Problem sheet 8, 23-06-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de or by accessing the book online.

0. (Drill)

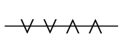
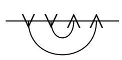
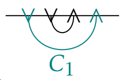
1. Write down the elements T_S of $\mathrm{TL}^{\uparrow\downarrow}_{2+2}(q)$ for the four paths $S = U_2^1 U_3^1 U_1^1 U_2^1$; $U_2^1 U_3^1 U_1^1 U_2^0$; $U_2^1 U_3^0 U_1^0 D_2^0$; $U_2^1 U_3^0 U_1^0 D_2^1$ for $\mathfrak{S}_2 \times \mathfrak{S}_2 \leq \mathfrak{S}_4$ and compute their degree (see Example 7.3.5).
2. Draw the bottom edge of the oriented Temperley–Lieb diagrams of Figure 7.12 (for the parabolic $\mathfrak{S}_2 \times \mathfrak{S}_3 \leq \mathfrak{S}_5$)

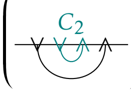
1. Proof of Proposition 7.5.7 Read the proof of Proposition 7.5.7 stating that for any reduced path T_μ, the Temperley–Lieb element $E_{T_\mu} = e_\mu$. This follows from the (alternative) proof of the correspondence between the diagrammatic and its generators and relations presentations of Temperley–Lieb algebras done in Theorem 5.2.3.

2. Complete the proof of Theorem 7.5.10 Follow the proof of Theorem 7.5.10 and complete the other diagrammatic verification that the degree is preserved by the map.

3. More on Proposition 7.5.18 This exercise gives the details of a combinatorial property of the Kazhdan–Lusztig polynomials. Let μ be a m, n diagram and $\overline{\mu}$ be the cup diagram generated by closing each $\vee$ with its closed left $\wedge$ neighbour. Then for each cup C, define the width of the cup $w(C)$ as twice the number of cups inside C, understanding that C is inside itself.

For example,

$\mu =$ , $\overline{\mu} =$ , $w\Big($ 

$\Big) = 2 \times 2 = 4$, $w\Big($ 

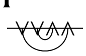
$\Big) = 1 \times 2 = 2.$

Proposition 7.5.18 *Given μ a (m, n)-diagram, the following column-sum of the Kazhdan–Lusztig polynomial matrix is palindromic and unimodal and equal*

$$\sum_{\lambda \subseteq \mu} q^{\ell(\mu)-\ell(\lambda)} n_{\lambda,\mu}(q) = \prod_{C \text{ a cup in } \overline{\mu}} (1 + q^{w(C)}).$$

For example, let us do the first and second-to-last columns of Figure 7.13. We have:

$$\begin{aligned}\sum_{\lambda \subseteq \mu} q^{\ell(\mu)-\ell(\lambda)} n_{\lambda,\mu}(q) &= q^{4-4}q^0 + q^{4-3}q + q^{4-1}q + q^{4-0}q^2 \\ &= 1 + q^2 + q^4 + q^6 \\ &= (1+q^4)(1+q^2) \\ &= \prod_{C \text{ a cup in }} (1 + q^{w(C)}).\end{aligned}$$



4.