NUMERICAL INVESTIGATIONS OF NONLINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS

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OVERVIEW

- Introductions to Numerical Schemes
  - Standard Finite Difference
  - Nonstandard Finite Difference
- Cubic Equations $u' = -u^3 + F(u)$
  - Cubic Decay $F(u) = 0,$
  - Bernoulli $F(u) = u,$
  - Combustion $F(u) = u^2$
- Bratu IVP
- Conclusion

RESEARCH GOAL AND CONTEXT

- ODE's can be used to model many real-world phenomena
- Solutions of ODE's are continuous functions often found through integration
- Exact solutions are rare and numerical approximations are necessary
- We are trying to find numerical solutions of ODE's that outperform standard methods
- Our research involves investigation of nonstandard finite difference schemes (NSFD's) to solve $u' = -u^3 + F(u),\ u(0) = u_0$

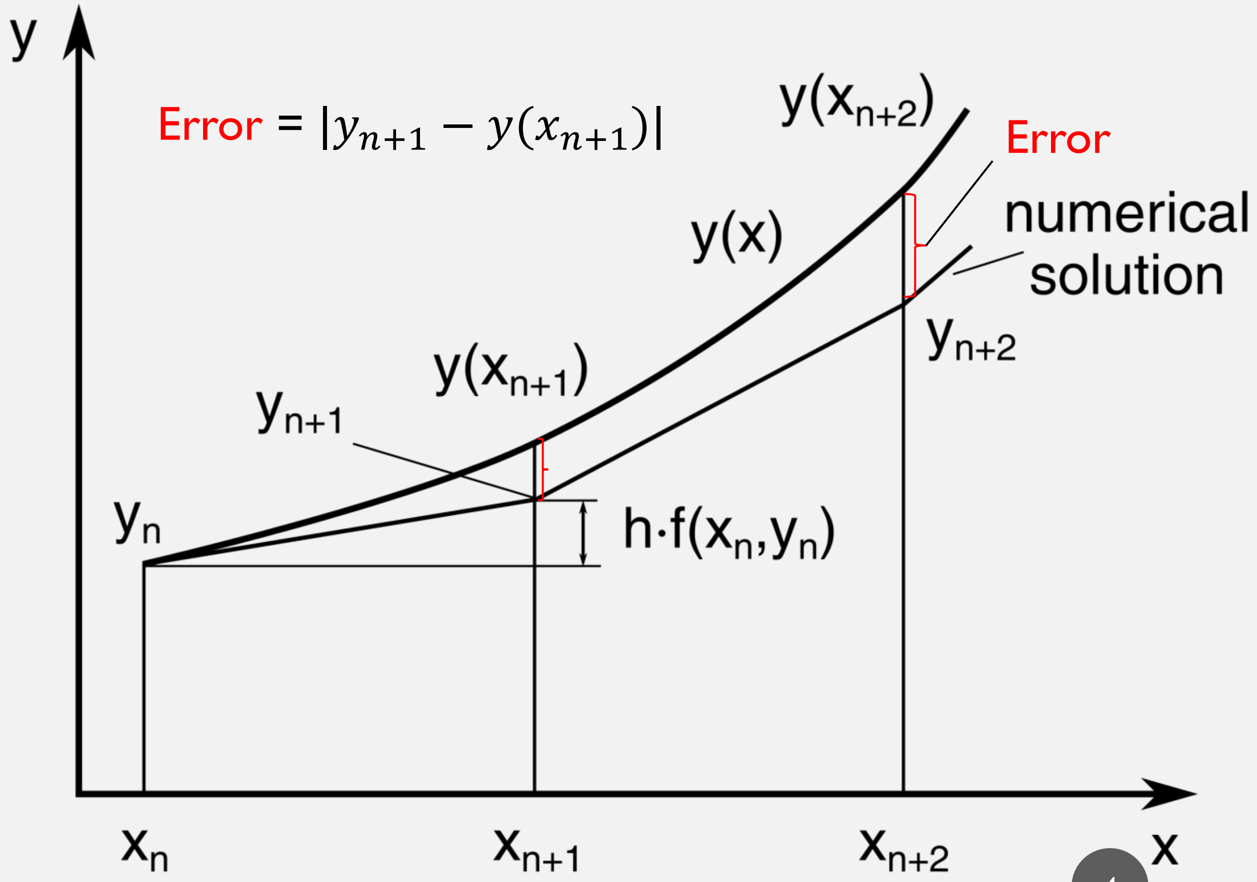
CONTINUOUS TO DISCRETE

- Partition the domain into n parts
- $n+1$ points will be needed
- At each point we want to evaluate the function $y(x_n) = y_n$
- After evaluating the function, we get corresponding values (x_n, y_n)
- $n+1$ ordered pairs of (x_n, y_n)

Euler's Method:

$$y'(x_n) = f(x_n, y_n) \approx \frac{y_{n+1} - y_n}{h}$$

$$y_{n+1} \approx y_n + hf(x_n, y_n)$$



INTRODUCTION TO NON-STANDARD FINITE DIFFERENCES (NSFDs)

RONALD E. MICKENS

- Born February 7, 1943, in Petersburg, Virginia
- HBCU: Fisk University – Bachelor's Degree in Mathematics and Physics
- Vanderbilt University – PhD in Theoretical Physics
- Returned to Fisk University to teach
- HBCU: Clark Atlanta University - became a Callaway Professor
- Author of 12+ books and 300+ research articles

NSFD'S IMPORTANCE AND PURPOSE

$$\frac{du}{dt} = \frac{u_{k+1} - \psi u_k}{\phi}$$

$$\boldsymbol{\phi}(\boldsymbol{h}) = h + O(h^2)$$
$$\boldsymbol{\psi}(\boldsymbol{h}) = 1 + O(h)$$

NSFD's may yield numerical solutions that are better than SFD's

NSFD's may preserve important properties of the ODE

NSFD's may yield exact solutions (zero error!)

ODE TO OΔE
$u' = -u^3$

SFD

$$\frac{u_{k+1} - u_k}{h} = -{u_k}^3$$

- Only one way to do standard

Method:

- Non-standard denominator
- Non-local term
- All together

NSFD

- $\frac{u_{k+1} - u_k}{1-e^{-h}} = -{u_k}^3$
- $\frac{u_{k+1} - u_k}{h} = -{u_k}^2 u_{k+1}$
- $\frac{u_{k+1} - u_k}{1-e^{-h}} = -{u_k}^2 u_{k+1}$

Goal is to find u_{k+1} in terms of u_k

ODE TO OΔE

$$u' = -u^3$$

SFD

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NSFD

- $\frac{u_{k+1} - u_k}{1-e^{-h}} = -u_k{}^3$
- $\frac{u_{k+1} - u_k}{h} = -u_k{}^2 u_{k+1}$
- $\frac{u_{k+1} - u_k}{1-e^{-h}} = -u_k{}^2 u_{k+1}$

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ODE TO OΔE
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NSFD

- $\frac{u_{k+1} - u_k}{1-e^{-h}} = -u_k{}^3$
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Goal is to find u_{k+1} in terms of u_k

ODE TO OΔE

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SFD

$$\frac{u_{k+1} - u_k}{h} = -{u_k}^3$$

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Method:

- Non-standard denominator
- Non-local term
- All together

NSFD

- $\frac{u_{k+1} - u_k}{1-e^{-h}} = -{u_k}^3$
- $\frac{u_{k+1} - u_k}{h} = -{u_k}^2 u_{k+1}$
- $\frac{u_{k+1} - u_k}{1-e^{-h}} = -{u_k}^2 u_{k+1}$

Goal is to find u_{k+1} in terms of u_k

OUR RESEARCH: NSFD UNITY APPROXIMATIONS

- Simple unity approximations
- 2-point averaging approximations
- Square averaging approximations
- Cubic averaging approximations

$$1 = \frac{u}{u} \approx \frac{u_k}{u_{k+1}} \approx \frac{u_{k+1}}{u_k}$$

$$\approx \frac{u_k + \ u_{k+1}}{2u_k} \approx \frac{u_k + \ u_{k+1}}{2u_{k+1}}$$

$$\approx \frac{{u^2}_{k+1}}{{u^2}_k} \approx \frac{{u^2}_k}{{u^2}_{k+1}}$$

CUBIC DECAY EQUATION

$$u' = -u^3$$

$$u(0) = 1$$

INVESTIGATING THE CUBIC DECAY DIFFERENTIAL EQUATION

Consider the Cubic Decay initial-value problem

$$\frac{du}{dt} = -u^3, u(0) = u_0 > 0.$$

The exact general solution is:

$$u(t) = \frac{1}{\sqrt{2t + C}} \tag{1}$$

Assuming $u(0) = 1$ we have

$$u(t) = \frac{1}{\sqrt{2t + 1}}$$

Applying Euler's method gives us the following:

$$\frac{u_{k+1} - u_k}{h} = -u_k{}^3 \rightarrow u_{k+1} = -u_k\left(hu_k{}^2 - 1\right) \tag{2}$$

LET'S SEE IF OUR NSFD UNITY APPROXIMATION IS BETTER

Cubic Average

$$\frac{u_{k+1} - u_k}{h} = -u_k^3$$

Cubic Average

$$\frac{u_{k+1} - u_k}{h} = -u_k^3 \cdot \color{red}{1}$$

Cubic Average

$$\frac{u_{k+1} - u_k}{h} = -u_k^3 \cdot \frac{\left(u_{k+1}^3 + u_k^3\right)}{2u_k^3}$$

Cubic Average

$$\frac{u_{k+1} - u_k}{h} = -u_k^3 \cdot \frac{\left(u_{k+1}^3 + u_k^3\right)}{2u_k^3}$$

Cubic Average

$$\frac{u_{k+1} - u_k}{h} = -u_k^3 \cdot \frac{\left(u_{k+1}^3 + u_k^3\right)}{2u_k^3}$$

Cubic Average

$$\frac{u_{k+1} - u_k}{h} = -\frac{1}{2} \cdot \left(u_{k+1}^3 + u_k^3\right)$$

WolframAlpha®

WolframAlpha®

$$u_{k+1} = \frac{\sqrt[3]{-27h^3u_k^3 + 54h^2u_k + \sqrt{864h^3 + \left(54h^2u_k - 27h^3u_k^3\right)^2}}}{3\sqrt[3]{2}h} - \frac{2\sqrt[3]{2}}{\sqrt[3]{-27h^3u_k^3 + 54h^2u_k + \sqrt{864h^3 + \left(54h^2u_k - 27h^3u_k^3\right)^2}}}$$

ERROR ANALYSIS: SHOWING HOW OUR METHOD IS BETTER

Error Analysis: Cubic Average

k	h				
$k = 1$	$\frac{1}{2^2}$				
$k = 2$	$\frac{1}{2^3}$				
$k = 3$	$\frac{1}{2^4}$				
$k = 4$	$\frac{1}{2^5}$				
$k = 5$	$\frac{1}{2^6}$				

Error Analysis: Cubic Average

k	Δt_k				
$k = 1$	$\frac{1}{2^2}$				
$k = 2$	$\frac{1}{2^3}$				
$k = 3$	$\frac{1}{2^4}$				
$k = 4$	$\frac{1}{2^5}$				
$k = 5$	$\frac{1}{2^6}$				

Error Analysis: Cubic Average

k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$			
$k = 1$	$\frac{1}{2^2}$	$\sim$			
$k = 2$	$\frac{1}{2^3}$				
$k = 3$	$\frac{1}{2^4}$				
$k = 4$	$\frac{1}{2^5}$				
$k = 5$	$\frac{1}{2^6}$				

Error Analysis: Cubic Average

k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$			
$k = 1$	$\frac{1}{2^2}$	$\sim$			
$k = 2$	$\frac{1}{2^3}$	$\frac{1/2^3}{1/2^2}$			
$k = 3$	$\frac{1}{2^4}$				
$k = 4$	$\frac{1}{2^5}$				
$k = 5$	$\frac{1}{2^6}$				

Error Analysis: Cubic Average

k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$			
$k = 1$	$\frac{1}{2^2}$	$\sim$			
$k = 2$	$\frac{1}{2^3}$	$\frac{1/2^3}{1/2^2}$			
$k = 3$	$\frac{1}{2^4}$				
$k = 4$	$\frac{1}{2^5}$				
$k = 5$	$\frac{1}{2^6}$				

Error Analysis: Cubic Average

k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$			
$k = 1$	$\frac{1}{2^2}$	$\sim$			
$k = 2$	$\frac{1}{2^3}$	$\frac{1}{2}$			
$k = 3$	$\frac{1}{2^4}$	$\frac{1}{2}$			
$k = 4$	$\frac{1}{2^5}$	$\frac{1}{2}$			
$k = 5$	$\frac{1}{2^6}$	$\frac{1}{2}$			

Error Analysis: Cubic Average

k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$	l^∞ Error		
$k = 1$	$\frac{1}{2^2}$	$\sim$	$7.64464041\text{e} - 03$		
$k = 2$	$\frac{1}{2^3}$	$\frac{1}{2}$	$1.83461895\text{e} - 03$		
$k = 3$	$\frac{1}{2^4}$	$\frac{1}{2}$	$4.55164060\text{e} - 04$		
$k = 4$	$\frac{1}{2^5}$	$\frac{1}{2}$	$1.13543772\text{e} - 04$		
$k = 5$	$\frac{1}{2^6}$	$\frac{1}{2}$	$2.83715248\text{e} - 05$		

Error Analysis: Cubic Average

k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$	l^∞ Error	Ratio of Errors	
$k = 1$	$\frac{1}{2^2}$	~	7.64464041e − 03	~	
$k = 2$	$\frac{1}{2^3}$	$\frac{1}{2}$	1.83461895e − 03		
$k = 3$	$\frac{1}{2^4}$	$\frac{1}{2}$	4.55164060e − 04		
$k = 4$	$\frac{1}{2^5}$	$\frac{1}{2}$	1.13543772e − 04		
$k = 5$	$\frac{1}{2^6}$	$\frac{1}{2}$	2.83715248e − 05		

Error Analysis: Cubic Average

k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$	l^∞ Error	Ratio of Errors	
$k = 1$	$\frac{1}{2^2}$	$\sim$	$7.64464041\mathrm{e} - 03$	$\sim$	
$k = 2$	$\frac{1}{2^3}$	$\frac{1}{2}$	$1.83461895\mathrm{e} - 03$	$\frac{1.83461895\mathrm{e} - 03}{7.64464041\mathrm{e} - 03}$	
$k = 3$	$\frac{1}{2^4}$	$\frac{1}{2}$	$4.55164060\mathrm{e} - 04$		
$k = 4$	$\frac{1}{2^5}$	$\frac{1}{2}$	$1.13543772\mathrm{e} - 04$		
$k = 5$	$\frac{1}{2^6}$	$\frac{1}{2}$	$2.83715248\mathrm{e} - 05$		

Error Analysis: Cubic Average

k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$	l^∞ Error	Ratio of Errors	
$k = 1$	$\frac{1}{2^2}$	~	7.64464041e − 03	~	
$k = 2$	$\frac{1}{2^3}$	$\frac{1}{2}$	1.83461895e − 03	$\frac{1.83461895\mathrm{e}-03}{7.64464041\mathrm{e}-03}$	
$k = 3$	$\frac{1}{2^4}$	$\frac{1}{2}$	4.55164060e − 04		
$k = 4$	$\frac{1}{2^5}$	$\frac{1}{2}$	1.13543772e − 04		
$k = 5$	$\frac{1}{2^6}$	$\frac{1}{2}$	2.83715248e − 05		

Error Analysis: Cubic Average

k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$	l^∞ Error	Ratio of Errors	
$k = 1$	$\frac{1}{2^2}$	$\sim$	$7.64464041\text{e} - 03$	$\sim$	
$k = 2$	$\frac{1}{2^3}$	$\frac{1}{2}$	$1.83461895\text{e} - 03$	0.2310	
$k = 3$	$\frac{1}{2^4}$	$\frac{1}{2}$	$4.55164060\text{e} - 04$	0.2481	
$k = 4$	$\frac{1}{2^5}$	$\frac{1}{2}$	$1.13543772\text{e} - 04$	0.2495	
$k = 5$	$\frac{1}{2^6}$	$\frac{1}{2}$	$2.83715248\text{e} - 05$	0.2499	

Error Analysis: Cubic Average

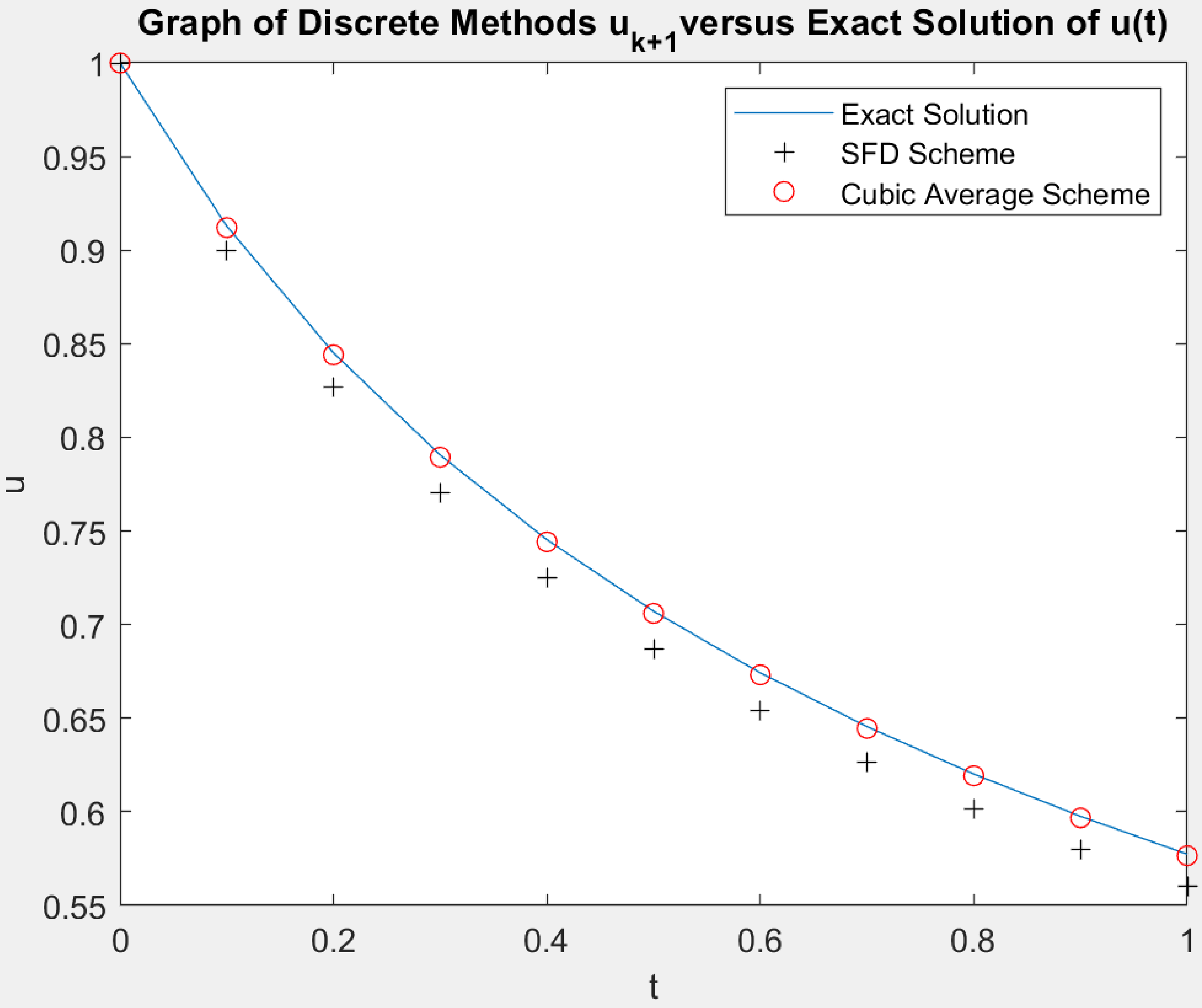
k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$	l^∞ Error	Ratio of Errors	Order
$k = 1$	$\frac{1}{2^2}$	~	$7.64464041e - 03$	~	~
$k = 2$	$\frac{1}{2^3}$	$\frac{1}{2}$	$1.83461895e - 03$	0.2310	$\frac{\log_2 0.2310}{\log_2 0.5}$
$k = 3$	$\frac{1}{2^4}$	$\frac{1}{2}$	$4.55164060e - 04$	0.2481	
$k = 4$	$\frac{1}{2^5}$	$\frac{1}{2}$	$1.13543772e - 04$	0.2495	
$k = 5$	$\frac{1}{2^6}$	$\frac{1}{2}$	$2.83715248e - 05$	0.2499	

Error Analysis: Cubic Average

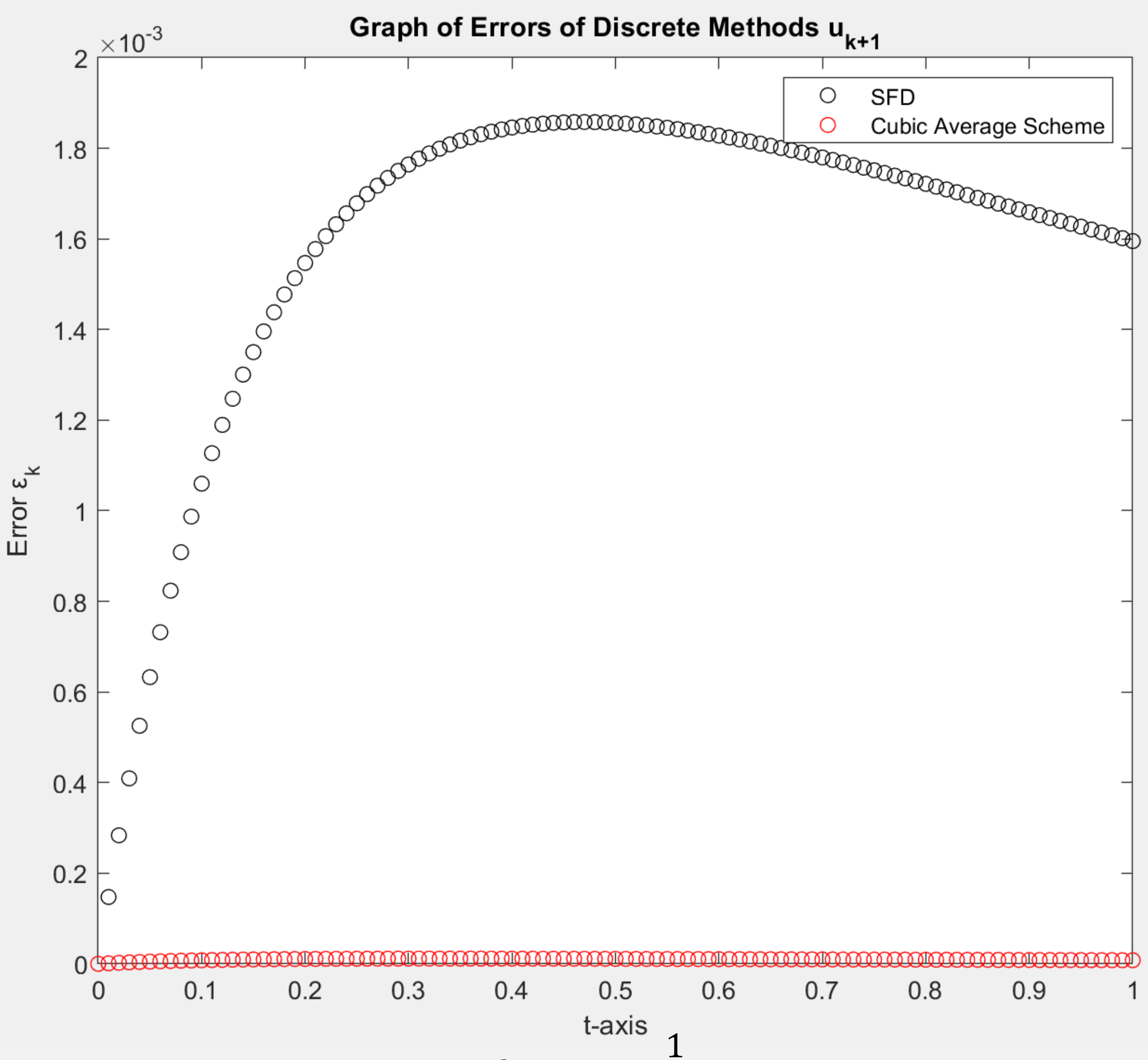
k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$	l^∞ Error	Ratio of Errors	Order
$k=1$	$\frac{1}{2^2}$	~	7.64464041e − 03	~	~
$k=2$	$\frac{1}{2^3}$	$\frac{1}{2}$	1.83461895e − 03	0.2310	$\frac{\log_2 0.2310}{\log_2 0.5}$
$k=3$	$\frac{1}{2^4}$	$\frac{1}{2}$	4.55164060e − 04	0.2481	
$k=4$	$\frac{1}{2^5}$	$\frac{1}{2}$	1.13543772e − 04	0.2495	
$k=5$	$\frac{1}{2^6}$	$\frac{1}{2}$	2.83715248e − 05	0.2499	

Error Analysis: Cubic Average

k	Δt_k	$\frac{\Delta t_k}{\Delta t_{k-1}}$	l^∞ Error	Ratio of Errors	Order
$k = 1$	$\frac{1}{2^2}$	~	7.64464041e − 03	~	~
$k = 2$	$\frac{1}{2^3}$	$\frac{1}{2}$	1.83461895e − 03	0.2310	2.1140
$k = 3$	$\frac{1}{2^4}$	$\frac{1}{2}$	4.55164060e − 04	0.2481	2.0110
$k = 4$	$\frac{1}{2^5}$	$\frac{1}{2}$	1.13543772e − 04	0.2495	2.0029
$k = 5$	$\frac{1}{2^6}$	$\frac{1}{2}$	2.83715248e − 05	0.2499	2.0006



Graphical Representation of Cubic Average Scheme & SFD Scheme



$$h = \Delta t = \frac{1}{100}$$

ERROR ANALYSIS: CUBIC AVERAGE SCHEME & SFD SCHEME

BERNOULLI EQUATION

$$u' = -u^3 + u$$

$$u(0) = 0.5$$

BERNOULLI EQUATION

Bernoulli Differential Equations take the form of :

$$u' + p(t)u = q(t)u^n \qquad (3)$$

When $p(t) = -1, q(t) = -1,\ n = 3$

*For all Bernoulli equations there exists an exact NSFD

$$\begin{aligned} u' - u &= -u^3 \\ u' &= -u^3 + u \end{aligned} \qquad (4)$$

Explicit solution to Bernoulli IVP: $u(t) = \dfrac{\frac{u_0}{\sqrt{1-u_0^2}} \cdot e^t}{\sqrt{1 + \left(\frac{u_0}{\sqrt{1-u_0^2}}\right)^2 \cdot e^{2t}}}$ (5)

TWO TERM AVERAGING SCHEME

With the knowledge gained from the cubic decay equation, we found applying an averaging scheme of the same power as the term produced a 2nd order NSFD Scheme,

Recall:

$$\frac{u_{k+1} - u_k}{h} = -u_k^3 \cdot \frac{\left(u_{k+1}^3 + u_k^3\right)}{2u_k^3}$$

As for the Bernoulli equation we chose to use the same logic in hopes of it also producing a 2nd order scheme

Bernoulli Two-Term Averaging

$$\frac{u_{k+1} - u_k}{h} = -u_k^3 + u_k$$

Bernoulli Two-Term Averaging

$$\frac{u_{k+1} - u_k}{h} = -u_k^3 \cdot 1 + u_k \cdot 1$$

Bernoulli Two-Term Averaging

$$\frac{u_{k+1} - u_k}{h} = -u_k^3 \cdot \frac{\left(u_{k+1}^3 + u_k^3\right)}{2u_k^3} + u_k \cdot \frac{\left(u_{k+1} + u_k\right)}{2u_k}$$

$$u_{k+1} = \frac{1}{3\sqrt[3]{2}h}\left(\left(-27h^3u_k^3 + 27h^3u_k + 54h^2u_k + \sqrt{108\left(2-h\right)^3h^3 + \left(-27h^3u_k^3 + 27h^3u_k + 54h^2u_k\right)^2}\right)^{1/3}\right)$$

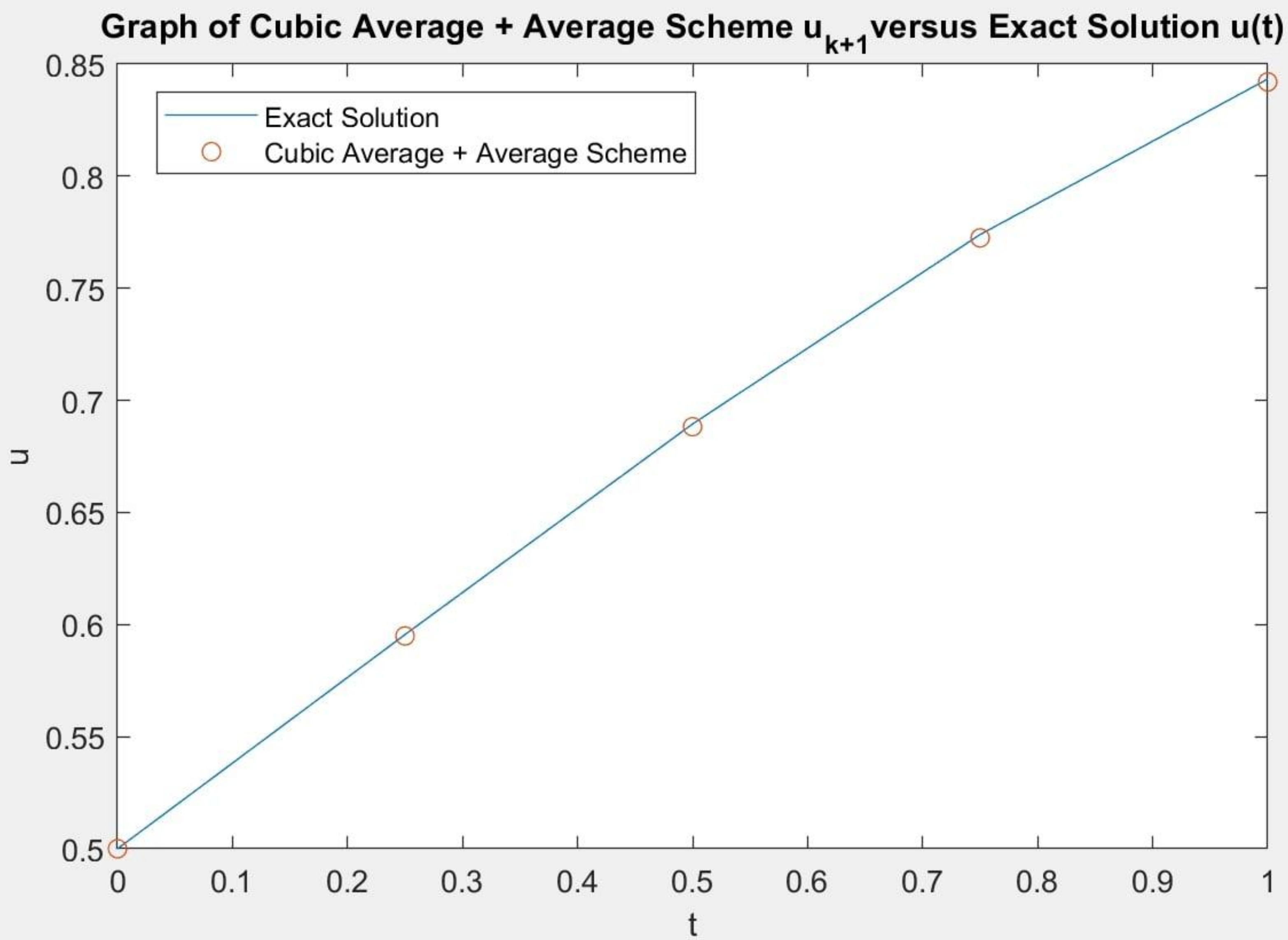
$$- \frac{\sqrt[3]{2}(2-h)}{\left(\left(-27h^3u_k^3 + 27h^3u_k + 54h^2u_k + \sqrt{108\left(2-h\right)^3h^3 + \left(-27h^3u_k^3 + 27h^3u_k + 54h^2u_k\right)^2}\right)^{1/3}\right)}$$

BERNOULLI TWO-TERM AVERAGING SCHEME

Error Analysis: Bernoulli Two-Term Averaging

k	Δt_k	$\frac{\Delta t_{k+1}}{\Delta t_k}$	l^∞ Error	Ratio of Errors	Order
1	$\frac{1}{2^2}$	0.50000	$1.49874077\text{e}-03$	~	~
2	$\frac{1}{2^3}$	0.50000	$3.75058277\text{e}-04$	0.2502	1.9988
3	$\frac{1}{2^4}$	0.50000	$9.37847180\text{e}-05$	0.2500	2.0000
4	$\frac{1}{2^5}$	0.50000	$2.34473894\text{e}-05$	0.2500	2.0000
5	$\frac{1}{2^6}$	0.50000	$5.86466333\text{e}-06$	0.2501	1.9994

GRAPH OF TWO TERM AVERAGING SCHEME



COMBUSTION EQUATION

$$u' = -u^3 + u^2$$

$$u(0) = 2$$

INVESTIGATING THE COMBUSTION EQUATION

Consider the Combustion Equation

$$\frac{du}{dt} = u^2 - u^3, u(t_0) = u_0 > 0.$$

The implicit solution is represented as:

$$\ln(u) - \ln(u-1) + \frac{1}{u} = t + C \quad (6)$$

Unfortunately, an explicit solution of the differential equation does not exist. A built-in function in MATLAB called ODE78 which implements a 7th order Runge-Kutta method will be used for our comparisons, along with using the initial condition $u(0) = 2$.

Applying Euler's Method gives us the following:

$$\frac{u_{k+1} - u_k}{h} = {u_k}^2 - {u_k}^3 \ \rightarrow \ u_{k+1} = h\left({u_k}^2 - {u_k}^3\right) + u_k \quad (7)$$

Cubic Average + Square Average

$$\frac{u_{k+1} - u_k}{h} = -u_k^3 + u_k^2$$

Cubic Average + Square Average

$$\frac{u_{k+1} - u_k}{h} = -u_k^3 \cdot 1 + u_k^2 \cdot 1$$

Cubic Average + Square Average

$$\frac{u_{k+1} - u_k}{h} = -u_k^3 \cdot \frac{\left(u_{k+1}^3 + u_k^3\right)}{2u_k^3} + u_k^2 \cdot \frac{\left(u_{k+1}^2 + u_k^2\right)}{2u_k^2}$$

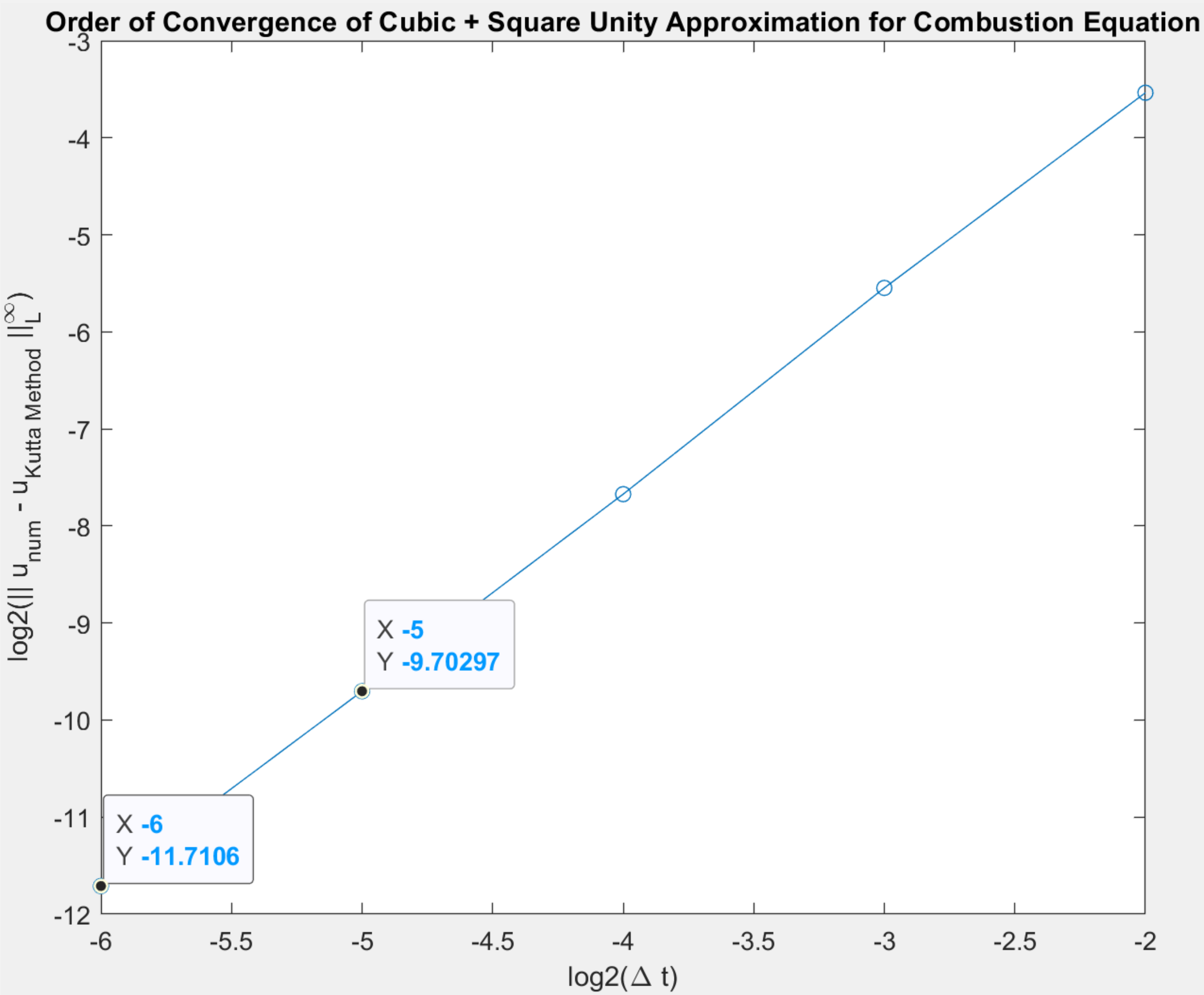
Cubic Average + Square Average

$$u_{k+1} = \frac{\sqrt[3]{2}(6h-h^2)}{3h\left(-27h^3u_k^3+27h^3u_k^2+2h^3+54h^2u_k-18h^2+\sqrt{\left(4(6h-h^2)^3+\left(-27h^3u_k^3+27h^3u_k^2+2h^3+54h^2u_k-18h^2\right)^2\right)}\right)^{\frac{1}{3}}}$$

$$-\frac{1}{3\sqrt[3]{2}h}\left(\left(-27h^3u_k^3+27h^3u_k^2+2h^3+54h^2u_k-18h^2+\sqrt{(4(6h-h^2)^3+\left(-27h^3u_k^3+27h^3u_k^2+2h^3+54h^2u_k-18h^2\right)^2}\right)^{\frac{1}{3}}\right)+\frac{1}{3}$$

Error Analysis: Cubic Average + Square Average

k	Δt_k	Δt_k Ratio	l^∞ Error	Ratio of Errors	Order
1	$\frac{1}{2^2}$	~	$8.62138310\mathrm{e}-02$	~	~
2	$\frac{1}{2^3}$	$\frac{1}{2}$	$2.13858740\mathrm{e}-02$	0.2481	2.0110
3	$\frac{1}{2^4}$	$\frac{1}{2}$	$4.90232643\mathrm{e}-03$	0.2292	2.1253
4	$\frac{1}{2^5}$	$\frac{1}{2}$	$1.19981796\mathrm{e}-03$	0.2447	2.0309
5	$\frac{1}{2^6}$	$\frac{1}{2}$	$2.98363181\mathrm{e}-04$	0.2487	2.0075



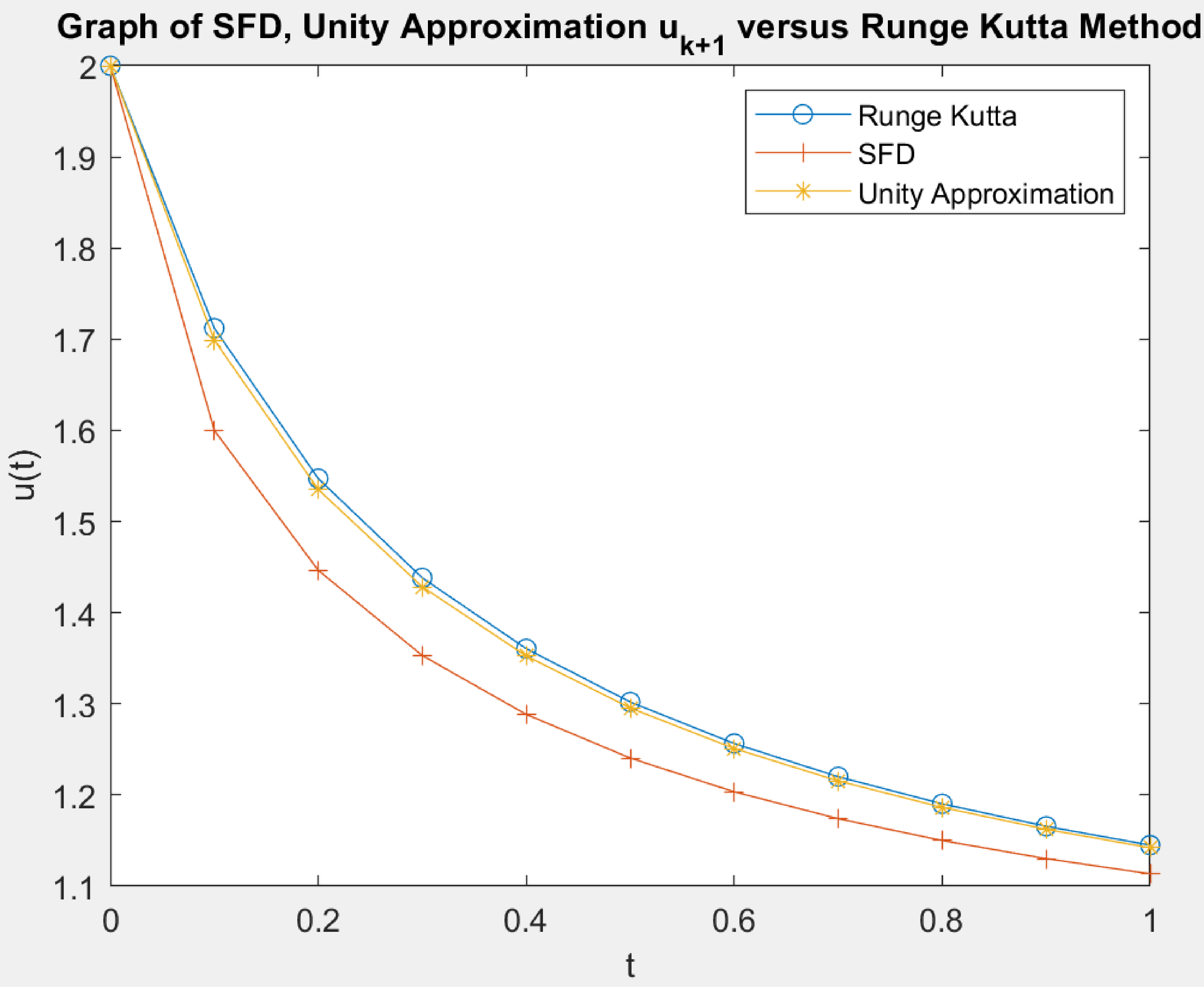
$$\frac{(-9.70297)-(-11.7106)}{(-5)-(-6)} \approx 2.00763 \approx 2$$

Order of Convergence of Unity Approximation

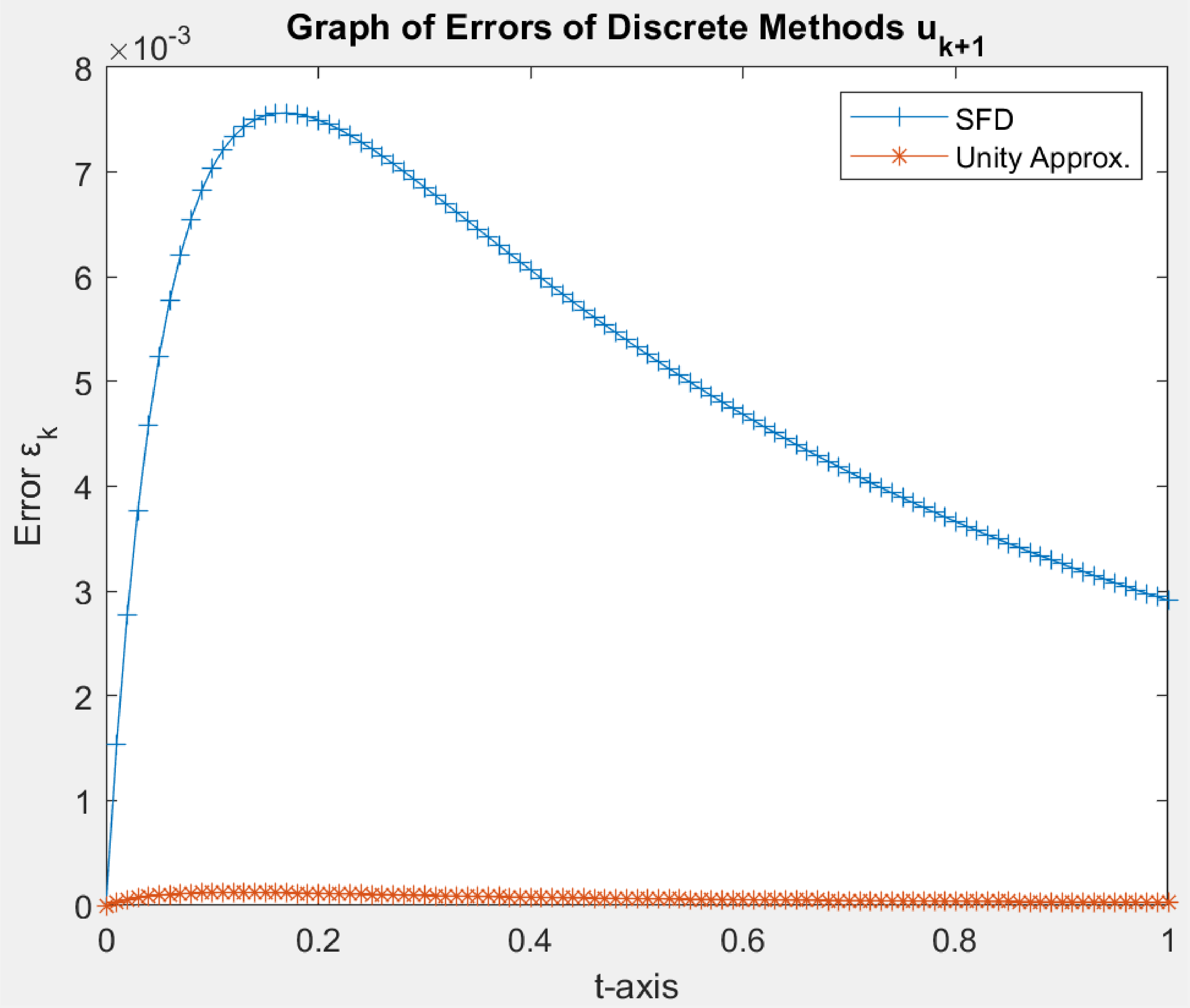
Definition of Order

Error $\approx C\Delta t^{p}$**, where** p **is the order**

$$\log_2\left(\frac{Error_2}{Error_1}\right) \approx A + p\log_2\left(\frac{\Delta t_2}{\Delta t_1}\right)$$



Graphical Representation of Cubic + Square Average Scheme
(Unity Approximation)
& SFD Scheme

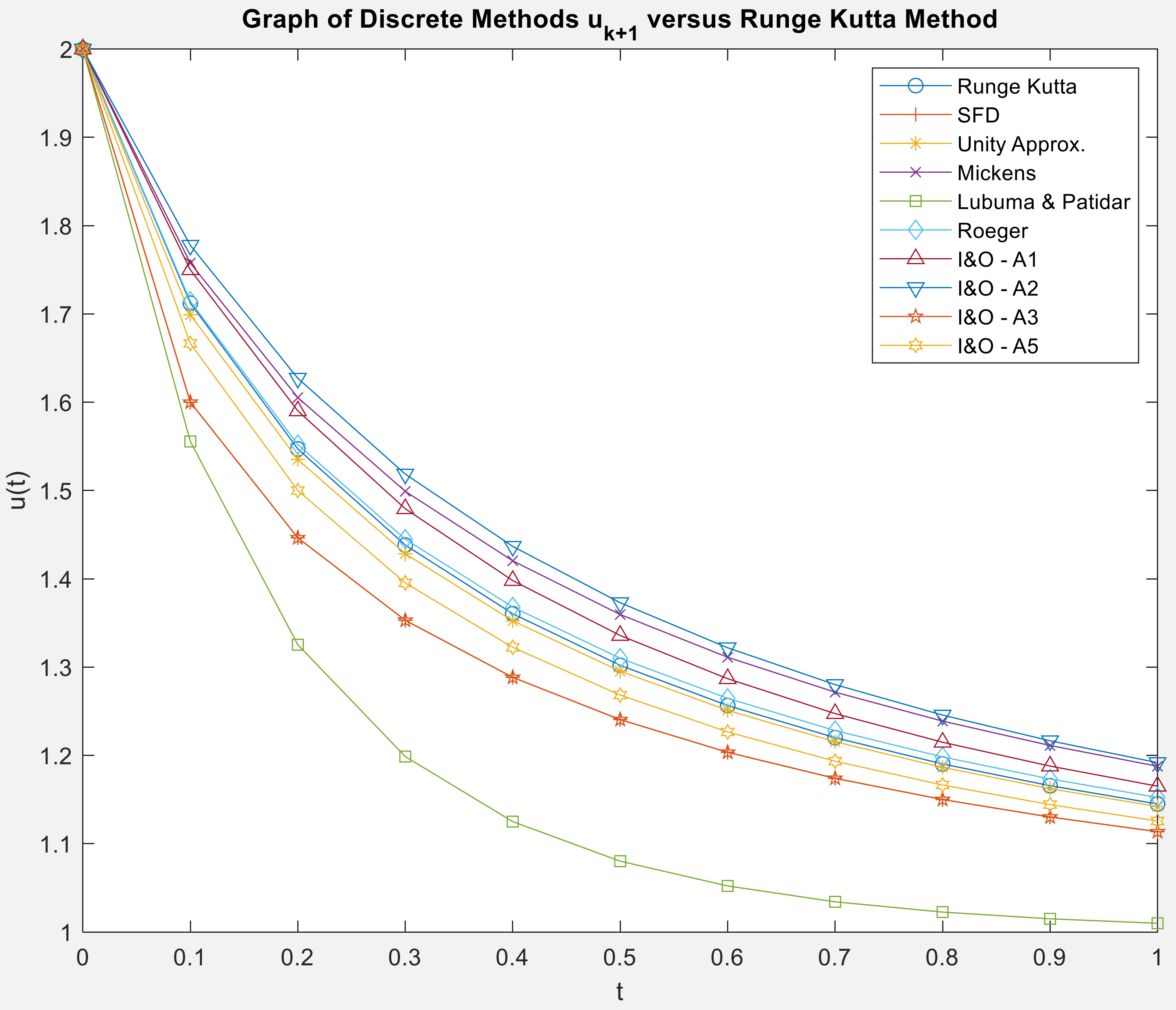


$$h = \Delta t = \frac{1}{100}$$

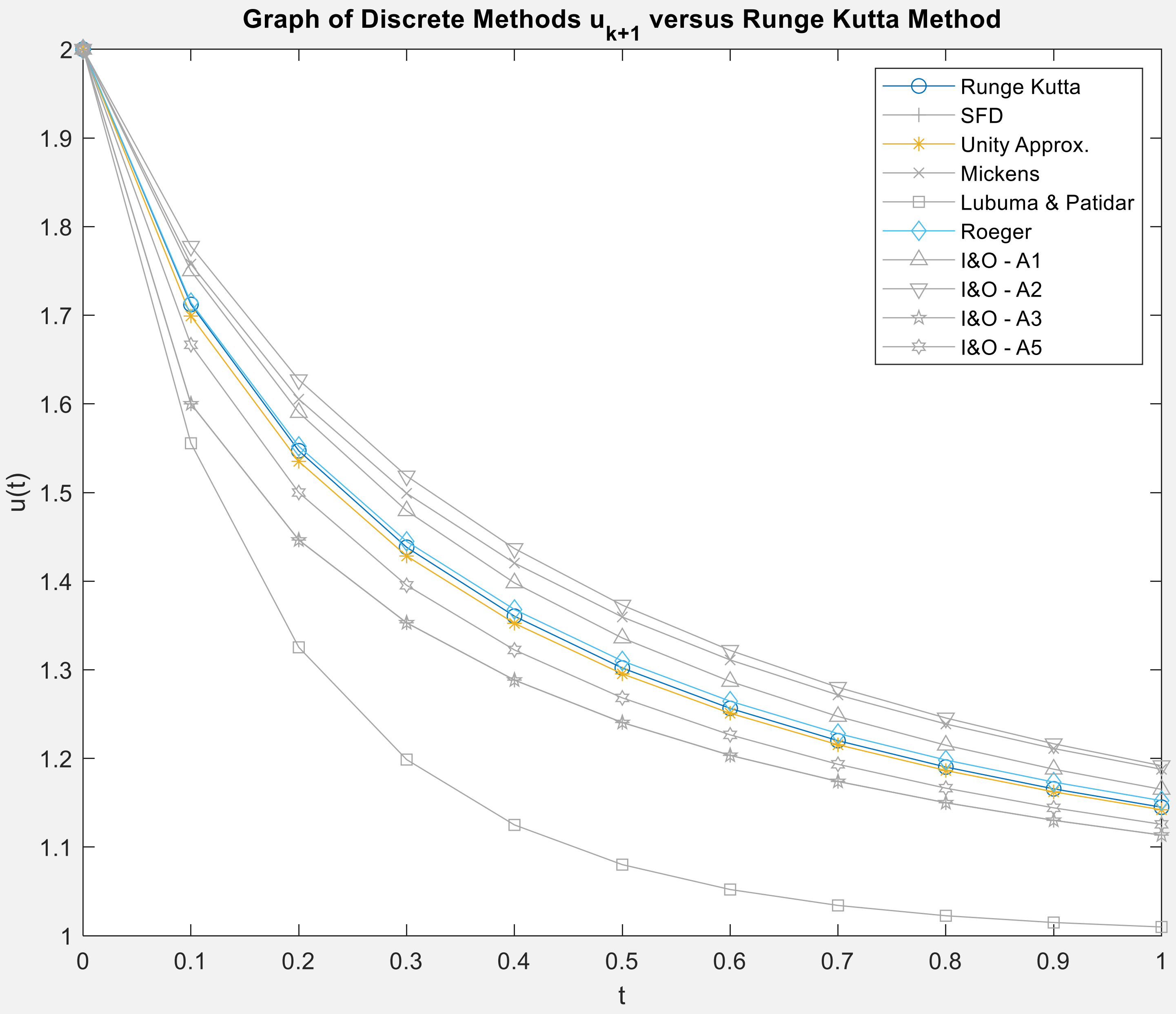
Graphical Representation of Errors of Numerical Schemes against Runge-Kutta Method

WORK FROM OTHER RESEARCHERS

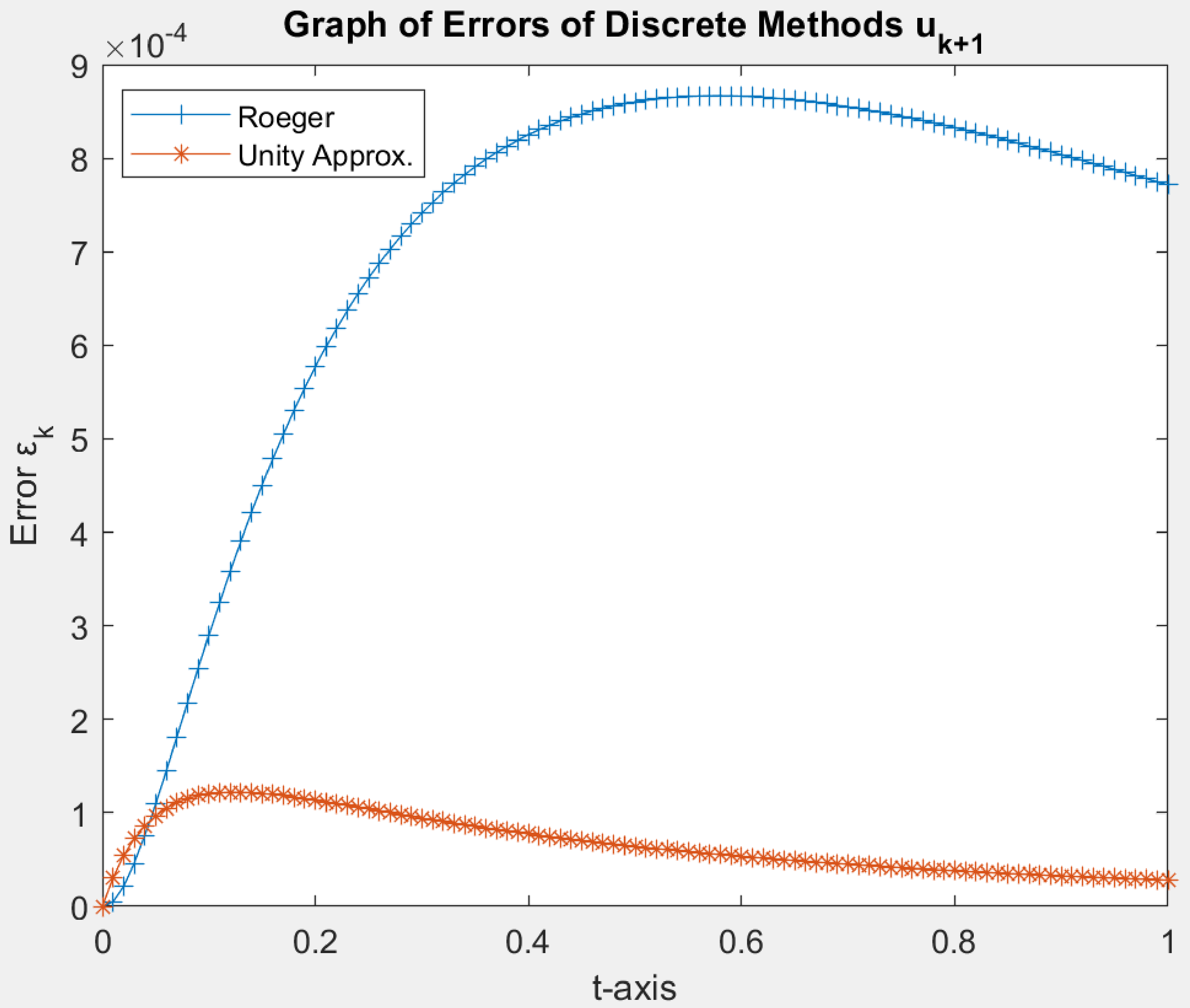
Researcher	Non-Standard Finite Difference (NSFD) Scheme
Mickens, R. E.	$\frac{u_{k+1} - u_k}{1 - e^{-h}} = 2u_k^2 - u_{k+1}u_k - u_{k+1}u_k^2$
Lubuma, M.-S. & Patidar, K. C.	$\frac{u_{k+1}-u_k}{\phi} = u_k + (u_k^2 - u_k^3)\left(\frac{\varepsilon^{-1}\lambda\phi}{1+\varepsilon^{-1}\lambda\phi(\alpha-1)u_k+\varepsilon^{-1}\lambda\phi(1-\beta)u_k^2}\right)$
Roeger, L.-I.W.	$\frac{u_{k+1} - u_k}{\phi} = \frac{(1 + \alpha\phi u_k + \phi u_k)u_k}{1 + \alpha\phi + \phi u_k^2}$
Ibijola, E.A. & Obayomi, A.A. (A1)	$\frac{u_{k+1} - u_k}{h} = u_k^2(1 - a - bu_k) + u_{k+1}(au_k - (1 - b)u_k^2)$
Ibijola, E.A. & Obayomi, A.A. (A2)	$\frac{u_{k+1} - u_k}{h} = u_k^2(a - bu_k) + u_{k+1}((1 - a)u_k - (1 - b)u_k^2)$
Ibijola, E.A. & Obayomi, A.A. (A3)	$\frac{u_{k+1} - u_k}{e^h - 1} = u_k^2 - u_k^3$
Ibijola, E.A. & Obayomi, A.A. (A5)	$\frac{u_{k+1} - u_k}{h} = u_k u_{k+1} - u_k^2 u_k$



Graphical Representation of Numerical Schemes for Combustion Equation



Graphical Representation of Numerical Schemes for Combustion Equation



$$h = \Delta t = \frac{1}{100}$$

Graphical Representation of Errors of Numerical Schemes against Runge-Kutta Method

BRATU IVP

$$u'' = 2e^{u}$$

$$u(0) = 0, u'(0) = 0$$

SUMMARY OF RESEARCH ON BRATU IVP

The Bratu IVP $u'' = 2e^u \; u(0) = 0, u'(0) = 0$ has a known true solution of $u(t) = -2\ln(\cos(t))$

Scalar Standard Finite Difference Scheme of the Bratu IVP is as follows

$$u_{k+1} = 2h^2 e^{u_k} + 2u_k - u_{k-1} \tag{8}$$

2D Standard Finite Difference Scheme of the Bratu IVP is as follows

$$\begin{aligned} u' &= v, & u_0 &= 0 \\ v' &= 2e^u, & v_0 &= 0 \\ u_{k+1} &= v_k h + u_k \\ v_{k+1} &= 2e^{u_k} h + v_k \end{aligned} \tag{9}$$

Summary:

- This is a second order non linear initial value problem
- We are looking for second order NSFD schemes
- We investigated many schemes in both the scalar and 2D, and we found 2 second order 2D systems and only first order for the scalar version

CONCLUSION

- Standard and Non-Standard Finite Difference Schemes are utilized to numerically approximate solutions of nonlinear first-order ODE's $u' = -u^3 + F(u)$
- We used Unity Approximations to find NSFD Schemes that can produce higher-order schemes
- Our NSFD schemes are comparable to other NSFDs that rely on more complicated approaches to approximate differential equations
- We were able to create second-order numerical schemes for the Cubic Decay equation, the Bernoulli equation, and the Combustion equation
- We constructed an exact NSFD unity approximation scheme for the Cubic Decay equation
- We found two second-order numerical schemes for the 2D system version of the Bratu IVP

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THANK YOU FOR ATTENDING!

QUESTIONS?