‘MATH+ECON+CODE’ MASTERCLASS ON MATCHING MODELS, OPTIMAL TRANSPORT AND APPLICATIONS

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Day 6, January 20 2018: “Matching models beyond quasilinearity”

Section 1

Block 16: matching function equilibria

References for block 16

- G, Kominers and Weber (2017). Costly concessions. An Empirical Framework for Matching with Imperfectly Transferable Utility. Mimeo.
- G and Weber (2018). "Matching function equilibrium". Mimeo.
- Mourifié and Siow (2014). Cohabitation versus marriage: Marriage matching with peer effects. Mimeo.

- Matching function equilibrium
- Generalized IPFP

- A matching function is a function $M_{xy}\left(\mu_{x0},\mu_{0y}\right)$ which is isotone and such that
$$\mu_{xy}=M_{xy}\left(\mu_{x0},\mu_{0y}\right).$$
- A matching function equilibrium (MFE) is the set of equations
$$\left\{\begin{array}{c} n_x=\mu_{x0}+\sum_{y\in\mathcal{Y}}M_{xy}\left(\mu_{x0},\mu_{0y}\right)\\ m_y=\mu_{0y}+\sum_{x\in\mathcal{X}}M_{xy}\left(\mu_{x0},\mu_{0y}\right)\end{array}\right.$$
- Today, we will first provide examples of models that reformulate as MFEs; then we will discuss existence, computation, and uniqueness of a MFE, and then we'll discuss comparative statics.

- This model was seen yesterday. Consider a labor market where $\mathcal{X}$ are the types of the workers and $\mathcal{Y}$ are the types of the firms. There are n_x workers of type x, and m_y firms of type y.
- Let w_{xy} be the equilibrium salary of a worker of type x working for a firm y. Assume that worker $x \in \mathcal{X}$ has utility for matching with firm of type y equal to

$$\alpha_{xy} + w_{xy} + \varepsilon_y$$

  and ε_0 if remains unemployed, where the random utility vector ε is a vector of i.i.d. Gumbel distributions drawn by each worker, and α is a term that captures job amenity.
- Similarly, the profit of the firm is

$$\gamma_{xy} - w_{xy} + \eta_y$$

  and η_0 if it does not hire, where the random utility vector η is a vector of i.i.d. Gumbel distributions drawn by each worker, and γ is a term that captures job productivity.

- The quantities (α, γ, n, m) as well as the distributions of ε and η are exogenous: they are primitives of the model. The equilibrium quantities are the matching patterns (μ_{xy}), as well as the equilibrium wages w_{xy}.
- The conditional choice probabilities on the side of workers is $\mu_{y|x} = \mu_{xy} / n_x$, and the CCPs on the side of firms is $\mu_{x|y} = \mu_{xy} / m_y$. Because we are in a logit model, we have by the log-odds ratio formula that

$$\alpha_{xy} + w_{xy} = \ln \frac{\mu_{xy}}{\mu_{x0}} \text{ and } \gamma_{xy} - w_{xy} = \ln \frac{\mu_{xy}}{\mu_{0y}},$$

so by summation, $\alpha_{xy} + \gamma_{xy} = 2 \ln \mu_{xy} - \ln \mu_{x0} - \ln \mu_{0y}$, thus $\mu_{xy} = \sqrt{\mu_{x0}\mu_{0y}} K_{xy}$, where $K_{xy} = \exp\left(\frac{\alpha_{xy}+\gamma_{xy}}{2}\right)$, that is

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \sqrt{\mu_{x0}\mu_{0y}} K_{xy}. \tag{1}$$

- Note that TU models with general heterogeneities do not have a MMF formulation.

- We consider a dynamic model of matching with search frictions where a pair of employers and employees drawn from the population of unmatched agents decide whether to match or not.
- As before, the number of xy pairs is μ_{xy}, the number of unassigned workers of type x is μ_{x0}, and the number of unassigned firms of type y is μ_{0y}.
- The continuation value of an unassigned worker of type x (resp. firm of type y) is U_{x0} (resp. V_{0y}). If x and y match, then x gets $U_{xy} + \varepsilon_y$, and y gets $V_{xy} + \eta_x$. Note that U and V are endogenous.
- Let $a_{xy} = \Pr\left(U_{xy} + \varepsilon_y \geq U_{x0} \text{ and } V_{xy} + \eta_x \geq V_{0y}\right)$ be the probability that if x and y meet, they decide to match.
    - In the original model, $\varepsilon = 0$ and $\eta = 0$, so that $a_{xy} = 1\left\{U_{xy} + \varepsilon_y \geq U_{x0} \text{ and } V_{xy} + \eta_x \geq V_{0y}\right\}$.
    - In general, a_{xy} is an increasing function of both $U_{xy} - U_{x0}$ and $V_{xy} - V_{0y}$. See e.g. Goussé, Jacquemet and Robin (2016).

- There is an an exogenous destruction of xy matches with intensity δ, so that the flow of dissolutions of xy matches is

$$\delta\mu_{xy}.$$

- The flow of new xy matches created is equal to

$$\lambda\mu_{x0}\mu_{0y}a_{xy}$$

- At steady state, the flow of matched created is equal to the flow of matches destructed, thus

$$\delta\mu_{xy} = \lambda\mu_{x0}\mu_{0y}a_{xy}$$

which leads to matching function

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \mu_{x0}\mu_{0y}K_{xy},$$

where $K_{xy} = \left(\lambda/\delta\right)a_{xy}$.

- The previous analysis is incomplete as U_{xy}, U_{x0}, V_{xy} and V_{0y} are endogenous. To close the model, one shall need:
  - feasibility: relate U_{xy} and V_{xy}, typically by $U_{xy} + V_{xy} = \Phi_{xy}$
  - bargaining: e.g. if fair (Nash), $U_{xy} - U_{x0} = V_{xy} - V_{0y}$
  - Bellman equations relating U_{x0} and U_{xy}; for instance

$$U_{x0} = \rho \sum_y a_{xy} \left(U_{xy} - U_{x0} \right) \mu_{0y}$$

$$V_{0y} = \rho \sum_x a_{xy} \left(V_{xy} - V_{0y} \right) \mu_{x0}$$

- Full equilibrium therefore involves $\mu_{x0}, \mu_{0y}, \mu_{xy}, U_{x0}, U_{xy}, V_{0y}, V_{xy}$ as unknowns, and an equal number of equations.

- Dagsvik-Menzel's model (seen later today) is a model of matching without transfers and with logit heterogeneity. If worker i of type $x \in \mathcal{X}$ matches with firm j of type $y \in \mathcal{X}$, then worker and firm get respectively

$$\alpha_{xy} + \varepsilon_{ij} \text{ and } \gamma_{xy} + \eta_{ij}$$

  where ε_{ij} and η_{ij} are iid Gumbel. If unassigned, get $\alpha_{x0} + \varepsilon_{i0}$ and $\gamma_{0y} + \eta_{0j}$.
- Wages are decided exogenously: there are no possible equilibrium adjustments of wage. Hence, worker i chooses preferred firm among firms who find her acceptable. At equilibrium, one has

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \mu_{x0}\mu_{0y}K_{xy}$$

  with $K_{xy} = \exp\left(\alpha_{xy} + \gamma_{xy}\right)$.
- We will see why this afternoon.

- The gravity equation can be reformulated

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \mu_{x0}\mu_{0y}K_{xy}$$

  where $\ln \mu_{x0}$=importer x's fixed effect and $\ln \mu_{0y}$=exporter y's fixed effect.
- See block 14.

- A *matching function equilibrium* is a solution of the following system with unknowns μ_{x0} and μ_{0y}:

$$\left\{ \begin{array}{c} \mu_{x0} + \sum_{y \in \mathcal{Y}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = n_x \\ \mu_{0y} + \sum_{x \in \mathcal{X}} M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = m_y \end{array} \right. . \tag{2}$$

- In the sequel we will consider the following questions:
  - Existence of an equilibrium
  - Algorithms for the determination of an equilibrium
  - Uniqueness of an equilibrium

We will assume the following about the matching functions:

ASSUMPTION

M is such that for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$:
(i) Map $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$ is continuous.
(ii) Map $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$ is weakly isotone, i.e. if $a \leq a'$ and $b \leq b'$, then $M_{xy}(a, b) \leq M_{xy}(a', b')$.
(iii) For each $a > 0$, $\lim_{b \to 0^+} M_{xy}(a, b) = 0$, and for each $b > 0$, $\lim_{a \to 0^+} M_{xy}(a, b) = 0$.

ALGORITHM

Step 0. Fix the initial value of μ_{0y} *at* $\mu_{0y}^{0} = m_y$.

Step $2t+1$. *Keep the values* μ_{0y}^{2t} *fixed. For each* $x \in \mathcal{X}$, *solve for the value,* μ_{x0}^{2t+1}, *of* μ_{x0} *so that*

$$\sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}^{2t}) + \mu_{x0} = n_x.$$

Step $2t+2$. *Keep the values* μ_{x0}^{2t+1} *fixed. For each* $y \in \mathcal{Y}$, *solve for which is the value,* μ_{0y}^{2t+2}, *of* μ_{0y} *so that*

$$\sum_{x \in \mathcal{X}} M_{xy}(\mu_{x0}^{2t+1}, \mu_{0y}) + \mu_{0y} = m_y.$$

The following theorem from [GKW] ensures that the algorithm converges to a matching function equilibrium.

THEOREM

Under Assumptions (i)–(iii) above, there exists a matching function equilibrium which is the limit of $\left(\mu_{x0}^{2t+1}, \mu_{0y}^{2t+2}\right)$ *defined in the algorithm.*

- We show that the construction of μ_{x0}^{2t+1} and μ_{0y}^{2t+2} at each step is well defined. Consider step $2t+1$. For each $x \in \mathcal{X}$, the equation to solve is

$$\sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}) + \mu_{x0} = n_x$$

but the right-hand side is a continuous and increasing function of μ_{x0}, tends to 0 when $\mu_{x0} \to 0$ and tends to $+\infty$ when $\mu_{x0} \to +\infty$. Hence μ_{x0}^{2t+1} is well defined and is in $(0, +\infty)$. The map $\left(\mu_{0y}^{2t}\right) \to \left(\mu_{x0}^{2t+1}\right)$ is antitone, meaning that $\mu_{0y}^{2t} \leq \tilde{\mu}_{0y}^{2t}$ for all $y \in \mathcal{Y}$ implies $\tilde{\mu}_{x0}^{2t+1} \leq \mu_{x0}^{2t+1}$ for all $x \in \mathcal{X}$.
- By the same token, the map $\left(\mu_{x0}^{2t+1}\right) \to \left(\mu_{0y}^{2t+2}\right)$ is well defined and antitone. Thus, the map $\left(\mu_{0y}^{2t}\right) \to \left(\mu_{0y}^{2t+2}\right)$ is isotone. But $\mu_{0y}^{2} \leq m_y = \mu_{0y}^{0}$ implies that $\mu_{0.}^{2t+2} \leq \mu_{0.}^{2t}$. Hence $\left(\mu_{0y}^{2t}\right)_{t \in \mathbb{N}}$ is a decreasing sequence, bounded from below by 0. As a result $\left(\mu_{0y}^{2t}\right)$ converges. Letting $(\bar{\mu}_{0y})$ be its limit, and letting $(\bar{\mu}_{x0})$ be the limit of $\left(\mu_{x0}^{2t+1}\right)$, it is not hard to see that $(\bar{\mu}_{0x}, \bar{\mu}_{0y})$ is a solution to (2).

- We illustrate the algorithm on Dagsvik-Menzel's model. Recall that in this model $M_{xy}\left(\mu_{x0},\mu_{0y}\right)=\mu_{x0}\mu_{0y}K_{xy}$. Loop consists in:

$$\left\{\begin{array}{c}\mu_{x0}^{2t+1}=\frac{n_x}{\sum_{y\in\mathcal{Y}}K_{xy}\mu_{0y}^{2t}+1}\\ \mu_{0y}^{2t+2}=\frac{m_y}{\sum_{x\in\mathcal{X}}\mu_{x0}^{2t+1}K_{xy}+1}\end{array}\right. .$$

- In R, this is implemented as:

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mux0 = n / (K %*% mu0y + 1)
mu0y = m / (t(K) %*% mux0 + 1)
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THEOREM

Under Assumptions (i)–(iii) above, and under the additional assumption that the domain of M_{xy} is $\mathbb{R}^2_+$ for each x and y, the matching function equilibrium is unique.

The proof of this theorem (proven more generally in [GKW]) is based on reformulating the equilibrium as a demand system and applying the result of Berry, Gandhi and Haile (2013).

- Let $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$ be the set of goods, and let $u_x = \mu_{x0}$ if $x \in \mathcal{X}$, and $u_z = -\mu_{0y}$ if $y \in \mathcal{X}$. Consider the map

$$\sigma_x(u) = u_x + \sum_{y \in \mathcal{Y}} M_{xy}(u_x, -u_y) - n_x$$
$$\sigma_y(u) = m_y + u_y - \sum_{x \in \mathcal{Y}} M_{xy}(u_x, -u_y)$$

- Let $\emptyset$ be a zero good defined by

$$\sigma_\emptyset(u) = 1 + \sum_{x \in \mathcal{Y}} n_x - \sum_{y \in \mathcal{Y}} m_y - \sum_{y \in \mathcal{Y}} u_y - \sum_{x \in \mathcal{Y}} u_x.$$

and let

$$\mathcal{Z}_\emptyset = \mathcal{Z} \cup \{\emptyset\}.$$

- We can show that, under minimal assumptions on M_{xy}, the map σ satisfies the assumptions in this lecture, guaranteeing the existence and uniqueness of the equilibrium matching μ.
  - Existence is constructive (IPFP)
  - Uniqueness follows from a result of Berry, Gandhi and Haile (2013), see GKW (2017).

EXERCISE

Write down the Jacobian on the map on the left handside of equations (2). When can one interpret these equations as the first order conditions to a convex minimization problem?

(a) The map σ is such that

$$\left\{ \begin{array}{c} \sigma_x(\mu) = \mu_{x0} + \sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}) - n_x \\ \sigma_y(\mu) = \mu_{0y} + \sum_{x \in \mathcal{X}} M_{xy}(\mu_{x0}, \mu_{0y}) - m_y \end{array} \right.$$

and its Jacobian J is written blockwise with

$J_{11} = diag\left(1 + \sum_{y \in \mathcal{Y}} \partial_{\mu_{x0}} M_{xy}(\mu_{x0}, \mu_{0y})\right)$

$J_{12} = \left(\partial_{\mu_{0y}} M_{xy}(\mu_{x0}, \mu_{0y})\right)_{xy}$ and $J_{21} = \left(\partial_{\mu_{x0}} M_{xy}(\mu_{x0}, \mu_{0y})\right)_{xy}$

$J_{22} = diag\left(1 + \sum_{y \in \mathcal{Y}} \partial_{\mu_{x0}} M_{xy}(\mu_{x0}, \mu_{0y})\right)$.

(b) This problem can be interpreted as the first order conditions associated to a convex optimization problem when there are maps $M_{x0}\left(u_x\right)$ and $M_{0y}\left(v_y\right)$ such that $\sigma\left(M\left(u,v\right)\right)$ is a gradient. In particular, this implies that the off-diagonal blocks of the Jacobian of $u \rightarrow \sigma\left(M\left(u,v\right)\right)$ should be transpose of eachother. Thus

$$\partial_{u_x} M_{xy}\left(M_{x0}\left(u_x\right), M_{0y}\left(v_y\right)\right) = \partial_{v_y} M_{xy}\left(M_{x0}\left(u_x\right), M_{0y}\left(v_y\right)\right)$$

therefore $M_{xy}\left(M_{x0}\left(u_x\right), M_{0y}\left(v_y\right)\right) = F_{xy}\left(u_x + v_y\right)$ and hence

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = F_{xy}\left(M_{x0}^{-1}\left(\mu_{x0}\right) + M_{0y}^{-1}\left(\mu_{0y}\right)\right)$$

Assume M_{x0}, M_{0y} and F_{xy} are decreasing. Letting $\mathfrak{M}_{x0}$, $\mathfrak{M}_{0y}$, and $\mathfrak{F}_{xy}$ be primitives of M_{x0}, M_{0y}, and F_{xy} respectively, these are therefore concave functions. Consider the optimization problem

$$\min_{u,v} \sum_{x\in\mathcal{X}} n_x u_x + \sum_{y\in\mathcal{Y}} m_y v_y - \sum_{x\in\mathcal{X}} \mathfrak{M}_{x0}\left(u_x\right) - \sum_{y\in\mathcal{Y}} \mathfrak{M}_{0y}\left(v_y\right) - \sum_{\substack{x\in\mathcal{X}\\ y\in\mathcal{Y}}} \mathfrak{F}_{xy}\left(u_x + v_y\right)$$

whose first order conditions are

$$\begin{cases} M_{x0}\left(u_x\right) + \sum_{y\in\mathcal{Y}} F_{xy}\left(u_x + v_y\right) = n_x \\ M_{0y}\left(v_y\right) + \sum_{x\in\mathcal{X}} F_{xy}\left(u_x + v_y\right) = m_y. \end{cases}$$

Section 2

Blocks 17: Matching with imperfectly transferable utility

- Galois connections, distance-to-frontier function
- nonlinear complementary slackness
- equilibrium transport
- collective models, sharing rule, Pareto weights

- [OTME], Ch. 10.4
- Nöldeke and Samuelson (2016). The implementation duality. Mimeo.
- G, Kominers and Weber (2016). Costly concessions. An Empirical Framework for Matching with Imperfectly Transferable Utility. Mimeo.
- Dupuy, G, Jaffe and Kominers (2016). Taxation in matching markets. Mimeo.

- Recall the interpretation of Optimal Transport as a model of the labor market. A population of *workers* is characterized by their type $x \in \mathcal{X}$, where $\mathcal{X} = \mathbb{R}^d$ for simplicity. There is a distribution P over the workers, which is assumed to sum to one.
- A population of *firms* is characterized by their types $y \in \mathcal{Y}$ (say $\mathcal{Y} = \mathbb{R}^d$), and their distribution Q. It is assumed that there is the same total mass of workers and firms, so Q sums to one.
- Each worker must work for one firm; each firm must hire one worker. Let $\pi(x, y)$ be the probability of observing a matched (x, y) pair. π should have marginal P and Q, which is denoted

$$\pi \in \mathcal{M}(P, Q).$$

- In the simplest case, the utility of a worker x working for a firm y at wage $w(x,y)$ will be
$$\alpha(x,y) + w(x,y)$$
while the corresponding profit of firm y is
$$\gamma(x,y) - w(x,y).$$
- In this case, the total surplus generated by a pair (x,y) is
$$\alpha(x,y) + w + \gamma(x,y) - w = \alpha(x,y) + \gamma(x,y) =: \Phi(x,y)$$
which does not depend on w (no transfer frictions). A central planner may thus like to choose assignment $\pi \in \mathcal{M}(P,Q)$ so to
$$\max_{\pi \in \mathcal{M}(P,Q)} \int \Phi(x,y)\, d\pi(x,y).$$
But as it turns out, this is also **the equilibrium solution**.

- The equilibrium assignment is determined by an important quantity: the **wages**. Let $w(x,y)$ be the wage of employee x working for firm of type y.
- Let the indirect surpluses of worker x and firm y be respectively

$$u(x) = \max_y \{\alpha(x,y) + w(x,y)\}$$
$$v(y) = \max_x \{\gamma(x,y) - w(x,y)\}$$

so that (π, w) is an equilibrium when

$$u(x) \geq \alpha(x,y) + w(x,y) \text{ with equality if } (x,y) \in Supp(\pi)$$
$$v(y) \geq \gamma(x,y) - w(x,y) \text{ with equality if } (x,y) \in Supp(\pi)$$

- By summation,

$$u(x) + v(y) \geq \Phi(x,y) \text{ with equality if } (x,y) \in Supp(\pi).$$

- One can show that the equilibrium outcome (π, u, v) is such that π is solution to the primal Monge-Kantorovich Optimal Transportation problem

$$\max_{\pi \in \mathcal{M}(P,Q)} \int \Phi\left(x,y\right) d\pi\left(x,y\right)$$

and (u, v) is solution to the dual OT problem

$$\begin{aligned} &\min_{u,v} \int u\left(x\right) dP\left(x\right) + \int v\left(y\right) dQ\left(y\right) \\ &s.t.\ u\left(x\right) + v\left(y\right) \geq \Phi\left(x,y\right) \end{aligned}$$

- Feasibility+Complementary slackness yield the desired equilibrium conditions

$$\begin{aligned} &\pi \in \mathcal{M}\left(P,Q\right) \\ &u\left(x\right) + v\left(y\right) \geq \Phi\left(x,y\right) \\ &\left(x,y\right) \in Supp\left(\pi\right) \Longrightarrow u\left(x\right) + v\left(y\right) = \Phi\left(x,y\right) \end{aligned}$$

Here, **optimum=equilibrium**. "Second welfare theorem", "invisible hand", etc.

- Consider the same setting as above, but introduce (possibly nonlinear) taxes.
- Instead of assuming that workers' and firm's payoffs are linear in wages, assume

$$u(x) = \max_y \{\alpha_{xy} + N(w(x,y))\}$$
$$v(y) = \max_x \{\gamma_{xy} - w(x,y)\}$$

  where $N(w)$ is indecreasing and continuous, interpreted as the net wage if w if the gross wage.
- Of course, OT is recovered when $N(w) = w$ (no tax).
- Linear taxes: $N(w) = (1-\theta)w$, where $\theta \in (0,1)$ is the (flat) tax rate.
- Progressive tax schedule:

$$N(w) = \min_{k \in \{0,1,\ldots,K\}} \{(1-\theta_k)(w - w_k) + n_k\},$$

  where $n_0 = 0 < \ldots < n_k$, are the net income at the start of bracket k, $\theta_0 = 0 < \theta_1 < \ldots < \theta_k$ are the marginal tax rates in bracket k. $[n^{k+1} = n^k + (1-\theta^k)(w^{k+1} - w^k).]$

- Let $\mathcal{F}_{xy}$ be the set of feasible utilities that x and y can achieve through some wage w. One has

$$\mathcal{F}_{xy} = \left\{ (u, v) : \exists w \in \mathbb{R}, u \leq \alpha_{xy} + N\left(w\right), v \leq \gamma_{xy} - w \right\},$$

which rewrites

$$\mathcal{F}_{xy} = \left\{ (u, v) : u - \alpha_{xy} \leq N\left(\gamma_{xy} - v\right) \right\}.$$

- The interior of this set, denoted $\mathcal{F}^0_{xy}$, is the set such that this ineqality holds true.
- In the case of OT,

$$\mathcal{F}_{xy} = \left\{ (u, v) : u + v \leq \alpha_{xy} + \gamma_{xy} \right\}.$$

- In the case of linear taxes.

$$\mathcal{F}_{xy} = \left\{ (u, v) : (u - \alpha_{xy}) + (1 - \theta)(v - \gamma_{xy}) \leq 0 \right\}.$$

- In the case of a convex tax schedule,
- Back to the tax example: we have

$$N(w) = \min_{k=0,\dots,K} \left\{ n^k + \left(1 - \theta^k\right)\left(w - w^k\right) \right\}$$

thus $\mathcal{F}_{xy}$ can be written as the intersection of the $\mathcal{F}_{xy}^k$, where

$$\mathcal{F}_{xy}^k = \left\{ (u, v) : u - \alpha_{xy} - n^k + \left(1 - \theta^k\right)\left(\gamma_{xy} - v - w^k\right) \leq 0 \right\}.$$

- Is equilibrium always the solution to an optimization problem?
- **It is not**. This is why we introduce “Equilibrium Transport,” which contains, but is strictly more general than “Optimal Transport”. Equilibrium transport is a framework for general matching problems.
- Surprisingly, the real difficulty is not so much with taxes, but with nonlinear taxes.

- The reference for the following is [DGJK]. We want to understand the effect of a raise in income tax. Employee x's and firm y's indirect surpluses are now respectively

$$u_x = \max_y \left\{ \alpha_{xy} + (1-t)\, w_{xy}, 0 \right\},$$
$$v_y = \max_x \left\{ \gamma_{xy} - w_{xy}, 0 \right\}.$$

- Rewrites this as

$$\lambda u_x = \max_y \left\{ \lambda \alpha_{xy} + w_{xy}, 0 \right\},$$
$$v_y = \max_x \left\{ \gamma_{xy} - w_{xy}, 0 \right\}.$$

where

$$\lambda = \frac{1}{1-t} > 1.$$

THEOREM (DGJK)

The equilibrium matching π_{xy} is optimal for fictitious surplus

$$\tilde{\Phi}_{xy} = \lambda \alpha_{xy} + \gamma_{xy}.$$

That is, π_{xy} is solution to

$$\begin{aligned} & \max_{\pi \geq 0} \sum_{xy} \pi_{xy} \tilde{\Phi}_{xy} \\ s.t. \ & \sum_{y} \pi_{xy} \leq n_x \\ & \sum_{y} \pi_{xy} \leq n_x. \end{aligned}$$

Note that this is not the only instance where equilibrium is the solution of an optimization problem; Wardorp equilibria in congested network flow problems are another one. Later on we shall see instances where equilibrium has no formulation as an optimization problem.

- We have therefore that (π, u, v) is an equilibrium outcome when

$$\left\{ \begin{array}{l} (PF) : \pi \in \mathcal{M}(P, Q) \\ (DF) : (u(x), v(y)) \notin \mathcal{F}^0_{xy} \\ (NC) : (x, y) \in Supp(\pi) \Longrightarrow (u(x), v(y)) \in \mathcal{F}_{xy}. \end{array} \right.$$

- This is an *equilibrium transport problem*, a.k.a. *matching with imperfecly transferable utility*. It is a **Nonlinear Complementarity Problem** (NCP) with much structure.
- Problem: existence of an equilibrium outcome? yes in the discrete case ($\mathcal{X}$ and $\mathcal{Y}$ finite): Kelso-Crawford, Alkan and Gale.

- Link with Galois connection, see Noeldeke and Samuelson (2017). Let

$$G_{xy}(v) = \alpha_{xy} + N(\gamma_{xy} - v).$$

One has $(u_x, v_y) \notin \mathcal{F}^0_{xy}$ if and only if $u_x \geq G_{xy}(v_y)$ which is equivalent to $v_y \geq G^{-1}_{xy}(u_x)$.

- By condition (DF) and (NC), we get that if (π, u, v) is the solution to an Equilibrium transportation problem

$$u(x) = \max_{y \in \mathcal{Y}} G_{xy}(v(y)) \text{ and } v(y) = \max_{x \in \mathcal{X}} G^{-1}_{xy}(u(x))$$

- OT is a special case (Φ-conjugacy, see Villani's book):

$$u(x) = \max_{y \in \mathcal{Y}} \{\Phi_{xy} - v(y)\} \text{ and } v(y) = \max_{x \in \mathcal{X}} \{\Phi_{xy} - u(x)\}$$

and even more special: $\Phi_{xy} = x.y$ (Legendre-Fenchel conjugacy)

$$u(x) = \max_{y \in \mathcal{Y}} \{x.y - v(y)\} \text{ and } v(y) = \max_{x \in \mathcal{X}} \{x.y - u(x)\}.$$

- [GKW] introduce the *distance-to frontier* (DTF) function: for $(u,v) \in \mathbb{R}^2$, let

$$D_{xy}(u,v) = \min\left\{t \in \mathbb{R} : (u-t, v-t) \in \mathcal{F}_{xy}\right\}$$

which is the distance along the diagonal between (u,v) and the frontier of $\mathcal{F}_{xy}$, with a minus sign if (u,v) is in the set.

- Economic intepretation: what is the quantity of utility that we can give or remove to x and y *in the same amount* such that they reach the efficient frontier?
- This object has nice properties:
  - $D_{xy}(u,v) \leq 0$ iff $(u,v) \in \mathcal{F}_{xy}$
  - $D_{xy}(u,v) < 0$ iff $(u,v) \in \mathcal{F}_{xy}^0$
  - $D_{xy}(u+t, v+t) = D_{xy}(u,v) + t$
- Note that in the case of OT,

$$D_{xy}(u,v) = \frac{u+v-(\alpha_{xy}+\gamma_{xy})}{2}.$$

- Note that D associated with the intersection of D^k is equal to the $\max_k D^k$. One has

$$D^k\left(u,v\right)=\frac{u-\alpha_{xy}-n^k-\left(1-\theta^k\right)\left(\gamma_{xy}-v-w^k\right)}{2-\theta^k},$$

and as a result

$$D\left(u,v\right)=\max_{k=1,\ldots,K}\left\{\frac{u-\alpha_{xy}-n^k-\left(1-\theta^k\right)\left(\gamma_{xy}-v-w^k\right)}{2-\theta^k}\right\}.$$

- (π, u, v) is an equilibrium outcome when

$$\begin{cases} (PF) : \pi \in \mathcal{M}(P, Q) \\ (DF) : D_{xy}(u(x), v(y)) \geq 0 \\ (NC) : (x, y) \in Supp(\pi) \Longrightarrow D_{xy}(u(x), v(y)) = 0. \end{cases}$$

- In the rest of the talk, I will argue that these objects can be useful in the study of the Equilibrium Transport problem.

- More generally, the following operations on DTF functions correspond to geometric operations on feasible sets:
  - $\max\left\{D^1, D^2\right\}$: intersection
  - $\min\left\{D^1, D^2\right\}$: union
  - $D\left(u-\alpha, v-\gamma\right)$: translation
  - $T\Psi\left(u/T, v/T\right)$: homothety
  - $\lambda\Psi^1 + \left(1-\lambda\right)D^2$: intepolation
- These operations are exploited in the TraME project (https://github.com/TraME-Project/), a software for flexible computation of equilibrium transportation problems.

- Look at the individual rationality conditions:

$$\begin{cases} u_x \geq \mathcal{U}_{xy}\left(v_y\right) \\ \pi_{y|x} > 0 \Longrightarrow u_x = \mathcal{U}_{xy}\left(v_y\right) \end{cases}$$

where $\pi_{y|x} = \pi_{xy} / p_x$ is the conditional distribution of Y given X under π.

- A regularized version of these is provided by the Gibbs distribution

$$\pi_{y|x} = \frac{\exp \mathcal{U}_{xy}\left(w_{xy}\right) / T}{\exp u_x / T}$$

where the max has been replaced by the smooth-max. Hence, letting $a_x = u_x - T \ln p_x$, we get

$$T \ln \pi_{xy} + a_x = \mathcal{U}_{xy}\left(w_{xy}\right).$$

- Similarly, individual rationality on the side of firms relaxes into

$$\pi_{x|y} = \frac{\exp \mathcal{V}_{xy}\left(w_{xy}\right) / T}{\exp v_y / T}$$

and thus, after letting $b_y = v_y - T \ln q_y$,

$$T \ln \pi_{xy} + b_y = \mathcal{V}_{xy}\left(w_{xy}\right).$$

- We get
$$\begin{cases} T \ln \pi_{xy} + a_x = \mathcal{U}_{xy}\left(w_{xy}\right) \\ T \ln \pi_{xy} + b_y = \mathcal{V}_{xy}\left(w_{xy}\right) \end{cases}$$
- Thus, applying D term by term yields
$$D_{xy}\left(T \ln \pi_{xy} + a_x, T \ln \pi_{xy} + b_y\right) = 0$$
hence
$$\pi_{xy} = \exp\left(-D_{xy}\left(a_x, b_y\right)/T\right)$$
where a_x and b_y solve the system
$$\begin{cases} \sum_y \exp\left(-D_{xy}\left(a_x, b_y\right)/T\right) = p_x \\ \sum_x \exp\left(-D_{xy}\left(a_x, b_y\right)/T\right) = q_y \end{cases}.$$
- By substraction, we have
$$a_x - b_y = w_{xy}.$$

Section 3

BLOCK 18: MATCHING WITH NON-TRANSFERABLE UTILITY

- [D] Dagsvik (2000). "Aggregation in matching markets." International Economic Review.
- [M] Menzel (2015). "Large Matching Markets as Two-Sided Demand Systems." Econometrica.
- [GKW] Galichon, Kominers and Weber (2016). "Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility." Preprint.
- [GH] Galichon and Hsieh (2016). "A theory of decentralized matching markets without transfers." Preprint.

- This talk is about aggregate equilibria in **decentralized matching market without prices**, aka decentralized non-transferable utility (NTU) markets.
- Decentralized NTU markets arises in a number of situations (taxis, health care, rent-controlled housing...)
- Need for analytical tools for regulatory purposes (fix prices to optimize efficiency/fairness tradeoff).
- Also, operators doing dynamic pricing need consider the market as NTU over short time scales. Eg. Uber sets fixed prices for some time span–during that time span, the market is NTU.

- Assume that there are 2 identical passengers and 1 driver. The value of being unmatched (for the passengers and the driver alike) is 0. The value of being matched is 1, both for the passengers and driver.
- In a model with prices (Uber model–transferable utility), the price of the ride will be 1, so that the driver's payoff is 2, and both passengers' payoffs is zero. Thus, passengers are indifferent between being matched and unmatched.
- In a classical model without transfers (taxi model–nontransferable utility), there are two stable matchings in each of which the matched passenger gets one, while the unmatched gets zero. Thus in this Gale-Shapley solution, one passenger is happier than the other one.

- However, people don't like to be unhappier than their peers!
  - Sometimes passengers will fight for the only available taxi...
  - ... or they will wait in line, and the length of the line will make each passenger indifferent between waiting in line and opting out.
  - In both cases, the driver is not better off, but both passengers have destroyed utility so that they are indifferent between being matched or unmatched, and both passengers have the same payoff (i.e., zero) at equilibrium.
- If, on the contrary, there are two drivers and one passengers, the story is reversed: drivers will fight / wait in line, and destroy utility so that both drivers get zero payoff; in this case, the passenger gets surplus one.

- A philosophical debate:
  - waiting lines: arguably the most inefficient way to clear the market. Agents (passengers or taxis) pay in a numeraire which is not transfered to the other side of the market.
  - but: also arguably the fairest rationing system as time endowment is same for every individual. Cf. Michael Sandel's book *What Money Can't Buy*.
- Many other dimensions not taken into account here. e.g.:
  - stochasticity of the rationing process
  - lines as a signal of desirability
  - surge pricing creates uncertainty at consumer level
  - etc.
- Goal for present paper: build an assignment model parallel to Shapley-Shubik but where the market is cleared by waiting lines.

- We will build a model of stable matchings with perfectly inefficient "transfers".
- "Transfers" here means something that will regulate competitive forces on one side of the market, but cannot be appropriate by the other side of the market. Eg. time waited in line.
- By two-sided stability, there cannot be a line on the buyer's side and on the seller's side. Let $\tau^D \geq 0$ be the time waited on demand' side, and $\tau^S \geq 0$ be the time waited on supply's side, one has by two-sided stability:

$$\min\left(\tau^D, \tau^S\right) = 0$$

- Introduce *directional waiting time* = (time waited by buyers - time waited by sellers), i.e.

$$\tau := \tau^D - \tau^S$$

so that $\tau^D = \tau^+ := \max(\tau, 0)$ and $\tau^S = \tau^- := \max(-\tau, 0)$.

- Assume prices p are free to adjust. Let $D(p)$ be the demand, $S(p)$ be the supply. Then the prices adjust to balance demand and suppy:

$$Z(p) := D(p) - S(p) = 0. \tag{3}$$

- Now assume that prices are not free to adjust, and are set to $\overline{p}$. Then a queue (either on the supply or on the demand side) will form. Assume demand and supply are then given by

$$\text{demand} = D\left(\overline{p} + \tau^D\right) = D\left(\overline{p} + \tau^+\right)$$
$$\text{supply} = S\left(\overline{p} - \tau^S\right) = S\left(\overline{p} - \tau^-\right)$$

so that $\tau = \tau^D - \tau^S$ is determined at equilibrium by

$$Z(\tau; \overline{p}) := D\left(\overline{p} + \tau^+\right) - S\left(\overline{p} - \tau^-\right) = 0, \tag{4}$$

and waiting times have replaced prices as a market-clearing numéraire.

- Today, we will study a matching model based on rationing-by-waiting in the spirit of equilibrium equation (4).

- Assume that sellers' types are represented by seller's types $x \in \mathcal{X}$ and buyers' types by $y \in \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are finite.
- There are n_x sellers of type x and m_y buyers of type y. Within a type, individuals (buyers or sellers) are indistinguishable.
- If x and y match, x enjoys utility α_{xy}, while y enjoys γ_{xy}. If they remain unmatched, get utility normalized to 0.
- A matching is the number μ_{xy} of $x - y$ matches. Letting μ_{x0} be the number of unmatched sellers, and μ_{y0} the number of unmatched buyers, one has

$$\mu \in \mathcal{M} \Leftrightarrow_{def} \begin{cases} \sum_{y \in \mathcal{Y}} \mu_{xy} + \mu_{x0} = n_x \\ \sum_{x \in \mathcal{X}} \mu_{xy} + \mu_{0y} = m_y \end{cases}$$

- Letting τ^S_{xy} be time that sellers of type of type x would have to wait for buyers of type y, and τ^D_{xy} be the time that buyers of type y would have to wait for sellers of type x, one has in general

$$\min\left(\tau^S_{xy},\tau^D_{xy}\right)\geq 0.$$

- By pairwise stability, if there are matches between x and y, then there cannot be a waiting line on both sides, thus

$$\mu_{xy}>0 \implies \min\left(\tau^S_{xy},\tau^D_{xy}\right)=0.$$

- Letting u_x and v_y be the payoff of x and y at equilibrium, one has

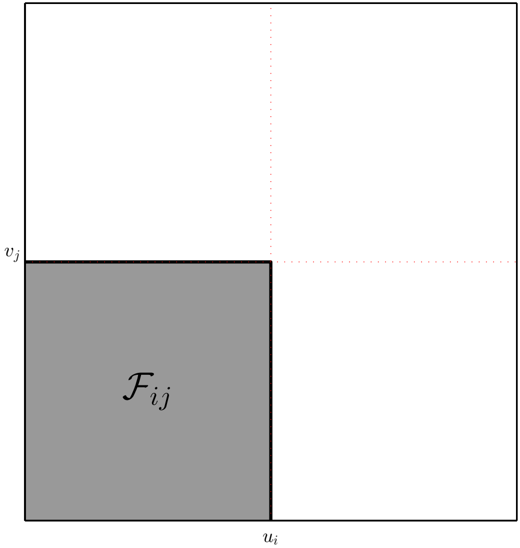
$$u_x=\max_{y\in\mathcal{Y}}\left\{\alpha_{xy}-\tau^S_{xy},0\right\} \text{ and } v_y=\max_{x\in\mathcal{X}}\left\{\gamma_{xy}-\tau^D_{xy},0\right\}.$$

- An outcome is a triple (μ, u, v).
- A *stable outcome with rationing-by-waiting* is such that

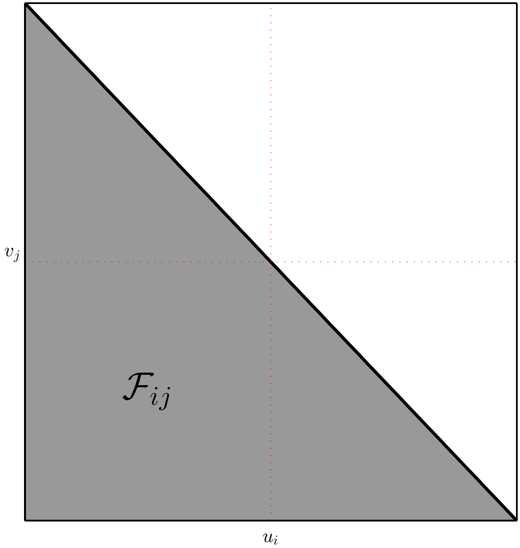
$$\begin{cases} \mu \in \mathcal{M} \\ \max\{u_x - \alpha_{xy}, v_y - \gamma_{xy}\} \geq 0 \text{ with equality if } \mu_{xy} > 0 \\ u_x \geq 0 \text{ with equality if } \mu_{x0} > 0 \\ v_y \geq 0 \text{ with equality if } \mu_{0y} > 0 \end{cases}$$

- Note that replacing max by sum yields TU stability. More generally, can replace $\max(u, v)$ by large class of function $D(u, v)$ and get notions of stability with imperfectly transferable utility, see G, Kominers and Weber (2017).

FEASIBLE UTILITY SETS, NTU



Feasible utility sets, TU



- Theorem 1: There exists a stable outcome with rationing-by-waiting.
  - Proof by an adaptation of Gale and Shapley where instead of increasing "set of rejection", increase τ^S_{xy}.
- Theorem 2: The set of $(u, -v)$ for which there is μ such that (u, v, μ) is stable is a lattice.
  - Proof by an adaptation of Demange and Gale (1985).
- Next, we connect stability with rationing-by-waiting to classical stability.

- Theorem 3: Assume $n_x = 1$ and $m_y = 1$ (one individual per type). Then:
  (i) Let $\mu \in \mathcal{M}$ be stable in the classical (Gale-Shapley) sense. Defining $u_x^\mu = \sum_{y \in \mathcal{Y}} \mu_{xy} \alpha_{xy}$ and $v_y^\mu = \sum_{x \in \mathcal{X}} \mu_{xy} \gamma_{xy}$, we get that (μ, u^μ, v^μ) is stable in our sense.
  (ii) Let (μ, u, v) be stable in our sense, then μ is stable in the classical sense.
- Interpretation: when there is one individual of each type, no need to regulate supply/demand by waiting lines. The efficient point on the frontier of the feasible utility set can be attained.

- Recall $u_x^\mu = \sum_{y \in \mathcal{Y}} \mu_{xy} \alpha_{xy}$ and $v_y^\mu = \sum_{x \in \mathcal{X}} \mu_{xy} \gamma_{xy}$.
- Proof of (i) is straightforward as there cannot be x and y such that $\max \left\{ u_x^\mu - \alpha_{xy}, v_y^\mu - \gamma_{xy} \right\} < 0$, and if x and y are matched, then $u_x^\mu = \alpha_{xy}$ and $v_y^\mu = \gamma_{xy}$.
- Proof of (ii): assume (μ, u, v) is stable in our sense. Assume x and y is a blocking pair, in the sense that $u_x^\mu < \alpha_{xy}$ and $v_y^\mu < \gamma_{xy}$.
  Let x' be the partner of y under μ and y' be the partner of x under μ, hence $\alpha_{xy} > \alpha_{xy'} = u_x^\mu$ and $\gamma_{xy} > \gamma_{x'y} = v_y^\mu$. One has $u_x - \alpha_{xy'} > u_x - \alpha_{xy}$ and $v_y - \gamma_{x'y} > v_y - \gamma_{xy}$, hence $\max \left(u_x - \alpha_{xy'}, v_y - \gamma_{x'y} \right) > \max \left(u_x - \alpha_{xy}, v_y - \gamma_{xy} \right) \geq 0$.
  Thus either $u_x > \alpha_{xy'}$ or $v_y > \gamma_{x'y}$; without loss of generality assume that $u_x > \alpha_{xy'}$. But because x and y' are matched under μ, it follows that $\max \left(u_x - \alpha_{xy'}, v_{y'} - \gamma_{xy'} \right) = 0$, hence $u_x \leq \alpha_{xy'}$, a contradiction.

- Assume $\alpha_{ij} = \alpha_{x_i y_j} + \varepsilon_{iy_j}$ and $\gamma_{ij} = \gamma_{x_i y_j} + \eta_{x_i j}$, ε and η iid Gumbel, as Choo and Siow (2006) and G. and Salanié (2016), but here in the NTU case.
- As in the TU case, one can show that the equilibrium waiting times τ^S_{xy} and τ^D_{xy} only depend on the observable characteristics x and y.
- Letting $U_{xy} = \alpha_{xy} - \tau^S_{xy}$ and $V_{xy} = \gamma_{xy} - \tau^D_{xy}$, we have

$$u_i = \max_{y \in \mathcal{Y}} \left\{ U_{xy} + \varepsilon_{iy}, \varepsilon_{i0} \right\}$$

$$v_j = \max_{y \in \mathcal{Y}} \left\{ V_{xy} + \eta_{xj}, \eta_{0j} \right\}$$

so the equilibrium relates U_{xy} to the supply-side conditional choice probabilities μ_{xy} / n_x, V_{xy} to the demand-side ccp μ_{xy} / m_y, and U_{xy} to V_{xy} by

$$\max \left\{ U_{xy} - \alpha_{xy}, V_{xy} - \gamma_{xy} \right\} = 0.$$

- Assume ε and η iid Gumbel, as Choo and Siow, but here in the NTU case. In this case, the ccp inversion is explicit, as $U_{xy} = \ln \mu_{xy} / \mu_{x0}$ and $V_{xy} = \ln \mu_{xy} / \mu_{0y}$.
- Thus one has existence and uniqueness of an equilibrium, and

$$\mu_{xy} = \min \left(\mu_{x0} e^{\alpha_{xy}}, \mu_{0y} e^{\gamma_{xy}} \right). \tag{5}$$

  where

$$\begin{cases} \mu_{x0} + \sum_{y \in \mathcal{Y}} \min \left(\mu_{x0} e^{\alpha_{xy}}, \mu_{0y} e^{\gamma_{xy}} \right) = n_x \\ \mu_{0y} + \sum_{x \in \mathcal{X}} \min \left(\mu_{x0} e^{\alpha_{xy}}, \mu_{0y} e^{\gamma_{xy}} \right) = m_y \end{cases}$$

  and this system can be efficiently solved with a nonlinear version of the IPFP (a.k.a. RAS/Sinkhorn/matrix scaling) algorithm.
- Computationally, scales extremely well ("embarrassingly parallelizable").
- Note contrast between (5) and Dagsvik-Menzel equilibrium, which assumes $\alpha_{ij} = \alpha_{x_i y_j} + \varepsilon_{ij}$ and $\gamma_{ij} = \gamma_{x_i y_j} + \eta_{ij}$ and where

$$\mu_{xy} = \mu_{x0} \mu_{0y} e^{\alpha_{xy}} e^{\gamma_{xy}}. \tag{6}$$

- Take a slight variant with $\alpha_{ij} = \alpha_{x_i y_j} + \sigma \varepsilon_{i y_j}$ and $\gamma_{ij} = \gamma_{x_i y_j} + \sigma \eta_{x_i j}$ where $\sigma > 0$. Equilibrium matching μ^σ is now determined by

$$\begin{cases} \mu_{x0} + \sum_{y \in \mathcal{Y}} \min \left(\mu_{x0} e^{\alpha_{xy}/\sigma}, \mu_{0y} e^{\gamma_{xy}/\sigma} \right) = n_x \\ \mu_{0y} + \sum_{x \in \mathcal{X}} \min \left(\mu_{x0} e^{\alpha_{xy}/\sigma}, \mu_{0y} e^{\gamma_{xy}/\sigma} \right) = m_y \end{cases}$$

  and the average utilities are given by $u_x^\sigma = \sigma \log \left(\mu_{xy} / n_x \right)$ and $v_y^\sigma = \sigma \log \left(\mu_{xy} / m_y \right)$.
- One can show that when $\sigma \to 0$, the equilibrium $(\mu^\sigma, u^\sigma, v^\sigma)$ then converges to a stable matching with rationing-by-waiting and no heterogeneity.
- It is an "interior" matching in the sense that it is neither supply-preferred, nor demand-preferred matching. "Most central" solution in some sense.

- The total inefficiency is computed as

$$L(\alpha,\gamma,n,m)=\sum_{xy}\mu_{xy}\left(l^{S}\left(\tau_{xy}^{S}\right)+l^{D}\left(\tau_{xy}^{D}\right)\right)$$

where $l^{S}(.)$ (resp. $l^{D}(.)$) is a loss function depending on the time waited on each segment on the supply (resp. demand) side.

- In the logit model, when the loss functions are exponential, that is when $l^{S}\left(\tau_{xy}^{S}\right)=\exp\left(\tau_{xy}^{S}\right)-1$ and $l^{D}\left(\tau_{xy}^{D}\right)=\exp\left(\tau_{xy}^{D}\right)-1$ for all x and y, then the total welfare loss is expressed as $L(\alpha,\gamma,n,m)=$

$$\sum_{x\in\mathcal{X}}\mu_{x0}\left(1+\sum_{y\in\mathcal{Y}}e^{\alpha_{xy}}\right)+\sum_{y\in\mathcal{Y}}\mu_{0y}\left(1+\sum_{x\in\mathcal{X}}e^{\gamma_{xy}}\right)-\sum_{x\in\mathcal{X}}n_{x}-\sum_{y\in\mathcal{Y}}m_{y}$$
$$=\sum_{xy}\mu_{x0}\left(e^{\alpha_{xy}}-e^{U_{xy}}\right)+\mu_{0y}\left(e^{\gamma_{xy}}-e^{V_{xy}}\right).$$

- Dagsvik-Menzel's model ([D], [M]) is a model of matching without transfers and with logit heterogeneity. If worker i of type $x \in \mathcal{X}$ matches with firm j of type $y \in \mathcal{X}$, then worker and firm get respectively
$$\alpha_{xy} + \varepsilon_{ij} \text{ and } \gamma_{xy} + \eta_{ij}$$
where ε_{ij} and η_{ij} are iid Gumbel. If unassigned, get $\alpha_{x0} + \varepsilon_{i0}$ and $\gamma_{0y} + \eta_{0j}$.
- Wages are decided exogenously: there are no possible equilibrium adjustments of wage. Hence, worker i chooses preferred firm among firms who find her acceptable. Let u_i (resp. v_j) be the payoffs of worker i (resp. firm j) at equilibrium. One has
$$u_i = \max\left(\max_j \left\{\alpha_{x_i y_j} + \varepsilon_{ij} : v_j \leq \gamma_{x_i y_j} + \eta_{ij}\right\}, \alpha_{x0} + \varepsilon_{i0}\right)$$
$$v_j = \max\left(\max_i \left\{\gamma_{x_i y_j} + \eta_{ij} : u_i \leq \alpha_{x_i y_j} + \varepsilon_{ij}\right\}, \gamma_{0y} + \eta_{0j}\right).$$

- Because of the Gumbel assumption, if i is of type x,

$$\begin{cases} u_i = e^{\alpha_{x0}} + \sum_j e^{\alpha_{xy_j}} 1\left\{ v_j \leq \gamma_{xy_j} + \eta_{ij} \right\} + \hat{\varepsilon}_i \\ v_j = e^{\gamma_{0y}} + \sum_i e^{\gamma_{x_i y}} 1\left\{ u_i \leq \alpha_{x_i y} + \varepsilon_{ij} \right\} + \hat{\eta}_j \end{cases}$$

where ϵ_i and η_j are standard Gumbel.

- Hence $u_i = u_x + \hat{\varepsilon}_i$ and $v_j = v_y + \hat{\eta}_j$, where

$$\begin{cases} u_x = \log\left(e^{\alpha_{x0}} + \sum_y e^{\alpha_{xy}} m_y \Pr\left(v_y + \hat{\eta}_j \leq \gamma_{xy} + \eta_{ij} \right) \right) \\ v_y = \log\left(e^{\gamma_{0y}} + \sum_x e^{\gamma_{xy}} n_x \Pr\left(u_x + \hat{\varepsilon}_i \leq \alpha_{xy} + \varepsilon_{ij} \right) \right). \end{cases}$$

- Because $\hat{\eta}_j$ and η_{ij} are Gumbel and (nearly) independent, in the large population limit,

$$\Pr\left(v_y + \hat{\eta}_j \leq \gamma_{xy} + \eta_{ij}\right) = \frac{e^{\gamma_{xy}}}{e^{v_y} + e^{\gamma_{xy}}} \simeq e^{\gamma_{xy} - v_y}$$
$$\Pr\left(u_x + \hat{\varepsilon}_i \leq \alpha_{xy} + \varepsilon_{ij}\right) = \frac{e^{\alpha_{xy}}}{e^{u_x} + e^{\alpha_{xy}}} \simeq e^{\alpha_{xy} - u_x}$$

- Hence,

$$\left\{ \begin{array}{l} e^{u_x} = e^{\alpha_{x0}} + \sum_y m_y e^{\alpha_{xy} + \gamma_{xy} - v_y} \\ e^{v_y} = e^{\gamma_{0y}} + \sum_x n_x e^{\alpha_{xy} + \gamma_{xy} - u_x} \end{array} \right.$$

- Thus

$$\left\{ \begin{array}{l} n_x = n_x e^{\alpha_{x0} - u_x} + \sum_y n_x m_y e^{\alpha_{xy} + \gamma_{xy} - u_x - v_y} \\ m_y = m_y e^{\gamma_{0y} - v_y} + \sum_x n_x m_y e^{\alpha_{xy} + \gamma_{xy} - u_x - v_y} \end{array} \right.$$

- Setting $\mu_{x0} = n_x e^{\alpha_{x0} - u_x}$ and $\mu_{0y} = m_y e^{\gamma_{0y} - v_y}$ thus yields $\mu_{xy} = M_{xy}\left(\mu_{x0}, \mu_{0y}\right)$, where

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \mu_{x0}\mu_{0y}K_{xy}$$

with $K_{xy} = \exp\left(\alpha_{xy} + \gamma_{xy}\right)$.

- Note the resemblances and the differences with Choo-Siow model:
  - In both models, $\alpha + \gamma$ is a primitive of the model.
  - In the Choo-Siow model, M is positive homogenous of degree one (constant returns to scale), while in the Dagsvik-Menzel model, is positive homogenous of degree 2 (increasing returns to scale). Why?

Section 4

CODING

- We will explore TraME, a toolbox for computing equilbrium and performingn estimation in ITU matching problems.
- To install, run:
```
install.packages("devtools")
library(devtools)
install_github("TraME-Project/TraME-Rcpp")
```
- Start by doing:
```
tests_TraME()
```

- To compute MFE, we first need to define a MMF. In R, this will be an object 'themmf' for which we need to define a number of functions. For instance, Choo-Siow (TU)'s MMF is defined here https://github.com/TraME-Project/TraME-R/blob/master/R/mmfs.R#L92
- At the very least, one should specify the matching function

$$\mu_{xy} = M_{xy}\left(\mu_{x0}, \mu_{0y}\right)$$

  which is implemented by M.themmf.
- For example, Choo-Siow's MMF is coded as:

```
M.geommfs <- function(mmfs, mux0s, mu0ys,
xs=1:length(mmfs$n), ys=1:length(mmfs$m))
{term_1 = mmfs$K[xs,ys]
term_2 = sqrt(mux0s %*% t(mu0ys))
ret = term_1 * term_2
return(ret)}
```

- When no more information is available, one has a default method for solving for μ_{x0} in

$$n_x = \mu_{x0} + \sum_y M_{xy}\left(\mu_{x0}, \mu_{0y}\right)$$

  which is done by dichotomy in `margxInv.default`, see https://github.com/TraME-Project/TraME-R/blob/master/R/mmfs.R#L34
- However, in some cases, we have direct methods to solve for μ_{x0}. For instance, in the case of Choo and Siow, we saw that this inversion was done using quadratic equation solving. Hence, we define a specific inversion rule; in our Choo-Siow example, this is done by '`margxInv.geommfs`', cf. https://github.com/TraME-Project/TraME-R/blob/master/R/mmfs.R#L163

- The IPFP in TraME is therefore a generic tool that calls iteratively margxInv and margyInv, taking advantage of the (possible) simplification. See https://github.com/TraME-Project/TraME-R/blob/master/R/0_equilibrium.R#L100
- There is also a parallel version of the IPFP here https://github.com/TraME-Project/TraME-R/blob/master/R/0_equilibrium.R#L174

- Usually the MMF is parameterized. For instance in the case of Choo and Siow's MMF $M_{xy}\left(\mu_{x0},\mu_{0y}\right)=\left(\mu_{x0}\mu_{0y}\right)^{1/2}\exp\left(\Phi_{xy}/2\right)$, the parameter is $K_{xy}=\exp\left(\Phi_{xy}/2\right)$.
- Usually, this parameter is itself parameterized. E.g. $K_{xy}^{\theta}=\exp\left(\sum_k \theta^k \phi_{xy}^k/2\right)$.
- For that reason, we shall provide TraME with the information of how $M_{xy}\left(\mu_{x0},\mu_{0y}\right)$ depends on K, in order to be able to solve how the equilibrium matching μ_{xy} depends on K (and by composition, on the value of the parmaeter θ), which will allow us to compute the gradient of the log-likelihood.
  - In the case of Choo-Siow, cf. https://github.com/TraME-Project/TraME-R/blob/master/R/mmfs.R#L152

- Here is how we code a few MMF's in TraME:
  - CES (ETU models): https://github.com/TraME-Project/TraME-R/blob/master/R/market_MFE.R#L82
  - Cobb-Douglas (models with linear taxes): https://github.com/TraME-Project/TraME-R/blob/master/R/market_MFE.R#L105
  - min (NTU models): https://github.com/TraME-Project/TraME-R/blob/master/R/mmfs.R#L198
- The tool is setup so that one can input user-defined MMFs. For instance, one could create a MMF associated with nonlinear taxes in a few lines of code.