Exercices

Alfred Galichon
New York University, Economics Department and Courant Institute

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Abstract

Many of these exercises are taken from my monograph *Optimal Transport Methods in Economics* (Princeton University Press), hereafter [OTME]. When it is the case, they are indicated as such, and the solutions can be found there. Exercises denoted respectively (M), (E) and (C) have a focus on the mathematical, economic and computational aspects.

1 Optimal transport

Exercise 1.1 (M)**. *The Wasserstein distance*.** *([OTME], ex. 2.2.) Assume $\mathcal{X} = \mathcal{Y} = \mathbb{R}^d$, and that P and Q have second moments. Show that the value of*

$$B = \max_{\pi \in \mathcal{M}(P,Q)} \mathbb{E}_\pi \left[X'Y \right] \tag{1.1}$$

can be related to the value of

$$W = \min_{\pi \in \mathcal{M}(P,Q)} \mathbb{E}_\pi \left[\left\| x - y \right\|^2 / 2 \right] \tag{1.2}$$

up to constants to be characterized, and show that the optimal π is the same in the two problems. The value of Program (1.2) is called the squared Wasserstein distance *between P and Q. Write down the dual problem associated with with both Problems (1.1) and (1.2) and relate the solution to these two problems.*

2 Equilibrium

Exercise 2.1 (E)**.** ***Walrasian wages***. *([OTME], ex. 2.5) Assume worker characteristics are drawn from a population distribution P, and firm characteristics are drawn from a population distribution Q. Assume worker x has utility surplus $\alpha(x,y)+w$ of working for firm y at wage w, and firm y has surplus $\gamma(x,y)-w$. Let π be the equilibrium assignment, and $w(x,y)$ be the equilibrium wage.*

(i) Show that the optimum assignment of firms to workers is a solution to the primal Monge-Kantorovich problem, with $\Phi(x,y)=\alpha(x,y)+\gamma(x,y)$.

(ii) Assuming P, Q and Φ are such that both the primal and dual Monge-Kantorovich problems have solutions, show that $w(x,y)$ is an equilibrium wage if and only there is a solution (u,v) of the dual problem such that for every x and y in the support of P and Q,

$$\gamma(x,y)-v(y)\leq w(x,y)\leq u(x)-\alpha(x,y).$$

Exercise 2.2 (E)**.** ***The Becker-Coase theorem***. *([OTME], ex. 2.6) Assume a new bill requires landlords to pay the broker's fees, which were previously customarily paid by tenants. Rent prices are not controlled (and thus are adjusted by the market), and it is assumed that there is a slight excess supply of rental houses in the market under consideration. Using the formalism developed in this lecture, and more particularly the result of Exercise 2.1, argue why the bill may be inefficient.*

3 Discrete optimal transport

Exercise 3.1 (M)**.** ***Matching with 0–1 costs***. *([OTME], ex. 3.2) Assume $c_{xy}\in\{0,1\}$ and consider the problem*

$$\begin{aligned} V &= \max_{u,v}\sum_x u_x-\sum_y v_y. \\ & \quad s.t.\ u_x-v_y\leq c_{xy} \end{aligned} \tag{3.1}$$

(i) Show that the value of problem (3.1) is unchanged if one restricts the entries of u and v to be between 0 and 1.

(ii) Show that if $(u_x,v_y)\in[0,1]^2$, then

$$(u_x,v_y)=\int_0^1\left(1\{t\leq u_x\},1\{t\leq v_y\}\right)dt.$$

(iii) Assume (u, v) is feasible for problem (3.1). Show that for each $t \in [0,1]$, the vectors (u^t, v^t) defined by $u^t_x = 1\{t \leq u_x\}$ and $v^t_y = 1\{t \leq v_y\}$ are feasible for problem (3.1).

(iv) Show that the value of problem (3.1) is unchanged if one restricts the entries of u and v to be in $\{0,1\}$.

Exercise 3.2 (C)**.** ***Purifying a matching.*** *([OTME], ex. 3.3) Write a program that takes as an input a doubly stochastic matrix solution to the optimal assignment problem and returns a permutation matrix solution to the same problem.*

Exercise 3.3 (C)**.** ***A numerical experiment.*** *([OTME], ex. 3.4) Generate 100 sample points from two bivariate Gaussian distribution of mean 0 and the variance-covariance matrix of your choice. Take for $\theta \in [0, 2\pi]$, let $\Phi_\theta(x, y) = x_1 x_2 \cos\theta + y_1 y_2 \sin\theta$, and let π_θ be the solution to the optimal assignment problem between the sample points and surplus Φ_θ. Let*

$$C_\theta = \left(\mathbb{E}_{\pi_\theta}[X_1 X_2], \mathbb{E}_{\pi_\theta}[Y_1 Y_2]\right) \in \mathbb{R}^2$$

be the corresponding correlations. For each value of θ on a grid compute π_θ (you can use the program in the programming example for this purpose), and plot C_θ in the plane. What do you notice?

Exercise 3.4 (E)**.** ***Matching with singles.*** *([OTME], ex. 3.5) Consider a variant of the optimal assignment problem when the constraints are replaced by inequality constraints*

$$\sum_{y=1}^{M} \pi_{xy} \leq p_x \text{ and } \sum_{x=1}^{N} \pi_{xy} \leq q_y.$$

Write the dual associated with the new problem, and provide the economic interpretation.

Exercise 3.5 (E)**.** ***Only types matter.*** *([OTME], ex. 3.6) Consider a population of individual workers and firms $i \in \mathcal{I}$ and $j \in \mathcal{J}$. The type of worker i (respectively, firm j) is $x_i \in \mathcal{X}$ (resp. $y_j \in \mathcal{Y}$). Assume that the matching surplus between i and j only depends on i and j through their respective type, that is $\Phi_{ij} = \Phi_{x_i y_j}$. The reservation utilities are all set to zero. Let (π_{ij}, u_i, v_j) be a stable outcome. Show that u_i (resp. v_j) only depends on x_i (resp. on y_j), that is, there are vectors (u_x) and (v_y) such that $u_i = u_{x_i}$ and $v_j = v_{y_j}$.*

Exercise 3.6 (E)**. *Matching with taxes*.** *Consider a setting where if a worker of type $x \in \mathcal{X}$ is matched to a firm of type $y \in \mathcal{Y}$ with a gross wage w_{xy}, then the worker and the firm respectively get utility*

$$\alpha_{xy} + (1 - \tau) w_{xy} \text{ and } \gamma_{xy} - w_{xy}$$

where $\tau \in (0, 1)$ is a linear tax rate, $\alpha_{xy} \leq 0$ is the job amenity, and $\gamma_{xy} \geq 0$ is the economic output productivity. Show that the equilibrium matching μ^{τ}_{xy} is the optimal matching associated to surplus $\Phi^{\tau}_{xy} = \alpha_{xy} + (1 - \tau) \gamma_{xy}$. How is the Walrasian vector of wages w_{xy} determined based on the dual lp associated to the latter problem?

4 One-dimensional optimal transport

Exercise 4.1 (M)**. *The RUSC model*.** *([OTME], ex. 4.2) We study a Monge-Kantorovich problem when the distribution P of X is the uniform distribution over $[0, 1]$, and Q is a finitely supported distribution with $M + 1$ atom points y_j, $0 \leq j \leq M$, and the mass of y_j is q_j. Without loss of generality one may assume that $y_j < y_{j'}$ for $j < j'$, and $\Phi(x, y) = xy$. The social surplus is given by the value of the Monge-Kantorovich problem, that is*

$$\max_{\pi \in M(P,Q)} \mathbb{E}_{\pi}[XY].$$

This model is called the Random Uniform Scalar Coefficient (RUSC) model in Galichon and Salanié (2015).

(i) Show that this value is $\frac{1}{2} \sum_{k,j} \max(y_k, y_j) q_j q_k$.

(ii) Show that the solution $(u(x), v_j)$ of the dual Kantorovich problem is given by $u(x) = \max_{j=0,\ldots,M} \{xy_j - v_j\}$, $v_j = \sum_{k \neq 0} A_{jk} q_k + b_j$, where $A_{jk} = \max(y_k, y_j) - \max(y_k, y_0) - \max(y_j, y_0) + y_0$ and $b_k = \max(y_k, y_0) - y_0$.

Exercise 4.2 (C)**. *Simulation vs. closed-form*.** *([OTME], ex. 4.3) Let $\Phi(x, y) = xy$. Assume the distribution P is $\mathcal{U}([0, 1])$, and the distribution Q is a uniform distribution over {0.1, 0.3, 0.6,1}, whose points have probability 0.25 each.*

(i) Draw a random sample of $N = 10,000$ points from P and compute the optimal assignment between samples $\{x_1, ..., x_N\}$ and $\{y_1, ..., y_4\}$, as well as the prices v_j $(j = 1, ..., 4)$.

(ii) Using the results of Exercise 4.1, compare the prices obtained in (i) and the prices v_j obtained in closed form.

Exercise 4.3 (E). ***Capital vs. Labor***. *([OTME], ex. 4.5) The production function is assumed to satisfy $\Phi \geq 0$, $\partial_x \Phi \geq 0$, $\partial_y \Phi \geq 0$, and $\partial^2_{xy} \Phi \geq 0$. Assume workers are in excess supply, and that the reservation wage is zero, so that the lowest paid worker receives wage zero.*

(i) What is the effect of a technological shock $f(x) \geq 0$ in the productivity, so that production function $\Phi(x, y)$ is replaced by $\Phi(x, y) + f(x)$? Does this change necessarily increase the salary of the worker?

(ii) What is the effect of a homogenous technological shock $g(y) \geq 0$ in the productivity, so that production function $\Phi(x, y)$ is replaced by $\Phi(x, y) + g(y)$?

Exercise 4.4 (E). ***Fiscal gain from marriage***. *([OTME], ex. 4.6) Assume that individuals (men and women) marry only for fiscal purposes. Assume the tax levied on a single individual with income x is $\tau(x)$, where the tax schedule τ is convex, while the tax levied on a household where x and y are the man and woman's incomes is $2\tau((x + y)/2)$. Write down the fiscal gain from marriage $\Phi(x, y)$. What are the properties of the optimal matching? How will the fiscal gain from marriage be shared?*

Exercise 4.5 (M). ***The Knothe-Rosenblatt map.*** *([OTME], ex. 6.2) Let P and Q be two continuous distributions over $\mathbb{R}^2$ with finite second moments. For $\lambda > 0$, let $T^\lambda(x)$ be the optimal transport map between P and Q relative to surplus $\Phi^\lambda(x, y) = x_1 y_1 + \lambda x_2 y_2$. Let $\bar{T}(x) = \lim_{\lambda \to 0^+} T^\lambda(x)$. Give a heuristic argument to explain that*

$$\bar{T}(x_1, x_2) = \left(\bar{T}_1(x_1), \bar{T}_2(x_1, x_2)\right)$$

where $\bar{T}$ is called Knothe-Rosenblatt map *between distributions P and Q, and is defined by*

$$\begin{aligned}
\bar{T}_1(x_1, x_2) &= F^{-1}_{Y_1}(F_{X_1}(x_1)), \text{ and} \\
\bar{T}_2(x_1, x_2) &= F^{-1}_{Y_2|Y_1}\left(F_{X_2|X_1}(x_2|X_1 = X_1)\,|Y_1 = F^{-1}_{Y_1}(F_{X_1}(x_1))\right)
\end{aligned}$$

(note that $\bar{T}_1(x_1, x_2)$ does not depend on x_2).

5 Semi-discrete optimal transport

Exercise 5.1 (M). ***Power diagram as a Voronoi tesselation***. *([OTME], ex. 5.1) Show that any power diagram in $\mathbb{R}^d$ can be reexpressed as a Voronoi diagram in $\mathbb{R}^{d+1}$ projected onto a the d-dimensional space.*

Exercise 5.2 (M)**. *Logit as a Power diagram*.** *([OTME], ex. 5.2) Assume that $d = M$, i.e. the number of y's is equal to the dimension of the ambient space, and assume that y_j is the j-th element of the canonical basis of $\mathbb{R}^M$, that is all the components of y_j are zero except for the jth entry, whose value is one. Assume that P is the distribution of the centered Gumbel (extreme value type 1) distribution, that is*

$$F_P(x) = \prod\nolimits_{j=1}^{m} \exp\left(-\exp\left(-\left(x+\gamma\right)\right)\right)$$

where $\gamma \simeq 0.5772...$ is Euler's constant. Show that W, defined by

$$W(v) := \mathbb{E}_P\left[\max_{j \in \{1,...,M\}} \{X'y_j - v_j\}\right]. \tag{5.1}$$

is given by $W(v) = \log \sum_{k=1}^{M} \exp(-v_k)$, and compute the equilibrium price vector v, as well as the value of the social planner's problem. Interpret in terms of discrete choice models.

Exercise 5.3 (E)**. *Matching is a game*.** *([OTME], ex. 5.3) This exercise requires some notions about supermodular games. Set $v_1 = 0$ and consider the $(M-1)$-player game where the profit of agent $k \in \{2, ..., M\}$ is given by*

$$\Pi_k(v_k; v_{-k}) := -W(v) - q_k v_k,$$

where W is defined by (5.1).

(i) Show that $\Pi_k(v_k; v_{-k})$ is concave in v_k and supermodular.

(ii) Show that the best reply dynamics given by

$$v_k^{t+1} = \arg\min_{\bar{v} \in \mathbb{R}} W\left(\bar{v}; v_{-k}^t\right) + q_k \bar{v} \tag{5.2}$$

converges isotonically to the equilibrium price $(v_k^*)_k$ if the v_k^0's are chosen high enough. This suggests an algorithm for computing equilibrium prices which is an alternative to the one described in class. Discuss possible advantages of the present method.

6 Convex analysis

Exercise 6.1 (M)**. *Legendre-Fenchel transforms*.** *Compute the Legendre-Fenchel transforms of the following convex functions:*

1. $f(x) = \iota_C(x)$, *which is defined as* 0 *if* $x \in C$ *and* $+\infty$ *otherwise, where* C *is a closed convex set.*

2. $f(x) = \frac{1}{2}|x|^2$.

3. $f(x) = \frac{1}{2}\sum_i \lambda_i x_i^2$, $\lambda_i > 0$.

4. $f(x) = \frac{1}{2}x'\Sigma x$, *where* Σ *is a positive-definite matrix.*

5. $f(x) = \sum_i x_i \ln x_i$ *for* $x \geq 0$, $\sum_i x_i = 1$, $+\infty$ *otherwise.*

6. $f(x) = \|x\|^p/p$ *where* $p > 1$, *and* $\|\cdot\|$ *denotes the Euclidian norm.*

Exercise 6.2 (E)**.** ***CES function***. *Consider the constant elasticity of substitution (CES) function:*

$$f(x) = \begin{cases} (\sum_i x_i^{(\alpha-1)/\alpha})^{\alpha/(\alpha-1)} \text{ for } x \geq 0 \\ +\infty \text{ otherwise,} \end{cases}$$

where $r = (\alpha - 1)/\alpha > 1$.

(i) What is the limit of $f(x)$ *when* $r \to +\infty$*? when* $r \to -\infty$*?*

(ii) Compute the Hessian of f *and show that* f *is convex for* $r \geq 1$ *and concave for* $r \leq 1$.

(iii) What is the Legendre transform of f *when* $r > 1$ *and of* $-f$ *when* $r < 1$*?*

Exercise 6.3 (M)**.** *Let* f *and* g *be two convex functions defined on* $\mathbb{R}^n$. *Show that* $\nabla f \circ \nabla g$ *is in general not the gradient of a convex function, unless* $n = 1$.

Exercise 6.4 (C)**.** *Assume* f *is* C^1 *and consider a solution to the differential equation*

$$\frac{dx_t}{dt} = -\lambda \frac{\nabla f(x_t)}{|\nabla f(x_t)|^2}.$$

Show that on the domain where x_t *is defined,* $f(x_t) = f(x_0) - \lambda t$.

Exercise 6.5 (C)**.** *Assume* f *is* C^2 *and consider a solution to the differential equation*

$$\frac{dx_t}{dt} = -\lambda \left(D^2 f(x_t)\right)^{-1} \nabla f(x_t).$$

Show that on the domain where x_t *is defined,* $\nabla f(x_t) = \nabla f(x_0) \exp(-\lambda t)$.

Exercise 6.6 (M)**.** *Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a convex function and $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be a convex and increasing function. Show that $F = \varphi \circ f$ is a convex function.*

Exercise 6.7 (M)**.** ***Quasiconvex functions****. A function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is called quasiconvex if for every $\alpha \in \mathbb{R}$, the set $\left\{x \in \mathbb{R}^d : f(x) \leq \alpha\right\}$ is convex whenever it is not empty.*

(i) Show that a convex function is quasiconvex.

(ii) Show that if $f : \mathbb{R}^d \mapsto \mathbb{R}$ is convex and if $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is increasing, then $\varphi \circ f$ is quasi-convex.

(iii) Show that if f is quasi-convex and C^2, then

$$\forall z \in \nabla f(x)^{\perp}, z^{\intercal} D^2 f(x) z \geq 0.$$

7 Discrete choice models

Exercise 7.1. *Compute explicitely $D^2 G$ and $D^2 G^*$ in the case of the logit model.*

8 Matching with heterogeneities

Exercise 8.1. *Consider the case $M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \sqrt{\mu_{x0}\mu_{0y}} K_{xy}$, where $K_{xy} > 0$. [This is the Choo-Siow model discussed in class]. Write an algorithm for determining the equilibrium.*

Exercise 8.2. *In the setting of the previous exercise, assume that $K_{xy} = \exp\left(\frac{\Phi_{xy}}{2T}\right)$, for $T \to 0$. Show that the solution μ_{xy} converges when $T \to 0$ to the solution of a linear programming problem. Interpret. [NB: it is useful to have solved the two previous exercises for this one].*