M+E+C: Shortest Path Algorithms

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Shortest Path Algorithms for Directed Graphs

- Shortest path problem: find the minimum distance path from a source node to other nodes in a network.

1. **Dijkstra**: Begin at the source node and choose the unvisited vertex with minimal distance. Repeat.

   $$O(|\mathcal{X}|^2)$$

   Requires non-negative weights along edges.

   (Worst case cost can be reduced to $O(|\mathcal{A}| + |\mathcal{X}| \ln(|\mathcal{X}|))$.)

2. **Bellman-Ford**: Iterate over all nodes and adjacent vertices.

   $$O(|\mathcal{A}||\mathcal{X}|)$$

   Allows for negative weights along edges.

Dijkstra's Algorithm

Algorithm 1 Dijkstra

procedure DIJKSTRA$(s, \mathcal{X}, \mathcal{A})$ $\triangleright$ Solve for $\mathbf{d}^*$
  Define the pair $(u_j, d_j) \in (\mathcal{X}, \mathbb{R})$, where $d_j := \text{dist}(u_s, u_j)$.
  $u_s \in \mathcal{X}$, $\mathcal{V} = \mathcal{X}$; $\mathbf{d}^* = \boldsymbol{\infty}$, $d_s^* = 0$ $\triangleright$ Initialization
  while $\mathcal{V} \neq \{\emptyset\}$ **do**
    Choose $u_i := \{u_j \in \mathcal{V} : d_j^* = \min_k\{d_k^*\}\}$.
    if $d_j^* = \infty$ **then**
      break;
    $\mathcal{V} = \mathcal{V} \backslash \{u_i\}$ $\triangleright$ Remove u_i
    Define the adjacent network to u_i by $N(u_i)$.
    For each $v_j \in N(u_i) \cap \mathcal{V}$, with arc weights $\{a_{i,j}\}$, calculate

$$c_j = d_i^* + a_{i,j}$$

    if $c_j < d_j^*$ **then**
      $d_j^* = c_j$ $\triangleright$ Update d_j^*
  return $\mathbf{d}^* = \{d_j^*\}_{j \in \mathcal{X}}$. $\triangleright$ Solution

Dijkstra in Words

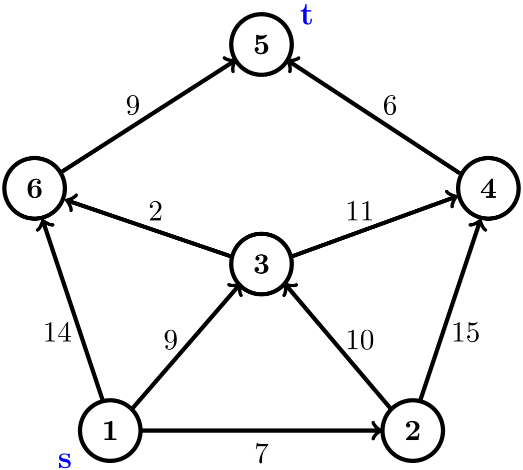
1 Begin at the source node u_s

- initialize the (minimum) distance vector: $\{d_j^*\}_{j=1}^{|\mathcal{X}|} = \infty$ and $d_s = 0$
- insert the source node into a vertex queue $\mathcal{V}$

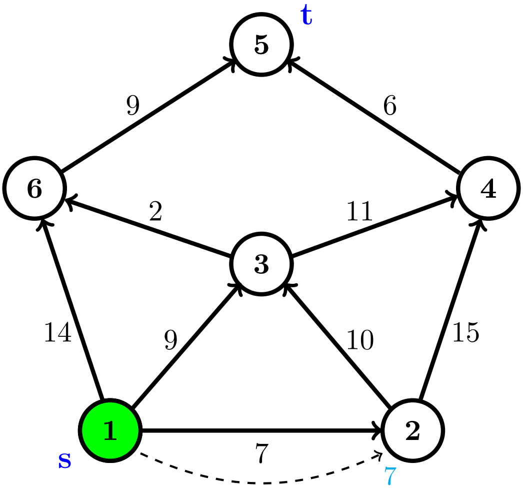
2 While the vertex queue is nonempty:

- select the node at the beginning of the vertex queue; call it u_i
- for every node adjacent to u_i, labelled $\{v_j\}_{j \in N(i)}$:
  - ★ calculate the distance to each v_j node through u_i: $\{c_j(u_i)\}_{j \in N(i)}$
  - ★ if $c_j(u_i)$ is less than the minimum distance observed thus far (d_j^*), set $d_j^* = c_j(u_i)$
- update the vertex queue from $N(i)$; remove u_i from the queue

Dijkstra in Action



Dijkstra in Action



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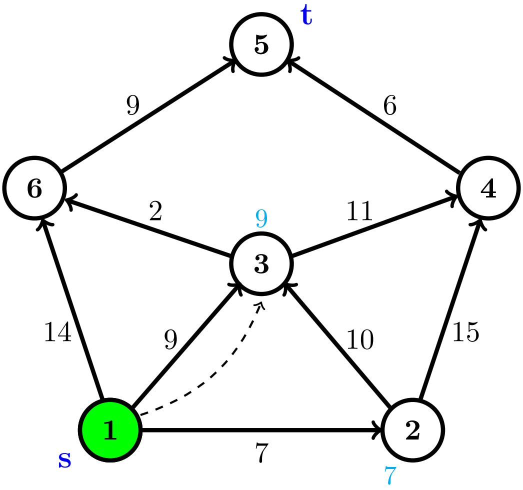

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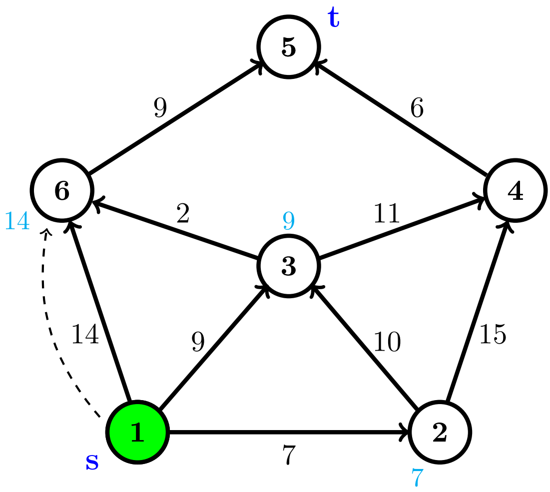

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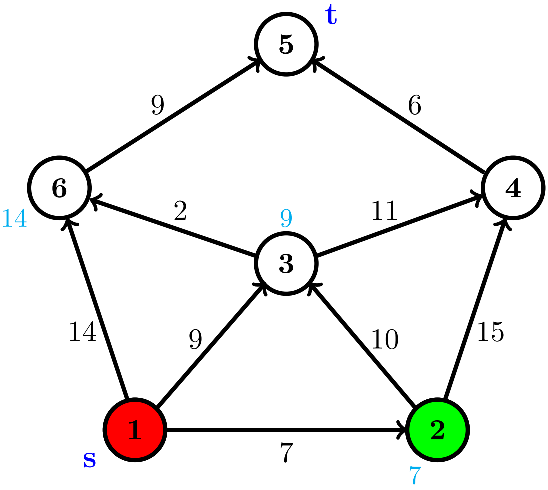

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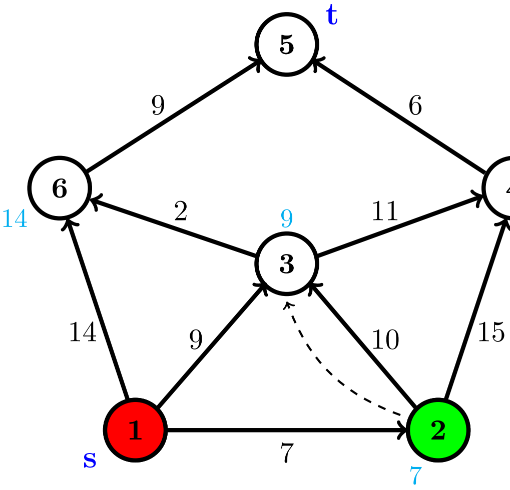

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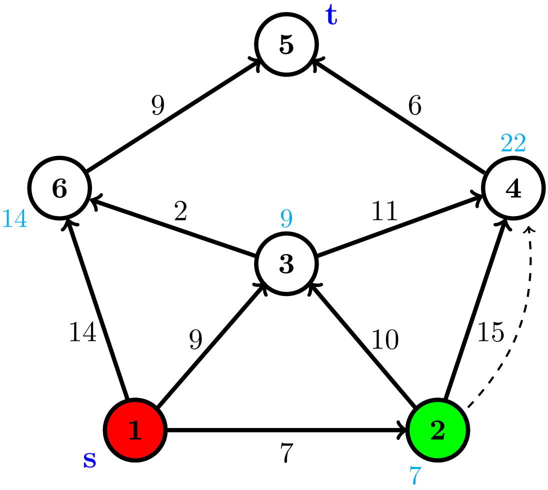

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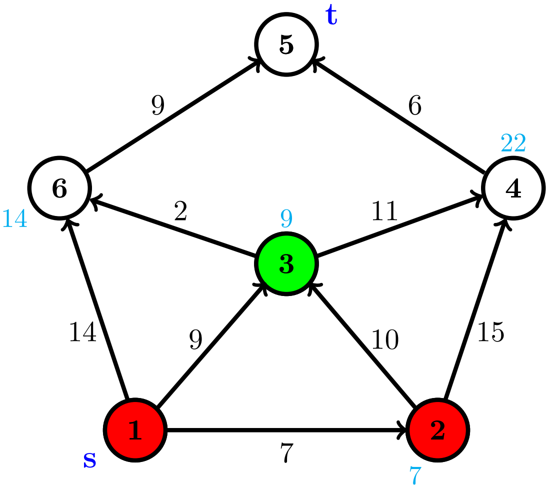

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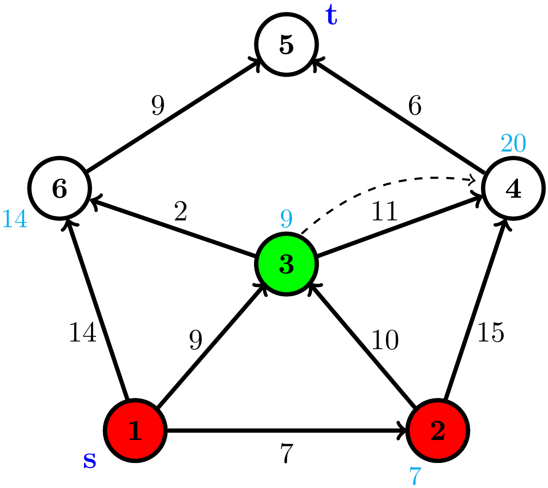

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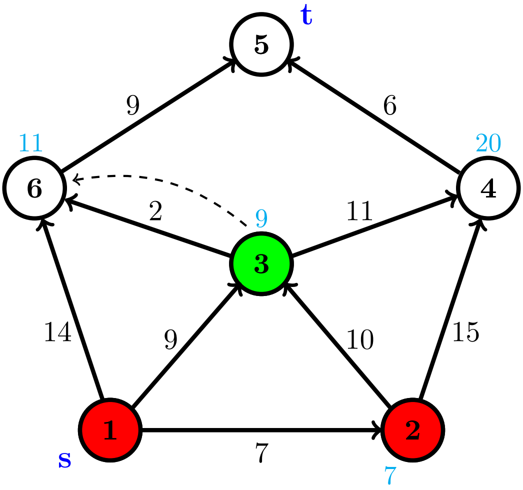

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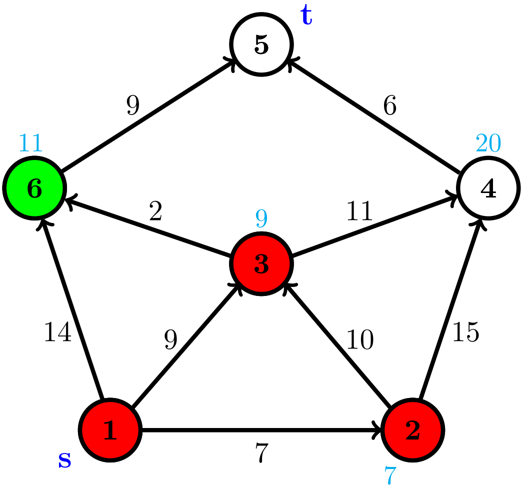

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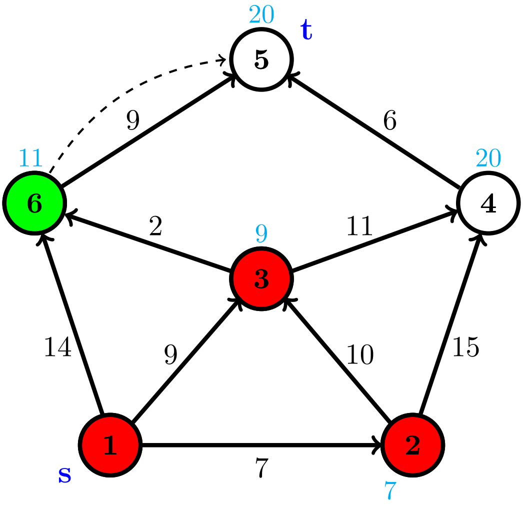

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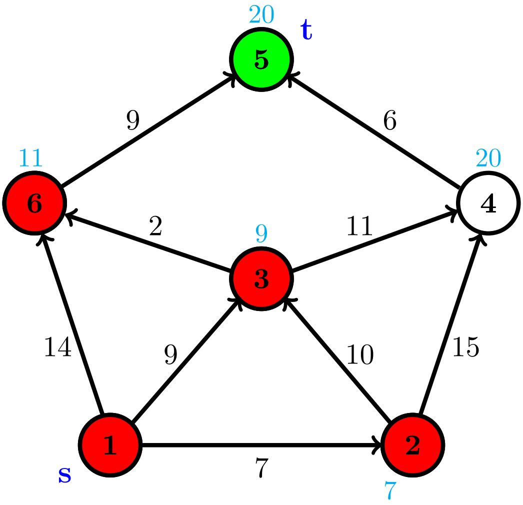

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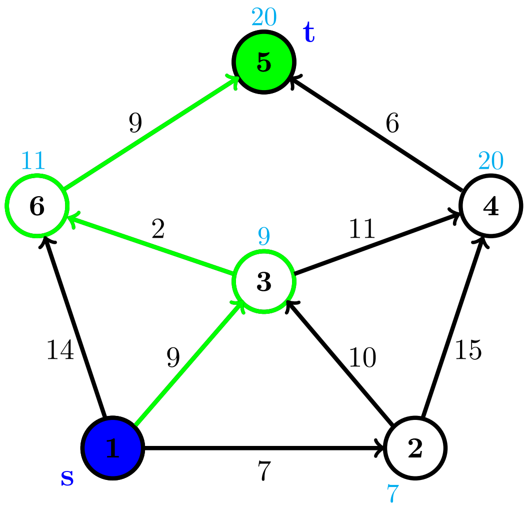


Dijkstra in Code

- We've written a lightweight **R** package for this: SPR. It acts as a wrapper to C++ code.

```
library(SPR)

arcs <- rbind(c(1,2,7),
              c(1,3,9),
              c(1,6,14),
              c(2,3,10),
              c(2,4,15),
              c(3,4,11),
              c(3,6,2),
              c(4,5,6),
              c(6,5,9))

nbNodes <- max(max(arcs[,1]),max(arcs[,2]))

sol <- dijkstra(nbNodes,1,arcs)
sp <- get_shortest_path(5,sol$path_list)

sol <- dijkstra(nbNodes,1,arcs,5) # 1 step
```

Bellman-Ford Algorithm

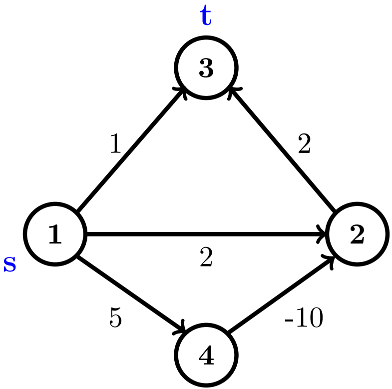
Algorithm 2 Bellman-Ford

procedure $\textsc{BellmanFord}(s, \mathcal{X}, \mathcal{A})$ $\triangleright$ Solve for $\mathbf{d}^*$
    Define the pair $(u_j, d_j) \in (\mathcal{X}, \mathbb{R})$, where $d_j := \text{dist}(u_s, u_j)$.
    $u_s \in \mathcal{X}$; $\mathbf{d}^* = \boldsymbol{\infty}$, $d_s^* = 0$ $\triangleright$ Initialization
    for $i = 1, \ldots, |\mathcal{X}| - 1$ **do**
        for $j = 1, \ldots, |\mathcal{X}|$ **do**
            for $v_k \in N(j)$ **do**
                Calculate

$$c_k = d_j^* + a_{j,k}$$

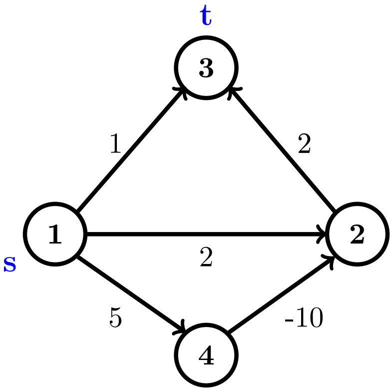
                if $c_k < d_k^*$ **then**
                    $d_k^* = c_k$ $\triangleright$ Update d_k^*
    return $\mathbf{d}^* = \{d_j^*\}_{j \in \mathcal{X}}$. $\triangleright$ Solution

Example with Negative Weights



- What would Dijkstra do?

Example with Negative Weights



- What would Dijkstra do?
- 1: $\{0, 2, 1, 5\}$. 3: $\{0, 2, 1, 5\}$. 2: $\{0, 2, 1, 5\}$. 4: $\{0, 2, 1, 5\}$. End.