Computation of kappa

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Remember that

$$\kappa_n := \sup_{t:\, |t| \geq a_n/\sqrt{n}} |f_{X_n/\sigma_n}(t)|.$$

with $a_n := 2t_1^* \pi \sqrt{n}/(2K_{3,n})$ for large n.

Furthermore, $t_1^* \pi \approx= 2.0106$

```
0.64 * pi
```

```
## [1] 2.010619
```

Therefore,

$$\kappa_n \leq \sup_{t:\, |t| \geq 2/K_{3,n}} |f_{X_n/\sigma_n}(t)|.$$

Normal distribution

For the normal distribution $K_{3,n} = \sqrt{2/pi}$ and $|f_{X_n/\sigma_n}(t)| = \exp(-t^2/2)$. Therefore κ_n is smaller than

```
t = 2 / sqrt(2 / pi)
exp( - t^2 / 2)
```

```
## [1] 0.04321392
```

Laplace distribution

For the Laplace distribution $K_{3,n} = 6/2^{3/2}$ and $|f_{X_n/\sigma_n}(t)| = 1/(1 + t^2/2)$. Therefore κ_n is smaller than

```
t = 2 / (6 / 2^{3/2})
1 / (1 + t^2/2)
```

```
## [1] 0.6923077
```

Student distribution

```
# Setting parameters
Nrep = 100000

t1 <- 0.64 # universal constant from paper
n<- 5000
nu <- 8 # df for student
K4_exp <- 9
K4_gauss <- 3
```

```r
# Absolute third order moment of exp distr upper bounded while that of
# standard normal distr explicitly computed

K3_exp <- K4_exp^(3/4)
K3_gauss <- 2^(3/2)*gamma(2)/sqrt(pi)
```

```r
# Function to compute a_n (cf article)

compute_an <- function(t1,n,K3) {

  build_an_1 <- 2*t1*pi*sqrt(n)/(K3+1)
  build_an_2 <- 16*pi^3*n^2/(K3+1)^4

  return( min(build_an_1,build_an_2)/sqrt(n) )
}


# Compute kappa_n for exp and standard normal

carac_exp_at_an <- 1 / sqrt(1 + compute_an(t1=t1,n=n,K3=K3_exp)^2)

carac_gauss_at_an <- exp( - compute_an(t1=t1,n=n,K3=K3_gauss)^2/2)


# Function to evaluate the characteristic function of
# the Student distribution at some point of interest (here a_n)

stud_carac_fun <- function(t1, n, nu) {

  sigma <- sqrt(nu/(nu-2))
  K4 <- 6/(nu-4)+3
  an <- compute_an(t1=t1, n=n, K3= mean(abs(scale(rt(Nrep, df = nu)))^3) )
  temp1 <- besselK(nu = nu/2, x = sqrt(nu)*abs(an)/sigma)
  temp2 <- (sqrt(nu)*abs(an)/sigma)^(nu/2) / (gamma(nu/2)*2^(nu/2-1))

  return(temp1 * temp2)
}
```

```r
# compute kappa_n for the Student distribution

stud_carac_fun(t1=t1,n=n,nu=8)
```

```
## [1] 0.428395
```

```r
stud_carac_fun(t1=t1,n=n,nu=5)
```

```
## [1] 0.5364768
```

Exponential distribution

The characteristic function of the exponential distribution is $1/(1-it)$. So the one we need is $1/(1-it)*\exp(it)$ (which has the same modulus).

```r
an_Gamma = compute_an(t1=t1, n = 5000, K3 = 2.31)
```

```r
sqrt((1 / (1 + an_Gamma^2)^2 )^2 +
       (an_Gamma^2 / (1 + an_Gamma^2)^2 )^2 )
```

```
## [1] 0.2908215
```