[SP5] MINIMUM COVER

INSTANCE: Collection C of subsets of a finite set S, positive integer $K \leq |C|$.
QUESTION: Does C contain a cover for S of size K or less, i.e., a subset $C' \subseteq C$ with $|C'| \leq K$ such that every element of S belongs to at least one member of C'?

Reference: [Karp, 1972]. Transformation from X3C.
Comment: Remains NP-complete even if all $c \in C$ have $|c| \leq 3$. Solvable in polynomial time by matching techniques if all $c \in C$ have $|c| \leq 2$.

[SP6] MINIMUM TEST SET

INSTANCE: Collection C of subsets of a finite set S, positive integer $K \leq |C|$.
QUESTION: Is there a subcollection $C' \subseteq C$ with $|C'| \leq K$ such that for each pair of distinct elements $u, v \in S$, there is some set $c \in C'$ that contains exactly one of u and v?

Reference: [Garey and Johnson, ——]. Transformation from 3DM.
Comment: Remains NP-complete if all $c \in C$ have $|c| \leq 3$, but is solvable in polynomial time if all $c \in C$ have $|c| \leq 2$. Variant in which C' can contain unions of subsets in C as well as subsets in C is also NP-complete [Ibaraki, Kameda, and Toida, 1977].

[SP7] SET BASIS

INSTANCE: Collection C of subsets of a finite set S, positive integer $K \leq |C|$.
QUESTION: Is there a collection B of subsets of S with $|B| = K$ such that, for each $c \in C$, there is a subcollection of B whose union is exactly c?

Reference: [Stockmeyer, 1975]. Transformation from VERTEX COVER.
Comment: Remains NP-complete if all $c \in C$ have $|c| \leq 3$, but is trivial if all $c \in C$ have $|c| \leq 2$.

[SP8] HITTING SET

INSTANCE: Collection C of subsets of a finite set S, positive integer $K \leq |S|$.
QUESTION: Is there a subset $S' \subseteq S$ with $|S'| \leq K$ such that S' contains at least one element from each subset in C?

Reference: [Karp, 1972]. Transformation from VERTEX COVER.
Comment: Remains NP-complete even if $|c| \leq 2$ for all $c \in C$.

[SP9] INTERSECTION PATTERN

INSTANCE: An $n \times n$ matrix $A = (a_{ij})$ with entries in Z_0^+.
QUESTION: Is there a collection $C = \{C_1, C_2, \ldots, C_n\}$ of sets such that for all i,j, $1 \leq i,j \leq n$, $a_{ij} = |C_i \cap C_j|$?

Reference: [Chvátal, 1978]. Transformation from GRAPH 3-COLORABILITY.
Comment: Remains NP-complete even if all $a_{ii} = 3$, $1 \leq i \leq m$ (and hence all C_i must have cardinality 3). If all $a_{ii} = 2$, it is equivalent to edge graph recognition and hence can be solved in polynomial time (e.g., see [Harary, 1969]).

[SP10] COMPARATIVE CONTAINMENT

INSTANCE: Two collections $R = \{R_1, R_2, \ldots, R_k\}$ and $S = \{S_1, S_2, \ldots, S_l\}$ of subsets of a finite set X, weights $w(R_i) \in Z^+$, $1 \leq i \leq k$, and $w(S_j) \in Z^+$, $1 \leq j \leq l$.
QUESTION: Is there a subset $Y \subseteq X$ such that

$$\sum_{Y \subseteq R_i} w(R_i) \geq \sum_{Y \subseteq S_j} w(S_j) \ ?$$

Reference: [Plaisted, 1976]. Transformation from VERTEX COVER.
Comment: Remains NP-complete even if all subsets in R and S have weight 1 [Garey and Johnson, ——].

[SP11] 3-MATROID INTERSECTION

INSTANCE: Three matroids $(E,F_1), (E,F_2), (E,F_3)$, positive integer $K \leq |E|$. (A matroid (E,F) consists of a set E of elements and a non-empty family F of subsets of E such that (1) $S \in F$ implies all subsets of S are in F and (2) if two sets $S, S' \in F$ satisfy $|S| = |S'| + 1$, then there exists an element $e \in S - S'$ such that $(S' \cup \{e\}) \in F$.)
QUESTION: Is there a subset $E' \subseteq E$ such that $|E'| = K$ and $E' \in (F_1 \cap F_2 \cap F_3)$?

Reference: Transformation from 3DM.
Comment: The related 2-MATROID INTERSECTION problem can be solved in polynomial time, even if the matroids are described by giving polynomial time algorithms for recognizing their members, and even if each element $e \in E$ has a weight $w(e) \in Z^+$, with the goal being to find an $E' \in (F_1 \cap F_2)$ having maximum total weight (e.g., see [Lawler, 1976a]).

A3.2 WEIGHTED SET PROBLEMS

[SP12] PARTITION

INSTANCE: Finite set A and a size $s(a) \in Z^+$ for each $a \in A$.
QUESTION: Is there a subset $A' \subseteq A$ such that $\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)$?

Reference: [Karp, 1972]. Transformation from 3DM (see Section 3.1.5).
Comment: Remains NP-complete even if we require that $|A'| = |A|/2$, or if the elements in A are ordered as $a_1, a_2, \ldots, a_{2n}$ and we require that A' contain exactly one of a_{2i-1}, a_{2i} for $1 \leq i \leq n$. However, all these problems can be solved in pseudo-polynomial time by dynamic programming (see Section 4.2).

[SP13] SUBSET SUM

INSTANCE: Finite set A, size $s(a) \in Z^+$ for each $a \in A$, positive integer B.
QUESTION: Is there a subset $A' \subseteq A$ such that the sum of the sizes of the elements in A' is exactly B?

Reference: [Karp, 1972]. Transformation from PARTITION.
Comment: Solvable in pseudo-polynomial time (see Section 4.2).

tone non-decreasing concave cost functions, if these can be computed in polynomial time.

[SS22] DEADLOCK AVOIDANCE

INSTANCE: Set $\{P_1, P_2, \ldots, P_m\}$ of "process flow diagrams" (directed acyclic graphs), set Q of "resources," state S of system giving current "active" vertex in each process and "allocation" of resources (see references for details).
QUESTION: Is S "unsafe," i.e., are there control flows for the various processes from state S such that no sequence of resource allocations and deallocations can enable the system to reach a "final" state?

Reference: [Araki, Sugiyama, Kasami, and Okui, 1977], [Sugiyama, Araki, Okui, and Kasami, 1977]. Transformation from 3SAT.
Comment: Remains NP-complete even if allocation calls are "properly nested" and no allocation call involves more than two resources. See references for additional complexity results. See also [Gold, 1978] for results and algorithms for a related model of the deadlock problem.

A6 MATHEMATICAL PROGRAMMING

[MP1] INTEGER PROGRAMMING

INSTANCE: Finite set X of pairs $(\bar{x}, b)$, where $\bar{x}$ is an m-tuple of integers and b is an integer, an m-tuple $\bar{c}$ of integers, and an integer B.
QUESTION: Is there an m-tuple $\bar{y}$ of integers such that $\bar{x}\cdot\bar{y} \leqslant b$ for all $(\bar{x}, b) \in X$ and such that $\bar{c}\cdot\bar{y} \geqslant B$ (where the dot-product $\bar{u}\cdot\bar{v}$ of two m-tuples $\bar{u} = (u_1, u_2, \ldots, u_m)$ and $\bar{v} = (v_1, v_2, \ldots, v_m)$ is given by $\sum_{i=1}^{m} u_i v_i$)?

Reference: [Karp, 1972], [Borosh and Treybig, 1976]. Transformation from 3SAT. The second reference proves membership in NP.
Comment: NP-complete in the strong sense. Variant in which all components of $\bar{y}$ are required to belong to $\{0,1\}$ (ZERO-ONE INTEGER PROGRAMMING) is also NP-complete, even if each b, all components of each $\bar{x}$, and all components of $\bar{c}$ are required to belong to $\{0,1\}$. Also NP-complete are the questions of whether a $\bar{y}$ with non-negative integer entries exists such that $\bar{x}\cdot\bar{y} = b$ for all $(\bar{x}, b) \in X$, and the question of whether there exists any $\bar{y}$ with integer entries such that $\bar{x}\cdot\bar{y} \geqslant 0$ for all $(\bar{x}, b) \in X$ [Sahni, 1974].

[MP2] QUADRATIC PROGRAMMING (*)

INSTANCE: Finite set X of pairs $(\bar{x}, b)$, where $\bar{x}$ is an m-tuple of rational numbers and b is a rational number, two m-tuples $\bar{c}$ and $\bar{d}$ of rational numbers, and a rational number B.
QUESTION: Is there an m-tuple $\bar{y}$ of rational numbers such that $\bar{x}\cdot\bar{y} \leqslant b$ for all $(\bar{x}, b) \in X$ and such that $\sum_{i=1}^{m} (c_i y_i^2 + d_i y_i) \geqslant B$, where c_i, y_i, and d_i denote the i^{th} components of $\bar{c}$, $\bar{y}$, and $\bar{d}$ respectively?

Reference: [Sahni, 1974]. Transformation from PARTITION.
Comment: Not known to be in NP, unless the c_i's are all non-negative [Klee, 1978]. If the constraints are quadratic and the objective function is linear (the reverse of the situation above), then the problem is also NP-hard [Sahni, 1974]. If we add to this last problem the requirement that all entries of $\bar{y}$ be integers, then the problem becomes undecidable [Jeroslow, 1973].

[MP3] COST-PARAMETRIC LINEAR PROGRAMMING

INSTANCE: Finite set X of pairs $(\bar{x}, b)$, where $\bar{x}$ is an m-tuple of integers and b is an integer, a set $J \subseteq \{1, 2, \ldots, m\}$, and a positive rational number q.
QUESTION: Is there an m-tuple $\bar{c}$ with rational entries such that $(\bar{c}\cdot\bar{c})^{1/2} \leqslant q$ and such that, if Y is the set of all m-tuples $\bar{y}$ with non-negative rational entries satisfying $\bar{x}\cdot\bar{y} \geqslant b$ for all $(\bar{x}, b) \in X$, then the minimum of $\sum_{j \in J} c_j y_j$ over all $\bar{y} \in Y$ exceeds

$$\tfrac{1}{2} \max \{|c_j| : j \in J\} + \sum_{j \in J} \min \{0, c_j\} \ ?$$

Reference: [Jeroslow, 1976]. Transformation from 3SAT.
Comment: Remains NP-complete for any fixed $q > 0$. The problem arises from first order error analysis for linear programming.

[MP4] FEASIBLE BASIS EXTENSION

INSTANCE: An $m \times n$ integer matrix A, $m < n$, a column vector $\bar{a}$ of length m, and a subset S of the columns of A with $|S| < m$.
QUESTION: Is there a *feasible basis* B for $A\bar{x} = \bar{a}$, $\bar{x} \geqslant 0$, i.e., a nonsingular $m \times m$ submatrix B of A such that $B^{-1}\bar{a} \geqslant 0$, and such that B contains all the columns in S?

Reference: [Murty, 1972]. Transformation from HAMILTONIAN CIRCUIT.

[MP5] MINIMUM WEIGHT SOLUTION TO LINEAR EQUATIONS

INSTANCE: Finite set X of pairs $(\bar{x}, b)$, where $\bar{x}$ is an m-tuple of integers and b is an integer, and a positive integer $K \leqslant m$.
QUESTION: Is there an m-tuple $\bar{y}$ with rational entries such that $\bar{y}$ has at most K non-zero entries and such that $\bar{x} \cdot \bar{y} = b$ for all $(\bar{x}, b) \in X$?

Reference: [Garey and Johnson, ——]. Transformation from X3C.
Comment: NP-complete in the strong sense. Solvable in polynomial time if $K = m$.

[MP6] OPEN HEMISPHERE

INSTANCE: Finite set X of m-tuples of integers, and a positive integer $K \leqslant |X|$.
QUESTION: Is there an m-tuple $\bar{y}$ of rational numbers such that $\bar{x} \cdot \bar{y} > 0$ for at least K m-tuples $\bar{x} \in X$?

Reference: [Johnson and Preparata, 1978]. Transformation from MAXIMUM 2-SATISFIABILITY.
Comment: NP-complete in the strong sense, but solvable in polynomial time for any fixed m, even in a "weighted" version of the problem. The same results hold for the related CLOSED HEMISPHERE problem in which we ask that $\bar{y}$ satisfy $\bar{x} \cdot \bar{y} \geqslant 0$ for at least K m-tuples $\bar{x} \in X$ [Johnson and Preparata, 1978]. If $K = 0$ or $K = |X|$, both problems are polynomially equivalent to linear programming [Reiss and Dobkin, 1976].

[MP7] K-RELEVANCY

INSTANCE: Finite set X of pairs $(\bar{x}, b)$, where $\bar{x}$ is an m-tuple of integers and b is an integer, and a positive integer $K \leqslant |X|$.
QUESTION: Is there a subset $X' \subseteq X$ with $|X'| \leqslant K$ such that, for all m-tuples $\bar{y}$ of rational numbers, if $\bar{x} \cdot \bar{y} \leqslant b$ for all $(\bar{x}, b) \in X'$, then $\bar{x} \cdot \bar{y} \leqslant b$ for all $(\bar{x}, b) \in X$?

Reference: [Reiss and Dobkin, 1976]. Transformation from X3C.
Comment: NP-complete in the strong sense. Equivalent to linear programming if $K = |X| - 1$ [Reiss and Dobkin, 1976]. Other NP-complete problems of this form, where a standard linear programming problem is modified by asking that the desired property hold for some subset of K constraints, can be found in the reference.

[MP8] TRAVELING SALESMAN POLYTOPE NON-ADJACENCY

INSTANCE: Graph $G = (V, E)$, two Hamiltonian circuits C and C' for G.
QUESTION: Do C and C' correspond to non-adjacent vertices of the "traveling salesman polytope" for G?

Reference: [Papadimitriou, 1978a]. Transformation from 3SAT.
Comment: Result also holds for the "non-symmetric" case where G is a directed graph and C and C' are directed Hamiltonian circuits. Analogous polytope non-adjacency problems for graph matching and CLIQUE can be solved in polynomial time [Chvátal, 1975].

[MP9] KNAPSACK

INSTANCE: Finite set U, for each $u \in U$ a size $s(u) \in Z^+$ and a value $v(u) \in Z^+$, and positive integers B and K.
QUESTION: Is there a subset $U' \subseteq U$ such that $\sum_{u \in U'} s(u) \leqslant B$ and such that $\sum_{u \in U'} v(u) \geqslant K$?

Reference: [Karp, 1972]. Transformation from PARTITION.
Comment: Remains NP-complete if $s(u) = v(u)$ for all $u \in U$ (SUBSET SUM). Can be solved in pseudo-polynomial time by dynamic programming (e.g., see [Dantzig, 1957] or [Lawler, 1976a]).

[MP10] INTEGER KNAPSACK

INSTANCE: Finite set U, for each $u \in U$ a size $s(u) \in Z^+$ and a value $v(u) \in Z^+$, and positive integers B and K.
QUESTION: Is there an assignment of a non-negative integer $c(u)$ to each $u \in U$ such that $\sum_{u \in U} c(u) \cdot s(u) \leqslant B$ and such that $\sum_{u \in U} c(u) \cdot v(u) \geqslant K$?

Reference: [Lueker, 1975]. Transformation from SUBSET SUM.
Comment: Remains NP-complete if $s(u) = v(u)$ for all $u \in U$. Solvable in pseudo-polynomial time by dynamic programming. Solvable in polynomial time if $|U| = 2$ [Hirschberg and Wong, 1976].

[MP11] CONTINUOUS MULTIPLE CHOICE KNAPSACK

INSTANCE: Finite set U, for each $u \in U$ a size $s(u) \in Z^+$ and a value $v(u) \in Z^+$, a partition of U into disjoint sets $U_1, U_2, \ldots, U_m$, and positive integers B and K.
QUESTION: Is there a choice of a unique element $u_i \in U_i$, $1 \leqslant i \leqslant m$, and an assignment of rational numbers r_i, $0 \leqslant r_i \leqslant 1$, to these elements, such that $\sum_{i=1}^{m} r_i \cdot s(u_i) \leqslant B$ and $\sum_{i=1}^{m} r_i \cdot v(u_i) \geqslant K$?

Reference: [Ibaraki, 1978]. Transformation from PARTITION.
Comment: Solvable in pseudo-polynomial time, but remains NP-complete even if $|U_i| \leqslant 2$, $1 \leqslant i \leqslant m$. Solvable in polynomial time by "greedy" algorithms if $|U_i| = 1$, $1 \leqslant i \leqslant m$, or if we only require that the $r_i \geqslant 0$ but place no upper bound on them. [Ibaraki, Hasegawa, Teranaka, and Iwase, 1978].

[MP12] PARTIALLY ORDERED KNAPSACK

INSTANCE: Finite set U, partial order $<$ on U, for each $u \in U$ a size $s(u) \in Z^+$ and a value $v(u) \in Z^+$, positive integers B and K.
QUESTION: Is there a subset $U' \subseteq U$ such that if $u \in U'$ and $u' < u$, then $u' \in U'$, and such that $\sum_{u \in U'} s(u) \leqslant B$ and $\sum_{u \in U'} v(u) \geqslant K$?

Reference: [Garey and Johnson, ——]. Transformation from CLIQUE. Problem is discussed in [Ibarra and Kim, 1975b].