About the state variable diffusion of a Cheyette model in the context of swaption pricing

Alexis HOUSSARD

October 15, 2021

Summary

Introduction

Interest rate modelling represents a central part of financial mathematics in the sens that it is needed in the pricing step of many financial products. Indeed it is crucial to question which interest rate model to consider in order to evaluate the right way the price of a product whose payoff depends on the interest rate curve at one or several instants. The main purpose is therefore to simplify the valuation method from a mathematical and practical point of view without altering the capacity to reproduce observable market phenomena such as skews and smiles of the implied volatility curve of benchmark instruments.

This document deals with a study of the so-called "Cheyette" interest rate model, a specification of the HJM framework which provides a continuous forward curve with a Markovian short rate. The main idea of this work is to understand the construction, the parameterization and the capacity to reproduce market's prices of this model. In order to highlight the properties of this forward curve characterization, one will carry out all the applications in one of the most common contexts of interest rate modelling: swaption pricing. In reason of their important liquidity and the form of their payoff, swaptions proved to be good candidates to calibrate an interest rate model. Our aim here is to introduce briefly the theory of the generic model and to specify its uni-dimensional version with a parametric local volatility of the short rate. Afterwards, some numerical tests will be provided to find out the best way to calibrate the Cheyette model on implied volatility and to price swaptions. For this purpose, our work will consist in three essentials steps: a classical swaptions valuation, an approximated swaptions pricing and the calibration of the Cheyette model. This study represents a base which can be extended with a non-parametric local or stochastic volatility function of the short rate.

1 A brief presentation of the uni-dimensional Cheyette model

First of all, one will pay attention to the construction of the Cheyette model. It deals with a term structure model which is based on the Heath-Jarrow-Morton approach. One will explain succinctly in this section the reasoning of this approach.

1.1 The HJM framework

Knowing the initial forward curve $T \to f(0,T)$, one aims to describe the entire forward curve $t \to f(t,T)$ in a continuous way for all the horizons T. To begin with, one will write the following generic dynamic of the forward rate:

$$df(t,T) = \mu_f(t,T)\,dt + \sigma_f(t,T)^\top\, dW_t \qquad , t \leq T$$

where W is a Brownian motion under the historical probability measure $\mathbb{P}$.

To ensure the absence of arbitrage, it is sufficient to assume the existence of a risk neutral probability $\mathbb{Q}$, a Brownian motion $W^{\mathbb{Q}}$ on $\mathbb{Q}$ and a volatility σ_P differentiable with respect to T so that

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt - \sigma_P(t,T)^\top dW_t^{\mathbb{Q}}.$$

One recall the link between forward rates and the zero coupon bond $f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}$. After an application of the Itô's lemma and some considerations, it yields:

$$\boxed{\begin{aligned} df(t,T) &= \sigma_f(t,T)^\top \left(\int_t^T \sigma_f(t,u)\,du \right) dt + \sigma_f(t,T)^\top\, dW_t^{\mathbb{Q}} \\ r(t) &= f(t,t) \end{aligned}}$$

1.2 The Cheyette specification

Despite the generality of the model, it requires in some cases, for example for financial instruments with long maturities and regular maturities, to simulate a dynamic for the forward rate at each maturity. Similarly, we note that the previously induced forward rate modelling is path dependent, which can complicate the simulations. Cheyette evokes a space of infinite dimensional dynamics.

To overcome these problems, O. Cheyette proposes (in [1]) a redesign of the HJM framework that emphasizes the Markovian character of the short rate. In general, working with a short rate which is expressed as a deterministic function of a Markov process allows to facilitate the calculation of conditional expectation and to circumvent the problems linked to path dependence. Cheyette's specification lies in the use of a forward volatility structure with separable variables. In other terms,

$$\boxed{\sigma_f(t,T,\omega) = g(t,\omega)\,h(T) \qquad \begin{aligned} g &: \mathbb{R}^+ \times \Omega \to \mathbb{R} \\ h &: \mathbb{R}^+ \to \mathbb{R}^+ \end{aligned}}$$

This model is called "One-factor quasi Gaussian". On the one hand, the model is mono-factor because one considers a one dimension Brownian motion to drive the forward rate. On the other hand, it is said to be quasi Gaussian because of the stochastic nature of g. If it had been deterministic, the short rate would have been a Gaussian random variable.

These considerations make the HJM model more tractable. In particular, this implies the following important property:

$$\boxed{\begin{aligned} &\forall 0 \leq t \leq T, \quad r(t) = f(0,t) + x(t) \\ &\text{And} \quad P(t,T,x(t),y(t)) = \tfrac{P(0,T)}{P(0,t)}\, e^{-G(t,T)\,x(t) - \frac{1}{2}G(t,T)^2\,y(t)} \\ &\text{Where} \quad \begin{cases} dx(t) = (y(t) - \chi(t)x(t))\,dt \;+\; \sigma_r(t,\omega)\,dW_t^{\mathbb{Q}}, \\ dy(t) = (\sigma_r(t,\omega))^2 - 2\chi(t)y(t))\,dt, \\ x(0) = y(0) = 0 \end{cases} \end{aligned}} \tag{1}$$

One notes $\chi(t) = -\frac{h'(t)}{h(t)}$ and $G(t,T) = \frac{1}{h(t)} \int_t^T h(s)\,ds$.

This proposition allows us to conclude that, in the Cheyette model, the forward curve and, even more so, the zero-coupon curve, are entirely characterised by two state variables x and y. The latter will be at the heart of our study since they control the price of swaptions through that of zero-coupon bonds. One will be particularly interested in the effect of the volatility structure $\sigma_r(t) = \sigma_f(t,t)$ on the implied volatilities of european swaptions.

In our study, one will focus on a specific form of the volatility structure, a local volatility which is linear in the state variable x, that's:

$$\sigma_r(t,x,y) = \lambda(t)\big(a(t) + b(t)\,x\big).$$

With this choice, the calibration will be made easier since it will only involve two functions, λa and λb. Moreover one will see that this form simplifies the equations and confers a freedom that will lead to a number of useful approximations. Note that y plays the role of an adjustment variable, that's why it does not seem aberrant to consider volatility as a function of x only.

2 Swaption Pricing under the risk neutral measure

We propose to develop in this section the most classical method to price a swaption, namely the Monte Carlo approximation under the risk neutral measure.

2.1 Swaption price formula

One considers in this paper a swaption on an underlying swap of strike K and maturity T_0 which pays at $T_1 < ... < T_N$. His price V_0 writes:

$$\begin{aligned} V_0 &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^{T_0} r(s)\,\mathrm{d}s} Swap(T_0, x(T_0), y(T_0))^+\right] \\ &= P(0,T_0)\,\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^{T_0} x(s)\,\mathrm{d}s}\Big(1 - P(T_0,T_N,x(T_0),y(T_0))\right. \\ &\left. -K\textstyle\sum_{n=0}^{N-1}\tau_n P(T_0,T_{n+1},x(T_0),y(T_0))\Big)^+\right] \end{aligned} \quad (2)$$

Our aim is to estimate the value of this expectation by a Monte Carlo estimator $\hat{V}_0$ computed with M simulations $(S_m)_{m=1,M}$ so that:

$$\begin{cases} \hat{V}_0 = \frac{1}{M}\sum_{m=1}^{M} S_m \\ \forall m,\ S_m = e^{I_m}\Big(P(0,T_0) - P(0,T_0)P(T_0,T_N,x_m,y_m) \\ -K\,P(0,T_0)A(T_0,x_m,y_m)\Big)^+ \end{cases}$$

x_m, y_m, I_m is the m-th simulation of the random variables $x(T_0), y(T_0), I(T_0) = -\int_0^{T_0} x(s)\,\mathrm{d}s$ resulting from a discretization scheme of the SDE in (1) such as the Euler scheme.
The term noted A is the function called "**Annuity**" which corresponds to the strictly positive quantity

$$A(t,x,y) = \sum_{n=0}^{N-1}\tau_n \frac{P(0,T_{n+1})}{P(0,t)} e^{-G(t,T_{n+1})x - \frac{1}{2}G(t,T_{n+1})^2 y}$$

$\forall n \in [|0, N-1|], \quad \tau_n = T_{n+1} - T_n.$

2.2 Numerical methods

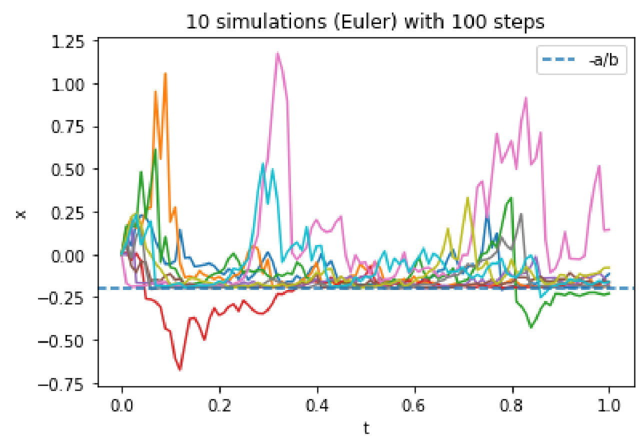
First of all, let $\chi(t) = \chi \in \mathbb{R}$ for the simplicity of interpretation.
The classical scheme is the following Euler scheme:

$$\begin{cases} x_{t_{i+1}} &= x_{t_i} + (y_{t_i} - \chi x_{t_i})\,\Delta_i + \lambda_{t_i}(a_{t_i} + b_{t_i}x_{t_i})\sqrt{\Delta_i}Z_i \quad ,Z_i \sim \mathcal{N}(0,1) \\ y_{t_{i+1}} &= y_{t_i} + (\lambda_{t_i}^2(a_{t_i} + b_{t_i}x_{t_i})^2 - 2\chi y_{t_i})\,\Delta_i \\ I_{t_{i+1}} &= I_{t_i} - x_{t_i}\,\Delta_i \end{cases}$$

The classical Euler scheme involves Gaussian increments, that's why the approximated diffusion of x can take its values in $]-\infty,+\infty[$. What's more, the volatility of x under its linear form is not necessary positive. If one thinks to the case of a CIR process, one knows that the volatility $\sqrt{\lambda(a+bx(t))}$ is always positive under the well known Feller condition. In our case, it is possible to show a similar condition described below:

$$\text{If } \begin{cases} \bullet\ x(0) + \frac{a(0)}{b(0)} \geq 0 \\ \bullet\ t \to \frac{a(t)}{b(t)} e^{\int_0^t \chi(u)\,\mathrm{d}u} \text{ is increasing} \end{cases} \quad \text{then} \quad \forall t \geq 0,\ x(t) + \frac{a(t)}{b(t)} \geq 0$$

This phenomenon is observable on the following graphic simulation with a constant bound.



Proof
To begin with, we will express y under an explicit form. Let $Y_t = \exp\big(2\int_0^t \chi(u)\,\mathrm{d}u\big)y(t)$.
One has

$$\begin{aligned} \mathrm{d}\big(Y_t\big) &= 2\chi(t)Y_t\,\mathrm{d}t + \exp\big(2\textstyle\int_0^t \chi(u)\,\mathrm{d}u\big)\,\mathrm{d}y(t) \\ &+\mathrm{d}\langle \exp\big(2\textstyle\int_0^{\cdot} \chi(u)\,\mathrm{d}u\big), y(\cdot)\rangle_t \\ &= \exp\big(2\textstyle\int_0^t \chi(u)\,\mathrm{d}u\big)\sigma_r(t)^2\,\mathrm{d}t \end{aligned}$$

By integration and because $y(0) = 0$,

$$\begin{aligned} &Y_t = \int_0^t \exp\big(2\int_0^s \chi(u)\,\mathrm{d}u\big)\sigma_r(s)^2\,\mathrm{d}s \\ \iff\ &y(t) = \int_0^t \exp\big(-2\int_s^t \chi(u)\,\mathrm{d}u\big)\sigma_r(s)^2\,\mathrm{d}s \end{aligned}$$

Hence $\forall t \geq 0,\ y(t) \geq 0$.
Consider x with the dynamic

$$\mathrm{d}x(t) = \big(y(t) - \chi(t)x(t)\big)\,\mathrm{d}t + \big(a(t) + b(t)x(t)\big)\,\mathrm{d}W_t^{\mathbb{Q}}$$

Furthermore let $X_t = \exp\Big(\int_0^t \chi(u)\,\mathrm{d}u + \frac{1}{2}\int_0^t b(s)^2\,\mathrm{d}s - \int_0^t b(s)\,\mathrm{d}W_s\Big)$.
By Itô's lemma, it yields

$$\begin{aligned} \mathrm{d}X_t &= X_t\Big[\chi(t)\mathrm{d}t + \tfrac{1}{2}b(t)^2\mathrm{d}t - b(t)\mathrm{d}W_t^{\mathbb{Q}} \\ &+\tfrac{1}{2}\mathrm{d}\langle\textstyle\int_0^{\cdot}\chi(u)\,\mathrm{d}u + \frac{1}{2}\int_0^{\cdot} b(s)^2\,\mathrm{d}s - \int_0^{\cdot} b(s)\,\mathrm{d}W_s\rangle_t\Big] \\ &= X_t\Big[\chi(t)\mathrm{d}t + b(t)^2\mathrm{d}t - b(t)\mathrm{d}W_t^{\mathbb{Q}}\Big] \end{aligned} \quad (3)$$

By integration by parts with (3), we have:

$$\begin{aligned}
\mathrm{d}\Big(X_t\, x(t)\Big) &= x(t)\mathrm{d}X_t + X_t\mathrm{d}x(t) + \mathrm{d}\langle X_., x(\cdot)\rangle_t \\
&= x(t)X_t\Big(\chi(t)\mathrm{d}t + b(t)^2\mathrm{d}t - b(t)\mathrm{d}W_t^{\mathbb{Q}}\Big) \\
&+X_t\Big(\big(y(t) - \chi(t)x(t)\big)\mathrm{d}t + \big(a(t) + b(t)x(t)\big)\ \mathrm{d}W_t^{\mathbb{Q}}\Big) \\
&-b(t)X_t\big(a(t) + b(t)x(t)\big)\ \mathrm{d}t \\
&= X_t y(t)\mathrm{d}t - \tfrac{a(t)}{b(t)}X_t\Big(b(t)^2\mathrm{d}t - b(t)\mathrm{d}W_t^{\mathbb{Q}}\Big) \\
&= X_t y(t)\mathrm{d}t - \tfrac{a(t)}{b(t)}\Big(\mathrm{d}X_t - X_t\chi(t)\mathrm{d}t\Big) \\
&= X_t y(t)\mathrm{d}t - \tfrac{a(t)}{b(t)}\exp\Big(\int_0^t \chi(u)\,\mathrm{d}u\Big)\mathrm{d}\Big(\exp\big(-\int_0^t \chi(u)\,\mathrm{d}u\big)X_t\Big)
\end{aligned}$$

A new integration by parts leads to

$$\begin{aligned}
X_t x(t) &= x(0) + \int_0^t X_u y(u)\mathrm{d}u \\
&-\int_0^t \tfrac{a(s)}{b(s)}\exp\Big(\int_0^s \chi(u)\,\mathrm{d}u\Big)\,\mathrm{d}\Big(\exp\big(-\int_0^s \chi(u)\,\mathrm{d}u\big)\ X_s\Big) \\
&= x(0) + \int_0^t X_u y(u)\mathrm{d}u + \tfrac{a(0)}{b(0)} - \tfrac{a(t)}{b(t)}X_t \\
&+\int_0^t \exp\big(-\int_0^s \chi(u)\,\mathrm{d}u\big)\ X_s\ \mathrm{d}\Big(\tfrac{a(s)}{b(s)}\exp\Big(\int_0^s \chi(u)\,\mathrm{d}u\Big)\Big)
\end{aligned}$$

Hence

$$\begin{aligned}
X_t\Big(x(t) + \tfrac{a(t)}{b(t)}\Big) &= x(0) + \tfrac{a(0)}{b(0)} + \int_0^t X_u y(u)\mathrm{d}u \\
&+\int_0^t \exp\big(-\int_0^s \chi(u)\,\mathrm{d}u\big)X_s\ \mathrm{d}\Big(\tfrac{a(s)}{b(s)}\exp\Big(\int_0^s \chi(u)\,\mathrm{d}u\Big)\Big)
\end{aligned}$$

Or

$$\begin{aligned}
x(t) + \tfrac{a(t)}{b(t)} &= \tfrac{1}{X_t}\Big[x(0) + \tfrac{a(0)}{b(0)} + \int_0^t X_u y(u)\mathrm{d}u \\
&+\int_0^t \exp\big(-\int_0^s \chi(u)\,\mathrm{d}u\big)X_s\ \mathrm{d}\Big(\tfrac{a(s)}{b(s)}\exp\Big(\int_0^s \chi(u)\,\mathrm{d}u\Big)\Big)\Big]
\end{aligned}$$

Conclusion comes directly from the positivity of all terms under the hypothesis.

To produce an approximation scheme of x which satisfies the last property, it is possible to implement the **log-Euler scheme**. This method lies in an application of the Euler scheme on an invertible transformation of x. To do so, one let

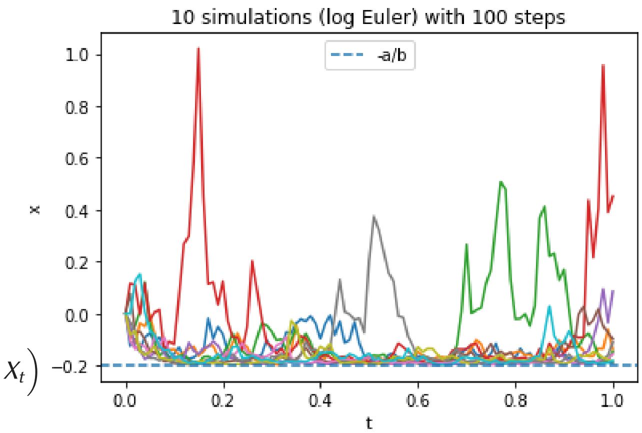
$$x(t) = e^{u(t)} - \frac{a(t)}{b(t)}. \qquad \Longleftrightarrow \quad u(t) = \ln\Big(x(t) + \frac{a(t)}{b(t)}\Big)$$

Finally, by Itô's lemma on u and the inverse transformation, one has:

$$\boxed{\forall i \in [|0, N_{Euler} - 1|],\ \hat{x}_{i+1} = -\frac{a(t_{i+1})}{b(t_{i+1})} + \Big(\hat{x}_i + \frac{a(t_i)}{b(t_i)}\Big)e^{\alpha(t_i,\hat{x}_i)\ \Delta_i + \gamma(t_i,\hat{x}_i)\ \sqrt{\Delta_i}Z_i}}$$

Where $\alpha(t, x(t)) = \frac{b(t)\lambda(t)}{\sigma_r(t,x(t))}\Big[y(t) - \chi x(t) + \Big(\frac{a(t)}{b(t)}\Big)'\Big] - \frac{(b(t)\lambda(t))^2}{2}$.

It is then straightforward to see that $\forall i, \hat{x}_i \geq -\frac{a(t_i)}{b(t_i)}$ mathematically and visually.



In reason of the exponential function computed at each iteration, this scheme provides an approximation which can diverge quickly, that's why we recommend to use a discretization scheme with an artificial chosen bound. By the way, one notes that the coefficients of the diffusion of u are non globally Lipschitz, which explains that the classical conditions for a strong convergence of the scheme are not satisfied here. To finish with, one will generate its diffusions with $N_{Euler} = 250$ steps and approximate swaption prices with $M = 10^5$ simulations.

Alternative scheme

With the aid of the previous proof, it is possible to find a more stable scheme which provides a simulation of x with the asked properties.

Indeed, one will simulate trajectories of x by applying the Euler scheme on the triplet (X,y,U) where

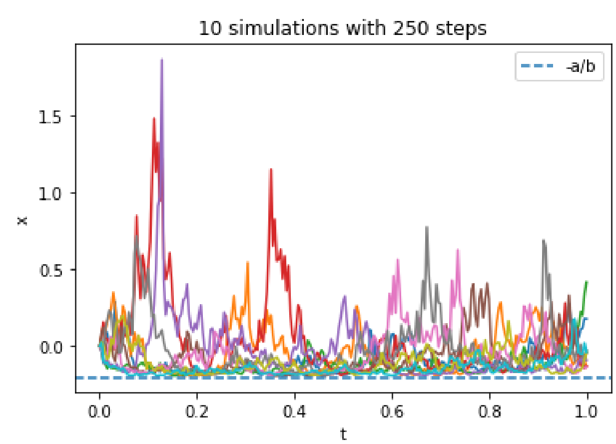
$$\begin{cases} X_t &= \exp\Big(\int_0^t \chi(u)\,\mathrm{d}u + \frac{1}{2}\int_0^t b(s)^2\,\mathrm{d}s - \int_0^t b(s)\,\mathrm{d}W_s\Big) \\ U_t &= X_t\big(x(t) + \frac{a(t)}{b(t)}\big)\end{cases}$$

It is straightforward to see that the Euler scheme on $\ln(X_t)$ writes

$$\ln(X_{t_{i+1}}) = \ln(X_{t_i}) + \big(\chi(t_i) + \frac{1}{2}b(t_i)^2\big)\Delta_i - b(t_i)\sqrt{\Delta_i}Z_i.$$

So the scheme we suggest here comes directly from the penultimate equation of the proof on the diffusion of x. Hence

$$\begin{cases} U_{t_{i+1}} &= U_{t_i} + \Delta_i X_{t_i}\Big[y(t_i) + \big(\frac{a(t_i)}{b(t_i)}\big)' + \frac{a(t_i)}{b(t_i)}\chi(t_i)\Big] \\ X_{t_{i+1}} &= X_{t_i}\exp\Big(\big(\chi(t_i) + \frac{1}{2}b(t_i)^2\big)\Delta_i - b(t_i)\sqrt{\Delta_i}Z_i\Big)\end{cases}$$



3 A displaced log-normal approximation

The Monte Carlo method is a simple and accurate way to compute an option price. However, this method requires a significant computational effort in reason of her high complexity. Therefore it seems unreasonable to use this pricing method in the calibration algorithm. In this section, one will present a pricing technique from [2] with an explicit formula based on a Markovian projection and some approximations.

3.1 Markovian projection

If one introduces the annuity measure $\mathbb{Q}^A$ related to the annuity A as the numeraire asset, one sees that a swaption is Call option on the swap rate

$$S(t,x,y) = \frac{P(t,T_0,x,y) - P(t,T_N,x,y)}{A(t,x,y)}.$$

Indeed,

$$V_0 = A(0,0,0)\mathbb{E}_{\mathbb{Q}^A}\left[\left(S(T_0,x(T_0),y(T_0)) - K\right)^+\right].$$

Now, one has to determine the dynamic of S under the new measure, by Itô's lemma, it yields:

$$\boxed{\mathrm{d}S(t,x(t),y(t)) = \frac{\partial S}{\partial x}(t,x(t),y(t))\,\sigma_r(t,x(t),y(t))\,\mathrm{d}W_t^A}. \quad (4)$$

Where W^A is a Brownian motion under $\mathbb{Q}^A$.

By using the result of [3], the swap rate dynamic can be converted to the form below without changing the value of the expectation.

$$\begin{cases} \mathrm{d}S(t,x(t),y(t)) = \phi(t,S(t,x(t),y(t)))\,\mathrm{d}W_t^A \\ \phi(t,s)^2 = \mathbb{E}_{\mathbb{Q}^A}\left[\left(\frac{\partial S}{\partial x}(t,x(t),y(t))\,\sigma_r(t,x(t),y(t))\right)^2 | S(t,x(t),y(t)) = s\right] \end{cases}$$

Moreover, we aim to approximate the factor y by a deterministic function $\bar{y}$ and express the factor x as a function of the swap rate so that $x(t) = X(t,S(t))$. To access the value of $X(t,s)$, it is possible to use a Taylor formula in the relation $S(t,X(t,s),\bar{y}(t)) = s$ around a deterministic point $\bar{x}(t)$. This gives:

$$S(t,\bar{x}(t),\bar{y}(t)) + \frac{\partial S}{\partial x}(t,\bar{x}(t),\bar{y}(t))\left(X(t,s) - \bar{x}(t)\right) + \frac{1}{2}\frac{\partial^2 S}{\partial x^2}(t,\bar{x}(t),\bar{y}(t))\left(X(t,s) - \bar{x}(t)\right)^2 = s.$$

3.2 Deterministic approximations

Now, one will pay attention to the functions $\bar{x}$ and $\bar{y}$. The idea is to use the expectation $\mathbb{E}_{\mathbb{Q}^A}\left[y(t)\right] = \bar{y}(t)$ as a benchmark point for $y(t)$. With the help of the approximation

$$\mathbb{E}_{\mathbb{Q}^A}\left[\sigma_r(t,x(t),y(t))^2\right] \approx \sigma_r(t,0,0)^2,$$

one has

$$\boxed{\bar{y}(t) = h(t)^2\int_0^t \sigma_r(t,0,0)^2 h(s)^{-2}\,\mathrm{d}s}$$

To estimate the expectation of $x(t)$ under the annuity measure, it is possible to use a second order Taylor formula which leads to the following quantity:

$$\bar{x}(t) = x_0(t) + \frac{1}{2}\frac{\partial^2 X}{\partial s^2}(t,S(0)) \times Var^A[S(t)]$$

With $x_0(t) = X(t,S(0))$. One can take also

$$Var^A[S(t)] \approx \int_0^t \frac{\partial S}{\partial x}(s,0,0)^2\sigma_r(s,0)^2\,\mathrm{d}s.$$

3.3 An explicit pricing formula

This part will be focused on the special case where λ, a, b are constants. With the above in mind, the dynamic of the swap rate can be approached by:

$$\mathrm{d}S(t) \approx \lambda_S(t)\big(b_S(t)S(t) + (1 - b_S(t))S(0)\big)\,\mathrm{d}W^A(t)$$

$$\text{Where}\quad \begin{cases} \lambda_S(t) = \lambda\frac{1}{S(0)}\frac{\partial S}{\partial x}(t,\bar{x}(t),\bar{y}(t))\,(a + b\bar{x}(t)) \\ b_S(t) = \frac{S(0)}{(a+b\bar{x}(t))}\frac{b}{\frac{\partial S}{\partial x}(t,\bar{x}(t),\bar{y}(t))} + \frac{S(0)\frac{\partial^2 S}{\partial x^2}(t,\bar{x}(t),\bar{y}(t))}{\left(\frac{\partial S}{\partial x}(t,\bar{x}(t),\bar{y}(t))\right)^2} \end{cases}$$

It deals with a well-known displaced log-normal dynamic with time dependent parameters. To get rid of this difficulty, it is convenient to use time-averaging methods. The averaging method suggested in [2] gives:

$$\bar{\lambda}_S = \left(\frac{1}{T_0}\int_0^{T_0}\lambda_S(t)^2 dt\right)^{1/2}$$

$$\bar{b}_S = \int_0^{T_0} b_S(t)w_S(t)dt$$

$$w_S(t) = \frac{\lambda_S(t)^2\int_0^t \lambda_S(s)^2 ds}{\int_0^{T_0}\left(\lambda_S(u)^2\int_0^u \lambda_S(s)^2 ds\right)du}$$

This brings us back to a framework in which the Call price has an explicit formula. Finally, one has the approximated swaption price:

$$\boxed{\begin{aligned} V_0 &= A(0)\left(\left(S(0) + \tfrac{1-\bar{b}_S}{\bar{b}_S}S(0)\right)\Phi(d^+) - \left(K + \tfrac{1-\bar{b}_S}{\bar{b}_S}S(0)\right)\Phi(d^-)\right) \\ d^{+/-} &= \frac{\ln\left(\frac{S(0)+S(0)\frac{1-\bar{b}_S}{\bar{b}_S}}{K+S(0)\frac{1-\bar{b}_S}{\bar{b}_S}}\right) +/- \frac{\bar{\lambda}_S^2\bar{b}_S^2}{2}T_0}{\bar{\lambda}_S\bar{b}_S\sqrt{T_0}} \end{aligned}}.$$

4 Calibration of the linear local volatility

In this section, one will consider a swaption strip composed of $N-1$ swaptions with a maturity grid $0 = T_0 < T_1 < \ldots < T_N$. That is a set of swaptions, the n-th of which expires at T_n and its underlying swap pays at $T_n, T_n + 6M, \ldots, T_N$. One notes S_n the swap rate associated to the n-th swaption.

4.1 Parameterization

We aim to calibrate the linear local volatility of the short rate with piecewise constants parameters λ, a, b. To do so, one lets the following target variables:

$$\begin{aligned} \lambda(t) &= \sum_{n=1}^{N-1} \lambda_n \, 1_{]T_{n-1},T_n]}(t) \\ a(t) &= \sum_{n=1}^{N-1} S_n(0) \, 1_{]T_{n-1},T_n]}(t) \\ b(t) &= \sum_{n=1}^{N-1} b_n \, D_n \, 1_{]T_{n-1},T_n]}(t) \end{aligned}$$

Neglecting the small terms in the expression of b_{S_n}, one deduces

$$b_{S_n}(t) \approx \frac{S_n(0)}{a(t)} \frac{b(t)}{\frac{\partial S}{\partial x}(t, \bar{x}(t), \bar{y}(t))}$$

It is generally judicious to calibrate a function of the same order than the calibrated parameter. So we chose $a_n = S_n(0)$ and $b(t) = b_n D_n, \ t \in]T_{n-1}, T_n]$ with $D_n = \frac{\partial S_n}{\partial x}(t, 0, 0)$.

Note that the pricing formula from section 3.3 requires to compute λ_S and b_S at each time t with the right values $\lambda_n, b_n, S_n(0)$.

4.2 Data set construction

The specific Cheyette model considered will be calibrated on an implied volatility surface $\hat{\sigma}$, in other terms our observed data are:

$$\hat{\sigma}(T_n, K_m), \quad n = 1, N-1 \quad m = 1, M.$$

It is preferable to calibrate on the implied volatility rather than on the swaption prices in reason of the weak level of prices for some parameters.

To build our data set, one has to determine the parameters $(\hat{\lambda_n}, \hat{b}_n)_{n=1,N-1}$ so that the implied volatility of the n-th swaption equals the implied volatility of a swaption in the displaced log-normal model of parameters $(\hat{\lambda}_n, \hat{b}_n)$. We will do it by minimising with the least square method the function below:

$$\forall n, \ (\hat{\lambda_n}, \hat{b}_n) = argmin_{(\lambda,b)\in\mathbb{R}\times\mathbb{R}^*} \sum_{m=1}^{M} \left(IV^{BS}\left(Swaption^{LD}(\lambda, b, T_n, K_m), T_n, K_m\right) - \hat{\sigma}(T_n, K_m)\right)^2$$

$IV^{BS}(\Pi, T, K)$ is the implied volatility computed from the price Π in the Black Scholes model and $Swaption^{LD}$ is the pricing function of a swaption in the displaced log-normal model.

It is legit to ask whether this construction is necessary. Indeed, one could calibrate directly the Cheyette model on the observed implied volatility surface but this would lead to a huge computational time. The approximation from section 3 is therefore very useful here.

4.3 Algorithm

For $n = 1, ..., N-1$:
One notes $(\lambda_i)_{i=1,n-1} = \lambda^{(n-1)}$ et $(b_i)_{i=1,n-1} = b^{(n-1)}$ the already calibrated parameters at step n.

- Compute $\bar{x}, \bar{y}$ with $\lambda^{(n-1)}, b^{(n-1)}$ on $[0, T_{n-1}]$
- Compute $\bar{\lambda}_{Sn}$ and $\bar{b}_{S_n}$ with $\lambda^{(n-1)}, b^{(n-1)}$ on $[0, T_{n-1}]$
- Solve

$$(\lambda_n, b_n) = \underset{\lambda,b}{argmin} \left\| \left(\psi\left(\lambda^{(n-1)}, b^{(n-1)}, \lambda, b, T_n\right), \phi\left(\lambda^{(n-1)}, b^{(n-1)}, \lambda, b, T_n\right)\right) - \left(\hat{\lambda}_n, \hat{b}_n\right) \right\|^2$$

Where
$$\begin{aligned} \psi &: \left(\lambda^{(n-1)}, b^{(n-1)}, \lambda, b, T_n\right) \to \bar{\lambda}_{S_n} \\ \phi &: \left(\lambda^{(n-1)}, b^{(n-1)}, \lambda, b, T_n\right) \to \bar{b}_{S_n} \end{aligned}$$

Conclusion

In this paper, we have tackled the problem of the calibration of a Cheyette interest rate model. Knowing a swaption's implied volatility surface, the proposed algorithm allows us to price swaptions while being consistent with the implied volatility curve of the market. While the exact swaption price expression can only be reduced to a conditional expectation or the solution of a partial differential equation, we have highlighted here a way to avoid classical numerical methods during calibration such as Monte Carlo simulation and Euler scheme. With the aid of some approximations and considerations, one has approached the swaption price thanks to a closed formula in order to make the calibration algorithm more efficient.

Once the volatility is calibrated, one can model an entire forward curve $t \to f(t, T)$ with the important Markov property of the short rate. This point justify the use of a Cheyette model because it allows to circumvent some pricing difficulties.

To finish with, we invite the reader to look at the multiple extensions of the model described here. On the one hand, the natural extension of this work is the calibration of the multidimensional Cheyette model with a linear local volatility structure. One remark furthermore that the mean reversion parameter χ remained constant here, but it is possible to calibrate it separately. On the other hand, it is interesting to focus on a stochastic volatility structure by adding to σ_r a stochastic factor (a CIR process) z such that $dz(t) = \theta\left(z(0) - z(t)\right)dt + \nu(t)\sqrt{z(t)}dW_t$. Finally, we can think to a Cheyette model with a non-parametric volatility structure which would be calibrated with a Dupire method for instance.

References

[1] O. Cheyette. Markov representation of the heath-jarrow-morton model. 1995.

[2] V. Piterbarg Leif B.G Andersen. *Interest rate modeling*, volume 2. 2010.

[3] Vladimir Piterbarg. Markovian projection method for volatility calibration. 2006.

A propos d'Awalee

Cabinet de conseil indépendant spécialiste du secteur de la Finance.

Nous sommes nés en 2009 en pleine crise financière. Cette période complexe nous a conduits à une conclusion simple : face aux exigences accrues et à la nécessité de faire preuve de souplesse, nous nous devions d'aider nos clients à se concentrer sur l'essentiel, à savoir leur performance.

Pour accomplir cette mission, nous nous appuyons sur trois ingrédients : habileté technique, savoir-faire fonctionnel et innovation.

Ceci au service d'une ambition : dompter la complexité pour simplifier la vie de nos clients.

«*Run the bank*» avec Awalee !

Contactez-nous

Ronald LOMAS
Partner
rlomas@awaleeconsulting.com
06 62 49 05 97

awalee

est une marque de

canopee group