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PhD Candidate | Ghent University
alexislangloisremillard@gmail.com
https://alexisl-r.github.io/

# Diagrammatics and representations of algebras related to Howe and Schur–Weyl dualities

## General context

My main research field is concerned with representation theory, a subfield of algebra. Algebra is concerned about the abstract structures that govern mathematics. For an algebraist, the symmetries of a molecule and the shuffling of a deck of cards are two examples of the same structure: group theory. The force of this abstraction is to enable simultaneous advancements in many seemingly unrelated fields. A single result in group theory can then be applied to make magic tricks, or to study the development of crystals.

One of the main problems an algebraist faces when studying abstract structures is the sheer complexity of the objects. That is where representation theory plays a rôle, its main leitmotiv is to study a structure by considering its action on known objects, most commonly on vector spaces. Then a question in abstract algebra is transformed into one of *linear* algebra, and we, or computers, are very adept at linear algebra. For a class of representations, called faithful, what happens at the level of the representation is guarantee to represent what is happening at the abstract level.

Not all algebras have the same ease of approach. My research concentrates on algebras admitting a type of graphical calculus. This enables the statement of complex conditions by simple topological rules. Figure 1 below presents an example on the Jones–Wenzl idempotents [Jon83].

$$\mathsf{P}_n = \mathsf{P}_n^2,$$

$$e_i \mathsf{P}_n = 0 = \mathsf{P}_n e_i,$$

$$\mathsf{P}_n = \mathsf{P}_{n-1} - \frac{[n-1]_q}{[n]_q} \mathsf{P}_{n-1} e_1 \mathsf{P}_{n-1},$$

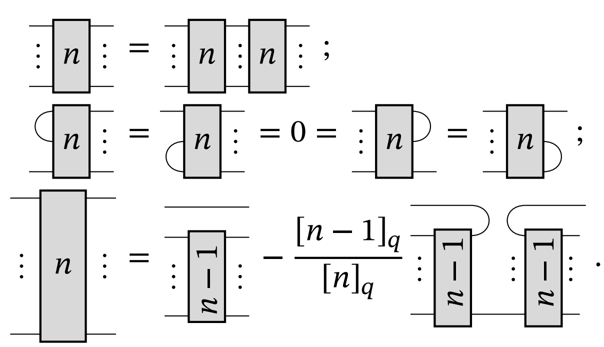

Figure 1: On the left, the algebraic properties of the elements $\mathsf{P}_n$ in the Temperley–Lieb calculus, and their diagrammatic counterparts.

## Statement of research problems

**Main research aim**

Construct faithful representations and new diagrammatic calculus of algebras related to Howe and Schur–Weyl dualities.

The concept of **Howe duality** originates from the influential work of Howe [How89]. In its most classical inception, it relates the representations of a dual pair $G, G'$ that are mutually centralising subgroups of the double cover of a symplectic group. The methods of Howe have been applied over the last 30 years in a vast array of cases [CW12].

The original **Schur–Weyl duality** states that the actions of $\mathrm{GL}_m$ and the symmetric group $S_n$ on the tensor product of a fundamental representation $(\mathbb{C}^m)^{\otimes n}$ are each other centraliser. So we can decompose $(\mathbb{C}^m)^{\otimes n}$ into a direct sum of simple $S_n$ module tensor simple $\mathrm{GL}_n$-modules.

Many algebras can be studied via these methods; I will focus first on three concrete problems to initiate the research, and after will consider links between them.

**First concrete projects**

1. Study the representation theory of the total angular momentum algebra for specific group and define a diagrammatic calculus.
2. Study modular versions of the untangled affine Temperley–Lieb algebras.
3. Define an infinite symmetric webs calculus to study LKB representations.

## Proposed research plan

**Total angular momentum algebra** The first problem is a continuation of my PhD. The algebra studied can be defined by generators and relations [DOV18a] or as the supercentraliser of an $\mathfrak{osp}(1|2)$ realisation inside the tensor product of a rational Cherednik algebra [EG02] and a Clifford algebra. This means that the algebra depends on a reflection group $W$ and a weight function $\varkappa$ invariant on $W$-orbits. It is motivated from the Howe dual pair $(\mathrm{Pin}(d), \mathfrak{osp}(1|2))$ [ØSS09], but other Howe dualities have also been studied [CD20; Ciu+20]. Relatively little was known over the representation theory of this algebra, only the groups $W = \mathbb{Z}_2^N$ [DGV16] and $W = S_3$ [DOV18b] had been studied. As part of my PhD, we did $W = D_{2m} \times \mathbb{Z}_2$ [De +22a] and $W = D_{2m} \times D_{2n}$.

An ongoing collaboration with Marcelo De Martino and Roy Oste aims to extend the two results to a stack of dihedral groups, and to consider exotic $\varkappa$. Our preliminary computations hint that the general case will divide into 4-dimensional "slices" and, for odd dimension, with an extra 3-dimensional "slice"; the two cases we already studied, leaving only the question on how to coordinate the slices. Furthermore, in most previous works, we avoided values of $\varkappa$ that do not permit unitarity. In the low-dimensional cases, the values we avoided did not result in interesting behaviour, but we expect that having many values simultaneously conflicting could allow for remarkable types of representation.

The second part of the project will go on to investigate general $W$. We know from [De +22b] that generalised symmetries can be used to create a basis for a realisation of an important representation: the polynomial null-solutions of a Dirac operators in which the derivatives are changed to Dunkl derivatives [Dun89]. First I will try to define a deformation of the conformal algebra defined in [CD15] and use a reduction to the total angular momentum algebra. The second promising direction I wish to investigate is to create diagrammatics for this algebra by combining Webster's diagrammatics for rational Cherednik algebras [Web17] with a modification of Brundan's, Comes's and Kujawa's diagrammatics for Brauer–Clifford supercategory [BCK19]. A hint that these algebras encode interesting diagrammatics was already pointed out in [FH15] where crossing relations that could be represented via Temperley–Lieb algebras elements were found.

**The untangled affine Temperley–Lieb algebras** The affine Temperley–Lieb algebra is an algebra of high relevance for physicists and algebraist. It is an infinite-dimensional algebra that appears in conformal field theory and is linked to Virasoro algebras. Since the influential work of Graham and Lehrer [JG98], its representation theory has been a central object of interest, mainly via the study of its monoidal category. Recently, Martin and Spencer proved a modular version of the famed Jones–Wenzl projectors [MS22]. It enabled Spencer to generalise our work [LS20] and the work of Flores and Peltola [SE18] on the boundary seam algebra [AJD15] to the modular case [Spe21].

The goal of this project would be to approach the affine Temperley–Lieb algebra via quotients making it finite-dimensional. The chain of algebras we obtained are called untangled affine Temperley–Lieb algebra. It was motivated by a question of Tubbenhauer coming from their recent work with Khovanov and Sitaraman [KST22] where they used representation theory of specific algebras to make cryptographic protocols.

At the moment, we have defined the algebras and proved it is sandwich cellular [TV22], a generalisation of cellularity [JG96]. This gives ways to a study of its representation theory via its cell modules. Furthermore, we are able to compute the Jones–Wenzl projectors for characteristic 0. The first step will be to complete the representation theory before diving in the modular case. This is part of an ongoing collaboration with Alexi Morin-Duchesne and Robert Spencer.

**LKB representations and infinite web calculus.** In a recent preprint, Lacabanne, Tubbenhauer and Vaz gave a formulation of Verma Howe duality [LTV22] with the pair $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_n)$. As such, it gives a double centraliser formulation with the action of both quantum enveloping algebras on a tensor product of quantum Verma modules. In it, they found that it realises the Lawrence–Krammer–Bigelow (LKB) representations [JK11].

The last problem stems from a question of Tubbenhauer: *is it possible to find a diagrammatic calculus mimicking symmetric webs to replace $U_q(\mathfrak{sl}_2)$ in the duality?* A motivation to investigate lies in the fact that it is the case in the finite-dimensional case, in the quantum Howe duality outside Verma [RT16]. Symmetric webs offer a diagrammatic calculus for the category of the representation of $U_q(\mathfrak{sl}_n)$ and its presentation by generators and relation was proven in [CKM14]. Furthermore, it is of interest to note that this is somehow an extension of the Temperley–Lieb algebra calculus linked to $U_q(\mathfrak{sl}_2)$.

The goal of this project would be to extend this calculus outside finite-dimensional modules. This would be the subject of a future collaboration with Daniel Tubbenhauer.

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