1-D Scalar Wave Equation with Finite Differences

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Background and Motivation

The 1-D scalar wave equation is given by

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0$$

IC: $u(x,0) = f(x)$ $\frac{\partial u}{\partial t}(x,0) = g(x)$ BC: $u(0,t) = u(L,t) = 0$ for $t > 0$

- Insights into complex systems
- Testing numerical methods

The second order nature of the scalar wave equation is crucial for several reasons:

- Harmonic oscillator analogy
- Numerical stability and convergence

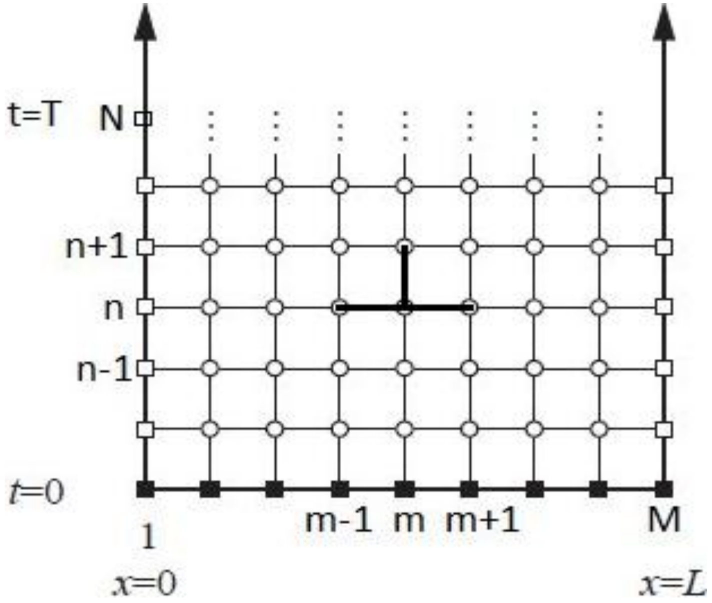
Theory

- Finite Difference Method
- Taylor Series Expansion
- Discretization and the Mesh
- Von Neumann Stability Analysis
- Limitations
- Examples

Numerical Method: Finite Difference Approximations

- Finite Difference Approximation
  - Numerical Solution
  - Discretization of PDE

- Mesh of Numerical Method
  - Spatial Grid & Temporal Grid
  - Spatial-Temporal Grid
    - Step sizes : $\boldsymbol{h}$ and $\boldsymbol{\Delta t}$
    - $\boldsymbol{N_x} + 1$ Nodes
    - $\boldsymbol{N_t}$ Time steps

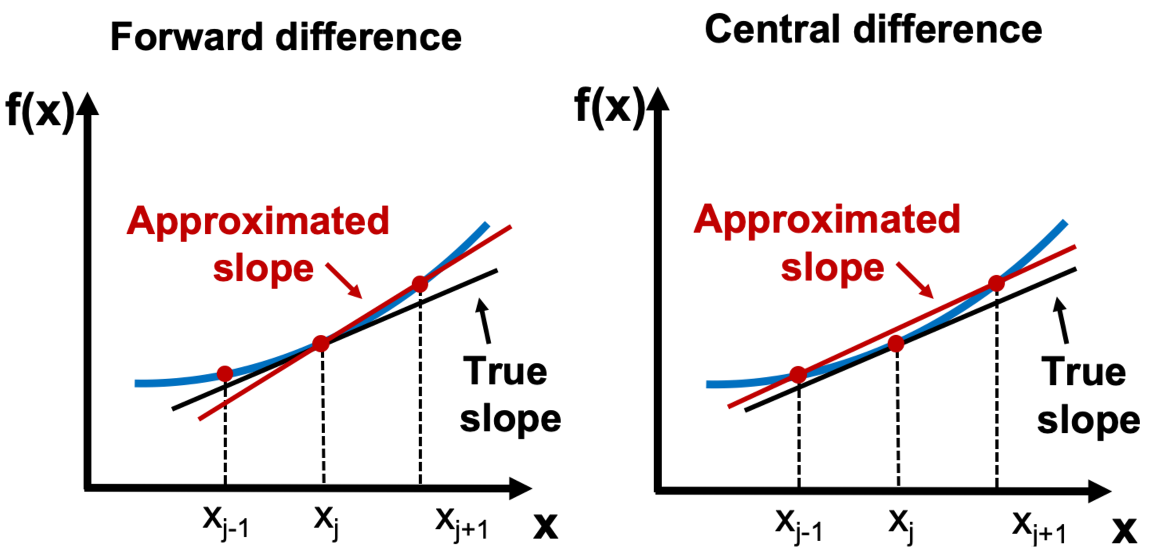


<u>Spatial Grid:</u>
$x_i = x_l + i * h \ , \ i = 0,1,\dots,N_x$
with $N_x = \frac{x_r - x_l}{h}$ for $x \in [x_l, x_r]$

<u>Temporal Grid:</u>
$t_n = t_0 + n * \Delta t \ , \ n = 0,1,\dots,N_t$
with $N_t = \frac{t_f - t_0}{\Delta t}$ for $t \in [t_0, t_f]$

Numerical Method:
Forward and Centered Finite Differences

- Approximation of Derivatives
  - **Main Idea**
  - **Finite Difference Formulas**
  - Continuous PDE → Discrete PDE
  - Recursive Formula
  - Absolute Error of Method
    - Error = | True – Approximate |



<u>Forward FD</u>: $\frac{d}{dx}u(x_i) \approx \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} = \frac{u(x_{i+1}) - u(x_i)}{h}$

<u>Centered FD</u>: $\frac{d}{dx}u(x_i) \approx \frac{u(x_{i+1}) - u(x_{i-1})}{x_{i+1} - x_{i-1}} = \frac{u(x_{i+1}) - u(x_{i-1})}{2h}$

Numerical Method:
Forward and Centered Finite Differences

- Approximation of Derivatives
  - Main Idea
  - Finite Difference Formulas
  - **Continuous PDE → Discrete PDE**
  - Recursive Formula
  - Absolute Error of Method
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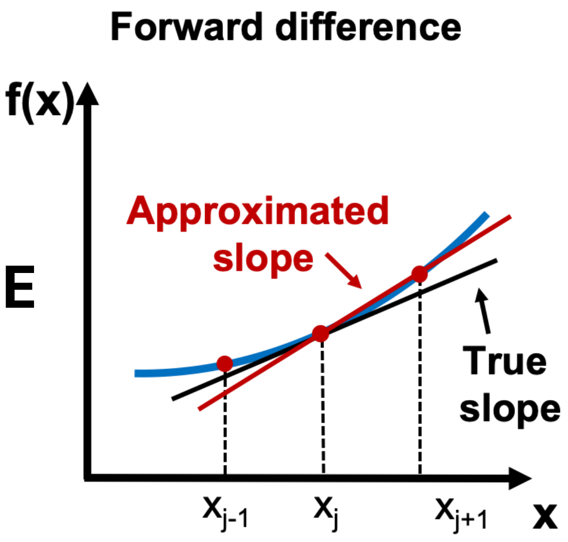

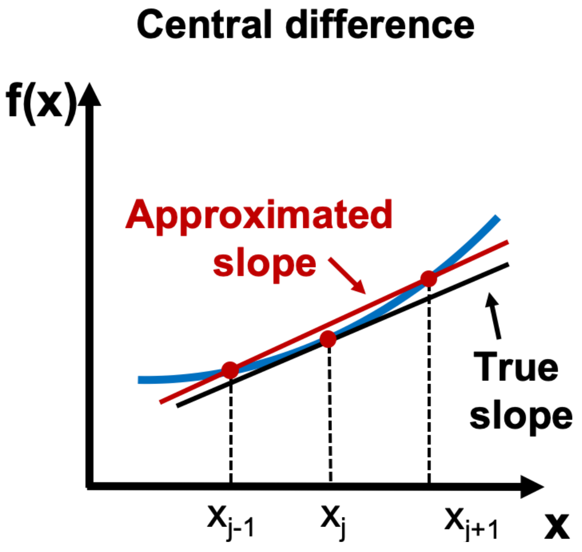

$$\frac{d}{dx}u(x) = f(x) \;\; \rightarrow \;\; \frac{d}{dx}u(x_i) = f(x_i)$$

Numerical Method: Forward and Centered Finite Differences

- Approximation of Derivatives
  - Main Idea
  - Finite Difference Formulas
  - Continuous PDE → Discrete PDE
  - **Recursive Formula**
  - **Absolute Error of Method**
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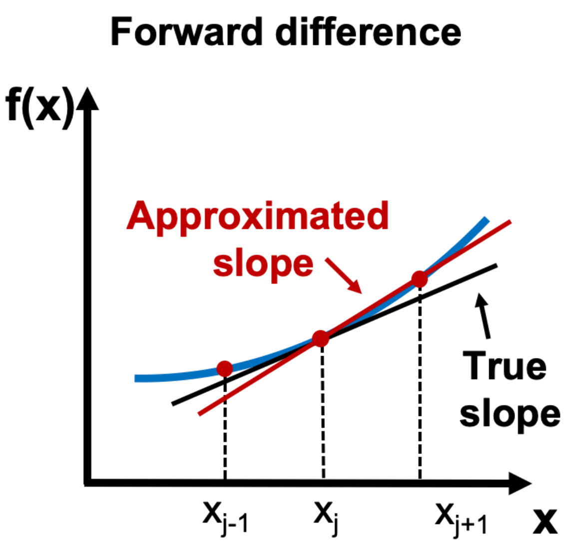

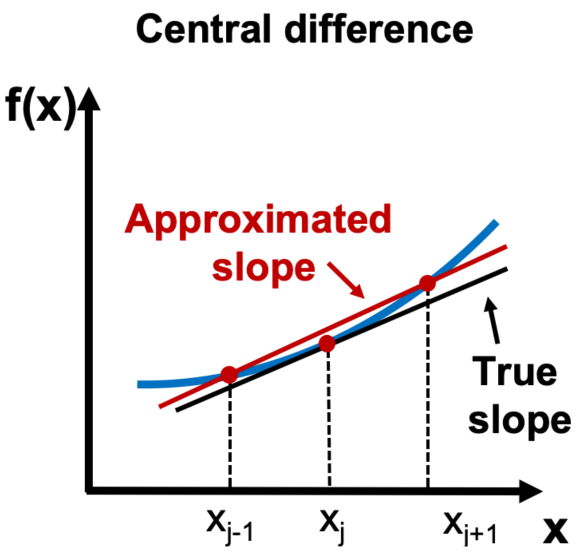

Forward FD: $\frac{d}{dx}u(x_i) = f(x_i) \approx \frac{u(x_{i+1}) - u(x_i)}{h}$

$$\boldsymbol{u(x_{i+1}) \approx u(x_i) + h * f(x_i)}$$

Centered FD: $\frac{d}{dx}u(x_i) = f(x_i) \approx \frac{u(x_{i+1}) - u(x_{i-1})}{2h}$

$$\boldsymbol{u(x_{i+1}) \approx u(x_{i-1}) + 2h * f(x_i)}$$

Applying Finite Difference Approximation to 1-D Scalar Wave Equation

- Approximation of 2nd Order Centered Finite Differences

2nd Order Forward FD: $\frac{\partial^2 u}{\partial x^2}(x_i) \approx \frac{u(x_{i+2})-2u(x_{i+1})+u(x_i)}{h^2}$

2nd Order Centered FD: $\frac{\partial^2 u}{\partial x^2}(x_i) \approx \frac{u(x_{i+1})-2u(x_i)+u(x_{i-i})}{h^2}$

Applying Finite Difference Approximation to 1-D Scalar Wave Equation

- Approach to 1-D Scalar Wave Equation
  - **Mesh**
  - Finite Difference Approximations of Derivatives
  - PDE Discretization
  - Recursive Formula

Spatial Grid:

$$x_i = x_l + i * h \ , \ i = 0,1,\dots,N_x$$

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Temporal Grid:

$$t_n = t_0 + n * \Delta t \ , \ n = 0,1,\dots,N_t$$

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Applying Finite Difference Approximation to 1-D Scalar Wave Equation

- Approach to 1-D Scalar Wave Equation
  - Mesh
  - **Finite Difference Approximations of Derivatives**
  - PDE Discretization
  - Recursive Formula

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) \approx \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{\Delta t^2}$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) \approx \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2}$$

Applying Finite Difference Approximation to 1-D Scalar Wave Equation

- Approach to 1-D Scalar Wave Equation
  - Mesh
  - Finite Difference Approximations of Derivatives
  - **PDE Discretization**
  - Recursive Formula

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0 \quad \rightarrow \quad \frac{\partial^2 u}{\partial t^2}\left(x_i, t_j\right) - \alpha^2 \frac{\partial^2 u}{\partial x^2}\left(x_i, t_j\right) = 0$$

Applying Finite Difference Approximation to 1-D Scalar Wave Equation

- Approach to 1-D Scalar Wave Equation
  - Mesh
  - Finite Difference Approximations of Derivatives
  - PDE Discretization
  - **Recursive Formula**

$$\frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{\Delta t^2} - \alpha^2 \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} = 0$$

$$u(x_i, t_{j+1}) = 2(1 - \lambda^2)\, u(x_i, t_j) + \lambda^2 \left(u(x_{i+1}, t_j) + u(x_{i-1}, t_j)\right) - u(x_i, t_{j-1}) \text{ where } \lambda = \frac{\alpha \Delta t}{h}$$

Initial Approximation of $u(x_i, 0)$

- **Issue Encountered**
  - **First time step**
- Approach
  - Method 1: Initial Velocity
    - Forward Finite Difference
  - Method 2: Initial Displacement & Velocity
    - Centered Finite Difference

$$u(x_i, t_{j+1}) = 2(1 - \lambda^2)\, u(x_i, t_j) + \lambda^2 \left(u(x_{i+1}, t_j) + u(x_{i-1}, t_j) \right) - u(x_i, t_{j-1}) \text{ where } \lambda = \frac{\alpha \Delta t}{h}$$

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$$\frac{\partial u}{\partial t}(x, 0) = g(x) \quad \rightarrow \quad \frac{\partial u}{\partial t}(x_i, 0) = g(x_i)$$

$$\frac{\partial u}{\partial t}(x_i, 0) \approx \frac{u(x_i, t_1) - u(x_i, 0)}{\Delta t} \quad \rightarrow \quad u(x_i, t_1) \approx u(x_i, 0) + \Delta \mathrm{t} * g(x_i)$$

Initial Approximation of $u(x_i, 0)$

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  - Method 1: Initial Velocity
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    - **Centered Finite Difference**

$$u(x_i, t_1) \approx u(x_i, 0) + \Delta\mathrm{t} * g(x_i) + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2}(x_i, \mu_i) \text{ for some } \mu_i \in (0, t_1)$$

<u>Note</u>: $\frac{\partial^2 u}{\partial t^2}(x_i, 0) = \alpha^2 \ \frac{\partial^2 u}{\partial x^2}(x_i, 0) = \alpha^2 \ \frac{d^2 f}{dx^2}(x_i) = \alpha^2 f''(x_i)$

Initial Approximation of $u(x_i, 0)$

- Issue Encountered
  - First time step
- Approach
  - Method 1: Initial Velocity
    - Forward Finite Difference
  - **Method 2: Initial Displacement & Velocity**
    - **Centered Finite Difference**

$$u(x_i, t_1) \approx u(x_i, 0) + \Delta\mathrm{t} * g(x_i) + \frac{\alpha^2 \Delta t^2}{2} f''(x_i) \text{ if } f'' \text{ exists}$$

$$u(x_i, t_1) \approx \left(1 - \lambda^2\right) f(x_i) + \frac{\lambda^2}{2}\left(f(x_{i+1}) + f(x_{i-1})\right) + \Delta t * g(x_i) \text{ where } \lambda = \frac{\alpha \Delta t}{h}$$

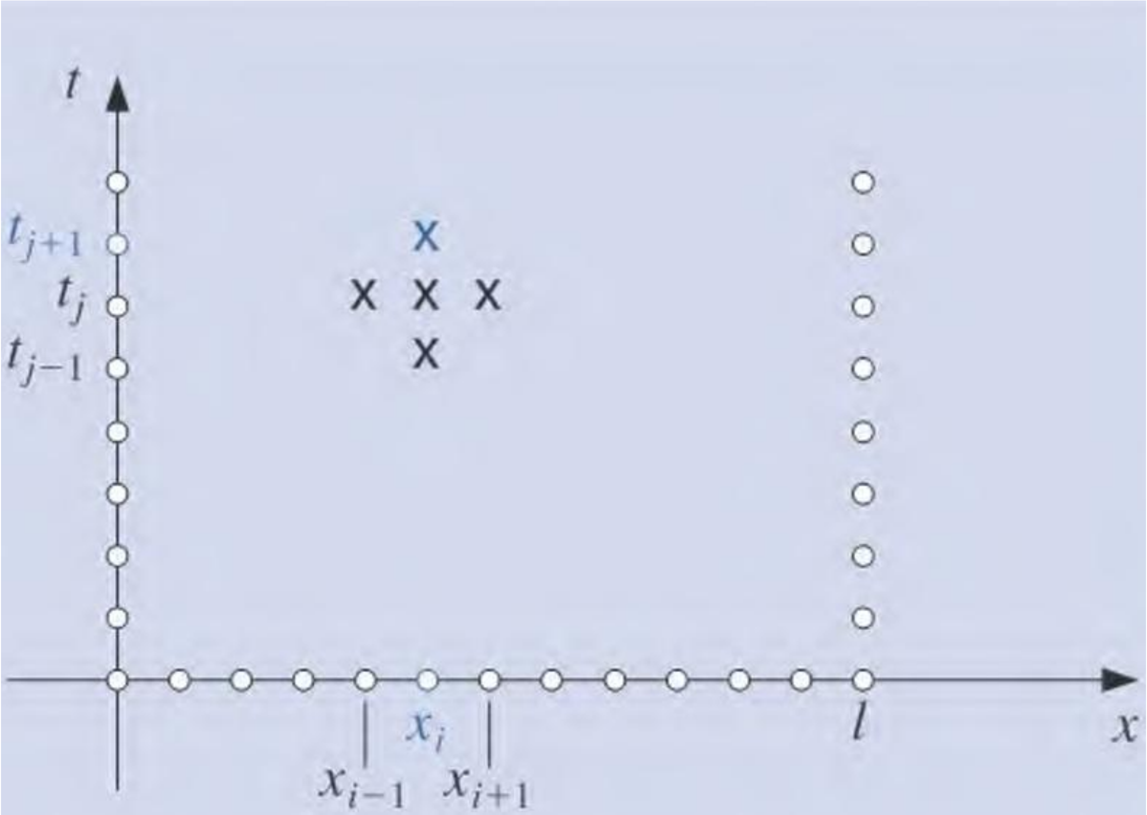
Initial Approximation of $u(x_i, 0)$

- Issue Encountered
  - First time step
- Approach
  - Method 1: Initial Velocity
    - Forward Finite Difference
  - Method 2: Initial Displacement & Velocity
    - Centered Finite Difference

Method 1: $u(x_i, t_1) \approx u(x_i, 0) + \Delta \mathrm{t} * g(x_i)$

Method 2: $u(x_i, t_1) \approx \left(1 - \lambda^2\right) f(x_i) + \frac{\lambda^2}{2}\left(f(x_{i+1}) + f(x_{i-1})\right) - \Delta t * g(x_i)$ where $\lambda = \frac{\alpha \Delta t}{h}$

Stencil of Method



Implementation

- 1-D Scalar Wave Equation Function
  - Inputs
  - Tasks
  - Outputs

Implementation

- 1-D Scalar Wave Equation Function
  - **Inputs**
    - Spatial Step Size h
    - CFL condition λ
    - Wave Speed α
    - Length of domain L
    - Final Time T
    - Initial Displacement Function $f(x)$
    - Initial Velocity Function $g(x)$
    - Whether to animate solution is_movie
  - Tasks
  - Outputs

$$\lambda = \frac{\alpha \Delta t}{h}$$

B.C: $u(0,t) = u(L,t) = 0$ for $t > 0$

I.C.: $u(x,0) = f(x)$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \text{ for } 0 \le x \le L$$

Implementation

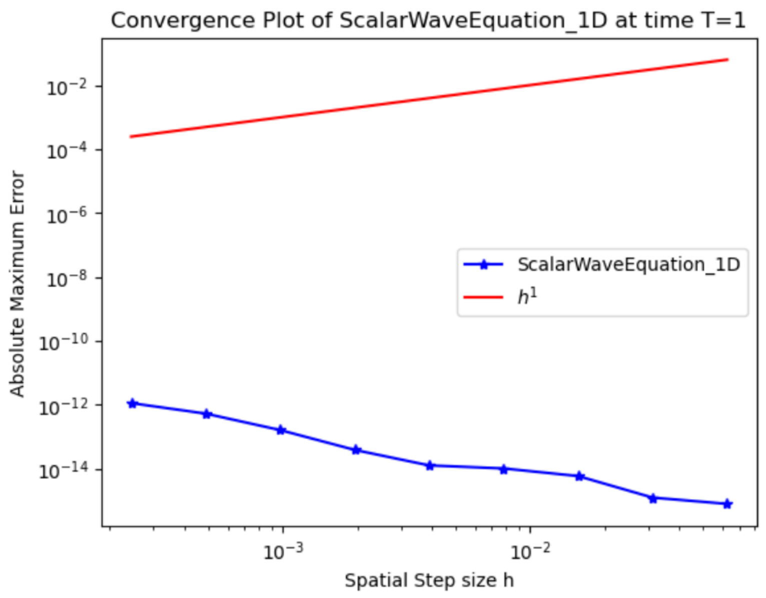
- 1-D Scalar Wave Equation Function
  - Inputs
  - **Tasks**
    - Define Δt
    - Define Number of Spatial & Time Grid Points
    - Initialize Spatial Grid, Time Grid, & Approximate Solutions
    - Define Boundary Conditions & Initial Conditions
    - Compute First Time Step using <u>Method 1</u> or <u>Method 2</u>
    - Compute 2nd to Last Time Steps
    - Animate Approximate Solutions of u
  - Outputs

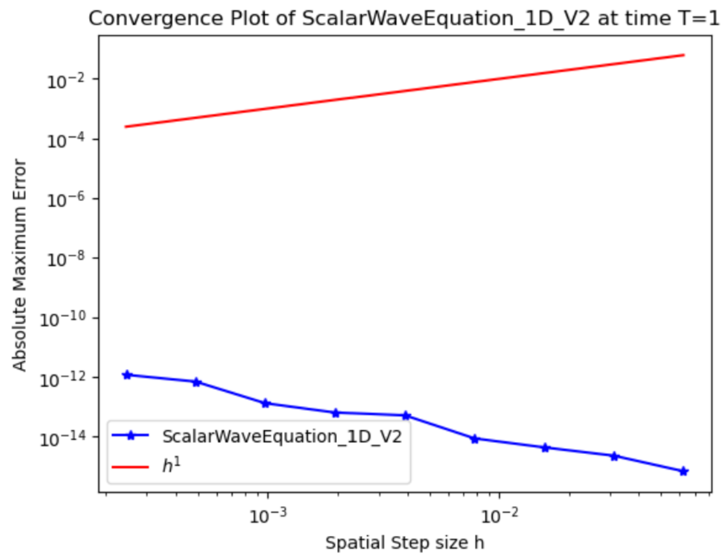
Implementation

- 1-D Scalar Wave Equation Function
  - Inputs
  - Tasks
  - **Outputs**
    - Approximate Solution Δt
    - Spatial Grid x
    - Temporal Grid t

Verification

- Study Case 1
  - Inputs
    - $h = 2^{-k}$ for $k = 4, \ldots, 12$
    - $\lambda = 1$
    - $\alpha = 2$
    - $L = 1$
    - $T = 1$
    - $f(x) = \sin(\pi x)$
    - $g(x) = 0$
  - Exact Solution
    - $u(x,t) = \sin(\pi x)\cos(2\pi t)$



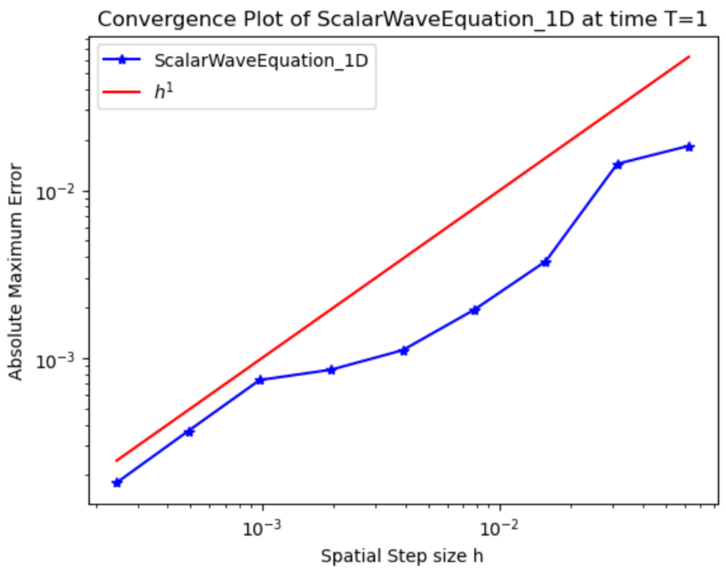


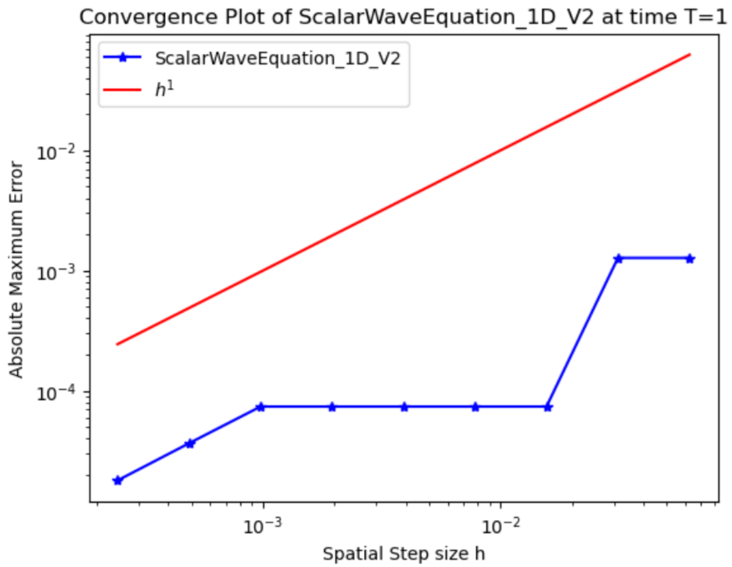
Verification

- Study Case 2
  - Inputs
    - $h = 2^{-k}$ for $k = 4, \dots, 12$
    - $\lambda = 1$
    - $\alpha = \frac{1}{4\pi}$
    - $L = 0.5$
    - $T = 0.5$
    - $f(x) = 0$
    - $g(x) = \sin(4\pi \mathrm{x})$
  - Exact Solution
    - $u(x, t) = \sin(t) \sin(4\pi x)$

Verification

- Study Case 3
  - Inputs
    - $h = 2^{-k}$ for $k = 4, \ldots, 13$
    - $\lambda = \frac{0.5}{\pi}$
    - $\alpha = 1$
    - $L = \pi$
    - $T = 0.5$
    - $f(x) = \sin(x)$
    - $g(x) = 0$
  - Exact Solution
    - $u(x, t) = \cos(t)\sin(x)$





Conclusion

- Von Neumann Multi-Step Method

- Strengths
  - Simple to implement
  - Easy to obtain higher-order approximations
- Weaknesses
  - Not able to handle discontinuities & shocks

References

- Burden, R.L., Faires, J.D., Burden A.M (2016) *Numerical Analysis*. 10th Edition, Cengage Learning, Boston, 757-765.
- Strang, G. (2007). *Computational science and engineering*. 1st Edition, Wellesley-Cambridge Press, 485-489.