A brief introduction to predicate logic *

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Predicates

1. Predicates are logical formulas that contain variables and constants. Variables are usually given by x, y, z, whereas constants are integers in context of Mini-Java.

2. Besides the property of propositional logical formulas, predicates are usually definied with relations. $P(x)$ is a predicate describing the property of variable x. $x = 0$, $x + 1 < 2$, and $x + y = 0 \wedge x < 0$ are all predicates of x.

3. A predicate may contain more than one variables. The number of the variables in the predicate is called the arity of the predicate. For example $P(x, y, z)$ has the arity of 3.

4. With the existence of $<$ relation, we can also apply the related rules to simplify the logical formula. For example

$$\begin{aligned} & (a > 0 \vee x + y > 0) \wedge (a \leq 0 \vee x + y > 0) \\ \equiv\ & (a > 0 \wedge a \leq 0) \vee x + y > 0 \\ \equiv\ & \text{true} \vee x + y > 0 \\ \equiv\ & x + y > 0 \end{aligned}$$

*This is unofficial material generated by tutors. Please follow the lecture notes for the accurate information. The materials are based on the lecture notes of the course Logic(IN2049) by Prof. Esparza and Prof. Nipkow

Equivalent formulas can be exchanged as one sees fit. This includes (but is not limited to) rules like **reflexivity, transitivity, associativity** etc. whenever applicable

$$\exists a.b < a \wedge a < c \Longrightarrow b < c$$

Quantifiers

1. There are two different quantifiers, the universal quantifier $\forall$ and the existential quantifier $\exists$. Each quantifier is always followed by a predicate.

2. Quantifiers bind variables. The scope of the quantifier contains all of the subformula on the right side.

formulas	explanations
$\forall x.x > 0$	x is bound by the universal quantifier.
$\forall x \exists y.x > y$	x is bound by the universal quantifier, y is bound by the existential quantifier.
$\forall x \exists x.x > 0$	x is bound by the existential quantifier, the rightmost quantifier binds the strongest.

Table 1: Boundness of formulas

3. Rectified formula. A formula is rectified if no variable occurs bound and free and if all quantifiers in the formula bind different variables. If

formulas	rectifiedness
$\forall x \forall y.x + y > c$	Yes
$x > 0 \wedge \forall x.x < 42$	No
$\forall x.x < 42 \wedge \forall x.x > 13$	No

Table 2: Rectifiedness of formulas

you find a logical formula hard to read, you may convert the formula to a rectified formula. The convertion is possible by renaming the duplicated binding with a new variable, which does not exist in the current formula.

$$\forall x \exists x . x > 0 \longrightarrow \forall x \exists x_1 . x_1 > 0$$
$$x > 0 \wedge \forall x . x < 42 \longrightarrow x > 0 \wedge \forall x_1 . x_1 < 42$$
$$\forall x . x < 42 \wedge \forall x . x > 13 \longrightarrow \forall x . x < 42 \wedge \forall x_1 . x_1 > 13$$

Substitutions

- Substitutions replace **free** variables with terms. [1]
- The bound variables are not substituted, for example

$$(x > 0 \wedge \forall x . x > 0)[1/x] \equiv 1 > 0 \wedge \forall x . x > 0$$

- You cannot substitute a constant for a variable or another constant.

Problem set

You may use this problem set to test your knowledge about the predicate logic. The problems are **not** directly related to the exam, but you may find them useful for understanding the contents.

Predicates and Relations

1. Let x be a variable. What is the **minimal** arity of the predicate $x = y$ when y is a variable and when y is not a variable?
2. Can a predicate have an arity of **zero**?
3. What is the **maximum** of the arity of a predicate?
4. What are the **reflexity, symmetry** and **transitivity** of a relation?
5. Given an order $a \leq b \leq c$ with **transitivity**, convert it into a conjunction of relations.

[1] $A[a/y]$ reads substitute y for a in A.

6. Let $\sim$ be a reflexive, symmetric and transitive relation over $\mathbb{N} \times \mathbb{N}$ and $\circ$ be a reflexive, antisymmetric relation over $\mathbb{Z} \times \mathbb{Z}$. $\leq$ and $<$ are the ordinary less equal and less relations. Simplify the following formulas. For each one, determine which sets the variables must belong to so that the formula is well defined and then determine what set satisfies the formula.
   Example: $a \sim b$ dictates that $a, b \in \mathbb{N}$ as $\sim$ is only defined over natural numbers. The formula is satisfied iff $a \sim b$, which we cannot narrow down any further since we know nothing about the $\sim$ relation.
   (a) $a \leq b \wedge b \sim a$
   (b) $a < b \leq a$
   (c) $a \sim b \wedge a \sim c \Longrightarrow c \sim b$
   (d) $a \sim b \sim c \sim a \Longrightarrow c \circ b$
   (e) $c \circ d \wedge d \circ c \equiv c \sim d \wedge d \sim c$
   (f) $a = -3 \wedge a^2 \sim a \sim b \Longrightarrow a^2 \sim b$

Quantifiers

In the following formulas, name by which quantifier is each x bound and convert them into rectified formulas.

1. $\forall x \exists x \forall x . x + 1 > 0$
2. $x > 0 \wedge \forall x . x < 0 \wedge (\exists x . x \neq 0) \vee x < 0$
3. $x > 0 \wedge \forall x \exists x \forall x . x < 0 \wedge \forall x . x > 42$

Substitutions

For each formulae F given in the previous exercise, perform the substitution $F[1/x]$ and simplify it as far as you can. **(Do not simplify any subformulas that contain quantifiers)**

Advanced

This part includes some advanced questions, which are not related to the **FPV** lecture. They may be inspiring if you are interested.

1. Sometimes, we want to state a property or relation about a function, for example, $\forall x.f(x) > 0$. How can we elaborate the rectifying rule with the existence of a function?

2. In rectified formulas, the quantifiers are also existent inside the subformulas. The formula are hence less readable. Do you think of any way to improve this? You may compare the following examples.

   $(\forall x.x = 0 \vee x \neq 0) \wedge \forall y.(\forall z.z > 0 \vee z < y) \wedge y > 0$ **(Rectified)**

   $\forall x \forall y \forall z.(x = 0 \vee x \neq 0) \wedge (z > 0 \vee z < y) \wedge y > 0$ **(Rectified Prenex)**

Sample solutions

Predicates and Substitutions

1. 1 if y is a variable, otherwise 2.

2. Yes. For arity 0, we write P instead of $P()$. For example, $P :\equiv 1 = 1$ is a trivial property without the existence of any variable.

3. There is no upperbound, thus ∞. See an example, $P(x, y) :\equiv x > 0$. There is no existence of y in P, but you can still include it. Using the similar construction, we may add as many variables as we want.

4. For a relation $\sim$

   - reflexity: $a \sim a$
   - symmetry: $a \sim b \iff b \sim a$
   - transitivity $a \sim b \wedge b \sim c \Longrightarrow a \sim c$

   The symbol of the relation does not matter, you may replace it with $\geq, \neq, \Longrightarrow$ or whatever you want/define.

5. $a \leq b \wedge b \leq c \wedge a \leq c$

6. (a) Not simplifiable. $a, b \in \mathbb{N}$ for it to be defined and it holds true for $(a, b) \in \{(a, a + k) | a \in \mathbb{N} \wedge k \in \mathbb{N}_0 \wedge a \sim a + k\}$

   (b) $a < b \leq a \equiv$ **false** $\forall a, b \in \mathbb{Z}$

   (c) The statement is equivalent to **true** $\forall a, b, c \in \mathbb{N}$:

   $$a \sim b \wedge a \sim c \overset{symmetry}{\Longrightarrow} b \sim a \wedge a \sim c \overset{transitivity}{\Longrightarrow} b \sim c \overset{symmetry}{\Longrightarrow} c \sim b$$

   (d) $\forall a, b, c \in \mathbb{N}$ the statement can be simplified to $a \sim b \sim c \Longrightarrow c \circ b$, as symmetry and reflexivity already cover all the statements that the extra $\sim a$ would give us. If $\sim$ were antisymmetric, we could follow that $b = c$ and consequently $c \circ b$, but as it is, we can't say anything further. The statement therefore holds iff $\neg(a \sim b \sim c) \vee c \circ b$.

(e) The statement holds if and only if $c = d$. Thus, it is valid only over $(c, d) \in \{(a, a) | a \in \mathbb{N}\}$. Interestingly, this is only necessary for the right hand side (rhs) to imply the lhs. The other direction (lhs implies rhs) holds true due to antisymmetry of $\circ$ and reflexivity of $\sim$.

(f) The statement is not defined as $a = -3 \notin \mathbb{N}$ which would be required for $a \sim b$.

Quantifiers and Substitutions

1. Bound by the third quantifier from the left.

$$\forall x \exists x_1 \forall x_2 . x_2 + 1 > 0$$

   Identical after the substitution.

2. The first and the last x are free. The second is bound by the universal quantifier, and the third is bound by the existential quantifier.

$$\begin{aligned} & x > 0 \land \forall x_1 . x_1 < 0 \land (\exists x_2 . x_2 \neq 0) \lor x < 0 \\ & (x > 0 \land \forall x_1 . x_1 < 0 \land (\exists x_2 . x_2 \neq 0) \lor x < 0)[1/x] \\ \equiv\ & 1 > 0 \land \forall x_1 . x_1 < 0 \land (\exists x_2 . x_2 \neq 0) \lor 1 < 0 \\ \equiv\ & \forall x_1 . x_1 < 0 \land (\exists x_2 . x_2 \neq 0) \end{aligned}$$

3. The first x is free. the second is bound by the third quantifier from the left. The third is bound by the fourth quantifier from the left.

$$\begin{aligned} & x > 0 \land \forall x_1 \exists x_2 \forall x_3 . x_3 < 0 \land \forall x_4 . x_4 > 42 \\ & (x > 0 \land \forall x_1 \exists x_2 \forall x_3 . x_3 < 0 \land \forall x_4 . x_4 > 42)[1/x] \\ \equiv\ & 1 > 0 \land \forall x_1 \exists x_2 \forall x_3 . x_3 < 0 \land \forall x_4 . x_4 > 42 \\ \equiv\ & \forall x_1 \exists x_2 \forall x_3 . x_3 < 0 \land \forall x_4 . x_4 > 42 \end{aligned}$$

Advanced

You may refer to the slides of the **Logic(IN2049)** lecture for the answer. The answers are accessible via the link for question 1) and this the link for question 2).