Category $\mathcal{O}$

Basics on Lie algebras, the semisimple case.

$\mathfrak{g}$ is a ss Lie algebra, we take $\mathfrak{h} \subset \mathfrak{g}$ to be its Cartan matrix, $\underset{\mathrm{ad}_x^n \mathfrak{h} = 0}{\underline{\text{maximal nilpotent}}}$

$\underset{[x,y] \in \mathfrak{h} \text{ for all } x \in \mathfrak{h} \Rightarrow y \in \mathfrak{h}}{\text{self-}\underline{\text{normalizing}}}$ algebra of $\mathfrak{g}$.

For the cartan subalgebra, we have a root system $\Phi \subset \mathfrak{h}^*$ and we call the dual of $\mathfrak{h}$, the weights of $\mathfrak{h}$, noted $\mathfrak{h}^*$.
For each root of Φ, we denote

$$\mathfrak{g}_\alpha := \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h)\, x, \forall h \in \mathfrak{h} \}$$

We have the triangle decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^- \quad \text{With} \quad \mathfrak{n} = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha, \ \mathfrak{n}^- = \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha.$$

For $\mathfrak{sl}_n$: $\mathfrak{n}$ is strict upper triangular matrices
$\mathfrak{h}$ is traceless diagonal matrices
$\mathfrak{n}^-$ is strict lower triangular matrices

We call $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$

The Borel subalgebra.

We take the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$
And then, if we have ordered bases for $\mathfrak{n}, \mathfrak{h}$ and $\mathfrak{n}^-$, the Poincaré Birkhoff-Witt (PBW) theorem state that first $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n})\mathcal{U}(\mathfrak{h})\mathcal{U}(\mathfrak{n}^-)$

and that the basis of

$$\{ \underbrace{x_1^{i_1} \cdots x_n^{i_n}}_{\text{basis for } \mathfrak{n}} \underbrace{h_1^{j_1} \cdots h_m^{j_m}}_{\text{basis for } \mathfrak{h}} \underbrace{y_1^{k_1} \cdots y_n^{k_n}}_{\text{basis for } \mathfrak{n}^-} \} \text{ are a basis for } \mathcal{U}(\mathfrak{g})$$

We note $\mathrm{Mod}\,\mathcal{U}(\mathfrak{g})$ for module, and $Z(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$ as the center of $\mathcal{U}(\mathfrak{g})$.

For a weight $\lambda \in \mathfrak{h}^*$, we denote for $M \in \mathrm{Mod}(\mathcal{U}(\mathfrak{g}))$

$$M_\lambda = \{ v \in M \mid h \cdot v = \lambda(h) v \text{ for all } h \in \mathfrak{h} \}$$

and we call them weight space of M of multiplicity $\dim M_\lambda$ for the weight λ.

Category $\mathcal{O}$, arrival

→ Introduced by Bernstein Gel'fand Gel'fand, also known as the BGG category (1976), after work by Verma (68).
→ It's a bit everywhere, especially in Kazhdan-Lusztig theory.

Definition $\mathcal{O} \subset \mathrm{Mod}(\mathcal{U}(\mathfrak{g}))$ respect

$(\mathcal{O}_1)$ M is finitely generated

$(\mathcal{O}_2)$ M is $\mathfrak{h}$-semisimple $\left(M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda\right)$, so it is also a weight module

$(\mathcal{O}_3)$ M is locally $\mathfrak{n}$-finite (for $v \in M$ $\underline{\mathcal{U}(\mathfrak{n}) \cdot v} \subset M$ is finite-dimensional, so $\mathcal{U}(\mathfrak{n})$ acts locally nilpotently)

We can deduce two others directly

$(\mathcal{O}_4)$ All weight space of M are finite dimensional

$(\mathcal{O}_5)$ (technical) $\Pi(M) = \{ \lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0 \}$ is contained in the union of finitely many $\lambda - \Gamma$, with Γ the root lattice generated by Φ^+.

Properties

a) $\mathcal{O}$ is noetherian, so $M \in \mathrm{Obj}\,\mathcal{O}$ is noetherian
Descending chain condition.

b) $\mathcal{O}$ is closed under submodule, quotient, finite direct sums

c) $\mathcal{O}$ is abelian category

d) all finite-dimensional modules are in $\mathcal{O}$

e) $M \longmapsto L \otimes M$ is an exact endofunctor when L is finite dimensional

f) $M \in \mathcal{O}$ is $Z(\mathfrak{g})$-finite

g) M is finitely generated as a $U(\mathfrak{n}^-)$ module.

We look at highest weight module in $\mathcal{O}$ for a nice subset. Another nice subset will be the Verma modules

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If you have $v^+ \in M$, it is a maximal vector of weight $\lambda \in \mathfrak{h}^*$ if $v^+ \in M_\lambda$ and $\mathfrak{n}^+ \cdot v^+ = 0$. From $\mathcal{O}_2$ $\mathcal{O}_3$, all M have at least one maximal vector.

M is a highest weight module if $M = U(\mathfrak{g})\cdot v^+$ for a certain maximal vector v^+

Properties of HWM

M is a HWM

a) $M \in \mathcal{O}$, M_μ of M is finite-dimensional $\forall \mu$

b) A quotient of M is also a HWM

c) M has a unique maximal submodule and simple quotient.

d) all simple HWM of weight λ are isomorphic.

$\rightarrow$ For $M \in \mathcal{O}$, there exists a finite filtration of M

$$0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \cdots \hookrightarrow M_n := M$$

such that $M_i/M_{i-1} \simeq N_i$ for N_i HWM.

Verma module.

Take a weight $\lambda \in \mathfrak{h}^*$, a nonzero vector v_λ.
and $\mathbb{C}_\lambda := \mathbb{C} \cdot v_\lambda$.

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda \quad \text{Verma module.}$$

The Verma module have a unique simple quotient $L(\lambda)$.

Prop Each simple module of $\mathcal{O}$ is isomorphic to one of $L(\lambda)$.

Gap!!!

Prop $\mathcal{O}$ is artinian category and so $M \in \mathcal{O}$ is of finite Jordan-Hölder length

there exist a composition series

$$0 \hookrightarrow M_0 \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow M_m = M$$

with $M_i / M_{i-1} \simeq L(\mu)$.

As a consequence, we get the first step towards categorification:

Prop The Grothendieck group of $\mathcal{O}$ is free and abelian

$$K(\mathcal{O}) = \{ [L(\mu)] \mid \mu \in \mathfrak{h}^* \}$$

$$K(M) = \sum_{\lambda \in \mathfrak{h}^*} \underbrace{[M : L(\lambda)]}_{} \, [L(\lambda)]$$

$\hookrightarrow$ Number of copies of $L(\lambda)$ in the composition series

$\to$ We get the "$\mathbb{Z}$" part of $\mathbb{Z}[S_n]$.

As a consequence of $\mathcal{O}$ being artinian and noetherian, $\mathcal{O}$ has enough projective, each $M \in \mathcal{O}$ has a projective cover

$$P \to M \to 0.$$

For the Verma module $M(\lambda)$, it has a unique simple module $L(\lambda)$, so it has a unique projective cover

$$\underline{P(\lambda)} \to M(\lambda) \to L(\lambda)$$

BGG Reciprocity

1) Each $M \in \mathcal{O}$ has a Verma flag or standard filtration

$$0 \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow M_n = M$$

where $\quad M_i/M_{i-1} \simeq \underbrace{M(\lambda)}_{\text{Verma module}}$

and we ~~denote~~ denote $(M : M(\lambda)) = \#$ of ~~q~~ quotient $\simeq M(\lambda)$ in Verma flag

2) $$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$$

Blocks

We say that two simple module M_1, M_2 are in the same block if there exist non-split ~~exact~~ exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

$$0 \longrightarrow M_2 \longrightarrow M \longrightarrow M_1 \longrightarrow 0$$

M is in a block if all its simple factor (in composition series) are in the same block.

Usually, M decompose as a direct sum of submodule belonging ~~to~~ to a single. M indecomposable means that M is in a single block.

$\downarrow$

Characters on $Z(\mathfrak{g})$?

Humphreys.

$\downarrow$

$\mathcal{O}$ decomposes as sum of block via central characters. $\mathcal{O}_0$ is the principal block.

→ Each block is isomorphic to a category of finite-dim. modules for a C algebra. (Hard to describe)

→ All regular blocks are isomorphic (under conditio on the stabilizer of the action of the Weyl group)

Takeaway: you can study the principal block linked with a trivial representation of W without worrying too much.

Denote $\mathcal{O}_{\chi_\lambda}$ the block linked to character of λ.

Gap

Translation functors

Take $\lambda, \mu \in \mathfrak{h}^*$. $\quad pr_\lambda : \mathcal{O} \longrightarrow \mathcal{O}_{\chi_\lambda}$ $\quad$ Suppose $\mu - \lambda \in \Lambda$ ← lattice

$pr_\mu : \mathcal{O} \longrightarrow \mathcal{O}_{\chi_\mu}$ $\quad$ So they are compatible

The translation functor $T_\lambda^\mu : \mathcal{O}_{\chi_\lambda} \longrightarrow \mathcal{O}_{\chi_\mu}$

$$M \longmapsto pr_\mu\left(L(\overline{\mu-\lambda}) \otimes M\right) \qquad \overline{\mu-\lambda} \in \Lambda^+$$

→ in the W-orbit of $\mu - \lambda$.

or $\quad T_\lambda^\mu : \mathcal{O} \longrightarrow \mathcal{O}$

$$M \longrightarrow inc_\mu\left(pr_\mu\left(L(\lambda-\mu) \otimes pr_\lambda(M)\right)\right)$$

Prop

a) sends projective to projective

b) $\mathrm{Hom}_{\mathcal{O}}(T_\lambda^\mu M, N) \simeq \mathrm{Hom}(M, T_\mu^\lambda N)$

c) Induces endomorphism on the Grothendieck group

d) They are exact.

Theorem (Assuming conditions technical)

$$T_\lambda^\mu(M(w\cdot\lambda)) = M(w\cdot\mu) \qquad 7.6 - 7.9$$

(→ "dot" action of W)

$$T_\lambda^\mu(L(w\cdot\lambda)) = \begin{cases} 0 \\ \simeq L(w\cdot\mu) \end{cases}$$

Induce an equivalence of category between the blocks $\quad \mathcal{O}_{\chi_\lambda} \xrightarrow{T_\lambda^\mu} \mathcal{O}_{\chi_\mu}$

$$\xleftarrow{T_\mu^\lambda}$$

$$T_\mu^\lambda (P(w \cdot \mu)) = P(w \cdot \lambda)$$

We will use lastly something called the <u>Translation through the wall</u>

$$\Theta_s = T_\mu^\lambda T_\lambda^\mu \qquad \text{for } \lambda, \mu \text{ in a } s\text{-wall of the root system.}$$

$$\Theta_s \cdot [(w \cdot \lambda)] = [(w \cdot \lambda)] \oplus [(w \cdot \lambda)]$$

$$\mathrm{Hom}(T_\lambda^\mu M, T_\lambda^\mu N) \simeq \mathrm{Hom}(M, \Theta_s N)$$

$$\Theta_s (M(w \cdot \lambda)) \simeq P(w \cdot s - \lambda)$$

$\hookrightarrow$ switch the chamber in the root system

They are used in the projective functors classification