Chapter 6: Extensions and Resolutions

§ 6.1 Fix $\lambda \in \Lambda^+$. Want to realise

$$\mathrm{ch}\, L(\lambda) = \sum_{w} (-1)^{\ell(w)}\, \mathrm{ch}\, M(w\cdot\lambda) \qquad (2.4)$$

Def A BGG resolution of $L(\lambda)$ is an exact sequence

$$(*) \quad 0 \to C_m \xrightarrow{\delta_m} C_{m-1} \to \dots \to C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\varepsilon} L(\lambda) \to 0$$

with $\displaystyle C_k := \bigoplus_{w \in W^{(k)}} M(w\cdot\lambda)$, $k = 0, 1, \dots, m = |\Phi^+|$

$(C_m = M(w_0\cdot\lambda), \; C_0 = M(\lambda))$.

Goals:
(1) BGG resolutions exist
(2) Uniqueness?
(3) Applications (LA cohomology, Homology of $\mathcal{O}$).

Exercise: $(C_\bullet, \delta_\bullet) \to L(\lambda)$ BGG resol $\Rightarrow \delta|_{M(w\cdot\lambda)} \neq 0$

Sketch: $\lambda \in \Lambda^+ \Rightarrow w_0 \cdot \lambda = \mu$ is anti dom regular.

$M(w\cdot\lambda) = \langle v_+ \rangle$, $\quad w \in W^{(k)}$

$\delta(v_+) = 0 \quad \Rightarrow \quad \exists\, v \in C_{k+1}$ s.t. $\delta(v) = v_+$

$\Rightarrow \quad C_{k+1} \twoheadrightarrow M(w\cdot\lambda)$

$\Rightarrow \quad [M(u\cdot\lambda) : L(w\cdot\lambda)] \neq 0$, $\exists\, u \in W^{(k+1)}$

$\Rightarrow \quad M(w\cdot\lambda) \hookrightarrow M(u\cdot\lambda)$

$\Leftrightarrow \quad ww_0\cdot\mu \leq uw_0\cdot\mu$

$\Leftrightarrow \quad ww_0 \leq uw_0$

$\Rightarrow k = \ell(w) \geq \ell(u) = k+1$ □

§ 6.2

Thm $\lambda \in \Lambda^+$. There is an exact seq. $(D_\bullet^\lambda, \partial_\bullet) \to L(\lambda)$ s.t. D_k^λ has std filt with $(D_k^\lambda : M(w\cdot\lambda)) = 1$, $\forall w \in W^{(k)}$.

Sketch: Assume $\lambda = 0$

$\mathfrak{b}$-cosets: $v_j = y_j + \mathfrak{b}$

(A) Let $\mathfrak{n}^- \cong V = \mathfrak{g}/\mathfrak{b} \supseteq \{v_1, \dots, v_m\}$ basis

Wts V : $-\alpha_i \leftrightarrow v_i$

Wts $\Lambda^k V$: $-\sum \alpha_{i_j} \leftrightarrow v_{i_1} \wedge \dots \wedge v_{i_k}$

(B) $D_k := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\Lambda^k V)$, has std filt. (3.6)

$\Rightarrow D_0 = M(0)$ $(\Lambda^0 V = \text{triv})$

$D_m = M(w_0 \cdot 0)$ $(h . v_1 \wedge \dots \wedge v_m = -2\rho(h)\, v_1 \wedge \dots \wedge v_m)$

(C) Introduce $\partial_k : D_k \to D_{k-1}$, ε (general construction)

(D) $D_k^0 := D_k \cap \mathcal{O}^{\chi_0}$ (principal block)

(E) Apply T_0^λ to pass from $L(0) \to L(\lambda)$ $\square$

Details are in § 6.3 – § 6.5

Claim Each D_k has std filtration

Pf: $M = \Lambda^k V \supseteq \{z_1, \dots, z_N\}$

$\mathrm{Wts}(M) \supseteq \{\mu_1, \dots, \mu_N\}$

$\mu_i \leqslant \mu_j \Rightarrow i \leqslant j$

$$\rightsquigarrow \quad 0 \subseteq M_N \subseteq \cdots \subseteq M_2 \subseteq M_1 = D_k$$

$$M_j := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \langle z_j, \ldots, z_N \rangle$$

s.t. $M_j / M_{j+1} \cong M(\mu_j)$ $\square$

Example $\mathfrak{g} = \mathfrak{sl}(3)$, $\Phi^+ = \{\alpha, \beta, \gamma = \alpha+\beta\}$

$$V \supseteq \{ v_\alpha, \quad v_\beta, \quad v_\gamma \}$$

wts: $-\alpha$, $-\beta$, $-\alpha-\beta$

$$M := \Lambda^2 V \supseteq \{ v_\alpha \wedge v_\gamma, \quad v_\beta \wedge v_\gamma, \quad v_\alpha \wedge v_\beta \}$$

wts:

$$\begin{tikzcd} -(2\alpha+\beta) \arrow[rr, bend right, "+\alpha"'] & -(\alpha+2\beta) & -(\alpha+\beta) \end{tikzcd}$$

Now: $\mu_3 - \mu_1 = \alpha > 0$

$$x_\alpha (v_\alpha \wedge v_\gamma) = x_{\underline{\alpha}} v_\alpha \wedge v_\gamma + v_\alpha \wedge x_{\underline{\alpha}} v_\gamma \in \langle v_\alpha \wedge v_\beta \rangle$$

$$x_\beta (v_\alpha \wedge v_\gamma) = x_\beta v_\alpha \wedge v_\gamma + v_\alpha \wedge x_\beta v_\gamma = 0$$

$$x_\gamma (v_\alpha \wedge v_\gamma) = x_\gamma v_\alpha \wedge v_\gamma + v_\alpha \wedge x_\gamma v_\gamma = 0$$

$\therefore$ $v_\alpha \wedge v_\gamma$ is HWV modulo M_2.

Further $W^{(2)} \cdot 0 = \{ s_\alpha s_\beta \cdot 0 = -(2\alpha+\beta),$
$s_\beta s_\alpha \cdot 0 = -(\alpha+2\beta) \}$

$\therefore$ $W^{(k)} \cdot 0 \neq \mathrm{Wts}(\Lambda^k V)$

Recall $M = \bigoplus_{\chi} M^{\chi} \quad \forall M \in \mathcal{O}$

Claim: $M, N \in \mathcal{O}$, $\varphi \in \mathrm{Hom}_{\mathcal{O}}(M,N) \Rightarrow \varphi(M^{\chi}) \subseteq N^{\chi}$

Pf: $v \in M^{\chi}$, $z \in Z(\mathfrak{g}) \Rightarrow (z-\chi(z))^n v = 0, \exists n$

$\Rightarrow 0 = (z-\chi(z))^n \varphi(v)$ □

$\therefore (D_\bullet, \partial_\bullet) \to L(\lambda)$ exact $\Rightarrow (D_\bullet^{\chi_\lambda}, \partial_\bullet^{\chi_\lambda}) \longrightarrow L(\lambda)$ exact

Def: $\Pi_w := \Phi^+ \cap w(\Phi^-)$

$\Gamma_w := \Phi^+ \cap w(\Phi^+)$

Notation: $\Pi \subseteq \Phi^+ \Rightarrow \overline{\Pi} = \sum_{\alpha \in \Pi} \alpha \in \mathfrak{h}^*$.

Lemma: $\mu = w \cdot 0$ occurs in $\Lambda^{\ell(w)} V$

Pf: $\Phi^+ = \Phi^+ \cap (w\Phi^+ \cup w\Phi^-)$

$= \Gamma_w \cup \Pi_w$

$w\Phi^+ = w\Phi^{\overset{-}{+}} \cap (\Phi^+ \cup \Phi^{\overset{+}{-}})$

$= \Gamma_w \cup (-\Pi_w)$

$\Rightarrow w \cdot 0 = w\rho - \rho$

$= \frac{1}{2}(\overline{\Gamma_w} - \overline{\Pi_w}) - (\overline{\Gamma_w} + \overline{\Pi_w})$ □

$= -\overline{\Pi_w}$

Lemma $\mu = w.0$ occurs only once in $\Lambda^{\bullet} V$

Pf: We show: $\Pi \subset \Phi^+$ s.t. $\overline{\Pi} = \overline{\Pi}_w \Rightarrow \Pi = \Pi_w$.

Clear $\ell(w) = 0$

Suppose $\ell(w) = k > 0$

$\Rightarrow \ell(s_\alpha w) = k - 1$, $\exists \alpha \in \Delta$

(0.3) $\Rightarrow w^{-1}\alpha < 0$

$\Rightarrow \begin{cases} \alpha \in \Pi_w \\ (w')^{-1}\alpha > 0 \end{cases}$ $\qquad w' = s_\alpha w$

$\Rightarrow \underline{\alpha \notin \Pi_{w'}}$

Claim $\Pi_w = s_\alpha \Pi_{w'} \cup \{\alpha\}$ (*)

Pf: Have

$$s_\alpha \Pi_{w'} = s_\alpha(\Phi^+ \cap w'\Phi^-)$$

$$= (\Phi^+ \setminus \{\alpha\} \cup \{-\alpha\}) \cap \overset{s_\alpha w'}{w}\Phi^-$$

$(\alpha \in \Pi_w \subseteq w\Phi^-)$ $= (\Phi^+ \setminus \{\alpha\}) \cap w\Phi^-$

$\Rightarrow \{\alpha\} \cup s_\alpha(\Pi_{w'}) = \Phi^+ \cap w\Phi^- = \Pi_w$. □

Back to the Lemma:

$$\Pi \subseteq \Phi^+, \quad \overline{\Pi} = \Pi_w = \rho - w\rho$$

$$\Rightarrow \underline{s_\alpha \overline{\Pi}} = (\rho - \alpha) - s_\alpha w\rho = (\rho - w'\rho) - \underline{\alpha}$$

$$= \underline{\overline{\Pi}_{w'} - \alpha}$$

$$\alpha \notin \Pi \quad \Rightarrow \quad s_\alpha \Pi \subseteq \Phi^+$$

$$\Rightarrow \quad s_\alpha \Pi \cup \{\alpha\} \subseteq \Phi^+$$

$$\Rightarrow \quad \overline{s_\alpha \Pi \cup \{\alpha\}} = \overline{\Pi_{w'}}$$

$$\text{(IH)} \quad \Rightarrow \quad \Pi_{w'} = s_\alpha \Pi \cup \{\alpha\}$$

$$\Rightarrow \quad \alpha \in \Pi_{w'} \qquad \text{(contr.)}$$

$\therefore \quad \alpha \in \Pi$

Let $\Pi' = s_\alpha(\Pi \setminus \{\alpha\}) \subseteq \Phi^+$

$$\Rightarrow \overline{\Pi'} = \overline{\Pi_{w'}} \qquad\qquad \overline{\Pi} = \Pi_w$$

$$\text{(IH)} \Rightarrow \underline{\Pi'} = \underline{\Pi_{w'}}$$

$$\Rightarrow \Pi = s_\alpha(\Pi_{w'}) \cup \{\alpha\} = \Pi_w \qquad \square$$

So far we showed:

- $\mathrm{Wts}(D_k^0) = \{w \cdot 0, \ w \in W^{(k)}\}$
- D_k^0 has a std filtration.

Question: $\mathrm{Ext}_{\mathcal{O}}(M(w \cdot \lambda), M(w' \cdot \lambda)) = ?$

if $\lambda \in \Lambda^+$, $\ell(w) = \ell(w')$.

Thm (~~6.3~~) 6.5 $\lambda \in \mathfrak{h}^*$

(a) $\mathrm{Ext}_{\mathcal{O}}(M(\mu), M(\lambda)) \neq 0 \Rightarrow \mu \uparrow \lambda, \ \mu \neq \lambda$

(b) $\lambda \in \Lambda^+, \ w, w' \in W$. Then

$$\mathrm{Ext}_{\mathcal{O}}(M(w\cdot\lambda), M(w'\cdot\lambda)) \neq 0 \Rightarrow w' < w$$
$$\Rightarrow \ell(w) < \ell(w')$$

Pf: **(a)** $(3.1\,a) \Rightarrow \mu \neq \lambda$. Given

$$\begin{tikzcd}
 & & & P(\mu) \arrow[dl, "\varphi"'] \arrow[d, "\pi"] & & \\
0 \arrow[r] & M(\lambda) \arrow[r, "f"] & M \arrow[r, "g"] & M(\mu) \arrow[r] \arrow[l, dashed, bend left, "\sigma"] & 0 & (*)
\end{tikzcd}$$

if $v \in M(\mu)$, $x, y \in P(\mu)$ s.t. $v = \pi x = \pi y$

$$0 = \pi(x-y) = g\varphi(x-y)$$
$$\Rightarrow \varphi(x-y) \in \mathrm{im}(f) \Leftrightarrow \varphi(x-y) \in M(\lambda)$$

__if:__ $\varphi P(\mu) \cap \mathrm{im} f = 0 \Rightarrow \sigma(v) = \varphi(x), \ \exists x \in \pi^{-1} v$

$\Rightarrow (*)$ splits

(3.10) $0 \subseteq P_{\underline{0}} \subseteq P_{\underline{1}} \subset \cdots \subset P_{\underline{n}} = P(\mu), \ P_i/P_{i-1} \cong M(\mu_i)$

(3.11) $(P(\mu) : M(\mu_i)) = [M(\mu_i) : L(\mu)] > 0$

(5.1) $\underline{\mu \uparrow \mu_i}$ $\forall i$

$f(M(\lambda)) \cap \varphi(P(\mu)) \neq 0 \Rightarrow \varphi P_i \cap f(M(\lambda))$, $\exists i$ min.

$$\Rightarrow \begin{tikzcd} \varphi|P_i : P_i \arrow[r] \arrow[dr] & M(\lambda) \\ & P_i/P_{i-1} \arrow[u] \cong M(\mu_i) \end{tikzcd} \Rightarrow \mu \uparrow \lambda$$

$\Rightarrow [M(\lambda) : L(\mu_i)] \neq 0 \Rightarrow \underline{\exists \mu_i \uparrow \lambda}$.

(b) $\lambda \in \Lambda^+ \Rightarrow \mu = w_0 \cdot \lambda$ is antidom regular

$\therefore \ \mathrm{Ext}_{\mathcal{O}}(M(w\cdot\lambda), M(w'\cdot\lambda)) \neq 0$

$(a) \implies ww_0\cdot\mu \uparrow w'w_0\cdot\mu$ & $w \neq w'$

$(5.2) \implies ww_0 < w'w_0$

$\implies w > w'$ $\square$

General construction of δ_k's [Hilton – Stamb.]

From LA cohomology: M $\mathfrak{g}$-module, $V(\mathfrak{g}) = \mathrm{triv}$

$$H^n(\mathfrak{g}, M) = \mathrm{Ext}^n_{\mathcal{U}}(V(\mathfrak{g}), M)$$

$\underline{P}$ proj. resol. of triv.

$$= H^n(\mathrm{Hom}_{\mathcal{U}}(\underline{P}, M))$$

Notations: $\underline{e}_k := e_1 \wedge \dots \wedge e_k$ $\quad \exists\, e_1, \dots, e_k \in \mathfrak{g}$

$\hat{\underline{e}}_k^i := e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_k$

$(i<j)$ $\quad \hat{\underline{e}}_k^{i,j} := e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_k$

Def: $P_k := \mathcal{U} \otimes \Lambda^k \mathfrak{g}$

$$\partial_k(u \otimes \underline{e}_k) = \sum_i (-1)^{i+1} u e_i \otimes \hat{\underline{e}}_k^i + \sum_{i<j} u \otimes [e_i, e_j] \wedge \hat{\underline{e}}_k^{i,j}$$

Prop: $\partial_{k-1}\partial_k = 0 \quad \forall k.$

Introduce filtrations on $\bigoplus_k P_k = \mathcal{U} \otimes \Lambda \mathfrak{g}$, $q \geqslant 0$:

$F^q := \operatorname{span}\{ \underline{e}_m \otimes \underline{e}_n \mid m+n = q \}$

PBW basis w.r.t. $\{ e_1, \dots, e_d \mid d = \dim \mathfrak{g} \}$

$$F^q P_n = F^q \cap P_n$$

Def: $\underline{W}^q = (W^q_\bullet , \partial^q_\bullet)$ a complex with.

$$W^q_n := F^q P_n / F^{q-1} P_n$$

$$\partial^q_n (u \otimes \underline{e}_n) = \partial_n (u \otimes \underline{e}_n) \quad \text{mod } F^{q-1}$$

$$\equiv \sum_i (-1)^{i+1} u e_{\ell_i} \otimes \hat{\underline{e}}^i_n$$

Thm: $\underline{W}^q$ exact $\forall q \geqslant 0$

Cor: $\underline{P} = (P_\bullet , \partial_\bullet)$ is a free-resol. of triv.

Pf: From SES $F^{q-1} \underline{P} \hookrightarrow F^q \underline{P} \twoheadrightarrow \underline{W}^q$

$H_n(\underline{W}^q) = 0 \overset{\text{LES}}{\Rightarrow} H_n(F^{q-1}\underline{P}) \cong H_n(F^q \underline{P})$, $\forall n$

$$F^0 \underline{P} = 0 \to V(\mathfrak{g}) \to V(\mathfrak{g}) \to 0$$

$$\Rightarrow H_n(F^0 \underline{P}) = 0 \quad \forall n$$

(induction) $\Rightarrow H_n(F^q \underline{P}) = 0 \quad \forall n, q$

$$\Rightarrow H_n(\underline{P}) = 0$$

□

Moral of the story: all goes through relatively [BGG]

The relative version for $(\mathfrak{g}, \mathfrak{b})$:

$$D_k = \mathcal{U} \underset{\mathcal{U}(\mathfrak{b})}{\otimes} \Lambda^k(\mathfrak{g}/\mathfrak{b})$$

$$\delta_k(u \otimes \underline{v}_k) = \sum_i (-1)^i \, u e_i \otimes \hat{\underline{v}}_k^{\,i} + \sum_{i,j} (-1)^{i+j} \, u \otimes \overset{\text{in } \mathfrak{g}/\mathfrak{b}}{\overline{[e_i, e_j]}} \wedge \hat{\underline{v}}_k^{\,i,j}$$

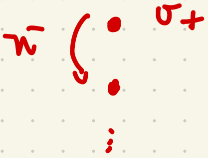
with $\underline{v}_k = v_1 \wedge \dots \wedge v_k \in \Lambda^k(\mathfrak{g}/\mathfrak{b})$

$e_i \in \mathfrak{g}$ a represent. of v_i

Is an exact complex. □

§6.6. Thm (Bott) $\lambda \in \Lambda^+$: $\dim H^k(\mathfrak{n}^-, L(\lambda)) = |W^{(k)}|$

Sketch: $H^k(\mathfrak{n}^-, L(\lambda)) = \mathrm{Ext}^k_{\mathfrak{n}_-}(\mathbb{C}, L(\lambda))$



$$\cong \mathrm{Ext}^k_{\mathfrak{n}_-}(L(\lambda)^\vee, \mathbb{C}^\vee)$$

$$= H^k(\mathrm{Hom}_{\mathfrak{n}-}(\underline{M}, \mathbb{C}))$$

take $\underline{M}$ to be BGG resol!

$\underline{M}$ any $\mathcal{U}(\mathfrak{n}^-)$ - proj. resol. of $L(\lambda)$.

$$\Rightarrow \mathrm{Hom}_{\mathfrak{n}-}(M(\mu), \mathbb{C}) \cong (M(\mu)/\mathfrak{n}^- M(\mu))^* \cong \mathbb{C}_{-\mu}$$

$$\Rightarrow \mathrm{Hom}_{\mathfrak{n}-}(D^\lambda_k, \mathbb{C}) \cong \bigoplus_{w \in W^{(k)}} \mathbb{C}_{-w.\lambda}$$

$$\Rightarrow H^k(\mathrm{Hom}_{\mathfrak{n}-}(\underline{C}_\bullet, \mathbb{C})) \cong \bigoplus_{w \in W^{(k)}} \mathbb{C}_{-w.\lambda}$$ □

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{Hom}(D_k^{\lambda}, \mathbb{C}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mathrm{Hom}(D_{k-1}^{\lambda}, \mathbb{C}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \cdots \\
\cdots \arrow[r, "0"] & \bigoplus_{w \in W^{(k)}} \mathbb{C}_{-w\cdot\lambda} \arrow[r, "0"] & \bigoplus_{u \in W^{(k-1)}} \mathbb{C}_{-u\cdot\lambda} \arrow[r, "0"] & \cdots
\end{tikzcd}$$

Remarks on Uniqueness of BGG-resolutions (6.7, 6.8).

Let $\underline{C} = (\underline{C}_\bullet, \delta_\bullet) \xrightarrow{\varepsilon} L(\lambda)$ be a BGG-res

Rewrite it as $\underline{C} = (C^0_\bullet, \varepsilon_\bullet)$

$$C^0_k = \bigoplus_{w \in W^{(k)}} M(w\cdot\lambda^0) \quad , \quad \lambda^0 := w_0\cdot\lambda$$

$$\varepsilon_k : C^0_k \longrightarrow C^0_{k+1}$$

(Note: $C^0_k = C_{m-k}$, $\varepsilon_k = \delta_{m-k}$)

Now:

$$\varepsilon_k\big|_{M(w\cdot\lambda^0)} \neq 0 \Rightarrow M(w\cdot\lambda^0) \hookrightarrow M(w'\cdot\lambda^0),$$
$$\Leftrightarrow \quad w < w'$$

Notation / Def

$w \longrightarrow w'$: map $M(w\cdot\lambda^0) \hookrightarrow M(w'\cdot\lambda^0)$

$w \xrightarrow{\alpha} w'$: when $w' = s_\alpha w$, $\exists\, \alpha > 0$

Remarks:

(1) $w \longrightarrow w'$ is defined up to a scalar call it $e(w, w') \in \mathbb{C}$

(2) $e(w,w') = 0$ if $\nexists \; w \longrightarrow w'$

(3) $\underline{C} = (C^{\circ}_{\bullet}, \varepsilon_{\bullet})$ defines a matrix

$$E = \big(e(w,w') \big)_{w,w' \in W}$$

Def: Let $(w_1, w_2, w_3, w_4) \in W^4$ s.t.

$$\begin{tikzcd} & w_2 \arrow[dr] & \\ w_1 \arrow[ur] \arrow[dr] & & w_4 \\ & w_3 \arrow[ur] & \end{tikzcd}$$

These elements are said to form a square.

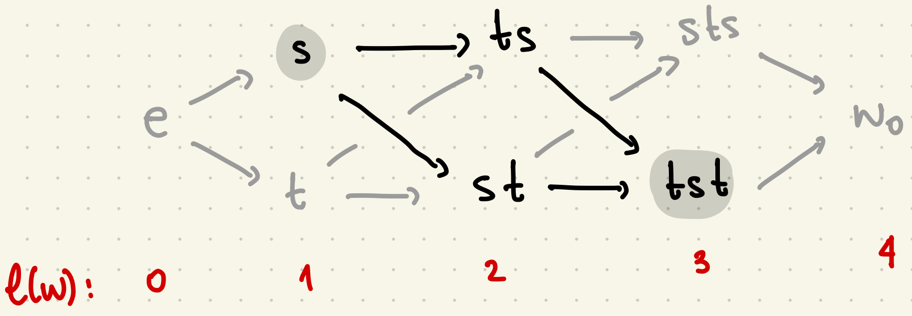
Fact: (BLACK BOX)

Suppose $\ell(w_4) = \ell(w_1) + 2$. Then, $\exists$ exactly two $w, w' \in W$ s.t.

$$\begin{tikzcd} & w \arrow[dr] & \\ w_1 \arrow[ur] \arrow[dr] & & w_4 \\ & w' \arrow[ur] & \end{tikzcd}$$

$\square$

Example: If $W = W(I_2(4)) = \langle s, t \rangle$ dihedral



Claim: The matrix $E = E(\subseteq)$ satisfies

$$(*) \quad e(w_2, w_4)\, e(w_1, w_2) + e(w_3, w_4)\, e(w_1, w_3) = 0$$

whenever (w_1, w_2, w_3, w_4) form a square

Sketch:

Have $\begin{cases} w \to w' \iff w < w' \\ \varepsilon_K \text{ are determined by } \varepsilon / M(w.\lambda^0), \end{cases}$ $\forall w \in W^{(k)}$

From Fact

$$\begin{tikzcd} & w_2 \arrow[dr] & \\ w_1 \arrow[ur] \arrow[dr] & & w_4 \\ & w_3 \arrow[ur] & \end{tikzcd}$$

We apply that $\varepsilon^2 = 0$

$\square$

Thm (6.8) Given $(C^\circ_\bullet, \varepsilon_\bullet)$ a BGG-resol. all $e(w,w') \neq 0$ when $\begin{cases} \ell(w) = k \\ \ell(w') = k+1 \end{cases}$ and $w < w'$.

Pf: Downward induction on $k = \ell(w)$.

$k=m$, $(m-1)$ are clear:

$$M(w_0 \cdot \lambda^\circ) = M(\lambda) \xrightarrow{\varepsilon} L(\lambda) \quad \text{non-zero}$$

$$\underbrace{\bigoplus_{\alpha\in\Delta} M(s_\alpha \cdot \lambda)}_{N(\lambda)} \longrightarrow M(\lambda) \quad \underline{\text{non-zero}}$$

For the inductive hypothesis:

Lemma (6.7): $\alpha \in \Delta$, $\beta > 0$, $\alpha \neq \beta$. Then

$\exists$ diagram (L) $\Rightarrow$ $\exists$ diagram (R)

and vice-versa:

(L)
$$\begin{tikzcd} & w' \\ s_\alpha w \arrow[ur, "\beta"] \arrow[dr, "\alpha"'] & \\ & w \end{tikzcd}$$

$$\begin{tikzcd} w' \arrow[dr, "\alpha"] & \\ & s_\alpha w' \\ w \arrow[ur, "\gamma = s_\alpha \beta"'] & \end{tikzcd}$$
(R)

Pf: (L) means: $\begin{cases} w' = \underline{s_\beta} s_\alpha w \\ \ell(w') = \ell(w) \\ \quad = \ell(s_\alpha w) + 1 \end{cases}$

$$\gamma = s_\alpha \beta \Rightarrow s_\gamma = s_\alpha s_\beta s_\alpha$$

$$\Rightarrow \underline{s_\gamma w} = s_\alpha(s_\beta s_\alpha w) \overset{(L)}{=} \underline{s_\alpha w'}$$

It remains to show $\underline{\ell(s_\alpha w')} = \underline{\ell(w')}\,\underline{+1}$

$$(L) \Rightarrow \ell(s_\beta w') < \ell(w')$$

$$\Rightarrow (w')^{-1}\beta < 0$$

$$\Leftrightarrow (w')^{-1} s_\alpha \gamma < 0 \qquad \text{as } \gamma = s_\alpha \beta$$

$$\Leftrightarrow (s_\alpha w')\gamma < 0$$

$$\Rightarrow \ell(\underbrace{s_\gamma \underbrace{s_\alpha w'}_{s_\gamma w}}_{w}) < \ell(s_\alpha w')$$

$$\overset{(L)}{\Leftrightarrow} \underline{\ell(w')} = \underline{\ell(w)} < \underline{\ell(s_\alpha w')}$$

□

Back to the Theorem :