1)} + \ln \dfrac{P(X_1,X_2|Y=0)}{P(X_1,X_2|Y=1)}\right)} \\
&= \frac{1}{1+\exp\left(\ln \dfrac{1-\pi}{\pi} + \sum_{i=3}^{d} \ln \dfrac{\theta_{i0}^{X_i}(1-\theta_{i0})^{(1-X_i)}}{\theta_{i1}^{X_i}(1-\theta_{i1})^{(1-X_i)}} + \ln \dfrac{(\beta_{110})^{X_1X_2}\,(\beta_{100})^{X_1(1-X_2)}\,(\beta_{010})^{(1-X_1)X_2}\,(\beta_{000})^{(1-X_1)(1-X_2)}}{(\beta_{111})^{X_1X_2}\,(\beta_{101})^{X_1(1-X_2)}\,(\beta_{011})^{(1-X_1)X_2}\,(\beta_{001})^{(1-X_1)(1-X_2)}}\right)} \\
&= \frac{1}{1+\exp\left(\ln \dfrac{1-\pi}{\pi} + \sum_{i=3}^{d}\left(X_i\left(\ln\dfrac{\theta_{i0}}{\theta_{i1}} + \ln\dfrac{1-\theta_{i1}}{1-\theta_{i0}}\right) + \ln\dfrac{1-\theta_{i1}}{1-\theta_{i0}}\right) + \ln\dfrac{\beta_{000}}{\beta_{001}} + w_1X_1 + w_2X_2 + w_{1,2}X_1X_2\right)}
\end{aligned}
$$

with

$$
\begin{aligned}
w_0 &= \ln \frac{1-\pi}{\pi} + \sum_{i=3}^{d} \ln \frac{1-\theta_{i1}}{1-\theta_{i0}} + \ln \frac{\beta_{000}}{\beta_{001}} \\
w_1 &= \ln \frac{\beta_{100}}{\beta_{101}} + \ln \frac{\beta_{001}}{\beta_{000}} \\
w_2 &= \ln \frac{\beta_{010}}{\beta_{011}} + \ln \frac{\beta_{001}}{\beta_{000}} \\
w_{1,2} &= \ln \frac{\beta_{110}}{\beta_{111}} + \ln \frac{\beta_{000}}{\beta_{001}} \ln \frac{\beta_{101}}{\beta_{100}} + \ln \frac{\beta_{011}}{\beta_{010}} \\
w_i &= \ln \frac{\theta_{i0}}{\theta_{i1}} + \ln \frac{1-\theta_{i1}}{1-\theta_{i0}} \ \textbf{for}\ i = 3, \ldots, d.
\end{aligned}
$$

Gaussian Baysian Classification
Some properties

Proving

the relationship between the decision rules for

Gaussian Naive Bayes **and the** ***Logistic Regression*** **algorithm**

when the covariance matrices are diagonal and identical

i.e., $\sigma_{0i}^2 = \sigma_{1i}^2$ **for** $i = 1, \ldots, d$

CMU, 2009 spring, Ziv Bar-Joseph, HW2, pr. 2

Assume a two-class ($Y \in \{0, 1\}$) Naive Bayes model over the d-dimensional real-valued input space $\mathbb{R}^d$, where the input variables $X|Y = 0 \in \mathbb{R}^d$ are distributed as

$$Gaussian(\mu_0 =< \mu_{01}, \ldots, \mu_{0d} >, \ \sigma =< \sigma_1, \ldots, \sigma_d >)$$

and $X|Y = 1 \in \mathbb{R}^d$ as

$$Gaussian(\mu_1 =< \mu_{11}, \ldots, \mu_{1d} >, \ \sigma =< \sigma_1, \ldots, \sigma_d >)$$

i.e., the inputs given the class have different means but identical variance for both classes.

Prove that, given the conditions stated above, the conditional probability $P(Y = 1|X = x)$**, where** $X = (X_1, \ldots, X_d)$ **and** $x = (x_1, \ldots, x_d)$ **can be written in a simiar form to Logistic Regression:**

$$\frac{1}{1 + \exp(w_0 + w \cdot x)}$$

with the parameters $w_0 \in \mathbb{R}$ **and** $w = (w_1, \ldots, w_d) \in \mathbb{R}^d$ **chosen in a suitable way.**

As a consequence, the decision rule for the Gaussean Bayes classifier supported by this model the desion rule has a linear form.

Solution

$$
\begin{aligned}
P(Y=1|X=x) &\stackrel{B.F.}{=} \frac{P(X=x|Y=1)\,P(Y=1)}{\sum_{y'\in\{0,1\}} P(X=x|Y=y')\,P(Y=y')} \\
&= \frac{1}{1+\dfrac{P(X=x|Y=0)P(Y=0)}{P(X=x|Y=1)P(Y=1)}} \\
&= \frac{1}{1+\exp\left(\ln\dfrac{P(X=x|Y=0)P(Y=0)}{P(X=x|Y=1)P(Y=1)}\right)} \\
&= \frac{1}{1+\underbrace{\exp\left(\ln\dfrac{P(X_1=x_1,\ldots,X_d=x_d|Y=0)P(Y=0)}{P(X_1=x_1,\ldots,X_d=x_d|Y=1)P(Y=1)}\right)}_{exponent}}
\end{aligned}
$$

$$
\begin{aligned}
exponent &\overset{cond.\ indep.}{=} \ln\frac{P(Y=0)}{P(Y=1)} + \sum_{i=1}^{d} \ln\frac{P(X_i=x_i|Y=0)}{P(X_i=x_i|Y=1)} \\
&= \ln\frac{P(Y=0)}{P(Y=1)} + \sum_{i=1}^{d} \ln\left(\frac{\frac{1}{\sqrt{2\pi}\sigma_i}\exp\left(-\frac{(x_i-\mu_{i0})^2}{2\sigma_i^2}\right)}{\frac{1}{\sqrt{2\pi}\sigma_i}\exp\left(-\frac{(x_i-\mu_{i1})^2}{2\sigma_i^2}\right)}\right) \\
&= \ln\frac{P(Y=0)}{P(Y=1)} + \sum_{i=1}^{d}\left(\frac{(x_i-\mu_{i1})^2}{2\sigma_i^2} - \frac{(x_i-\mu_{i0})^2}{2\sigma_i^2}\right) \\
&= \ln\frac{P(Y=0)}{P(Y=1)} + \sum_{i=1}^{d}\frac{2x_i(\mu_{0i}-\mu_{1i}) + (\mu_{1i}^2-\mu_{0i}^2)}{2\sigma_i^2} \\
&= \ln\frac{P(Y=0)}{P(Y=1)} + \sum_{i=1}^{d}\left(\frac{x_i(\mu_{0i}-\mu_{1i})}{\sigma_i^2} + \frac{(\mu_{1i}^2-\mu_{0i}^2)}{2\sigma_i^2}\right) \\
&= \underbrace{\ln\frac{P(Y=0)}{P(Y=1)} + \sum_{i=1}^{d}\frac{(\mu_{1i}^2-\mu_{0i}^2)}{2\sigma_i^2}}_{w_0} + \sum_{i=1}^{d}\underbrace{\frac{\mu_{0i}-\mu_{1i}}{\sigma_i^2}}_{w_i} x_i
\end{aligned}
$$

In conclusion,

$$P(Y=1|X=x) = \frac{1}{1+e^{(w\cdot x+w_0)}}$$

with

$$w_0 = \ln\frac{P(Y=0)}{P(Y=1)} + \sum_{i=1}^{d}\frac{(\mu_{1i}^2-\mu_{0i}^2)}{2\sigma_i^2} \textbf{ and } w_i = \frac{\mu_{0i}-\mu_{1i}}{\sigma_i^2}, i=1,\ldots,d$$

Note that

$$P(Y=0|X=x) = \frac{e^{(w\cdot x+w_0)}}{1+e^{(w\cdot x+w_0)}}$$

and

$$P(Y=1|X=x) > P(Y=0|X=x) \Leftrightarrow w\cdot x+w_0 < 0$$

Since the coefficients w_i for $i = 1, \ldots, d$ do not depend on x_i, it follows that this *decison rule* of Gaussian Naive Bayes [in the conditions stated in the beginning of this problem] is a linear rule, like in Logistic Regression.

However, this relationship does not mean that there is a one-to-one correspondence between the parameters w_i of Gaussian Naive Bayes (GNB) and the parameters w_i of logistic regression (LR) because LR is discriminative and therefore doesn't model $P(X)$, while GNB does model $P(X)$.

To be more specific, note that the coefficients w_i in the GNB decision rules should be devided by $P(x_1, \ldots, x_d)$ in order to correspond to $P(Y = 1|X = x)$, which means that then they will not anymore be independent of x_i, like the LR coefficients.

Estimating the parameters for Gaussian Naive Bayes and Full/Joint Gaussian Naive Bayes algorithms

CMU, 2014 fall, W. Cohen, Z. Bar-Joseph, HW2, pr. 5.ab

Consider a Gaussian Naive Bayes model, where the conditional distribution of each feature is a one-dimensional Gaussian, $X^{(j)}|Y \sim N(\mu_Y^{(j)}, (\sigma_Y^{(j)})^2)$**,** $j = 1, \cdots, d$**.**

a. Given n **independent training data points,** $\{(X_1, Y_1), \cdots, (X_n, Y_n)\}$**, give a maximum-likelihood estimate (MLE) of the conditional distribution of feature** $X^{(j)}, j = 1, \ldots, d$**.**

Solution:

The likelihood of the samples in Class 0 is

$$\begin{aligned} L(X_{i,0}^{(j)}|\mu_0^{(j)},(\sigma\mu_0^{(j)})^2) &= \prod_{i=1}^{n_0}\frac{1}{\sqrt{2\pi}\sigma_0^{(j)}}\exp\left(-\frac{(X_{i,0}^{(j)}-\mu_0^{(j)})^2}{2(\sigma_0^{(j)})^2}\right) \\ &= \left(\frac{1}{\sqrt{2\pi}\sigma_0^{(j)}}\right)^{n_0}\exp\left(-\sum_{i=1}^{n_0}\frac{(X_{i,0}^{(j)}-\mu_0^{(j)})^2}{2(\sigma_0^{(j)})^2}\right) \end{aligned}$$

and the log-likelihood is

$$\ln L = -n_0 \ln \sigma_0^{(j)} - \frac{1}{2(\sigma_0^{(j)})^2}\sum_{i=1}^{n_0}(X_{i,0}^{(j)}-\mu_0^{(j)})^2 + constant$$

Taking the partial derivatives of the log-likelihood, we have

$$\frac{\partial \ln L}{\partial \mu_0^{(j)}} = 0 \;\Leftrightarrow\; \sum_{i=1}^{n_0}(X_{i,0}^{(j)}-\mu_0^{(j)}) = 0 \Leftrightarrow \mu_0^{(j)} = \frac{1}{n_0}\sum_{i=1}^{n_0}X_{i,0}^{(j)}$$

$$\frac{\partial \ln L}{\partial \sigma_0^{(j)}} = 0 \;\Leftrightarrow\; -\frac{n_0}{\sigma_0^{(j)}} + \frac{1}{(\sigma_0^{(j)})^3}\sum_{i=1}^{n_0}(X_{i,0}^{(j)}-\mu_0^{(j)})^2 = 0 \Leftrightarrow (\sigma\mu_0^{(j)})^2 = \frac{1}{n_0}\sum_{i=1}^{n_0}(X_{i,0}^{(j)}-\hat{\mu}_0^{(j)})^2$$

Similarly, one can derive the MLE for the parameters in Class 1.

b. Suppose the prior of Y is already given. How many parameters do you need to estimate in Gaussian Naive Bayes model?

Solution:

For each class, there are 2 parameters (the mean and variance) for each feature, therefore there are $2 \cdot 2d = 4d$ parameters for all features in the two classes.

c. In a full/Joint Gaussian Bayes model, we assume that the conditional distribution $\Pr(X|Y)$ is a multidimensional Gaussian, $X|Y \sim \mathcal{N}(\mu_Y, \Sigma_Y)$, where $\mu \in \mathbb{R}^d$ is the mean vector and $\Sigma \in \mathbb{R}^{d \times d}$ is the covariance matrix.

Again, suppose the prior of Y is already given. How many parameters do you need to estimate in a full/Joint Gaussian Bayes model?

Solution:

For each class, there are d parameters for the mean, $d(d+1)/2$ parameters for the covariance matrix, because the covariance matrix is symmetric. Therefore, the number of parameters is $2 \cdot (d + d(d+1)/2) = d(d+3)$ in total for the two classes.

Proving

the relationship between

The full Gaussian Bayes **algorithm and** ***Logistic Regression***

when $\Sigma_0 = \Sigma_1$

CMU, 2011 spring, Tom Mitchell, HW2, pr. 2.2

Let's make the following *assumptions*:

1. Y **is a boolean variable following a Bernoulli distribution, with parameter** $\pi = P(Y = 1)$ **and thus** $P(Y = 0) = 1 - \pi$**.**

2. $X =< X_1, X_2, \ldots, X_n >$ **is a vector of random variables *not* conditionally independent given** Y**, and** $P(X|Y = k)$ **follows a *multivariate normal distribution*** $N(\mu_k, \Sigma)$**.**

Note that μ_k **is the** $n \times 1$ **mean vector depending on the value of** Y**, and** Σ **is the** $n \times n$ **covariance matrix, which does not depend on** Y**. We will write/use the density of the multivariate normal distribution in vector/matrix notation.**

$$\mathcal{N}(x; \mu\Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu)^\top \Sigma^{-1}(x - \mu)\right)$$

Is the form of $P(Y|X)$ **implied by such this [not-so-naive] Gaussian Bayes classifier [LC: similar to] the form used by logistic regression?**
Derive the form of $P(Y|X)$ **to prove your answer.**

We start with:

$$
\begin{aligned}
P(Y=1|X) &= \frac{P(X|Y=1)\,P(Y=1)}{P(X|Y=1)\,P(Y=1)+P(X|Y=0)\,P(Y=0)} \\
&= \frac{1}{1+\dfrac{P(Y=0)\,P(X|Y=0)}{P(Y=1)\,P(X|Y=1)}} = \frac{1}{1+\exp\left(\ln\dfrac{P(Y=0)\,P(X|Y=0)}{P(Y=1)\,P(X|Y=1)}\right)} \\
&= \frac{1}{1+\exp\left(\ln\dfrac{P(Y=0)}{P(Y=1)}+\ln\dfrac{P(X|Y=0)}{P(X|Y=1)}\right)}
\end{aligned}
$$

Next we will focus on the term $\ln\dfrac{P(X|Y=0)}{P(X|Y=1)}$**:**

$$
\ln\frac{P(X|Y=0)}{P(X|Y=1)} = \ln\frac{\dfrac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}}}{\dfrac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}}} + \ln\exp[(\star)] = \ln\exp[(\star)] = (\star)
$$

where $(\star)$ **is the formulation obtained as the difference between the exponential parts of two multivariate Gaussian densities** $P(X|Y=0)$ **and** $P(X|Y=1)$**.**

$$
\begin{aligned}
(\star) &= \frac{1}{2}[(X-\mu_1)^\top \Sigma^{-1}(X-\mu_1) - (X-\mu_0)^\top \Sigma^{-1}(X-\mu_0)] \\
&= (\mu_0^\top - \mu_1^\top)\Sigma^{-1}X + \frac{1}{2}\mu_1^\top \Sigma^{-1}\mu_1 - \frac{1}{2}\mu_0^\top \Sigma^{-1}\mu_0
\end{aligned}
$$

As a result, we have:

$$
\begin{aligned}
P(Y=1|X) &= \frac{1}{1+\exp\left(\ln\frac{1-\pi}{\pi} + \frac{1}{2}\mu_1^\top\Sigma^{-1}\mu_1 - \frac{1}{2}\mu_0^\top\Sigma^{-1}\mu_0 + (\mu_0^\top - \mu_1^\top)\Sigma^{-1}X\right)} \\
&= \frac{1}{1+\exp(w_0 + w^\top X)}
\end{aligned}
$$

where $w_0 = \ln\frac{1-\pi}{\pi} + \frac{1}{2}\mu_1^\top\Sigma^{-1}\mu_1 - \frac{1}{2}\mu_0^\top\Sigma^{-1}\mu_0$ **is a scalar,**
and $w = \Sigma^{-1}(\mu_0 - \mu_1)$ **is a** $d \times 1$ **a parameter vector.**

Note that $((\mu_0^\top - \mu_1^\top)\Sigma^{-1})^\top = ((\mu_0 - \mu_1)^\top\Sigma^{-1})^\top = (\Sigma^{-1})^\top((\mu_0-\mu_1)^\top)^\top = \Sigma^{-1}(\mu_0 - \mu_1)$ **because** Σ^{-1} **is symmetric.**
(Σ **is symmetric because it is a covariance matrix, and therefore** Σ^{-1} **is also symmetric.)**

In conclusion, $P(Y|X)$ **has the form of the logistic regression (in vector and matrix notation).**

Gaussian Baysian Classification

Some exercises

Exemplifying the Gaussian [Naive] Bayes algorithm on data from $\mathbb{R}$

CMU, 2001 fall, Andrew Moore, midterm, pr. 3.a

X	Y
0	A
2	A
3	B
4	B
5	B
6	B
7	B

Suppose you have the nearby training set with one real-valued input X and a categorial output Y that has two values.

a. You must learn from this data the parameters of the Gaussian Bayes classifer. Write your answer in the following table.

$\mu_A =$	$\sigma_A^2 =$	$P(Y = A) =$
$\mu_B =$	$\sigma_B^2 =$	$P(Y = B) =$

b. Using the notation $\alpha = p(X = 2|Y = A)$ and $\beta = p(X = 2|Y = B)$,

– **What is $p(X = 2, Y = A)$? (Answer in terms of α.)**

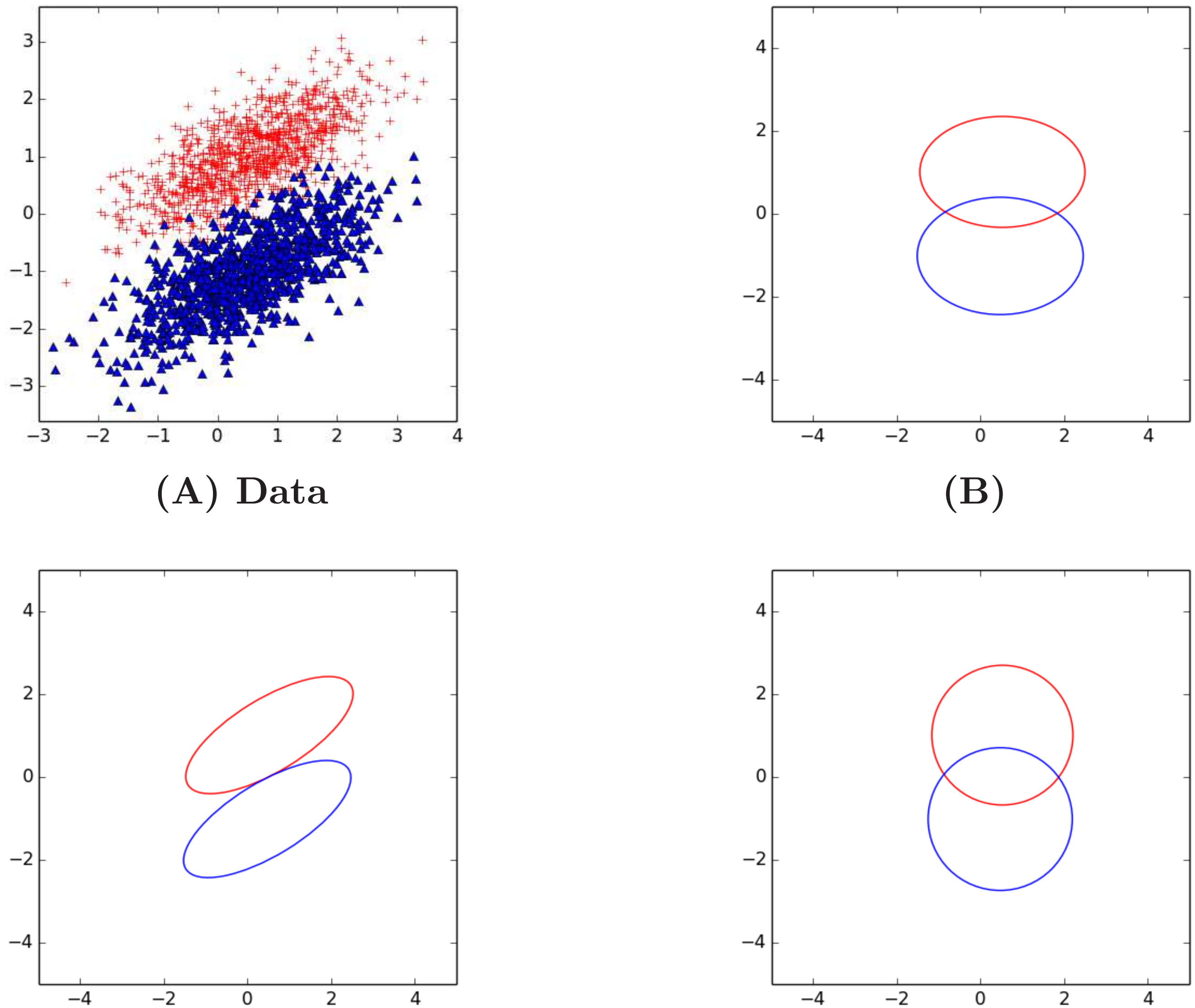
– **What is $p(X = 2, Y = B)$? (Answer in terms of β.)**

– **What is $p(X = 2)$? (Answer in terms of α and β.)**

– **What is $p(Y = A|X = 2)$? (Answer in terms of α and β.)**

– **How would the point $X = 2$ be classified by the Gaussian Bayes algorithm? (Answer in terms of α and β.)**

54.



(A) Data

(B)

(C)

(D)

Solution:

a. (C) is the truth.

b. (B) corresponds to the Gaussian Naive Bayes estimates. [LC: Here follows the explanation:]
Because the Gaussian Naive Bayes model assume independence of the two features conditioned on the class label, the estimated model should be aligned with the axies. Both (B) and (D) satisfy this, but only in (B) the width and height of the oval, which are proportional to the standard deviation of each axis, matched the data.

c. (C) gives the lowest training error.

Exemplifying the Gaussian [Naive] Bayes algorithm on data from $\mathbb{R}^2$

CMU, 2014 fall, W. Cohen, Z. Bar-Joseph, HW2, pr. 5.c

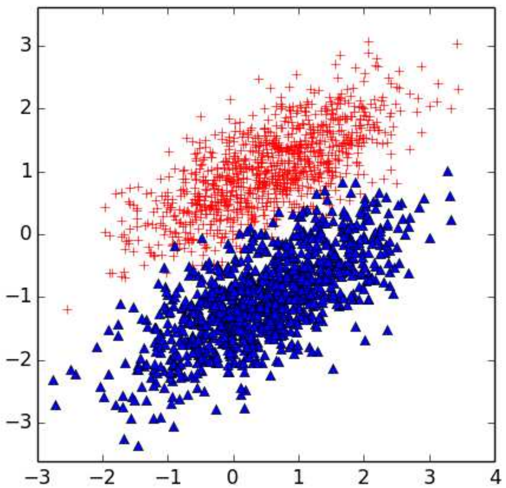
In a two dimensional case, we can visualize how Gaussian Naive Bayes behaves when input features are correlated. A data set is shown in Figure (A), where red points are in Class 0, blue points are in Class 1. The conditional distributions are two-dimensional Gaussians. In (B), (C) and (D), the ellipses represent conditional distributions for each class. The centers of ellipses show the means, and the contours show the boundary of two standard deviations.

a. Which of them is most likely to be the true conditional distribution?

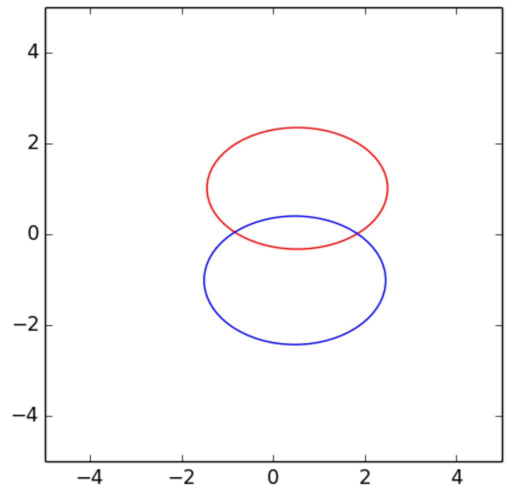
b. Which of them is most likely to be estimates by a Gaussian Naive Bayes model?

c. If we assume the prior probabilities for both classes are equal, which model will achieve a higher accuracy on the training data?

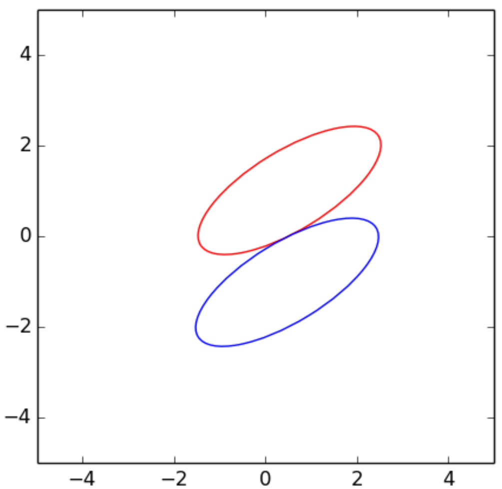
58.



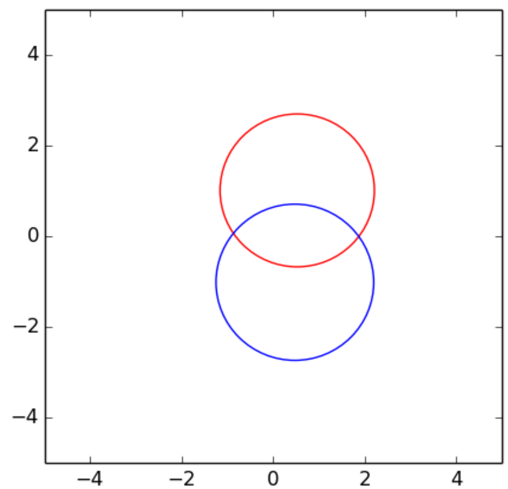
(A) Data



(B)



(C)



(D)

Solution:

a. (C) is the truth.

b. (B) corresponds to the Gaussian Naive Bayes estimates. [LC: Here follows the explanation:]
Because the Gaussian Naive Bayes model assume independence of the two features conditioned on the class label, the estimated model should be aligned with the axies. Both (B) and (D) satisfy this, but only in (B) the width and height of the oval, which are proportional to the standard deviation of each axis, matched the data.

c. (C) gives the lowest training error.