Topics in representation theory of the Dunkl total angular momentum algebra

Onderwerpen in representatietheorie van de Dunkl totaalimpulsmoment algebra

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Dissertation submitted to the Faculty of Sciences of Ghent University to obtain the degree of Doctor of Science: Mathematics.

Ghent University
Faculty of Sciences
Department of Applied Mathematics,
Computer Science and Statistics
Academic year 2022-2023

Acknowledgements anthology

Maybe I should begin with a comment on the title. On a sunny afternoon of the summer 2020, outside at good distance of each others, I could clearly hear, for the distance had to be accounted for in the volume of our voices, my supervisor Hendrik comparing the writing styles of his students, how he enjoyed each of their idiosyncrasies. Curiosity prevailed and I joined the discussion, a feat ever so easily done in the dance of sphere packing we were still following, and rejoicing at the sight of my approach from the many meters, he took me as example: “for example, never have I had a student whose writing I could describe as ‘flowery’ before Alexis!” As he had had his first encounter with my scientific prose recently, I could not be so surprised: such aesthetic judgments have been made times before in various variations, not the least by my previous supervisor with whom I had just spent the previous summer and fall trying to circumspect my writing. I thought to have at the least improved my ailment. Not yet apparently, and so I strove the next years to improve this unfortunate trait. For the readers that shall go past this preface, we can only hope that I have been successful. I shall also say, for my own sake, I can only hope to improve it.

Nonetheless, I felt this flowery style might well be the most suited for a section dedicated to acknowledge the support and help I have received during the last four years and so, to the dismay maybe only of Hendrik, shall it be the voice in which they are written, and hence why this *anthology* of acknowledgments, an etymological quip I hope the reader will forgive. If not only to make a practical joke, this anthology, deze bloemlezing, ce florilège, shall also serve to help me solve another conundrum: what to do with the books that have came to share my flat? I could certainly not leave behind such nice flatmates, never complaining, never messing around, always available when needed and discrete when should be; yet, alas, it is

too big a burden to let them travel with me, and their nature, most home-bound, would not be suited for such uncertain trips. And so, from the books I read during my stay in Ghent, shall a selection make their way into the hands of the people who, like them, helped make it a more pleasant place.

Naturellement, peut-être à l'image de mes dernières années passées dans un mélange perpétuel de langues: English at work and with new friends, Nederlands naast het werk met het leven en met nieuwe vriend·inn·en puis français avec la famille et dans les solides amitiés qui se sont préservées dans le fil ténu des rencontres et des appels, et aussi pour laisser sortir une voix plus humaine, je ferai un petit mélange langagier, sans indiquer les changements, au rythme de ce qui me vient; petit plaisir de ne pas nécessairement avoir à être compris.

Over supervision and jury

It is only fair that I begin with those who supervised this work and those who accepted to supervise it. I will begin by the jury, but before, it might do good to state how the jury is made, for the curious readers who might be reading the acknowledgments of other people.

In UGent, the thesis is evaluated by a jury composed of: the supervisors, Hendrik De Bie, Roy Oste and Joris Van der Jeugt; external examiners, here Martina Balagović and Kieran Calvert; members of UGent outside the faculty, here Martino De Martino and Hennie De Schepper, and finally the president of the jury, Bart De Bruyn. The jury received the thesis on January 17th. A first defence, closed to the public, was held on March 6th after the jury could read and comment on the manuscript. This is were I answered the many technical questions the jury found suitable to ask. After this, I incorporated the remarks of the jury to the current manuscript at the best of my capacity and we went to the public defence on April 24th, where I presented the subject to the greater public and answered the last questions of the jury.

J'espère que ceci explique le déroulement de la fin d'un doctorat à Gand et passe sans plus de commentaire aux premiers remerciements. They shall proceed in the reverse order of appearance in the previous enumeration.

Aan de president Bart De Bruyn, bedankt voor een goede organisatie en om mijn vragen te hebben beantworden. Ik heb het boek *Raad eens wie boven ligt* interessant gevonden, en ik hoop dat u ook.

Aan Hennie De Schepper, bedankt om de tijd te hebben nemen om mijn proefschrijft te lezen. We hebben elkaar ampers gezien deze jaar, wat ik jammer vind want ik heb vaak over u gehoord en altijd was het positief! Omdat ik weet dat u weinig tijd, denk ik dat dit klein boek, *Willem* ,over een briefdiefstal zal lukken.

To Marcelo De Martino, I wish to express my appreciation for your constant enthusiasm and my admiration for your passion in every aspect of your life. I do not think I have heard someone laugh so much while doing mathematics, or simply in general. One last time, I want to reassure you that it was always a pleasure to do mathematics with you. The book *The three-body problem* left me thinking of you, as I could see you dive as far as the author in the consequences of small advances.

To Kieran Calvert, it was a pleasure to talk with you in Manchester, and your insightful comments really helped me improve this research. One of the aspect I appreciate the most of mathematics, and that Covid sadly prevented for a while, is to meet interesting people with a shared interest. I am happy to have been able to meet you and hope to be able to do so again in the near future. I bought this book, *North and South*, in Manchester during my visit, and I hope I will be able to let it find its way to you soon.

To Martina Balagović, I wish to express my thanks for the thorough reading of the thesis and the many comments given. Reading the annotated manuscript proved a most instructive incursion in how one reads mathematics and extracts important information, and I am still trying to parse how to pursue some of the ideas you suggested. To thank you, I would like to share an exposition booklet of the artist Hiroshi Sugimoto, *Conceptual forms and mathematical models*.

If the jury spends more of their share of time reading this thesis, or more precisely, a version of the thesis that still had not benefited of their insights, what to say of the supervisors who had to endure me for the past 4 years, and read the drafts of article with their frustrating slow progress. Now that the time to close this adventure arrives, I realise that I have not taken enough advantage of their wisdom and experience, as we say, au temps pour moi! but it will not prevent me from expressing my gratitude.

Dear Joris, thank you for always being available for questions. You always made me feel trusted and valued during my time in Ghent and I really enjoyed your approach to work: even when work was taking over, you always found a way to make time for me. As a token, the first Belgian book I read in Ghent: *Bruxelles, here I come*.

Dear Hendrik, thank you for your advice and your pragmatic views, always useful to ground for someone who has a tendency to loose himself in the ideas and not the realisation of the ideas. I toyed with the idea of letting you my edition of *La recherche du temps perdu*, but even if I think you would appreciate its bad influence away from me, I think the simple and direct poetry of Cees Nooteboom in *Vos* would be more adapted.

Dear Roy, van academische peter tot promotor, je was altijd bij voor alle mijn vragen en het was altijd best boeiend om met jou onderzoek te doen. Ik ben erg dankbaar dat ik met jou heb kunnen werken. Maar ook, het was een plezier om jou een vriend te kunnen noemen. Comme cadeau, je pensais à *La vouivre* de Marcel Aymé, un de mes auteurs préférés d'adolescence, en référence au dragon de Gand. I hope our paths will cross many times in the following years!

On friendship and family circles

Bien futile la comparaison des amitiés, dans les incomparables dynamiques des relations qui se développent au fil de la vie, on ne peut chercher à hiérarchiser sans tomber dans un distinction qui ne tient plus la route dès que changent nos conditions. S'il faut une hiérarchie, qu'elle soit basée sur des faits. So to decide who to mention; despite my statement about flowers and style, I do want to keep this at a pace, let only because it will be printed and that I can only plant so many trees, so I will restrain myself and consider factually how to decide whom and how to mention them. Of course, the first fact that I cannot avoid is the help receive: all my friends share a part in my hearth, but some of them really warmed it, especially when I looked through my office windows to enjoy yet another rainy Sunday dawn, those are the ones who actually read this manuscript beforehand. To those I have a special circle in my hearth.

With the highest distinction

Du haut d'une chaire, Jean-Philippe Chassé, tu as su tonitruer sur ma prose, injustement bien sûr, mais j'imagine qu'à toi peut être attribuée une certaine constance de ponctuation au travers de ce manuscrit. Petite pique exceptée, je suis fort content que tu as passé un trajet de train Gand–Zürich sur mes deux premiers chapitres et y as investi ton temps. Bien entendu, j'accorde plus de valeur à notre amitié que simplement sur le (très bon) travail d'édition que tu as pu faire. Nos multiples appels et notre correspondance assidue m'ont soutenu tout au long de ces quatre dernières années, et j'espère bien continuer à connaître ton adresse! Tu ne seras pas surpris·e de mon choix de livre, *Miracle de la rose* de Genet.

A full week you spend, Asmus Kjær Bisbo, to read and correct this thesis. As you thought me, a week is 37.56 hours and I would certainly want to spend a full week thanking you, your comments were the fuel that motivated my last days writing. As if not enough, you were also my companion, of office, of covid, of travel, of boardgames, long walks and an explainer of the true sense, and pronunciation, of *hygge*. In the spirit of our discussions full of "what if" I think *Never let me go* from Ishiguro might be interesting for you.

With high distinction

I also would like to reserve a small place to those, not mentioned yet, with whom I collaborated during my research, also in the extended vision of research.

Cher Alexi, ta venue à Gand la dernière année de mon doctorat a été une surprise des plus agréable. Après avoir passé ma maîtrise à poursuivre tes travaux, travailler avec, et côtoyer, l'homme derrière la prose a été révélateur et a été certainement une des périodes favorites de mon doctorat. Pour commémorer notre projet d'algèbres diagrammatiques, *L'écorché* me semblait adapté.

Très cher Charles, ce fut un plaisir véritable que de de côtoyer ces trop rares moments où nous nous trouvions à proximité; je te le dis assez pour ne pas avoir besoin de le redire. Merci aussi d'avoir embarqué dans ce projet de vulgarisation mathématico-échiquéenne au milieu de la covid. *La défense Loujine*, de mon toujours apprécié Nabokov, me semble des plus appropriés en rapport à ce projet.

Beste Toon, je was altijd mee in alle projecten dat ik met jou wilde bespreken. Je weet waarschijnlijk al alles erin, maar *X-factor, 20 verhalen over de onzichtbare kracht van wiskunde* was bijna bedoeld voor jou!

Cher Yvan, tu t'inclus encore des les remerciements de cette thèse, comme nous avons terminé notre recherche durant les premiers mois de mon séjour, mais aussi comme l'impact de ta supervision s'est fait sentir tout au long de cette deuxième aventure. Un livre belge, *Le passeur de lumière : Nivard de Chassepierre maître verrier* semble approprié et devrait t'atteindre en temps et lieux.

With distinction

Naturally, I cannot ask to everyone I know to read my thesis or collaborate with me, but there are many more ways to support, and I wish to highlight a few.

À mes parents, qui m'ont accompagné dans cette aventure sans pouvoir m'y suivre, je penche sur la bande dessinée *Oleg* : la forme pour rappeler la Belgique qui m'a accueilli, mais aussi comme on y voit quelqu'un se perdre dans l'écriture en pouvant compter sur son entourage, thématique donc.

À mes soeurs Catherine, Sophie et Sylvie, j'espère que les quelques bandes dessinées belges que je rapporterai vous donneront un air de ces dernières années.

À Hadewijch, merci de m'avoir accueilli à mon arrivée en Belgique. Nous avons pu partager maintes discussions, glaces, clémentines, j'espère que ce petit livre québécois, *Les falaises*, te fera penser à ton ami québécois!

Aan Sven, bedankt voor de gezelschaft en voor de vele discussies rond literatuur, taal, gewoontes en eekhoorn. Je hebt wel mijn wankele Nederlands moeten verdragen. Ik heb al vele boeken aangeraden, maar ik hoop dat deze gedichtenverzameling van Joëlle Sambi, *Caillasses* zal voldoen.

Aan Charlotte, de bureau zou anders geweest zijn zonder jou. Tautologie, maar ook waar. Bedankt voor de vele dingen die je om onze werkvloer leuker te maken hebt gedaan. De groeten aan Kero Kero ook, en ik hoop dat uit *De avond is ongemak* zal je de poëzie halen uit de hard wereld van Marieke Lucas Rijneveld.

À Eleonora, merci de nos discussions littéraires, poétiques, sur le langage et donc sur tout. Elles m'ont aidé à quitter quelques fois l'univers mathématiques pour la grande vie. Une poétesse québécoise, Chloé Savoie-Bernard et son recueil *Sainte Chloé de l'amour* comme cadeau en l'honneur de ces moments.

Aan Niko, nieuwe laatkomer in ons bureau. Ik dacht direct aan jou en je standvastigheid toen ik het boek *Rituelen* van Cees Nooteboom in mijn bibliotheek terugvond, misschien zal je het aanvaren.

To Sigiswald, we sadly did not speak literature enough, but let me share with you one of my favorite French author, Michel Tournier, and his masterpiece *De elzenkoning* with the hope to meet again to speak about it.

Aan Ann, die altijd hulpzaam was voor alle administratieve taken, ik heb het boek *Klank van jasmijn* gevonden en ik denk dat je het zou kunnen apprecieren.

To Lea, you really became a friend in those years. Your hospitality made me discover the true sense of this word and I will deeply miss it. I had only French books to recommend you, so I will continue pushing my French agenda by recommending *Moi, Tituba sorcière...*

To Srdan, always you made an effort to put your best out regardless of how difficult it was at that moment and I hope to be able to learn from that trait. *Sense and sensibility* to represent your strive for perfection and your sensibility to your surrounding.

To Juan, *Jamrach's Menagerie* is a story of adventures, with its sad and terrible moments, but also with the awe it comes with it.

Aan Felix, met wie ik heb mijn eerste Nederlandse boeken gedeeld, ik geef jou *Magiër* van Feist. You have it already in English, but then, a second version is also an invitation to share as you did.

À Rémi qui est venu maintes fois me rendre visite, j'aimerais te partager *Berlin Alexanderplatz*, un de mes livres préférés qui saura te plaire je pense.

Aan Jorg en Louise, hun kippen en katten, denk ik dat met deze albums van Moebius, *De wereld van Edena* een deel van mijn geest zal blijven in je bibliotheek.

Aan Oliver Lenz, ik heb met jou veel leuke gesprekken gehad en je was een van mijn eerste contact met Nederlands, het boek *Een vis verdrinken* is een gedacht over Nederland, taal en kijken.

Dear Jari, I know it will be difficult for you to loose one you had so much fun teasing, so I think *Het leven gaat verder* will have good lessons for you.

Dieter, je hebt op mijn eerste dag voor de resto uitgenodigd en misschien is het raar, maar dat was iets dat ik toen buitengewoon vond. Misschien is het best uitgelegd in een van de eerste boeken die ik in het Nederlands gelezen hebt: *Het zigzagkind*.

Steven, ik vond het erg fijn om over "extra"-onderzoek met jou bezig te zijn, en denk dat *Things to make and do in the fourth dimension* verder in dit idee met jou kan gaan.

Aan Rien, nieuwekomer van de bureau, een gesprek was altijd leuk met jou en ik raad een reisverhaal van Huxley, *Beyond the Mexique Bay*.

Chère Shophika, tu m'as accompagné dans cette aventure à distance avec toujours du support, je t'ai déjà transmis bien des livres dans les années, mais je pense que tu aimeras le recueil de poème *Caillasses* de Joëlle Sambi.

Dear Carol, you were an always interesting partner of conversation and I wish we had more time to have had more of them! The beeldkunstige poetry of Annemarie Estor in her bundle *Onster Target* evoked some of the painting you had in our first shared office en I think you might appreciate it.

À Daniel, mon tandem de langue, het was te interessant met jou te praten om bezig te zijn met verbeteren, jammer voor onze talen, maar zeer plezant! Totdat je kan het in het Frans lezen, *De buik van Parijs* van mijn lieveling Zola.

Nicolas, you had access to all these books during our cohabitation, so maybe recommending one here is a bit pointless and instead I will let you grab what you prefer.

Aan Benjamin, heel leuk om kennis te maken; je hebt me ooit verteld dat je geen fysiek boek wil, dus zal ik er hier zeker geen geven.

And now comes the difficult moment: I simply do not have the time or the space to name everyone that had an impact, and here I must cut short this (in retrospective, too ambitious—again we see I should have listened to the good advice of Hendrik) idea. Luckily, the comment of Benjamin just above helps me by letting me think that maybe in any case, the "gift" received would be more burden. So, Adnan, Ali, Arne, Astrid, Bart, Bryan, Georgi, Gunnar, Hans, Jan, Jonathan, Kelly, Kris, Marnix, Michèle, Michiel, Nico, Oliver, Peter, Pieter, Robbert, Robin, Sam, Steven, Teppo, Tibo, Tim, Tom, Veerle, Wout, Yang, Zoe and anyone the pressing time made me forget, you made the environment of the departments nice for me and I appreciate it. Alexis, Camille, Élise, François, Gabriel, Geoffroy, Ismaël, Julien, Jonathan, Marc-Antoine, Olivier, P-A, Théo, Vanessa, Victoire, vous avez rendu mes retours à Montréal des plus plaisants, merci de votre support continué!

On public support

Knowledge seldom appears out of thin air, it needs a very privileged environment to grow. I was very fortunate to have been supported by scholarships during my stay and here is my acknowledgment of their support. This research was supported by PhD scholarships from the Fonds de recherche du Québec – Nature et technologies (270527) and the Excellence of Science Research Project PRIMA (30889451)

Contents

1

Introduction

1.1 Remembrance of things past, stories

Mathematics does not happen in a vacuum. Mathematicians, those who do mathematics, are the people infusing life in the theory. Subjects are studied and established from sparks and with reason; be it for their intrinsic beauty, in the search of a solution to concrete problems, as they are stumbled upon while searching for something else, or even as the result of an obsession.

The subject of this thesis thus does not exist in a vacuum; it follows ideas established in the last two centuries and reinterpreted by contemporary research.

To lay the foundation whose ground this thesis builds on, the next few pages will quickly survey a selection of encounters that paved the way for us. The stories come mostly from the historical notes of Bourbaki [Bou07a], Humphreys [Hum90], Coxeter [Cox73] and van der Waerden [Wae85].

When Felix Klein and Sophus Lie met in Berlin in 1869, they shared an interest in geometric structures that would shape the landscape of mathematics for the next half-century. In Berlin, the great news at that time was the publication by Jordan of his commentary on the work of Galois [Jor69]. The ideas of Galois were already known to the young Lie and the publication was the occasion for him to share them with Klein and work together. They met again in Paris the next year to follow the course of Jordan from his famous *Traité* [Jor70] on groups and substitution. The trend, started by Descartes, of mixing geometry and algebra would be pushed forward by them. Klein would go on to attempt to unify the different geometries with his Erlangen program and Lie would go on to study continuous groups. Their next (non)-meeting would be at the end of the 19th century when Klein arranged the appointment of Lie in Leipzig to replace him; even if they would not work together again after that, it would set up the scene for the ideas of the young Norwegian to reach the rest of Europe (for more on the life of Klein and Lie see respectively [Tob19; Fri99]).

The ideas of Lie bloomed in Leipzig with the help of Engel [LE93], and their school would become a motor for young mathematicians. Studying transformation groups, now known as Lie groups, had lead them to the study of their infinitesimal transformations, what we now call Lie algebras. Even that early in their study, they recognised the importance of the correspondence between the geometric differentiable world of Lie groups and the algebraic linear one of Lie algebras. The focus was put on semisimple Lie algebras.

The teaching of Lie was highly influential for the new generation of mathematicians and would blossom after the work of Killing [Kil90] and the systemic study of Cartan [Car94]. Their main idea was to use a subalgebra that is invariant under a group specific to the algebra. This group is now known as the Weyl group of a classical Lie algebra, and the subalgebra, as the Cartan subalgebra. They were able to classify the simple complex Lie algebras by focussing on the properties of a geometric structure: the root system associated with the Cartan subalgebra. But it was really van der Waerden [Wae33] and especially Coxeter [Cox34] that understood the intrinsic potential of root systems and started a systematic approach to reflection groups.

Another side of this picture resides in the study of special functions and hypergeometric functions. Klein and Fricke [FK97], and Poincaré, following the works of Riemann and Möbius, would develop the theory of automorphic functions, and recognised its similarities with the search of discrete subgroups of the motion group of the hyperbolic plane.

Classical orthogonal polynomials have a lot of symmetries. But just how central their symmetries were to their study became really apparent in the work of MacDonald on Hall polynomials [Mac98] (now more commonly called MacDonald polynomials) that sparked research in the eighties. These polynomials generalise many families of orthogonal polynomials and the insight of MacDonald was to conjecture they would also share many of properties of classical orthogonal polynomials, most notably a conjecture of a constant formula for their norm [Mac82].

The interest in orthogonal functions and polynomials and their symmetries also led Charles Dunkl to investigate the relations of these objects with reflection groups, leading to his now famous definition of differential-difference operators, nowadays most commonly known as Dunkl operators [Dun89]. This allowed him to retrieve classes of orthogonal polynomials as solution to differential equations deformed with the Dunkl operators.

The norm conjecture of MacDonald was only proven in full generality by Cherednik [Che95], who defined to this goal double affine Hecke algebras, now often called Cherednik algebras. The idea of Cherednik was to show that the MacDonald polynomials appear in the representation theory of an algebraic object: the deformation of an affine Hecke algebra. It enabled the use of results on the structure of the representation of the algebra to gain knowledge on the polynomials. To this goal, he used crucially the newly-defined Dunkl operators and an insight of Heckman [Hec91a] linking them to shift operators.

The viewpoint of Cherednik was highly influential, and it has inspired an active research program. Stemming from this, while investigating symplectic algebras, Etingof and Ginzburg [EG02] defined the rational Cherednik algebra, having in mind the study of integrable systems, specifically Calogero–Moser systems [Eti07; Fei12]. For a given reflection group and a parameter function, rational Cherednik algebras are the algebras encoding the structure generated by the Dunkl operators linked to the group, the group algebra, and multiplication by the variables. In general, the representation theory of the algebra changes at some singular points of the parameter function. For example, in the case of the root system A_n, the parameter function is a constant and there are finite-dimensional representations only when the constant is some rational number [BEG03].

The representation theory of rational Cherednik algebras has been studied for most cases: for A_n [BEG03; Rou08], for the dihedral root systems [Chm06], for the exceptional root system H_3 [BP14], and then the remaining exceptional cases in a series of papers [Nor14; Nor16a; Nor16b; LS18] and for the unitary case B_n [Gri18] with link to crystal combinatorics [Nor21]. General results were also obtained by studying a category equivalent to the Bernstein–Gel'fand–Gel'fand category $\mathcal{O}$ [BGG71] in the rational Cherednik algebras setting [Gin+03; Eti12].

The last story we will tell goes back in time to follow a subject that developed in parallel: Clifford algebras. Introduced by Clifford in his study of Grassmann exterior algebras [Cli78], Clifford algebras (or as Clifford called them, geometric algebras) entered the first stages of mathematical physics in the works of Pauli [Pau27] and Dirac [Dir28], on the nonrelativistic and the relativistic wave equation of the electron, respectively. Brauer and Weyl connected Clifford algebras with Lie theory [BW35], joining the work edified by Élie Cartan in his study of Lie algebra representations, which included spinors [Car38].

The place of Clifford algebras in mathematics folklore has been cemented by the seminal work of Atiyah, Bott and Shapiro [ABS64] for their use in K-theory. They remain an important tool for mathematics and mathematical physics. See [Tra06; BDS82] for more on Clifford theory.

1.2 Present work

In this thesis, we study an algebra: the (Dunkl) total angular momentum algebra via its representation theory. The representations of this algebra motivating its study are the polynomial null-solutions to the Dunkl–Dirac equation, the so-called Dunkl monogenic polynomials. This algebra is the supercentraliser of a realisation of the Lie superalgebra $\mathfrak{osp}(1|2)$ inside the tensor product of a rational Cherednik algebra and a Clifford algebra.

This algebra appeared before in the study of the Bannai–Ito algebra [DGV16a] and the higher-rank Bannai–Ito algebra [DGV16b]. The even part of the algebra, named the Dunkl angular momenta algebra, appeared in the study of a deformation of the quantum angular momentum algebra as a quadratic subalgebra of the rational Cherednik algebra [FH15].

A related abstract algebra generated by symmetries of a generalised Dirac operator was given afterwards [DOV18a], and further properties were given in [Ost22]. Note that the full set of relations of the abstract algebra is not yet known apart from low dimensions.

In contrast to the Dunkl angular momenta algebra and rational Cherednik algebra, the Dunkl total angular momentum algebra does not have a PBW-like basis theorem. This is due to the embedding of the reflection group being a double covering interchanging the positive and negative subspace. The Dunkl total angular momentum algebra also differs from the Cherednik algebra by having a family of finite-dimensional unitary irreducible representations for any positive parameter function. A realisation of this family of representations in the Dunkl representation is given by the Dunkl monogenic polynomials: the null-solutions of the Dunkl–Dirac operator.

Only partial results on the representation theory of the Dunkl total angular momentum algebras were known before the start of this thesis. Namely, it had been studied only for the reflection groups $W = \mathbb{Z}_2^d \subset \mathcal{O}(d)$ [DGV16a; DGV16b] and $W = S_3 \subset \mathcal{O}(3)$ [DOV18b].

The main results of this thesis are described briefly as follows. We show a construction of Dunkl monogenic polynomials using generalised symmetries for any reflection group and positive parameter function (Theorem 3.4.4). We give the full classification of unitary finite-dimensional representations of the Dunkl total angular momentum algebra for the groups $W = \mathsf{D}_{2m} \times \mathbb{Z}_2$ in three dimensions (Theorems 5.6.1, 5.6.2). Finally, we study finite-dimensional representations of the Dunkl total angular momentum algebra for the groups $W = \mathsf{D}_{2m} \times \mathsf{D}_{2n}$ in four dimensions (Theorem 6.5.9 and Proposition 6.5.14).

1.3 Structure of the thesis

The thesis collects results published or submitted. Most of the content is taken from the published versions, but we often provide more detail when possible. Furthermore, to avoid repetitions, we merged some results together, making use of new insights gained over the years.

Section 2 will give the necessary knowledge to allow the readers to dive right away into the core research. Each subsequent chapter is related to one of the articles written during my doctoral studies, see the next section for the bibliographical details. They will be reviewed in more detail here.

Chapter 3 contains article [DLOV22b]. It explores one specific basis for the Dunkl total angular momentum algebra constructed from generalised symmetries. This basis exists for all reflection groups W. One of the main advantages of the basis lies in the fact that the double covering $\widetilde{W}$ of the group has a tractable action on it. On small rank or reducible examples, it also can be used to relate the algebra to the theory of special functions.

Chapter 4 presents a proceedings paper [LO20]. This short note covers some of the properties of the total angular momentum algebra linked to the exceptional root system G_2. It is, in a sense, superseded by Chapter 5, but it is our belief that it can be of use as an example of the elementary algebraic approach to the symmetry algebra taken in the following chapters. The embedding of the root system in the space was chosen to conform to the more usual embedding of root system A_2 in $\mathbb{R}^3$, it is not the one used in Chapter 5.

Chapter 5 takes the main part of the thesis. It contains an extended version of article [DLOV22a], including some further work done in a proceedings paper [Lan22]. It contains a classification of all finite-dimensional representations of the symmetry algebra linked to dihedral groups of the form $\mathsf{D}_{2m} \times \mathbb{Z}_2$.

The results in that chapter are obtained purely algebraically using the minimal number of assumptions, and they are linked to special functions at the end of the paper as polynomial null-solutions to the Dunkl–Dirac equation.

The results of the proceedings paper [Lan22] were concerned with generalising the signature of the Clifford algebra used. When relevant, the generalised result is put in the chapter. However, the main theorems are left with the convention of [DLOV22a]: changing the general Clifford signature would represent a lot of careful sign-checking to gain very little insight, and so we decided against it.

Chapter 6 contains the preprint [DLO23]. In this chapter we consider the total angular momentum algebra related to the groups of the form $W = \mathsf{D}_{2m} \times \mathsf{D}_{2n}$. It is a stepping stone to the general cases of a product of an arbitrary number of dihedral groups.

In this chapter, we present general properties of the algebra using the specificities of the group, and then go on to prove a coarse classification of the possible representations. The main striking feature we were able to use is the presence of a subalgebra with a triangular decomposition. We focus on a class of representations that contains the Dunkl monogenics and provide a classification for them. An example closes the chapter.

1.4 Articles written

During the course of my doctoral research, I have written the following preprints, articles and proceedings contributions.

Preprints

[DLO23] M. De Martino, A. Langlois-Rémillard, and R. Oste. *Double dihedral total angular momentum algebra*. Work in progress. 2023 (cit. on pp. 7, 8, 9, 143).

[LM23] A. Langlois-Rémillard and A. Morin-Duchesne. *Uncoiled affine Temperley–Lieb algebras and their Wenzl–Jones projectors*. 2023. arXiv: 2302.12782 (cit. on p. 8).

[LMR22] A. Langlois-Rémillard, C. Müßig, and É. Roldán-Roa. *Complexity of Chess Domination Problems*. 2022. arXiv: 2211.05651 (cit. on pp. 8, 9).

Published and accepted articles

[DLOV22a] H. De Bie, A. Langlois-Rémillard, R. Oste, and J. Van der Jeugt. "Finite-dimensional representations of the symmetry algebra of the dihedral Dunkl–Dirac operator". *J. Algebra* 591 (2022), pp. 170–216. DOI: 10.1016/j.jalgebra.2021.09.025. arXiv: 2010.03381 (cit. on pp. 6, 7, 8, 9, 73).

[DLOV22b] H. De Bie, A. Langlois-Rémillard, R. Oste, and J. Van der Jeugt. "Generalised symmetries and bases for Dunkl monogenics". *Rocky Mt. J. Math.* (2022). To appear, 18p. arXiv: 2203.01204 (cit. on pp. 6, 8, 9, 37).

[LS20] A. Langlois-Rémillard and Y. Saint-Aubin. "The representation theory of seam algebras". *SciPost Phy.* 8.2.019 (2020), 34p. DOI: 10.21468/SciPostPhys.8.2.019 (cit. on p. 9).

Published and accepted proceedings contributions

[Lan21] A. Langlois-Rémillard. *Deforming algebras with anti-involution via twisted associativity*. To appear in Proceedings in Mathematics & Statistics, SPAS II Västerås 2019. 2021. arXiv: 2106.01855 (cit. on p. 8).

[Lan22] A. Langlois-Rémillard. "The dihedral Dunkl–Dirac symmetry algebra with negative Clifford signature". *Lie Theory and Its Applications in Physics, LT 2021*. Ed. by V. Dobrev. Vol. 396. Springer Proceedings in Mathematics & Statistics. 2022, 7p. DOI: 10.1007/978-981-19-4751-3_50. arXiv: 2209.06599 (cit. on pp. 6, 7, 8, 73, 94).

[LO20] A. Langlois-Rémillard and R. Oste. "An exceptional symmetry algebra for the 3D Dirac–Dunkl operator". *Lie Theory and Its Applications in Physics, LT 2019*. Ed. by V. Dobrev. Springer Proceedings in Mathematics & Statistics. 2020, pp. 399–405. arXiv: 2009.13904 (cit. on pp. 6, 8, 9, 65, 77).

The thesis contains items [DLOV22b], [DLOV22a], [Lan22], [LO20] and [DLO23] The preprints [LMR22; LM23] and the proceedings contribution [Lan21] reached subjects tangential to my main area of research and are thus omitted from the text for coherence and concision. The article [LS20] was finished during my first months in Ghent, but is concerned with the research I did during my master's degree at Université de Montréal; it should therefore not be taken into account for evaluation purposes.

To finish, a note on authorship. Most of the articles are written in collaboration. It is hard, and mostly meaningless, to try to quantify contribution in sciences and specially in mathematics, but for the purpose of evaluation I can be considered the main author for papers [DLOV22b; DLOV22a] and have contributed equally with the other co-authors for contributions [LO20; DLO23; LMR22].

2
Preliminary notions

This chapter has one goal: to give the readers the tools that they need to start reading any of the remaining chapters right away. We would recommend a chronological reading, since the choice has been made to gradually increase the complexity and abstraction needed.

We begin by reviewing briefly reflection groups and root systems, and presenting Dunkl operators. We then proceed to introduce the main subject of this thesis: the (Dunkl) total angular momentum algebra. Before it can come to the front of the scene, however, a cast of algebraic notions must be presented.

We hope the readers will find the sources we give at the beginning of each section useful should they wish to go deeper in the subject. For more historically oriented references, we refer to the previous chapter.

2.1 Reflection groups and Dunkl operators

No objects are more central to this thesis than reflection groups. As such, they deserve the first place in this presentation that has no pretension to honour them at their just value. We begin with generalities, present common properties the chapters will use, and end the section by considering in detail Dunkl operators, one of the main tools to realise the abstract algebras we study. We are following the book of Humphreys [Hum90] for reflection groups and root systems, and that of Dunkl and Xu for Dunkl operators [DX14].

2.1.1 Reflection groups and root systems

Let V be a Euclidean space with bilinear form $\langle -,- \rangle$. In most of the thesis, we will take $V = \mathbb{R}^d$ with the canonical inner product. On Euclidean spaces, the notion of *reflection* takes place naturally. For a non-zero vector $\alpha \in V$, the reflection with respect to the hyperplane normal to α is denoted by

$$\sigma_\alpha(x) := x - 2\frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle}\alpha. \tag{2.1}$$

It is exemplified geometrically in Figure 2.1 in the case $V = \mathbb{R}^2$.

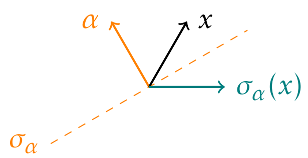


Figure 2.1: The reflection of x by σ_α.

From the notions of reflections and of Euclidean spaces, it is already possible to introduce the definition of root systems.

Definition 2.1.1 (Root systems)**.** *Let $\Phi \subset V - \{0\}$ be a finite collection of non-zero vectors. It is called a* root system *if*

1. *Φ spans V as a vector space;*
2. *$-\alpha \in \Phi$, for every $\alpha \in \Phi$;*
3. *$\sigma_\alpha(\beta) \in \Phi$, for every $\alpha \in \Phi$ and all $\beta \in \Phi$.*

A root system is called reduced *if*

4. *$\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$, for every $\alpha \in \Phi$,*

and crystallographic *if furthermore*

5. $2\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$, *for all* $\alpha, \beta \in \Phi$.

The rank *of a root system is the dimension of* V*, and the elements of a root system are called* roots.

In this thesis, we will always consider reduced root systems, and specifically focus on one family of non-crystallographic root systems.

A root system Φ is called *reducible* if it is the sum of two non-trivial root systems, and it is called *irreducible* if it cannot be expressed as the sum of two root systems.

Let $\mathrm{H} \subset V$ be an arbitrary hyperplane that does not contain any vector of a root system Φ. The hyperplane then divides the roots of Φ into two: the positive roots Φ_+ and the negative roots Φ_-:

$$\Phi_+ := \{\alpha \in \Phi \mid \langle \mathrm{H}, \alpha\rangle > 0\},\ \Phi_- := \{\alpha \in \Phi \mid \langle \mathrm{H}, \alpha\rangle < 0\}. \tag{2.2}$$

It is a remarkable fact of the geometry of root systems that they can be generated by a subset of their positive roots with a positivity constraint.

Proposition 2.1.2. *Let* $\Phi \subset V$ *be a root system of rank* n*, and let* Φ_+ *be the subset of positive roots with respect to a hyperplane* H*. Then there exists a set of* n *positive roots* $\{\alpha_1, \dots, \alpha_n\} \subset \Phi_+$ *such that any root* $\beta \in \Phi$ *can be expressed as*

$$\beta = \sum_{i=1}^{n} c_i \alpha_i, \quad \textit{with the } c_i\textit{'s either all positive or all negative.} \tag{2.3}$$

Such a set of roots is called a set of simple roots.

The properties of root systems we will use are independent of the choice of positive roots. So let us always fix a certain set of positive and simple roots when a root system is defined.

Definition 2.1.1, simple as it is, offers a very restricted realm of possibilities. There are four infinite families of irreducible crystallographic reduced root systems and five exceptional ones. The four

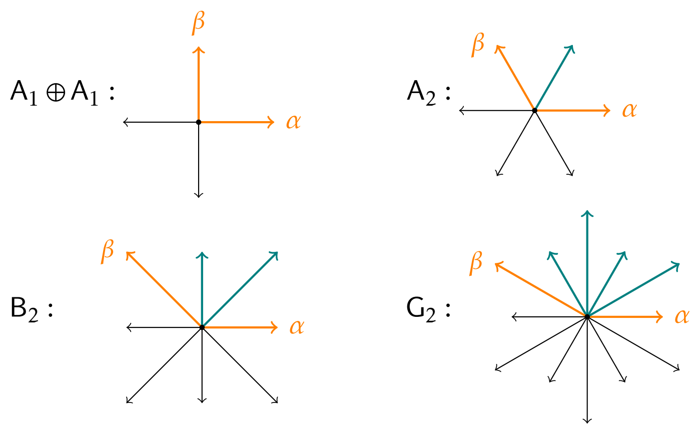


Figure 2.2: The four crystallographic reduced root systems of rank 2. The orange roots are the two simple roots and the teal roots are the remaining positive roots for a certain choice of hyperplane H.

crystallographic reduced root systems of rank two are presented in Figure 2.2.

Removing the crystallographic condition adds one infinite family of rank 2 irreducible root systems and two more exceptional ones. The core of this thesis will focus on this infinite family: the dihedral root systems.

Before continuing with the properties, we include a nomenclature for all the possibilities of irreducible root systems for completeness. The complete classification, with concrete realisations, can be found in the classical references [Bou07b; Hum90].

Theorem 2.1.3. *Let V be a Euclidean space. An irreducible reduced root system $\Phi \subset V$ lies in one of the following crystallographic families:* $\mathsf{A}_{n-1}(n \geq 2)$, $\mathsf{B}_n(n \geq 2)$, $\mathsf{C}_n(n \geq 3)$, $\mathsf{D}_n(n \geq 4)$*; one of the following exceptional crystallographic root systems:* E_6, E_7, E_8, F_4, G_2*, or is one of the following non-crystallographic root systems:* $\mathsf{H}_3, \mathsf{H}_4, \mathsf{I}_2(m)(m \geq 5, m \neq 6)$.

Remark 2.1.4. *For ease of notation, we will extend the family* $\mathsf{I}_2(m)$ *to*

$m \geq 2$, *understanding that the following identifications happen:* $\mathsf{I}_2(2) \simeq \mathsf{A}_1 \oplus A_1$, $\mathsf{I}_2(3) \simeq \mathsf{A}_2$, $\mathsf{I}_2(4) \simeq \mathsf{B}_2$, $\mathsf{I}_2(6) \simeq \mathsf{G}_2$.

The study of crystallographic root systems came hand in hand with that of Lie algebras. The crystallographic families label the families of simple complex Lie algebras. The correspondence is as follows:

1. A_n corresponds to $\mathfrak{sl}(n+1)$;
2. B_n corresponds to $\mathfrak{so}(2n+1)$;
3. C_n corresponds to $\mathfrak{sp}(2n)$;
4. D_n corresponds to $\mathfrak{so}(2n)$.

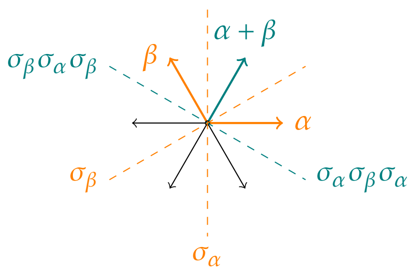
The exceptional crystallographic root systems are also linked to simple complex Lie algebras.

Denote by $\mathcal{O}(V)$ the group of endomorphisms of V that preserves the norm induced by $\langle -,- \rangle$. The reflections generated by the elements of the root systems form a finite subgroup of $\mathcal{O}(V)$, specifically a *reflection group*. We denote it by

$$W := W(\Phi) = \langle \sigma_\alpha \mid \alpha \in \Phi \rangle \subset \mathcal{O}(V). \tag{2.4}$$

It is usually named the *Weyl group* of the root system when the root system is of type A–G. It can be proven that it is generated by the reflections corresponding to simple roots of Φ. An example is carried out in Figure 2.3.

$\mathsf{I}_2(3) \simeq \mathsf{A}_2$:



$W(\Phi) = S_3 \simeq \mathsf{D}_6.$

Figure 2.3: The root system $\mathsf{A}_2 \simeq \mathsf{I}_2(3)$ and its reflection group; we can see the defining braid relation $\sigma_\alpha \sigma_\beta \sigma_\alpha = \sigma_\beta \sigma_\alpha \sigma_\beta$ of S_3.

The root system $\mathsf{I}_2(m)$ is normally called the dihedral root system since its reflection group is isomorphic to a dihedral group[1] D_{2m} of size $2m$.

We end this short section by presenting in more detail the root systems and the reflection groups that we will tackle in Chapters 5 and 6. We refer the readers to [Bou07b; Hum90] for more on the many properties that general root systems enjoy.

In Chapter 5, we choose the root system $\mathsf{A}_1 \oplus \mathsf{I}_2(m)$, for $m \geq 3$ with roots given by

$$\alpha_0 = (0,0,1) \qquad \alpha_j = (\sin(\tfrac{j\pi}{m}), -\cos(\tfrac{j\pi}{m}), 0), \quad j = 1, \dots, 2m, \tag{2.5}$$

with the set of positive roots given by $R_+ = \{\alpha_0, \alpha_1, \dots, \alpha_m\}$. The root system is presented in Figure 2.4. We decided to follow the same convention as Dunkl [Dun89] and Humphreys [Hum90] for the dihedral root system. The associated reflections σ_j are given in matrix form by

$$\sigma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \sigma_j = \begin{pmatrix} \cos(\frac{2j\pi}{m}) & \sin(\frac{2j\pi}{m}) & 0 \\ \sin(\frac{2j\pi}{m}) & -\cos(\frac{2j\pi}{m}) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.6}$$

The reflection group is generated by the reflections corresponding to the three simple roots $\sigma_0 := \sigma_{\alpha_0}$, $\sigma_1 := \sigma_{\alpha_1}$ and $\sigma_m := \sigma_{\alpha_m}$, with Coxeter presentation

$$W = \left\langle \sigma_0, \sigma_1, \sigma_m \mid \sigma_0^2 = \sigma_1^2 = \sigma_m^2 = (\sigma_0\sigma_1)^2 = (\sigma_0\sigma_m)^2 = (\sigma_1\sigma_m)^m = 1 \right\rangle. \tag{2.7}$$

This is the group $W = \mathbb{Z}_2 \times \mathsf{D}_{2m}$.

In Chapter 6, we choose the standard root system $\Phi \subset \mathbb{R}^4$ of type $\mathsf{I}_2(m) \oplus \mathsf{I}_2(n)$, explicitly realised as

$$\alpha_p = (\sin(\tfrac{p\pi}{m}), -\cos(\tfrac{p\pi}{m}), 0, 0), \quad \beta_q = (0, 0, \sin(\tfrac{q\pi}{n}), -\cos(\tfrac{q\pi}{n})), \tag{2.8}$$

[1] Although the same notation is used for a root system of type D_n and a dihedral group D_{2m}, there cannot be any confusion since in the rest of the thesis, root systems of type D_n no longer appear.

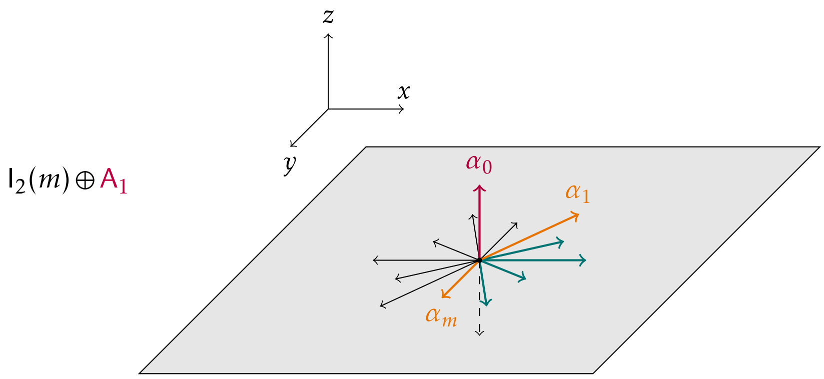


Figure 2.4: The root system $\mathsf{A}_1 \oplus \mathsf{I}_{2m}$ studied in Chapter 5, here illustrated for $m = 6$.

for $p = 1,\dots,2m$ and $q = 1,\dots,2n$. We fix the set of positive roots to be $\Phi_+ = \{\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n\}$. The associated reflections s_p, for $p = 1,\dots,m$ and t_q, for $q = 1,\dots,n$, are given in matrix form by

$$\sigma_p = \begin{pmatrix} \cos(\frac{2p\pi}{m}) & \sin(\frac{2p\pi}{m}) & 0 & 0 \\ \sin(\frac{2p\pi}{m}) & -\cos(\frac{2p\pi}{m}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; \sigma_q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\frac{2q\pi}{n}) & \sin(\frac{2q\pi}{n}) \\ 0 & 0 & \sin(\frac{2q\pi}{n}) & -\cos(\frac{2q\pi}{n}) \end{pmatrix}.$$

The reflection group associated with these reflections is $W = \mathsf{D}_{2m} \times \mathsf{D}_{2n}$.

2.1.2 Dunkl operators

Reflection groups and root systems boast a realm of combinatorial properties, which can be used to define rich mathematical structures. Inspired by properties of special functions, Dunkl discovered that one can modify the definition of partial derivatives to create new operators sharing similar properties: they reduce the degree by 1, they commute amongst themselves and the common differential operators can be defined with them.

We will present the definition of these Dunkl operators and give some of their properties. They will be used throughout the thesis to construct concrete examples of the abstract algebraic objects encountered.

Let W be a reflection group acting on $\mathbb{R}^d$, and let $\langle -,-\rangle$ be the canonical bilinear form of $\mathbb{R}^d$. Let $\Phi \subset \mathbb{R}^d$ denote the root system linked to W, and let Φ_+ be a fixed set of positive roots. The group W is generated as a Coxeter group by the reflections σ_α for $\alpha \in \Phi$, and its elements act on functions of $x \in \mathbb{R}^d$ by

$$wf(x) = f(w^{-1}x), \quad w \in W. \tag{2.9}$$

It will be often useful to start from a reflection group and associate to it the roots of the root system, but then normalise them, as is often done in Dunkl theory; see [Dun89; Hec91b]. Indeed only the orientation of the vectors is useful to define reflections; the orientation and the hyperplane arrangements are what matter, not the length of roots. Hence, from now on, we will assume that the roots of Φ are normalised.

We consider a W-invariant function $\kappa : \Phi \to \mathbb{C}$, that is, an assignment $\alpha \mapsto \kappa_\alpha \in \mathbb{C}$ such that $\kappa_{w(\alpha)} = \kappa_\alpha$, for all $\alpha \in \Phi$ and $w \in W$. We will usually assume κ to be a positive real function.

For an algebra A, we will denote by $[a,b] = ab - ba$ and $\{a,b\} = ab + ba$ the commutator and anti-commutator, for any elements $a, b \in A$.

Let $\xi_1, \dots, \xi_d$ denote the canonical basis of $\mathbb{R}^d$.

Definition 2.1.5 ([Dun89])**.** *The* Dunkl operator *(or* differential-difference operator*) associated with ξ_j is defined by its action on multivariate functions*

$$\mathrm{D}_j f(x) = \partial_{x_j} f(x) + \sum_{\alpha \in R^+} \kappa(\alpha) \frac{f(x) - \sigma_\alpha f(x)}{\langle \alpha, x\rangle} \left\langle \alpha, \xi_j \right\rangle, \tag{2.10}$$

where ∂_{x_j} is the partial derivative on variable $x_j := \left\langle x, \xi_j \right\rangle$. For a vector $\mu = (\mu_1, \dots, \mu_d)$, the Dunkl operator is given by

$$\mathrm{D}_\mu = \sum_{j=1}^{d} \mu_j \mathrm{D}_j. \tag{2.11}$$

Theorem 2.1.6 ([Dun89, Thm 1.9])**.** *Let* D_j *and* D_k *be two Dunkl operators. They commute,*

$$\left[\mathrm{D}_j, \mathrm{D}_k\right] = \mathrm{D}_j\mathrm{D}_k - \mathrm{D}_k\mathrm{D}_j = 0. \tag{2.12}$$

The proof is non-trivial and can be found either in the original reference [Dun89], or the lecture notes of Etingof and Ma [EM10].

With normalised roots, the commutation relations between the variables and the Dunkl operators are given by

$$\left[\mathrm{D}_i, x_j\right] = \delta_{ij} + 2\sum_{\alpha\in\Phi^+}\kappa(\alpha)\langle\alpha,\xi_i\rangle\left\langle\alpha,\xi_j\right\rangle\sigma_\alpha, \quad \text{where } \delta_{ij} = \begin{cases}1 & i=j,\\ 0 & \text{else,}\end{cases} \tag{2.13}$$

and one readily sees that $\left[\mathrm{D}_i, x_j\right] = \left[\mathrm{D}_j, x_i\right]$.

The algebra generated by $x_1,\ldots,x_d$, $\mathrm{D}_1,\ldots,\mathrm{D}_d$ and the group algebra $\mathbb{C}W$ can be seen as a deformation of the Weyl algebra of partial derivatives with the group algebra $\mathbb{C}W$; it is called the *Dunkl–Weyl algebra*. This is an example of the algebra $\mathcal{A}$ considered in [DOV18a, Ex. 4.2].

When restricted to a radial function, that is, a function f constant on the sets $\{x, \mid |x| = K\}$ for $K \in \mathbb{R}_{\geq 0}$, the Dunkl operators satisfy the following *Dunkl–Leibniz* rule for any function g:

$$\mathrm{D}_j f g = (\partial_{x_j} f)g + f(\mathrm{D}_j g). \tag{2.14}$$

The *Dunkl–Laplace operator* Δ_κ, the squared norm and the norm of a vector $x \in \mathbb{R}^d$ are respectively given by

$$\Delta_\kappa := \sum_{j=1}^d \mathrm{D}_j^2, \qquad |x|^2 := \sum_{j=1}^d x_j^2, \qquad |x| := \sqrt{\sum_{j=1}^d x_j^2}, \tag{2.15}$$

which are all invariant under the action of W. As a consequence of this invariance, and by an application of the Dunkl–Leibniz rule for radial functions (2.14), we get that

$$\left[\mathrm{D}_j, |x|^a\right] = a|x|^{a-2}x_j, \quad \text{for } a \in \mathbb{R}. \tag{2.16}$$

The classical Euler operator $\mathbb{E}$, which measures the degree of a homogeneous polynomial, is also W-invariant. We define another operator, often known as the Casimir akin to it in this setting:

$$H := \frac{1}{2}\sum_{j=1}^{d}\left\{\mathrm{D}_j, x_j\right\} = \frac{1}{2}\sum_{j=1}^{d}\mathrm{D}_j x_j + x_j\mathrm{D}_j = \mathbb{E} + d/2 + \gamma, \tag{2.17}$$

$$\text{where}\quad \mathbb{E} := \sum_{j=1}^{d} x_j\partial_{x_j} \quad\text{and}\quad \gamma := \sum_{\alpha\in R^+}\kappa(\alpha).$$

Proposition 2.1.7 ([Hec91b])**.** *The operators Δ_κ, $|x|^2$ and H form an $\mathfrak{sl}(2)$ triple, that is, they respect the following relations:*

$$\left[H, |x|^2\right] = 2|x|^2, \qquad [H, \Delta_\kappa] = -2\Delta_\kappa, \qquad \left[\Delta_\kappa, |x|^2\right] = 4H. \tag{2.18}$$

Moreover, we also have the relations

$$\begin{aligned} \left[H, x_j\right] &= x_j, & \left[H, \mathrm{D}_j\right] &= -\mathrm{D}_j, \\ \left[\Delta_\kappa, x_j\right] &= 2\mathrm{D}_j, & \left[|x|^2, \mathrm{D}_j\right] &= -2x_j. \end{aligned} \tag{2.19}$$

Proof. Direct computations using (2.13) (see for example [DOV18a, Theorem 2.2]). ∎

Throughout the thesis, we make use of the following shorthand notations:

$$C_{ij} := \left[\mathrm{D}_i, x_j\right] \overset{(2.13)}{=} \left[\mathrm{D}_j, x_i\right], \qquad C_{ij} = C_{ji}, \tag{2.20}$$

$$L_{ij} := x_i\mathrm{D}_j - x_j\mathrm{D}_i, \qquad L_{ij} = -L_{ji}, \qquad L_{ii} = 0. \tag{2.21}$$

The following theorem relates the L_{ij}'s and the C_{ij}'s for general root systems. In particular, it will be used in Chapter 5 for the proof of Proposition 5.5.5.

Theorem 2.1.8 ([DOV18a, Thm 2.5] and [FH15, Prop. 3.1])**.** *Let i, j, k, l be four non-necessarily distinct integers in $\{1,\ldots,d\}$. The commutation relation between L_{ij} and L_{kl} is given by*

$$\left[L_{ij}, L_{kl}\right] = L_{il}C_{jk} + L_{jk}C_{il} + L_{ki}C_{lj} + L_{lj}C_{ki} \tag{2.22}$$

$$= C_{jk}L_{il} + C_{il}L_{jk} + C_{lj}L_{ki} + C_{ki}L_{lj}, \tag{2.23}$$

and the following identities hold:

$$\left\{L_{ij}, L_{kl}\right\} + \left\{L_{ki}, L_{jl}\right\} + \left\{L_{jk}, L_{il}\right\} = 0, \tag{2.24}$$

$$\left[L_{ij}, C_{kl}\right] + \left[L_{ki}, C_{jl}\right] + \left[L_{jk}, C_{il}\right] = 0, \tag{2.25}$$

$$L_{ij}L_{kl} + L_{ki}L_{jl} + L_{jk}L_{il} = L_{ij}C_{kl} + L_{ki}C_{jl} + L_{jk}C_{il}. \tag{2.26}$$

The last relation (2.26) comes from [FH15] and is the crossing relation that defines the angular momentum algebra.

2.2 The algebraic actors

In this section, we present the main algebraic objects that will make an appearance during the thesis.

2.2.1 Rational Cherednik algebras

The Dunkl–Weyl algebra and the corresponding reflection group generate an algebraic structure, or to be more precise, form a representation of an abstract algebraic structure. The above-mentioned algebraic structure is named a rational Cherednik algebra and denoted $H_\kappa(V, W)$. It will be the algebraic framework in which the total angular momentum algebra will be defined. Rational Cherednik algebras were introduced by Etingof and Ginzburg [EG02]. We will follow partially the lecture notes of Etingof and Ma [EM10].

In fact, one can define the rational Cherednik algebra $H_\kappa(V, W)$ via its Dunkl polynomial representation in the complex vector space $\mathbb{C}[V]$ in which $w \in W$ acts on V as a reflection in the representation of W in V^*; elements $x \in V$ act by multiplication, and elements $\xi \in V^*$ act as the Dunkl operator D_ξ [EG02, Prop. 45].

For our purpose, we will only need this way of viewing the rational Cherednik algebra. Hence, we will use this as our definition and present the general definition after for the interested readers.

Definition 2.2.1 ([EG02])**.** *Let V be a $d \in \mathbb{N}$, $W \in \mathcal{O}(d)$ be a reflection group with root system Φ, and $\kappa : \Phi \to \mathbb{C}$ be a W-invariant parameter function. The algebra generated by the d commutating variables $x_1, \ldots, x_d$,* *the d Dunkl operators $\mathrm{D}_1, \dots, \mathrm{D}_d$ and $\mathbb{C}W$ is called the rational Cherednik algebra denoted by H_κ.*

Using the relations (2.28), it is possible to prove a Poincaré–Birkhoff–Witt theorem for rational Cherednik algebras.

Theorem 2.2.2. *The elements*

$$\prod_{i=1}^{d} \mathrm{D}_i^{m_i} \prod_{j=1}^{d} x_j^{n_j} \prod_{w \in W} w, \tag{2.27}$$

form a basis of H_κ.

For the sake of completeness, the general definition of rational Cherednik goes as follows.

Definition 2.2.3 ([EG02]). *Let V be a Euclidean space. Let $W \in \mathcal{O}(V)$ be a reflection group. The rational Cherednik algebra $H_\kappa = H_\kappa(V, W)$ is the quotient of the smash-product algebra $T(V \oplus V^*) \rtimes \mathbb{C}[W]$ by the relations $[\xi, \eta] = 0 = [x, y]$ and*

$$[\xi, x] = \xi(x) + \psi_\kappa(B^{-1}(\xi), x), \tag{2.28}$$

for all $\xi, \eta \in V^*$ and $x, y \in V$, where $\psi_\kappa(x, y) = \sum_{\alpha \in \Phi_+} \frac{2\kappa_\alpha}{\langle \alpha, \alpha \rangle} \langle x, \alpha \rangle \langle y, \alpha \rangle \sigma_\alpha$. and $B : V \to V^*$ is the linear isomorphism induced by the Euclidean structure.

Let $\mathrm{gr}(H_\kappa)$ be the graded algebra coming from H_κ by setting $\deg \xi = 1$, for all $\xi \in V^*$ and $\deg w = 0 = \deg x$ for all $w \in W$ and all $x \in V$. Then Theorem 2.2.2 means that there is an isomorphism of graded algebra between

$$\mathbb{C}[V \otimes V^*] \rtimes \mathbb{C}W \to \mathrm{gr}(H_\kappa). \tag{2.29}$$

2.2.2 Clifford algebras and superalgebras

We have already seen from the last sections that there is a meaningful interpretation of harmonic analysis in the context of rational Cherednik algebras or Dunkl operators by taking the operator Δ_κ as the appropriate generalisation of the Laplace operator. It is a natural question then to ask if a similar meaningful generalisation of monogenic analysis exists, that is, if we have a proper generalisation of the Dirac operator. For this, we turn to Clifford algebras and will define the relevant algebraic context.

Our source for this material is the book of Cheng and Wang [CW12]. We begin by some notations and usual definition.

Definition 2.2.4. *A $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is called a* vector superspace*. In particular, the vector superspace with even subspace $\mathbb{C}^m$ and odd subspace $\mathbb{C}^n$ is denoted by $\mathbb{C}^{m|n}$. The* parity *of a homogeneous element $a \in V_{\bar{i}}$ for $i \in \mathbb{Z}_2$ is denoted by $|a| = \bar{i}$.*

Definition 2.2.5. *A vector superspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is called a* superalgebra*, or a $\mathbb{Z}_2$-graded algebra, if it is equipped with a multiplication such that the grading $A \to \mathbb{Z}_2$ is respected by the multiplication, that is, $A_c A_d \subset A_{c+d}$ for $c, d \in \mathbb{Z}_2$.*

If A is a unital associative superalgebra, and if $a, b \in A$ are homogeneous elements of respective degree $|a|, |b|$, we shall use the notation $[\![a, b]\!] := ab - (-1)^{|a||b|} ba$, called the supercommutator.

We are now ready to define Clifford algebras. Let $\varepsilon \in \{-1, +1\}$ be a sign, and let $Cl(d)$ be the Clifford algebra associated with $\mathbb{R}^d$ and $\langle -, - \rangle$. The Clifford algebra is generated by $e_1, \ldots, e_d$, the images of the canonical basis, $\{\xi_1, \ldots \xi_d\}$, of $\mathbb{R}^d$,

$$\gamma : \xi_j \mapsto e_j, \tag{2.30}$$

subject to the following anticommutation relations

$$\left\{e_i, e_j\right\} = e_i e_j + e_j e_i = 2\varepsilon \delta_{ij}. \tag{2.31}$$

Remark 2.2.6. *We will often specialise to one value of ε. To this end, we will use the notation Cl^{ε} for the Clifford algebra with Clifford signature ε. Specifically, Chapter 3 and most of Chapter 5 will have general signature ε, but the remaining will be specific to $\varepsilon = 1$. Furthermore, Chapter 6 will also have a slightly more general definition of Clifford algebras.*

For an ordered subset $A = \{a_1, \ldots, a_p\} \subseteq \{1, \ldots, d\}$, we put

$$e_A := \overrightarrow{\prod}_{j=1}^{p} e_{a_j} = e_{a_1} e_{a_2} \cdots e_{a_p}.$$

We warn the reader that the order of the product matters as indicated by the arrow on top of the product. The set $\{e_A \mid A \subseteq \{1, \dots d\}\}$ forms a linear basis of $Cl(d)$.

Clifford algebras have a natural $\mathbb{Z}_2$-grading given by setting the degree of the generators e_i to $\bar{1}$. It is thus a superalgebra. We write $Cl = Cl_{\bar{0}} \oplus Cl_{\bar{1}}$ for the induced $\mathbb{Z}_2$-grading. We also let $|\cdot|$be the degree map defined on homogeneous elements by

$$|e| = \begin{cases} 0, & \text{if } e \in Cl_{\bar{0}} \\ 1, & \text{if } e \in Cl_{\bar{1}}. \end{cases}$$

We also define an involution by $\bar{s} = s$ if $s \in Cl_{\bar{0}}$ and $\bar{s} = -s$ if $s \in Cl_{\bar{1}}$. We denote by $*$ the anti-involution defined by $s^* := \bar{s}$ and $(st)^* := t^* s^*$.

2.2.3 Double coverings

In this section, the results concerning double covering groups and their representation theory are recalled. The main source for this material is the important work of Schur [Sch11] and Morris [Mor76; Mor80]. We have decided to focus only on the specific cases used in this thesis, where we have an explicit presentation by generators and relations. We refer the reader to [Kar68], [HH92] or [ABS64] for further references on the general theory.

Suppose that W is a real reflection group of arbitrary rank d and that $V_{\mathbb{R}} \cong \mathbb{R}^d$ is its reflection representation. There are generally two double coverings of W, a positive W^+ and a negative W^-, reflecting the two possibilities for a definite symmetric bilinear form on $V_{\mathbb{R}}$.

If W has Coxeter presentation given by some $m_{ij} \in \mathbb{N}$

$$W = \left\langle \sigma_1, \dots, \sigma_n \;\middle|\; (\sigma_i \sigma_j)^{m_{ij}} = 1;\, m_{ii} = 1 \right\rangle, \tag{2.32}$$

then one gets in general two double coverings of the Coxeter group W and their generators and relations presentations.

Theorem 2.2.7 (Morris [Mor76, Thm 3.6]). *The two double coverings* *of W are given by*

$$\widetilde{W}^{+} = \left\langle z, \tilde{\sigma}_1, \dots, \tilde{\sigma}_n \;\middle|\; z^2 = 1, \begin{array}{ll} (\tilde{\sigma}_i \tilde{\sigma}_j)^{m_{ij}} = 1, & m_{ij} \text{ odd}, \\ (\tilde{\sigma}_i \tilde{\sigma}_j)^{m_{ij}} = z, & m_{ij} \text{ even} \end{array} \right\rangle, \tag{2.33}$$

$$\widetilde{W}^{-} = \left\langle z, \tilde{\sigma}_1, \dots, \tilde{\sigma}_n \;\middle|\; z^2 = 1, (\tilde{\sigma}_i \tilde{\sigma}_j)^{m_{ij}} = z \right\rangle. \tag{2.34}$$

Corollary 2.2.8 ([Mor76, Prop. 3.5])**.** *The two sequences*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{W}^{+} \longrightarrow W \longrightarrow 0 \tag{2.35}$$

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{W}^{-} \longrightarrow W \longrightarrow 0 \tag{2.36}$$

are exact, and $\widetilde{W}^{+}$ and $\widetilde{W}^{-}$ are central extensions of W.

Remark that $\widetilde{W}^{+}$ might not be a *non-trivial* central extension of the group W, that is, W might not have a projective representation that is not equivalent to a linear representation. For example, this is the case when $W = \mathsf{D}_{2m}$ for m odd [Sch07]. When considering representations of $\widetilde{W}^{-}$ and $\widetilde{W}^{+}$, those where the commuting element z is acting as the identity are in correspondence with the representations of W. Those where z acts as -1 are in correspondence with projective representations of W and are called *spin representations* [Mor76].

This last statement necessitates a bit more explanation. Let $e_{\pm} := \frac{1 \pm z}{2}$ denote the canonical idempotents of $\mathbb{C}\widetilde{W}$ and put $\mathbb{C}\widetilde{W}_{\pm} := e_{\pm}(\mathbb{C}\widetilde{W})$. This is not the group algebra of the positive or negative double coverings of the previous paragraph. Then

$$\mathbb{C}\widetilde{W} = \mathbb{C}\widetilde{W}_{+} \oplus \mathbb{C}\widetilde{W}_{-}. \tag{2.37}$$

Moreover, $\mathbb{C}\widetilde{W}_{+} \cong \mathbb{C}W$. Indeed, note that if $\{\tilde{w}, z\tilde{w}\} = \pi^{-1}(w)$ for $w \in W$, then $e_{+}\tilde{w} = e_{+}(z\tilde{w})$; so the assignment $e_{+}\tilde{w} \mapsto \pi(\tilde{w})$ defined on the canonical generators of $\mathbb{C}\widetilde{W}$ is well-defined and induces the isomorphism.

In light of (2.37), the representations of $\widetilde{W}$ are split in two types: the first are the linear representations that factor through the action of W, and the second are the spin representations.

Let $\mathrm{Irr}(\widetilde{W})$ denote the set of equivalence classes of irreducible representations of $\widetilde{W}$. We can decompose

$$\mathrm{Irr}(\widetilde{W}) = \mathrm{Irr}(W) \sqcup \mathrm{sIrr}(\widetilde{W}), \tag{2.38}$$

where $\mathrm{sIrr}(\widetilde{W})$ is the set of equivalence classes of irreducible spin representations of $\widetilde{W}$. Since

$$\sum_{V \in \mathrm{Irr}(W)} \dim(V)^2 = |W| \text{ and } \sum_{V \in \mathrm{Irr}(\widetilde{W})} \dim(V)^2 = 2|W|, \tag{2.39}$$

we get

$$\sum_{U \in \mathrm{sIrr}(\widetilde{W})} \dim(U)^2 = |W|. \tag{2.40}$$

If $\rho : \mathbb{C}\widetilde{W} \to \mathbb{C}W \otimes Cl$ is the diagonal algebra homomorphism defined on the generators $\tilde{w}, z \in \widetilde{W}$ by $\rho(\tilde{w}) = \pi(\tilde{w}) \otimes \tilde{w}$, $\rho(z) = 1 \otimes (-1)$ and extended linearly, then [CDO22, Proposition 2.5]

$$\rho(\mathbb{C}\widetilde{W}) \cong \mathbb{C}\widetilde{W}_-. \tag{2.41}$$

Since we can realise $\widetilde{W}$ as a subgroup of the group of units in a Clifford algebra, we can use the $\mathbb{Z}_2$-grading of the latter to decompose into even and odd parts:

$$\widetilde{W} = \widetilde{W}_{\bar{0}} \cup \widetilde{W}_{\bar{1}}. \tag{2.42}$$

2.2.4 A detour through some Lie superalgebras

We will now make a small detour to the theory of Lie superalgebras. We have seen in Definition 2.2.5 what a superalgebra is. The super version of a Lie algebra will be a superalgebra with a superbracket respecting super-versions of the skew-symmetry and Jacobi identity.

Definition 2.2.9. *A superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ equipped with a bracket $[\![-, -]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a* Lie superalgebra *if the two following axioms are satisfied, for $a, b, c \in \mathfrak{g}$ homogeneous,*

1. *skew-supersymmetry: $[\![a, b]\!] = -(-1)^{|a|\cdot|b|}[\![b, a]\!]$;*

2. *super Jacobi identity:*

$$[\![a, [\![b, c]\!]]\!] = [\![[\![a, b]\!], c]\!] + (-1)^{|a|\cdot|b|}[\![b, [\![a, c]\!]]\!]. \tag{2.43}$$

The usual definitions of homomorphisms, subalgebras, ideals and quotients go through categorically.

The even part of a Lie superalgebra is a Lie algebra in the classical sense. Any associative superalgebra can be made into a Lie superalgebra by defining the supercommutator

$$[\![a, b]\!] := ab - (-1)^{|a|\cdot|b|}ba. \tag{2.44}$$

The *adjoint map*

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}) \\ \mathrm{ad}(a)(b) &:= [\![a, b]\!], \end{aligned} \tag{2.45}$$

is a homomorphism of Lie superalgebras, and the action on $\mathfrak{g}$ it induces is called the *adjoint action*.

The classification of simple complex Lie superalgebras was completed by Kac [Kac77], and we refer the reader to this original source for a complete overview.

Theorem 2.2.10 (Kac)**.** *The simple complex finite-dimensional Lie superalgebras that are not Lie algebras are in one of the followings families:*

1. *the classical Lie superalgebras* $\mathsf{A}(m,n)$*,* $\mathsf{B}(m,n)$*,* $\mathsf{C}(n)$ *and* $\mathsf{D}(m,n)$*;*
2. *the exceptional classical Lie algebras* $\mathsf{F}(4)$ *and* $\mathsf{G}(3)$ *or one of the deformations of* $\mathsf{D}(2,1)$*:* $\mathsf{D}(2,1;\alpha)$*, for* $\alpha \in \mathbb{C}$*;*
3. *the strange classical Lie superalgebras* $\mathsf{P}(n)$ *and* $\mathsf{Q}(n)$*;*
4. *the Lie superalgebras of Cartan type* $\mathsf{W}(n)$*,* $\mathsf{S}(n)$*,* $\mathsf{H}(n)$*,* $\widetilde{\mathsf{S}}(n)$*.*

We will only need one specific algebra and so shall restrict ourselves to introduce two of the main families of Lie superalgebras and one "strange" one to give the readers a sense of these objects.

General and special linear Lie superalgebras. For a vector superspace V, equip $\mathrm{End}(V)$ with the supercommutator (2.44). This forms a Lie superalgebra called the *general linear Lie superalgebra* denoted by $\mathfrak{gl}(V)$. Specifically for $V = \mathbb{C}^{m|n}$, it is denoted by $\mathfrak{gl}(m|n)$.

Let us parametrise a chosen set of basis elements for $V_{\bar{0}}$ and $V_{\bar{1}}$ by $I(m|n) = \{\bar{1}, \ldots, \bar{m}; 1, \ldots, \bar{n}\}$. The indices $\bar{1}, \ldots, \bar{m}$ correspond to vectors in $V_{\bar{0}}$ and the indices $1, \ldots, \bar{n}$ to those of $V_{\bar{1}}$. Then $\mathfrak{gl}(V)$ can be realised as $(m+n) \times (m+n)$ complex matrices of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.46}$$

with a, a $m \times m$ matrix; b, a $m \times n$ matrix; c, a $n \times m$ matrix, and d, a $n \times n$ matrix.

The *supertrace* is defined on the matrix g as

$$\mathrm{str}(g) := \mathrm{tr}(a) - \mathrm{tr}(d). \tag{2.47}$$

The subspace of supertraceless matrices is a subalgebra of $\mathfrak{gl}(m|n)$ called the special linear Lie superalgebra and denoted by

$$\mathfrak{sl}(m|n) := \{g \in \mathfrak{gl}(m|n) \mid \mathrm{str}(g) = 0\}. \tag{2.48}$$

Note that if $m = n$, it contains a one-dimensional ideal generated by the scalar multiple of the identity matrix I_{2n}. The family $\mathsf{A}(m,n)$ of Theorem 2.2.10 is identified with

$$\begin{cases} \mathfrak{sl}(m+1 \mid n+1) & m \neq n; \\ \mathfrak{sl}(n+1 \mid n+1)/\langle I_{2n+2} \rangle & m = n. \end{cases} \tag{2.49}$$

Orthosymplectic Lie superalgebras We begin by defining the *supertranspose* of a matrix A. For $\bar{j} \in \mathbb{Z}_2$ and for $i \in I_{\bar{j}}$, denote $|i| := \bar{j}$. The elementary matrices are denoted by E_{ij}, $i, j \in I$. Then a matrix

$$A = \sum_{i,j \in I} a_{ij} E_{ij}, \qquad a_{ij} \in \mathbb{C},$$

has a supertranspose defined as

$$A^{\mathrm{st}} := \sum_{i,j \in I} (-1)^{|j|(|i|+|j|)} a_{ij} E_{ji}. \tag{2.50}$$

Definition 2.2.11. *A bilinear form on a vector superspace V*

$$B(-,-) : V \times V \to V \tag{2.51}$$

is called even *if $B(V_{\bar{\imath}}, V_{\bar{\jmath}}) = 0$ unless $\bar{\imath} + \bar{\jmath} = \bar{0}$ and* odd *if $B(V_{\bar{\imath}}, V_{\bar{\jmath}}) = 0$ unless $\bar{\imath} + \bar{\jmath} = \bar{1}$. An even bilinear form is called* supersymmetric *if its restriction to $V_{\bar{0}} \times V_{\bar{0}}$ is symmetric and its restriction to $V_{\bar{1}} \times V_{\bar{1}}$ is skew-symmetric. It is called* skew-supersymmetric *if its restriction to $V_{\bar{0}} \times V_{\bar{0}}$ is skew-symmetric and its restriction to $V_{\bar{1}} \times V_{\bar{1}}$ is symmetric.*

The vector superspace V can only have a non-degenerate supersymmetric even bilinear form if $\dim V_{\bar{1}}$ is even.

Definition 2.2.12. *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace with* $\dim V_{\bar{1}}$ *even and B be a non-degenerate even supersymmetric bilinear form. We call the following Lie superalgebra, with $j \in \mathbb{Z}_2$,*

$$\begin{aligned} \mathfrak{osp}_{\bar{j}} &:= \{g \in \mathfrak{gl}(V)_{\bar{j}} \mid B(g(x), y) = -(-1)^{j|x|} B(x, g(y)), \forall x, y \in V\}, \\ \mathfrak{osp}(V) &:= \mathfrak{osp}(V)_{\bar{0}} \oplus \mathfrak{osp}(V)_{\bar{1}}, \end{aligned}$$

the orthosymplectic Lie superalgebra.

It is the subalgebra of $\mathfrak{gl}(V)$ that preserves the non-degenerate supersymmetric bilinear form B. When $V = \mathbb{C}^{m|2n}$, we denote $\mathfrak{osp}(V)$ by $\mathfrak{osp}(m|2n)$. For $m = 0$, we retrieve the Lie algebra $\mathfrak{sp}(2n)$ and for $n = 0$, the Lie algebra $\mathfrak{so}(m)$.

If B is instead a non-degenerate odd skew-supersymmetric bilinear form on $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with $\dim V_{\bar{0}}$ even, we denote the subalgebra of $\mathfrak{gl}(V)$ preserving it by $\mathfrak{spo}(V)$. Note that, as Lie superalgebras, $\mathfrak{osp}(m|2n) \simeq \mathfrak{spo}(2n|m)$.

In the notation of Theorem 2.2.10, the families $\mathsf{B}(m,n)$, $\mathsf{C}(n)$ and $\mathsf{D}(m,n)$ are defined as

$$\begin{aligned} \mathsf{B}(m,n) &= \mathfrak{osp}(2m+1|2n) & m \geq 0, n > 0; \\ \mathsf{D}(m,n) &= \mathfrak{osp}(2m|2n) & m \geq 2, n > 0; \\ \mathsf{C}(n) &= \mathfrak{osp}(2|2n-2) & n \geq 2. \end{aligned} \tag{2.52}$$

It will be more convenient to work with generators and relations for the modest Lie superalgebra that will appear again and again in this thesis: $\mathfrak{osp}(1|2)$. With only two odd generators, the Lie superalgebra can be generated via triple relations [GP80].

Definition 2.2.13. *The Lie superalgebra* $\mathfrak{osp}(1|2)$ *is generated by two odd generators* C^+, C^- *and with triple relations*

$$[\![[\![C^-, C^+]\!], C^\pm]\!] = \pm 2C^\pm. \tag{2.53}$$

The queer Lie superalgebras The first two cases of Theorem 2.2.10 might give the impression that Lie superalgebras share more similarities with classical Lie algebras than they do. The following is a short example to illustrate to the readers how the extra grading gives more possibilities before we close this detour.

Definition 2.2.14. *Let* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ *with* $\dim V_{\bar{0}} = \dim V_{\bar{1}}$. *For any* $P \in \mathrm{End}(V)$ *such that* $P^2 = Id_{n|n}$, *the subspace*

$$\mathfrak{q}(V) = \{T \in \mathrm{End}(V) \,|\, [\![T, P]\!] = 0\}, \tag{2.54}$$

is called the queer Lie superalgebra.

The queer Lie superalgebra is not isomorphic to the special linear Lie superalgebra. There is no Lie algebra similar to this superalgebra.

2.2.5 Total angular momentum algebra

In our last section, we introduced the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$. We now will show that an instance of this Lie superalgebra appears in the algebraic setting we developed during the chapter.

We will work in the tensor product $H_\kappa \otimes Cl$. The $\mathbb{Z}_2$-grading of the Clifford algebra Cl naturally induces a $\mathbb{Z}_2$-grading on $H_\kappa \otimes Cl$, by declaring $(H_\kappa \otimes Cl)_{\bar{j}} = H_\kappa \otimes Cl_{\bar{j}}$, for $j \in \{0,1\}$. To ease notation, we will identify $H_\kappa \cong H_\kappa \otimes \mathbb{C} \subset H_\kappa \otimes Cl$ and $Cl \cong \mathbb{C} \otimes Cl \subset H_\kappa \otimes Cl$.

We will often not make explicit mention of the tensor product for ease of notation when evident from the context.

Since $\mathbb{C}W \hookrightarrow H_\kappa$, we can define $\rho : \mathbb{C}\widetilde{W} \to H_\kappa \otimes Cl$ as the map used in (2.41).

Notation 2.2.15. *For a root* $\alpha \in \Phi$, *we will denote* $\underline{\alpha} := \sum_{j=1}^{d} \left\langle \alpha, \xi_j \right\rangle \otimes e_j$ *and* $\tilde{\sigma}_\alpha := \rho(\underline{\alpha}) = \underline{\alpha}\sigma_\alpha \in \rho(\mathbb{C}\widetilde{W})$.

Note that $\underline{\alpha} \in Cl(d)$, where we identified $Cl(d)$ with $1 \otimes Cl(d) \subset H_\kappa \otimes Cl(d)$.

Let us denote

$$\underline{x} := \sum_{j=1}^{d} x_j \otimes e_j, \qquad \underline{\mathrm{D}} := \sum_{j=1}^{d} \mathrm{D}_j \otimes e_j, \tag{2.55}$$

where $\underline{x}$ is the *vector variable* and $\underline{\mathrm{D}}$ is the *Dunkl–Dirac operator*.

Up to the Clifford signature ε, the square of the Dunkl–Dirac operator is the Dunkl–Laplace operator, and its dual operator is the square of the vector variable:

$$\Delta_\kappa := \sum_{j=1}^{d} \mathrm{D}_j^2 = \varepsilon \underline{\mathrm{D}}^2, \qquad |x|^2 := \sum_{j=1}^{d} x_j^2 = \varepsilon \underline{x}^2. \tag{2.56}$$

Moreover, we have by direct computation

$$\left\{\underline{\mathrm{D}}, \underline{x}\right\} = 2\varepsilon H = 2\varepsilon(\mathbb{E} + d/2 + \gamma). \tag{2.57}$$

With this, we have proved that $\underline{\mathrm{D}}$ and $\underline{x}$ are the odd generators of a realisation of the Lie superalgebra $\mathfrak{osp}(1|2)$ containing the Lie algebra $\mathfrak{sl}(2)$ as an even subalgebra realised by (2.18) [DOV18a, Theorem 3.4]:

$$\begin{aligned} \left[\underline{\mathrm{D}}, |x|^2\right] &= 2\underline{x}, \quad [\underline{x}, \Delta] = -2\underline{\mathrm{D}}, \\ \left[\underline{\mathrm{D}}, H\right] &= \underline{\mathrm{D}}, \quad [\underline{x}, H] = -\underline{x}. \end{aligned} \tag{2.58}$$

We can see they are equivalent to the presentation (2.53) by direct computations. This is summarised in the following result.

Theorem 2.2.16 ([ØSS09, Lemma 4.2])**.** *Let* $\mathfrak{g}_{\bar{0}} = \mathrm{span}\{\Delta_\kappa, |x|^2, E\}$ *and* $\mathfrak{g}_{\bar{1}} = \mathrm{span}\{\underline{\mathrm{D}}, \underline{x}\}$. *The vector subspace* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *of* $H_\kappa \otimes Cl$ *spanned by* $\{\Delta_\kappa, |x|^2, H, \underline{\mathrm{D}}, \underline{x}\}$ *has the structure of a Lie superalgebra isomorphic to* $\mathfrak{osp}(1|2)$.

We are now ready to give the definition of the *(Dunkl) total angular momentum algebra*, the central object of study of our thesis.

Definition 2.2.17. *The (Dunkl) total angular momentum algebra* $\mathfrak{O}_\kappa = \mathfrak{O}_\kappa(V, W)$ *is defined as the graded centraliser in* $H_\kappa \otimes Cl$ *of the Lie superalgebra* $\mathfrak{g} \simeq \mathrm{span}(\underline{x}, \underline{\mathrm{D}}, \Delta_\kappa, H, |x|^2) \subset H_\kappa \otimes Cl(d)$ *isomorphic to* $\mathfrak{osp}(1|2)$.

For the rank two and three, another presentation of the total angular momentum algebra will be used in this thesis in Chapters 4 and 5. It is given by generators and relations and was defined in [DOV18a]. It will be called the *Dunkl–Dirac symmetry algebra*. This name is a slight misnomer, since it is really the symmetry algebra of the $\mathfrak{osp}(1|2)$ realisation generated by abstract elements, so not just the Dirac operator is involved, and it admits more possibilities than the Dunkl deformation.

The algebra considered in [DOV18a] is more general. In particular, it does not require a reflection group. The role of the group elements is played by the one-index symmetries introduced in (2.65). However, for our purposes, we will keep the convention of the total angular momentum algebra and always see it inside the tensor product of a rational Cherednik algebra and a Clifford algebra.

We begin by expressing the commutation relations between the elements $\underline{x}$ and $\underline{\mathrm{D}}$ and the elements of $H_\kappa \otimes Cl(d)$. For ease of notation, denote $\langle \mathrm{D}, \alpha \rangle := \sum_{j=1}^{d} \left\langle \alpha, \xi_j \right\rangle \mathrm{D}_j$.

Lemma 2.2.18. *The Dunkl–Dirac operator* $\underline{\mathrm{D}}$ *and its dual symbol* $\underline{x}$ *respect the following relations:*

$$\{\tilde{\sigma}_\alpha, \underline{x}\} = 0 = \left\{\tilde{\sigma}_\alpha, \underline{\mathrm{D}}\right\}; \tag{2.59}$$

$$\left[\mathrm{D}_j, \underline{x}\right] = \left[\underline{\mathrm{D}}, x_j\right] = e_j + 2 \sum_{\alpha \in R^+} \kappa(\alpha) \left\langle \alpha, \xi_j \right\rangle \tilde{\sigma}_\alpha; \tag{2.60}$$

$$\left[\underline{\mathrm{D}}, \sigma_\alpha\right] = 2 \langle \mathrm{D}, \alpha \rangle \tilde{\sigma}_\alpha, \qquad [\underline{x}, \sigma_\alpha] = 2 \langle x, \alpha \rangle \tilde{\sigma}_\alpha. \tag{2.61}$$

Proof. Equations (2.59) follow using (2.13) and the Clifford algebra relations. We focus now on (2.60). This is a direct application of the commutation relation between Dunkl operators and variables (2.13)

$$\begin{aligned}
\underline{\mathrm{D}} x_j &= \mathrm{D}_j x_j e_j + \sum_{i \neq j} \mathrm{D}_i x_j e_i \\
&= e_j + 2 \sum_{\alpha \in \Phi^+} \kappa(\alpha) \left\langle \alpha, \xi_j \right\rangle^2 \sigma_\alpha e_j + x_j \mathrm{D}_j e_j \\
&\quad + \sum_{i \neq j} (2 \sum_{\alpha \in \Phi^+} (\kappa(\alpha) \langle \alpha, \xi_i \rangle \left\langle \alpha, \xi_j \right\rangle \sigma_\alpha) e_i + x_j \mathrm{D}_j) e_i
\end{aligned}$$

$$= x_j\underline{\mathrm{D}} + e_j + 2\sum_{\alpha\in\Phi^+}\kappa(\alpha)\big\langle\alpha,\xi_j\big\rangle(\big\langle\alpha,\xi_j\big\rangle e_j + \sum_{i\neq j}\langle\alpha,\xi_i\rangle)\sigma_\alpha e_i$$
$$= x_j\underline{\mathrm{D}} + e_j + 2\sum_{\alpha\in\Phi^+}\kappa(\alpha)\big\langle\alpha,\xi_j\big\rangle\underline{\alpha}\sigma_\alpha.$$

For the two equations of (2.61), proving one will suffice. We do the first. Recall that the action of σ_α on the Dunkl operators is as that on vectors of $\mathbb{R}^d$, with $\langle\alpha,\alpha\rangle = 1$ since the roots are assumed to be normalised. Therefore,

$$\sigma_\alpha\underline{\mathrm{D}} = \sigma_\alpha\sum_{j=1}^{d}\mathrm{D}_j e_j = \sum_{j=1}^{d}\sigma_\alpha\mathrm{D}_j e_j = \sum_{j=1}^{d}(\mathrm{D}_j - 2\langle\alpha,\xi_i\rangle\langle\mathrm{D},\alpha\rangle)\sigma_\alpha e_j$$
$$= \underline{\mathrm{D}}\sigma_\alpha - 2\sum_{j=1}^{d}\langle\alpha,\xi_i\rangle\langle\mathrm{D},\alpha\rangle\sigma_\alpha e_i = \underline{\mathrm{D}}\sigma_\alpha - 2\langle\mathrm{D},\alpha\rangle\underline{\alpha}\sigma_\alpha.$$

∎

Let $A\subset\{1,\dots,d\}$ be an ordered subset. Define

$$\underline{\mathrm{D}}_A := \sum_{a\in A}\mathrm{D}_a e_a \qquad\qquad \underline{x}_A := \sum_{a\in A}x_a e_a. \tag{2.62}$$

Theorem 2.2.19 ([DOV18a, Thm 3.7])**.** *Let $A\subset\{1,\dots,d\}$ be a list of distinct elements. The following operators,*

$$O_A := \frac{1}{2}(\underline{\mathrm{D}}_A\underline{x}_A e_A - e_A\underline{x}_A\underline{\mathrm{D}}_A - \varepsilon e_A), \tag{2.63}$$

supercommute with $\underline{\mathrm{D}}$ and $\underline{x}$, that is,

$$\underline{\mathrm{D}}O_A = (-1)^{|A|}O_A\underline{\mathrm{D}} \quad \textit{and} \quad \underline{x}O_A = (-1)^{|A|}O_A\underline{x}. \tag{2.64}$$

The symmetries can be used to denote more compactly specific linear combinations of elements in $\mathbb{C}\widetilde{W}$, such as the one appearing in the right-hand side of (2.60), we write (see also [DOV18a, Ex. 4.2])

$$O_j := \frac{\varepsilon}{2}(\big[\underline{\mathrm{D}}, x_j\big] - e_j) = \varepsilon\sum_{\alpha\in R^+}\kappa(\alpha)\big\langle\alpha,\xi_j\big\rangle\tilde{\sigma}_\alpha. \tag{2.65}$$

We call the elements O_j *one-index symmetries*.

Using the one-index symmetries, we have simpler expression of the general index symmetries

$$O_{ij} = L_{ij} + \frac{\varepsilon}{2} e_i e_j + O_i e_j - O_j e_i; \tag{2.66}$$

$$O_A = (\varepsilon \frac{|A|-1}{2} + \varepsilon \sum_{a \in A} O_a e_a - \sum_{\{i,j\} \subset A} L_{ij} e_i e_j) e_A. \tag{2.67}$$

The following lemma shows how Clifford elements interact with the one-index symmetries.

Lemma 2.2.20 ([DOV18a, Lem. 3.10])**.** *For any two indices i, j the following relation holds*

$$\left\{e_i, O_j\right\} = \left\{e_j, O_i\right\}. \tag{2.68}$$

Lemma 2.2.21. *The following holds*

$$\sum_{j=1}^{d} O_j e_j = \sum_{\alpha \in R^+} \kappa(\alpha) \sigma_\alpha. \tag{2.69}$$

Proof. Recall that the roots are normalised and so, in particular, $\underline{\alpha}^2 = \varepsilon$. Replacing O_j by its expression (2.65) and using the anticommutation of Clifford elements then yield

$$\begin{aligned} \sum_{j=1}^{d} O_j e_j &= \varepsilon \sum_{j=1}^{d} \sum_{\alpha \in R^+} \kappa(\alpha) \left\langle \alpha, \xi_j \right\rangle \underline{\alpha} \sigma_\alpha e_j \\ &= \varepsilon \sum_{\alpha \in R^+} \kappa(\alpha) \underline{\alpha}^2 \sigma_\alpha = \sum_{\alpha \in R^+} \kappa(\alpha) \sigma_\alpha. \end{aligned}$$

∎

The total angular momentum algebra $\mathcal{O}_\kappa$ can then be defined as the associative unital $\mathbb{C}$-algebra generated by O_A [DOV18a].

Remark 2.2.22. *Note that this definition makes sense only because we are seeing $\mathcal{O}_\kappa$ as a subalgebra of $H_\kappa \otimes Cl$. For the abstract version, we would need the commutation relations.*

It was proven recently that only the algebra $\mathbb{C}\widetilde{W}$, the two-index symmetries O_{ij} and the three-index symmetries and O_{ijk} are required to generate $\mathfrak{O}_\kappa$ [Ost22].

Let u_j be an index in $\{1,\dots,d\}$. The four- and five-index symmetries $O_{u_1u_2u_3u_4}$ and $O_{u_1u_2u_3u_4u_5}$ have expressions in terms of two- and three-index symmetries [Ost22, Lem. 4.11]:

$$\begin{aligned} O_{u_1u_2u_3u_4} &= 6\mathscr{A}(O_{u_1u_2}O_{u_3u_4}) - 8\mathscr{A}(O_{u_1u_2u_3}O_{u_4}) \\ O_{u_1u_2u_3u_4u_5} &= 4\mathscr{A}(O_{u_1u_2u_3}O_{u_4u_5}) + 48\mathscr{A}(O_{u_1u_2u_3}O_{u_4}O_{u_5}) \\ &\quad -36\mathscr{A}(O_{u_1u_2}O_{u_3u_4}O_{u_5}), \end{aligned}$$

where the anti-symmetriser $\mathscr{A}$ is defined on a multivariate expression f by

$$\mathscr{A}(f(u_1,\dots,u_n)) := \frac{1}{n!}\sum_{s\in S_n} \operatorname{sign}(s) f(u_{s(1)},\dots,u_{s(n)}). \tag{2.70}$$

The remaining commutation relations are given in the following theorem extracted from [Ost22]. Note that the possible other higher-order relations are not yet fully known.

Theorem 2.2.23 ([Ost22, Prop. 4.9, Props 4.12–14]). *The algebra $\mathfrak{O}_\kappa$ is generated by $\mathbb{C}\widetilde{W}$, and multilinear symbols O_{ij} and O_{ijk} with commutation relations, for distinct $a,b,c \in \{1,\dots,d\}$ and distinct $u,v,w \in \{1,\dots,d\}$, given by*

$$\begin{aligned} [O_{ab}, O_{uv}] &= \delta_{bu}(O_{av} + \{O_a, O_v\}) - \delta_{au}(O_{bv} + \{O_b, O_v\}) \\ &\quad - \delta_{bv}(O_{au} + \{O_a, O_u\}) + \delta_{av}(O_{bu} + \{O_b, O_u\}) \\ &\quad + (\{O_a, O_{buv}\} - \{O_b, O_{auv}\} + \{O_{abu}, O_v\} - \{O_{abv}, O_u\})/2; \end{aligned}$$

denoting $\hat{x} := \delta_{bx}a - \delta_{ax}b$, for $x \in \{u,v,w\}$,

$$\begin{aligned} [O_{ab}, O_{uvw}] &= O_{\hat{u}vw} + O_{u\hat{v}w} + O_{uv\hat{w}} \\ &\quad + \{O_{\hat{u}}, O_{vw}\} - \{O_{\hat{v}}, O_{uw}\} + \{O_{\hat{w}}, O_{uv}\} \\ &\quad + [O_a, O_{buvw}] - [O_b, O_{auvw}]. \end{aligned}$$

Furthermore, supposing that $\delta_{ac} = 0 = \delta_{bc}$ and that the only possibly non-zero pairings between $\{a,b,c\}$ and $\{u,v,w\}$ are $\delta_{au}, \delta_{bv}, \delta_{cw}$,

$$\{O_{abc}, O_{uvw}\} = \delta_{au}(O_{bcvw} + \{O_{bc}, O_{vw}\}) + [O_a, O_{bcuvw}]$$

$$
\begin{aligned}
&+ \delta_{bv}(O_{acuw} + \{O_{ac}, O_{uw}\}) + [O_b, O_{acuvw}] \\
&+ \delta_{cw}(O_{abuv} + \{O_{ab}, O_{uv}\}) + [O_c, O_{abuvw}] \\
&+ \delta_{bv}\delta_{cw}[O_a, O_u] + \delta_{au}\delta_{cw}[O_b, O_v] \\
&+ \delta_{au}\delta_{bv}[O_c, O_w] - \delta_{au}\delta_{bv}\delta_{cw}/2.
\end{aligned}
$$

From this result, we see that the commutation relations between the symmetries are trivial in dimension two. The first time that non-trivial commutation relations happen is in dimension three, and the full relations of the three-index symmetries happen only in dimension four. This will be the focus of Chapter 5 and Chapter 6, respectively.

Remark 2.2.24. *We emphasise that Theorem 2.2.23 only concerns the commutation relations of the algebra. Finding* all *relations of the algebra for spaces of dimension more than 3 is, as of yet, unknown, and it is generally a very difficult problem to tackle. This is subject to ongoing investigations.*

3

Generalised symmetries and bases for Dunkl monogenics

The content of this chapter is extracted from the article:

Hendrik De Bie, Alexis Langlois-Rémillard, Roy Oste, Joris Van der Jeugt (2022+). To appear in *Rocky Mountain Journal of Mathematics* [DLOV22b].

3.1 Introduction

Since their introduction by Dunkl in 1989 [Dun89], the family of commutative differential-difference operators associated with a reflection group W, nowadays known as Dunkl operators, have enjoyed a great deal of interest in the mathematical and mathematical physics communities. Due to their properties, it is possible to replace partial derivatives with Dunkl operators in classical differential equations and operators appearing in many physical systems. A great deal of work has been done in the study of the resulting differential operators, most notably on the Dunkl version of the Laplace operator and its harmonic functions.

This work focuses on the kernel of the Dunkl version of the Dirac operator, which, like its classical analogue, is a square root of the Dunkl-Laplace operator. Polynomials in the kernel of the Dunkl–Dirac operator are called Dunkl monogenics and they form solutions of the Dunkl version of the homogeneous Dirac equation.

Symmetries play an important role in our study. We call an operator S a *symmetry* of an operator A if $[S,A] = SA - AS = 0$; a *generalised symmetry* of A if $[S,A] = fA$ for a certain operator f, and a *supersymmetry* if $[\![S,A]\!] = 0$. Finally, S anticommutes with A if $\{S,A\} := SA + AS = 0$.

The study of the Dunkl–Dirac operator $\underline{D}$ and its kernel can take many forms. A recent fruitful path to its understanding resides in the consideration of the symmetry algebra linked to the $\mathfrak{osp}(1|2)$ realisation generated by $\underline{D}$ and its dual symbol $\underline{x}$ [ØSS09]. This symmetry algebra, which is the total angular momentum algebra $\mathcal{O}_\kappa$ presented in Section 2.2.5, consists of elements supercommuting with $\underline{D}$, and the Dunkl monogenics form natural representation spaces for $\mathcal{O}_\kappa$. The study of its representation theory is furthered in the following chapters.

The goal of this chapter is to introduce a class of generalised symmetries of the Dunkl–Dirac operator. Since these generalised symmetries preserve the kernel of $\underline{D}$, they can be used to construct natural bases for the spaces of monogenic polynomials. Constructing bases of Dunkl monogenic polynomials with tractable expressions is not an easy feat. Formulas are only known for the groups $W = \mathsf{D}_{2m}$ [Dun89], $W = \mathbb{Z}_2^N$ [DGV16a; DGV16b], $W = \mathsf{D}_{2m} \times \mathbb{Z}_2$ (Theorem 5.7.7–5.7.8), and partially for S_n [Dun16].

The generalised symmetries are related to the Maxwell representation in harmonic analysis [Mül98, p.69], which was translated to Dunkl harmonic analysis by Xu [Xu00] and to Dunkl–Clifford analysis in [FCK09; Yac11]. Similar operators were also considered in the study of the conformal symmetries of the super Dirac operator [CD15] and on the radially deformed Dirac operator [DDE17]. The last two were presented via Kelvin inverses; the generalised symmetries defined here are valid also in the abstract context of [DOV18a] with abstract generators and commutation relations, but admit a presentation using a Clifford–Kelvin type transform when specialised to the Dunkl setting.

As an application, we use these generalised symmetries to give a new interpretation of the basis previously obtained by means of a Dunkl version of the Cauchy–Kovalevskaya (CK) extension Theorem in [DGV16b].

We now go through the structure of the chapter and highlight the main results. In Section 3.2, we introduce the preliminaries on Dunkl operators and rewrite some results of Xu [Xu00] on Dunkl harmonics in terms of generalised symmetries of the Dunkl–Laplace operator. Section 3.3 goes from the Dunkl harmonics to the Dunkl monogenics. We introduce a class of operators and prove their main properties. They are generalised symmetries of the Dunkl–Dirac operator (Proposition 3.3.4), they commute with each other (Proposition 3.3.8), they can be written by means of a Dunkl–Clifford–Kelvin transform (Proposition 3.3.7) and they are related with a monogenic projection operator (Propositions 3.3.10 and 3.3.11). A basis of the monogenic representation for any reflection group is then constructed in Section 3.4 (Theorem 3.4.4). Finally, we study in Section 3.5 the case of the group $W = \mathbb{Z}_2^d$ and retrieve a known basis (Proposition 3.5.11).

3.2 Dunkl operators

Recall the definitions of the previous chapter. We fix $d \in \mathbb{N}$ and work in $\mathbb{R}^d$ with canonical bilinear form $\langle -, - \rangle$. We denote the Dunkl operator (2.10) in coordinate j by D_j and the Dunkl degree operator by $H := \mathbb{E} + d/2 + \gamma$ (2.17). In this chapter, we will identify the rational Cherednik algebra H_κ with its Dunkl representation.

3.2.1 Dunkl harmonics

We will denote by $\mathcal{P} := \mathcal{P}(\mathbb{R}^d)$ the space of complex-valued polynomials on $\mathbb{R}^d$ and by $\mathcal{P}_n := \mathcal{P}_n(\mathbb{R}^d)$ the space of homogeneous polynomials of degree n. The space $\mathcal{H}$ of Dunkl harmonic polynomials consists of all polynomials in the kernel of the Dunkl–Laplace operator Δ_κ, introduced in (2.15). We further denote $\mathcal{H}_n := \mathcal{H} \cap \mathcal{P}_n$.

In a classical construction of harmonic analysis, the Maxwell representation [Mül98] allows one to construct bases of polynomial harmonics by means of the Kelvin transformation. This was extended by Xu to Dunkl harmonics [Xu00]. For $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$, Xu considered the harmonic polynomials $\mathsf{Y}_\beta(x)$ given by (we define them directly with Kelvin transform, compare with [Xu00, Def. 2.2])

$$\mathsf{Y}_\beta(x) := \mathrm{K}_\kappa \mathrm{D}_1^{\beta_1} \dots \mathrm{D}_d^{\beta_d} \mathrm{K}_\kappa(1), \tag{3.1}$$

where a Dunkl version of the Kelvin transform is used:

$$\mathrm{K}_\kappa f(x) := |x|^{-(2\gamma+d-2)} f\left(\frac{x}{|x|^2}\right), \quad \mathrm{K}_\kappa \mathrm{K}_\kappa f(x) = f(x). \tag{3.2}$$

They satisfy $\Delta_\kappa \mathsf{Y}_\beta(x) = 0$.

3.2.2 Generalised symmetries

It is possible to express Xu's construction by means of generalised symmetries of the Dunkl–Laplace operator. The definition of these operators is inspired by [Xu00, Thm 2.3]. They are related to the adjoints of a Dunkl operator, see [Dun89, Thm 2.1 and Prop. 2.3].

Definition 3.2.1. *We define $\mathfrak{m}_j \in H_\kappa$ to be*

$$\mathfrak{m}_j := 2x_j(H-1) - |x|^2 \mathrm{D}_j, \tag{3.3}$$

where H is defined in (2.17). For a multi-index $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$, we write $\mathfrak{m}^\beta := \mathfrak{m}_1^{\beta_1} \cdots \mathfrak{m}_d^{\beta_d}$.

Proposition 3.2.2. *The operator $\mathfrak{m}_j$ is a generalised symmetry of the Dunkl–Laplace operator:*

$$\left[\Delta_\kappa, \mathfrak{m}_j\right] = 4x_j \Delta_\kappa. \tag{3.4}$$

Proof. It follows from the relations (2.18) and (2.19):

$$\begin{aligned}
\Delta_\kappa \mathfrak{m}_j &= \Delta_\kappa (2x_j H - 2x_j - |x|^2 \mathrm{D}_j) \\
&= 2x_j \Delta_\kappa H + 4\mathrm{D}_j H - 2x_j \Delta_\kappa - 4\mathrm{D}_j - |x|^2 \mathrm{D}_j \Delta_\kappa - 4H\mathrm{D}_j \\
&= (2x_j H - 2x_j - |x|^2 \mathrm{D}_j)\Delta_\kappa + 4x_j \Delta_\kappa + 4\left[\mathrm{D}_j, H\right] - 4\mathrm{D}_j \\
&= \mathfrak{m}_j \Delta_\kappa + 4x_j \Delta_\kappa.
\end{aligned}$$

■

The next result gives the correspondence $\mathfrak{m}^\beta(1) = (-1)^n \mathrm{Y}_\beta(x)$ for $\beta \in \mathbb{N}^d$ with $|\beta|_1 = n$, where $|\cdot|_1$ is the 1-norm of β, that is, the sum of its components.

Proposition 3.2.3. *For $\beta \in \mathbb{N}^d$ with $|\beta|_1 = m$, when acting on $\mathcal{P}$,*

$$\mathfrak{m}_j = -\mathrm{K}_\kappa \mathrm{D}_j \mathrm{K}_\kappa, \quad \text{and} \quad \mathfrak{m}^\beta = (-1)^m \mathrm{K}_\kappa \mathrm{D}^\beta \mathrm{K}_\kappa. \tag{3.5}$$

Proof. By linearity, it is sufficient to prove it for a homogeneous polynomial $p \in \mathcal{P}_n$. We apply the Dunkl–Leibniz rule (2.16) to get

$$\begin{aligned} \mathrm{D}_j \mathrm{K}_\kappa p(x) &= \mathrm{D}_j |x|^{-(2\gamma+d-2+2n)} p(x) \\ &= |x|^{-(2\gamma+d-2+2n)} \mathrm{D}_j p(x) \\ &\quad - (2\gamma + d - 2 + 2n)|x|^{-(2\gamma+d+2n)} x_j p(x). \end{aligned}$$

Both terms have degree of homogeneity $-2\gamma - d + 1 - n$, so we can apply again the Kelvin transform K_κ on the two sides to obtain

$$\begin{aligned} \mathrm{K}_\kappa \mathrm{D}_j \mathrm{K}_\kappa p(x) &= |x|^{-(2\gamma+d-2-2\gamma-2d+2+2n)} |x|^{-(2\gamma+d-2+2n)} \mathrm{D}_j p(x) \\ &\quad - (2\gamma + d + 2n - 2) \frac{|x|^{-(2\gamma+d+2n)}}{|x|^{2\gamma+d-2-4\gamma-2d+2+2n}} x_j p(x) \\ &= |x|^2 \mathrm{D}_j p(x) + 2x_j p - (2\gamma + d + 2n) x_j p(x), \end{aligned}$$

and this is precisely $-\mathfrak{m}_j p(x) = -(2x_j H - 2x_j - |x|^2 \mathrm{D}_j) p(x)$. ∎

Proposition 3.2.4. *The generalised symmetries $\mathfrak{m}_j$ commute amongst themselves when acting on $\mathcal{P}$*

$$\left[\mathfrak{m}_j, \mathfrak{m}_k\right] = 0. \tag{3.6}$$

Proof. By Proposition 3.2.3, when acting on $\mathcal{P}$,

$$\begin{aligned} \mathfrak{m}_j \mathfrak{m}_k &= \mathrm{K}_\kappa \mathrm{D}_j \mathrm{K}_\kappa \mathrm{K}_\kappa \mathrm{D}_k \mathrm{K}_\kappa = \mathrm{K}_\kappa \mathrm{D}_j \mathrm{D}_k \mathrm{K}_\kappa \\ &= \mathrm{K}_\kappa \mathrm{D}_k \mathrm{D}_j \mathrm{K}_\kappa = \mathrm{K}_\kappa \mathrm{D}_k \mathrm{K}_\kappa \mathrm{K}_\kappa \mathrm{D}_j \mathrm{K}_\kappa = \mathfrak{m}_k \mathfrak{m}_j. \end{aligned} \tag{3.7}$$

∎

Let $\operatorname{proj}_{\mathcal{H}}^{\mathcal{P}}\colon \mathcal{P} \to \mathcal{H}$ denote the projection operator that, when restricted to $\mathcal{P}_n$, reduces to $\operatorname{proj}_{\mathcal{H}_n}^{\mathcal{P}_n}$ given by [Xu00, (2.5)]

$$\operatorname{proj}_{\mathcal{H}_n}^{\mathcal{P}_n} p(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{|x|^{2j} \Delta_\kappa^j p(x)}{2^{2j} j! (-n - d/2 - \gamma + 2)_j}, \tag{3.8}$$

where the notation for the Pochhammer symbol is used, which is defined as $(a)_0 = 1$, and $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$, with the Gamma function Γ.

The Dunkl harmonic $\mathsf{Y}_\beta(x) = (-1)^n \mathfrak{m}^\beta(1)$ is related to the projection (3.8) as follows.

Theorem 3.2.5 ([Xu00, Theorem 2.4])**.** *For $\beta \in \mathbb{N}^d$ with $|\beta|_1 = n$,*

$$\mathsf{Y}_\beta(x) = (-1)^n 2^n (\gamma - 1 + d/2)_n \operatorname{proj}_{\mathcal{H}_n}^{\mathcal{P}_n}(x^\beta). \tag{3.9}$$

We conclude this section with a result relating the operator $\mathfrak{m}_j$ with the projection (3.8).

Proposition 3.2.6. *With H given by (2.17) and x_j, the operator that multiplies a polynomial by x_j, we have, when acting on $\mathcal{H}$,*

$$\mathfrak{m}_j = 2(H-2) \circ \operatorname{proj}_{\mathcal{H}}^{\mathcal{P}} \circ x_j. \tag{3.10}$$

Proof. Let $h_{n-1} \in \mathcal{H}_{n-1}$, then $\Delta_\kappa^k x_j(h_{n-1}) = 0$ for $k \geq 2$, so using (3.8) we have

$$\begin{aligned} 2(H-2)\operatorname{proj}_{\mathcal{H}}^{\mathcal{P}}(x_j h_{n-1}) &= 2(H-2)\operatorname{proj}_{\mathcal{H}_n}^{\mathcal{P}_n}(x_j h_{n-1}) \\ &= 2(H-2)x_j h_{n-1} \\ &\quad - \frac{2(\mathbb{E} + d/2 + \gamma - 2)|x|^2 \Delta_\kappa (x_j h_{n-1})}{(4(\gamma + n - 2 + d/2))} \\ &= 2x_j(H-1)h_{n-1} - |x|^2 \mathrm{D}_j h_{n-1}, \end{aligned}$$

where we used $\left[\Delta_\kappa, x_j\right] = 2\mathrm{D}_j$ and $\Delta_\kappa h_{n-1} = 0$. The last line is precisely (3.3). ∎

3.3 Dunkl monogenic analysis

In this section we introduce the Dunkl monogenic space, define the generalised symmetries and the Dunkl–Clifford–Kelvin inverse used in the chapter, and present results pertaining to the projection operator.

We recall the Dunkl–Dirac operator $\underline{\mathrm{D}}$ and its dual symbol $\underline{x}$ (2.55). We refer the readers to Section 2.2 for the background.

3.3.1 Dunkl monogenics

Let V be an irreducible representation of $Cl(d)$, also called a spinor representation. There is a natural action of $H_\kappa \otimes Cl(d)$ on the space $\mathcal{P} \otimes V$. The space of Dunkl monogenic polynomials consists of the elements of $\mathcal{P} \otimes V$ that are in the kernel of the Dunkl–Dirac operator, and will be denoted by $\mathcal{M} := \mathcal{M}(\mathbb{R}^d; V)$. We denote $\mathcal{M}_n := \mathcal{M}_n(\mathbb{R}^d; V) = \mathcal{M} \cap (\mathcal{P}_n \otimes V)$ for the $\mathcal{M}$-subspace of (spinor-valued) homogeneous polynomials of degree n, and we have $\mathcal{M}(\mathbb{R}^d; V) = \bigoplus_{n\geq 0} \mathcal{M}_n(\mathbb{R}^d; V)$.

There is a projection $\mathrm{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V} : \mathcal{P}(\mathbb{R}^d) \otimes V \longrightarrow \mathcal{M}(\mathbb{R}^d; V)$ that, when restricted to $\mathcal{H}_n \otimes V$, is given by [ØSS09, Lem. 4.6]

$$\begin{aligned} \mathrm{proj}_{\mathcal{M}_n}^{\mathcal{P}_n\otimes V} &: \mathcal{P}_n(\mathbb{R}^d) \otimes V \longrightarrow \mathcal{M}_n(\mathbb{R}^d; V) \\ p &\longmapsto p - \varepsilon \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} \frac{(-1)^j \underline{x}^{2j+1} \underline{\mathrm{D}}^{2j+1} p}{2^{2j+1} j! (n-j-1+d/2+\gamma)_{j+1}} \\ &\quad + \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{(-1)^j |x|^{2j} \Delta_\kappa^j p}{2^{2j} j! (n-j+d/2+\gamma)_j}. \end{aligned} \tag{3.11}$$

Remark 3.3.1. *Each Dunkl monogenic polynomial is a highest weight vector for the $\mathfrak{osp}(1|2)$ realisation containing the Dunkl–Dirac operator as positive root vector. For the $\mathfrak{osp}(1|2)$ extremal projector $\mathrm{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V}$, we have [BT81, (3.8a)]*

$$\mathrm{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V} = \mathrm{proj}_{\mathcal{M}}^{\mathcal{H}\otimes V} \mathrm{proj}_{\mathcal{H}\otimes V}^{\mathcal{P}\otimes V}, \tag{3.12}$$

where $\mathrm{proj}_{\mathcal{H}\otimes V}^{\mathcal{P}\otimes V} = \mathrm{proj}_{\mathcal{H}}^{\mathcal{P}}$ as considered above (3.8) for the $\mathfrak{sl}(2)$ part, and

$\mathrm{proj}_{\mathcal{M}}^{\mathcal{H}\otimes V}$, *when restricted to* $\mathcal{P}_n \otimes V$, *is given by*

$$\mathrm{proj}_{\mathcal{M}_n}^{\mathcal{H}_n\otimes V} = \left(1 - \frac{\varepsilon \underline{x}\,\underline{\mathrm{D}}}{2(n-1+d/2+\gamma)}\right). \tag{3.13}$$

3.3.2 Generalised symmetries

Recall that $\varepsilon \in \{-1,+1\}$ and that the Clifford generators e_j satisfy anti-commutation (2.31). We will define a similar generalised symmetry $\mathfrak{z}_j$ mimicking the harmonic case $\mathfrak{m}_j$.

Definition 3.3.2. *We define* $\mathfrak{z}_j \in H_\kappa \otimes Cl(d)$ *for* $1 \le j \le d$ *by*

$$\mathfrak{z}_j := 2\varepsilon x_j H - \underline{x}\mathrm{D}_j\underline{x}. \tag{3.14}$$

For a multi-index $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$, *we write* $\mathfrak{z}^\beta := \mathfrak{z}_1^{\beta_1} \dots \mathfrak{z}_d^{\beta_d}$.

We begin by giving alternative formulations of $\mathfrak{z}_j$ in terms of elements of $H_\kappa \otimes Cl(d)$ that follow from Lemma 2.2.18 and the expressions (2.65) of O_j.

Lemma 3.3.3. *The operator* $\mathfrak{z}_j := 2\varepsilon x_j H - \underline{x}\mathrm{D}_j\underline{x}$ *has the following expressions*

$$\mathfrak{z}_j = x_j\left\{\underline{\mathrm{D}}, \underline{x}\right\} - \underline{x}\left[\underline{\mathrm{D}}, x_j\right] - \varepsilon|x|^2\mathrm{D}_j; \tag{3.15}$$

$$\mathfrak{z}_j = 2\varepsilon x_j(\mathbb{E} + d/2 + \gamma) - \underline{x}(e_j + 2\varepsilon O_j) - \varepsilon|x|^2\mathrm{D}_j. \tag{3.16}$$

We now consider some of the main properties of $\mathfrak{z}_j$ that are useful for our purposes.

Proposition 3.3.4. *The operator* $\mathfrak{z}_j$ *is a generalised symmetry of the Dunkl–Dirac operator*

$$\left[\underline{\mathrm{D}}, \mathfrak{z}_j\right] = 2\varepsilon x_j\underline{\mathrm{D}}. \tag{3.17}$$

Proof. First we anticommute $\underline{\mathrm{D}}$ and $\underline{x}$ by (2.57) and commute $\underline{\mathrm{D}}$ and x_j by (2.60)

$$\begin{aligned}\underline{\mathrm{D}}\,\mathfrak{z}_j &= \underline{\mathrm{D}}(x_j\left\{\underline{\mathrm{D}}, \underline{x}\right\} - \underline{x}\mathrm{D}_j\underline{x}) \\ &= (x_j\underline{\mathrm{D}} + \left[\underline{\mathrm{D}}, x_j\right])\left\{\underline{\mathrm{D}}, \underline{x}\right\} - (-\underline{x}\,\underline{\mathrm{D}} + \left\{\underline{x}, \underline{\mathrm{D}}\right\})\mathrm{D}_j\underline{x},\end{aligned}$$

now we use the commutation of D_j

$$= x_j \underline{\mathrm{D}} \left\{\underline{\mathrm{D}}, \underline{x}\right\} + \left[\mathrm{D}_j, \underline{x}\right]\left\{\underline{\mathrm{D}}, \underline{x}\right\} + \underline{x}\mathrm{D}_j \underline{\mathrm{D}}\,\underline{x} - \left\{\underline{\mathrm{D}}, \underline{x}\right\}\mathrm{D}_j \underline{x},$$

then we apply (2.57) and (2.19) to send $\underline{\mathrm{D}}$ right to $\left\{\underline{\mathrm{D}}, \underline{x}\right\}$, and anticommute a second time $\underline{\mathrm{D}}$ and $\underline{x}$

$$\begin{aligned}
&= x_j \left\{\underline{\mathrm{D}}, \underline{x}\right\}\underline{\mathrm{D}} + 2\varepsilon x_j \underline{\mathrm{D}} + \left[\mathrm{D}_j, \underline{x}\right]\left\{\underline{\mathrm{D}}, \underline{x}\right\} \\
&\quad - \underline{x}\mathrm{D}_j \underline{x}\,\underline{\mathrm{D}} + \underline{x}\mathrm{D}_j \left\{\underline{\mathrm{D}}, \underline{x}\right\} - \left\{\underline{\mathrm{D}}, \underline{x}\right\}\mathrm{D}_j \underline{x},
\end{aligned}$$

finally, $\left\{\underline{\mathrm{D}}, \underline{x}\right\}$ commutes with $\mathrm{D}_j\underline{x}$ because of $\left[H, \mathrm{D}_j\right] = -\mathrm{D}_j$, $[H, \underline{x}] = \underline{x}$ and $\varepsilon^2 = 1$ so

$$\begin{aligned}
&= (x_j \left\{\underline{\mathrm{D}}, \underline{x}\right\} - \underline{x}\mathrm{D}_j \underline{x} + 2\varepsilon x_j)\underline{\mathrm{D}} + \left[\mathrm{D}_j, \underline{x}\right]\left\{\underline{\mathrm{D}}, \underline{x}\right\} \\
&\quad - \mathrm{D}_j \underline{x}\left\{\underline{\mathrm{D}}, \underline{x}\right\} + \underline{x}\mathrm{D}_j \left\{\underline{\mathrm{D}}, \underline{x}\right\} \\
&= \mathbb{z}_j \underline{\mathrm{D}} + 2\varepsilon x_j \underline{\mathrm{D}}.
\end{aligned}$$

■

Proposition 3.3.5. *The operator $\mathbb{z}_k$ respects the following commutation relations*

$$\begin{aligned}
[\underline{x}, \mathbb{z}_k] &= -2\varepsilon x_k \underline{x} + \underline{x}(e_k + 2\varepsilon O_k)\underline{x}; && (3.18)\\
\left[x_j, \mathbb{z}_k\right] &= -2\varepsilon x_j x_k - \underline{x}\left[x_j, \mathrm{D}_k\right]\underline{x}; && (3.19)\\
\left[e_j, \mathbb{z}_k\right] &= 2\varepsilon(\underline{x}\mathrm{D}_k x_j - x_j \mathrm{D}_k \underline{x}); && (3.20)\\
\left[\mathrm{D}_j, \mathbb{z}_k\right] &= 2\varepsilon(x_k \mathrm{D}_j - x_j \mathrm{D}_k) + 2\varepsilon\left[\mathrm{D}_j, x_k\right]H + e_j\left[\underline{x}, \mathrm{D}_k\right] \\
&\quad - 2\varepsilon(O_j \mathrm{D}_k \underline{x} + \underline{x}\mathrm{D}_k O_j); && (3.21)\\
\tilde{\sigma}_\alpha \mathbb{z}_k &= \mathbb{z}_{\sigma_\alpha(\xi_k)}\tilde{\sigma}_\alpha, \quad \textit{with} \quad \mathbb{z}_{\sigma_\alpha(\xi_k)} := \sum_{j=1}^{d} \left\langle \sigma_\alpha(\xi_k), \xi_j \right\rangle \mathbb{z}_j. && (3.22)
\end{aligned}$$

Proof. Equation (3.18) follows from a small computation using equations (2.19) and (2.58)

$$\begin{aligned}
[\underline{x}, \mathbb{z}_k] &= 2\varepsilon \underline{x} x_k H - \underline{x}\,\underline{x}\mathrm{D}_k \underline{x} - \mathbb{z}_k \underline{x} = 2\varepsilon x_k \underline{x} H - \underline{x}\mathrm{D}_k \underline{x}\,\underline{x} - \underline{x}[\underline{x}, \mathrm{D}_k]\underline{x} - \mathbb{z}_k \underline{x} \\
&= -2\varepsilon x_k \underline{x} + \underline{x}(e_k + 2\varepsilon O_k)\underline{x}.
\end{aligned}$$

Equation (3.19) follows from the commutation relation (2.19) between x_j and H:

$$\begin{aligned}\left[x_j, z_k\right] &= 2\varepsilon x_j x_k H - x_j \underline{x} \mathrm{D}_k \underline{x} - z_k x_j \\ &= 2\varepsilon x_k H x_j - 2\varepsilon x_k x_j - (\underline{x}\mathrm{D}_k x_j \underline{x} + \underline{x}\left[x_j, \mathrm{D}_k\right]\underline{x}) - z_k x_j \\ &= -2\varepsilon x_j x_k - \underline{x}\left[x_j, \mathrm{D}_k\right]\underline{x}.\end{aligned}$$

Equation (3.20) comes from $\left\{e_j, \underline{x}\right\} = 2\varepsilon x_j$:

$$\begin{aligned}e_j z_k &= 2\varepsilon e_j x_k H - e_j \underline{x}\mathrm{D}_k \underline{x} = 2\varepsilon x_k H e_j + \underline{x} e_j \mathrm{D}_k \underline{x} - 2\varepsilon x_j \mathrm{D}_k \underline{x} \\ &= z_k e_j + 2\varepsilon \underline{x}\mathrm{D}_k x_j - 2\varepsilon x_j \mathrm{D}_k \underline{x}.\end{aligned}$$

Slightly more tedious computations yield equation (3.21). First develop

$$\begin{aligned}\left[\mathrm{D}_j, z_k\right] &= 2\varepsilon \mathrm{D}_j x_k H - \mathrm{D}_j \underline{x}\mathrm{D}_k \underline{x} - z_k \mathrm{D}_j \\ &= 2\varepsilon x_k \mathrm{D}_j H + 2\varepsilon\left[\mathrm{D}_j, x_k\right]H - (\underline{x}\mathrm{D}_j \mathrm{D}_k \underline{x} + \left[\mathrm{D}_j, \underline{x}\right]\mathrm{D}_k \underline{x}) \\ &\quad - (2\varepsilon x_k H \mathrm{D}_j - \underline{x}\mathrm{D}_k \mathrm{D}_j \underline{x} - \underline{x}\mathrm{D}_k\left[\underline{x}, \mathrm{D}_j\right]),\end{aligned}$$

then employ $\left[H, \mathrm{D}_j\right] = -\mathrm{D}_j$ and cancel some terms with (2.65)

$$\begin{aligned}&= 2\varepsilon x_k \mathrm{D}_j + 2\varepsilon\left[\mathrm{D}_j, x_k\right]H - \left[\mathrm{D}_j, \underline{x}\right]\mathrm{D}_k \underline{x} + \underline{x}\mathrm{D}_k\left[\underline{x}, \mathrm{D}_j\right] \\ &= 2\varepsilon x_k \mathrm{D}_j + 2\varepsilon\left[\mathrm{D}_j, x_k\right]H - (e_j + 2\varepsilon O_j)\mathrm{D}_k \underline{x} - \underline{x}\mathrm{D}_k(e_j + 2\varepsilon O_j),\end{aligned}$$

now use $\left\{\underline{x}, e_j\right\} = 2\varepsilon x_j$ to get

$$\begin{aligned}&= 2\varepsilon x_k \mathrm{D}_j + 2\varepsilon\left[\mathrm{D}_j, x_k\right]H - e_j \mathrm{D}_k \underline{x} + e_j \underline{x}\mathrm{D}_k \\ &\quad - 2\varepsilon x_j \mathrm{D}_k - 2\varepsilon(O_j \mathrm{D}_k \underline{x} + \underline{x}\mathrm{D}_k O_j) \\ &= 2\varepsilon(x_k \mathrm{D}_j - x_j \mathrm{D}_k) + 2\varepsilon\left[\mathrm{D}_j, x_k\right]H + e_j\,[\underline{x}, \mathrm{D}_k] \\ &\quad - 2\varepsilon(O_j \mathrm{D}_k \underline{x} + \underline{x}\mathrm{D}_k O_j).\end{aligned}$$

Relation (3.22) follows from (2.59), and from the action of W on x_j and D_j. ∎

Lemma 3.3.6. *The operator* $\underline{z} := \sum_{j=1}^{d} z_j e_j$ *can be written as*

$$\underline{z} = 2\varepsilon\underline{x}(\mathbb{E} + \gamma - \sum_{\alpha\in R^+} \kappa(\alpha)\sigma_\alpha) - \varepsilon|x|^2\underline{\mathrm{D}}. \tag{3.23}$$

Proof. We express z_j by (3.16) and use Lemma 2.2.21:

$$\begin{aligned}
\underline{z} &= \sum_{j=1}^{d}(2\varepsilon x_j(\mathbb{E} + d/2 + \gamma) - \underline{x}(e_j + 2\varepsilon O_j) - \varepsilon|x|^2 \mathrm{D}_j)e_j \\
&= 2\varepsilon\underline{x}(\mathbb{E} + d/2 + \gamma) - \varepsilon\underline{x}d - 2\varepsilon\underline{x}\sum_{j=1}^{d} O_j e_j - \varepsilon|x|^2\underline{\mathrm{D}} \\
&= 2\varepsilon\underline{x}(\mathbb{E} + \gamma - \sum_{\alpha\in\Phi^+} \kappa(\alpha)\sigma_\alpha) - \varepsilon|x|^2\underline{\mathrm{D}}.
\end{aligned}$$

■

3.3.3 Kelvin transformation

Define the Dunkl–Clifford–Kelvin transform I_κ as

$$\mathrm{I}_\kappa f(x) = \underline{x}|x|^{-(2\gamma+d)} f\left(\frac{x}{|x|^2}\right). \tag{3.24}$$

Since $\kappa \geq 0$, the sum $\gamma = \sum_{\alpha\in\Phi^+}\kappa(\alpha)$ is non-negative and the expression $|x|^{-(2\gamma+d)}$ is thus well-defined. The operator I_κ is ε-idempotent, that is $\mathrm{I}_\kappa^2 = \varepsilon$. Indeed, using $\underline{x}\,\underline{x} = \varepsilon|x|^2$,

$$\begin{aligned}
\mathrm{I}_\kappa \mathrm{I}_\kappa f(x) &= \mathrm{I}_\kappa\left(\underline{x}|x|^{-(2\gamma+d)} f\left(\frac{x}{|x|^2}\right)\right) \\
&= \underline{x}|x|^{-(2\gamma+d)}\left(\frac{\underline{x}}{|x|^2}\frac{|x|^{2(2\gamma+d)}}{|x|^{(2\gamma+d)}} f\left(\frac{x}{|x|^2}\frac{|x|^4}{|x|^2}\right)\right) = \varepsilon f(x).
\end{aligned}$$

The relation between the two Kelvin-type transforms (3.2) and (3.24) is

$$\mathrm{I}_\kappa f = \varepsilon\underline{x}|x|^{-2}\mathrm{K}_\kappa f. \tag{3.25}$$

Remark that for $p(x) \in \mathcal{P}_n(\mathbb{R}^d)$ we have $p(x/|x|^2) = |x|^{-2n}p(x)$, and thus the action of the Dunkl–Clifford–Kelvin transform becomes

$$\mathrm{I}_\kappa p(x) = |x|^{-(2\gamma+d+2n)}\underline{x}p(x). \tag{3.26}$$

The transform (3.24) was considered before, for example see [Yac11] and [FCK09]. One of the main results of those two papers is to prove that, for any polynomial monogenic f, also $\mathrm{I}_\kappa \mathrm{D}_j \mathrm{I}_\kappa(f)$ is a polynomial monogenic. We give an interpretation in terms of generalised symmetries of the Dunkl–Dirac operator.

Proposition 3.3.7. *For $\beta \in \mathbb{N}^d$ with $|\beta|_1 := \sum_{j=1}^d \beta_j = m$, when acting on $\mathcal{P} \otimes V$,*

$$\mathfrak{z}_j = -\mathrm{I}_\kappa \mathrm{D}_j \mathrm{I}_\kappa, \quad \text{and} \quad \mathfrak{z}^\beta = (-1)^m \varepsilon^{m-1} \mathrm{I}_\kappa \mathrm{D}^\beta \mathrm{I}_\kappa. \tag{3.27}$$

Proof. Let $p \in \mathcal{P}_n(\mathbb{R}^d)$ be a homogeneous polynomial of degree n. Apply equation (2.16) to get

$$\begin{aligned} \mathrm{D}_j \mathrm{I}_\kappa p(x) &= \mathrm{D}_j |x|^{-(2\gamma+d+2n)} \underline{x} p(x) \\ &= -(2\gamma + d + 2n)|x|^{-(2\gamma+d+2n+2)} x_j \underline{x} p(x) \\ &\quad + |x|^{-(2\gamma+d+2n)} \mathrm{D}_j \underline{x} p(x). \end{aligned}$$

Remark now that the first and second terms have degree of homogeneity $-2\gamma - d - n$. Thus applying another time the Dunkl–Clifford–Kelvin transform yields

$$\begin{aligned} \mathrm{I}_\kappa \mathrm{D}_j \mathrm{I}_\kappa p(x) &= -(2\gamma + d + 2n)\underline{x} \frac{|x|^{-(2\gamma+d+2n+2)}}{|x|^{2\gamma+d-4\gamma-2d-2n}} x_j \underline{x} p(x) \\ &\quad + \underline{x}|x|^{-(2\gamma+d-4\gamma-2d-2n)} |x|^{-(2\gamma+d+2n)} \mathrm{D}_j \underline{x} p(x) \\ &= -(2\gamma + d + 2n)\underline{x}^2 |x|^{-2} x_j p(x) + \underline{x} \mathrm{D}_j \underline{x} p(x) \\ &= -2\varepsilon(n + d/2 + \gamma) x_j p(x) + \underline{x} \mathrm{D}_j \underline{x} p(x), \end{aligned}$$

which equals $-\mathfrak{z}_j p(x) = -(2\varepsilon x_j(\mathbb{E} + d/2 + \gamma) - \underline{x} \mathrm{D}_j \underline{x}) p(x)$. ∎

The commutativity of the Dunkl operators implies that of the $\mathfrak{z}_j$.

Proposition 3.3.8. *The operators $\mathfrak{z}_j$ commute amongst themselves, namely*

$$\left[\mathfrak{z}_j, \mathfrak{z}_\ell\right] = 0. \tag{3.28}$$

Proof. Apply the commutation relations of Lemma 2.2.18 and equation (2.58). First expand the product of the two generalised symmetries $\mathfrak{z}_j$ and $\mathfrak{z}_\ell$:

$$\mathfrak{z}_j \mathfrak{z}_\ell = 4\varepsilon^2 x_j H x_\ell H - 2\varepsilon x_j H \underline{x} \mathrm{D}_\ell \underline{x} - 2\varepsilon \underline{x} \mathrm{D}_j \underline{x} x_\ell H + \varepsilon \underline{x} \mathrm{D}_j |x|^2 \mathrm{D}_\ell \underline{x}.$$

Now work out the commutation of the four terms and combine the results. The first one can be switched directly using $[H, x_i] = x_i$ and $\left[x_i, x_j\right] = 0$

$$\begin{aligned} x_j H x_\ell H &= H x_j x_\ell H - x_j x_\ell H \\ &= x_\ell H x_j H + x_\ell x_j H - x_j x_\ell H = x_\ell H x_j H. \end{aligned} \tag{3.29}$$

Send the $x_j H$ of the second term to the right using equations (2.58) and (2.19)

$$\begin{aligned} x_j H \underline{x} \mathrm{D}_\ell \underline{x} &= \underline{x} x_j H \mathrm{D}_\ell \underline{x} + \underline{x} x_j \mathrm{D}_\ell \underline{x} = \underline{x} x_j \mathrm{D}_\ell H \underline{x} - \underline{x} x_j \mathrm{D}_\ell + \underline{x} x_j \mathrm{D}_\ell \underline{x} \\ &= \underline{x} x_j \mathrm{D}_\ell H \underline{x} = \underline{x} x_j \mathrm{D}_\ell \underline{x} H + \underline{x} x_j \mathrm{D}_\ell \underline{x} \\ &= \underline{x} \mathrm{D}_\ell \underline{x} x_j H + \underline{x}\left[x_j, \mathrm{D}_\ell\right] \underline{x} H + \underline{x} x_j \mathrm{D}_\ell \underline{x}. \end{aligned} \tag{3.30}$$

Now send the $x_\ell H$ of the third term to the left using the same equations

$$\begin{aligned} \underline{x} \mathrm{D}_j \underline{x} x_\ell H &= \underline{x} \mathrm{D}_j x_\ell \underline{x} H = x_\ell \underline{x} \mathrm{D}_j \underline{x} H + \underline{x}\left[\mathrm{D}_j, x_\ell\right] \underline{x} H \\ &= x_\ell \underline{x} \mathrm{D}_j H \underline{x} - x_\ell \underline{x} \mathrm{D}_j \underline{x} + \underline{x}\left[\mathrm{D}_j, x_\ell\right] \underline{x} H \\ &= x_\ell \underline{x} H \mathrm{D}_j \underline{x} + x_\ell \underline{x} \mathrm{D}_j \underline{x} - x_\ell \underline{x} \mathrm{D}_j \underline{x} + \underline{x}\left[\mathrm{D}_j, x_\ell\right] \underline{x} H \\ &= x_\ell H \underline{x} \mathrm{D}_j \underline{x} - \underline{x} x_\ell \mathrm{D}_j \underline{x} - \underline{x}\left[x_\ell, \mathrm{D}_j\right] \underline{x} H. \end{aligned} \tag{3.31}$$

Finally, developing the fourth term uses only $\left[|x|^2, \mathrm{D}_i\right] = -2x_i$ and $\left[\mathrm{D}_j, \mathrm{D}_\ell\right] = 0$:

$$\begin{aligned} \underline{x} \mathrm{D}_j |x|^2 \mathrm{D}_\ell \underline{x} &= (\underline{x} |x|^2 \mathrm{D}_j \mathrm{D}_\ell \underline{x} + 2 \underline{x} x_j \mathrm{D}_\ell \underline{x}) \\ &= \underline{x} \mathrm{D}_\ell |x|^2 \mathrm{D}_j \underline{x} - 2 \underline{x} x_\ell \mathrm{D}_j \underline{x} + 2 \underline{x} x_j \mathrm{D}_\ell \underline{x}. \end{aligned} \tag{3.32}$$

Combining equations (3.29) to (3.32) cancels the superfluous terms and yields $z_\ell z_j$, thus proving the commutation $z_j z_\ell = z_\ell z_j$. ■

Remark 3.3.9. *Proposition 3.3.8 holds in $H_\kappa \otimes Cl(d)$. When acting on $\mathcal{P} \otimes V$ we can use Proposition 3.3.7 to have a shorter proof. This was to avoid any problem due to specific values of κ.*

Apply Proposition 3.3.7 and the ε-idempotence of I_κ:

$$\begin{aligned} z_j z_\ell &= \mathrm{I}_\kappa \mathrm{D}_j \mathrm{I}_\kappa \mathrm{I}_\kappa \mathrm{D}_\ell \mathrm{I}_\kappa = \varepsilon \mathrm{I}_\kappa \mathrm{D}_j \mathrm{D}_\ell \mathrm{I}_\kappa \\ &= \varepsilon \mathrm{I}_\kappa \mathrm{D}_\ell \mathrm{D}_j \mathrm{I}_\kappa = \varepsilon^2 \mathrm{I}_\kappa \mathrm{D}_\ell \mathrm{I}_\kappa \mathrm{I}_\kappa \mathrm{D}_j \mathrm{I}_\kappa = z_\ell z_j. \end{aligned} \tag{3.33}$$

3.3.4 Projection operator relation

Proposition 3.3.10. *With H given by (2.17) and x_j the operator that multiplies a polynomial by x_j we have, when acting on $\mathcal{M}$,*

$$\mathfrak{z}_j = 2\varepsilon(H-1)\circ \operatorname{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V} \circ x_j. \tag{3.34}$$

Proof. Let $M_n \in \mathcal{M}_n$, then $\mathrm{D}^k x_j M_n = 0$ for $k \geq 3$, since $\underline{\mathrm{D}}^3 x_j M_n = \varepsilon\underline{\mathrm{D}}\Delta_\kappa x_j M_n = \underline{\mathrm{D}} x_j \Delta_\kappa M_n - \underline{\mathrm{D}}\,\overline{\mathrm{D}}_j M_n = 0$, so using the first three terms of the projector (3.11), we have

$$\begin{aligned}
2\varepsilon H \operatorname{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V}(x_j M_n) &= 2\varepsilon H(\operatorname{proj}_{\mathcal{M}_{n+1}}^{\mathcal{P}_{n+1}\otimes V}(x_j M_n)) \\
&= 2\varepsilon H(x_j M_n - \frac{\underline{x}\,\underline{\mathrm{D}}}{2(n+d/2+\gamma)} x_j M_n \\
&\quad - \frac{|x|^2\Delta_\kappa}{4(n+d/2+\gamma)} x_j M_n) \\
&= 2\varepsilon(n+1+\tfrac{d}{2}+\gamma)x_j M_n - \underline{x}\left[\underline{\mathrm{D}}, x_j\right] M_n - \varepsilon |x|^2 \mathrm{D}_j M_n \\
&\quad - \frac{\underline{x}\,\underline{\mathrm{D}}}{(n+d/2+\gamma)} x_j M_n - \frac{\varepsilon |x|^2\Delta_\kappa}{2(n+d/2+\gamma)} x_j M_n
\end{aligned}$$

where we evaluated H on polynomials of degree $n+1$ and used $\left[\Delta_\kappa, x_j\right] = 2\mathrm{D}_j$ and $\underline{\mathrm{D}} M_n = 0 = \Delta_\kappa M_{n-1}$. From this, we use (3.15) to see that indeed

$$2\varepsilon H \operatorname{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V}(x_j M_n) = \mathfrak{z}_j M_n + 2\varepsilon \operatorname{proj}_{\mathcal{M}}^{\mathcal{P}\otimes V}(x_j M_n),$$

proving the proposition. ∎

The next result is related to [Yac11, Prop 4.2] where the Dunkl–Clifford–Kelvin transform is used.

Proposition 3.3.11. *Let $\beta \in \mathbb{N}^d$ with $|\beta|_1 = n$ and $x^\beta \in \mathcal{P}_n$. Then, acting on V,*

$$\mathfrak{z}^\beta = \varepsilon^n 2^n (\gamma + d/2)_n \operatorname{proj}_{\mathcal{M}}^{\mathcal{P}_n\otimes V} \circ x^\beta. \tag{3.35}$$

Proof. By (3.12), we write

$$\operatorname{proj}_{\mathcal{M}_n}^{\mathcal{P}_n\otimes V} = \operatorname{proj}_{\mathcal{M}_n}^{\mathcal{H}_n\otimes V} \operatorname{proj}_{\mathcal{H}_n\otimes V}^{\mathcal{P}_n\otimes V}. \tag{3.36}$$

Let $s \in V$, then by (3.9) (or [Xu00, Theorem 2.4])

$$\mathrm{proj}_{\mathcal{H}_n \otimes V}^{\mathcal{P}_n \otimes V}(x^\beta s) = (-1)^n \mathrm{Y}_\beta\, s/(2^n(\gamma - 1 + d/2)_n).$$

Next, we apply (3.13) and make use of (2.16)

$$\begin{aligned}
\mathrm{proj}_{\mathcal{M}_n}^{\mathcal{H}_n \otimes V} \mathrm{Y}_\beta s &= \mathrm{Y}_\beta s - \frac{\varepsilon \underline{x}}{2n + d + 2\gamma - 2} \underline{\mathrm{D}} \mathrm{K}_\kappa \mathrm{D}^\beta \mathrm{K}_\kappa(s) \\
&= \mathrm{Y}_\beta s - \frac{\varepsilon \underline{x}\, \underline{\mathrm{D}}}{2n + d + 2\gamma - 2} |x|^{2\gamma + d - 2 + 2n} \mathrm{D}^\beta |x|^{-2\gamma - d + 2} s \\
&= \mathrm{Y}_\beta s - \frac{\varepsilon (2\gamma + d - 2 + 2n) \underline{x}^2 |x|^{2\gamma + d - 2 + 2n}}{(2n + d + 2\gamma - 2)|x|^{-2}} \mathrm{D}^\beta |x|^{-2\gamma - d + 2} s \\
&\quad + \frac{\varepsilon}{2n + d + 2\gamma - 2} \underline{x} |x|^{2\gamma + d - 2 + 2n} \mathrm{D}^\beta \underline{\mathrm{D}} |x|^{-2\gamma - d + 2} s \\
&= \mathrm{Y}_\beta s - |x|^{2\gamma + d - 2 + 2n} \mathrm{D}^\beta |x|^{-2\gamma - d + 2} s \\
&\quad + \frac{2\varepsilon (\gamma + d/2 - 1) \underline{x}}{2n + d + 2\gamma - 2} |x|^{2\gamma + d - 2 + 2n} \mathrm{D}^\beta \underline{x} |x|^{-2\gamma - d + 2 - 2} s \\
&= \frac{\varepsilon(\gamma + d/2 - 1)}{n + d/2 + \gamma - 1} \mathrm{I}_\kappa \mathrm{D}^\beta \mathrm{I}_\kappa(s),
\end{aligned}$$

where we used (3.25) in the last line. The result now follows by Proposition 3.3.7. ∎

3.4 Monogenic bases

The goal of this section is to give a basis for the polynomial monogenics using the generalised symmetries of the previous section. Note that we will here also assume κ to be a positive real function.

The idea is that acting with a generalised symmetry on a monogenic polynomial gives a monogenic polynomial of a higher degree. We can thus construct a generating set for $\mathcal{M}_n$ starting from the space of monogenics of degree 0. The strategy we employ to reduce the generated set to a basis is inspired by the one applied by Xu in the Dunkl harmonic case [Xu00]. Note that for the case of harmonic polynomials, the degree 0 harmonics $\mathcal{H}_0$ are the constant polynomials, while the space of monogenics of degree 0 is given by the spinor space: $\mathcal{M}_0 = V$.

For each multi-index $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$ with $|\beta|_1 = n$ and each spinor $s \in V$, we define a polynomial monogenic of degree n by

$$Z_s^\beta := \mathfrak{z}^\beta s = \mathfrak{z}_1^{\beta_1} \cdots \mathfrak{z}_d^{\beta_d} s. \tag{3.37}$$

It is direct to see from (3.27) that

$$Z_s^\beta = (-1)^n \varepsilon^{n-1} \mathrm{I}_\kappa \mathrm{D}^\beta \mathrm{I}_\kappa(s) = (-1)^n \varepsilon^{n-1} \mathrm{I}_\kappa \mathrm{D}_1^{\beta_1} \ldots \mathrm{D}_d^{\beta_d} \mathrm{I}_\kappa(s). \tag{3.38}$$

Recall that $\xi_j = (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0) \in \mathbb{N}^d \subset \mathbb{R}^d$, and so we define $\beta + \xi_j = (\beta_1, \ldots, \beta_j + 1, \ldots, \beta_d)$.

Lemma 3.4.1. *Let $\beta \in \mathbb{N}^d$ with $|\beta|_1 = n$ and $s \in V$. The spinor-valued polynomial Z_s^β respects*

$$Z_s^{\beta+\xi_j} = (2\varepsilon(n + d/2 + \gamma)x_j - 2\varepsilon \underline{x} O_j - \underline{x} e_j - \varepsilon |x|^2 \mathrm{D}_j) Z_s^\beta. \tag{3.39}$$

Proof. It follows from the commutativity of the $\mathfrak{z}_j$ along with their expression (3.16). ∎

We now turn our attention to the construction of bases for the monogenics.

Proposition 3.4.2. *Let ν be a basis of V, the spinor representation of $Cl(d)$. The set*

$$\mathcal{C}_n = \{ Z_s^\beta \mid \beta \in \mathbb{N}^d, |\beta|_1 = n, \ s \in \nu \} \tag{3.40}$$

is a generating set for $\mathcal{M}_n(\mathbb{R}^d; V)$. Moreover, the following relations hold

$$\sum_{j=1}^d Z_{e_j \cdot s}^{\eta+\xi_j} = 0, \text{ for every } \eta \in \mathbb{N}^d, \text{ with } |\eta|_1 = n-1, \text{ and } s \in V. \tag{3.41}$$

Proof. Proposition 3.3.4 states that every $\mathfrak{z}_j$ is a generalised symmetry of $\underline{\mathrm{D}}$, and thus $\underline{\mathrm{D}} Z_s^\beta = 0$, so $\mathrm{span}_{\mathbb{C}}(\mathcal{C}_n) \subset \mathcal{M}_n(\mathbb{R}^d; V)$. The monomials x^β for $\beta \in \mathbb{N}^d$ with $|\beta|_1 = n$, together with the spinors $s \in \nu$, form a basis of $\mathcal{P}_n \otimes V$. The projection $\mathrm{proj}_{\mathcal{M}_n}^{\mathcal{P}_n \otimes V}$ sends the element $x^\beta \otimes s$ to a multiple of $Z_s^\beta = \mathfrak{z}^\beta s \in \mathcal{C}_n$ by Proposition 3.3.11, so $\mathrm{span}(\mathcal{C}_n)$ contains

$\mathrm{im}(\mathrm{proj}_{\mathcal{M}_n}^{\mathcal{P}_n \otimes V})$. The projection is surjective, therefore $\mathrm{span}_{\mathbb{C}}(\mathcal{C}_n) \supset \mathcal{M}_n(\mathbb{R}^d; V)$. We have thus shown that the set $\mathcal{C}_n$ generates $\mathcal{M}_n(\mathbb{R}^d; V)$.

We now show that the relations of the form (3.41) hold. Let $\eta \in \mathbb{N}^d$ with $|\eta|_1 = n-1$ and $s \in V$. We now use Lemma 3.3.6, and the relations $\mathbb{E}s = 0$, $\sigma_\alpha s = s$ and $\underline{\mathrm{D}}s = 0$ to find

$$\sum_{j=1}^{d} \mathrm{Z}_{e_j \cdot s}^{\eta+\xi_j} = \sum_{j=1}^{d} \mathrm{z}^{\eta} \mathrm{z}_j e_j s = \mathrm{z}^{\eta} \underline{\mathrm{z}} s$$
$$= \mathrm{z}^{\eta}(2\varepsilon \underline{x}(\mathbb{E} + \gamma - \sum_{\alpha \in \Phi^+} \kappa(\alpha)\sigma_\alpha) - \varepsilon |x|^2 \underline{\mathrm{D}})s = 0. \qquad \blacksquare$$

Remark 3.4.3. *Let $\eta \in \mathbb{N}^d$ with $|\eta|_1 = n-2$. Xu exhibited the corresponding relations satisfied by the harmonics of equation (3.1) [Xu00, p. 500] (see also [DX14, pp. 212-213]):*

$$\sum_{j=1}^{d} \mathrm{Y}_{\eta+2\xi_j} = \mathrm{K}_\kappa \mathrm{D}^{\eta} \Delta_\kappa \mathrm{K}_\kappa(1) = 0. \tag{3.42}$$

The monogenics satisfy the same relation, as can be seen by applying twice relation (3.41), or by viewing the relation in the Dunkl–Clifford–Kelvin transform

$$\sum_{j=1}^{d} \mathrm{Z}_s^{\eta+2\xi_j} = (-1)^n \varepsilon^{n-1} \mathrm{I}_\kappa \mathrm{D}^{\eta} \Delta_\kappa \mathrm{I}_\kappa(s) = 0. \tag{3.43}$$

The relations (3.41) can be used to reduce the generating set $\mathcal{C}_n$ to a basis. For instance, the next theorem shows that if we consider only multi-indices $\mathbf{j} \in \mathbb{N}^d$ with zero as last index, we get a basis of the polynomial monogenics. Other bases can be constructed by following the same strategy but excluding other elements from $\mathcal{C}_n$ using the relations (3.41). The proof of the following theorem also shows that for a fixed $n \in \mathbb{N}$ and $s \in \nu$, with ν a basis of V, the relations (3.41) are all independent.

Theorem 3.4.4. *Let ν be a basis of V, the spinor representation of $Cl(d)$. The set*

$$\mathcal{B}_n = \{\mathrm{Z}_s^{\mathbf{j}} \mid \mathbf{j} = (j_1, \ldots, j_{d-1}, 0) \in \mathbb{N}^d, |\mathbf{j}|_1 = n, \ s \in \nu\} \tag{3.44}$$

is a basis of $\mathcal{M}_n(\mathbb{R}^d;V)$.

Proof. For every $s \in \nu$, let $s' = \varepsilon e_d \cdot s$ and, for these s', consider the relations (3.41) for all η with $|\eta|_1 = n-1$. These relations can be used to go from $\mathcal{C}_n$ to $\mathcal{B}_n$ by removing all polynomials Z_s^β with $\beta_d \neq 0$. We will show that each relation will remove exactly one polynomial, after which the result follows by a dimension argument. We order the multi-indices $I := \{\eta \in \mathbb{N}^d \mid |\eta|_1 = n-1\}$ by reverse lexicographic order, so $\eta_1 = (0,\dots,0,n-1)$, $\eta_2 = (0,\dots,0,1,n-2),\dots,\eta_{|I|} = (n-1,0,\dots,0)$, with $|I| = \dim \mathcal{P}_{n-1}(\mathbb{R}^d) = \binom{n+d-2}{d-1}$.

Since $e_d \cdot s' = s$ for $s \in \nu$, we can write relation (3.41) for η_i and $s' \in V$ as

$$Z_s^{\eta_i+\xi_d} = -\sum_{j=1}^{d-1} Z_{e_j\cdot s'}^{\eta_i+\xi_j}. \tag{3.45}$$

For each η_i, we can use this to exclude the polynomials $\{Z_s^{\eta_i+\xi_d} \mid s \in \nu\}$, since the right-hand side of (3.45) is a sum of polynomials strictly lower in the ordering. Therefore, starting from the set $\mathcal{C}_n$, doing this procedure in the reverse lexicographic order for all η_i each step will exclude $\dim V$ polynomials from the set.

This results in a basis, as can be seen from the dimensions of the spaces involved. Indeed, the cardinality of the spanning set is $|\mathcal{C}_n| = \dim \mathcal{P}_n(\mathbb{R}^d) \times \dim V$, and there are $\dim \mathcal{P}_{n-1}(\mathbb{R}^d) \times \dim V$ different linear relations between its members; $\dim \mathcal{M}_n(\mathbb{R}^d;V) = \dim \mathcal{P}_n(\mathbb{R}^{d-1}) \times \dim V$ and

$$\begin{aligned}(\dim \mathcal{P}_n(\mathbb{R}^d) - \dim \mathcal{P}_{n-1}(\mathbb{R}^d)) \times \dim V &= \binom{n+d-2}{d-2} \times \dim V \\ &= \dim \mathcal{M}_n(\mathbb{R}^d;V) = |\mathcal{B}_n|. \quad \blacksquare\end{aligned}$$

3.5 Examples: the abelian cases

3.5.1 Reducible reflection groups

In this section, we consider the cases when the reflection group is reducible: $W = W_\Psi \times W_{\Psi'}$, for Ψ, Ψ' two root subsystems of lower rank with $\Phi = \Psi \sqcup \Psi'$ the disjoint union of the two.

Let M be the rank of Ψ. For simplicity, assume that Ψ is restricted to the first M coordinates. We have an $\mathfrak{osp}(1|2)$ realisation given by the following operators

$$\underline{D}_{[M]} := \sum_{a=1}^{M} D_a e_a, \qquad \underline{x}_{[M]} := \sum_{a=1}^{M} x_a e_a.$$

The odd elements $\underline{D}_{[M]}$ and $\underline{x}_{[M]}$ generate a realisation of the superalgebra $\mathfrak{osp}(1|2)$ with the following commutation relation

$$\left\{\underline{D}_{[M]}, \underline{x}_{[M]}\right\} = 2\varepsilon H_{[M]} = 2\varepsilon(\mathbb{E}_{[M]} + M/2 + \gamma_{[M]}), \tag{3.46}$$

where

$$\mathbb{E}_{[M]} := \sum_{a=1}^{M} x_a \partial_{x_a}, \qquad \gamma_{[M]} := \sum_{\alpha\in\Psi^+} \kappa(\alpha).$$

Thus one can also define generalised symmetries in this "smaller" $\mathfrak{osp}(1|2)$-realisation.

Definition 3.5.1. *Let $1 \leq j \leq M$. The partial generalised symmetry linked to $\underline{D}_{[M]}$ is given by*

$$z_{j,[M]} := 2\varepsilon x_j H_{[M]} - \underline{x}_{[M]} D_j \underline{x}_{[M]}. \tag{3.47}$$

Naturally, since it is also in an $\mathfrak{osp}(1|2)$-realisation, $z_{j,[M]}$ satisfies the equivalent relations of Proposition 3.3.8, Lemma 3.3.3, Proposition 3.3.5 and Proposition 3.3.7.

3.5.2 The abelian cases

We turn to the study of the abelian case. This means the Dunkl–Dirac symmetry algebra for the group $W = \mathbb{Z}_2^d$ acting on $\mathbb{R}^d$ with a W-invariant function given by the d-tuple of non-negative constants $(\kappa_1, \ldots, \kappa_d)$. In the abelian case, the reflection σ_j sends x_j to $-x_j$ and leaves the other variables invariant. The Dunkl operators in this case are given by

$$D_i = \partial_{x_i} + \kappa_i \frac{1-\sigma_i}{x_i}, \tag{3.48}$$

and the commutation relation (2.13) becomes

$$\left[\mathrm{D}_i, x_j\right] = \delta_{ij}(1 + 2\kappa_i\sigma_j). \tag{3.49}$$

Albeit Theorem 3.4.4 gives a basis of $\mathcal{M}_n$, the specificity of the group studied call for a slightly different approach. The complete reducibility of $W = \mathbb{Z}_2^d$ was used in [DGV16a; DGV16b] to construct a basis from the Cauchy–Kovalevskaya extension Theorem. We will retrieve this construction from the partial generalised symmetries of Definition 3.5.1.

3.5.3 The Cauchy–Kovalevskaya basis

In the abelian case, there exists a generalisation of the Cauchy–Kovalevskaya map. It can be used to construct a basis of the polynomial monogenics.

Proposition 3.5.2 ([DGV16b, Eq. (31)]). *Let V be an irreducible representation of the Clifford algebra $Cl(d)$. There is an isomorphism between the space of spinor-valued polynomials of degree n over $k-1$ variables and the monogenics of degree n over k variables given by*

$$\begin{aligned} \mathbf{CK}_{x_k}^{\kappa_k} : \mathcal{P}_n(\mathbb{R}^{k-1}) \otimes V &\longrightarrow \mathcal{M}_n(\mathbb{R}^k; V) \\ p \longmapsto \mathbf{CK}_{x_k}^{\kappa_k}(p) &= \sum_{a=0}^{\lfloor n/2 \rfloor} \frac{x_k^{2a}\underline{\mathrm{D}}_{[k-1]}^{2a}}{2^{2a}a!(\kappa_k+1/2)_a} p \\ &\quad - \varepsilon \frac{e_k x_k \underline{\mathrm{D}}_{[k-1]}}{2} \sum_{a=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{x_k^{2a}\underline{\mathrm{D}}_{[k-1]}^{2a}}{2^{2a}a!(\kappa_k+1/2)_{a+1}} p. \end{aligned} \tag{3.50}$$

Note that in [DGV16b], the proposition is given for $\varepsilon = -1$. The proof for the two signs is the same up to minor modifications.

The map $\mathbf{CK}_{x_k}^{\kappa_k}$ is an isomorphism and has an inverse given by the map evaluating the last variable to 0:

$$\begin{aligned} R_k : \mathcal{M}_n(\mathbb{R}^k; V) &\longrightarrow \mathcal{P}_n(\mathbb{R}^{k-1}) \otimes V \\ f(x_1, \ldots, x_k) &\longmapsto R_k(f)(x_1, \ldots, x_{k-1}) = f(x_1, \ldots, x_{k-1}, 0). \end{aligned} \tag{3.51}$$

Consider now the Fischer decomposition of the polynomial space.

Proposition 3.5.3 (Fischer decomposition [BDS82; ØSS09]). *The space of spinor-valued polynomials decomposes in monogenic spaces as*

$$\mathcal{P}_n(\mathbb{R}^d) \otimes V = \bigoplus_{k=0}^{n} \underline{x}_{[d]}^{n-k} \mathcal{M}_k(\mathbb{R}^d; V). \tag{3.52}$$

When κ is positive, the same also holds for Dunkl polynomial spaces.

$$\begin{tikzcd}
 & \mathcal{P}_n(\mathbb{R}^2) \otimes V \arrow[r, "\mathbf{CK}_{x_3}^{\kappa_3}"] & \mathcal{M}_n(\mathbb{R}^3; V) \\
 & = & \\
\mathcal{P}_n(\mathbb{R}) \otimes V \arrow[r, "\mathbf{CK}_{x_2}^{\kappa_2}"] & \mathcal{M}_n(\mathbb{R}^2; V) & \\
 & \oplus & \\
\mathcal{P}_{n-1}(\mathbb{R}) \otimes V \arrow[r, "\underline{x}_{[2]}\mathbf{CK}_{x_2}^{\kappa_2}"] & \underline{x}_{[2]}\mathcal{M}_{n-1}(\mathbb{R}^2; V) & \\
 & \oplus & \\
 & \vdots & \\
 & \oplus & \\
\mathcal{P}_0(\mathbb{R}) \otimes V \arrow[r, "\underline{x}_{[2]}^n\mathbf{CK}_{x_2}^{\kappa_2}"] & \underline{x}_{[2]}^n\mathcal{M}_0(\mathbb{R}^2; V) &
\end{tikzcd}$$

Figure 3.1: Tower of CK extension and Fischer decomposition for $d = 3$, see also [DGV16b, Eq. (32)]

From this proposition, and the tower of CK extensions and Fischer decompositions (see Figure 3.1), we get a basis of the space of monogenics.

Proposition 3.5.4 ([DGV16b, Prop. 6]). *Let $\{s\}_{s\in\nu}$ be a basis of the spinor representation V. The set of functions defined, for all multi-index with last entry zero, $\mathbf{j} = (j_1, \dots, j_{d-1}, 0) \in \mathbb{N}^d$ with $|\mathbf{j}|_1 = n$, by*

$$\begin{aligned}\psi_s^{\mathbf{j}}(x_1, \dots, x_d) = \mathbf{CK}_{x_d}^{\kappa_d}\Big(\underline{x}_{d-1}^{j_{d-1}} \mathbf{CK}_{x_{d-1}}^{\kappa_{d-1}}\Big(\cdots \\ \cdots \mathbf{CK}_{x_3}^{\kappa_3}\Big(\underline{x}_2^{j_2}\mathbf{CK}_{x_2}^{\kappa_2}(x_1^{j_1})\Big)\dots\Big)\Big)s,\end{aligned} \tag{3.53}$$

is a basis of $\mathcal{M}_n(\mathbb{R}^d; V)$.

3.5.4 A new basis

We will use the partial generalised symmetries of Subsection 3.5.1 to make full use of the completely reducible nature of $\mathbb{Z}_2^d$. The crucial point of the abelian case $W = \mathbb{Z}_2^d$ is a chain of inclusions

$$\mathbb{Z}_2 \subset \mathbb{Z}_2^2 \subset \cdots \subset \mathbb{Z}_2^{d-1} \subset \mathbb{Z}_2^d. \tag{3.54}$$

This gives in turn a tower of $\mathfrak{osp}(1|2)$ algebra realisations given by the pairs $(\underline{D}_{[k]}, \underline{x}_{[k]})$ for each $1 \le k \le d$. We remark that the constant $\gamma_{[k]}$ appearing in the anticommutation relation (3.46) is $\gamma_{[k]} := \kappa_1 + \kappa_2 + \cdots + \kappa_k$ in the abelian case.

This feature of the group $\mathbb{Z}_2^d$ was used in Proposition 3.5.4 to give a basis. We give a basis proportional to the CK basis by replacing the operators $\mathfrak{z}_j$ in Theorem 3.4.4 by the partial ones $\mathfrak{z}_{j,[j]}$. This is done by linking the Cauchy–Kovalevskaya extension of each level to one partial generalised symmetries.

An important note, the commutation of the $\mathfrak{z}_{j,[j]}$ requires that they stay on the same level. Indeed, two partial generalised symmetries at a different level in the tower do not commute in general. So in the basis of the following proposition, the order of application matters.

Proposition 3.5.5. *The set of polynomials of the form*

$$\begin{gathered}\phi_s^{\mathbf{j}} := \mathfrak{z}_{d,[d]}^{j_{d-1}} \mathfrak{z}_{d-1,[d-1]}^{j_{d-2}} \cdots \mathfrak{z}_{2,[2]}^{j_1} s,\\ \textit{for } \mathbf{j} = (j_1, \dots j_{d-1}, 0) \in \mathbb{N}^d,\ |\mathbf{j}|_1 = n, \textit{ and } s \in \nu,\end{gathered} \tag{3.55}$$

constitutes a basis of $\mathcal{M}_n(\mathbb{R}^d; V)$.

The proof of this proposition will follow from Proposition 3.5.11, as it exhibits a change of basis from the $\psi_s^{\mathbf{j}}$ to the $\phi_s^{\mathbf{j}}$. The remaining of the section is dedicated to proving this change of basis.

Propositions 3.5.7 and 3.5.10 show that $\mathbf{CK}_{x_k}^{\kappa_k} \underline{x}_{[k-1]}^j$ and $\mathfrak{z}_{k,[k]}^j$ are Clifford proportional as operators, meaning that they differ only by a Clifford number. There is a small difference between $k = 2$ and $k > 2$, and thus we separate the proof in two steps. We begin by showing what will constitute the hard part of the induction proof of Proposition 3.5.7.

Lemma 3.5.6. *The $z_{2,[2]}$ operator and the $\mathbf{CK}_{x_2}^{\kappa_2}$ extension are linked by*

$$z_{2,[2]}\mathbf{CK}_{x_2}^{\kappa_2}(x_1^m s) = A_m \mathbf{CK}_{x_2}^{\kappa_2}(x_1^{m+1} e_2 e_1 s), \tag{3.56}$$

with

$$A_m = 1 + m + (1 - (-1)^m)\kappa_1 + 2\kappa_2. \tag{3.57}$$

Proof. Recall that R_2 is the inverse of $\mathbf{CK}_{x_2}^{\kappa_2}$. Acting with R_2 on (3.56) thus yields

$$R_2 z_{2,[2]}\mathbf{CK}_{x_2}^{\kappa_2} x_1^m s = A_m x_1^{m+1} e_2 e_1 s, \tag{3.58}$$

so it suffices to compute the left-hand side of (3.58). Begin by using the expression (3.16) of $z_{2,[2]}$

$$\begin{aligned} R_2(z_{2,[2]}\mathbf{CK}_{x_2}^{\kappa_2}(x_1^m s)) = R_2((&2\varepsilon x_2(\mathbb{E}_{[2]} + 1 + \gamma_{[2]}) \\ &- \underline{x}_2(1 + 2\kappa_2\sigma_2)e_2 - \varepsilon |x|_{[2]}^2 \mathrm{D}_2)\mathbf{CK}_{x_2}^{\kappa_2}(x_1^m s)), \end{aligned}$$

and since R_2 sends x_2 to 0, this reduces to

$$\begin{aligned} &= (-x_1 e_1 (1 + 2\kappa_2\sigma_2) e_2) x_1^m s \\ &\quad + \varepsilon^2 x_1^2 \mathrm{D}_2 \frac{x_2 e_2 \mathrm{D}_1 e_1}{2} \frac{x_1^m s}{(1/2 + \kappa_2)} \\ &= ((1 + 2\kappa_2\sigma_2) e_2 e_1) x_1^{m+1} s \\ &\quad + x_1^2 (1 + 2\kappa_2\sigma_2) \frac{\mathrm{D}_1 x_1^m e_2 e_1 s}{(1 + 2\kappa_2)}, \end{aligned}$$

and now we apply $\mathrm{D}_1 x_1^m s = (m + \kappa_1(1 - (-1)^m)) x_1^{m-1} s$, since $\mathrm{D}_1 s = 0$, to obtain

$$= (1 + m + 2\kappa_2 + (1 - (-1)^m)\kappa_1) x_1^{m+1} e_2 e_1 s. \qquad \blacksquare$$

Using this lemma, we can prove the general proposition.

Proposition 3.5.7. *Acting on a spinor s, we have*

$$z_{2,[2]}^j s = a_2^j \mathbf{CK}_{x_2}^{\kappa_2} x_1^j (e_2 e_1)^j s, \tag{3.59}$$

with

$$a_2^j := 2^j (\kappa_2 + 1/2)_{\lfloor (j+1)/2 \rfloor} (\gamma_{[2]} + 1)_{\lfloor j/2 \rfloor}. \tag{3.60}$$

Proof. We proceed by induction on j, the case $j=1$ being covered by Lemma 3.5.6 with $m=0$. Assume the induction hypothesis holds up to $j=m$. Now we consider the $(m+1)$-th step and apply the induction hypothesis

$$\mathbb{z}_{2,[2]}^{m+1}s = \mathbb{z}_{2,[2]}\mathbb{z}_{2,[2]}^{m}s = \mathbb{z}_{2,[2]}a_2^m\mathbf{CK}_{x_2}^{\kappa_2}x_1^m(e_2e_1)^m s), \tag{3.61}$$

then we apply Lemma 3.5.6 to get $\mathbb{z}_{2,[2]}^{m+1}s = a_2^{m+1}\mathbf{CK}_{x_2}^{\kappa_2}x_1^{m+1}(e_2e_1)^{m+1}s$ since $A_m a_2^m = a_2^{m+1}$. ∎

In general, for $k>2$, there is one additional difficulty: the CK map includes not only Dunkl derivatives, but also partial Dunkl–Dirac operators. We will thus need a small lemma.

Lemma 3.5.8 ([DGV16b, Lem. 13])**.** *Let $f\in\mathcal{M}_n(\mathbb{R}^k;V)$. The action of $\underline{D}_{[k]}$ on $\underline{x}_{[k]}^m f$ is given by*

$$\underline{D}_{[k]}(\underline{x}_{[k]}^m f) = \varepsilon(m + \frac{(1-(-1)^m)}{2}(2n+k-1+2\gamma_{[k]}))\underline{x}_{[k]}^{m-1}f. \tag{3.62}$$

Proof. Proceed by induction on m, first for even m. The base case $m=2$ comes from the commutation relation $\left[\underline{D}_{[k]}, |x|_k^2\right] = 2|x|_k^2$, equation (2.58). The induction step follows then from

$$\underline{D}_{[k]}\underline{x}_{[k]}^m f = \underline{D}_{[k]}\underline{x}_{[k]}^2\underline{x}_{[k]}^{m-2}f = \underline{x}_{[k]}^2\underline{D}_{[k]}\underline{x}_{[k]}^{m-2}f + 2\varepsilon\underline{x}_{[k]}^{m-1}f = \varepsilon m\underline{x}_{[k]}^{m-1}f. \tag{3.63}$$

The base case for odd m follows from the anticommutation relation $\left\{\underline{D}_{[k]}, \underline{x}_{[k]}\right\} = 2\varepsilon(\mathbb{E}_k + k/2 + \gamma_{[k]})$, $\underline{D}_{[k]}f = 0$ and $\mathbb{E}_k f = nf$. The induction step is then achieved by one application of equation (2.58)

$$\begin{aligned}\underline{D}_{[k]}\underline{x}_{[k]}^m f &= \underline{D}_{[k]}\underline{x}_{[k]}^2\underline{x}_{[k]}^{m-2}f = \underline{x}_{[k]}^2\underline{D}_{[k]}\underline{x}_{[k]}^{m-2}f + 2\varepsilon\underline{x}_{[k]}^{m-1}f \\ &= \varepsilon(2n+m+k-1+2\gamma_{[k]})\underline{x}_{[k]}^{m-1}f.\end{aligned} \tag{3.64}$$

∎

Now to prove the relation between the partial generalised symmetry and the CK map for the other levels of the tower, we introduce a lemma that takes care of the difficult induction step.

Lemma 3.5.9. *Let $f \in \mathcal{M}_n(\mathbb{R}^{k-1}; V)$ be a monogenic of degree n in the first $k-1$ variables. Then*

$$\mathbb{z}_{k,[k]} \mathbf{CK}_{x_k}^{\kappa_k} \underline{x}_{[k-1]}^m f = B_{k,n}^m \mathbf{CK}_{x_k}^{\kappa_k} \underline{x}_{[k-1]}^{m+1} e_k f, \tag{3.65}$$

with

$$B_{k,n}^m = (-1)^{m+1}(m + 1 + \frac{(1-(-1)^m)}{2}(2n + k - 2 + 2\gamma_{[k-1]}) + 2\kappa_k). \tag{3.66}$$

Proof. The proof proceeds in the same fashion as the one of Lemma 3.5.6, using $\mathbb{E}_{[k-1]} f = nf$, $\underline{\mathrm{D}}_{k-1} f = 0$ and $\mathrm{D}_k f = 0$ instead of $\mathbb{E}s = 0$, $\mathrm{D}_1 s = 0$ and $\mathrm{D}_2 s = 0$ in the corresponding steps.

The map $\mathbf{CK}_{x_k}^{\kappa_k}$ has an inverse R_k defined as evaluating x_k to 0. We compute, using (3.50),

$$\begin{aligned}
R &:= R_k(\mathbb{z}_{k,[k]} \mathbf{CK}_{x_k}^{\kappa_k} \underline{x}_{[k-1]}^m f) \\
&= R_k\Big(2\varepsilon x_k(\mathbb{E}_{[k]} + k/2 + \gamma_{[k]}) \\
&\quad - \underline{x}_{[k]}(1 + 2\kappa_k \sigma_k) e_k - \varepsilon |x|_k^2 \mathrm{D}_k\Big) \mathbf{CK}_{x_k}^{\kappa_k} \underline{x}_{[k-1]}^m f \\
&= -\underline{x}_{[k-1]}(1 + 2\kappa_k \sigma_k) e_k \underline{x}_{[k-1]}^m f + \varepsilon^2 |x|_{k-1}^2 \mathrm{D}_k \frac{e_k x_k \underline{\mathrm{D}}_{[k-1]}}{2(\kappa_k + 1/2)} \underline{x}_{[k-1]}^n f \\
&= (-1)^{m+1}(1 + 2\kappa_k) \underline{x}_{[k-1]}^{m+1} e_k f + |x|_{k-1}^2 e_k \frac{[\mathrm{D}_k, x_k]}{2(\kappa_k + 1/2)} \underline{\mathrm{D}}_{[k-1]} \underline{x}_{[k-1]}^n f
\end{aligned}$$

and we use Lemma 3.5.8 for $f \in \mathcal{M}_n(\mathbb{R}^{k-1}; V)$ on the rightmost term to get

$$\begin{aligned}
&= (-1)^{m+1}(1 + 2\kappa_k) \underline{x}_{[k-1]}^{m+1} e_k f \\
&\quad + \varepsilon(m + \frac{(1-(-1)^m)}{2}(2n + k - 2 + 2\gamma_{[k-1]}))|x|_{k-1}^2 e_k \underline{x}_{[k-1]}^{m-1} f \\
&= (-1)^{m+1}\Big(1 + 2\kappa_k + \varepsilon^2 m \\
&\quad + \varepsilon^2 \frac{(1-(-1)^m)}{2}(2n + k - 2 + 2\gamma_{[k-1]})\Big) \underline{x}_{[k-1]}^{m+1} e_k f.
\end{aligned}$$

This allows to determine the constant $B_{k,n}^m$ by comparing with (3.65). ∎

Proposition 3.5.10. *Let $f \in \mathcal{M}_n(\mathbb{R}^{k-1}; V)$ be a monogenic in $k-1$ variables of degree n. For $k > 2$,*

$$\mathbb{z}^j_{k,[k]} f = b^j_{k,n} \mathbf{CK}^{\kappa_k}_{x_k} \underline{x}^j_{[k-1]} e^j_k f, \tag{3.67}$$

with

$$b^j_{k,n} = (-1)^{\lfloor (j+1)/2 \rfloor} 2^j (\kappa_k + 1/2)_{\lfloor (j+1)/2 \rfloor} (\gamma_{[k]} + n + k/2)_{\lfloor j/2 \rfloor}. \tag{3.68}$$

Proof. We proceed by induction on j. The base case follows from Lemma 3.5.9 with $m = 0$. Assume the induction hypothesis holds up to j. The induction step follows from the induction hypothesis and Lemma 3.5.9

$$\begin{aligned} \mathbb{z}^{j+1}_{k,[k]} f &= \mathbb{z}_{k,[k]} \mathbb{z}^j_{k,[k]} f = \mathbb{z}_{k,[k]} b^j_{k,n} \mathbf{CK}^{\kappa_k}_{x_k} \underline{x}^j_{k-1} e^j_k f \\ &= B^j_{k,n} b^j_{k,n} \mathbf{CK}^{\kappa_k}_{x_k} \underline{x}^{j+1}_{k-1} e^{j+1}_k f. \end{aligned} \tag{3.69}$$

This shows the result since $B^j_{k,n} b^j_{k,n} = b^{j+1}_{k,n}$. ∎

Connecting this to the CK basis, we get the following correspondence, proving Proposition 3.5.5.

Proposition 3.5.11. *Let $\mathbf{j} = (j_1, \ldots, j_{d-1}, 0) \in \mathbb{N}^d$ with $|\mathbf{j}|_1 = n$ and $s \in \nu$ be a spinor. The partial generalised symmetry basis is linked to the CK basis by*

$$\phi^{\mathbf{j}}_s = c_{\mathbf{j}} \psi^{\mathbf{j}}_{\mathbf{j}\cdot s}, \tag{3.70}$$

where the action $\mathbf{j} \cdot s$ in $\psi^{\mathbf{j}}_{\mathbf{j}\cdot s}$ denotes the action $e^{j_{d-1}}_d \ldots e^{j_2}_3 (e_2 e_1)^{j_1} s$ on the spinor space in the expression of the polynomial ψ, and where the proportionality constant is given by

$$\begin{aligned} c_{\mathbf{j}} = &\left(\prod_{k=3}^{d-1} \prod_{l=2}^{k-1} (-1)^{j_k j_l} \right) 2^n (1/2 + \kappa_2)_{\lfloor (j_1+1)/2 \rfloor} (1 + \gamma_{[2]})_{\lfloor j_1/2 \rfloor} \times \\ &\prod_{i=2}^{d-1} ((-1)^{\lfloor (j_i+1)/2 \rfloor} \times \\ &(1/2 + \kappa_{i+1})_{\lfloor (j_i+1)/2 \rfloor} ((i+1)/2 + \gamma_{[i+1]} + \sum_{k=1}^{i-1} j_k)_{\lfloor j_i/2 \rfloor}). \end{aligned} \tag{3.71}$$

Proof. The first steps are to apply once Proposition 3.5.7 for the $\mathbb{z}_{2,[2]}$ contribution and multiple times Proposition 3.5.10 for the remaining contributions of the $\mathbb{z}_{k,[k]}$. This will give $c_{\mathbf{j}}$ up to the first sign. This sign is obtained when Clifford elements go to the right from their interaction with the vector variables. Note that e_k commutes with $\mathbf{CK}_{x_l}^{\kappa_l}$ when $k > l$, as can be clearly seen from the expression (3.50), so the only sign to consider is from the vector variable crossing. Step by step, this gives

$$\begin{aligned}
\phi_{\mathbf{j}}^{s} &= \mathbb{z}_{d,[d]}^{j_{d-1}}\mathbb{z}_{d-1,[d-1]}^{j_{d-2}}\cdots\mathbb{z}_{2,[2]}^{j_1}s\\
\text{(Prop. 3.5.7)}\quad &= a_2^{j_1}\mathbb{z}_{d,[d]}^{j_{d-1}}\mathbb{z}_{d-1,[d-1]}^{j_{d-2}}\cdots\mathbb{z}_{3,[3]}^{j_2}\mathbf{CK}_{x_2}^{\kappa_2}(x_1^{j_1}(e_2e_1)^{j_1}s\\
\text{(Prop. 3.5.10)}\quad &= a_2^{j_1}\prod_{k=2}^{d-1}(b_{k+1,\sum_{j=1}^{k-1}j_k}^{j_k})\mathbf{CK}_{x_d}^{\kappa_d}(\underline{x}_{[d-1]}^{j_d}e_d^{j_{d-1}}\mathbf{CK}_{x_{d-1}}^{\kappa_{d-1}}\cdots\\
&\quad e_3^{j_2}\mathbf{CK}_{x_2}^{\kappa_2}(x_1^{j_1}(e_2e_1)^{j_1}s))\\
&= \left(\prod_{k=3}^{d-1}\prod_{l=2}^{k-1}(-1)^{j_kj_l}\right)a_2^{j_1}\prod_{k=2}^{d-1}(b_{k+1,\sum_{j=1}^{k-1}j_k}^{j_k})\mathbf{CK}_{x_d}^{\kappa_d}(\underline{x}_{[d-1]}^{j_d}\cdots\\
&\quad \mathbf{CK}_{x_2}^{\kappa_2}(x_1^{j_1}e_d^{j_{d-1}}\ldots(e_2e_1)^{j_1}s))\\
&= c_{\mathbf{j}}\psi_{\mathbf{j}}^{\mathbf{j}\cdot s}.
\end{aligned}$$

∎

Thus, the generalised symmetries can be used to retrieve the **CK** basis, and Proposition 3.5.11 gives the change of basis.

4

An Exceptional Symmetry Algebra for the 3D Dirac–Dunkl Operator

The content of this chapter is extracted from the contribution:

Alexis Langlois-Rémillard and Roy Oste (2020). *Proceedings of the conference Lie Theory and Its Applications in Physics, LT-XIII Varna 2019* [LO20].

In the present chapter, we initiate the study of an algebra of symmetries for the Dirac–Dunkl operator associated with the exceptional root system G_2. The latter is primarily known from the classification of simple Lie algebras. The associated Lie group and algebra continue to spark interest, see for instance the recent paper of Dobrev [Dob19] and references therein. Our purpose is related instead to the action of the Weyl group associated with G_2 on a (two-dimensional subspace of a) three-dimensional space. Though G_2 is indeed a root system of rank 2, the arising symmetry algebra associated with a three-dimensional space portrays interesting non-trivial relations, which are not present when considering the two-dimensional analogue.

We will briefly recall how the symmetry algebra in question arises.

See Section 2.2.5 for more detail. For a finite reflection group W acting on a finite dimensional vector space, there exists a rational Cherednik algebra (RCA) [EM10] that can be viewed as a deformation of the algebra of polynomial differential operators on the vector space. An explicit realisation is given by means of differential-difference operators called Dunkl operators [Dun89]. A generalisation of the Dirac operator is defined abstractly inside the tensor product of the RCA and a Clifford algebra, or explicitly by using Dunkl operators in lieu of partial derivatives in the ordinary definition of the Dirac operator.

In this way, the Dirac–Dunkl operator squares to a Dunkl version of the Laplace operator whose invariance is restricted to the group W as opposed to the full orthogonal invariance of its classical counterpart. Moreover, together with its dual partner, the Dirac–Dunkl operator generates a Lie superalgebra isomorphic to $\mathfrak{osp}(1|2)$. The latter's (super)centraliser, $\mathcal{O}_\kappa$ (Definition 2.2.17) inside the tensor product of RCA and Clifford algebra gives an algebra of symmetries (super)commuting with the Dirac–Dunkl operator. Structurally it can be seen as a deformation of the orthogonal Lie algebra representing total angular momentum in the non-deformed case.

The full analysis of the representation theory goes beyond the scope of this chapter and will be presented in Chapter 5. Here, we will present some preliminary results pertaining to three-dimensional spaces and focus in particular on the exceptional root system G_2, embedded herein. Note that this chapter follows the same convention for the root system as previous work on $W = S_3$ [DOV18a], whereas the next chapter will use a different one.

In Section 4.1, the required definitions of the exceptional root system G_2 and the Dirac–Dunkl operator are introduced and we present the symmetry algebra both abstractly and as an explicit realisation. In Section 4.2, we prove an intermediate result for arbitrary root systems in $\mathbb{R}^3$ and show that this leads to the existence of ladder operators for the symmetry algebra associated with G_2.

4.1 An exceptional symmetry algebra

We consider the Euclidean space $\mathbb{R}^3$ with coordinates x_1, x_2, x_3. The 2-dimensional root system G_2 is realised in a plane and is generated by two simple roots $\alpha_1 = (0,1,-1)$ and $\alpha_2 = (1,-2,1)$. The Coxeter group linked to G_2 is the dihedral group $\mathsf{D}_{2\cdot 6}$ that we will present by: $\mathsf{D}_{12} = \langle \sigma_1, \sigma_2 \mid \sigma_1^2 = \sigma_2^2 = (\sigma_1\sigma_2)^6 = (\sigma_2\sigma_1)^6 = 1\rangle$ with the reflections σ_1 connected to the short root α_1, and σ_2 to the long root α_2. Their actions on $\mathbb{R}^3$ are expressed matricially by:

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}. \tag{4.1}$$

A set of positive roots is given by

$$\begin{aligned}\Phi_+ = \{&\alpha_1 = (0,1,-1), \alpha_2 = (1,-2,1), \alpha_3 = (1,-1,0), \\ &\alpha_4 = (1,1,-2), \alpha_5 = (1,0,-1), \alpha_6 = (2,-1,-1)\}.\end{aligned} \tag{4.2}$$

To each root α_i, a reflection σ_i is paired. The remaining reflections have the following decompositions in terms of the simple reflections σ_1, σ_2:

$$\begin{aligned} &\sigma_3 = \sigma_2\sigma_1\sigma_2, && \sigma_4 = \sigma_1\sigma_2\sigma_1, \\ &\sigma_5 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1, && \sigma_6 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2. \end{aligned} \tag{4.3}$$

We introduce a D_{12}-invariant weight function $\kappa : \mathsf{G}_2 \to \mathbb{C}$, which is defined by two complex numbers κ_1 and κ_2 linked respectively to the short and long roots. With this, it is possible to define Dunkl operators (2.10) for the root system G_2; for example the one associated with the coordinate x_2 is given by

$$\begin{aligned}\mathrm{D}_2 = \frac{\partial}{\partial x_2} &+ \kappa_1\left(\frac{1-\sigma_1}{x_2 - x_3} - \frac{1-\sigma_3}{x_1 - x_2}\right) \\ &+ \kappa_2\left(-2\frac{1-\sigma_2}{x_1 - 2x_2 + x_3} + \frac{1-\sigma_4}{x_1 + x_2 - 2x_3} - \frac{1-\sigma_6}{2x_1 - x_2 - x_3}\right),\end{aligned}$$

while D_1 and D_3 are defined similarly.

Fix a sign $\varepsilon \in \{+1,-1\}$. We consider the Clifford algebra with three anticommuting generators e_1, e_2, e_3 that all square to ε. The Dirac–Dunkl operator associated with our embedding of G_2 in $\mathbb{R}^3$ is $\underline{\mathrm{D}} :=$

$\mathrm{D}_1 e_1 + \mathrm{D}_2 e_2 + \mathrm{D}_3 e_3$. Together with its dual partner $\underline{x} := x_1 e_1 + x_2 e_2 + x_3 e_3$, it generates a realisation of $\mathfrak{osp}(1|2)$. For ease of notation, we shall not make explicit mention of the tensor product, trusting the reader to add it whenever Clifford elements e_i are involved.

The elements of the symmetry algebra $\mathcal{O}_\kappa$ were obtained in previous work [DOV18a] (that they indeed generate the full centraliser is the subject of [Ost22]) and we will go over them now. First, we need a double cover of the Weyl group D_{12}. The orthogonal group $O(3)$ has two non-isomorphic double covers. These correspond to the two choices of ε in the definition of the Clifford algebra [Mor76]. For either choice of ε, we obtain a double cover $\widetilde{\mathsf{D}}^{\varepsilon}_{12}$ from Theorem 2.2.7. In this way, we obtain the elements of $\mathbb{C}[\widetilde{\mathsf{D}}^{\varepsilon}_{12}]$ (together with their additive inverses):

$$\tilde{\sigma}_1 = \frac{\sigma_1(e_2 - e_3)}{\sqrt{2}}, \qquad \tilde{\sigma}_2 = \frac{\sigma_2(e_1 - 2e_2 + e_3)}{\sqrt{6}},$$
$$\tilde{\sigma}_3 = \frac{\sigma_3(e_1 - e_2)}{\sqrt{2}}, \qquad \tilde{\sigma}_4 = \frac{\sigma_4(e_1 + e_2 - 2e_3)}{\sqrt{6}},$$
$$\tilde{\sigma}_5 = \frac{\sigma_5(e_1 - e_3)}{\sqrt{2}}, \qquad \tilde{\sigma}_6 = \frac{\sigma_6(2e_1 - e_2 - e_3)}{\sqrt{6}}.$$

Note that the group relations depend on the choice of ε. By direct computation we find $\widetilde{\mathsf{D}}^{\varepsilon}_{12} = \langle -1, \tilde{\sigma}_1, \tilde{\sigma}_2 \mid -1^2 = 1, \tilde{\sigma}_1^2 = \tilde{\sigma}_2^2 = \varepsilon, (\tilde{\sigma}_1\tilde{\sigma}_2)^6 = (\tilde{\sigma}_2\tilde{\sigma}_1)^6 = -1\rangle$, which also follows from Theorem 2.2.7. The order of this group is 24, and for $\varepsilon = +1$ it is again a dihedral group, while for $\varepsilon = -1$ it is a dicyclic group. Regardless of the choice of ε, all elements of $\widetilde{\mathsf{D}}^{\varepsilon}_{12}$ will supercommute with the Dunkl–Dirac operator when taking into account the $\mathbb{Z}_2$-grading inherited from the Clifford algebra. Both $\underline{\mathrm{D}}$ and $\pm\sigma_i$ are odd elements with respect to this grading, so they will in fact anticommute.

Furthermore, there are three analogues of the total angular momentum operators that commute with the Dirac operator: O_{12}, O_{23}, O_{13}, see (2.66). Classically (non-Dunkl) they generate a realisation of the orthogonal Lie algebra $\mathfrak{so}(3)$, though here it will be a deformation of the product $\mathfrak{so}(3) \rtimes \mathbb{C}\widetilde{W}$.

For this group, the one-index symmetries (2.65) will be explicitly the following:

$$\begin{aligned} O_1 &= \kappa_1(\tilde{\sigma}_3 + \tilde{\sigma}_5) + \kappa_2(\tilde{\sigma}_2 + \tilde{\sigma}_4 + 2\tilde{\sigma}_6), \\ O_2 &= \kappa_1(\tilde{\sigma}_1 - \tilde{\sigma}_3) + \kappa_2(-2\tilde{\sigma}_2 + \tilde{\sigma}_4 - \tilde{\sigma}_6), \\ O_3 &= \kappa_1(-\tilde{\sigma}_1 - \tilde{\sigma}_5) + \kappa_2(\tilde{\sigma}_2 - 2\tilde{\sigma}_4 - \tilde{\sigma}_6). \end{aligned} \tag{4.4}$$

It is immediate to see that the sum $O_1 + O_2 + O_3 = 0$. Moreover, we will denote $\mathcal{E} = [O_1, O_2]$, and by direct but slightly tedious computations, we can also see that $[O_2, O_3] = \mathcal{E} = -[O_1, O_3]$. From the realisation (2.66), it is clear that $O_{ij} = -O_{ji}$.

The interaction of the two simple reflections $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ with the two-index symmetries are given by:

$$\begin{aligned} \tilde{\sigma}_1 O_{12} &= O_{13}\tilde{\sigma}_1, & \tilde{\sigma}_2 O_{12} &= (-2/3 O_{12} + 2/3 O_{13} + 1/3 O_{23})\tilde{\sigma}_2, \\ \tilde{\sigma}_1 O_{13} &= O_{12}\tilde{\sigma}_1, & \tilde{\sigma}_2 O_{13} &= (2/3 O_{12} + 1/3 O_{13} + 2/3 O_{23})\tilde{\sigma}_2, \\ \tilde{\sigma}_1 O_{23} &= -O_{23}\tilde{\sigma}_1, & \tilde{\sigma}_2 O_{23} &= (1/3 O_{12} + 2/3 O_{13} - 2/3 O_{23})\tilde{\sigma}_2, \end{aligned} \tag{4.5}$$

from which the entire action of $\widetilde{\mathsf{D}}^{\varepsilon}_{12}$ on $\mathfrak{O}_\kappa$ follows.

The final generator of our symmetry algebra is a central element O_{123}, of which an explicit realisation is given by

$$O_{123} = \varepsilon e_1 e_2 e_3 + O_1 e_2 e_3 - O_2 e_1 e_3 + O_3 e_1 e_2 + L_{12} e_3 - L_{13} e_2 + L_{23} e_1.$$

As a consequence of the relations in the general case presented in Section 2.2.5, the two-index symmetries O_{ij} respect

$$\begin{aligned} [O_{13}, O_{12}] &= O_{23} + 2O_{123}O_1 + \mathcal{E}; \\ [O_{23}, O_{12}] &= -O_{13} + 2O_{123}O_2 + \mathcal{E}; \\ [O_{23}, O_{13}] &= O_{12} + 2O_{123}O_3 + \mathcal{E}. \end{aligned} \tag{4.6}$$

In the right-hand sides of (4.6), the linear combinations of elements of $\widetilde{\mathsf{D}}^{\varepsilon}_{12}$ given by (4.4) and $\mathcal{E}$ appear. When the deformation parameters κ_1, κ_2 are chosen to be zero, these all vanish and the relations (4.6) reduce to those of the orthogonal Lie algebra $\mathfrak{so}(3)$.

4.2 Ladder operators

The result we prove next holds for any arbitrary root system in $\mathbb{R}^3$. Hereto, one should use the appropriate definitions for O_1, O_2, O_3 as given in [DOV18a, eq. (3.8) and Ex. 4.2] and the relations analogous to (4.6) given by [DOV18b, eq. (1.7)]. What we obtain in this way are not yet the desired ladder operators, though we will show that they do lead to ladder operators for the G_2 case at hand.

Proposition 4.2.1. *Let $\omega = e^{2i\pi/3}$ and consider the following linear combinations:*

$$\begin{aligned} O_0 &:= -i/\sqrt{3}(O_{12} + O_{23} - O_{13}), \\ O_+ &:= -i\sqrt{2/3}(O_{12} + \omega O_{23} - \omega^2 O_{13}), \\ O_- &:= -i\sqrt{2/3}(O_{12} + \omega^2 O_{23} - \omega O_{13}). \end{aligned} \tag{4.7}$$

Denoting $\omega^+ := \omega$ and $\omega^- := \omega^2$, the linear combinations satisfy

$$\begin{aligned} [O_0, O_\pm] &= \pm O_\pm \mp i\sqrt{2/3}(2O_{123}(O_3 + \omega^\pm O_1 + \omega^\mp O_2) \\ &\qquad + [O_1, O_2] + \omega^\pm [O_2, O_3] + \omega^\mp [O_3, O_1]); \\ [O_+, O_-] &= 2O_0 - 2i/\sqrt{3}(2O_{123}(O_1 + O_2 + O_3) \\ &\qquad + [O_1, O_2] + [O_2, O_3] + [O_3, O_1]). \end{aligned} \tag{4.8}$$

Proof. Using the definitions (4.7) and grouping the terms appropriately we obtain

$$\begin{aligned} [O_0, O_\pm] = -\sqrt{2}/3\Big(&(1-\omega^\pm)[O_{23}, O_{12}] \\ &+ (\omega^\mp - 1)[O_{12}, O_{31}] + (\omega^\pm - \omega^\mp)[O_{31}, O_{23}]\Big). \end{aligned}$$

Noticing that $(\omega^\pm - \omega^\mp) = \pm i\sqrt{3}$, and $(1-\omega^\pm) = 3/2 \mp i\sqrt{3}/2 = \pm i\sqrt{3}\omega^\mp$, and $(\omega^\mp - 1) = -3/2 \mp i\sqrt{3}/2 = \pm i\sqrt{3}\omega^\pm$, and applying [DOV18b, eq. (1.7)] result in

$$\begin{aligned} = \mp i\sqrt{2}/\sqrt{3}\Big(&\omega^\mp(O_{31} + \{O_{123}, O_2\} + [O_3, O_1]) \\ &+ \omega^\pm(O_{23} + \{O_{123}, O_1\} + [O_2, O_3]) \\ &+ O_{12} + \{O_{123}, O_3\} + [O_1, O_2]\Big), \end{aligned}$$

and finally using again the definitions (4.7) one arrives at the desired expression. In the same manner for the second equation, we find

$$[O_+, O_-] = -2/3(\omega - \omega^2)([O_{23}, O_{12}] + [O_{12}, O_{31}] + [O_{31}, O_{23}])$$

$$\begin{aligned}
&= -2i/\sqrt{3}\,(O_{31} + \{O_{123}, O_2\} + [O_3, O_1] + O_{23} + \{O_{123}, O_1\} \\
&\quad + [O_2, O_1] + O_{12} + \{O_{123}, O_1\} + [O_1, O_2]) \\
&= 2O_0 - 2i/\sqrt{3}\,(\{O_{123}, O_1 + O_2 + O_3\} \\
&\quad + [O_1, O_2] + [O_2, O_3] + [O_3, O_1]).
\end{aligned}$$

As O_{123} is central, this proves the second equality. ∎

When the root system satisfies some specific properties, we can use the previous result to obtain ladder operators.

Proposition 4.2.2. *For the root system* G_2*, the elements* O_0*,* O_+ *and* O_- *satisfy*

$$\begin{aligned}
&[O_0, O_\pm] = \pm O_\pm \mp 2i\sqrt{2/3}\, O_{123}\left(O_3 + \omega^{\pm} O_1 + \omega^{\mp} O_2\right); \\
&[O_+, O_-] = 2O_0 - 2i\sqrt{3}\mathcal{E}.
\end{aligned} \tag{4.9}$$

Moreover, the quadratic elements

$$K_+ := \frac{1}{2}\{O_0, O_+\} \quad K_- := \frac{1}{2}\{O_0, O_-\} \tag{4.10}$$

fulfill the ladder operator relations

$$[O_0, K_\pm] = \pm K_\pm. \tag{4.11}$$

Proof. Starting from the relations (4.8), we can use $1 + \omega + \omega^2 = 0$, and $O_1 + O_2 + O_3 = 0$, while $[O_1, O_2] = [O_2, O_3] = [O_3, O_1] = \mathcal{E}$, to arrive at (4.9).

In addition, $[O_0, K_\pm] = 1/2\,[O_0, \{O_0, O_\pm\}] = 1/2\,\{O_0, [O_0, O_\pm]\}$. By the first relation (4.9), this becomes

$$\begin{aligned}
[O_0, K_\pm] &= \pm 1/2\,\{O_0, O_\pm\} \mp i\sqrt{2/3}\left\{O_0, O_{123}(O_3 + \omega^{\pm} O_1 + \omega^{\mp} O_2)\right\} \\
&= \pm K_\pm.
\end{aligned}$$

In the last step we used the fact that O_{123} is central, and that all elements of $\widetilde{\mathsf{D}}_{12}^{\varepsilon}$ anticommute with O_0, which is clear from the action (4.5). ∎

These ladder operators can now be used in the study of the representation theory of the symmetry algebra in a similar vein as what was done in the S_3 case [DOV18b]. This is the subject of the following chapter.

5

Finite-dimensional representations of the dihedral Dunkl–Dirac symmetry algebra

The content of this chapter is extract from the article and the contribution:

Hendrik De Bie, Alexis Langlois-Rémillard, Roy Oste, and Joris Van der Jeugt. (2022) *Journal of Algebra* [DLOV22a].

Alexis Langlois-Rémillard. (2022+) *Proceedings of Lie Theory and Its Applications in Physics, LT-XIV 2021*. [Lan22].

5.1 Forewords

The following chapter contains work extracted from the two publications mentioned above. To adapt them to the notation of this thesis, we have modified the results of the first sections to work with both Clifford signatures. However, Section 5.6 has kept the convention $\varepsilon = 1$.

5.2 Introduction

The aim of this work is to continue the inquiry of the representation theory of the symmetry algebra, generated by symmetries supercommuting with an $\mathfrak{osp}(1|2)$ realisation linked to a Dirac-like operator deformed by a reflection group [DOV18a], namely the Dunkl total angular momentum algebra $\mathcal{O}_\kappa$. We consider the case of a Dunkl–Dirac operator acting on a three-dimensional space as this is the lowest dimension where the symmetry algebra portrays interesting behaviour. This work enlarges the class of studied systems to all the reducible rank 3 root systems in a three-dimensional setting.

Dunkl operators are a generalisation of partial derivatives introduced by Dunkl [Dun89] in the context of orthogonal polynomials in several variables. The constituents of those operators are: a reflection group $W \subset \mathcal{O}(d)$ acting on the Euclidean space $\mathbb{R}^d$ and its root system Φ, a W-invariant function $\kappa : \Phi \to \mathbb{C}$, and the algebra of polynomials in d variables $x_1,\ldots,x_d$. Then, as in Section 2.2.1, we identify the rational Cherednik algebra H_κ to its Dunkl representation generated by the group algebra $\mathbb{C}W$, the d Dunkl operators $\mathrm{D}_1,\ldots,\mathrm{D}_d$, and the d multiplication operators $x_1,\ldots,x_d$. The symmetry algebra we consider is the centralizer of an $\mathfrak{osp}(1|2)$ realisation inside the tensor product of a Clifford algebra and H_κ. The symmetry algebra is in fact the full centralizer of the $\mathfrak{osp}(1|2)$ algebra realisation. This is considered in generality in [Ost22]. It is related to the $(\mathrm{Pin}(d), \mathfrak{osp}(1|2))$ Howe duality. Other deformations of Howe dualities were recently studied in the rational Cherednik algebra context [CD20; Ciu+20].

We recall here all the possibilities for rank three root systems.

(i) Three rank 1 root systems: $\mathsf{A}_1 \oplus \mathsf{A}_1 \oplus \mathsf{A}_1$, with Weyl group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) The sum of a rank 1 and a rank 2 root systems, so the infinite family of dihedral root systems with an A_1 part: $\mathsf{A}_1 \oplus \mathsf{I}_2(m)$ ($m \geq 3$). Their Coxeter group is $\mathbb{Z}_2 \times \mathrm{D}_{2m}$.

(iii) The irreducible rank 3 root systems A_3, B_3, C_3 and H_3 of respective Weyl groups S_4, $S_3 \rtimes \mathbb{Z}_2^3$, $S_3 \rtimes \mathbb{Z}_2^3$ and $A_5 \times \mathbb{Z}_2$.

This chapter addresses the reducible root systems of rank 3 (cases (i) and (ii)) as they can all be studied using the same method. The study of the irreducible root systems A_3, B_3 and H_3 (C_3 has the same Weyl group and is indiscernible in our context) requires different methods and is subject to ongoing investigations.

Dihedral groups in the Dunkl operators context are well-studied examples. They offer a tractable non-trivial behaviour and already divide into two different cases depending on whether the parameter m for the dihedral group is odd or even. The harmonic polynomials of the dihedral Dunkl–Laplacian were already given in the original paper of Dunkl [Dun89]. The complete finite-dimensional representation theory in the dihedral case for rational Cherednik algebras is also known [Chm06]. They are often the first non-trivial examples one can hope to consider completely. Recent investigations on the dihedral case include: closed formulas for intertwining operators [Xu19; DL21], the geometrical properties of the Calogero–Moser space associated with dihedral groups [Bon18] and the complete description of the deformed unitary Howe dual pairs [Ciu+20].

Here we study the finite-dimensional irreducible representations of the symmetry algebra of the dihedral Dunkl–Dirac operator acting on a three-dimensional space. Albeit dihedral root systems are of rank 2, the symmetry algebra becomes interesting only for dimension three and higher. Adding the extra dimension by means of an A_1 root system is the most general approach. Note that the symmetry algebra associated with the reflection group D_{2m} is a subalgebra of the one associated with $\mathbb{Z}_2 \times \mathsf{D}_{2m}$ and is not simply obtained by setting the function κ to zero on the $\mathbb{Z}_2$ part. This is a difference with the study of the S_3 case [DOV18b]: the extra $\mathbb{Z}_2$ component adds more constraints on the forms the representations can take. Another difference is that restricting representations of the symmetry algebra to representations of W now gives rise to projective representations of W. Projective representations are representations of the double covering of the group that are not isomorphic to representation of the group. For a reflection group, considered as a subgroup of $\mathcal{O}(d)$, there are in general two double coverings $\widetilde{W}^+$ and $\widetilde{W}^-$; see Section 2.2.3. In general, as group algebras over $\mathbb{C}$, they are isomorphic, and hence their module categories are equivalent. However the whole picture was not present for the root system A_2 because the group S_3 does not admit two non-trivial double coverings. We give the detailed construction of the representation of the double coverings of the direct sum of a dihedral group with $\mathbb{Z}_2$ in Section 5.4 following Morris [Mor76].

The main results of this chapter and its structure are now reviewed. Section 5.3 reviews the background of Dunkl operators in general and in the dihedral cases. Section 5.5 first presents the general results and definitions on the symmetry algebra, proves a useful proposition on the square of the central symmetry for any reflection group acting on a three-dimensional space (Proposition 5.5.5), then constructs a new set of generators (Proposition 5.5.8), continues with the construction of a pair of ladder operators (Proposition 5.5.10) and finishes with a discussion on the possible unitary structures. Section 5.6 states and proves the main results: a classification of the finite-dimensional irreducible and unitary representations (Theorems 5.6.1 and 5.6.2 for the odd and even case respectively). The main ideas of the proofs are explained in Subsection 5.6.1 and the details are given in Subsections 5.6.2 and 5.6.3. Finally, Section 5.7 studies the important example of the monogenic representation families (Propositions 5.7.7, and 5.7.8). For the convenience of the reader, we also included Section 5.4 to recall results on the double coverings of reflections groups and write down the complete construction for $W = \mathbb{Z}_2 \times \mathsf{D}_{2m}$ of the irreducible finite-dimensional representations of the double coverings $\widetilde{W}^-$ and $\widetilde{W}^+$ (Theorem 5.4.2).

5.3 The dihedral Dunkl–Dirac equation

In this section, the required theory of Dunkl operators is recalled, both in the general case and in the specific dihedral case. The book of Dunkl and Xu [DX14] contains the material for Dunkl operators. The definitions of the generalized Dunkl–Laplace and Dunkl–Dirac operators can be found in [DOV18a].

5.3.1 The dihedral Dunkl operators

Let $\mathsf{I}_2(m)$ denote the root system associated with the dihedral group D_{2m} of order $2m$. For $m = 1,2,3,4,6$, it is a crystallographic root system, respectively A_1, $\mathsf{A}_1 \oplus \mathsf{A}_1$, A_2, B_2 and G_2.

We will consider for the remainder of the chapter $m \geq 2$ and $d = 3$ and put $W = \mathbb{Z}_2 \times \mathsf{D}_{2m}$ acting on $\mathbb{R}^3$. Recall its Coxeter presentation (2.7)

$$W = \left\langle \sigma_0, \sigma_1, \sigma_m \mid \sigma_0^2 = \sigma_1^2 = \sigma_m^2 = (\sigma_0\sigma_1)^2 = (\sigma_0\sigma_m)^2 = (\sigma_1\sigma_m)^m = 1 \right\rangle,$$

where σ_0 is the generator for $\mathbb{Z}_2$. We choose the standard root system Φ of $\mathsf{A}_1 \oplus \mathsf{I}_2(m)$ as

$$\alpha_0 = (0, 0, 1) \quad \alpha_j = (\sin(j\pi/m), -\cos(j\pi/m), 0), \quad j = 1, \ldots, 2m, \tag{5.1}$$

with the set of positive roots given by $\Phi_+ = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$.

Before proceeding any further, we warn the reader that in previous works on $\mathsf{A}_2 = \mathsf{I}_2(3)$ [DOV18b] and $\mathsf{G}_2 = \mathsf{I}_2(6)$ [LO20], the root system used is the natural embedding of the roots in the $(1,1,1)$-hyperplane. In the A_2 case, the change of variables to u, v and w corresponds to $x_1 = v$, $x_2 = u$ and $w = x_3$ in this chapter. Here we decided to follow the same convention for the dihedral groups as Dunkl [Dun89] and Humphreys [Hum90]. The associated reflections σ_j are given in matrix form by

$$\sigma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \sigma_j = \begin{pmatrix} \cos(2j\pi/m) & \sin(2j\pi/m) & 0 \\ \sin(2j\pi/m) & -\cos(2j\pi/m) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

The structure of dihedral groups depends on whether m is even or odd. When it is even, the elements σ_1 and σ_m are in two different conjugacy classes; when m is odd, they are in the same. This has an impact on the double coverings (see Section 5.4) of the group and will impact slightly the representation theory.

With this in mind, a W-invariant function κ is defined by at most three constants $\kappa_0 := \kappa(\alpha_0)$, $\kappa_1 := \kappa(\alpha_1)$ and $\kappa_m := \kappa(\alpha_m)$ linked to the W-orbits of α_0, α_1 and α_m respectively. Understand that $\kappa_1 = \kappa_m$ when m is odd. To this effect then, for positive j, $\kappa(\alpha_j) = \kappa_1$ when m is odd; and $\kappa(\alpha_{2j}) = \kappa_m$, $\kappa(\alpha_{2j+1}) = \kappa_1$ when m is even. The Dunkl operators (2.10) are then given, when m is odd, by

$$\begin{aligned}
\mathrm{D}_1(f(x)) &= \partial_{x_1} f(x) + \kappa_1 \sum_{j=1}^{m} \frac{\sin(\frac{j\pi}{m})(f(x) - \sigma_j f(x))}{\sin(\frac{j\pi}{m})x_1 - \cos(\frac{j\pi}{m})x_2};\\
\mathrm{D}_2(f(x)) &= \partial_{x_2} f(x) - \kappa_1 \sum_{j=1}^{m} \frac{\cos(\frac{j\pi}{m})(f(x) - \sigma_j f(x))}{\sin(\frac{j\pi}{m})x_1 - \cos(\frac{j\pi}{m})x_2};\\
\mathrm{D}_3(f(x)) &= \partial_{x_3} f(x) + \kappa_0 \frac{f(x) - \sigma_0 f(x)}{x_3};
\end{aligned} \tag{5.3}$$

and, when m is even, by

$$\begin{aligned}
\mathrm{D}_1(f(x)) &= \partial_{x_1} f(x) + \kappa_1 \sum_{\substack{j=1\\ j \text{ odd}}}^{m-1} \frac{\sin(\frac{j\pi}{m})(f(x) - \sigma_j f(x))}{\sin(\frac{j\pi}{m})x_1 - \cos(\frac{j\pi}{m})x_2}\\
&\quad + \kappa_m \sum_{\substack{j=1\\ j \text{ even}}}^{m} \frac{\sin(\frac{j\pi}{m})(f(x) - \sigma_j f(x))}{\sin(\frac{j\pi}{m})x_1 - \cos(\frac{j\pi}{m})x_2};\\
\mathrm{D}_2(f(x)) &= \partial_{x_2} f(x) - \kappa_1 \sum_{\substack{j=1\\ j \text{ odd}}}^{m-1} \frac{\cos(\frac{j\pi}{m})(f(x) - \sigma_j f(x))}{\sin(\frac{j\pi}{m})x_1 - \cos(\frac{j\pi}{m})x_2}\\
&\quad - \kappa_m \sum_{\substack{j=1\\ j \text{ even}}}^{m} \frac{\cos(\frac{j\pi}{m})(f(x) - \sigma_j f(x))}{\sin(\frac{j\pi}{m})x_1 - \cos(j\pi/m)x_2};\\
\mathrm{D}_3(f(x)) &= \partial_{x_3} f(x) + \kappa_0 \frac{f(x) - \sigma_0 f(x)}{x_3}.
\end{aligned} \tag{5.4}$$

For this reflection group $W = \mathbb{Z}_2{\times}\mathrm{D}_{2m}$, equation (2.57) becomes

$$\left\{\underline{\mathrm{D}}, \underline{x}\right\} = 2\varepsilon\left(\mathbb{E} + \frac{3}{2} + \frac{m}{2}(\kappa_1 + \kappa_m) + \kappa_0\right). \tag{5.5}$$

5.4 Double coverings

The general material has been presented in Section 2.2.3. In this section, we will apply it to the group $W = \mathbb{Z}_2 \times \mathrm{D}_{2m}$. We have presentations by generators and relations of its double coverings from Theorem 2.2.7.

Corollary 5.4.1. *The two central extensions of $\widetilde{W}^+$ and $\widetilde{W}^-$ of W have the following presentations by generators and relations depending on the parity of m.*

- *(m odd)*

$$\widetilde{W}^+ = \left\langle z, \tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_m \;\middle|\; \begin{array}{c} z^2 = \tilde{\sigma}_0^2 = \tilde{\sigma}_1^2 = \tilde{\sigma}_m^2 = 1; \\ (\tilde{\sigma}_1\tilde{\sigma}_m)^m = 1;\ (\tilde{\sigma}_0\tilde{\sigma}_1)^2 = (\tilde{\sigma}_0\tilde{\sigma}_m)^2 = z \end{array} \right\rangle,$$

$$\widetilde{W}^- = \left\langle z, \tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_m \;\middle|\; \begin{array}{c} z^2 = 1;\ \tilde{\sigma}_0^2 = \tilde{\sigma}_1^2 = \tilde{\sigma}_m^2 = z \\ (\tilde{\sigma}_0\tilde{\sigma}_1)^2 = (\tilde{\sigma}_0\tilde{\sigma}_m)^2 = (\tilde{\sigma}_1\tilde{\sigma}_m)^m = z \end{array} \right\rangle.$$

- *(m even)*

$$\widetilde{W}^+ = \left\langle z, \tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_m \;\middle|\; \begin{array}{c} z^2 = \tilde{\sigma}_0^2 = \tilde{\sigma}_1^2 = \tilde{\sigma}_m^2 = 1; \\ (\tilde{\sigma}_0\tilde{\sigma}_1)^2 = (\tilde{\sigma}_0\tilde{\sigma}_m)^2 = (\tilde{\sigma}_1\tilde{\sigma}_m)^m = z \end{array} \right\rangle,$$

$$\widetilde{W}^- = \left\langle z, \tilde{\sigma}_0, \tilde{\sigma}_1, \tilde{\sigma}_m \;\middle|\; \begin{array}{c} z^2 = 1;\ \tilde{\sigma}_0^2 = \tilde{\sigma}_1^2 = \tilde{\sigma}_m^2 = z \\ (\tilde{\sigma}_0\tilde{\sigma}_1)^2 = (\tilde{\sigma}_0\tilde{\sigma}_m)^2 = (\tilde{\sigma}_1\tilde{\sigma}_m)^m = z \end{array} \right\rangle.$$

For all presentations, the element z commutes with the rest.

From this corollary, we can construct all the finite-dimensional irreducible representations for $\widetilde{W}^+$ and $\widetilde{W}^-$ in the odd and even cases.

The classical idea followed here is to give all the conjugacy classes and then construct as many non-equivalent irreducible finite-dimensional representations, thus exhibiting them all. The results are summarised in Theorem 5.4.2 at the end of the section. We included all the details for $\widetilde{W}^+$ as this material is hard to find in recent literature and seems to us of good pedagogical value.

5.4.1 Irreducible representations for the odd case

When $m = 2p+1$ is odd, there are $4p+5 = 2m+3$ conjugacy classes for $\widetilde{W}^+$. For ease of notation, let $\tilde{\tau} := \tilde{\sigma}_1\tilde{\sigma}_m$ be the even element of order m. We start by listing the conjugacy classes of $\widetilde{W}^+$:

$$\begin{gathered}\{1\}, \quad \{z\}, \quad \{\tilde{\sigma}_0, z\tilde{\sigma}_0\},\\ \{\tilde{\tau}, \tilde{\tau}^{2p}\}, \{\tilde{\tau}^2, \tilde{\tau}^{2p-1}\}, \dots, \{\tilde{\tau}^p, \tilde{\tau}^{p+1}\},\\ \{z\tilde{\tau}, z\tilde{\tau}^{2p}\}, \{z\tilde{\tau}^2, z\tilde{\tau}^{2p-1}\}, \dots, \{z\tilde{\tau}^p, z\tilde{\tau}^{p+1}\},\\ \{\tilde{\sigma}_0\tilde{\tau}, z\tilde{\sigma}_0\tilde{\tau}^{2p}\}, \{\tilde{\sigma}_0\tilde{\tau}^2, z\tilde{\sigma}_0\tilde{\tau}^{2p-1}\}, \dots, \{\tilde{\sigma}_0\tilde{\tau}^p, z\tilde{\sigma}_0\tilde{\tau}^{p+1}\},\\ \{z\tilde{\sigma}_0\tilde{\tau}, \tilde{\sigma}_0\tilde{\tau}^{2p}\}, \{z\tilde{\sigma}_0\tilde{\tau}^2, \tilde{\sigma}_0\tilde{\tau}^{2p-1}\}, \dots, \{z\tilde{\sigma}_0\tilde{\tau}^p, \tilde{\sigma}_0\tilde{\tau}^{p+1}\},\\ \{\tilde{\sigma}_m, \tilde{\tau}\tilde{\sigma}_m, \tilde{\tau}^2\tilde{\sigma}_m, \dots, \tilde{\tau}^{2p}\tilde{\sigma}_m, z\tilde{\sigma}_m, z\tilde{\tau}\tilde{\sigma}_m, \dots, z\tilde{\tau}^{2p}\tilde{\sigma}_m\},\\ \{\tilde{\sigma}_0\tilde{\sigma}_m, \tilde{\sigma}_0\tilde{\tau}\tilde{\sigma}_m, \tilde{\sigma}_0\tilde{\tau}^2\tilde{\sigma}_m, \dots, \tilde{\sigma}_0\tilde{\tau}^{2p}\tilde{\sigma}_m, z\tilde{\sigma}_0\tilde{\sigma}_m, z\tilde{\sigma}_0\tilde{\tau}\tilde{\sigma}_m, \dots, z\tilde{\sigma}_0\tilde{\tau}^{2p}\tilde{\sigma}_m\}.\end{gathered}$$

And indeed, counting the elements in the conjugacy classes gives:

$$1+1+2+p\times2+p\times2+p\times2+p\times2+(4p+2)+(4p+2) = 16p+8 = 8m = |\widetilde{W}^+|.$$

We now construct the $4p+5$ non-equivalent irreducible finite-dimensional representations.

Let V be a finite-dimensional vector space and $X : \widetilde{W}^+ \longrightarrow Gl(V)$ be a non-trivial finite-dimensional irreducible representation of $\widetilde{W}^+$. Consider an eigenvector for $\tilde{\tau}$, z and $\tilde{\sigma}_0$, denoted $v_1 \in V$. As $z^2 = \tilde{\sigma}_0^2 = \tilde{\tau}^m = 1$, we have that

$$zv_1 = \epsilon v_1, \qquad \epsilon \in \{-1,+1\}; \tag{5.6}$$

$$\tilde{\sigma}_0 v_1 = \delta v_1, \qquad \delta \in \{-1,+1\}; \tag{5.7}$$

$$\tilde{\tau} v_1 = \zeta^{2\ell} v_1, \qquad \zeta := e^{\pi i/m},\ \ell \in \{0,1,\dots,m-1\}. \tag{5.8}$$

Put $v_2 := \tilde{\sigma}_m v_1$. It follows from $\tilde{\tau}\tilde{\sigma}_m\tilde{\tau} = \tilde{\sigma}_m$ that

$$\begin{aligned}\tilde{\tau} v_2 &= \tilde{\tau}\tilde{\sigma}_m v_1 = \tilde{\sigma}_m\tilde{\tau}^{-1} v_1 = \zeta^{-2\ell} v_2, & \tilde{\sigma}_m v_2 &= \tilde{\sigma}_m^2 v_1 = v_1,\\ zv_2 &= \tilde{\sigma}_m z v_1 = -\tilde{\sigma}_m v_1 = -v_2, & \tilde{\sigma}_0 v_2 &= \tilde{\sigma}_0\tilde{\sigma}_m v_1 = z\tilde{\sigma}_m\tilde{\sigma}_0 v_1 = \epsilon\delta v_2.\end{aligned}$$

This means that $\langle v_1, v_2\rangle$ is a submodule of V; as V is irreducible, $\langle v_1, v_2\rangle = V$ and thus $\dim V \leq 2$.

The process divides in two cases whether $\zeta^{2\ell} = \zeta^{-2\ell}$ or not. First, assume $\zeta^{2\ell} \neq \zeta^{-2\ell}$. Then $\ell \in \{1,\dots,m-1\}$. In this case, v_1 and v_2 have two different eigenvalues, and they are therefore linearly independent, so the dimension of V is 2.

The matrices of the representation X in the basis $\{v_1, v_2\}$ are given by

$$
\begin{aligned}
X(\tilde{\tau}) &= \begin{pmatrix} \zeta^{2\ell} & 0 \\ 0 & \zeta^{-2\ell} \end{pmatrix}, & X(\tilde{\sigma}_m) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
X(z) &= \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, & X(\tilde{\sigma}_0) &= \begin{pmatrix} \delta & 0 \\ 0 & \epsilon\delta \end{pmatrix}.
\end{aligned}
\tag{5.9}
$$

Now count how many non-equivalent representations this gives. A first remark is that we can ask of the imaginary part of $\zeta^{2\ell}$ to be positive. Indeed, if it is not the case then for $\epsilon = 1$, switching v_1 and v_2 will make it so, and for $\epsilon = -1$, changing v_1 and v_2 and sending δ to $-\delta$ will give an equivalent representation. With the positivity condition on the imaginary part of $\zeta^{2\ell}$, such repetitions are avoided. This condition results in a restriction on the values ℓ can take: $\ell \in \{1, \ldots, p\}$. The values of δ and ϵ are independent in the set $\{-1, +1\}$ and so there are a total of $4p$ non-equivalent irreducible representations of dimension 2.

Second, assume $\zeta^{2\ell} = \zeta^{-2\ell}$. As $m = 2p+1$, this forces $\zeta^{2\ell} = 1$, so $\ell = 0$. Then the matrices in the generating set $\{v_1, v_2\}$ are given by

$$
\begin{aligned}
X(\tilde{\tau}) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & X(\tilde{\sigma}_m) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
X(z) &= \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, & X(\tilde{\sigma}_0) &= \begin{pmatrix} \delta & 0 \\ 0 & \epsilon\delta \end{pmatrix}.
\end{aligned}
\tag{5.10}
$$

Further divide according to the value of ϵ. If $\epsilon = 1$, then notice that the actions on $v_1 + v_2$ and $v_1 - v_2$ are given by

$$
\begin{aligned}
\tilde{\tau}(v_1 + v_2) &= v_1 + v_2, & \tilde{\sigma}_m(v_1 + v_2) &= v_2 + v_1, \\
\tilde{\tau}(v_1 - v_2) &= v_1 - v_2, & \tilde{\sigma}_m(v_1 - v_2) &= -(v_1 - v_2), \\
z(v_1 + v_2) &= v_1 + v_2, & \tilde{\sigma}_0(v_1 + v_2) &= \delta(v_1 + v_2), \\
z(v_1 - v_2) &= v_1 - v_2, & \tilde{\sigma}_0(v_1 - v_2) &= \delta(v_1 - v_2).
\end{aligned}
$$

Therefore, both $\langle v_1 + v_2 \rangle$ and $\langle v_1 - v_2 \rangle$ are submodules of V. The irreducibility of V forces one of them to be trivial and the other, to generate V. So $v_2 = v_1$ or $v_2 = -v_1$. Adding the choice of value for δ, this gives 4 one-dimensional irreducible representations.

When $\epsilon = -1$, the vectors v_1 and v_2 have different eigenvalues for $\tilde{\sigma}_0$, so they are linearly independent, and thus v_1 and v_2 are a basis for a two-dimensional representation. The two representations given by $\delta = -1$ and $\delta = 1$ are equivalent after switching v_1 and v_2 so there is only one more two-dimensional irreducible representation.

The total is $4p+5$ irreducible representations, the same number as conjugacy classes, so all of them have been found. The dimensions match as indeed the sum of the squares of the dimensions gives the order of the group: $4p \times 2^2 + 4 \times 1^2 + 1 \times 2^2 = 16p + 8 = |\widetilde{W}^+|$.

Similar steps will also give the $4p+5$ irreducible representations of $\widetilde{W}^-$. The only differences are that the extra relations constrict the values of ϵ according to the action of $\tilde{\tau}$ and that $\tilde{\sigma}_m$ possibly is of order 4, so δ takes values in $\{-1, +1, -i, +i\}$. All the representations are given in Theorem 5.4.2.

Consider the negative double covering $\widetilde{W}^-$. Its $4p+5$ conjugacy classes are listed below

$$
\begin{gathered}
\{1\}, \quad \{z\}, \quad \{\tilde{\sigma}_0, z\tilde{\sigma}_0\}, \\
\{\tilde{\tau}, \tilde{\tau}^{-1}\},\ \{\tilde{\tau}^2, \tilde{\tau}^{-2}\}, \ldots, \{\tilde{\tau}^p, \tilde{\tau}^{-p}\},\ \{z\tilde{\tau}, z\tilde{\tau}^{-1}\},\ \{z\tilde{\tau}^2, z\tilde{\tau}^{-2}\}, \ldots, \{z\tilde{\tau}^p, z\tilde{\tau}^{-p}\}, \\
\{\tilde{\sigma}_0\tilde{\tau}, \tilde{\sigma}_0\tilde{\tau}^{-1}\},\ \{\tilde{\sigma}_0\tilde{\tau}^2, \tilde{\sigma}_0\tilde{\tau}^{-2}\}, \ldots, \{\tilde{\sigma}_0\tilde{\tau}^p, \tilde{\sigma}_0\tilde{\tau}^{-p}\}, \\
\{z\tilde{\sigma}_0\tilde{\tau}, z\tilde{\sigma}_0\tilde{\tau}^{-1}\},\ \{z\tilde{\sigma}_0\tilde{\tau}^2, z\tilde{\sigma}_0\tilde{\tau}^{-2}\}, \ldots, \{z\tilde{\sigma}_0\tilde{\tau}^p, z\tilde{\sigma}_0\tilde{\tau}^{-p}\}, \\
\{\tilde{\sigma}_m, \tilde{\tau}\tilde{\sigma}_m, \ldots, \tilde{\tau}^{2p}\tilde{\sigma}_m, z\tilde{\sigma}_m, z\tilde{\tau}\tilde{\sigma}_m, \ldots, z\tilde{\tau}^{2p}\tilde{\sigma}_m\}, \\
\{\tilde{\sigma}_0\tilde{\sigma}_m, \tilde{\sigma}_0\tilde{\tau}\tilde{\sigma}_m, \ldots, \tilde{\sigma}_0\tilde{\tau}^{2p}\tilde{\sigma}_m, z\tilde{\sigma}_0\tilde{\sigma}_m, z\tilde{\sigma}_0\tilde{\tau}\tilde{\sigma}_m, \ldots, z\tilde{\sigma}_0\tilde{\tau}^{2p}\tilde{\sigma}_m\}.
\end{gathered}
$$

The process is very similar and the constructed representations are presented in Theorem 5.4.2.

Let V be a finite-dimensional vector space. Let $X : \widetilde{W}^- \to GL(V)$ be a finite-dimensional irreducible representation of $\widetilde{W}^-$. Let $v_1 \in v$ be an eigenvector for $\tilde{\tau}$, z and $\tilde{\sigma}_0$. Because $\tilde{\tau}^{2m} = z^2 = 1$ and $\tilde{\sigma}_0^4 = 1$, the values of the eigenvalues are restricted to

$$
\begin{aligned}
\tilde{\tau} v_1 &= \zeta^\ell v_1, && \zeta := e^{\pi i/m},\ \ell \in \{0, 1, \ldots, 2m\}; \\
z v_1 &= \epsilon v_1, && \epsilon \in \{-1, +1\}; \\
\tilde{\sigma}_0 v_1 &= \delta v_1, && \delta \in \{-1, +1, -i, +i\}.
\end{aligned}
$$

Furthermore, the two additional restrictions $\tilde{\tau}^m = \tilde{\sigma}_0^2 = z$ become

$$\epsilon = (-1)^\ell, \qquad \delta \in \begin{cases} \{-1,+1\}, & \epsilon = 1, \\ \{-i,+i\}, & \epsilon = -1. \end{cases}$$

Indeed, $\tilde{\tau}^m v_1 = \zeta^{m\ell} = (-1)^\ell v_1$, but also $\tilde{\tau}^m v_1 = z v_1 = \epsilon v_1$.

Define $v_2 := \tilde{\sigma}_m v_1$. The matrices of the action of the generators on the set $\{v_1, v_2\}$ are given by

$$\begin{aligned} X(\tilde{\tau}) &= \begin{pmatrix} \zeta^\ell & 0 \\ 0 & \zeta^{-\ell} \end{pmatrix}, & X(\tilde{\sigma}_m) &= \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}, \\ X(z) &= \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, & X(\tilde{\sigma}_0) &= \begin{pmatrix} \delta & 0 \\ 0 & \epsilon\delta \end{pmatrix}. \end{aligned} \tag{5.11}$$

As V is an irreducible $\widetilde{W}^-$ module, and as $\langle v_1, v_2 \rangle$ is a submodule, it means $\dim V \leq 2$. Distinguish between $\zeta^\ell = \zeta^{-\ell}$ ($\ell \in \{0, m\}$) and $\zeta^\ell \neq \zeta^{-\ell}$ ($\ell \in \{1, \ldots, m-1, m+1, \ldots, 2m-1\}$).

If $\ell = 0$, then $\epsilon = (-1)^0 = 1$ and by an argument similar to the positive case $\widetilde{W}^+$, $\langle v_1 + v_2 \rangle$ and $\langle v_1 - v_2 \rangle$ are two submodules of V, and thus it gives 4 one-dimensional non-equivalent irreducible representations given by the actions

$$\tilde{\tau} v_1 = v_1, \qquad \tilde{\sigma}_m v_1 = \beta v_1, \qquad \tilde{\sigma}_0 v_1 = \delta v_1, \qquad z v_1 = v_1, \tag{5.12}$$

with $\beta, \delta \in \{-1, +1\}$.

When $\ell = m$, then $\epsilon = (-1)^m = -1$ as m is odd. Then v_1 and v_2 have two different eigenvalues for $\tilde{\sigma}_0$ and so they are linearly independent vector and they form a basis for V. The two representations given by $\delta = -i$ and $\delta = +i$ are equivalent under the change of variables $v_2 \leftrightarrow v_1$. Thus, there is one two-dimensional irreducible representation given by the actions

$$\begin{aligned} X(\tilde{\tau}) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & X(\tilde{\sigma}_m) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ X(z) &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & X(\tilde{\sigma}_0) &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \end{aligned} \tag{5.13}$$

Finally, when $\zeta^\ell \neq \zeta^{-\ell}$, we can ask of ζ^ℓ to have its imaginary part positive as the cases with $\ell > m$ would be equivalent to those with $\ell < m$ under a switch of variables. So $\ell \in \{1, \ldots 2p\}$. As $\epsilon = (-1)^\ell$ is fixed by ℓ, and as there are two choices for δ, there are $4p$ non-equivalent irreducible two-dimensional representations given by the action expressed ins (5.11).

We have all the $4p+5$ irreducible non-equivalent representations and indeed

$$(4p+1) \times 2^2 + 4 \times 1^2 = 16p + 8 = |\widetilde{W}^-|.$$

5.4.2 Irreducible representations for the even case

When $m = 2p$ is even, there are $4p+6 = 2m+6$ conjugacy classes for $\widetilde{W}^+$. Still keeping the shorthand notation $\tilde{\tau} := \tilde{\sigma}_1 \tilde{\sigma}_m$ they go as

$$\begin{gathered}
\{1\}, \quad \{z\}, \quad \{\sigma_0, z\sigma_0\}, \\
\{\tilde{\tau}, z\tilde{\tau}^{2p-1}\}, \{\tilde{\tau}^2, z\tilde{\tau}^{2p-2}\}, \ldots, \{\tilde{\tau}^p, z\tilde{\tau}^p\}, \{\tilde{\tau}^{p+1}, z\tilde{\tau}^{p-1}\} \ldots, \{\tilde{\tau}^{2p-1}, -\tilde{\tau}\}, \\
\{\tilde{\sigma}_0\tilde{\tau}, \tilde{\sigma}_0\tilde{\tau}^{2p-1}\}, \{\tilde{\sigma}_0\tilde{\tau}^2, \tilde{\sigma}_0\tilde{\tau}^{2p-2}\}, \ldots, \{\tilde{\sigma}_0\tilde{\tau}^{p-1}, \tilde{\sigma}_0\tilde{\tau}^{p+1}\}, \{\tilde{\sigma}_0\tilde{\tau}^p\}, \\
\{z\tilde{\sigma}_0\tilde{\tau}, z\tilde{\sigma}_0\tilde{\tau}^{2p-1}\}, \{z\tilde{\sigma}_0\tilde{\tau}^2, z\tilde{\sigma}_0\tilde{\tau}^{2p-2}\}, \ldots, \{z\tilde{\sigma}_0\tilde{\tau}^{p-1}, z\tilde{\sigma}_0\tilde{\tau}^{p+1}\}, \{z\tilde{\sigma}_0\tilde{\tau}^p\}, \\
\{\tilde{\sigma}_m, \tilde{\tau}^2\tilde{\sigma}_m, \ldots, \tilde{\tau}^{2p-2}\tilde{\sigma}_m, z\tilde{\sigma}_m, z\tilde{\tau}^2\tilde{\sigma}_m, \ldots, z\tilde{\tau}^{2p-2}\tilde{\sigma}_m\}, \\
\{\tilde{\sigma}_0\tilde{\sigma}_m, \tilde{\sigma}_0\tilde{\tau}^2\tilde{\sigma}_m, \ldots, \tilde{\sigma}_0\tilde{\tau}^{2p-2}\tilde{\sigma}_m, -\tilde{\sigma}_0\tilde{\sigma}_m, z\tilde{\sigma}_0\tilde{\tau}^2\tilde{\sigma}_m, \ldots, z\tilde{\sigma}_0\tilde{\tau}^{2p-2}\tilde{\sigma}_m\}, \\
\{\tilde{\tau}\tilde{\sigma}_m, \tilde{\tau}^3\tilde{\sigma}_m, \ldots, \tilde{\tau}^{2p-1}\tilde{\sigma}_m, z\tilde{\tau}\tilde{\sigma}_m, z\tilde{\tau}^3\tilde{\sigma}_m, \ldots, z\tilde{\tau}^{2p-1}\tilde{\sigma}_m\}, \\
\{\tilde{\sigma}_0\tilde{\tau}\tilde{\sigma}_m, \tilde{\sigma}_0\tilde{\tau}^3\tilde{\sigma}_m, \ldots, \tilde{\sigma}_0\tilde{\tau}^{2p-1}\tilde{\sigma}_m, -\tilde{\sigma}_0\tilde{\tau}\tilde{\sigma}_m, z\tilde{\sigma}_0\tilde{\tau}^3\tilde{\sigma}_m, \ldots, z\tilde{\sigma}_0\tilde{\tau}^{2p-1}\tilde{\sigma}_m\}.
\end{gathered}$$

Adding the elements of the conjugacy classes gives the order of the group $\widetilde{W}^+$:

$$1+1+2+(2p-1)\times 2+(p-1)\times 2+1+(p-1)\times 2+1+2p+2p+2p+2p = 16p.$$

Let V be a finite-dimensional vector space and $X : \widetilde{W}^+ \longrightarrow GL(V)$ be a non-trivial finite-dimensional irreducible representation of $\widetilde{W}^+$. Take $v_1 \in V$ to be an eigenvector for $\tilde{\tau}$, z and $\tilde{\sigma}_0$. From $\tilde{\tau}^m = z$, $z^2 = 1$ and $\tilde{\sigma}_0^2 = 1$, it follows

$$zv_1 = \epsilon v_1, \qquad \epsilon \in \{-1, +1\}; \tag{5.14}$$

$$\tilde{\sigma}_0 v_1 = \delta v_1, \qquad \delta \in \{-1, +1\}; \tag{5.15}$$

$$\tilde{\tau} v_1 = \zeta^\ell v_1, \qquad \zeta := e^{\pi i/m},\ \ell \in \{0, 1, \ldots, 2m-1\}. \tag{5.16}$$

The main difference with the odd case is that $\tilde{\tau}$ now has order $2m$ instead of order m.

Define $v_2 := \tilde{\sigma}_m v_1$. We have

$$\begin{aligned} \tilde{\tau} v_2 &= \tilde{\tau}\tilde{\sigma}_m v_1 = \tilde{\sigma}_m \tilde{\tau}^{-1} v_1 = \zeta^{-\ell} v_2, & \tilde{\sigma}_m v_2 &= \tilde{\sigma}_m^2 v_1 = v_1, \\ z v_2 &= z\tilde{\sigma}_m v_1 = \tilde{\sigma}_m z v_1 = \epsilon v_2, & \tilde{\sigma}_0 v_2 &= \tilde{\sigma}_0 \tilde{\sigma}_m v_1 = z\tilde{\sigma}_m \tilde{\sigma}_0 v_1 = \epsilon\delta v_2. \end{aligned}$$

So $\langle v_1, v_2\rangle \subset V$ is a submodule and by irreducibility of V that means $V = \langle v_1, v_2\rangle$. We condition on whether $\zeta^{\ell} = \zeta^{-\ell}$; this only happens when $\ell \in \{0, m\}$.

When $\zeta^{\ell} \neq \zeta^{-\ell}$, then $\ell \in \{1, \dots m-1, m+1, \dots 2m-1\}$ and the matrices realising the actions of the elements in the basis $\{v_1, v_2\}$ are given by

$$\begin{aligned} X(\tilde{\tau}) &= \begin{pmatrix} \zeta^{\ell} & 0 \\ 0 & \zeta^{-\ell} \end{pmatrix}, & X(\tilde{\sigma}_m) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ X(z) &= \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, & X(\tilde{\sigma}_0) &= \begin{pmatrix} \delta & 0 \\ 0 & \epsilon\delta \end{pmatrix}. \end{aligned} \tag{5.17}$$

Notice that the condition $\tilde{\tau}^m = z$ restricts the possible values of ϵ to $\epsilon = (-1)^{\ell}$. Indeed, as ℓ is neither 0 nor m then $\tilde{\tau}^m v_1 = \zeta^{m\ell} v_1 = (-1)^{\ell} v_1$, but also $\tilde{\tau}^m v_1 = z v_1 = \epsilon v_1$.

$$\begin{aligned} \tilde{\tau} v_2 &= \tilde{\tau}\tilde{\sigma}_m v_1 = \tilde{\sigma}_m \tilde{\tau}^{-1} v_1 = \zeta^{-\ell} v_2 = \tilde{\sigma}_m \tilde{\tau}^{2m-1} v_1 \\ &= \tilde{\sigma}_m \tilde{\tau}^m \tilde{\tau}^{m-1} v_1 = (-1)\tilde{\sigma}_m \tilde{\tau}^{m-1} v_1 = \epsilon \zeta^{(m-1)\ell} v_2. \end{aligned}$$

And for $\zeta^{-\ell}$ to equal $\epsilon\zeta^{(m-1)\ell}$, it is required that $\epsilon = -1$.

We again demand that the imaginary part of ζ^{ℓ} is positive and thus $\ell \in \{1, \dots, m-1\}$. There are $2p-1$ choices for ℓ and, for each ℓ, 2 choices for δ: a total of $4p-2$ representations.

The cases following from $\zeta^{\ell} = \zeta^{-\ell}$ follow almost the same argument as the odd case. When $\ell = 0$, the actions are given by

$$X(\tilde{\tau}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ X(\tilde{\sigma}_m) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ X(z) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix},\ X(\tilde{\sigma}_0) = \begin{pmatrix} \delta & 0 \\ 0 & \epsilon\delta \end{pmatrix},$$

it gives 4 one-dimensional non-equivalent representations. The actions are then $\tilde{\sigma}_m v_1 = \beta v_1$ and $\tilde{\sigma}_0 v_1 = \delta v_1$ with $\beta, \delta \in \{-1, +1\}$.

There is no possible representation when $\epsilon = -1$ as $\tilde{\tau}^m v_1 = v_1$ and $\tilde{\tau}^m = z$.

When $\ell = m$, it gives 4 one-dimensional non-equivalent representations with $\epsilon = 1$. Namely $\tilde{\sigma}_m v_1 = \beta v_1$ and $\tilde{\sigma}_0 v_1 = \delta v_1$ with $\beta, \delta \in \{-1, +1\}$. There is again no possible representation when $\epsilon = -1$ as $\tilde{\tau}^m v_1 = v_1$ and $\tilde{\tau}^m = z$.

In total, we get $4p-2$ two-dimensional non-equivalent irreducible representations from the first case and 8 one-dimensional non-equivalent irreducible representations from the second and third cases, for a total of $4p+6$, which is equal to the number of conjugacy classes. Hence, all of them have been found. Furthermore, indeed $(4p-2) \times 2^2 + 8 \times 1^2 = 16p = |\widetilde{W}^+|$.

The construction of the irreducible representations of the negative double covering $\widetilde{W}^-$ follows in almost the same way, with only a slight difference on $\tilde{\sigma}_m$.

For the negative double covering $\widetilde{W}^-$, the $4p+6$ conjugacy classes are given by

$$
\begin{gathered}
\{1\}, \quad \{z\}, \quad \{\tilde{\sigma}_0, z\tilde{\sigma}_0\}, \\
\{\tilde{\tau}, \tilde{\tau}^{-1}\}, \{\tilde{\tau}^2, \tilde{\tau}^{-2}\}, \dots, \{\tilde{\tau}^p, \tilde{\tau}^{-p}\}, \ \{z\tilde{\tau}, z\tilde{\tau}^{-1}\}, \{z\tilde{\tau}^2, z\tilde{\tau}^{-2}\}, \dots, \{z\tilde{\tau}^p, z\tilde{\tau}^{-p}\}, \\
\{\tilde{\sigma}_0\tilde{\tau}, \tilde{\sigma}_0\tilde{\tau}^{-1}\}, \{\tilde{\sigma}_0\tilde{\tau}^2, \tilde{\sigma}_0\tilde{\tau}^{-2}\}, \dots, \{\tilde{\sigma}_0\tilde{\tau}^p, \tilde{\sigma}_0\tilde{\tau}^{-p}\}, \\
\{z\tilde{\sigma}_0\tilde{\tau}, z\tilde{\sigma}_0\tilde{\tau}^{-1}\}, \{z\tilde{\sigma}_0\tilde{\tau}^2, z\tilde{\sigma}_0\tilde{\tau}^{-2}\}, \dots, \{z\tilde{\sigma}_0\tilde{\tau}^p, z\tilde{\sigma}_0\tilde{\tau}^{-p}\}, \\
\{\tilde{\sigma}_m, \tilde{\tau}^2\tilde{\sigma}_m, \dots, \tilde{\tau}^{2p-2}\tilde{\sigma}_m, z\tilde{\sigma}_m, z\tilde{\tau}^2\tilde{\sigma}_m, \dots, z\tilde{\tau}^{2p-2}\tilde{\sigma}_m\}, \\
\{\tilde{\tau}\tilde{\sigma}_m, \tilde{\tau}^3\tilde{\sigma}_m, \dots, \tilde{\tau}^{2p-1}\tilde{\sigma}_m, z\tilde{\tau}\tilde{\sigma}_m, z\tilde{\tau}^3\tilde{\sigma}_m, \dots, z\tilde{\tau}^{2p-1}\tilde{\sigma}_m\}, \\
\{\tilde{\sigma}_0\tilde{\sigma}_m, \tilde{\sigma}_0\tilde{\tau}^2\tilde{\sigma}_m, \dots, \tilde{\sigma}_0\tilde{\tau}^{2p-2}\tilde{\sigma}_m, z\tilde{\sigma}_0\tilde{\sigma}_m, z\tilde{\sigma}_0\tilde{\tau}^2\tilde{\sigma}_m, \dots, z\tilde{\sigma}_0\tilde{\tau}^{2p-2}\tilde{\sigma}_m\}, \\
\{\tilde{\sigma}_0\tilde{\tau}\tilde{\sigma}_m, \tilde{\sigma}_0\tilde{\tau}^3\tilde{\sigma}_m, \dots, \tilde{\sigma}_0\tilde{\tau}^{2p-1}\tilde{\sigma}_m, z\tilde{\sigma}_0\tilde{\tau}\tilde{\sigma}_m, z\tilde{\sigma}_0\tilde{\tau}^3\tilde{\sigma}_m, \dots, z\tilde{\sigma}_0\tilde{\tau}^{2p-1}\tilde{\sigma}_m\}.
\end{gathered}
$$

Let V be a finite-dimensional vector space and $X : \widetilde{W}^- \longrightarrow GL(V)$ be a finite-dimensional irreducible representation of $\widetilde{W}^-$. Take an eigenvector for $\tilde{\tau}$, z and $\tilde{\sigma}_0$ and note it $v_1 \in V$. Define afterwards $v_2 := \tilde{\sigma}_m v_1$. For $\tilde{\tau}^{2m} = \tilde{\sigma}_0^4 = z^2 = 1$, and $\tilde{\sigma}_m^2 = z$ we have the following actions of the four generators on the set $\{v_1, v_2\}$:

$$\begin{aligned} X(\tilde{\tau}) &= \begin{pmatrix} \zeta^{\ell} & 0 \\ 0 & \zeta^{-\ell} \end{pmatrix}, \quad X(\tilde{\sigma}_m) = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}, \\ X(z) &= \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, \qquad X(\tilde{\sigma}_0) = \begin{pmatrix} \delta & 0 \\ 0 & \epsilon\delta \end{pmatrix}, \end{aligned} \tag{5.18}$$

with $\zeta = e^{\pi i/m}$, $\epsilon \in \{-1, +1\}$ and $\delta \in \{-1, +1, -i, +i\}$. The additional relations $\tilde{\tau}^m = z$ and $\tilde{\sigma}_0^2 = z$ further constrain the values of ϵ and δ to

$$\epsilon = (-1)^{\ell}, \qquad\qquad \delta \in \begin{cases} \{-1, +1\}, & \epsilon = 1, \\ \{-i, +i\}, & \epsilon = -1. \end{cases}$$

The cases differentiate according to whether $\zeta^{\ell} = \zeta^{-\ell}$ or not.

When $\zeta^{\ell} = \zeta^{-\ell}$, then $\ell \in \{0, m\}$. In both cases, because m is even, then $\zeta^{m\ell} = 1$. So $\epsilon = 1$ because $\tilde{\tau}^m v_1 = s v_1$. In this case, we already remarked multiple time that $v_1 - v_2$ and $v_1 + v_2$ then generate submodules of V and thus V is one-dimensional. There are 8 one-dimensional non-equivalent irreducible representations defined by the actions on v_1:

$$\tilde{\tau} v_1 = \mu v_1, \quad \tilde{\sigma}_m v_1 = \beta v_1 \quad \tilde{\sigma}_0 v_1 = \delta v_1, \quad z v_1 = v_1, \quad \mu, \beta, \delta \in \{-1, +1\}. \tag{5.19}$$

When $\zeta^{\ell} \neq \zeta^{-\ell}$, then we can ask of ζ^{ℓ} to have a positive imaginary part, because a change of variables would make those with negative imaginary part equivalent to their positive counterpart. The $4p - 2$ two-dimensional non-equivalent irreducible representations are then given by the actions (5.18) with $\ell \in \{1, \ldots, 2p - 1\}$, $\epsilon = (-1)^{\ell}$ and $\delta \in \{-1, +1\}$ if ℓ is even, and $\delta \in \{-i, +i\}$ if ℓ is odd. All the irreducible representations have thus been found.

In the following theorems, we summarise our results.

Theorem 5.4.2. *Let m be a positive integer and $W = \mathbb{Z}_2 \times \mathsf{D}_{2m}$. The complete sets of non-equivalent irreducible representations of the two double coverings $\widetilde{W}^+$ and $\widetilde{W}^-$ are given by the following tables.*

- ***(Odd*** $\mathbf{m = 2p+1}$***).*** *For the positive double covering* $\widetilde{W}^+$*, the* $4p+5$ *finite-dimensional non-equivalent irreducible representations are given by* 4 *one-dimensional representations* X_i *and* $4p+1$ *two-dimensional representations* $Y_j = Y_j(\epsilon, \delta)$ *with actions given on generators* z*,* $\tilde{\sigma}_0$*,* $\tilde{\sigma}_m$ *and* $\tilde{\tau} := \tilde{\sigma}_1\tilde{\sigma}_m$ *by*

$\widetilde{W}^+$	X_1	X_2	X_3	X_4
z	1	1	1	1
$\tilde{\sigma}_0$	1	−1	1	−1
$\tilde{\sigma}_m$	1	1	−1	−1
$\tilde{\tau}$	1	1	1	1

$\widetilde{W}^+$	Y_0	Y_1	$\cdots$	Y_j	$\cdots$	Y_p
z	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$
$\tilde{\sigma}_0$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \delta & 0 \\ 0 & \epsilon\delta \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \delta & 0 \\ 0 & \epsilon\delta \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \delta & 0 \\ 0 & \epsilon\delta \end{pmatrix}$
$\tilde{\sigma}_m$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\tilde{\tau}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^{-2} \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \zeta^{2j} & 0 \\ 0 & \zeta^{-2j} \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \zeta^{2p} & 0 \\ 0 & \zeta^{-2p} \end{pmatrix}$

where δ *and* ϵ *take values in the set* $\{-1, +1\}$ *and* $\zeta := e^{\pi i/m}$.

The $4p+5$ *non-equivalent finite-dimensional irreducible representations of the negative covering* $\widetilde{W}^-$ *are given by* 4 *one-dimensional representations* X_i *and* $4p+1$ *two-dimensional representations* $Y_j = Y_j(\delta)$*, with their actions on generators given in the next tables*

$\widetilde{W}^-$	X_1	X_2	X_3	X_4
z	1	1	1	1
$\tilde{\sigma}_0$	1	−1	1	−1
$\tilde{\sigma}_m$	1	1	−1	−1
$\tilde{\tau}$	1	1	1	1

$\widetilde{W}^-$	Y_m	Y_1	$\cdots$	Y_j	$\cdots$	Y_{2p}
z	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\cdots$	$\begin{pmatrix}(-1)^j&0\\0&(-1)^j\end{pmatrix}$	$\cdots$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
$\tilde{\sigma}_0$	$\begin{pmatrix}i&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}\delta&0\\0&-\delta\end{pmatrix}$	$\cdots$	$\begin{pmatrix}\delta&0\\0&(-1)^j\delta\end{pmatrix}$	$\cdots$	$\begin{pmatrix}\delta&0\\0&\delta\end{pmatrix}$
$\tilde{\sigma}_m$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$\cdots$	$\begin{pmatrix}0&(-1)^j\\1&0\end{pmatrix}$	$\cdots$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$\tilde{\tau}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}\zeta&0\\0&\zeta^{-1}\end{pmatrix}$	$\cdots$	$\begin{pmatrix}\zeta^j&0\\0&\zeta^{-j}\end{pmatrix}$	$\cdots$	$\begin{pmatrix}\zeta^{2p}&0\\0&\zeta^{-2p}\end{pmatrix}$

where $\zeta := e^{\pi i/m}$ and δ takes value in $\{-1,+1\}$ if j is even, and in $\{-i,+i\}$ if j is odd.

- ***(Even* $\mathbf{m} = \mathbf{2p}$*).*** *For the positive covering $\widetilde{W}^+$, the $4p+6$ finite-dimensional non-equivalent irreducible representations are given by 8 one-dimensional representations X_i and $4p-2$ two-dimensional representations $Y_j = Y_j(\delta)$ with actions given on generators $z, \tilde{\sigma}_0, \tilde{\sigma}_m$ and $\tilde{\tau} := \tilde{\sigma}_1\tilde{\sigma}_m$ by:*

$\widetilde{W}^+$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
z	1	1	1	1	1	1	1	1
$\tilde{\sigma}_0$	1	−1	1	−1	1	−1	1	−1
$\tilde{\sigma}_m$	1	1	−1	−1	1	1	−1	−1
$\tilde{\tau}$	1	1	1	1	−1	−1	−1	−1

$\widetilde{W}^+$	Y_1	$\cdots$	Y_j	$\cdots$	Y_{2p-1}
z	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\cdots$	$\begin{pmatrix}(-1)^j&0\\0&(-1)^j\end{pmatrix}$	$\cdots$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$
$\tilde{\sigma}_0$	$\begin{pmatrix}\delta&0\\0&-\delta\end{pmatrix}$	$\cdots$	$\begin{pmatrix}\delta&0\\0&(-1)^j\delta\end{pmatrix}$	$\cdots$	$\begin{pmatrix}\delta&0\\0&-\delta\end{pmatrix}$
$\tilde{\sigma}_m$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\cdots$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\cdots$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$\tilde{\tau}$	$\begin{pmatrix}\zeta&0\\0&\zeta^{-1}\end{pmatrix}$	$\cdots$	$\begin{pmatrix}\zeta^j&0\\0&\zeta^{-j}\end{pmatrix}$	$\cdots$	$\begin{pmatrix}\zeta^{2p-1}&0\\0&\zeta^{-(2p-1)}\end{pmatrix}$

where $\delta \in \{-1,+1\}$ and $\zeta := e^{\pi i/m}$.

The $4p+6$ finite-dimensional non-equivalent irreducible representations of $\widetilde{W}^-$ are given by 8 one-dimensional representations X_i and $4p-2$ two-dimensional representations $Y_j = Y_j(\delta)$ presented *by their actions on generators in the next tables*

$\widetilde{W}^-$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
z	1	1	1	1	1	1	1	1
$\tilde{\sigma}_0$	1	−1	1	−1	1	−1	1	−1
$\tilde{\sigma}_m$	1	1	−1	−1	1	1	−1	−1
$\tilde{\tau}$	1	1	1	1	−1	−1	−1	−1

$\widetilde{W}^-$	Y_1	$\cdots$	Y_j	$\cdots$	Y_{2p-1}
z	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} (-1)^j & 0 \\ 0 & (-1)^j \end{pmatrix}$	$\cdots$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$\tilde{\sigma}_0$	$\begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \delta & 0 \\ 0 & (-1)^j\delta \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix}$
$\tilde{\sigma}_m$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} 0 & (-1)^j \\ 1 & 0 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$\tilde{\tau}$	$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \zeta^j & 0 \\ 0 & \zeta^{-j} \end{pmatrix}$	$\cdots$	$\begin{pmatrix} \zeta^{2p-1} & 0 \\ 0 & \zeta^{-(2p-1)} \end{pmatrix}$

where $\zeta := e^{\pi i/m}$ *and* $\delta \in \{-1,+1\}$ *if* j *is even, and* $\delta \in \{-i,+i\}$ *if* j *is odd.*

5.5 The symmetry algebra of the Dunkl–Dirac operator

In this section, we define the algebra we study by giving a generating set of elements supercommuting with the Dunkl–Dirac operator $\underline{D}$ and the vector variable $\underline{x}$. The definition is not restricted to the dihedral case and thus we take the opportunity to prove a result, Proposition 5.5.5, that holds for any reflection group W acting on $\mathbb{R}^3$. We then return to the dihedral case and prove the main result of the section, Proposition 5.5.10, that exhibits a pair of ladder operators and the factorisations of their products. The section ends with a small discussion on the unitary structure considered.

5.5.1 General symmetry algebra for 3D space

We study elements of the algebra $H_\kappa \otimes Cl(3)$ with general reflection group W and W-invariant function κ. We begin by presenting elements, called *symmetries*, that supercommute with the Dunkl–Dirac operator and the vector variable. Their commutation relations are presented in Theorem 2.2.23.

The group algebra of the (positive) double covering $\widetilde{W}^+$ of the reflection group W is the first instance of such symmetries since its elements supercommute with $\underline{\mathrm{D}}$ and $\underline{x}$ [DOV18a]. In the realisation $\widetilde{W} \subset H_\kappa \otimes Cl(3)$, its generators are obtained as

$$\tilde{\sigma}_\alpha := \sum_{j=1}^{3} \left\langle \alpha, \xi_j \right\rangle \sigma_\alpha \otimes e_j, \quad \text{for } \alpha \in \Phi. \tag{5.20}$$

And we write $\tilde{\sigma}_j := \tilde{\sigma}_{\alpha_j}$.

Alternatively, a definition in terms of abstract generators and relations is available in Section 5.4. The commuting element z in the abstract presentation is realised as -1 in $H_\kappa \otimes Cl(3)$, thus the group algebra in our realisation is in fact a quotient of the abstract group algebra by $z = -1$.

We continue with three types of symmetries linked to polynomial expressions in Clifford variables. The *one-index symmetries* have the following general expression (2.65):

$$O_i := \frac{1}{2}(\left[\underline{\mathrm{D}}, x_i\right] - e_i) = \frac{1}{2}([\mathrm{D}_i, \underline{x}] - e_i) = \frac{1}{2}\left(\sum_{k=1}^{3} C_{ki} \otimes e_k - e_i\right), \tag{5.21}$$

where C_{ki} is defined in 2.20.

The one-index symmetries are included in the group algebra $\mathbb{C}\widetilde{W}$. They are however useful in order to simplify future expressions. If the root system is normalized, they can be rewritten in terms of the elements $\tilde{\sigma}_j$ as [DOV18a, Ex. 4.2]

$$O_i = \sum_{j=0}^{m} \kappa(\alpha_j) \left\langle \xi_i, \alpha_j \right\rangle \tilde{\sigma}_j. \tag{5.22}$$

The *two-index symmetries* are defined below, with a second expression, see (2.66)

$$\begin{aligned} O_{ij} &:= L_{ij} + \tfrac{\varepsilon}{2} e_i e_j + O_i e_j - O_j e_i, & (5.23)\\ &= L_{ij} + \tfrac{\varepsilon}{2} e_i e_j + e_i O_j - e_j O_i. & (5.24) \end{aligned}$$

Finally, the *three-index symmetry* (2.67) has also two equivalent expressions given by

$$O_{123} := \tfrac{1}{2}(\big[\underline{\mathrm{D}}, \underline{x}\big] - \varepsilon)e_1e_2e_3 = -\tfrac{1}{2}e_1e_2e_3(\big[\underline{\mathrm{D}}, \underline{x}\big] - \varepsilon), \tag{5.25}$$

but the following expansions will be more useful to work with

$$\begin{aligned} O_{123} &= -\tfrac{\varepsilon}{2}e_1e_2e_3 - O_1e_2e_3 - O_2e_3e_1 - O_3e_1e_2 \\ &\quad + O_{12}e_3 + O_{31}e_2 + O_{23}e_1, \qquad (5.26) \\ &= -\tfrac{\varepsilon}{2}e_1e_2e_3 - e_2e_3O_1 - e_3e_1O_2 - e_1e_2O_3 \\ &\quad + e_3O_{12} + e_2O_{31} + e_1O_{23}. \qquad (5.27) \end{aligned}$$

Remark 5.5.1. *In* (5.25)*, the factor* $\frac{1}{2}(\big[\underline{\mathrm{D}}, \underline{x}\big] - \varepsilon)$ *is the super-Casimir of* $\mathfrak{osp}(1|2)$ *[FSS00].*

A word of warning: albeit the last equations make it look so, the Clifford elements do not in general commute or anticommute with the symmetries; only certain combinations of Clifford elements can commute following Lemma 2.2.20. However, the L_{ij}'s and C_{ij}'s, being purely elements of H_κ, commute freely with Clifford elements.

We are ready to define the algebra studied in this chapter. It is given as a subalgebra of $H_\kappa \otimes Cl(3)$ generated by the elements presented above. The name of the algebra is derived from the fact that all its elements supercommute with the $\mathfrak{osp}(1|2)$-realisation.

Definition 5.5.2. *The* symmetry algebra $\mathfrak{O}_\kappa$ *is the associative subalgebra of* $H_\kappa \otimes Cl(3)$ *generated by the symmetries* O_{12}, O_{31}, O_{23}, O_{123} *and the group algebra of* $\widetilde{W}$.

Remark 5.5.3. *In Definition 2.2.17,* $\mathfrak{O}_\kappa$ *was defined as the super-centraliser of the* $\mathfrak{osp}(1|2)$*-realisation inside* $H_\kappa \otimes Cl(3)$*. The previous definition comes from the other presentation of the total angular momentum algebra derived from Theorem 2.2.23. As it gives explicit elements to work with, it is better suited for the context of this chapter.*

The two algebras are isomorphic for this low dimension since all commutation relations are then given by Theorem 2.2.23. This is not the case for higher dimensions. A precise determination of the relation ideal in general is left for further investigation. This is a hard problem.

Instead of the commutation relations presented in Theorem 2.2.23, we will use another set of generators better suited for our purpose. The new set is given by equations (5.52) and (5.51), and the relations are found in Lemma 5.5.7 and Proposition 5.5.8. In the general case, we only need to say that the three-index symmetry O_{123} commutes with every element of the symmetry algebra and state the commutation rules of the two-index symmetries. The other relations needed in the proof of Proposition 5.5.5 are retrieved implicitly from the definitions of the elements.

The following proposition shows clearly that the two-index symmetry relations form an extension of the Lie algebra $\mathfrak{so}(3)$ commutation rules as taking κ to be the zero map sends the one-index symmetries to 0.

We now rewrite more explicitly the relations of Theorem 2.2.23 in this context.

Proposition 5.5.4 ([DOV18a]). *The two-index symmetry commutation rules are given by*

$$\begin{aligned}
[O_{12}, O_{31}] &= O_{23} + \{O_{123}, O_1\} + \varepsilon [O_2, O_3];\\
[O_{23}, O_{12}] &= O_{31} + \{O_{123}, O_2\} + \varepsilon [O_3, O_1];\\
[O_{31}, O_{23}] &= O_{12} + \{O_{123}, O_3\} + \varepsilon [O_1, O_2].
\end{aligned} \tag{5.28}$$

It will be useful later on to have an expression for the square of O_{123}. Indeed, as O_{123} is the product of the super Casimir of $\mathfrak{osp}(1|2)$ with the pseudo-scalar $e_1e_2e_3$ (see equation (5.25)), its square is the Casimir of $\mathfrak{osp}(1|2)$. Proposition 5.5.5 expresses O_{123}^2 as a sum of the squares of the other symmetries (considering a trivial symmetry $O_\emptyset := i/2$). This sum is thus central and furthermore, when κ is set to 0, it reduces to the Casimir of the undeformed $\mathfrak{so}(3)$ algebra. We emphasize that this result does not assume anything on W outside it acting on a three-dimensional space.

Proposition 5.5.5. *For any reflection group $W \subset \mathcal{O}(3)$, the three-index symmetry O_{123} squares to*

$$O_{123}^2 = -\frac{\varepsilon}{4} + O_1^2 + O_2^2 + O_3^2 + \varepsilon(O_{12}^2 + O_{31}^2 + O_{23}^2). \tag{5.29}$$

Proof. The proof will be done for $\varepsilon = 1$. For $\varepsilon = -1$, the reader is invited to look at [Lan22, Prop. 2]. The first step in the proof is to use the two expressions (5.26) and (5.27) for O_{123} to put the Clifford elements in the middle:

$$O_{123}^2 = \left(-\tfrac{1}{2}e_1e_2e_3 - O_1e_2e_3 - O_2e_3e_1 - O_3e_1e_2 + O_{12}e_3 + O_{31}e_2 + O_{23}e_1\right) \\ \times\left(-\tfrac{1}{2}e_1e_2e_3 - e_2e_3O_1 - e_3e_1O_2 - e_1e_2O_3 + e_3O_{12} + e_2O_{31} + e_1O_{23}\right).$$

In working out the 49 terms of this product, separate the 7 "diagonal terms" and the 42 "cross terms". Simplifying with the Clifford anticommutation relations gives

$$O_{123}^2 = -\frac{1}{4} - O_1^2 - O_2^2 - O_3^2 + O_{12}^2 + O_{31}^2 + O_{23}^2 + Q, \tag{5.30}$$

with Q consisting of the 42 cross terms, all of them with symmetries shouldering Clifford elements. The proof is completed once it is shown that Q reduces to $2(O_1^2 + O_2^2 + O_3^2)$.

For this purpose, replace the two-index symmetries at the left of the central Clifford elements by their definition with Clifford elements on the right, equation (5.23); and replace the two-index symmetries at the right of the Clifford elements by their definition with Clifford elements on the left, equation (5.24). After simplifications, this will give

$$\begin{aligned}
O_{123}^2 = &-\frac{1}{4} - O_1^2 - O_2^2 - O_3^2 + O_{12}^2 + O_{31}^2 + O_{23}^2 && (5.31)\\
&- \tfrac{1}{2}\left(\{O_1, e_1\} + \{O_2, e_2\} + \{O_3, e_3\}\right) && (5.32)\\
&- \tfrac{1}{2}\left(\{O_{12}, e_1e_2\} + \{O_{31}, e_3e_1\} + \{O_{23}, e_2e_3\}\right) && (5.33)\\
&- \left((O_1e_1e_2O_2 + O_2e_2e_1O_1) + (O_1e_1e_3O_3 + O_3e_3e_1O_1)\right.\\
&\quad \left.+ (O_2e_2e_3O_3 + O_3e_3e_2O_2)\right) && (5.34)\\
&+ \left((O_2e_1O_{12} - O_{12}e_1O_2) + (O_1e_3O_{31} - O_{31}e_3O_1)\right.\\
&\quad \left.+ (O_3e_2O_{23} - O_{23}e_2O_3)\right) && (5.35)\\
&+ \left((O_{12}e_2O_1 - O_1e_2O_{12}) + (O_{31}e_1O_3 - O_3e_1O_{31})\right.\\
&\quad \left.+ (O_{23}e_3O_2 - O_2e_3O_{23})\right) && (5.36)\\
&- \left((O_1e_1e_2e_3O_{23} + O_{23}e_1e_2e_3O_1)\right.\\
&\quad + (O_{31}e_1e_2e_3O_2 + O_2e_1e_2e_3O_{31})\\
&\quad \left.+ (O_{12}e_1e_2e_3O_3 + O_3e_1e_2e_3O_{12})\right) && (5.37)
\end{aligned}$$

$$+ ((O_{31}e_2e_3O_{12} - O_{12}e_2e_3O_{31}) + (O_{12}e_3e_1O_{23} - O_{23}e_3e_1O_{12})$$
$$+ (O_{23}e_1e_2O_{31} - O_{31}e_1e_2O_{23})). \tag{5.38}$$

It then remains to show that the summands (5.32) to (5.38) sum to $2(O_1^2 + O_2^2 + O_3^2)$. For this purpose, rewrite summand (5.38) using equation (5.23) for the leftmost two-index symmetries in the positive terms and equation (5.24) for the rightmost two-index symmetries in the negative terms:

$$\begin{aligned}
(5.38) = \Big(&L_{31}e_2e_3O_{12} + \tfrac{1}{2}e_1e_2O_{12} + O_3e_1e_2e_3O_{12} + O_1e_2O_{12} \\
&- (O_{12}e_2e_3L_{31} + \tfrac{1}{2}O_{12}e_1e_2 + O_{12}e_2O_1 - O_{12}e_1e_2e_3O_3)\Big) \\
+ \Big(&L_{12}e_3e_1O_{23} + \tfrac{1}{2}e_2e_3O_{23} + O_1e_1e_2e_3O_{23} + O_2e_3O_{23} \\
&- (O_{23}e_3e_1L_{12} - \tfrac{1}{2}O_{23}e_2e_3 + O_{23}e_3O_2 - O_{23}e_1e_2e_3O_1)\Big) \\
+ \Big(&L_{23}e_1e_2O_{31} + \tfrac{1}{2}e_3e_1O_{31} + O_2e_1e_2e_3O_{31} + O_3e_1O_{31} \\
&- (O_{31}e_1e_2L_{23} - \tfrac{1}{2}O_{31}e_3e_1 + O_{31}e_1O_3 - O_{31}e_1e_2e_3O_2)\Big).
\end{aligned}$$

Rearrange the terms to make some simplifications obvious

$$(5.38) = L_{31}e_2e_3O_{12} + L_{12}e_3e_1O_{23} + L_{23}e_1e_2O_{31} \tag{5.39}$$
$$- (O_{12}e_2e_3L_{31} + O_{23}e_3e_1L_{12} - O_{31}e_1e_2L_{23}) \tag{5.40}$$
$$+ \tfrac{1}{2}(\{O_{12}, e_1e_2\} + \{O_{23}, e_2e_3\} + \{O_{31}, e_3e_1\}) \tag{5.41}$$
$$\begin{aligned}
&+ ((O_1e_1e_2e_3O_{23} + O_{23}e_1e_2e_3O_1) \\
&\quad + (O_{31}e_1e_2e_3O_2 + O_2e_1e_2e_3O_{31}) \\
&\quad + (O_{12}e_1e_2e_3O_3 + O_3e_1e_2e_3O_{12}))
\end{aligned} \tag{5.42}$$
$$\begin{aligned}
&- ((O_{12}e_2O_1 - O_1e_2O_{12}) + (O_{31}e_1O_3 - O_3e_1O_{31}) \\
&\quad + (O_{23}e_3O_2 - O_2e_3O_{23})).
\end{aligned} \tag{5.43}$$

Already some simplifications are happening. Indeed, summand (5.33) is the opposite of summand (5.41), and the same can be said of the pairs (5.36) and (5.43), and (5.37) and (5.42).

Therefore the only part yet to be taken care of consists in the two first lines (5.39) and (5.40). They can further be simplified by another application of the different expressions for O_{ij}, equations (5.23) and (5.24):

$$(5.39) + (5.40) = L_{31}e_2e_3L_{12} + \tfrac{1}{2}L_{31}e_3e_1 + L_{31}e_1e_2e_3O_2 + L_{31}e_3O_1$$

$$
\begin{aligned}
&+ L_{12}e_3e_1L_{23} + \tfrac{1}{2}L_{12}e_1e_2 + L_{12}e_1e_2e_3O_3 + L_{12}e_1O_2\\
&+ L_{23}e_1e_2L_{31} + \tfrac{1}{2}L_{23}e_2e_3 + L_{23}e_1e_2e_3O_1 + L_{23}e_2O_3\\
&- (L_{12}e_2e_3L_{31} - \tfrac{1}{2}e_3e_1L_{31} + O_1e_3L_{31} - O_2e_1e_2e_3L_{31})\\
&- (L_{23}e_3e_1L_{12} - \tfrac{1}{2}e_1e_2L_{12} + O_2e_1L_{12} - O_3e_1e_2e_3L_{12})\\
&- (L_{31}e_1e_2L_{23} - \tfrac{1}{2}e_2e_3L_{23} + O_3e_2L_{23} - O_1e_1e_2e_3L_{23}),
\end{aligned}
$$

which can be rearranged into

$$
\begin{aligned}
(5.39)+(5.40) = {}& (L_{31}e_2e_3L_{12} - L_{12}e_2e_3L_{31})\\
&\quad + (L_{12}e_3e_1L_{23} - L_{23}e_3e_1L_{12})\\
&\quad + (L_{23}e_1e_2L_{31} - L_{31}e_1e_2L_{23})\\
&+ (\{L_{31}e_3e_1, +\}\{L_{12}, e_1e_2\} + \{L_{23}, e_2e_3\})\\
&+ (L_{31}e_3O_1 - O_1e_3L_{31}) + (L_{12}e_1O_2 - O_2e_1L_{12})\\
&\quad + (L_{23}e_2O_3 - O_3e_2L_{23})\\
&+ (L_{31}e_1e_2e_3O_2 + O_2e_1e_2e_3L_{31})\\
&\quad + (L_{12}e_1e_2e_3O_3 + O_3e_1e_2e_3L_{12})\\
&\quad + (L_{23}e_1e_2e_3O_1 + O_1e_1e_2e_3L_{23}).
\end{aligned}
\tag{5.44}
$$

Consider now line (5.35). Again by applying the two definitions of O_{ij}:

$$
\begin{aligned}
(5.35) = {}& O_1e_3L_{31} + \tfrac{1}{2}O_1e_1 + O_1^2 + O_1e_1e_3O_3\\
&\quad - (L_{31}e_3O_1 - 1/2e_1O_1 - O_3e_3e_1O_1 - O_1^2)\\
&+ O_2e_1L_{12} + \tfrac{1}{2}O_2e_2 + O_2^2 + O_2e_2e_1O_1\\
&\quad - (L_{12}e_1O_2 - 1/2e_2O_2 - O_1e_1e_2O_2 - O_2^2)\\
&+ O_3e_2L_{23} + \tfrac{1}{2}O_3e_3 + O_3^2 + O_3e_3e_2O_2\\
&\quad - (L_{23}e_2O_3 - 1/2e_3O_3 - O_2e_2e_3O_3 - O_3^2),
\end{aligned}
$$

and the terms can be placed in a more telling way

$$
\begin{aligned}
(5.35) = {}& 2(O_1^2 + O_2^2 + O_3^2)\\
&+ (O_1e_3L_{31} - L_{31}e_3O_1) + (O_2e_1L_{12} - L_{12}e_1O_2)\\
&\quad + (O_3e_2L_{23} - L_{23}e_2O_3)\\
&+ \tfrac{1}{2}(\{O_1, e_1\} + \{O_2, e_2\} + \{O_3, e_3\})\\
&+ (O_1e_2e_3O_3 + O_3e_3e_1O_1) + (O_2e_2e_1O_1 + O_1e_1e_2O_2)\\
&\quad + (O_3e_3e_2O_2 + O_2e_2e_3O_3).
\end{aligned}
\tag{5.45}
$$

With all this, the square is reduced to the following expression:

$$\begin{aligned}
O_{123}^2 = &-\frac{1}{4} + O_1^2 + O_2^2 + O_3^2 + O_{12}^2 + O_{31}^2 + O_{23}^2 \\
&+ \tfrac{1}{2}(\{L_{31}, e_3e_1\} + \{L_{12}, e_1e_2\} + \{L_{23}, e_2e_3\}) \\
&+ (L_{31}e_2e_3L_{12} - L_{12}e_2e_3L_{31}) + (L_{12}e_3e_1L_{23} - L_{23}e_3e_1L_{12}) \\
&\quad + (L_{23}e_1e_2L_{31} - L_{31}e_1e_2L_{23}) \\
&+ L_{31}e_1e_2e_3O_2 + O_2e_1e_2e_3L_{31} + L_{12}e_1e_2e_3O_3 + O_3e_1e_2e_3L_{12} \\
&\quad + L_{23}e_1e_2e_3O_1 + O_1e_1e_2e_3L_{23},
\end{aligned}$$

and this can be rewritten as

$$O_{123}^2 = -\frac{1}{4} + O_1^2 + O_2^2 + O_3^2 + O_{12}^2 + O_{31}^2 + O_{23}^2 \tag{5.46}$$

$$+\frac{1}{2}\begin{pmatrix} L_{12}e_1e_2(1 - e_3e_1L_{31} - e_2e_3L_{23} + 2e_3O_3) \\ + L_{31}e_3e_1(1 - e_1e_2L_{12} - e_2e_3L_{23} + 2e_2O_2) \\ + L_{23}e_2e_3(1 - e_1e_2L_{12} - e_3e_1L_{31} + 2e_1O_1) \end{pmatrix} \tag{5.47}$$

$$+\frac{1}{2}\begin{pmatrix} (1 - L_{31}e_3e_1 - L_{23}e_2e_3 + 2O_3e_3)e_1e_2L_{12} \\ + (1 - L_{12}e_1e_2 - L_{23}e_2e_3 + 2O_2e_2)e_3e_1L_{31} \\ + (1 - L_{12}e_1e_2 - L_{31}e_3e_1 + 2O_1e_1)e_2e_3L_{23} \end{pmatrix}. \tag{5.48}$$

The final step is to show that each of the two last components (5.47) and (5.48) are 0. In each of them, replace the one-index symmetries by their expression (5.21) in terms of Clifford elements and C_{ij}. As the L_{ij}'s and C_{ij}'s commute with Clifford elements, factor out the Clifford elements. The coefficient of the Clifford variables will be sums of L_{ij} and C_{ij}. They will cancel out by equation (2.26).

$$\begin{aligned}
(5.47) = &L_{12}e_1e_2(C_{33} + e_3e_1(C_{31} - L_{31}) - e_2e_3(L_{23} + C_{23})) \\
&+ L_{31}e_3e_1(C_{22} - e_1e_2(L_{12} + C_{12}) + e_2e_3(C_{23} - L_{23})) \\
&+ L_{23}e_2e_3(C_{11} + e_1e_2(C_{12} - L_{12}) - e_3e_1(L_{31} + C_{31})).
\end{aligned}$$

As the L_{ij} and the C_{ij} commute with Clifford elements, the commutations relations of the Clifford elements then give

$$(5.47) = e_1e_2L_{12}C_{33} + e_2e_3(L_{12}C_{31} - L_{12}L_{31}) + e_3e_1(L_{12}L_{23} + L_{12}C_{23})$$

$$
\begin{aligned}
&+e_3e_1L_{31}C_{22}+e_2e_3(L_{31}L_{12}+L_{31}C_{12})+e_1e_2(L_{31}C_{23}-L_{31}L_{23})\\
&+e_2e_3L_{23}C_{11}+e_3e_1(L_{23}C_{12}-L_{23}L_{12})+e_1e_2(L_{23}L_{31}+L_{23}L_{31}),
\end{aligned}
$$

and regrouping gives

$$
\begin{aligned}
&=e_1e_2\,(L_{12}C_{33}+L_{31}C_{23}+L_{23}C_{31}-L_{31}L_{23}+L_{23}L_{31})\\
&\quad+e_3e_1\,(L_{31}C_{22}+L_{23}C_{12}+L_{12}C_{23}+L_{12}L_{23}-L_{23}L_{12})\\
&\quad+e_2e_3\,(L_{23}C_{11}+L_{31}C_{12}+L_{12}C_{31}+L_{31}L_{12}-L_{12}L_{31}).
\end{aligned}
$$

Recall that $L_{ij}=-L_{ji}$, $C_{ij}=C_{ji}$ and $L_{ii}=0$. Each line is then zero by equation (2.26), replacing the tuplets (i,j,k,l) by $(1,2,3,3)$, $(3,1,2,2)$ and $(2,3,1,1)$ for the first, second and third line respectively.

This leaves only the terms of equation (5.29) and thus concludes the proof. ∎

Remark 5.5.6. *Using the projector 6.17 introduced in the next Chapter, Oste was able to prove this results in more generality [Ost22, Prop. 4.16] with a much shorter proof. However, this requires to work in a realisation, whereas the previous proof hold for the abstract algebra generated by the symmetries seen as symbols.*

5.5.2 Dihedral 3D symmetry algebra and ladder operators

Let $m\geq 2$. We now specify to $W=\mathbb{Z}_2\times \mathrm{D}_{2m}$. We denote the symmetry algebra linked to this W by $\mathfrak{O}_{m,\kappa}$ as opposed to the general symmetry algebra $\mathfrak{O}_\kappa$. We begin by giving the explicit expressions of the elements of $\mathbb{C}\widetilde{W}$:

$$
\tilde{\sigma}_0=e_3\sigma_0,\qquad \tilde{\sigma}_j=(\sin(j\pi/m)e_1-\cos(j\pi/m)e_2)\sigma_j. \tag{5.49}
$$

They generate the group algebra $\mathbb{C}\widetilde{W}$ with presentation given by

$$
\mathbb{C}\widetilde{W}=\left\langle \tilde{\sigma}_0,\tilde{\sigma}_1,\tilde{\sigma}_m\;\middle|\;\tilde{\sigma}_0^2=\tilde{\sigma}_1^2=\tilde{\sigma}_m^2=1,\;\begin{matrix}(\tilde{\sigma}_0\tilde{\sigma}_1)^2=(\tilde{\sigma}_0\tilde{\sigma}_m)^2=-1,\\ (\tilde{\sigma}_1\tilde{\sigma}_m)^m=(-1)^{m+1}\varepsilon^m\end{matrix}\right\rangle. \tag{5.50}
$$

Note that because the group is realised in $Cl(3)\otimes\mathbb{C}[W]$, there is no need to add -1 as a generator. It is the positive double covering of W (or the negative double covering if the Clifford elements square to -1 instead of $+1$, see Corollary 5.4.1). Section 5.4 studies the double coverings abstractly. A few things can immediately be said of $\widetilde{W}$ nonetheless. The group depends on the parity of m, and the generator $\tilde{\sigma}_0$ of the $\mathbb{Z}_2$ part *anti*-commutes with $\tilde{\sigma}_1$ and $\tilde{\sigma}_m$ whereas σ_0 commutes with σ_1 and σ_m; so $\widetilde{W} \not\cong W \times \mathbb{Z}_2$ in general. This is a difference with the case $W = S_3$ as S_3 does not have a non-trivial positive double covering [Sch07] (it can also readily be seen from Corollary 5.4.1), however $S_3 \times \mathbb{Z}_2$ does.

The remaining part of the section is dedicated to finding ladder operators, which will be crucial for the construction of representations of $\mathfrak{O}_{m,\kappa}$. Inspired by the construction of ladder operators in the $\mathfrak{so}(3)$ case [Dir81], we define

$$O_0 := -iO_{12}, \qquad O_+ := iO_{31} + O_{23}, \qquad O_- := iO_{31} - O_{23}. \tag{5.51}$$

The algebra $\mathfrak{O}_{m,\kappa}$ is also generated by O_0, O_+, O_-, O_{123} and $\mathbb{C}\widetilde{W}$.

Define the following combinations of elements of $\mathbb{C}\widetilde{W}$

$$\begin{gathered} T_0 := i\kappa_0\tilde{\sigma}_0, \\ T_+ := -i\sum_{j=1}^{m} \kappa(\alpha_j)e^{j\pi i/m}\tilde{\sigma}_j, \quad T_- := i\sum_{j=1}^{m} \kappa(\alpha_j)e^{-j\pi i/m}\tilde{\sigma}_j. \end{gathered} \tag{5.52}$$

They are linked with the one-index symmetries by equation (5.22):

$$T_0 = iO_3, \quad T_+ = O_1 + iO_2, \quad T_- = O_1 - iO_2, \quad [T_+, T_-] = -2i[O_1, O_2].$$

Furthermore, they can be expressed in another form according to the parity of m. Put $\zeta := e^{\pi i/m}$. When m is **odd**, then all the $\kappa(\alpha_j)$ are the same and so

$$T_+ = -i\kappa_1\sum_{j=1}^{m} \zeta^j\tilde{\sigma}_j, \qquad T_- = i\kappa_1\sum_{j=1}^{m} \zeta^{-j}\tilde{\sigma}_j, \tag{5.53}$$

and when $m = 2p$ is **even**, then it is expressed as

$$T_+ = -i(\kappa_1 T_+^1 + \kappa_m T_+^2), \qquad T_- = +i(\kappa_1 T_-^1 + \kappa_m T_-^2), \tag{5.54}$$

with $T_\pm^1$, and $T_\pm^2$ the sums over odd and even indices:

$$\begin{aligned} T_+^1 &= \sum_{j=1}^{p} \zeta^{2j-1}\tilde{\sigma}_{2j-1}, \quad T_+^2 = \sum_{j=1}^{p} \zeta^{2j}\tilde{\sigma}_{2j}, \\ T_-^1 &= \sum_{j=1}^{p} \zeta^{1-2j}\tilde{\sigma}_{2j-1}, \quad T_-^2 = \sum_{j=1}^{p} \zeta^{-2j}\tilde{\sigma}_{2j}. \end{aligned} \tag{5.55}$$

The next lemma gives useful commutation properties between the new generators.

Lemma 5.5.7. *The element* O_0 *has the following commutation relations with elements of* $\mathbb{C}\widetilde{W}$*:*

$$[O_0, T_0] = \{O_0, T_+\} = \{O_0, T_-\} = 0. \tag{5.56}$$

Furthermore, T_0*,* T_+ *and* T_- *interact with* O_- *and* O_+ *as follows:*

$$T_0O_+ = -O_+T_0, \qquad T_0O_- = -O_-T_0, \tag{5.57}$$
$$T_0T_+ = -T_+T_0, \qquad T_0T_- = -T_-T_0, \tag{5.58}$$
$$T_+O_- = -O_+T_-, \qquad T_-O_+ = -O_-T_+. \tag{5.59}$$

Proof. Equations (5.56) are directly equivalent to $[O_{12}, \tilde{\sigma}_0] = 0$ and $\left\{O_{12}, \tilde{\sigma}_j\right\} = 0$ for $1 \le j \le m$. For the first, $[O_0, T_0] = 0$, it follows from the action of σ_0 that $\tilde{\sigma}_0$, and so T_0, will commute with O_{12}. For the two others, $\{O_0, T_+\} = 0 = \{O_0, T_-\}$, expand $\tilde{\sigma}_j$ and O_{12} by their definition to obtain a product of Clifford elements with one-index symmetries and L_{12}. We give the computations for $\sigma_j L_{12}$,

$$\begin{aligned}
\sigma_j L_{12} &= \sigma_j(x_1\mathrm{D}_2 - x_2\mathrm{D}_1)\\
&= \Big(\cos(\tfrac{2\pi j}{m})x_1 + \sin(\tfrac{2\pi j}{m})x_2\Big)\Big(\sin(\tfrac{2\pi j}{m})\mathrm{D}_1 - \cos(\tfrac{2\pi j}{m})\mathrm{D}_2\Big)\sigma_j\\
&\quad - \Big(\sin(\tfrac{2\pi j}{m})x_1 - \cos(\tfrac{2\pi j}{m})x_2\Big)\Big(\cos(\tfrac{2\pi j}{m})\mathrm{D}_1 + \sin(\tfrac{2\pi j}{m})\mathrm{D}_2\Big)\sigma_j\\
&= (\sin^2(\tfrac{2\pi j}{m}) + \cos^2(\tfrac{2\pi j}{m}))L_{21}\sigma_j = -L_{12}\sigma_j.
\end{aligned}$$

The computations on the one-index symmetries follow from their definition (5.21) and the Clifford part is direct, so $\left\{O_{12}, \tilde{\sigma}_j\right\} = 0$.

For the next expressions (5.57), (5.58) and (5.59), remark that O_3 leaves O_{12} invariant and sends O_{31} and O_{23} to $-O_{31}$ and $-O_{23}$ respectively, so $O_3O_+ = -O_+O_3$ and $O_3O_- = -O_-O_3$. This proves equations (5.57). Additionally, $\tilde{\sigma}_0$ and $\tilde{\sigma}_j$ anticommute because e_3 anticommutes with e_1 and e_2, and because σ_0 commutes with σ_j. Therefore, $\{O_3, O_1\} = 0 = \{O_3, O_2\}$, and so, $\{O_3, T_\pm\} = 0$; proving equations (5.58).

Finally, equations (5.59) are proven from the expression (5.52). By direct computations, we have that $\tilde{\sigma}_j O_\pm = e^{\pm 2j\pi i/m} O_\mp \tilde{\sigma}_j$ and therefore

$$\begin{aligned} T_+ O_- &= -i \sum_{j=1}^{m} \kappa(\alpha_j) e^{\frac{j\pi i}{m}} \tilde{\sigma}_j O_- = -i \sum_{j=1}^{m} \kappa(\alpha_j) e^{\frac{j\pi i}{m}} e^{\frac{-2j\pi i}{m}} O_+ \tilde{\sigma}_j \\ &= -O_+ \left(i \sum_{j=1}^{m} \kappa(\alpha_j) e^{\frac{-j\pi i}{m}} \tilde{\sigma}_j \right) = -O_+ T_-, \end{aligned}$$

and

$$T_- O_+ = i \sum_{j=1}^{m} \kappa(\alpha_j) e^{\frac{-j\pi i}{m}} \tilde{\sigma}_j O_+ = i \sum_{j=1}^{m} \kappa(\alpha_j) e^{\frac{-j\pi i}{m}} e^{\frac{2j\pi i}{m}} O_- \tilde{\sigma}_j = -O_- T_+.$$

This concludes the proof. ■

With the new set of generators, Proposition 5.5.4 translates to the following.

Proposition 5.5.8. *The linear combinations O_0, O_+ and O_- satisfy*

$$\begin{aligned} [O_0, O_+] &= +O_+ + \{O_{123}, T_+\} + \varepsilon [T_0, T_+]; \\ [O_0, O_-] &= -O_- + \{O_{123}, T_-\} - \varepsilon [T_0, T_-]; \\ [O_+, O_-] &= 2O_0 - 2\{O_{123}, T_0\} + \varepsilon [T_+, T_-]. \end{aligned} \tag{5.60}$$

Proof. It follows from the commutation rules (5.28) and the definitions of T_0, T_+ and T_- that:

$$\begin{aligned} [O_0, O_+] &= [-iO_{12}, iO_{31} + O_{23}] \\ &= [O_{12}, O_{31}] - i[O_{12}, O_{23}] \\ &= O_{23} + 2O_{123}O_1 + \varepsilon [O_2, O_3] + i(O_{31} + 2O_{123}O_2 + \varepsilon [O_3, O_1]) \\ &= O_+ + \{O_{123}, O_1 + iO_2\} + \varepsilon [O_2, O_3] + i\varepsilon [O_3, O_1], \\ &= O_+ + \{O_{123}, T_+\} + \varepsilon [O_3, iO_1 - O_2] \\ &= O_+ + \{O_{123}, T_+\} + i\varepsilon [O_3, T_+], \end{aligned}$$

and

$$\begin{aligned} [O_0, O_-] &= [-iO_{12}, iO_{31} - O_{23}] \\ &= (O_{23} + 2O_{123}O_1 + \varepsilon[O_2, O_3]) - i(O_{31} + 2O_{123}O_2 + \varepsilon[O_3, O_1]) \end{aligned}$$

$$= -O_- + \{O_{123}, T_-\} - i\varepsilon [O_3, T_-].$$

Finally directly from the definitions of O_0, $O_\pm$, T_0 and $T_\pm$:

$$\begin{aligned}[O_+, O_-] &= [iO_{31} + O_{23}, iO_{31} - O_{23}]\\ &= -2i [O_{31}, O_{23}]\\ &= 2O_0 - 2\{O_{123}, T_0\} + [T_+, T_-].\end{aligned}$$

and similarly for the last equation. ∎

As a corollary of Proposition 5.5.5, the square O_{123}^2, and the products O_+O_- and O_-O_+ have new expressions.

Corollary 5.5.9. *The square of the three-index symmetry becomes in the new generators*

$$O_{123}^2 = -\frac{\varepsilon}{4} + T_+T_- - T_0^2 - \varepsilon(O_0^2 - O_0 + O_+O_- + 2O_{123}T_0), \tag{5.61}$$

$$= -\frac{\varepsilon}{4} + T_-T_+ - T_0^2 - \varepsilon(O_0^2 + O_0 - O_-O_+ - 2O_{123}T_0). \tag{5.62}$$

The products O_+O_- and O_-O_+ can be expressed as

$$O_+O_- = \varepsilon T_+T_- - (O_0 - 1/2)^2 - \varepsilon(\varepsilon O_{123} + T_0)^2, \tag{5.63}$$

$$O_-O_+ = \varepsilon T_-T_+ - (O_0 + 1/2)^2 - \varepsilon(\varepsilon O_{123} - T_0)^2. \tag{5.64}$$

Proof. The formulas are obtained from changing the variables in the expression of O_{123}^2 of Proposition 5.5.5. First note that $O_0^2 = -O_{12}^2$. Then compute

$$O_+O_- = (iO_{31} + O_{23})(iO_{31} - O_{23}) = -O_{31}^2 - O_{23}^2 - i [O_{31}, O_{23}],$$

and thus

$$\begin{aligned}O_{31}^2 + O_{23}^2 &= -O_+O_- - i [O_{31}, O_{23}]\\ &= -O_+O_- - iO_{12} - 2iO_{123}O_3 - i\varepsilon [O_1, O_2]\\ &= -O_+O_- + O_0 - 2iO_{123}O_3 + \tfrac{\varepsilon}{2} [T_+, T_-].\end{aligned}$$

Follow up with

$$T_+T_- = (O_1 + iO_2)(O_1 - iO_2) = O_1^2 + O_2^2 - i [O_1, O_2],$$

hence,

$$O_1^2 + O_2^2 = T_+T_- - \tfrac{1}{2}[T_+, T_-].$$

Using the expression for $[O_+, O_-]$ to invert O_+O_- and T_+T_-, we have the two equalities

$$\begin{aligned} O_{123}^2 &= -\varepsilon/4 + T_+T_- + O_3^2 - \varepsilon(O_+O_-O_0^2 - O_0 + 2iO_{123}O_3), \\ &= -\varepsilon/4 + T_-T_+ + O_3^2 - \varepsilon(O_0^2 + O_0 - O_-O_+ - 2iO_{123}O_3), \end{aligned}$$

replacing $iO_3 = T_0$ and factorising finish the proof. ■

The next proposition introduces the two ladder operators and their factorisations.

Proposition 5.5.10. *Consider*

$$L_+ := \tfrac{1}{2}\{O_0, O_+\}, \textit{ and } L_- := \tfrac{1}{2}\{O_0, O_-\}. \tag{5.65}$$

They form a pair of ladder operators:

$$[O_0, L_+] = L_+, \quad [O_0, L_-] = -L_-. \tag{5.66}$$

Furthermore, they have the following factorisations:

$$L_+L_- = -\Big((O_0 - 1/2)^2 + \varepsilon(\varepsilon O_{123} + T_0)^2\Big)\Big((O_0 - 1/2)^2 - \varepsilon T_+T_-\Big), \tag{5.67}$$

$$L_-L_+ = -\Big((O_0 + 1/2)^2 + \varepsilon(\varepsilon O_{123} - T_0)^2\Big)\Big((O_0 + 1/2)^2 - \varepsilon T_-T_+\Big). \tag{5.68}$$

Proof. Start by expanding equation (5.66) by the definition of $L_\pm$:

$$\begin{aligned} 2[O_0, L_\pm] &= [O_0, \{O_0, O_\pm\}] = \{O_0, [O_0, O_\pm]\} \\ &= \{O_0, \pm O_\pm\} + \{O_0, \{O_{123}, T_\pm\}\} \pm i\{O_0, \varepsilon[O_3, T_\pm]\}, \end{aligned}$$

and as O_{123} is central (Theorem 2.2.23), and O_3, T_- and T_+ anticommute with O_0 by Lemma 5.5.7, it reduces to

$$[O_0, L_\pm] = \pm L_\pm.$$

For the factorisation, we will only do the case $\varepsilon = 1$ so as to not hide the essence of the proof with the extra care the sign calls for.

The proof begins by proving the following claim

$$[O_+O_-, O_0] = 0 = [O_-O_+, O_0], \tag{5.69}$$

for the first equality; the second being similar and thus omitted. We use Proposition 5.5.8 to replace the commutators of O_+, O_- and O_0, and we employ Lemma 5.5.7 to send O_- in front

$$\begin{aligned}[O_+O_-, O_0] &= O_+[O_-, O_0] + [O_+, O_0]O_- \\ &= O_+(O_- - 2O_{123}T_- + 2iO_3T_-) \\ &\quad - (O_+ + 2iO_{123}T_+ + 2iO_3T_+)O_- = 0.\end{aligned}$$

The claim (5.69) is proven.

In order to give the factorisation, we replace the ladder operators by their definitions and we use the commutation relations of Proposition 5.5.8 and Lemma 5.5.7 to reach:

$$L_+L_- = (O_{123} + T_0)^2T_+T_- + (O_0 - 1/2)^2O_+O_-, \tag{5.70}$$
$$L_-L_+ = (O_{123} - T_0)^2T_-T_+ + (O_0 + 1/2)^2O_-O_+. \tag{5.71}$$

Since the actual computation is rather tricky, we will show the details to obtain equation (5.70) and trust the reader to do the second. For clarity, we will add a factor 4 to remove the fractions and underline certain terms to identify where we will apply the relations. Start by the definition of the ladder operators

$$\begin{aligned}4L_+L_- &= (O_0O_+ + O_+O_0)(O_0O_- + O_-O_0) \\ &= O_0\underline{O_+O_0}O_- + O_0\underline{O_+O_-O_0} + \underline{O_+O_0}O_0O_- + \underline{O_+O_0}O_-O_0.\end{aligned}$$

Apply the commutation relations of Proposition 5.5.8 pertaining to $[O_0, O_+]$ and equation (5.69) to the underlined terms to obtain

$$\begin{aligned}4L_+L_- = {} & O_0(O_0O_+ - O_+ - 2O_{123}T_+ - 2iO_3T_+)O_- + O_0^2O_+O_- \\ & + (O_0O_+ - O_+ - 2O_{123}T_+ - 2iO_3T_+)O_0O_- \\ & + (O_0O_+ - O_+ - 2O_{123}T_+ - 2iO_3T_+)O_-O_0 \\ = {} & 2O_0^2O_+O_- - O_0O_+O_- - 2O_0O_{123}T_+O_- - 2iO_0O_3T_+O_- \\ & + O_0\underline{O_+O_0}O_- - \underline{O_+O_0}O_- - 2O_{123}T_+O_0O_- - 2iO_3T_+O_0O_- \\ & + O_0\underline{O_+O_-O_0} - \underline{O_+O_-O_0} - 2O_{123}T_+O_-O_0 - 2iO_3T_+O_-O_0.\end{aligned}$$

Use again Proposition 5.5.8 and equation (5.69) on the underlined terms to get

$$\begin{aligned}4L_+L_- = {}&4O_0^2O_+O_- - 4O_0O_+O_- + O_+O_- - 2O_{123}T_+(O_0O_- + \underline{O_-O_0} - O_-)\\ &- 4O_0O_{123}T_+O_- - 2iO_3T_+(O_0O_- + \underline{O_-O_0} - O_-) - 4iO_0O_3T_+O_-.\end{aligned}$$

At this point, replace the last instances of O_-O_0 with Proposition 5.5.8 to obtain

$$\begin{aligned}4L_+L_- = {}&4O_0^2O_+O_- - 4O_0O_+O_- + O_+O_-\\ &- 2O_{123}T_+(2\underline{O_0}O_- - 2O_{123}T_- + 2i\underline{O_3}T_-)\\ &- 2iO_3T_+(2\underline{O_0}O_- - 2O_{123}T_- + 2i\underline{O_3}T_-)\\ &- 4O_0O_{123}T_+O_- - 4iO_0O_3T_+O_-.\end{aligned}$$

Make use of Lemma 5.5.7 to then send the underlined O_0 and O_3 in front, which will give the expanded equation (5.70)

$$\begin{aligned}4L_+L_- = {}&4O_0^2O_+O_- - 4O_0O_+O_- + O_+O_- + 4O_{123}^2T_+T_-\\ &+ 8iO_{123}O_3T_+T_- - 4O_3^2T_+T_-.\end{aligned}$$

With this, we proved the first factorisation (5.70). To conclude, employ Corollary 5.5.9 to replace O_+O_- in equation (5.70):

$$\begin{aligned}L_+L_- &= (O_{123} + iO_3)^2T_+T_-\\ &\quad + (O_0 - 1/2)^2(T_+T_- - (O_0 - 1/2)^2 - (O_{123} + iO_3)^2)\\ &= \Big((O_{123} + iO_3)^2 + (O_0 - 1/2)^2\Big)T_+T_-\\ &\quad - (O_0 - 1/2)^2\Big((O_0 - 1/2)^2 + (O_{123} + iO_3)^2\Big),\end{aligned}$$

and because O_0 commutes with O_{123} and O_3, it factorises as

$$= -\Big((O_0 - 1/2)^2 + (O_{123} + T_0)^2\Big)\Big((O_0 - 1/2)^2 - T_+T_-\Big).$$

The same process applies, of course, for equation (5.71), using the expression of O_-O_+ in Corollary 5.5.9. The two factorisations (5.67) and (5.68) have been exhibited, concluding the proof. ■

Put $\tilde{\tau} := \tilde{\sigma}_1\tilde{\sigma}_m$, $\tilde{\tau}^{-1} = \tilde{\sigma}_m\tilde{\sigma}_1$ and as before $\zeta = e^{\pi i/m}$. From Lemma 5.5.7, the actions of the reflections on the symmetries and on the ladder operators are given by

$$\begin{aligned}
&\tilde{\sigma}_0 O_0 = O_0 \tilde{\sigma}_0, && \tilde{\sigma}_1 O_0 = -O_0 \tilde{\sigma}_1, && \tilde{\sigma}_m O_0 = -O_0 \tilde{\sigma}_m,\\
&\tilde{\sigma}_0 O_+ = -O_+ \tilde{\sigma}_0, && \tilde{\sigma}_1 O_+ = \zeta^2 O_- \tilde{\sigma}_1, && \tilde{\sigma}_m O_+ = O_- \tilde{\sigma}_m,\\
&\tilde{\sigma}_0 O_- = -O_- \tilde{\sigma}_0, && \tilde{\sigma}_1 O_- = \zeta^{-2} O_+ \tilde{\sigma}_1, && \tilde{\sigma}_m O_- = O_+ \tilde{\sigma}_m,\\
&\tilde{\sigma}_0 L_+ = -L_+ \tilde{\sigma}_0, && \tilde{\sigma}_1 L_+ = -\zeta^2 L_- \tilde{\sigma}_1, && \tilde{\sigma}_m L_+ = -L_- \tilde{\sigma}_m,\\
&\tilde{\sigma}_0 L_- = -L_- \tilde{\sigma}_0, && \tilde{\sigma}_1 L_- = -\zeta^{-2} L_+ \tilde{\sigma}_1, && \tilde{\sigma}_m L_- = -L_+ \tilde{\sigma}_m.
\end{aligned} \tag{5.72}$$

$$\begin{aligned}
\tilde{\tau} L_+ &= \zeta^{-2} L_+ \tilde{\tau}, & \tilde{\tau} L_- &= \zeta^2 L_- \tilde{\tau},\\
\tilde{\tau}^{-1} L_+ &= \zeta^2 L_+ \tilde{\tau}^{-1}, & \tilde{\tau}^{-1} L_- &= \zeta^{-2} L_- \tilde{\tau}^{-1},\\
T_+ L_- &= L_+ T_-, & T_- L_+ &= L_- T_+.
\end{aligned} \tag{5.73}$$

5.5.3 Unitary structure

The commutation relations of the two-index symmetries (5.60) of the algebra $\mathfrak{O}_{m,\kappa}$ reduce to the commutation relations of $\mathfrak{so}(3)$ (or of $\mathfrak{sl}(2)$) when the map κ becomes zero. Any $*$-structure on $\mathfrak{O}_{m,\kappa}$ must then reduce to either $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$, of which only $\mathfrak{su}(2)$ admits finite-dimensional unitary representations. Let $*: \mathfrak{O}_{m,\kappa} \to \mathfrak{O}_{m,\kappa}$ be the anti-linear ($(aX+bY)^* = \overline{a}X^* + \overline{b}Y^*$) anti-involution ($(XY)^* = Y^*X^*$, $(X^*)^* = X$) defined on generators by

$$\begin{aligned}
O_0^* &= O_0, & O_\pm^* &= O_\mp, & O_{123}^* &= -O_{123},\\
T_0^* &= -T_0, & T_\pm^* &= T_\mp, & \tilde{\sigma}_j^* &= \tilde{\sigma}_j,
\end{aligned} \tag{5.74}$$

for any reflection $\tilde{\sigma}_j \in \mathfrak{O}_{m,\kappa}$. As a consequence, the ladder operators satisfy $L_\pm^* = L_\mp$.

Direct computations show that the relations are compatible with the commutation relations (5.60). Furthermore, sending κ to 0 indeed gives back the $*$-structure of $\mathfrak{su}(2)$.

It would be possible to study unitarity with another structure, say imposing $\tilde{\sigma}_j^\dagger = -\tilde{\sigma}_j$, $O_\pm^\dagger = -O_\mp$ and $T_\pm^\dagger = -T_\mp$, but this would not include the important monogenics example.

5.6 Finite-dimensional representations

This section classifies the finite-dimensional irreducible representations of $\mathfrak{O}_{m,\kappa}$ and verifies if they are unitary under the $*$-structure presented in Section 5.5.3. **It assumes $\varepsilon = 1$.**

The section is divided into three parts: first the theorems are presented, then the idea of the proofs are exhibited, and the last part gives the details of the proofs of Theorems 5.6.1 and 5.6.2.

The techniques employed share some similarities with the construction of standard modules for the representation theory of rational Cherednik algebras [Chm06; Dez03; Rou05]. We construct the irreducible finite-dimensional $\mathcal{O}_{m,\kappa}$-representations from a certain class of representations of $\widetilde{W}$. For the benefit of the reader, the complete construction of the finite-dimensional irreducible representations of the group $\widetilde{W}$ is included in Section 5.4, and they are presented in Theorem 5.4.2. The specific representations we need are those for which the commuting element $z \in \widetilde{W}$ acts as $-\mathrm{id}$ ($\varepsilon = -1$ in the notation of Theorem 5.4.2); this comes from the realisation of the group $\widetilde{W}$ in $\mathcal{O}_{m,\kappa}$. These specific representations are called *spin representations* [Mor76].

The existence of irreducible finite-dimensional $\mathcal{O}_{m,\kappa}$-representations, and their unitarity, is constrained by the map κ. For an integer N and a certain irreducible spin representation U of $\widetilde{W}$, Theorem 5.6.1 (m odd) and Theorem 5.6.2 (m even) present the conditions on κ for the existence of an irreducible $\mathcal{O}_{m,\kappa}$-representation $L_{\lambda,\Lambda}(U) := L_{\lambda,\Lambda,N,\kappa}(U)$ of dimension $2N+2$. Note that the theorems take U and N independently. Even if the conditions constraining κ depend on both of them, it means that the $\widetilde{W}$-representation U does not fix the dimension.

The theorems adopt some notational conventions. First, all indices $j,k \in \{1,\ldots,N\}$. Second, $a \equiv_m b$ is a short-hand for $a \equiv b \mod m$. The tables are divided in families according to some conditions linked to N, the data of U, and the values of λ and Λ. When m is even, we denote $m = 2p$. The constants κ_0, κ_1 and κ_m will be real and positive. Finally, the indices of λ in the tables indicate the several possibilities of the parameter.

Theorem 5.6.1 (Conditions for irreducible and unitary representations, the odd cases)**.** *Let κ_0 and κ_1 be positive constants. Let N be a non-negative integer and let $U \simeq Y_\ell(-1,\delta)$ be an irreducible spin representation of $\widetilde{W}$. If the conditions on κ_0 and κ_1 presented in the next* *tables are respected, then U extends to a $2N+2$ dimensional irreducible representation $L_{\lambda,\Lambda}(V)$ of $\mathfrak{O}_{m,\kappa}$. Moreover, the constant Λ lies in one of the two families*

$$\Lambda_1 \in \{\pm i(\lambda + 1/2 + \kappa_0\delta)\} \quad or \quad \Lambda_2 \in \{\mp i(\lambda + 1/2 - \kappa_0\delta)\}. \tag{5.75}$$

It is unitary if κ_0 and κ_1 satisfy further conditions. Furthermore, all irreducible finite-dimensional representations of $\mathfrak{O}_{m,\kappa}$ are of this form.

Case I: $2(N+\ell)+1 \equiv_m 0, \lambda = \lambda_1 = N + 1/2 + \kappa_1 m$

(Λ,δ)	*Irreducibility*	*Unitarity*
$(\Lambda_1,1)$, $(\Lambda_2,-1)$	*No restriction*	*No restriction*
$(\Lambda_2,1)$, $(\Lambda_1,-1)$	$\kappa_0 \notin \{k/2,\ \lambda_1 + 1/2 - k/2 \mid odd\ k\}$	$\kappa_0 < 1/2$ *or* $\kappa_0 > \lambda_1$

Case I: $2(N+\ell)+1 \equiv_m 0, \lambda = \lambda_2 = N + 1/2 - \kappa_1 m$

(Λ,δ)	*Irreducibility*	*Unitarity*
$(\Lambda_1,1)$, $(\Lambda_2,-1)$	$\kappa_0 \notin \{-\lambda_2 - 1/2 + k/2 \mid odd\ k\}$ $\kappa_1 \notin \left\{\frac{N-k+1}{2m}, \frac{N-j+1}{m} \,\middle|\, 2(k+\ell) \equiv_m 1, 2j+\ell \not\equiv_m -1\right\}$	$\kappa_1 < \frac{1}{2m}$
$(\Lambda_2,1)$, $(\Lambda_1,-1)$	$\kappa_0 \notin \{k/2,\ \lambda_2 + 1/2 - k/2 \mid odd\ k\}$ $\kappa_1 \notin \left\{\frac{N-k+1}{2m}, \frac{N-j+1}{m} \,\middle|\, 2(k+\ell) \equiv_m 1, 2j+\ell \not\equiv_m -1\right\}$	$\kappa_0 < 1/2$ *or* $\kappa_0 > \lambda_2$ $\kappa_1 < \frac{1}{2m}$

Case II: $2(N+\ell)+1 \not\equiv_m 0, \lambda = \lambda_3 = N + 1/2$

(Λ,δ)	*Irreducibility*	*Unitarity*
$(\Lambda_1,1)$, $(\Lambda_2,-1)$	$\kappa_1 \notin \left\{\frac{N+1-k}{m} \,\middle|\, 2(k+\ell) \equiv_m 1\right\}$	$\kappa_1 < \min\left(\frac{N+1-k}{m} \,\middle|\, 2k+\ell \equiv_m 1\right)$
$(\Lambda_2,1)$, $(\Lambda_1,-1)$	$\kappa_0 \notin \{k/2,\ \lambda_3 + 1/2 - k/2 \mid odd\ k\}$ $\kappa_1 \notin \left\{\frac{N+1-k}{m} \,\middle|\, 2(k+\ell) \equiv_m 1\right\}$	$\kappa_0 < 1/2$ *or* $\kappa_0 > \lambda_3$ $\kappa_1 < \min\left(\frac{N+1-k}{m} \,\middle|\, 2k+\ell \equiv_m 1\right)$

Case III: even $N\ \lambda = \lambda_4 = N/2 + \kappa_1\delta$

(Λ,δ)	*Irreducibility*	*Unitarity*
$(\Lambda_1,1)$, $(\Lambda_2,-1)$	$\kappa_0 \notin \left\{\frac{2k-N-1}{2} \,\middle|\, k > N/2, 2(k+\ell) \not\equiv_m 1\right\}$ $\kappa_1 \notin \left\{\frac{\lvert\lambda_4-k+1/2\rvert}{m} \,\middle|\, 2(k+\ell) \equiv_m 1\right\}$	$\kappa_1 < \min\left(\frac{\lvert\frac{N}{2}+\kappa_0+\frac{1}{2}-k\rvert}{m} \,\middle|\, (2k+\ell) \equiv_m 1\right)$
$(\Lambda_2,1)$, $(\Lambda_1,-1)$	$\kappa_0 \notin \left\{\frac{k}{2}, \frac{N-k+1}{4}, \frac{N+1-2j}{2} \,\middle|\, odd\ k, 2(j+\ell) \not\equiv_m 1\right\}$ $\kappa_1 \notin \left\{\frac{\lvert\lambda_4-k+1/2\rvert}{m} \,\middle|\, 2(k+\ell) \equiv_m 1\right\}$	$\kappa_0 < 1/2$ *or* $\kappa_0 > N/2 + 1/2$ $\kappa_1 < \min\left(\frac{\lvert\frac{N}{2}+\kappa_0+\frac{1}{2}-k\rvert}{m} \,\middle|\, (2k+\ell) \equiv_m 1\right)$

Theorem 5.6.2 (Conditions for irreducible and unitary representations, the even cases)**.** *Denote $m = 2p$ for a certain $p \in \mathbb{N}$. Let κ_0,*

κ_1 and κ_m be positive constants. Let N be a non-negative integer and $U \simeq Y_{2\ell+1}(-1,\delta)$ be an irreducible spin representation of $\widetilde{W}$. If the conditions on κ_0, κ_1 and κ_m of the following tables hold, then U extends to a $2N+2$ dimensional irreducible $\mathfrak{O}_{m,\kappa}$-representation $L_{\lambda,\Lambda}(V)$. Moreover, the constant Λ is of the form

$$\Lambda_1 \in \{\pm i(\lambda + 1/2 + \kappa_0\delta)\} \quad \text{or} \quad \Lambda_2 \in \{\mp i(\lambda + 1/2 - \kappa_0\delta)\}. \tag{5.76}$$

It is unitary if κ_0, κ_1 and κ_m satisfy more restrictive conditions presented thereafter. Furthermore, all irreducible finite-dimensional representations of $\mathfrak{O}_{m,\kappa}$ are of this form.

Case I.i: $N+\ell \equiv_m 1-p, \lambda = \lambda_1 = N + 1/2 + (\kappa_1+\kappa_m)p$

(Λ,δ)	*Irreducibility*	*Unitarity*
$(\Lambda_1,1)$, $(\Lambda_2,-1)$	*No restriction*	*No restriction*
$(\Lambda_2,1)$, $(\Lambda_1,-1)$	$\kappa_0 \notin \{k/2,\ \lambda_1 + 1/2 - k/2 \mid \textit{odd } k\}$	$\kappa_0 < 1/2$, *or* $\kappa_0 > \lambda_1$

Case I.i: $N+\ell \equiv_m 1-p, \lambda = \lambda_2 = N + 1/2 - (\kappa_1+\kappa_m)p$

(Λ,δ)	*Irreducibility*	*Unitarity*
$(\Lambda_1,1)$, $(\Lambda_2,-1)$	$\kappa_0 \notin \{-\lambda_2 - 1/2 + k/2 \mid \textit{odd } k\}$ $\kappa_1, \kappa_m \notin \left\{\frac{N-k+1}{m} \,\middle|\, 2-k-\ell \equiv_m 0\right\}$ $\kappa_1 + \kappa_m \notin \left\{\frac{N-k+1}{m}, \frac{N-j+1}{p} \,\middle|\, \begin{smallmatrix} 2-k-\ell \equiv_m p; \\ j \not\equiv_m 0,p \end{smallmatrix}\right\}$	$\kappa_1 + \kappa_m < 1/p$
$(\Lambda_2,1)$, $(\Lambda_1,-1)$	$\kappa_0 \notin \{k/2,\ \lambda_2 + 1/2 - k/2 \mid \textit{odd } k\}$ $\kappa_1, \kappa_m \notin \left\{\frac{N-k+1}{m} \,\middle|\, 2-k-\ell \equiv_m 0\right\}$ $\kappa_1 + \kappa_m \notin \left\{\frac{N-k+1}{m}, \frac{N-j+1}{p} \,\middle|\, \begin{smallmatrix} 2-k-\ell \equiv_m p; \\ j \not\equiv_m 0,p \end{smallmatrix}\right\}$	$\kappa_0 < 1/2$ *or* $\kappa_0 > \lambda_2$ $\kappa_1 + \kappa_m < 1/p$

Case I.ii: $N+\ell \equiv_m 1, \lambda = \lambda_3 = N + 1/2 + (\kappa_1-\kappa_m)p$

(Λ,δ)	*Irreducibility*	*Unitarity*
$(\Lambda_1,1)$, $(\Lambda_2,-1)$	$\kappa_m \notin \left\{\frac{N-k+1}{m} \,\middle|\, 2-k-\ell \equiv_m p\right\}$	$\kappa_m < \min\left(\frac{N-k+1}{m} \,\middle|\, 2-k-\ell \equiv_m p\right)$
$(\Lambda_2,1)$, $(\Lambda_1,-1)$	$\kappa_0 \notin \{k/2,\ \lambda_3 + 1/2 - k/2 \mid \textit{odd } k\}$ $\kappa_m \notin \left\{\frac{N-k+1}{m} \,\middle|\, 2-k-\ell \equiv_m p\right\}$	$\kappa_0 < 1/2$ *or* $\kappa_0 > \lambda_3$ $\kappa_m < \min\left(\frac{N-k+1}{m} \,\middle|\, 2-k-\ell \equiv_m p\right)$

Case I.ii: $N+\ell \equiv_m 1, \lambda = \lambda_4 = N + 1/2 - (\kappa_1-\kappa_m)p$

(Λ,δ)	*Irreducibility*	*Unitarity*
$(\Lambda_1,1)$, $(\Lambda_2,-1)$	$\kappa_1 \notin \left\{\frac{N-k+1}{m} \,\middle|\, 2-k-\ell \equiv_m p\right\}$ $\kappa_1 - \kappa_m \notin \left\{\frac{N-k+1}{m}, \frac{N-j+1}{p} \,\middle|\, \begin{smallmatrix} 2-k-\ell \equiv_m 0; \\ 2-j-\ell \not\equiv_m 0,p \end{smallmatrix}\right\}$	$\kappa_1 - \kappa_m < 1/p$ $\kappa_1 < \min\left(\frac{N-k+1}{m} \,\middle|\, 2-k-\ell \equiv_m p\right)$
$(\Lambda_2,1)$, $(\Lambda_1,-1)$	$\kappa_0 \notin \{k/2,\ \lambda_4 + 1/2 - k/2 \mid \textit{odd } k\}$ $\kappa_1 \notin \left\{\frac{N-k+1}{m} \,\middle|\, 2-k-\ell \equiv_m p\right\}$ $\kappa_1 - \kappa_m \notin \left\{\frac{N-k+1}{m}, \frac{N-j+1}{p} \,\middle|\, \begin{smallmatrix} 2-k-\ell \equiv_m 0; \\ 2-j-\ell \not\equiv_m 0,p \end{smallmatrix}\right\}$	$\kappa_0 < 1/2$ *or* $\kappa_0 > \lambda_4$ $\kappa_1 - \kappa_m < 1/p$ $\kappa_1 < \min\left(\frac{N-k+1}{m} \,\middle|\, 2-k-\ell \equiv_m p\right)$

Case II: $(N+\ell) \not\equiv_m 1,\ 1-p\lambda = \lambda_5 = N+1/2$

(Λ,δ)	*Irreducibility*	*Unitarity*
$(\Lambda_1,1)$, $(\Lambda_2,-1)$	$\kappa_1+\kappa_m \notin \left\{\frac{N-k+1}{p} \,\middle\vert\, 2-k-\ell \equiv_m p\right\}$ $\lvert\kappa_1-\kappa_m\rvert \notin \left\{\frac{N-k+1}{p} \,\middle\vert\, 2-k-\ell \equiv_m 0\right\}$	$\kappa_1+\kappa_m < \min\left(\frac{N-k+1}{m} \,\middle\vert\, 2-k-\ell \equiv_m p\right)$ $\lvert\kappa_1-\kappa_m\rvert < \min\left(\frac{N-k+1}{p} \,\middle\vert\, 2-k-\ell \equiv_m 0\right)$
$(\Lambda_2,1)$, $(\Lambda_1,-1)$	$\kappa_0 \notin \{k/2,\ \lambda_5+1/2-k/2 \mid \textit{odd } k\}$ $\kappa_1+\kappa_m \notin \left\{\frac{N-k+1}{p} \,\middle\vert\, 2-k-\ell \equiv_m p\right\}$ $\lvert\kappa_1-\kappa_m\rvert \notin \left\{\frac{N-k+1}{p} \,\middle\vert\, 2-k-\ell \equiv_m 0\right\}$	$\kappa_0 < 1/2$ *or* $\kappa_0 > N-1/2$ $\kappa_1+\kappa_m < \min\left(\frac{N-k+1}{m} \,\middle\vert\, 2-k-\ell \equiv_m p\right)$ $\lvert\kappa_1-\kappa_m\rvert < \min\left(\frac{N-k+1}{p} \,\middle\vert\, 2-k-\ell \equiv_m 0\right)$

Case III: even $N\ \lambda = \lambda_6 = N/2+\kappa_0\delta$

(Λ,δ)	*Irreducibility*	*Unitarity*
$(\Lambda_1,1)$, $(\Lambda_2,-1)$	$\kappa_0 \notin \left\{\frac{N-2k+1}{2} \,\middle\vert\, 2-k-\ell \not\equiv_m 0,p\right\}$ $\kappa_1+\kappa_m \notin \left\{\frac{\lvert\lambda_6-k+1/2\rvert}{p} \,\middle\vert\, 2-k-\ell \equiv_m p\right\}$ $\lvert\kappa_1-\kappa_m\rvert \notin \left\{\frac{\lvert\lambda_6-k+1/2\rvert}{p} \,\middle\vert\, 2-k-\ell \equiv_m 0\right\}$	$\kappa_1+\kappa_m < \min\left(\frac{\lvert\lambda_6-k+\frac{1}{2}\rvert}{p} \,\middle\vert\, 2-k-\ell \equiv_m p\right)$ $\lvert\kappa_1-\kappa_m\rvert < \min\left(\frac{\lvert\lambda_6-k+\frac{1}{2}\rvert}{p} \,\middle\vert\, 2-k-\ell \equiv_m 0\right)$
$(\Lambda_2,1)$, $(\Lambda_1,-1)$	$\kappa_0 \notin \left\{\frac{k}{2},\ \frac{N-k+1}{4},\ \frac{2j-N-1}{2} \,\middle\vert\, \begin{smallmatrix}\textit{odd } k;\\ 2-j-\ell\not\equiv_m 0,p\end{smallmatrix}\right\}$ $\kappa_1+\kappa_m \notin \left\{\frac{\lvert\lambda_6-k+1/2\rvert}{p} \,\middle\vert\, 2-k-\ell \equiv_m p\right\}$ $\lvert\kappa_1-\kappa_m\rvert \notin \left\{\frac{\lvert\lambda_6-k+1/2\rvert}{p} \,\middle\vert\, 2-k-\ell \equiv_m 0\right\}$	$\kappa_0 < 1/2$ *or* $\kappa_0 > N/2+1/2$ $\kappa_1+\kappa_m < \min\left(\frac{\lvert\lambda_6-k+\frac{1}{2}\rvert}{p} \,\middle\vert\, 2-k-\ell \equiv_m p\right)$ $\lvert\kappa_1-\kappa_m\rvert < \min\left(\frac{\lvert\lambda_6-k+\frac{1}{2}\rvert}{p} \,\middle\vert\, 2-k-\ell \equiv_m 0\right)$

5.6.1 Preliminary results and ideas of the proofs

The proofs are straightforward, but long. They are constructive: in doing them, all the irreducible finite-dimensional representations are found, and the conditions are naturally derived from the constructions. The idea behind them is akin to the standard module construction, so the first step is to study representations of $\mathfrak{O}_{m,\kappa}$ by starting from an irreducible spin representation of the group $\widetilde{W}$. Note that unlike a semisimple Lie algebra or a rational Cherednik algebra, the algebra $\mathfrak{O}_{m,\kappa}$ does not have a triangular decomposition because the action of the group $\widetilde{W}$ interchanges L_+ and L_-, see equations (5.72) and (5.73).

However, let $\widetilde{W_0}$ denote the subgroup of $\widetilde{W}$ generated by the elements commuting with O_0. Then the associative subalgebra of $\mathfrak{O}_{m,\kappa}$ generated by L_-, L_+, O_0, O_{123} and $\widetilde{W_0}$ does exhibit a triangular decomposition. It depends on the reducibility of W. Proposition 6.4.17

of the following chapter formalises this observation.

Let U be an irreducible $\widetilde{W}$-representation. It decomposes into irreducible $\widetilde{W}_0$-representations, and thus specifically for the case at hand, into two one-dimensional representations, see Section 5.4. The elements O_0 and O_{123} commute with $\widetilde{W}_0$ and they act thus by scalar multiplication on $\widetilde{W}_0$-representations.

From there, we use the triangular decomposition of the subalgebra generated by L_-, L_+, O_0, O_{123} and $\widetilde{W}_0$ and then work out the action of the rest of the symmetry algebra $\mathfrak{O}_{m,\kappa}$.

We show that all the representations $L_{\lambda,\Lambda}(U)$ of Theorem 5.6.1 and Theorem 5.6.2 are obtained from the sets of eigenvectors given by Lemma 5.6.3, and we give the restrictions on the function κ by examining the action of the ladder operators on them. As we cover all the possible cases, a complete set of finite-dimensional irreducible representations of $\mathfrak{O}_{m,\kappa}$ is exhibited.

Lemma 5.6.3. *Let $\mathcal{V}$ be a finite-dimensional irreducible representation of $\mathfrak{O}_{m,\kappa}$. There exists a set of eigenvectors of O_0 and O_{123}*

$$\mathcal{B} = \left\{ v_k^+, v_k^- \;\middle|\; 0 \le k \le N \right\} \tag{5.77}$$

that generates $\mathcal{V}$, with each pair $\langle v_k^-, v_k^+ \rangle$ generating an irreducible spin representation of $\widetilde{W}$ and $L_+ v_0^+ = 0$, $L_- v_N^+ = 0$.

Proof. We begin by decomposing $\mathcal{V}$ into irreducible spin representations of $\widetilde{W}$ and exhibit an O_0- and O_{123}-eigenvector v_0^+ from the further decomposition into $\widetilde{W}_0$-representations that satisfies the condition of the lemma. We then show that putting $v_k^+ := L_-^k v_0^+$ and $v_k^- := \tilde{\sigma}_m v_k^+$ in the set $\mathcal{B}$ proves the lemma.

As $\mathcal{V}$ is a $\mathfrak{O}_{m,\kappa}$-representation, it is also a $\widetilde{W}$-representation. Furthermore, in its realisation in $\mathfrak{O}_{m,\kappa}$, a representation of $\widetilde{W}$ must be a spin representation: abstractly $\widetilde{W}$ has a commuting element z that acts as -1 or $+1$ on the representation, but the realisation forces z to act as -1. By Maschke's Theorem, the $\widetilde{W}$-representation $\mathcal{V}$ is expressible as a direct sum of irreducible spin representations of $\widetilde{W}$. From Theorem 5.4.2, all the irreducible spin representation of $\widetilde{W}$ are two-dimensional.

Each of the irreducible spin representations further decomposes as the sum of two one-dimensional $\widetilde{W}_0$-representations. Since O_0 and O_{123} commute with $\widetilde{W}_0$, they act as multiples of the identity on $\widetilde{W}_0$-representations. Let v be any such generator. The element $\tilde{\sigma}_m v$ generates another $\widetilde{W}_0$-representation and the pair $(\tilde{\sigma}_m v, v)$ generates an irreducible spin $\widetilde{W}$-representation. As L_+ and L_- form a pair of ladder operators with respect to O_0, we have that

$$O_0 L_\pm^k v = \left(\left[O_0, L_\pm^k\right] + L_\pm^k O_0\right) v = (\pm k L_\pm^k + L_\pm^k O_0) v. \tag{5.78}$$

Hence, $L_+^k v$ and $L_-^k v$ are also eigenvectors of O_0, with their eigenvalues respectively raised or lowered by k. Since $\mathcal{V}$ is finite-dimensional, one of the generators in the $\widetilde{W}_0$-decomposition must be annihilated by L_+; denote it by v_0^+. Let λ and Λ be the eigenvalues for O_0 and O_{123} of this element

$$O_0 v_0^+ = \lambda v_0^+, \qquad O_{123} v_0^+ = \Lambda v_0^+. \tag{5.79}$$

Applying L_- lowers the eigenvalue and changes the $\widetilde{W}_0$-representation, as is seen by (5.72). In particular, $v_k^+ := L_- v_0^+$ is an eigenvector of O_0 of eigenvalue $\lambda - k$

$$O_0 v_k^+ = (-k L_-^k + L_-^k O_0) v_0^+ = (\lambda - k) v_k^+. \tag{5.80}$$

As all the eigenvalues are distinct, there must be a N such that $L_- v_N^+ = 0$ since the representation is finite-dimensional. Hence, we have exhibited two elements v_0^+ and v_N^+ satisfying

$$L_+ v_0^+ = 0, \qquad L_- v_N^+ = L_-^{N+1} v_0^+ = 0. \tag{5.81}$$

Furthermore, $v_k^- := \tilde{\sigma}_m v_k^+$ is also an eigenvector of O_0 and O_{123}, indeed

$$O_0 v_k^- = -\tilde{\sigma}_m O_0 v_k^+ = (k - \lambda) v_k^-, \quad O_{123} v_k^- = \tilde{\sigma}_m O_{123} v_k^+ = \Lambda v_k^-. \tag{5.82}$$

There is thus a set of linearly independent eigenvectors of O_{123} and O_0

$$\mathcal{B} = \{v_k^+ \mid k = 0, \ldots, N\} \cup \{v_k^- \mid k = 0, \ldots, N\}. \tag{5.83}$$

It is a spanning set of $\mathcal{V}$ because $\mathcal{V}$ is irreducible. ∎

Remark that the eigenvectors v_k^+ all have distinct O_0-eigenvalues; it is however possible that v_j^- has the same eigenvalue as one v_k^+. Note also that $L_+^k v_0^- = (-1)^k v_k^-$ and so we can also write $v_k^- = (-1)^k L_+^k v_0^-$.

The proof of the lemma gives for each finite-dimensional representation of $\mathfrak{O}_{m,\kappa}$ a set of data: two eigenvalues λ, Λ and a spin $\widetilde{W}$-representation $U = \langle v_0^+, v_0^- \rangle$. Hence, we denote irreducible representations of $\mathfrak{O}_{m,\kappa}$ by $L_{\lambda,\Lambda}(V)$.

Note however that the lemma does not impose a unique choice of data (λ, Λ, U) to identify the representation $\mathcal{V}$. Indeed, we see immediately from (5.82) that there could have been another choice for v_0^+ and v_N^+ since

$$L_+ v_N^- = 0, \quad L_- v_0^- = 0, \tag{5.84}$$

and the two sets of data (λ, Λ, U) and $(N - \lambda, \Lambda, U' = \langle v_N^-, v_N^+ \rangle)$ refer to the same $\mathfrak{O}_{m,\kappa}$-representation. This is taken into account into the classification, see the cases to solve the system (5.137).

Furthermore on a representation $L_{\lambda,\Lambda}(V)$, we define unitarity with respect to the $*$-structure from Section 5.5.3. We define abstractly on $L_{\lambda,\Lambda}(V)$ a sesquilinear form

$$\langle -, - \rangle : L_{\lambda,\Lambda}(V) \times L_{\lambda,\Lambda}(V) \to \mathbb{C} \tag{5.85}$$

extending the unitary structure of U normalized by $\left\langle v_0^+, v_0^+ \right\rangle = 1$, such that, for all $X \in \mathfrak{O}_{m,\kappa}$ and $v, w \in L_{\lambda,\Lambda}(V)$,

$$\langle Xv, w \rangle = \langle v, X^* w \rangle. \tag{5.86}$$

The next lemma gives a condition on unitarity assuming a specific form for the action of L_+. (It will be proven below that indeed L_+ acts like this.)

Lemma 5.6.4 (Unitarity condition)**.** *If L_+ acts on v_k^+ as $L_+ v_k^+ = A(k) v_{k-1}^+$, for certain constants $A(k)$, then $L_{\lambda,\Lambda}(V)$ is unitary when $A(k) > 0$ for $1 \leq k \leq N$.*

Proof. As $O_0^* = O_0$, we can use (5.80) and (5.82),

$$\left\langle v_k^+, v_l^+ \right\rangle = h_k \delta_{k,l} = \left\langle v_k^-, v_l^- \right\rangle,$$

$$\left\langle v_k^+, v_l^- \right\rangle = 0, \quad \langle v_0^+, v_0^+ \rangle = h_0 = 1.$$

The fact that $L_\pm^* = L_\mp$ gives a recursive structure for the h_k linked with $A(k)$:

$$\begin{aligned} h_k &:= \left\langle v_k^+, v_k^+ \right\rangle = \left\langle L_- v_{k-1}^+, v_k^+ \right\rangle = \left\langle v_{k-1}^+, L_+ v_k^+ \right\rangle \\ &= \left\langle v_{k-1}^+, A(k) v_{k-1}^+ \right\rangle = A(k) h_{k-1}. \end{aligned}$$

Therefore, to have an inner product and unitarity, it must be that $A(k) > 0$ for $1 \leq k \leq N$. ■

Assume that $L_+ v_k^+ = A(k) v_{k-1}^+$ and $L_- v_k^- = A(k) v_{k-1}^-$ for certain $A(k)$ (as will be proved later). For both even and odd case, the proof of Lemma 5.6.4 indicates that there is an orthonormal basis, provided the representation is unitary.

5.6.2 Proof of Theorem 5.6.1

We will show that the set $\mathcal{B}$ of Lemma 5.6.3 is a basis of a $2N+2$ irreducible representation of $\mathfrak{O}_{m,\kappa}$ under the conditions of Theorem 5.6.1 on κ and characterised by λ, Λ and a spin $\widetilde{W}$-representation.

Let $m = 2p+1$, $N \in \mathbb{N}$, $\ell \in \{0,1,\ldots,p\}$ and $\delta \in \{-1,+1\}$. Put $U = Y_\ell(-1,\delta)$, an irreducible spin representation of $\widetilde{W}$.

Consider the standard $\mathfrak{O}_{m,\kappa}$-representation $L_{\lambda,\Lambda}(V) := \mathfrak{O}_{m,\kappa} \otimes U$ from the proof of Lemma 5.6.3. Its generating set of eigenvectors of O_0 and O_{123} is

$$\mathcal{B} = \{v_k^+ := L_-^k v_0 \mid k = 0,\ldots,N\} \cup \{v_k^- := \tilde{\sigma}_m v_k^+ \mid k = 0,\ldots,N\}. \qquad (5.87)$$

Recall that $L_+^k v_0^- = (-1)^k v_k^-$ and so $v_k^- = (-1)^k L_+^k v_0^-$. The actions of O_{123} and O_0 on the generators are

$$O_0 v_k^\pm = \pm(\lambda - k) v_k^\pm, \qquad O_{123} v_k^\pm = \Lambda v_k^\pm. \qquad (5.88)$$

The action of $\tilde{\tau}$, $\tilde{\sigma}_m$ and $\tilde{\sigma}_0$ on U is given in Theorem 5.4.2. They extend to give the actions of $\tilde{\tau}$, $\tilde{\sigma}_m$ and $\tilde{\sigma}_0$ on the vectors $v_k^\pm$:

$$\tilde{\tau} v_k^+ = \zeta^{2k} L_-^k \tilde{\tau} v_0 = \zeta^{2k+2\ell} v_k^+; \qquad \tilde{\tau} v_k^- = (-1)^k \zeta^{-2k} L_+^k \tilde{\tau} v_0^- = \zeta^{-2(k+\ell)} v_k^-;$$
$$\tilde{\sigma}_m v_k^+ = (-1)^k L_+^k \tilde{\sigma}_m v_0^+ = v_k^-; \qquad \tilde{\sigma}_m v_k^- = (-1)^{2k} L_-^k \tilde{\sigma}_m v_0^- = v_k^+;$$

$$\tilde{\sigma}_0 v_k^+ = (-1)^k L_-^k \tilde{\sigma}_0 v_0 = (-1)^k \delta v_k^+; \quad \tilde{\sigma}_0 v_k^- = (-1)^{2k} L_+^k \tilde{\sigma}_0 v_0^- = (-1)^{k+1} \delta v_k^-.$$

Because $\tilde{\sigma}_j = (-1)^{j+1} \tilde{\tau}^j \tilde{\sigma}_m$, we have

$$\tilde{\sigma}_j v_k^\pm = (-1)^{j+1} \tilde{\tau}^j \tilde{\sigma}_m v_k^\pm = (-1)^{j+1} \zeta^{\mp 2(k+\ell)} v_k^\mp. \tag{5.89}$$

The actions of the operators T_0, T_+ and T_- are then given by

$$\begin{aligned}
T_0 v_k^\pm &= i O_3 v_k^\pm = \pm i(-1)^k \kappa_0 \delta v_k^\pm, \\
T_+ v_k^\pm &= -i\kappa_1 \left(\sum_{j=1}^{m} (-1)^{j+1} \zeta^{j(1 \mp 2(k+\ell))} \right) v_k^\mp, \\
T_- v_k^\pm &= i\kappa_1 \left(\sum_{j=1}^{m} (-1)^{j+1} \zeta^{j(-1 \mp 2(k+\ell))} \right) v_k^\mp.
\end{aligned}$$

The sum of roots of unity either gives m if the exponent of ζ is divisible by m, or 0 as then the sum covers all the roots of unity over the circle. Denote the function

$$1_m(2x+1) = \frac{1}{m} \sum_{j=1}^{m} (-1)^j \zeta^{j(2x+1)} = \begin{cases} 1, & 2x+1 \equiv_m 0; \\ 0, & \text{else.} \end{cases} \tag{5.90}$$

So the actions of T_+ and T_- are expressed by

$$T_+ v_k^\pm = -i\kappa_1 m 1_m(1 \mp 2(k+\ell)) v_k^\mp, \quad T_- v_k^\pm = i\kappa_1 m 1_m(-1 \mp 2(k+\ell)) v_k^\mp.$$

That gives expressions for the combinations T_+T_- and T_-T_+:

$$\begin{aligned}
T_-T_+ v_k^\pm &= \kappa_1^2 m^2 1_m(-1 \pm 2(k+\ell)) 1_m(1 \mp 2(k+\ell)) v_k^\pm \\
&= \kappa_1^2 m^2 1_m(1 \mp 2(k+\ell)) v_k^\pm, \\
T_+T_- v_k^\pm &= \kappa_1^2 m^2 1_m(1 \pm 2(k+\ell)) 1_m(-1 \mp 2(k+\ell)) v_k^\pm \\
&= \kappa_1^2 m^2 1_m(1 \pm 2(k+\ell)) v_k^\pm.
\end{aligned}$$

Use the factorisations (5.67) and (5.68) to give the actions of L_+ on v_k^+ and of L_- on v_k^-:

$$\begin{aligned}
L_+ v_k^+ &= L_+ L_- v_{k-1}^+ \\
&= -\Big((O_0 - 1/2)^2 + (O_{123} + iO_3)^2\Big)\Big((O_0 - 1/2)^2 - T_+T_-\Big) v_{k-1}^+
\end{aligned}$$

$$
\begin{aligned}
&= -((\lambda - k + 1/2)^2 + (\Lambda + (-1)^{k-1} i\kappa_0\delta)^2)\times \\
&\qquad ((\lambda - k + 1/2)^2 - \kappa_1^2 m^2 1_m (2(k+\ell) - 1))v_{k-1}^+,
\end{aligned}
$$

and for $L_- v_k^-$,

$$
\begin{aligned}
L_- v_k^- &= -L_- L_+ v_{k-1}^- \\
&= \Big((O_0 + 1/2)^2 + (O_{123} - iO_3)^2\Big)\times \\
&\qquad \Big((O_0 + 1/2)^2 - T_- T_+\Big) v_{k-1}^- \\
&= ((\lambda - k + 1/2)^2 + (\Lambda + (-1)^{k-1} i\kappa_0\delta)^2)\times \\
&\qquad ((\lambda - k + 1/2)^2 - \kappa_1^2 m^2 1_m (2(k+\ell) - 1))v_{k-1}^+.
\end{aligned}
$$

So denote

$$
\begin{aligned}
A(k) &:= A_{(1)}(k) A_{(2)}(k) \quad \text{with} \\
A_{(1)}(k) &:= -\Big((\lambda - k + 1/2)^2 + (\Lambda - (-1)^k i\kappa_0\delta)^2\Big) \\
A_{(2)}(k) &:= \Big((\lambda - k + 1/2)^2 - \kappa_1^2 m^2 1_m (2(k+\ell) - 1)\Big).
\end{aligned}
\tag{5.91}
$$

Then the actions of L_+ and L_- on v_k^+ and v_k^- are denoted succinctly as

$$
\begin{aligned}
L_+ v_k^+ &= A(k) v_{k-1}^+ = L_+ L_- v_{k-1}^+, & L_- v_k^- &= -A(k) v_{k-1}^- = -L_- L_+ v_{k-1}^-, \\
L_+ L_- v_k^- &= A(k) v_k^-, & L_- L_+ v_k^+ &= A(k) v_k^+.
\end{aligned}
$$

Remark that the conditions of Lemma 5.6.4 are satisfied.

The actions of $O_\pm$ are obtained from the expression of $L_\pm$ (5.65). Indeed, we have

$$
\begin{aligned}
L_\pm &= 1/2\,\{O_0, O_\pm\} = O_\pm O_0 + 1/2\,[O_0, O_\pm] \\
&= O_\pm O_0 \pm 1/2\, O_\pm + (O_{123} \pm iO_3) T_\pm.
\end{aligned}
\tag{5.92}
$$

And so, after some computations, the actions are

$$
O_- v_k^+ = \frac{v_{k+1}^+ - 2i(\Lambda - (-1)^{k+1} i\kappa_0\delta)\kappa_1 m 1_m (1 - 2(k+\ell)) v_k^-}{\lambda - k - 1/2}, \tag{5.93}
$$

$$
O_- v_k^- = \frac{-A(k) v_{k-1}^- - 2i(\Lambda - i(-1)^k \kappa_0\delta)\kappa_1 m 1_m (2(k+\ell) - 1) v_k^+}{k - \lambda - 1/2}, \tag{5.94}
$$

$$
O_+ v_k^+ = \frac{A(k) v_{k-1}^+ + 2i(\Lambda + i(-1)^{k+1} \kappa_0\delta)\kappa_1 m 1_m (1 - 2(k+\ell)) v_k^-}{\lambda - k + 1/2}, \tag{5.95}
$$

$$O_+v_k^- = \frac{-v_{k+1}^- + 2i(\Lambda + i(-1)^k\kappa_0\delta)\kappa_1 m 1_m(1+2(k+\ell))v_k^+}{k-\lambda+1/2}, \tag{5.96}$$

with the conditions $\lambda \neq k+1/2$ for (5.93); $\lambda \neq 1/2-k$ for (5.94); $\lambda \neq k-1/2$ for (5.95), and $\lambda \neq k+1/2$ for (5.96).

These actions enable us to conclude that $\mathcal{B}$ is a basis.

The representation $L_{\lambda,\Lambda}(V)$ will be irreducible if $A(k) \neq 0$ for $1 \leq k \leq N$ since it then means that all elements of $\mathcal{V}$ can be reached by acting with ladder operators. Furthermore, we know that $L_+v_0^+ = 0$ and $L_-v_N^+ = 0$ and this can be translated as conditions on $A(0)$ and $A(N+1)$:

$$\begin{cases} L_-L_+v_0^+ = A(0)v_0^+ = 0, \\ L_+L_-v_N^+ = A(N+1)v_N^+ = 0. \end{cases} \tag{5.97}$$

This gives the system of equations

$$\begin{cases} ((\lambda+1/2)^2 + (\Lambda - i\kappa_0\delta)^2)((\lambda+1/2)^2 - \kappa_1^2 m^2 1_m(2\ell-1)) = 0, \\ \Big((\lambda-N-1/2)^2 + (\Lambda+(-1)^N i\kappa_0\delta)^2\Big)\times \\ \qquad \Big((\lambda-N-1/2)^2 - \kappa_1^2 m^2 1_m(2(N+\ell)+1)\Big) = 0, \\ A(k) \neq 0, \quad 1 \leq k \leq N. \end{cases} \tag{5.98}$$

We now solve the system (5.98). There are three cases to be considered; the other possibilities reduce to one of these and the verification is left to the reader.

5.6.2.1 **Type I**: $1_m(2(N+\ell)+1) = 1$, $(\lambda+1/2)^2 + (\Lambda - i\kappa_0\delta)^2 = 0$, $(\lambda - N - 1/2)^2 - \kappa_1 m^2 = 0$;

5.6.2.2 **Type II**: $1_m(2(N+\ell)+1) = 0$, $(\lambda+1/2)^2 + (\Lambda - i\kappa_0\delta)^2 = 0$, $(\lambda - N - 1/2)^2 = 0$;

5.6.2.3 **Type III**: $(\lambda + 1/2)^2 + (\Lambda + i\kappa_0\delta)^2 = 0$, $(\lambda - N - 1/2)^2 + (\Lambda + (-1)^N i\kappa_0\delta)^2 = 0$.

The first two depend on the relation between N and U. The last one will only be possible when N is even. The value of Λ will always depend on λ and δ, in particular:

$$\Lambda_1 = i(\lambda + 1/2 + \kappa_0\delta), \qquad \text{or} \qquad \Lambda_2 = -i(\lambda + 1/2 - \kappa_0\delta). \tag{5.99}$$

Note that the pairs (Λ_1, δ) and $(-\Lambda_2, -\delta)$ yield the same irreducible representation. We can thus only consider Λ_1 in what follows.

5.6.2.1 Cases of type I

When $2(N+\ell)+1 \equiv_m 0$, the system (5.98) is solved through taking

$$(\lambda + 1/2)^2 + (\Lambda - i\kappa_0\delta)^2 = 0; \tag{5.100}$$
$$(\lambda - N - 1/2)^2 - \kappa_1^2 m^2 = 0. \tag{5.101}$$

There are two possibilities for λ, namely

$$\lambda_1 = N + 1/2 + \kappa_1 m, \qquad \text{or} \qquad \lambda_2 = N + 1/2 - \kappa_1 m. \tag{5.102}$$

First option: $\boldsymbol{\lambda_1 = N+1/2+\kappa_1 m}$**.** The O_0-eigenvalues are then

$$O_0 v_k^+ = (N + 1/2 + \kappa_1 m - k)v_k^+, \qquad O_0 v_k^- = (k - N - 1/2 - \kappa_1 m)v_k^-.$$

They are all distinct because $\kappa_1 > 0$.

Take $\Lambda_1 = i(\lambda_1 + 1/2 + \kappa_0\delta)$ and consider the equation $A(k) = 0$. It is evident that for this λ_1, the second factor $A_{(2)}(k) > 0$. Study the first factor $A_{(1)}(k)$. When k is even, then $(-1)^{k-1} = -1$ and

$$\begin{aligned} A_{(1)}(k) &= (\lambda_1 - k + 1/2)^2 + (\Lambda_1 - i\kappa_0\delta)^2 \\ &= (N - k + \kappa_1 m + 1)^2 - (N + \kappa_1 m + 1)^2 < 0, \end{aligned}$$

so $A(k) > 0$ and in particular it is not 0.

When k is odd however,

$$\begin{aligned} A_{(1)}(k) &= (\lambda_1 - k + 1/2)^2 + (\Lambda_1 + i\kappa_0\delta)^2 \\ &= (N - k + \kappa_1 m + 1)^2 - (\lambda_1 + 1/2 + 2\kappa_0\delta)^2, \end{aligned}$$

so as long as $\kappa_0 \neq -k/(2\delta)$ or $\kappa_0 \neq -(\lambda_1 - k/2 + 1/2)/\delta$, it is not zero.

For Λ_2, similar computations show that $A(k) \neq 0$ as long as $\kappa_0 \notin \{k/2\delta, (\lambda_1 - k/2 + 1/2)/\delta \mid \text{odd } k\}$.

Finally, unitarity requires that $A(k) > 0$ for all $k \in \{1, \ldots, N\}$ by Lemma 5.6.4. It is satisfied for even k. For odd k, remark first that it is different according to the value of δ. When $\delta = -1$ then the conditions

$$0 < \kappa_0 < 1/2 \qquad \text{or} \qquad \kappa_0 > \lambda_1. \tag{5.103}$$

ensures $(\lambda_1 - k + 1/2)^2 - (\lambda_1 + 1/2 - 2\kappa_0)^2 < 0$ for all $1 \le k \le N$, and so $A(k) > 0$ for all $1 \le k \le N$. When $\delta = 1$ then it will always be that $(\lambda - k + 1/2)^2 - (\lambda + 1/2 + 2\kappa_0)^2 < 0$ as $\kappa_0 > 0$, there are thus no restriction for unitarity here.

For Λ_2 the unitarity analysis condition switches: when $\delta = -1$, it is always unitary, and when $\delta = 1$, it is unitary provided $\kappa_0 < 1/2$ or $\kappa_0 > \lambda_1$.

Second option: $\boldsymbol{\lambda_2 = N + 1/2 - \kappa_1 m}$**.** In this case, the eigenvalues of O_0 may not be all distinct

$$O_0 v_k^+ = (N + 1/2 - \kappa_1 m - k) v_k^+, \qquad O_0 v_k^- = (k - N - 1/2 + \kappa_1 m) v_k^-.$$

If $\kappa_1 = (2N + 1 - k - j)/2m$, the elements v_k^+ and v_j^- share the same eigenvalue. They would be linearly independent if the action of $\tilde{\sigma}_0$ would be different, that is if k and j have the same parity. For the action of $\tilde{\tau}$, they share eigenvalue if $k + j \equiv_m 2\ell$. The value of κ_1 is thus restricted by

$$\kappa_1 \neq \left\{ \tfrac{2N+1-(k+j)}{2m} \;\middle|\; j + k \equiv_2 1;\ j + k \equiv_m 2\ell \right\}. \tag{5.104}$$

The condition $A(k) \neq 0$ prevents κ_0 from taking some values for odd $1 \le k \le N$ by studying $A_{(1)}(k)$:

$$\kappa_0 \neq k/2\delta, \qquad \text{or} \qquad \kappa_0 \neq (\lambda_2 + 1/2 - k)/2\delta. \tag{5.105}$$

It also prevents some values of κ_1 by studying the factor $A_{(2)}(k)$:

$$\begin{cases} \kappa_1 \neq \frac{N-k+1}{2m}, & \text{for } k \text{ such that } 2(k+\ell) \equiv_m 1; \\ \kappa_1 \neq \frac{N-k+1}{m}, & \text{for other } k. \end{cases} \tag{5.106}$$

The cases covered by these two inequations contain the values (5.104) of κ_1 for which the O_0-, $\tilde{\sigma}_0$- and $\tilde{\tau}$-eigenvalues of v_k^+ and v_j^- are the same. We now have all conditions to ensure $A(k) \neq 0$.

To find the additional restrictions necessary for unitarity, we concentrate only on $\Lambda_1 = i(\lambda_2 + 1/2 + \kappa_0 \delta)$ as the previous case has shown Λ_2 follows from it.

First, when $\delta = 1$, then always $A_{(1)}(k) < 0$ because $\kappa_0 > 0$. So $A(k) > 0$ whenever $\kappa_1 < (N + 1 - k)/m$ for k such that $2(k + \ell) \equiv_m 1$ or $\kappa_1 <$

$(N+1-k)/2m$ for k such that $(2k+\ell) \not\equiv_m 1$. So $A(k) > 0$ for all k if $\kappa_1 < 1/2m$.

When $\delta = -1$, if k is even, $A_{(1)}(k) < 0$ and $A(k) > 0$ when $\kappa_1 < (N+1-k)/m$, for k such that $2(k+\ell) \equiv_m -1$, or $\kappa_1 < (N+1-k)/2m$, for other k. As $2(N+\ell) \equiv_m -1$, then to be unitarity it is necessary that $\kappa_1 < 1/m$. This condition means that, for odd k, we do not need to consider the possibility of $A(k) > 0$ by virtue of $A_{(1)}(k) > 0$ and $A_{(2)}(k) < 0$: this would lead to a contradiction. So, for odd k, still $\kappa_1 < (N+1-k)/m$ for $2(k+\ell) \equiv_m 1$ and $\kappa_1 < (N-1-k)/2m$ for other k; but additionally κ_0 must be restricted to $\kappa_0 < k/2$ or $\kappa_0 > \lambda_2 + 1/2 - k/2$. Combining everything means that unitarity will follow from $\kappa_1 < 1/2m$, and $\kappa_0 < 1/2$ or $\kappa_0 > \lambda_2$.

5.6.2.2 Cases of type II

For this type, $2(N+\ell) \not\equiv_m -1$ and equations (5.98) are

$$(\lambda + 1/2)^2 + (\Lambda - i\kappa_0\delta)^2 = 0; \tag{5.107}$$

$$(\lambda - N - 1/2)^2 = 0. \tag{5.108}$$

The second equation gives $\lambda_3 = N + 1/2$ and all the O_0-eigenvalues of $v_k^{\pm}$ are distinct. We study the conditions for $\Lambda_1 = i(\lambda_3 + 1/2 + \kappa_0\delta)$.

Consider both factors $A_{(1)}(k)$ and $A_{(2)}(k)$ of $A(k)$. The first one for k even is

$$\begin{aligned} A_{(1)}(k) &= (\lambda_3 - k + 1/2)^2 + (\Lambda_1 + (-1)^{k-1} i\kappa_0\delta)^2 \\ &= (N+1-k)^2 - (N+1)^2 < 0, \end{aligned}$$

so it is never 0. For odd k,

$$\begin{aligned} A_{(1)}(k) &= (\lambda_3 - k + 1/2)^2 + (\Lambda_1 + (-1)^{k-1} i\kappa_0\delta)^2 \\ &= (N+1-k)^2 - (N+1+2\kappa_0\delta)^2 \end{aligned}$$

and so, as long as $\kappa_0 \neq -k/(2\delta)$ or $\kappa_0 \neq -(N+1-k/2)/\delta$, it will not be zero.

However, the second term $A_{(2)}(k)$ may be zero in either cases, adding constraints on κ_1. Indeed, if $2(k+\ell) - 1 \equiv_m 0$, then

$$A_{(2)}(k) = (\lambda_3 - k + 1/2)^2 - \kappa_1^2 m^2 = (N+1-k)^2 - \kappa_1^2 m^2$$

and so $\kappa_1 \neq (N+1-k)/m$ when $2(k+\ell) \equiv_m 1$. For other k, $A_{(2)}(k) > 0$ as $A_{(2)}(k) = (\lambda_3 - k + 1/2)^2 > 0$.

To achieve unitarity, the two factors $A(k)$ must have different signs. The situation depends on the parity of k and on δ.

Start with $\delta = 1$. Then $A_{(1)}(k) < 0$ and so $A_{(2)}(k)$ must be positive: this happens when $\kappa_1 < (N+1-k)/m$ for the k such that $2(k+\ell) \equiv_m 1$. Unitarity happens then when $\kappa_1 < \min((N+1-k)/m \mid 2(k+\ell) \equiv_m 1)$.

For $\delta = -1$, then even k implies $A_{(1)}(k) < 0$ and so $A(k) > 0$ with the additional condition $\kappa_1 < (N+1-k)/m$, for k such that $2(k+\ell) \equiv_m 1$. For odd k, it is further required that $\kappa_0 < k/2$ or $\kappa_0 > \lambda_3 + 1/2 - k/2$, with still the same condition on κ_1. Therefore, $A(k) > 0$ for all $1 \leq k \leq N$ if $\kappa_1 < \min((N+1-k)/m \mid 2(k+\ell) \equiv_m 1)$, and $\kappa_0 < 1/2$ or $\kappa_0 > \lambda_3$.

5.6.2.3 Cases of type III

For those cases, the solutions to equations (5.98) are given by

$$(\lambda + 1/2)^2 + (\Lambda - i\kappa_0\delta)^2 = 0; \tag{5.109}$$

$$(\lambda - N - 1/2)^2 + (\Lambda + (-1)^N i\kappa_0\delta)^2 = 0. \tag{5.110}$$

We study them for $\Lambda_1 = i(\lambda + 1/2 + \kappa_0\delta)$. Equation (5.110) divides according to the parity of N. When it is even,

$$(\lambda - N - 1/2)^2 + (\Lambda_1 + (-1)^N i\kappa_0\delta)^2 = (\lambda - N - 1/2)^2 - (\lambda + 1/2 + 2\kappa_0\delta)^2,$$

and so $\lambda_4 = N/2 + \kappa_0\delta$.

When N is odd,

$$(\lambda - N - 1/2)^2 + (\Lambda + (-1)^N i\kappa_0\delta)^2 = (\lambda - N - 1/2)^2 - (\lambda + 1/2)^2,$$

and so $\lambda_5 = N/2$ solves the equation.

In the study of the S_3 case [DOV18b, Sect. 4.3.2], namely the subalgebra of $m = 3$ case without A_1 part, there was no representation for odd N due to incompatibility with the relation involving $[O_0, O_-]$ of Proposition 5.5.8. For the same reason, there will also be none in the present chapter.

We begin by showing that $\lambda_5 = N/2$ for odd N is not possible and then take care of the case $\lambda_4 = N/2 + \kappa_0\delta$ for even N.

Odd N ***and*** $\lambda_5 = N/2$**.** In this case, the integer $j_0 = (N-1)/2$ is such that

$$\begin{aligned} O_0 v_{j_0}^+ &= \tfrac{1}{2} v_{j_0}^+, & O_0 v_{j_0+1}^+ &= -\tfrac{1}{2} v_{j_0+1}^+, \\ O_0 v_{j_0}^- &= -\tfrac{1}{2} v_{j_0}^-, & O_0 v_{j_0+1}^- &= \tfrac{1}{2} v_{j_0+1}^-. \end{aligned} \tag{5.111}$$

We will show that the relation (5.60) is not respected, thus showing the impossibility of representation. The actions of O_+ and O_- previously found (equations (5.93)–(5.96)) do not work here because their denominator is 0. We can circumvent this by studying

$$v_{j_0+1}^+ = L_- v_{j_0}^+ = (O_- O_0 + \tfrac{1}{2}[O_0, O_-]) v_{j_0}^+ \tag{5.112}$$

$$= O_-(O_0 - \tfrac{1}{2}) v_{j_0}^+ + 2i(O_{123} - T_0) T_- v_{j_0}^+ \tag{5.113}$$

$$= i\kappa_1 m(\Lambda_1 - (-1)^{j_0}\kappa_0) 1_m(-1 + 2(j_0 + \ell)) v_{j_0}^- \tag{5.114}$$

This forces $1 + (2j_0 + \ell) \equiv_m 0$ as $v_{j_0+1}^+$ must not be zero. So $v_{j_0}^-$ is a multiple of $v_{j_0+1}^+$. But if this is the case, then studying $[O_0, O_-]$ leads to a contradiction. Indeed, by (5.60)

$$[O_0, O_-] v_{j_0}^+ = -O_- v_{j_0}^+ + 2(O_{123} - T_0) T_- v_{j_0}^+ \tag{5.115}$$

$$O_0 O_- v_{j_0}^+ - \tfrac{1}{2} O_- v_{j_0}^+ = -O_- v_{j_0}^+ + 2\kappa_1 m(\Lambda_1 - (-1)^{j_0}\kappa_0) v_{j_0}^- \tag{5.116}$$

$$O_0 O_- v_{j_0}^+ = -\tfrac{1}{2} O_- v_{j_0}^+ + 2 v_{j_0+1}^+. \tag{5.117}$$

However, from the two factorisations (5.63–5.64) we get that $O_- v_{j_0}^+ = a v_{j_0+1}^+$, which would force $v_{j_0+1}^+ = 0$ from the last equation, a contradiction. There are thus no representation in this case.

Even N ***and*** $\lambda_4 = N/2 + \kappa_0\delta$**.** The eigenvectors for O_0 are

$$O_0 v_k^+ = (N/2 + \kappa_0\delta - k) v_k^+, \qquad O_0 v_j^- = (j - N/2 - \kappa_0\delta) v_j^- \tag{5.118}$$

and so the eigenvalues of v_k^+ and v_j^- are the same if $\kappa_0 = (k + j - N)/2\delta$.

If the integers k and j have the same parity, then they are linearly independent because the action of $\tilde{\sigma}_0$ differs by a sign on them: $\tilde{\sigma}_0 v_k^+ = (-1)^k \delta v_k^+$ and $\tilde{\sigma}_0 v_j^- = -(-1)^j \delta$. If they do not have the same parity, then those values of κ_0 will be prohibited by conditions stemming from $A(k) \neq 0$, and thus all the vectors $v_k^\pm$ are linearly independent.

Consider the values of κ_0 and κ_1 for which $A(k) = 0$. The first factor of $A(k) = -A_{(1)}(k)A_{(2)}(k)$ is

$$A_{(1)}(k) = (\lambda_4 - k + 1/2)^2 - (\lambda_4 + 1/2 + \kappa_0\delta - (-1)^k\kappa_0\delta)^2 \qquad (5.119)$$

It is zero only for odd k if $\kappa_0 = -k/(2\delta)$ or $\kappa_0 = -(\lambda_4 + 1/2 - k/2)/\delta$. So κ_0 cannot be a half-integer $k/2$ for odd $1 \leq k \leq N-1$.

The second factor is

$$A_{(2)}(k) = (\lambda_4 - k + 1/2)^2 - \kappa_1^2 m^2 1_m(2(k+\ell) - 1). \qquad (5.120)$$

This is zero when $\kappa_1 = \pm(\lambda_4 - k + 1/2)/m$ for k such that $2(k+\ell) \equiv_m 1$, or when $\kappa_0 = (k - N/2 - 1/2)/\delta$ for other k. So $A(k) \neq 0$ if κ_0 and κ_1 avoid the previously-mentioned values.

The analysis of unitarity by the condition $A(k) > 0$ of Lemma 5.6.4 is easier done by considering $\delta = 1$ and $\delta = -1$ separately. First $\delta = 1$. In this case, the first factor $A_{(1)}(k)$ is always negative. The positivity of the second factor requires $\kappa_1 < |N/2 + \kappa_0 - k + 1/2|/m$ when k is such that $2(k+\ell) \equiv_m 1$. So the unitarity is guaranteed when $\kappa_1 < \min(|N/2 + \kappa_0 - k + 1/2|/m \mid 2(k+\ell) \equiv_m 1)$, and those are the only cases.

When $\delta = -1$ then for even k, always $A_{(1)}(k) < 0$, but for odd k, it requires $|N/2 - \kappa_0 + 1/2 - k| < |N/2 - 3\kappa_0 + 1/2|$ and so, to be true for all odd k, then either $\kappa_0 < 1/2$ or $\kappa_0 > N/2 + 1/2$. Along with this, the second factor $A_{(2)}(k)$ is positive for k such that $2(k+\ell) \equiv_m 1$ only when $\kappa_1 < |N/2 + \kappa_0 - k + 1/2|/m$.

The three cases cover all possibility and thus we have proven Theorem 5.6.1. ■

5.6.3 Proof of Theorem 5.6.2

We prove that the set $\mathcal{B}$ of Lemma 5.6.3 is a basis of a $(2N+2)$-dimensional irreducible representation of $\mathcal{O}_{m,\kappa}$ characterized by two constants Λ and λ under the conditions on κ of Theorem 5.6.2.

Let $m = 2p$, $N \in \mathbb{N}$, $\ell \in \{0, 1, \ldots, p-1\}$ and $\delta \in \{-1, +1\}$. Put $U = Y_{2\ell+1}(\delta)$ an irreducible spin representation of $\widetilde{W}$. By Lemma 5.6.3 and the discussion following it, consider the $\mathcal{O}_{m,\kappa}$-representation

$L_{\lambda,\Lambda}(V)$ with its generating set of $2N+2$ eigenvectors of O_0 and O_{123}

$$\mathcal{B} = \{v_k^+ := L_+^k v_0^+,\ v_k^- := \tilde{\sigma}_m v_k^+ \mid k = 0,\ldots,N\}. \tag{5.121}$$

The representation U is generated by v_0^+ and v_0^- and the O_0- and O_{123}-eigenvalues on $v_k^\pm$ are

$$O_0 v_k^\pm = \pm(\lambda - k) v_k^\pm, \qquad O_{123} v_k^\pm = \Lambda v_k^\pm. \tag{5.122}$$

The actions of the group elements on $v_k^\pm$ are given below, with the action on $v_0^\pm$ extracted from Theorem 5.4.2. Recall that $\zeta = e^{\pi i/m}$:

$$\begin{aligned}
\tilde{\tau} v_k^+ &= \zeta^{2k} L_-^k \tilde{\tau} v_0^+ = \zeta^{2(k+\ell)+1} v_k^+, & \tilde{\tau} v_k^- &= \zeta^{-2(k+\ell)-1} v_k^-,\\
\tilde{\sigma}_0 v_k^+ &= (-1)^k L_-^k \tilde{\sigma}_0 v_0^+ = (-1)^k \delta v_k^+, & \tilde{\sigma}_0 v_k^- &= (-1)^{2k} L_+^k \tilde{\sigma}_0 v_0^- = (-1)^{k+1} \delta v_k^-,\\
\tilde{\sigma}_m v_k^+ &= (-1)^k L_+^k \tilde{\sigma}_m v_0^+ = v_k^-, & \tilde{\sigma}_m v_k^- &= v_k^+.
\end{aligned}$$

Recall that $T_0 = i\kappa_0\tilde{\sigma}_0$ and so its action on $v_k^\pm$ is simply given by $T_0 v_k^\pm = \pm i(-1)^k \kappa_0 \delta v_k^\pm$. The actions of T_+ and T_- are obtained from $\tilde{\sigma}_j = (-1)^{j+1}\tilde{\tau}^j\tilde{\sigma}_m$ and equation (5.54), because m is even:

$$T_\pm v_k^+ = \mp i(\kappa_1 T_\pm^1 + \kappa_m T_\pm^2) v_k^+, \qquad T_\pm v_k^- = \mp i(\kappa_1 T_\pm^1 + \kappa_m T_\pm^2) v_k^-.$$

On the odd root components T_+^1 and T_-^1, the action is given by

$$\begin{aligned}
T_+^1 v_k^\pm &= \sum_{j=1}^p \zeta^{2j-1} \tilde{\sigma}_{2j-1} v_k^\pm = \sum_{j=1}^p \zeta^{2j-1} (-1)^{2j} \tilde{\tau}^{2j-1} \tilde{\sigma}_m v_k^\pm\\
&= \sum_{j=1}^p \zeta^{(2j-1)(1\mp 1\mp 2(k+\ell))} v_k^\mp,\\
T_-^1 v_k^\pm &= \sum_{j=1}^p \zeta^{(2j-1)(-1\mp 1\mp 2(k+\ell))} v_k^\mp,
\end{aligned}$$

and on the even roots components T_+^2 and T_-^2, by

$$T_+^2 v_k^\pm = \sum_{j=1}^p \zeta^{2j+1} \tilde{\sigma}_{2j} v_k^\pm = \sum_{j=1}^p (-1)^{2j} \zeta^{2j} \tilde{\tau}^{2j} \tilde{\sigma}_m v_k^\pm$$

$$
\begin{aligned}
&= -\sum_{j=1}^{p} \zeta^{2j(1\mp 1\mp 2(k+\ell))} v_k^{\mp}, \\
T_-^2 v_k^{\pm} &= -\sum_{j=1}^{p} \zeta^{2j(-1\mp 1\mp 2(k+\ell))} v_k^{\mp}.
\end{aligned}
$$

Define

$$
G_\kappa(X) := p(\kappa_1 1'_m(X) - \kappa_m 1_p(X)); \tag{5.123}
$$

$$
1_p(X) := \begin{cases} 1, & X \equiv_p 0; \\ 0, & \text{else}; \end{cases} \qquad 1'_m(X) := \begin{cases} -1, & X \equiv_m p; \\ 1, & X \equiv_m 0; \\ 0, & \text{else}. \end{cases}
$$

The actions of T_+ and T_- on v_k^- and v_k^+ are then expressed with this shorthand notation as

$$
T_+ v_k^- = -iG_\kappa(1-k-\ell)v_k^+, \qquad T_+ v_k^+ = -iG_\kappa(k+\ell)v_k^-, \tag{5.124}
$$

$$
T_- v_k^- = iG_\kappa(k+\ell)v_k^+, \qquad T_- v_k^+ = iG_\kappa(1-k-\ell)v_k^-. \tag{5.125}
$$

For ease of notation, define

$$
H_\kappa(x) := p^2((\kappa_1^2 + \kappa_m^2)1_p(x) - 2\kappa_1\kappa_m 1'_m(x)) = \begin{cases} p^2(\kappa_1+\kappa_m)^2, & x \equiv_m p; \\ p^2(\kappa_1-\kappa_m)^2, & x \equiv_m 0; \\ 0, & \text{else}. \end{cases}
$$

The actions of T_+T_- and T_-T_+ on v_k^+ and v_k^- are given below

$$
T_-T_+ v_k^+ = H_\kappa(k+\ell)v_k^+, \qquad T_-T_+ v_k^- = H_\kappa(1-k-\ell)v_k^-, \tag{5.126}
$$

$$
T_+T_- v_k^+ = H_\kappa(1-k-\ell)v_k^+, \qquad T_+T_- v_k^- = H_\kappa(k+\ell)v_k^-. \tag{5.127}
$$

We now employ the factorisations (5.67) and (5.68) to get conditions on the actions of L_+ and L_- on the vectors v_k^+ and v_k^-:

$$
\begin{aligned}
L_+ v_k^+ &= L_+ L_- v_{k-1}^+ \\
&= -\Big((O_0 - 1/2)^2 + (O_{123} + iO_3)^2\Big)\Big((O_0 - 1/2)^2 - T_+T_-\Big)v_{k-1}^+ \\
&= -\Big(\Big((\lambda - k + 1/2)^2 + (\Lambda + (-1)^{k-1} i\kappa_0\delta)^2\Big) \times \\
&\qquad \Big((\lambda - k + 1/2)^2 - H_\kappa(2-(k+\ell))\Big)\Big)v_{k-1}^+,
\end{aligned}
$$

and

$$
\begin{aligned}
L_- v_k^- &= -L_- L_+ v_{k-1}^- \\
&= \Big((O_0+1/2)^2 + (O_{123} - iO_3)^2\Big)\Big((O_0+1/2)^2 - T_- T_+\Big) v_{k-1}^- \\
&= \Big(\Big((k-\lambda-1/2)^2 + (\Lambda + (-1)^{k-1} i\kappa_0\delta)^2\Big) \times \\
&\qquad \Big((k-\lambda-1/2)^2 - H_\kappa(2-(k+\ell))\Big)\Big) v_{k-1}^-.
\end{aligned}
$$

Put

$$A(k) := A_{(1)}(k) A_{(2)}(k), \quad \text{with} \tag{5.128}$$

$$A_{(1)}(k) := \Big((\lambda-k+1/2)^2 + (\Lambda - (-1)^k i\kappa_0\delta)^2\Big) \tag{5.129}$$

$$A_{(2)}(k) := \Big((\lambda-k+1/2)^2 - H_\kappa(2-k-\ell)\Big). \tag{5.130}$$

So the actions are simply

$$L_+ v_k^+ = A(k) v_{k-1}^+, \qquad L_- v_k^- = -A(k) v_{k-1}^-, \tag{5.131}$$

$$L_+ L_- v_k^- = A(k) v_k^-, \qquad L_- L_+ v_k^+ = A(k) v_k^+. \tag{5.132}$$

The actions of O_+ and O_- follow like the previous case from expressing $L_\pm = O_\pm O_0 + [O_0, O_\pm]/2$:

$$O_- v_k^+ = \frac{v_{k+1}^+ - 2i(\Lambda - (-1)^{k+1} i\kappa_0\delta) G_\kappa(1-k-\ell) v_k^-}{\lambda - k - 1/2}, \tag{5.133}$$

$$O_- v_k^- = \frac{-A(k) v_{k-1}^- - 2i(\Lambda - i(-1)^k \kappa_0\delta) G_\kappa(k+\ell) v_k^+}{k - \lambda - 1/2}, \tag{5.134}$$

$$O_+ v_k^+ = \frac{A(k) v_{k-1}^+ + 2i(\Lambda + i(-1)^{k+1} \kappa_0\delta) G_\kappa(k+\ell) v_k^-}{\lambda - k + 1/2}, \tag{5.135}$$

$$O_+ v_k^- = \frac{-v_{k+1}^- + 2i(\Lambda + i(-1)^k \kappa_0\delta) G_\kappa(1-k-\ell) v_k^+}{k - \lambda + 1/2}, \tag{5.136}$$

with the restrictions $\lambda \neq k + 1/2$ for (5.133); $\lambda \neq 1/2 - k$ for (5.134); $\lambda \neq k - 1/2$ for (5.135), and $\lambda \neq k + 1/2$ for (5.136).

The system to solve for the irreducibility of $L_{\lambda,\Lambda}(V)$ is

$$
\begin{cases}
\Big((\lambda+1/2)^2 + (\Lambda - i\kappa_0\delta)^2\Big)\Big((\lambda+1/2)^2 - H_\kappa(2-\ell)\Big) = 0, \\
\Big((\lambda-N-1/2)^2 + (\Lambda + (-1)^N i\kappa_0\delta)^2\Big) \\
\quad \Big((\lambda-N-1/2)^2 - H_\kappa(1-N-\ell)\Big) = 0, \\
A(k) \neq 0, \quad 1 \leq k \leq N.
\end{cases}
\tag{5.137}
$$

The values of ℓ and N influence the value of $H_\kappa(1-N-\ell)$ and justify the division in the following types. All other ways to solve the first two equations of (5.137) are equivalent to one of these by a renaming of the generators $v_k^\pm$, see the discussion around equation (5.84).

5.6.3.1 **Type I.i**: $1-N-\ell \equiv_m p$; $(\lambda+1/2)^2 = -(\Lambda - i\kappa_0\delta)^2$, and $(\lambda - N - 1/2)^2 = H_\kappa(1-N-\ell)$.

5.6.3.2 **Type I.ii**: $1-N-\ell \equiv_m 0$; $(\lambda+1/2)^2 = -(\Lambda - i\kappa_0\delta)^2$, and $(\lambda - N - 1/2)^2 = H_\kappa(1-N-\ell)$.

5.6.3.3 **Type II**: $H_\kappa(1-N-\ell) = 0$; $(\lambda+1/2)^2 = -(\Lambda - i\kappa_0\delta)^2$, and $N - \lambda + 1/2 = 0$.

5.6.3.4 **Type III**: $(\lambda+1/2)^2 = -(\Lambda - i\kappa_0\delta)^2$, and $(\lambda - N - 1/2)^2 = -(\Lambda + (-1)^N i\kappa_0\delta)^2$.

The choice of these specific types is simply to normalize the expressions of Λ as either

$$\Lambda_1 = i(\lambda + 1/2 + \kappa_0\delta), \qquad \text{or} \qquad \Lambda_2 = -i(\lambda + 1/2 - \kappa_0\delta). \tag{5.138}$$

The two possibilities exist for Λ, but the representations are the same under the switch $(\Lambda_1, \delta) \to (-\Lambda_2, -\delta)$ so we will always only consider $\Lambda_1 = i(\lambda + 1/2 + \kappa_0\delta)$.

5.6.3.1 Cases of type I.i

The condition on $H_\kappa(1-N-\ell)$ is equivalent to $N + \ell \equiv_m 1 - p$.

There are two possible values for λ:

$$\lambda_1 = N + 1/2 + (\kappa_1 + \kappa_m)p, \qquad \lambda_2 = N + 1/2 - (\kappa_1 + \kappa_m)p. \tag{5.139}$$

First option: $\lambda_1 = N + 1/2 + (\kappa_1 + \kappa_m)p$**.** All the eigenvectors $v_k^\pm$ have different eigenvalues. The condition $A(k) \neq 0$ is achieved with only some conditions on κ_0. Indeed, the positivity conditions $\kappa_1, \kappa_m > 0$ implies that $(\kappa_1 + \kappa_m)^2 > (\kappa_1 - \kappa_m)^2$ and thus the second factor $A_{(2)}(k)$ of $A(k)$ is always positive:

$$\begin{aligned} A_{(2)}(k) &= (\lambda_1 - k + 1/2)^2 - H_\kappa(2-k-\ell) \\ &= (N - k + 1 + (\kappa_1 + \kappa_m)p)^2 - H_\kappa(2-k-\ell) > 0. \end{aligned} \tag{5.140}$$

The first factor $A_{(1)}(k) = (\lambda_1 - k + 1/2)^2 + (\Lambda + (-1)^{k+1} i\kappa_0\delta)^2$ is always negative for even k, but it is zero if $\kappa_0 = -k/(2\delta)$ or $\kappa_0 = -(2\lambda - k + 1)/(2\delta)$ for odd k.

When $\delta = 1$, the representation will be unitary without restriction. When $\delta = -1$, it will be unitary if $\kappa_0 < 1/2$ or $\kappa_0 > \lambda_1$.

Second option: $\lambda_2 = N + 1/2 - (\kappa_1 + \kappa_m)p$**.** As

$$\begin{aligned} O_0 v_k^+ &= (N - k + /2 - (\kappa_1 + \kappa_m)p)v_k^+, \\ O_0 v_j^- &= (j - N - 1/2 + (\kappa_1 + \kappa_m)p)v_j^-, \end{aligned}$$

then v_k^+ and v_j^- will have the same O_0-eigenvalue when $\kappa_1 + \kappa_2 = (2N + 1 - k - j)/2p$, values. Then this will imply $A(k) = 0$.

In addition to the conditions on κ_0, $\kappa_0 < 1/2$ or $\kappa_0 > \lambda_1$ for $\delta = -1$, some conditions on κ_1 and κ_m appear from the irreducibility condition $A(k) \neq 0$. The factor $A_{(1)}(k)$ is not zero as long as $\kappa_0 \neq -k/2\delta$ or $\kappa_0 \neq -(N - k + 1/2)/\delta$ for odd k, but it might be the case that the second factor $A_{(2)}(k)$ becomes 0.

For $A_{(2)}(k) = 0$, the following equation must hold

$$(N - k + 1 - (\kappa_1 + \kappa_m)p)^2 - H_\kappa(2 - k - \ell) = 0.$$

According to the value of k, we have to solve the following system of equations

$$\begin{cases} N - k + 1 - (\kappa_1 + \kappa_m)p = \pm(\kappa_1 - \kappa_m)p, & 2 - k - \ell \equiv_m 0; \\ N - k + 1 - (\kappa_1 + \kappa_m)p = \pm(\kappa_1 + \kappa_m)p, & 2 - k - \ell \equiv_m p; \\ N - k + 1 - (\kappa_1 + \kappa_m)p = 0, & \text{else}; \end{cases} \tag{5.141}$$

and so $L_{\lambda,\Lambda}(V)$ is not irreducible when

$$\begin{cases} \kappa_1 = (N - k + 1)/(2p),\ \kappa_m = (N - k + 1)/(2p), & 2 - k - \ell \equiv_m 0; \\ \kappa_1 + \kappa_m = (N - k + 1)/(2p), & 2 - k - \ell \equiv_m p; \\ \kappa_1 + \kappa_m = (N - k + 1)/p, & \text{else}. \end{cases}$$

Lemma 5.6.4 states that the representation will be unitary when all the $A(k) > 0$. If $\delta = 1$, it suffices that $\kappa_0 > 0$ and $\kappa_1 + \kappa_m < 1/p$. When $\delta = -1$, sufficient conditions for that are: $0 < \kappa_0 < 1/2$ or $\kappa_0 > N - 1/2$

with $\kappa_1 + \kappa_m < 1/p$. Indeed, $A_{(2)}(k) > 0$, and so $A(k) > 0$, under the following restrictions:

$$\begin{cases} \kappa_1, \kappa_m > (N-k+1)/2p \text{ or } \kappa_1, \kappa_m < (N-k+1)/2p, & 2-k-\ell \equiv_m 0; \\ \kappa_1 + \kappa_m < (N-k+1)/2p, & 2-k-\ell \equiv_m p; \\ \kappa_1 + \kappa_m < (N-k+1)/p, & \text{else}; \end{cases}$$

and as $2-N-\ell \not\equiv_m p, 0$, then the condition $\kappa_1 + \kappa_m < 1/p$ is sufficient for the inequality $A(k) > 0$ to hold for all k.

5.6.3.2 Cases of type I.ii

The condition $H_\kappa(1-N-\ell) = (\kappa_1 - \kappa_m)p$ is equivalent to $N + \ell \equiv_m 1$. There are two possibilities for λ, namely $\lambda_3 = N + 1/2 + (\kappa_1 - \kappa_m)p$ or $\lambda_4 = N + 1/2 - (\kappa_1 - \kappa_m)p$, and for each of these, we study $\Lambda_1 = i(\lambda_j + 1/2 + \kappa_0\delta)$. We assume here that $\kappa_1 > \kappa_m$ as, if it is not the case, it suffices to switch the analysis of λ_3 and λ_4.

First option: $\lambda_3 = N + 1/2 + (\kappa_1 - \kappa_m)p$ There might be some O_0-eigenvectors with equal eigenvalues as

$$O_0 v_k^+ = N - k + 1/2 + (\kappa_1 - \kappa_m)p, \quad O_0 v_j^- = j - N - 1/2 - (\kappa_1 - \kappa_m)p,$$

and they are equal if $\kappa_m - \kappa_1 = (2N + 1 - k - j)/2p$.

The analysis on $A(k)$ proceeds similarly to the previous case, but with a slight difference on the second factor of $A(k)$. Indeed, if $2-k-\ell \equiv_m p$ then

$$\begin{aligned} A_{(2)}(k) &= (\lambda_3 - k + 1/2)^2 - H_\kappa(2-k-\ell) \\ &= (N-k+1+(\kappa_1-\kappa_m)p)^2 - (\kappa_1+\kappa_2)^2 p^2 \end{aligned}$$

and this is 0 if and only if

$$\kappa_1 = -(N-k+1)/2p, \qquad \kappa_m = (N-k+1)/2p.$$

Then $A_{(2)}(k)$ will be positive if $\kappa_1 > 0 > -(N-k+1)/2p$ and $0 < \kappa_m < (N-k+1)/2p$. For the other values of k, then always $A_{(2)}(k) > 0$. Naturally, $A_{(1)}(k) > 0$ for even k, and is zero when $\kappa_0 = -k/2\delta$ or $\kappa_0 = -(\lambda_3 - k + 1/2)/\delta$ for odd k.

When $\delta = 1$, the condition for unitarity is then that $\kappa_m < (N-k+1)/2p$ for the biggest k such that $2-k-\ell \equiv_m p$ and, if $\delta = -1$, that $\kappa_0 < 1/2$ or $\kappa_0 > \lambda_3$.

Second option: $\lambda_4 = N + 1/2 - (\kappa_1 - \kappa_m)p$**.** The eigenvalues of $v_k^\pm$ might be the same, as

$$O_0 v_k^+ = (N + \tfrac{1}{2} - k - (\kappa_1 - \kappa_m)p)v_k^+, \; O_0 v_j^- = (j - N - \tfrac{1}{2} + (\kappa_1 - \kappa_m)p)v_j^-,$$

and so they are the same if $\kappa_1 - \kappa_m = (2N - j - k + 1)/2p$.

We get the following conditions on κ_1 and κ_m by analysing when $A_{(2)}(k) = 0$, or equivalently when

$$\begin{aligned}(\lambda_4 - k + 1/2)^2 &= (N - k + 1 - (\kappa_1 - \kappa_m)p)^2 \\ &= \begin{cases}(\kappa_1 + \kappa_m)^2 p^2, & 2 - k - \ell \equiv_m p; \\ (\kappa_1 - \kappa_m)^2 p^2, & 2 - k - \ell \equiv_m 0; \\ 0, & \text{else.}\end{cases}\end{aligned}$$

It gives the following restrictions, for k such that $2 - k - \ell \equiv_m p$, j such that $2 - j - \ell \equiv_m 0$ and q such that $2 - q - \ell \not\equiv_m 0, p$

$$\begin{aligned}\kappa_1 &\neq (N - k + 1)/(2p), & \kappa_m &\neq -(N - k + 1)/(2p), \\ \kappa_1 - \kappa_m &\neq (N - j + 1)/(2p), & \kappa_1 - \kappa_m &\neq (N - q + 1)/p.\end{aligned}$$

It is unitary if furthermore $A(k) > 0$ for all k. This is achieved by requiring $\kappa_1 - \kappa_2 < 1/p$ and $0 < \kappa_1 < (N - k + 1)/2p$ for the biggest k such that $2 - k - \ell \equiv_m 0$, when $\delta = 1$, and by adding $0 < \kappa_0 < 1/2$ or $\kappa_0 > \lambda_4 - k + 1/2$, if $\delta = -1$. Note that the first condition ensures that $\lambda_4 > k$.

5.6.3.3 Cases of type II

This case results in

$$\lambda_5 = N + 1/2, \tag{5.142}$$

and we do the study for $\Lambda_1 = i(\lambda_5 + 1/2 + \kappa_0\delta)$. All the $v_k^\pm$ have different eigenvalues.

Similar analysis of $A(k) \neq 0$ gives $\kappa_0 \neq -k/2\delta$ and $\kappa_0 \neq -(\lambda - k + 1/2)/\delta$ for k odd if the representation is to be irreducible. Furthermore, $A_{(2)}(k)$ might be zero. Indeed, $(\lambda - k + 1/2)^2 = H_\kappa(2 - k - \ell)$ when

$$\begin{cases}(N - k + 1)^2 = (\kappa_1 + \kappa_2)^2 p^2, & 2 - k - \ell \equiv_m p; \\ (N - k + 1)^2 = (\kappa_1 - \kappa_2)^2 p^2, & 2 - k - \ell \equiv_m 0.\end{cases}$$

Or more precisely, we get the conditions

$$\begin{cases} \kappa_1 + \kappa_2 = (N-k+1)/p, & 2-k-\ell \equiv_m p; \\ |\kappa_1 - \kappa_2| = (N-k+1)/p, & 2-k-\ell \equiv_m 0. \end{cases} \tag{5.143}$$

The analysis proceeds in a similar fashion. The factor $A_{(2)}(k) > 0$ as soon as $\kappa_1 + \kappa_m < (N-k+1)/m$ for $2-k-\ell \equiv_m p$ or $|\kappa_1 - \kappa_m| < (N-k+1)/p$ for $2-k-\ell \equiv_m 0$, or without condition on κ_1 and κ_m for other k. For $\delta = 1$, then $A_{(1)}(k) < 0$. For $\delta = -1$ then $A_{(1)}(k) < 0$ for even k and if $\kappa_0 < k/2$ or $\kappa_0 > \lambda_5 + 1/2 - k/2$ for odd k. Taking the minimum, or the maximum, of those set will ensure that $A(k) > 0$ for all k.

5.6.3.4 Cases of type III

We study $\Lambda_1 = i(\lambda + 1/2 + \kappa_0\delta)$. The O_0-eigenvalue λ takes different values according to the parity of N. When N is even, then

$$(\lambda - N - 1/2)^2 + (\Lambda + i\kappa_0\delta)^2 = 0, \tag{5.144}$$

so $\lambda_6 = N/2 + \kappa_0\delta$. When N is odd, then $\lambda_7 = N/2$.

We begin by showing that $\lambda_7 = N/2$ for odd N does not happen by the same argument as the S_3 case [DOV18b, Sect. 4.3.2] and then continue with the case $\lambda_6 = N/2 + \kappa_0\delta$ for even N.

Odd N ***and*** $\lambda_7 = N/2$ If $\lambda_7 = N/2$, the integer $j_0 = (N-1)/2$ is such that

$$O_0 v_{j_0}^+ = \tfrac{1}{2} v_{j_0}^+, \quad O_0 v_{j_0+1}^+ = -\tfrac{1}{2} v_{j_0+1}^+, \quad O_0 v_{j_0}^- = -\tfrac{1}{2} v_{j_0}^-, \quad O_0 v_{j_0+1}^- = \tfrac{1}{2} v_{j_0+1}^-. \tag{5.145}$$

We will show that the commutation relation (5.60) involving $[O_0, O_-]$ is not respected, thus showing the impossibility of the existence of a representation. The actions of O_+ and O_- previously found (equations (5.133)–(5.136)) do not work here because their denominator is 0. We can circumvent this by noticing that

$$v_{j_0+1}^+ = L_- v_{j_0}^+ = (O_- O_0 + \tfrac{1}{2}[O_0, O_-]) v_{j_0}^+ \tag{5.146}$$

$$= O_-(O_0 - \tfrac{1}{2}) v_{j_0}^+ + 2(O_{123} - T_0) T_- v_{j_0}^+ \tag{5.147}$$

$$= i(\Lambda_1 - (-1)^{j_0}\kappa_0) G_\kappa (1 - j_0 - \ell) v_{j_0}^-. \tag{5.148}$$

This forces $G_\kappa(1-j_0-\ell) \neq 0$ as $v^+_{j_0+1}$ must not be zero. So $v^-_{j_0}$ is a multiple of $v^+_{j_0+1}$. But if this is the case, then studying the two sides of the commutator $[O_0, O_-]$ leads to a contradiction from the factorisations (5.63–5.64). Indeed, plugging in the last equation:

$$[O_0, O_-]v^+_{j_0} = -O_-v^+_{j_0} + 2(O_{123} - T_0)T_-v^-_{j_0} \tag{5.149}$$

$$O_0O_-v^+_{j_0} - \tfrac{1}{2}O_-v^+_{j_0} = -O_-v^+_{j_0} + 2v^+_{j_0+1} \tag{5.150}$$

$$O_0O_-v^+_{j_0} = -\tfrac{1}{2}O_-v^+_{j_0} + 2v^+_{j_0+1}, \tag{5.151}$$

which is an impossible equation because $v^+_{j_0+1} \neq 0$. There are thus no representations in this case.

***Even* N *and* $\lambda_6 = N/2 + \kappa_0\delta$.** The O_0-eigenvalues of v^+_k and v^-_j are

$$O_0v^+_k = N/2 + \kappa_0\delta - k, \qquad O_0v^-_j = j - N/2 - \kappa_0\delta. \tag{5.152}$$

So they are the same if $\kappa_0 = (k+j-N)/2\delta$. If j and k have the same parity, then v^+_k and v^-_j are distinguishable under the action of $\tilde{\sigma}_0$. If they have the same parity, then the value of κ_0 is restricted by the study of $A(k) \neq 0$.

We study $A(k) = 0$. The first factor of $A(k) = -A_{(1)}(k)A_{(2)}(k)$ is zero for some values of κ_0

$$A_{(1)}(k) = (\lambda_6 - k + 1/2)^2 - (\lambda_6 + 1/2 + \kappa_0\delta + (-1)^k\kappa_0\delta)^2 = 0. \tag{5.153}$$

For even k, then it forces $\kappa_0 = (k-N-1)/2\delta$. For odd k, it forces $\kappa_0 = -k/2\delta$ or $\kappa_0 = -(\lambda_6 - k/2 + 1/2)/\delta = -(N-k+1)/4\delta$.

The other factor can also cancel as

$$A_{(2)}(k) = (\lambda_6 - k + 1/2)^2 - H_\kappa(2-k-\ell) = 0. \tag{5.154}$$

For $2-k-\ell \equiv_m p$ then this happens if and only if $\kappa_1 + \kappa_m = \pm(\lambda_6 + k + 1/2)/p$. For $2-k-\ell \equiv_m 0$ if $\kappa_1 - \kappa_m = \pm(\lambda_6 - k + 1/2)/p$. For the other k, this also happens if $\kappa_0 = (N-2k+1)/2\delta$.

Unitarity is studied from the condition $A(k) > 0$, for $1 \leq k \leq N$, of Lemma 5.6.4. The first factor $A_{(1)}(k)$ is negative under the assumption that

$$\begin{cases} |N/2 + \kappa_0\delta + 1/2 - k| < |N/2 + \kappa_0\delta + 1/2|, & \text{even } k; \\ |N/2 + \kappa_0\delta + 1/2 - k| < |N/2 + 3\kappa_0\delta + 1/2|, & \text{odd } k. \end{cases} \tag{5.155}$$

The first is then $\delta\kappa_0 > (k-N-1)/2$. The second divides according to k:

$$\begin{cases} \kappa_0\delta > -k/2 \text{ or } \kappa_0\delta < (k-N-1)/4, & k < (N+1)/3, \\ \kappa_0\delta < -k/2 \text{ or } \kappa_0\delta > (k-N-1)/4, & k > (N+1)/3. \end{cases} \tag{5.156}$$

For $\delta = 1$, this is always the case. If $\delta = -1$ then, for all k, it will require $\kappa_0 < 1/2$ or $\kappa_0 > N/2 + 1/2$.

The second factor $A_{(2)}(k)$ is always positive when k is such that $2-k-\ell \not\equiv_m 0, p$. When $2-k-\ell \equiv_m p$ then it is positive if $\kappa_1 + \kappa_m < |\lambda_6 - k + 1/2|/p$. And for $2-k-\ell \equiv_m 0$, then $|\kappa_1 - \kappa_m| < |\lambda_6 - k + 1/2|/p$.

The cases considered complete the proof of Theorem 5.6.2. ∎

5.7 The monogenic representations

This section contains a concrete realisation of a family of representations of the symmetry algebra $\mathcal{O}_{m,\kappa}$. It consists of the spaces of monogenics of the Dunkl–Dirac operator. They are built with the Fischer decomposition (Theorem 5.7.3) from the two-dimensional Dunkl–Laplace harmonics given by Dunkl [Dun89; DX14] and takes the third dimension, with its A_1 root system contribution, into account via a Cauchy–Kovalevskaya extension (Theorem 5.7.4). **Note that this section assumes $\varepsilon = 1$.**

Let $\mathcal{P}_n(\mathbb{R}^d)$ denote the space of polynomials of degree n in d variables. Let $\mathcal{M}_n(\mathbb{R}^3, \mathbb{C}^2) := \ker \underline{D} \cap (\mathcal{P}_n(\mathbb{R}^3) \otimes \mathbb{C}^2)$ be the space of Dunkl monogenics of degree n in $\overline{3}$ variables. Here we make the identification of a spinor representation of the Clifford algebra $Cl(3)$ with $\mathbb{C}^2$ using Pauli matrices with an extra sign. Let $\delta \in \{-1, +1\}$. Realise the Clifford elements as one set of Pauli matrices:

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_2 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad e_3 \mapsto \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix}.$$

The difference given by δ is to account for the two two-dimensional irreducible representations of the Clifford algebra $Cl(3)$. The pseudoscalar $e_1e_2e_3$ acts as $i\delta$.

Using the Cauchy–Kovalevskaya extension and the Fischer decomposition theorem, we will construct a basis for the Dunkl monogenics

$\mathcal{M}_n(\mathbb{R}^3,\mathbb{C}^2)$ of degree n in 3 variables from the harmonics of the Dunkl–Laplace operator in two dimensions.

There is another realisation of $\mathfrak{osp}(1|2)$ inside the algebra obtained by restricting to the x_1 and x_2 coordinates. Here we use the fact that the root system is reducible. Let the following denote the 2D counterparts of the operators defined in Section 5.3

$$\begin{aligned} \widehat{\underline{\mathrm{D}}} &:= \mathrm{D}_1 e_1 + \mathrm{D}_2 e_2, & \widehat{\underline{x}} &:= x_1 e_1 + x_2 e_2, \\ \widehat{\underline{\mathrm{D}}}^2 &= \mathrm{D}_1^2 + \mathrm{D}_2^2 =: \widehat{\Delta_\kappa}, & \widehat{\underline{x}}^2 &= x_1^2 + x_2^2 =: \widehat{\mathbf{x}}^2, \\ \widehat{\mathbb{E}} &:= x_1\partial_{x_1} + x_2\partial_{x_2}, & \widehat{\gamma} &:= \frac{m}{2}(\kappa_1 + \kappa_m). \end{aligned} \tag{5.157}$$

The operators $\widehat{\underline{\mathrm{D}}}$ and $\widehat{\underline{x}}$ respect

$$\left\{\widehat{\underline{\mathrm{D}}}, \widehat{\underline{x}}\right\} = 2(\widehat{\mathbb{E}} + 1 + \widehat{\gamma}), \tag{5.158}$$

and they interact with the $\mathfrak{sl}(2)$ triple $\widehat{\mathbf{x}}^2, \widehat{\Delta_\kappa}, \widehat{\mathbb{E}}$ as follows:

$$\begin{aligned} \left[\widehat{\underline{\mathrm{D}}}, \widehat{\mathbf{x}}^2\right] &= 2\widehat{\underline{x}}, & \left[\widehat{\mathbb{E}}, \widehat{\underline{\mathrm{D}}}\right] &= -\widehat{\underline{\mathrm{D}}}, \\ \left[\widehat{\Delta_\kappa}, \widehat{\underline{x}}\right] &= 2\widehat{\underline{\mathrm{D}}}, & \left[\widehat{\mathbb{E}}, \widehat{\underline{x}}\right] &= \widehat{\underline{x}}. \end{aligned} \tag{5.159}$$

Recall some basic facts from hypergeometric analysis. The Pochhammer symbol $(a)_n$ is defined as $(a)_0 = 1$; $(a)_1 = a$ and $(a)_n = a_{n-1}(a + n - 1) = a(a+1)\dots(a+n-1)$. The hypergeometric series ${}_rF_s$ is given by

$${}_rF_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix} \,\middle|\, z\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_r)_k}{(b_1)_k \dots (b_s)_k} \frac{z^k}{k!}. \tag{5.160}$$

We will need some special orthogonal polynomials to present the results. The *Jacobi polynomial* of degree n is given for constants a, b as

$$P_n^{(a,b)}(x) := \frac{(a+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n,\ n+a+b+1 \\ a+1 \end{matrix} \,\middle|\, \frac{1-x}{2}\right) \tag{5.161}$$

and they fulfil the identity

$$(x+y)^n P_n^{(a,b)}\left(\frac{x-y}{x+y}\right) = \frac{(a+1)_n}{n!} x^n {}_2F_1\left(\begin{matrix} -n,\ -n-b \\ a+1 \end{matrix} \,\middle|\, -\frac{y}{x}\right). \tag{5.162}$$

Dunkl and Xu defined [DX14] the *generalized Gegenbauer polynomials* by

$$\begin{aligned} G_{2n}^{(\lambda,\mu)}(x) &= \frac{(\lambda+\mu)_n}{(\mu+1/2)_n} P_n^{(\lambda-1/2,\mu-1/2)}(2x^2-1), \\ G_{2n+1}^{(\lambda,\mu)}(x) &= \frac{(\lambda+\mu)_{n+1}}{(\mu+1/2)_{n+1}} x P_n^{(\lambda-1/2,\mu+1/2)}(2x^2-1). \end{aligned} \tag{5.163}$$

The following proposition is extracted from the original paper of Dunkl [Dun89] in the updated formulation of the book [DX14] and gives a basis for the harmonics of the Dunkl-Laplacian.

Proposition 5.7.1 ([Dun89, Sects 3.14 and 3.19])**.** *Let n be a natural number and D_{2m} be the dihedral group of order $2m$. There is a basis of the space of D_{2m} Dunkl harmonics $\mathcal{H}_n(\mathbb{R}^2) := \ker \widehat{\Delta_\kappa} \cap \mathcal{P}_n(\mathbb{R}^2)$ given by pairs of polynomials ϕ_n^+, ϕ_n^- of degree n depending on the parity of m. Denote $z := x_1 + ix_2$ and $\overline{z} := x_1 - ix_2$.*

- *(Odd m). Recall that then $\kappa_1 = \kappa_m$. Decompose n by Euclidean division as $n = km + \ell$ for $0 \le \ell < m$.*

$$\begin{aligned} \phi_n^+(x_1,x_2) &= z^\ell \sum_{j=0}^{n} \frac{(\kappa_1)_j(\kappa_1+1)_{n-j}}{j!(n-j)!} \overline{z}^{mj} z^{m(n-j)}; \\ \phi_n^-(x_1,x_2) &= \overline{\phi_n^+(x_1,x_2)}. \end{aligned} \tag{5.164}$$

- *(Even $m = 2p$). Let $n = kp + \ell$ with $0 \le \ell < p$. The harmonics polynomials are given by*

$$\begin{aligned} \phi_n^+(x_1,x_2) &= z^\ell f_k(z^p, \overline{z}^p), \\ \phi_n^-(x_1,x_2) &= \overline{z}^\ell f_k(\overline{z}^p, z^p), \end{aligned} \tag{5.165}$$

  with f expressed with Gegenbauer polynomials and polar decomposition $z = re^{i\theta}$

$$\begin{aligned} f_k(z,\overline{z}) = r^k \Big(& \frac{n + 2\kappa_m + (1+(-1)^n)\kappa_1}{2(\kappa_m+\kappa_1)} G_n^{(\kappa_m,\kappa_1)}(\cos(\theta)) \\ & + i\sin(\theta) G_{n-1}^{(\kappa_m+1,\kappa_1)}(\cos(\theta)) \Big). \end{aligned} \tag{5.166}$$

It is possible to rewrite $f_k(z,\overline{z})$ in a slightly more direct way for computational purposes [DX14]. Recall that $2x_1 = (z+\overline{z})$ and $2ix_2 = z - \overline{z}$

and rewrite f_k as

$$f_k(z,\overline{z}) = \begin{cases} \dfrac{(\kappa_m+\kappa_1+1)_t}{(\kappa_1+1/2)_t} g_{2t}(z,\overline{z}), & k=2t, \\[2ex] \dfrac{(\kappa_m+\kappa_1+1)_t}{(\kappa_1+1/2)_{t+1}} g_{2t+1}(z,\overline{z}), & k=2t+1, \end{cases} \tag{5.167}$$

with

$$\begin{aligned} g_{2t}(z,\overline{z}) = (-1)^t \sum_{j=0}^{t} & \frac{(-t+1/2-\kappa_m)_{t-j}(-t+1/2-\kappa_1)_j}{(t-j)!j!} x_1^{2t-2j}(ix_2)^{2j} \\ - (-1)^t \sum_{j=0}^{t-1} & \frac{(-t+\frac{1}{2}-\kappa_m)_{t-1-j}(-t+\frac{1}{2}-\kappa_1)_j}{(t-1-j)!j!} x_1^{2t-1-2j}(ix_2)^{(2j+1)}, \end{aligned}$$

and

$$\begin{aligned} g_{2t+1}(z,\overline{z}) = (-1)^{t+1} \sum_{j=0}^{t} & \frac{(-t+\frac{1}{2}-\kappa_m)_{t+1-j}(-t+\frac{1}{2}-\kappa_1)_j}{(t-j)!j!} x_1^{2t+1-2j}(ix_2)^{2j} \\ - (-1)^t \sum_{j=0}^{t} & \frac{(-t+\frac{1}{2}-\kappa_m)_{t-j}(-t+\frac{1}{2}-\kappa_1)_{j+1}}{(t-j)!j!} x_1^{2t-2j}(ix_2)^{(2j+1)}. \end{aligned}$$

Knowing the Dunkl harmonics let us deduce the Dunkl monogenics. Let χ^+ and χ^- be spinors here expressed as the first and second coordinate vector $\chi^+ = (1,0)^T$ and $\chi^- = (0,1)^T$. The following proposition states a basis of Dunkl monogenics in two dimensions $\mathcal{M}_n(\mathbb{R}^2,\mathbb{C}^2) = \ker \underline{\mathrm{D}} \cap (\mathcal{P}_n(\mathbb{R}^2)\otimes\mathbb{C}^2)$, where again we identify the spinor representation with $\mathbb{C}^2$.

Proposition 5.7.2. *Let n be a natural number. The polynomials*

$$\Phi_n^+(x_1,x_2) = \phi_n^+(x_1,x_2)\chi^+ \quad \textit{and} \quad \Phi_n^-(x_1,x_2) = \phi_n^-(x_1,x_2)\chi^- \tag{5.168}$$

are a basis for the Dunkl monogenics $\mathcal{M}_n(\mathbb{R}^2,\mathbb{C}^2)$ of degree n.

Proof. Applying $\widehat{\underline{\mathrm{D}}}$ on Φ_j^+ in the Pauli matrices realisation yields

$$\widehat{\underline{\mathrm{D}}}\,\Phi_n^+ = \mathrm{D}_1 e_1 \phi_n^+ \chi^+ + \mathrm{D}_2 e_2 \phi_n^+ \chi^+ = (\mathrm{D}_1 + i\mathrm{D}_2)\phi_n^+. \tag{5.169}$$

The polynomial ϕ_n^+ is a harmonic of the Dunkl–Laplace operator. The Dunkl–Laplace operator factors as $\widehat{\Delta} = (\mathrm{D}_1 + i\mathrm{D}_2)(\mathrm{D}_1 - i\mathrm{D}_2)$. Working out the properties of the function $f_k(z,\overline{z})$ shows that ϕ_n^+ is annihilated by the first factor [Dun89], thus showing that Φ_n^+ is a monogenic. Conjugating shows the result for the other polynomial Φ_n^-. ■

When the constants κ_0, κ_1 and κ_m are positive, there is a decomposition of the space of spinor valued polynomials by Dunkl monogenics.

Theorem 5.7.3 (Fischer decomposition [ØSS09])**.** *Let $n \in \mathbb{N}$ and $\kappa_1, \kappa_m > 0$. There exists a decomposition of the space of spinor valued polynomials given by*

$$\mathcal{P}_n(\mathbb{R}^2) \otimes \mathbb{C}^2 = \bigoplus_{j=0}^{n} \widehat{\underline{x}}^{n-j} \mathcal{M}_j(\mathbb{R}^2, \mathbb{C}^2). \tag{5.170}$$

Everything is in place for the Cauchy–Kovalevskaya extension Theorem. It establishes an isomorphism between the two-dimensional space and the three-dimensional monogenics taking into account the $\mathbb{Z}_2$ reflection group.

Theorem 5.7.4 (Cauchy–Kovalevskaya, [DGV16a])**.** *Let $\kappa_0 > 0$. There is an isomorphism between the spaces of spinor-valued polynomials in two dimensions $\mathcal{P}_n(\mathbb{R}^2) \otimes \mathbb{C}^2$ and the Dunkl monogenics in three dimensions $\mathcal{M}_n(\mathbb{R}^3, \mathbb{C}^2)$ given on polynomials by*

$$\mathbf{CK}_{x_3}^{\kappa_0} = {}_0F_1\left(\begin{matrix} - \\ \kappa_0 + 1/2 \end{matrix} \,\middle|\, -\frac{(x_3 \widehat{\underline{\mathrm{D}}})^2}{4} \right) - \frac{x_3 e_3 \widehat{\underline{\mathrm{D}}}}{2\kappa_0 + 1} {}_0F_1\left(\begin{matrix} - \\ \kappa_0 + 3/2 \end{matrix} \,\middle|\, -\frac{(x_3 \widehat{\underline{\mathrm{D}}})^2}{4} \right).$$

We are now ready to construct a basis of the monogenics.

Corollary 5.7.5. *Let $n \in \mathbb{N}$. A basis for the space $\mathcal{M}_n(\mathbb{R}^3, \mathbb{C}^2)$ is given by the $2n+2$ polynomials*

$$\psi_{n,k}^{\pm}(x_1, x_2, x_3) := \mathbf{CK}_{x_3}^{\kappa_0}(\underline{x}^{n-k} \Phi_k^{\pm}(x_1, x_2)), \quad k = 0, \ldots, n. \tag{5.171}$$

The polynomials $\psi_{n,k}^{\pm}$ can also be given explicitly. The next proposition works them out similarly to the $W = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ case done in [DGV16a].

Proposition 5.7.6. *An explicit basis of the space of Dunkl monogenics $\mathcal{M}_n(\mathbb{R}^3, \mathbb{C}^2)$ is given, for $k = 0, \dots, n$, by*

$$\begin{aligned} \psi^+_{n,k}(x_1, x_2, x_3) &= B_{n,k}(\widehat{\underline{x}}, x_3)\Phi^+_k(x_1, x_2), \\ \psi^-_{n,k}(x_1, x_2, x_3) &= B_{n,k}(\widehat{\underline{x}}, x_3)\Phi^-_k(x_1, x_2), \end{aligned} \tag{5.172}$$

with $B_{n,k}$ defined as

$$B_{n,k}(\widehat{\underline{x}}, x_3) = \frac{t!}{(\kappa_0 + 1/2)_t}\underline{x}^{2t} \times \tag{5.173}$$

$$\begin{cases} \Big(\widehat{\underline{x}} P_t^{(\kappa_0 - 1/2, k+1+\widehat{\gamma})}(\Upsilon(x)) \\ \quad -x_3 e_3 \dfrac{t + k + 1 + \widehat{\gamma}}{t + \kappa_0 + 1/2} P_t^{(\kappa_0 + 1/2, k + \widehat{\gamma})}(\Upsilon(x))\Big), & n - k = 2t + 1; \\ \Big(P_t^{(\kappa_0 - 1/2, k+\widehat{\gamma})}(\Upsilon(x)) - \dfrac{x_3 e_3 \widehat{\underline{x}}}{\underline{x}^2} P_{t-1}^{(\kappa_0 + 1/2, k+1+\widehat{\gamma})}(\Upsilon(x))\Big), & n - k = 2t; \end{cases}$$

and $\Upsilon(x) := (x_1^2 + x_2^2 - x_3^2)/(x_1^2 + x_2^2 + x_3^2)$.

Proof. Let $M_k \in \mathcal{M}_k(\mathbb{R}^2, \mathbb{C}^2)$. The commutation relations (5.158) and the fact that M_k is a monogenic imply that

$$\begin{aligned} \widehat{\underline{D}}^2(\widehat{\underline{x}} M_k) = 0, \quad \widehat{\underline{D}}(\widehat{\underline{x}}^{2\beta+1} M_k) &= 2(\beta + k + \widehat{\gamma})\widehat{\underline{x}}^{2\beta} M_k, \\ \widehat{\underline{D}}(\widehat{\underline{x}}^{2\beta} M_k) &= 2\beta \widehat{\underline{x}}^{2\beta - 1} M_k. \end{aligned} \tag{5.174}$$

From equations (5.174), a short computation generalizes to

$$\widehat{\underline{D}}^a(\widehat{\underline{x}}^b M_k) = \begin{cases} d^k_{a,b}\widehat{\underline{x}}^{b-a} M_k & a \leq b; \\ 0 & a > b; \end{cases} \tag{5.175}$$

with the value of $d^k_{a,b}$ given by

$$d^k_{a,b} = \begin{cases} 2^{2\alpha}(-\beta)_\alpha(-\beta - k - \widehat{\gamma})_\alpha, & a = 2\alpha,\ b = 2\beta; \\ -2^{2\alpha+1}(-\beta)_{\alpha+1}(-\beta - k - \widehat{\gamma})_\alpha, & a = 2\alpha + 1,\ b = 2\beta; \\ -2^{2\alpha+1}(-\beta)_\alpha(-\beta - k - 1 - \widehat{\gamma})_{\alpha+1}, & a = 2\alpha + 1,\ b = 2\beta + 1; \\ 2^{2\alpha}(-\beta)_\alpha(-\beta - k - 1 - \widehat{\gamma})_\alpha, & a = 2\alpha,\ b = 2\beta + 1. \end{cases}$$

We now use the anticommutation relation $\left\{\widehat{\underline{D}}, e_3\right\} = 0$, Corollary 5.7.5 and the identity (5.162) of Jacobi polynomials to indeed obtain

$$\psi^\pm_{n,k}(x_1, x_2, x_3) = B_{n-k}(\widehat{\underline{x}}, x_3)\Phi^\pm_k(x_1, x_2).$$

∎

For $W = \mathbb{Z}_2 \times \mathrm{D}_{2m}$, there is an integral formulation of the inner product introduced abstractly in (5.85). Take the adapted weight function [DX14] with $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$

$$h_{\kappa_0,\kappa_1,\kappa_m}(x_1,x_2,x_3) := \left|\frac{z^m + \overline{z}^m}{2}\right|^{\kappa_1} \cdot \left|\frac{z^m - \overline{z}^m}{2i}\right|^{\kappa_m} \cdot |x_3|^{\kappa_0}. \tag{5.176}$$

Let $X^\dagger$ be the transpose of X. Define an inner product by

$$\langle \psi_1, \psi_2 \rangle := \int_{S_2} (\psi_1^\dagger \psi_2) h^2_{\kappa_0,\kappa_1,\kappa_m}(x_1,x_2,x_3) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3. \tag{5.177}$$

The structure of the monogenic representations is given in the next two propositions.

Proposition 5.7.7. *Let $m = 2p+1$. For each $n \in \mathbb{N}$, the space of monogenics $\mathcal{M}_n(\mathbb{R}^3, \mathbb{C}^2)$ of the Dunkl–Dirac operator of degree n forms an irreducible representation of dimension $2n+2$ of the symmetry algebra $\mathfrak{O}_{m,\kappa}$ with basis*

$$\{\psi^\pm_{n,k} \mid k = 0, 1 \ldots, n\}. \tag{5.178}$$

The action of the symmetry algebra is given by

$$O_0 \psi^\pm_{n,k} = \pm(k + \tfrac{1}{2} + m\kappa_1)\psi^\pm_{n,k}; \quad O_{123}\psi^\pm_{n,k} = \delta i(n + 1 + \kappa_1 m + \delta\kappa_0)\psi^\pm_{n,k},$$

where $\delta \in \{-1,+1\}$ comes from the realisation of the Clifford algebra element e_3. Let $k = rm + \ell$ with $0 \le \ell \le m-1$ and $\zeta = e^{i\pi/m}$. The group $\widetilde{W}$ action is given by

$$\begin{aligned} \tilde{\sigma}_0 \psi^\pm_{n,k} &= \pm\delta(-1)^{n-k}\psi^\pm_{n,k}; & \tilde{\sigma}_1 \psi^\pm_{n,k} &= \mp i(-1)^{(n-k)}\zeta^{\pm 2\ell}\zeta^{\pm 1}\psi^\mp_{n,k}; \\ \tilde{\sigma}_m \psi^\pm_{n,k} &= \pm i(-1)^{n-k}\psi^\mp_{n,k}; & \tilde{\tau}\psi^\pm_{n,k} &= -\zeta^{\mp(2\ell+1)}\psi^\pm_{n,k}. \end{aligned} \tag{5.179}$$

Furthermore, the representation is unitary.

Proof. Recall $O_{123} = \frac{1}{2}(\left[\underline{\mathrm{D}}, \underline{x}\right] - 1)e_1e_2e_3$, see equation (5.25). On any monogenic $\psi_{n,k}$ of degree n we have

$$\begin{aligned} 1/2(\left[\underline{\mathrm{D}}, \underline{x}\right] - 1)\psi_{n,k} &= 1/2(\underline{\mathrm{D}}\,\underline{x} - 1)\psi_{n,k} \\ &= 1/2(\left\{\underline{\mathrm{D}}, \underline{x}\right\} - 1)\psi_{n,k} = 1/2(2\mathbb{E} + 3 + 2\gamma - 1)\psi_{n,k} \\ &= (n + 1 + \kappa_1 m + \delta\kappa_0)\psi_{n,k} \end{aligned}$$

and thus, according to the realisation of e_3,

$$O_{123}\psi_{n,k} = \delta i(n+1+\kappa_1 m+\delta\kappa_0)\psi^{\pm}_{n,k}. \tag{5.180}$$

Restricting to the plane x_1, x_2, we have that $O_0 = -\frac{i}{2}e_1e_2(\left[\underline{\widehat{D}}, \underline{\widehat{x}}\right]-1)$ is the Scasimir of the $\mathfrak{osp}(1|2)$ realisation (5.157) times the pseudo-scalar e_1e_2. On 2D monogenics, we thus have

$$\begin{aligned}(\left[\underline{\widehat{D}}, \underline{\widehat{x}}\right]-1)\Phi^{\pm}_k &= 1/2(\left\{\underline{\widehat{D}}, \underline{\widehat{x}}\right\}-1)\Phi^{\pm}_k = 1/2(2\mathbb{E}+2+2m\kappa_1-1)\Phi^{\pm}_k \\ &= (k+1/2+m\kappa_1)\Phi^{\pm}_k.\end{aligned}$$

With $-ie_1e_2\chi^{\pm} = \pm\chi^{\pm}$, we get

$$O_0\psi^{\pm}_{n,k} = \pm(k+1/2+m\kappa_1)\psi^{\pm}_{n,k}. \tag{5.181}$$

By direct computations on the explicit expression of $\phi^{\pm}_k$ we have

$$\sigma_0\phi^{\pm}_k = \phi^{\pm}_k, \qquad \sigma_m\phi^{\pm}_k = \overline{\phi^{\pm}_k} = \phi^{\mp}_k, \qquad \sigma_1\phi^{\pm}_k = \zeta^{\pm 2\ell}\phi^{\mp}_k. \tag{5.182}$$

By example, for σ_1, note that $\sigma_1(z) = e^{2\pi i/m}\overline{z}$ and $\sigma_1(\overline{z}) = e^{-2\pi i/m}z$, so

$$\sigma_1\phi^{+}_k = \sigma_1(z^{\ell})C^{(\kappa,\kappa+1)}_k(z^m,\overline{z}^m) = \zeta^{2\ell}\overline{z}^{\ell}C^{(\kappa,\kappa+1)}_k(\zeta^{2m}\overline{z},\zeta^{-2m}z^m) = \zeta^{2\ell}\phi^{-}_k.$$

Furthermore

$$\begin{aligned}\tilde{\sigma}_1\Phi^{\pm}_k &= \sigma_1(\sin(\pi/m)e_1-\cos(\pi/m)e_2)\phi^{\pm}_k\chi^{\pm} \\ &= \zeta^{\pm 2\ell}\phi^{\mp}_k(\sin(\pi/m)\mp i\cos(\pi/m))\chi^{\mp} = \mp i\zeta^{\pm 2\ell}\zeta^{\pm 1}\Phi^{\mp}_k.\end{aligned}$$

Adding $e_1\chi^{\pm} = \chi^{\mp}$, $e_2\chi^{\pm} = \pm i\chi^{\mp}$ and $e_3\chi^{\pm} = \delta\chi^{\pm}$, and the fact that both $\tilde{\sigma}_1$ and $\tilde{\sigma}_m$ anticommute with $\underline{\widehat{x}}$, we get what is needed.

To prove that it is irreducible, it is sufficient to prove that each $\psi^{\pm}_{n,k}$ generates the whole representation. From the previous computations, the representation is a renormalized version of the irreducible representation constructed in the no-restriction subcases of cases I of Theorem 5.6.1 with a switch from $v^{\pm}_k$ to $C(n,k)\psi^{\mp}_{n,n-k}$ for certain non-zero constants $C(n,k)$. Therefore, acting on $\psi^{\pm}_{n,k}$ with the operators $L_{\pm}$ will be enough to travel between indices of $\psi^{\pm}_{n,k}$.

The actions of O_+ and O_- follow from the proofs of Theorem 5.6.1.

Unitarity comes from the definition of the weight function (5.176), the Dunkl harmonics used to construct the monogenics and from the case I of Theorem 5.6.1. ∎

A similar proposition holds when m is even. This representation is a renormalized version of the no-restriction subcase of cases I.i in Theorem 5.6.2 when sending $v_k^\pm$ to $C'(n,k)\psi^\mp_{n,n-k}$ for some non-zero constants $C'(n,k)$.

Proposition 5.7.8. *Let $m = 2p$. For each $n \in \mathbb{N}$, the space of monogenics $M_n(\mathbb{R}^3, \mathbb{C}^2)$ of the Dunkl–Dirac operator of degree n form an irreducible and unitary representation of dimension $2n+2$ of the symmetry algebra $\mathfrak{O}_{m,\kappa}$ with basis*

$$\{\psi^\pm_{n,k} \mid k = 0, 1 \ldots, n\}. \tag{5.183}$$

The action of the symmetry algebra is given by

$$O_0\psi^\pm_{n,k} = \pm(k + 1/2 + p(\kappa_1 + \kappa_m))\psi^\pm_{n,k}; \tag{5.184}$$

$$O_{123}\psi^\pm_{n,k} = \delta i(n + 1 + p(\kappa_1 + \kappa_m) + \delta\kappa_0)\psi^\pm_{n,k}, \tag{5.185}$$

where $\delta \in \{-1,+1\}$ comes from the realisation of the Clifford algebra element e_3. Let $k = rm + \ell$ with $0 \le \ell \le m-1$ and $\zeta = e^{i\pi/m}$. The group $\widetilde{W}$ action is given by

$$\begin{aligned} \tilde{\sigma}_0\psi^\pm_{n,k} &= \pm\delta(-1)^{n-k}\psi^\pm_{n,k}; \quad \tilde{\sigma}_1\psi^\pm_{n,k} = \mp i(-1)^{(n-k)}\zeta^{\pm 2\ell}\zeta^{\pm 1}\psi^\mp_{n,k}; \\ \tilde{\sigma}_m\psi^\pm_{n,k} &= \pm i(-1)^{n-k}\psi^\mp_{n,k}; \quad \tilde{\tau}\psi^\pm_{n,k} = -\zeta^{\mp(2\ell+1)}\psi^\pm_{n,k}. \end{aligned} \tag{5.186}$$

5.8 Concluding remarks

We here recall the main results and present the scope of the methods used. On the general 3D symmetry algebra of the Dunkl–Dirac operator, we added Proposition 5.5.5 giving the square of the symmetry O_{123} for any root system. Specifically for all reducible root systems of rank 3, we listed all the finite-dimensional irreducible representations provided κ real and positive, as well as sufficient conditions for their unitarity. A polynomial family of irreducible and unitary representations was realised through the important example of monogenics.

The idea employed here will be difficult to apply to other higher-rank root systems as the ladder operators trick will be likely to fail. However, as it covers all the rank 2 cases, it can serve as a base case on which to support the jump for higher dimensions. Another point of this chapter was to work out completely the details given from adding a $\mathbb{Z}_2$ direct product to the reflection group.

We note that removing the condition κ real and positive require further work. To remove this assumption, even just to study the monogenic representation (type I in our notation) for the simplest case $W = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, required long and tedious case-by-case computations [Hua22]. Extending the approach of Huang to the general dihedral case seems possible, but would require extensive work. Another path to the problem should probably instead be investigated.

We gave one important realisation of the irreducible representation in Section 5.7. However, there are many more possible families of irreducible representations available, as readily seen from Theorems 5.6.1 and 5.6.2, but we do not know of concrete useful examples of them. As the values where irreducibility and unitarity fail resemble conditions appearing in related work, for example in [Chm06; DL21], it seems interesting to link them. It would also serve as a motivation to study the structure of the representations when they are reducible.

6

Finite-dimensional representations of the double dihedral total angular momentum algebra

The content of this chapter is extracted from the work:

Marcelo De Martino, Alexis Langlois-Rémillard, and Roy Oste. Double dihedral total angular momentum algebra. Work in progress (2022+) [DLO23]

6.1 Introduction

In this chapter, we study the finite-dimensional representations of the total angular momentum algebra for the group $W = \mathsf{D}_{2m} \times \mathsf{D}_{2n} \subset \mathcal{O}(4)$. We have seen already in Chapter 5 the case $\mathsf{D}_{2m} \times \mathbb{Z}_2 \subset \mathcal{O}(3)$, where we did the complete classification of the finite-dimensional irreducible representations, along with the suitable restrictions on κ for the existence of a unitarity structure.

The structural properties of the total angular momentum algebra $\mathfrak{O}_\kappa(W, V)$ change not only with the group W and the parameter function κ, but also crucially with the dimension of V. As such, the analysis of the total angular momentum algebra associated with a dihedral group $W = \mathsf{D}_{2m}$ differs whether it is considered inside a three-dimensional space, as in Chapter 5, or in a two-dimensional one [Ciu+20, Sect. 4]. A recent result of [Ost22] (Theorem 2.2.19) states that the total angular momentum algebra is generated by two- and three-index symmetries. Dimension four is the first time that the full picture happens: in dimension three, the sole three-index symmetry super-anticommutes with all the elements of the algebra, but in higher dimensions the many three-index symmetries have more involved commutation relations. This is one of the motivations to study the total angular momentum algebra in four dimensions.

The case studied in this chapter is then a stepping stone to a study of the general representation theory of a group $W = \mathsf{D}_{2m_1} \times \cdots \times \mathsf{D}_{2m_n} \subset \mathcal{O}(2n)$. Adding more dihedral groups will make the study more complex, but the full structure of the dihedral total angular momentum algebra is first caught by the product of two dihedral groups.

The main result of this chapter is a coarse classification of the finite-dimensional irreducible representations of $\mathfrak{O}_\kappa$ (Theorem 6.5.9). It gives a label of weights with relations between them that any irreducible finite-dimensional representation must respect. We then proceed to construct representations for certain classes of weights, in particular the ones that work for any positive parameter function κ (Proposition 6.5.14) and the ones reached for κ sufficiently small (Proposition 6.5.16).

To achieve these results, we focus on a subalgebra $\mathfrak{T}$ of $\mathfrak{O}_\kappa$ with a weight theory and a Poincaré–Birkhoff–Witt-type factorisation (Proposition 6.4.17). The particularity of the dihedral total angular momentum algebra lies in the possibility to employ this method. As such, it cannot directly be applied to other groups of higher rank, but it will work for any product of groups of rank one or two.

This chapter is written in a slightly more abstract style than the rest of the thesis specifically to ease the generalisation of the results to higher dimensions. An example is included in Section 6.5.6, with the explicit weight spaces constructed in Figures 6.4–6.6, and the readers are encouraged to have a look at it while following the arguments to develop their intuition.

We now go through the structure of the chapter. First, Section 6.2 introduces the needed conventions and notions, revisiting some of them from Chapter 2. It ends with the classification of the representations of $\widetilde{W}$ (Theorem 6.2.3). The total angular momentum algebra for the group $W = \mathsf{D}_{2m} \times \mathsf{D}_{2n}$ is then presented in Section 6.3. Section 6.4 defines ladder operators and the triangular subalgebra $\mathfrak{T}$. Finally, Section 6.5 contains the coarse classification of the finite-dimensional irreducible representations of $\mathfrak{O}_\kappa$ (Theorem 6.5.9), the constructions of specific branches (Propositions 6.5.14, 6.5.16 and 6.5.21) and concludes with a worked-out example.

Note that this chapter follows the convention $\varepsilon = 1$ for the Clifford signature.

6.2 Initial definitions and notational conventions

Throughout this work, we let $V_{\mathbb{R}} \cong \mathbb{R}^4$ denote a four-dimensional Euclidean space with the standard inner product $\langle \cdot, \cdot \rangle$. The orthogonal group $\mathcal{O}(4)$ consists of all endomorphisms of $\mathbb{R}^4$ that preserve the Euclidean norm. We consider a subgroup $W = \mathsf{D}_{2m} \times \mathsf{D}_{2n} \subset \mathcal{O}(4)$, where D_{2m} denotes the dihedral group of order $2m$ with Coxeter presentation given by

$$\mathsf{D}_{2m} = \left\langle \sigma_1, \sigma_m \;\middle|\; \sigma_1^2 = \sigma_m^2 = (\sigma_1 \sigma_m)^m = 1 \right\rangle \tag{6.1}$$

(the group D_{2n} has a similar presentation). Another presentation of the dihedral group D_{2m} is in terms of the rotation $r = \sigma_1 \sigma_m$ and the reflection $f = \sigma_m$

$$\mathsf{D}_{2m} = \left\langle r, f \;\middle|\; r^m = f^2 = (rf)^2 = 1 \right\rangle. \tag{6.2}$$

The even elements of D_{2m} form a cyclic group of m elements, generated by r. The odd elements of D_{2m} are reflections, given by $\sigma_p = r^p f$ for $p = 1, \ldots, m$.

We let $V \cong \mathbb{C}^4$ denote the complexification of $V_{\mathbb{R}}$ and note that this complex vector space V is the (complexified) reflection representation of W. We still denote by $\langle \cdot, \cdot \rangle$ the non-degenerate complexified symmetric bilinear form on V and denote by $B : V \to V^*$ the isometry induced by $\langle\cdot,\cdot\rangle$, which is defined by $B(x)(y) = \langle x, y\rangle$, for all $x, y \in V$.

We choose the standard root system $\Phi \subset V_{\mathbb{R}}$ of type $\mathsf{I}_2(m) \oplus \mathsf{I}_2(n)$. Recall the explicit realisation (2.8)

$$\alpha_p = (\sin(\tfrac{p\pi}{m}), -\cos(\tfrac{p\pi}{m}), 0, 0), \quad \beta_q = (0, 0, \sin(\tfrac{q\pi}{n}), -\cos(\tfrac{q\pi}{n})), \qquad (6.3)$$

for $p = 1, \ldots, 2m$ and $q = 1, \ldots, 2n$. We fix the set of positive roots to be $\Phi_+ = \{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\}$. Here, we follow the same convention for the dihedral groups as Dunkl [Dun89] and Humphreys [Hum90]. The associated reflections σ_p, $p = 1, \ldots, m$ and τ_q, $q = 1, \ldots, n$, were given in matrix form previously (2.1.1).

As is well known, the structure of a dihedral group D_{2m} depends on whether m is even or odd. When it is even, the reflections are in two different conjugacy classes; when m is odd, they are in the same. Furthermore, when m is even, there is a non-trivial central element, which acts as minus the identity on the reflection representation; see Theorem 6.3.9.

6.2.1 Superalgebras and Clifford algebras

We recall the definition and some properties of the Clifford algebra associated to $V_{\mathbb{R}}$. Let $Cl_{\mathbb{R}}$ denote the real Clifford algebra of the pair $(V_{\mathbb{R}}, 2\langle\cdot,\cdot\rangle))$, where $2\langle\cdot,\cdot\rangle$ is twice the bilinear form. Specifically, this algebra is the quotient of the tensor algebra $T(V_{\mathbb{R}}) = \oplus_{j\geq 0} T^j(V_{\mathbb{R}})$ by the nonhomogeneous quadratic ideal I generated by the set

$$\{x \otimes y + y \otimes x - 2\langle x, y\rangle)1 \mid x, y \in V_{\mathbb{R}}\}.$$

We let $\gamma : V_{\mathbb{R}} \to Cl_{\mathbb{R}}$ be the canonical embedding (2.30). The pair $(Cl_{\mathbb{R}}, \gamma)$ satisfies the following universal property: for any unital $\mathbb{R}$-algebra A and any linear map $L\colon V_{\mathbb{R}} \to A$ satisfying $L(x)L(y) + L(y)L(x) = 2\langle x, y\rangle$, there is a unique algebra homomorphism $\tilde{L}\colon Cl_{\mathbb{R}} \to A$ such that $\tilde{L}\gamma = L$:

$$\begin{tikzcd} V_{\mathbb{R}} \arrow[r, "L"] \arrow[d, "\gamma"] & A \\ C_{\mathbb{R}} \arrow[ur, dotted, "\tilde{L}"'] & \end{tikzcd} \qquad . \qquad (6.4)$$

If we denote the standard $\langle\cdot,\cdot\rangle$-orthonormal basis of $V_{\mathbb{R}}$ by $\{x_1,\ldots alr,x_4\}$, we let $e_j = \gamma(x_j)$ so that the Clifford algebra has the usual presentation as the unital associative algebra generated by $e_1,\ldots,e_4$ subject to the relations

$$e_je_k + e_ke_j = 2\delta_{jk}, \tag{6.5}$$

for all $j,k = 1,\ldots,4$. Note that here we are taking the positive Clifford signature $\varepsilon = 1$ from (2.31) and considering the positive double covering.

More generally, let

$$\bigwedge(V_{\mathbb{R}}) := \oplus_{p\geq 0}\bigwedge^p(V_{\mathbb{R}})$$

denote the exterior algebra on $V_{\mathbb{R}}$. We extend $\gamma : V_{\mathbb{R}} \to Cl_{\mathbb{R}}$ to a linear isomorphism $\gamma : \bigwedge(V_{\mathbb{R}}) \to Cl_{\mathbb{R}}$ by declaring $\gamma(1) := 1$ and for each $p > 0$

$$\gamma(v_1\wedge\cdots\wedge v_p) := \frac{1}{p!}\sum_{g\in S_p}\operatorname{sgn}(g)\gamma(v_{g(1)})\cdots\gamma(v_{g(p)}), \tag{6.6}$$

for any p-tuple $(v_1,\ldots,v_p)$ of elements of $V_{\mathbb{R}}$. In particular, for each (ordered) subset $A = \{a_1,\ldots,a_p\} \subseteq \{1,\ldots,4\}$, we let $x_A := x_{a_1}\wedge\cdots\wedge x_{a_p} \in \bigwedge^p(V_{\mathbb{R}})$, with $x_\emptyset = 1$ and put $e_A := \gamma(x_A) = \overrightarrow{\prod}_{j=1}^{p} e_{a_j}$. The set $\{e_A \mid A \subseteq \{1,\ldots 4\}\}$ forms a linear basis of $Cl_{\mathbb{R}}$.

We shall also denote by γ the complexified isomorphism $\gamma : \bigwedge(V) \to Cl$ defined in (6.6), and we note that $\{e_A \mid A \subseteq \{1,\ldots,4\}\}$ is also a linear basis for Cl.

6.2.2 Double coverings and realisations

In this section, we consider the double coverings of a product of dihedral groups. We will focus on the positive double covering, recalling that $\varepsilon = 1$, and study its representation theory.

6.2.2.1 Dihedral groups

We now return to the setting where $V_{\mathbb{R}} \cong \mathbb{R}^4$ and $W = G\times H$ is the product of two dihedral groups with $G = \mathsf{D}_{2m}$ and $H = \mathsf{D}_{2n}$. The (positive) double covering of a dihedral group $G = \mathsf{D}_{2m}$ of order $2m$ has the structure of a dihedral group of order $4m$: $\widetilde{G} \cong \mathsf{D}_{4m}$.

By [Mor80, Thm. 3.4], the group algebra of the double covering of the product of two Coxeter groups is a quotient of the graded tensor product of the individual group algebras of the double coverings of the groups, identifying the central extension elements.

As an abstract group, $W = \mathrm{D}_{2m} \times \mathrm{D}_{2n}$ has the following Coxeter presentation, for z a central element

$$W = \left\langle \sigma_1, \sigma_m, \tau_1, \tau_n \,\middle|\, \begin{array}{l} \sigma_1^2 = \sigma_m^2 = (\sigma_1\sigma_m)^m = 1, \quad (\sigma_p\tau_q)^2 = 1, \\ \tau_1^2 = \tau_n^2 = (\tau_1\tau_n)^n = 1, \ (p = 1, m;\ q = 1, n) \end{array} \right\rangle. \tag{6.7}$$

It follows from Theorem 2.2.7 that $\widetilde{W}^+$ has a presentation by generator and relations. To ease notation, we will often write $\widetilde{W}$ instead of $\widetilde{W}^+$ since we only work with the positive double covering in this chapter. The presentation, taking z as central element, is

$$\widetilde{W} = \left\langle \begin{array}{c} z, \tilde\sigma_1, \tilde\sigma_m, \\ \tilde\tau_1, \tilde\tau_n \end{array} \,\middle|\, \begin{array}{l} z^2 = \tilde\sigma_1^2 = \tilde\sigma_m^2 = 1, (\tilde\sigma_1\tilde\sigma_m)^m = z^{m+1}, (\tilde\sigma_p\tilde\tau_q)^2 = z, \\ \tilde\tau_1^2 = \tilde\tau_n^2 = 1, (\tilde\tau_1\tilde\tau_n)^n = z^{n+1}, (p = 1, m; q = 1, n) \end{array} \right\rangle, \tag{6.8}$$

or (similar to (6.2)), using $\tilde r_1 := z\tilde\sigma_1\tilde\sigma_m$, $\tilde r_2 := z\tilde\tau_1\tilde\tau_n$, $\tilde f_1 := \tilde\sigma_m$, $\tilde f_2 := \tilde\tau_n$, and writing $m_1 := m$, $m_2 := n$

$$\widetilde{W} = \left\langle \begin{array}{c} z, \tilde r_1, \tilde f_1, \\ \tilde r_2, \tilde f_2 \end{array} \,\middle|\, \begin{array}{l} \tilde r_j^{m_j} = z,\ \tilde f_j^2 = 1 = (\tilde r_j \tilde f_j)^2,\ (\tilde f_1 \tilde f_2)^2 = z,\ z^2 = 1, \\ \tilde r_j \tilde r_k = \tilde r_k \tilde r_j,\ \tilde r_j \tilde f_k = \tilde f_k \tilde r_j, \quad (j, k \in \{1, 2\}, j \neq k) \end{array} \right\rangle. \tag{6.9}$$

In light of (2.42), we can decompose both groups $\widetilde{G}$ and $\widetilde{H}$ into even and odd parts and note that

$$\widetilde{G}_{\bar 0} = \langle \tilde r_1 \rangle, \qquad \widetilde{G}_{\bar 1} = \widetilde{G}_{\bar 0}\tilde f_1 = \{(\tilde r_1)^j \tilde f_1 \mid 0 \le j \le 2m-1\} = \tilde f_1 \widetilde{G}_{\bar 0}, \tag{6.10}$$

and similarly for $\widetilde{H}$. We note that $\widetilde{G}_{\bar 0}$ is a subgroup of $\widetilde{G}$, and similarly for $\widetilde{H}_{\bar 0}$. These splittings yield a decomposition of $\widetilde{W}$ into four disjoint parts

$$\widetilde{W} = \bigcup_{(\bar\imath, \bar\jmath) \in \mathbb{Z}_2^2} \widetilde{W}_{(\bar\imath, \bar\jmath)}.$$

Here $\widetilde{W}_{(\bar\imath,\bar\jmath)}$ is defined as the image of the multiplication map $\widetilde{G}_{\bar\imath} \times \widetilde{H}_{\bar\jmath} \to \widetilde{W}$

$$(z_1^a \tilde u, z_2^b \tilde w) = z^{a+b} \tilde u \tilde w,$$

for $z_1^a\tilde{u} \in \widetilde{G}_{\bar{\imath}}$ and $z_2^b\tilde{w} \in \widetilde{H}_{\bar{\jmath}}$, where we denote z_1 and z_2 to be the central extension elements in the double coverings of G and H, respectively. It is straightforward to check that the maps $\widetilde{G}_{\bar{\imath}} \times \widetilde{H}_{\bar{\jmath}} \to \widetilde{W}_{(\bar{\imath},\bar{\jmath})}$ are all two-to-one and hence $|\widetilde{W}_{(\bar{\imath},\bar{\jmath})}| = 2mn$ by (6.10).

As it will play a role in what follows, let us specify more what the subgroup $\widetilde{W}_{(\bar{0},\bar{0})}$ is. It is the quotient of the abelian group $C_{2m} \times C_{2n}$ inside $\widetilde{W}$ by the identification $\tilde{r}_1^m = z = \tilde{r}_2^n$.

6.2.2.2 Representation theory

We conclude this section with a hands-on review of the representation theory of the group $\widetilde{W}$. For this purpose, the set of generators (6.9) of $\widetilde{W}$ will be useful.

We focus here on the spin representations as these will be relevant for the representation theory of the total angular momentum algebra in the last section. The classification of the representations depends on the parity of the dihedral parameters m and n. Abstractly, we will characterise representations by considering the action of $\tilde{r}_1$ and $\tilde{r}_2$.

The commuting elements $\tilde{r}_1$ and $\tilde{r}_2$ satisfy $\tilde{r}_1^{2m} = 1 = \tilde{r}_2^{2n}$, and thus generate $\widetilde{W}_{(\bar{0},\bar{0})}$. The $W_{(\bar{0},\bar{0})}$-modules are given by the $C_{2m} \times C_{2n}$-modules where r_1^m and r_2^n have the same action. A finite-dimensional $\widetilde{W}$-module thus decomposes into a direct sum of one-dimensional irreducible modules for $C_{2m} \times C_{2n}$.

Denote by $u(\ell,k)$, with $\ell \in \{0,\ldots,2m-1\}$ and $k \in \{0,\ldots,2n-1\}$, the irreducible $C_{2m}\times C_{2n}$-module with the following action on $u \in u(\ell,k)$:

$$\tilde{r}_1 \cdot u = \zeta^\ell u, \qquad \tilde{r}_2 \cdot u = \eta^k u; \qquad \zeta := e^{i\pi/m}, \qquad \eta := e^{i\pi/n}. \qquad (6.11)$$

Next, we consider the induced $\widetilde{W}$-representation

$$U := \operatorname{Ind}_{\widetilde{W}_{(\bar{0},\bar{0})}}^{\widetilde{W}}(u(\ell,k)).$$

For this to be a spin representation, $z = \tilde{r}_1^m = \tilde{r}_2^n$ has to act by -1, which restricts the values of ℓ and k to odd integers.

We define

$$u_1 := u, \qquad u_2 := \tilde{f}_1 u_1, \qquad u_3 := \tilde{f}_2 u_1, \qquad u_4 := \tilde{f}_1\tilde{f}_2 u_1. \qquad (6.12)$$

Using the relations $\tilde{f}_j\tilde{r}_j = \tilde{r}_j^{-1}\tilde{f}_j$ and (6.11), we have

$$\begin{aligned} \tilde{r}_1 u_2 &= \zeta^{-\ell} u_2, & \tilde{r}_1 u_3 &= \zeta^{\ell} u_3, & \tilde{r}_1 u_4 &= \zeta^{-\ell} u_4, \\ \tilde{r}_2 u_2 &= \eta^{k} u_2, & \tilde{r}_2 u_3 &= \eta^{-k} u_3, & \tilde{r}_2 u_4 &= \eta^{-k} u_4. \end{aligned} \tag{6.13}$$

Define $\widehat{U}$ as the free vector space on the four elements $\{\hat{u}_j\}_{j=1}^4$, so $\widehat{U} := \bigoplus_{j=1}^4 \mathbb{C}[\hat{u}_j]$. We give to $\widehat{U}$ the structure of a $\widetilde{W}$-representation in such a way that the following linear map

$$\phi : \widehat{U} \to U, \quad \hat{u}_j \mapsto u_j,\ j \in \{1,2,3,4\} \tag{6.14}$$

is a morphism of $\widetilde{W}$-modules.

Lemma 6.2.1. *The irreducible representations of $\widetilde{W}$ are of dimension at most 4.*

Proof. The map ϕ (6.14) is a morphism of $\widetilde{W}$-representations. It is an epimorphism by Schur's lemma, since U is irreducible, so $\dim U \le \dim \widehat{U} = 4$. ■

Lemma 6.2.2. *Let U be a spin irreducible representation. Then it is either of dimension 2 or 4.*

Proof. Suppose $U = \langle u \rangle$ is a one-dimensional spin representation. Then $\tilde{s}_m u = \alpha u$ and $\tilde{t}_n u = \beta u$ with $\alpha, \beta \in \{-1,+1\}$ since $\tilde{s}_m^2 = \tilde{t}_n^2 = 1$. But then, from $zu = -u$, we reach the two equalities $\tilde{s}_m\tilde{t}_n u = \alpha\beta u$ and $\tilde{s}_m\tilde{t}_n u = z\tilde{t}_n\tilde{s}_m u = -\beta\alpha u$, a contradiction. Note that if U is a spin representation, then so is $\widehat{U}$. Hence, $\dim U \neq 3$, since $\ker\phi \subset \widehat{U}$ would then be a one-dimensional spin representations. Irreducible spin representations are thus either of dimension 2 or 4. ■

We can now state the classification of irreducible spin representations of $\widetilde{W}$.

Theorem 6.2.3. *Let $\widetilde{W}$ be the positive double covering of $W = \mathsf{D}_{2m} \times \mathsf{D}_{2n}$. A list of all irreducible spin representations is given as follows.*

- *For positive odd integers ℓ and k, with $\ell < m$ and $k < n$, $U(\ell,k)$ is a four-dimensional irreducible representation with the actions of the elements given in matrix form by*

$$\tilde{f}_1 = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}, \qquad \tilde{f}_2 = \begin{pmatrix} 0&0&1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&-1&0&0 \end{pmatrix},$$
$$\tilde{r}_1 = \begin{pmatrix} \zeta^\ell&0&0&0\\ 0&\zeta^{-\ell}&0&0\\ 0&0&\zeta^\ell&0\\ 0&0&0&\zeta^{-\ell} \end{pmatrix}, \qquad \tilde{r}_2 = \begin{pmatrix} \eta^k&0&0&0\\ 0&\eta^k&0&0\\ 0&0&\eta^{-k}&0\\ 0&0&0&\eta^{-k} \end{pmatrix}.$$

  Note that $U(2m-\ell,k)$ or $U(\ell,2n-k)$ would result in a representation equivalent to $U(\ell,k)$. This gives $\lfloor m/2\rfloor \times \lfloor n/2\rfloor$ different irreducible spin representations of dimension 4.

- *If at least one of the dihedral parameters m and n is odd, there are also irreducible spin representations of dimension 2.*

  1. *If m is odd, there are n representations $U(m,k)$ for a positive odd integer k with $k < 2n$. The actions of the elements are given in matrix form by*

  $$\tilde{f}_1 = \begin{pmatrix} 1&0\\ 0&-1 \end{pmatrix}, \quad \tilde{f}_2 = \begin{pmatrix} 0&1\\ 1&0 \end{pmatrix}, \quad \tilde{r}_1 = \begin{pmatrix} -1&0\\ 0&-1 \end{pmatrix}, \quad \tilde{r}_2 = \begin{pmatrix} \eta^k&0\\ 0&\eta^{-k} \end{pmatrix}.$$

  2. *If n is odd, there are m representations $U(\ell,n)$ for a positive odd integer ℓ with $\ell < 2m$. The actions of the elements are given in matrix form by*

  $$\tilde{f}_1 = \begin{pmatrix} 0&1\\ 1&0 \end{pmatrix}, \quad \tilde{f}_2 = \begin{pmatrix} 1&0\\ 0&-1 \end{pmatrix}, \quad \tilde{r}_1 = \begin{pmatrix} \zeta^\ell&0\\ 0&\zeta^{-\ell} \end{pmatrix}, \quad \tilde{r}_2 = \begin{pmatrix} -1&0\\ 0&-1 \end{pmatrix}.$$

  3. *If both m and n are odd, it does good to emphasise that there is a representation $U(m,n)$ inside cases 1. and 2. with the actions of the elements given in matrix form by*

  $$\tilde{f}_1 = \begin{pmatrix} 1&0\\ 0&-1 \end{pmatrix}, \quad \tilde{f}_2 = \begin{pmatrix} 0&1\\ 1&0 \end{pmatrix} \quad \tilde{r}_1 = \begin{pmatrix} -1&0\\ 0&-1 \end{pmatrix}, \quad \tilde{r}_2 = \begin{pmatrix} -1&0\\ 0&-1 \end{pmatrix}.$$

Proof. Consider the representation U defined with action (6.13).

From the eigenvalues (6.13), it follows that U is four-dimensional except when either $\zeta^{-\ell} = \zeta^\ell$ or $\eta^{-k} = \eta^k$. This can only happen when $\ell \equiv 0 \mod m$ or $k \equiv 0 \mod n$.

We know from Lemma 6.2.2 that the spin irreducible representations are either of dimension 2 or 4.

We now specify to the three different possibilities of parities of m and n and we give for each of them the restrictions of k and ℓ. Recall that from the arguments before the results, we know ℓ and k to be odd integers.

The odd-odd cases. In this case, the only possibility for $\zeta^\ell = \zeta^{-\ell}$ is $\ell = m$. Assume $\ell = m$ and $k \neq n$, then we consider the following subsets of $\widehat{U}$

$$\widehat{U}_1 := \langle \hat{u}_1 + \hat{u}_2, \hat{u}_3 - \hat{u}_4 \rangle \qquad \widehat{U}_2 := \langle \hat{u}_1 - \hat{u}_2, \hat{u}_3 + \hat{u}_4 \rangle. \tag{6.15}$$

Then the actions of $\tilde{f}_1, \tilde{f}_2$, $\tilde{r}_1$ and $\tilde{r}_2$ fix $\widehat{U}_1$ and $\widehat{U}_2$. Details for $\widehat{U}_1$ are:

$$\begin{aligned} \tilde{f}_1(\hat{u}_1 + \hat{u}_2) &= \hat{u}_2 + \hat{u}_1, & \tilde{f}_1(\hat{u}_3 - \hat{u}_4) &= \hat{u}_4 - \hat{u}_3, \\ \tilde{f}_2(\hat{u}_1 + \hat{u}_2) &= \hat{u}_3 - \hat{u}_4, & \tilde{f}_2(\hat{u}_3 - \hat{u}_4) &= \hat{u}_2 + \hat{u}_1. \end{aligned}$$

We know U is irreducible, so we have either $\phi(\widehat{U}_1) = U$ and $\phi(\widehat{U}_2) = 0$ or $\phi(\widehat{U}_2) = U$ and $\phi(\widehat{U}_1) = 0$. We have thus found a representation $U(m,k)$ of dimension 2. Every k gives a different representation and so there are $n-1$ of them.

A similar process holds when $\eta^k = \eta^{-k}$, but with the subsets given this time by

$$\widehat{U}_1' := \langle \hat{u}_1 + \hat{u}_3, \hat{u}_2 + \hat{u}_4 \rangle \qquad \widehat{U}_2' := \langle \hat{u}_1 - \hat{u}_3, \hat{u}_2 - \hat{u}_4 \rangle. \tag{6.16}$$

We find $m-1$ non-equivalent irreducible representations $U(\ell, n)$.

Finally, if both $\ell = m$ and $k = n$, there is one 2-dimensional representation $U(n,m)$ as then the two possible kernels $\widehat{U}_1$ and $\widehat{U}_2$, or $\widehat{U}_1'$ and $\widehat{U}_2'$ yield equivalent irreducible spin representations.

If $\ell \neq m$ and $k \neq n$, then $U(k,\ell)$ is four-dimensional. Remark furthermore that there is an isomorphism between representations if we change η and ζ to their inverse, so we can restrict the values of the parameters ℓ and k to not count twice by demanding that both ζ^ℓ and η^k have positive imaginary part. This is reflected in the restriction to odd $\ell \in \{1,\dots,m-1\}$ and odd $k \in \{1,\dots,n-1\}$. This gives $(m-1)(n-1)/4$ irreducible representations of dimension 4.

Comparing with equation (2.40), we see that we found the proper number of spin irreducible representations since $2^2 + (n-1) \times 2^2 + (m-1) \times 2^2 + (m-1)(n-1)/4 \times 4^2 = 4mn$.

The odd-even cases. Assume m is odd and n is even. Then $\eta^k \neq \eta^{-k}$, since k must be odd.

When $\ell = m$, we will have n different irreducible representations $U(m,k)$ from all odd values of $k \in \{1,\dots,2n\}$. This follows from looking at the two subsets $\widehat{U}_1, \widehat{U}_2$ of the previous case and showing that they indeed generate two possible subrepresentations.

For odd $\ell \in \{1,\dots,m-1\}$ and for odd $k \in \{1,\dots,n\}$ we have one four-dimensional irreducible representation for each pair ℓ, k. Indeed, there it is possible to ask for η^k to have positive imaginary part since the switch η^k to η^{n-k} offers two equivalent representations. Hence there are indeed $(m-1)n/4$ representations of dimension 4.

Adding all possibilities get $(m-1)n/4 \times 4^2 + n \times 2^2 = 4nm = |\widetilde{W}|/2$, the number of spin representations.

The case m even and n odd if, of course, similar.

The even-even cases. In this instance, there are no two-dimensional representation since ℓ and k are odd. Hence $\zeta^\ell \neq \zeta^{-\ell}$ and $\eta^k \neq \eta^{-k}$.

We ask of ζ^ℓ and η^k to have positive imaginary part to get nonequivalent representations, and we thus find $mn/4$ irreducible representations $U(\ell,k)$ for odd $\ell \in \{1,\dots,m\}$ and odd $k \in \{1,\dots,n\}$. And indeed summing the irreducible representation gets $mn/4 \times 4^2 = |\widetilde{W}|/2$. ∎

6.3 Total angular momentum algebra

Recall from Section 2.2.1 the definition of the rational Cherednik algebra H_κ. We will work in the tensor product $H_\kappa \otimes Cl$.

Notation 6.3.1. *For a root $\alpha \in \Phi$, we will denote $\tilde{\sigma}_\alpha := \rho(\gamma(\alpha)) = \sigma_\alpha \otimes \gamma(\alpha) \in \rho(\mathbb{C}\widetilde{W})$. More specifically, in the notations of (6.3) we shall write*

$$\tilde{\sigma}_p := \tilde{\sigma}_{\alpha_p} = -\tilde{\sigma}_{\alpha_{p+m}} \qquad \text{and} \qquad \tilde{\tau}_q := \tilde{\tau}_{\beta_q},$$

for $1 \leq p \leq m$ and $1 \leq q \leq n$.

Remark 6.3.2. *When evident from the context, we will sometimes omit ρ, implicitly identifying elements of $\widetilde{W}$ with $\rho(\mathbb{C}\widetilde{W}) \subset \mathcal{O}_\kappa$.*

Recall the $\mathfrak{osp}(1|2)$ realisations given by $\mathfrak{g}_{\bar{0}} = \mathrm{span}\{\Delta_\kappa, |x|^2, H\}$, where $\Delta_\kappa, |x|^2$ and H are defined in (2.15) and $\mathfrak{g}_{\bar{1}} = \mathrm{span}\{\underline{\mathrm{D}}, \underline{x}\}$, with $\underline{\mathrm{D}}$ and $\underline{x}$ defined in (2.55).

We know from (2.58) that $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ generate a realisation of the orthosymplectic Lie superalgebra $\mathfrak{g} \simeq \mathfrak{osp}(1|2)$. The total angular momentum algebra $\mathcal{O}_\kappa$ (Definition 2.2.17) is the graded centraliser of this copy.

We shall now recall some structural properties of the algebra $\mathcal{O}_\kappa \subseteq H_\kappa \otimes Cl$ (see [Ost22]), specialised to our context of (V, W). To that end, let $P \in \mathrm{End}(H_\kappa \otimes Cl)$ be the element defined by

$$P = \mathrm{Id} - \tfrac{1}{2}\,\mathrm{ad}(\underline{\mathrm{D}})\,\mathrm{ad}(\underline{x}) = \mathrm{Id} + \tfrac{1}{2}\,\mathrm{ad}(\underline{x})\,\mathrm{ad}(\underline{\mathrm{D}}). \tag{6.17}$$

Here, the adjoint action $\mathrm{ad}\colon H_\kappa \otimes Cl \to \mathrm{End}(H_\kappa \otimes Cl)$ of (2.45) is given by $\mathrm{ad}(x)(y) = [\![x, y]\!]$. The relevance of the operator $P \in \mathrm{End}(H_\kappa \otimes Cl)$ is manifest by the following result.

Theorem 6.3.3 ([Ost22])**.** *When restricted to $\mathcal{A} = \mathrm{Cent}_{H_\kappa \otimes Cl}(\mathfrak{g}_{\bar{0}})$, we have $(P|_{\mathcal{A}})^2 = P|_{\mathcal{A}}$ and $\mathcal{O}_\kappa = P(\mathcal{A})$.*

Note that the Clifford algebra $Cl \cong \mathbb{C} \otimes Cl \subset H_\kappa \otimes Cl$ trivially commutes with $\mathfrak{g}_{\bar{0}} \subset H_\kappa \otimes \mathbb{C} \subset H_\kappa \otimes Cl$, so $Cl \subset \mathcal{A}$. Recalling the linear isomorphism $\gamma : \bigwedge(V) \to Cl$ of (6.6), there is a W-equivariant linear map $O : \bigwedge(V) \to \mathcal{O}_\kappa$, defined for $v \in \bigwedge(V)$ as

$$O(v) = -\tfrac{1}{2} P(\gamma(v)). \tag{6.18}$$

Because of the W-equivariance, we have the following interaction with $\rho(\mathbb{C}\widetilde{W})$:

$$\rho(\tilde{w}) O(v) = (-1)^{|\tilde{w}|k} O(\pi(\tilde{w})(v)) \rho(\tilde{w}), \tag{6.19}$$

for $v \in \bigwedge^k(V)$ and $\tilde{w} \in \widetilde{W}$, where (2.42) gives the $\mathbb{Z}_2$-grading of $\widetilde{W}$, and where π is the map $\mathrm{Pin}_+(d) \to \mathcal{O}(d)$ giving the action of $\widetilde{W}$ on V.

We revisit the results of Section 2.2.5 in the specific context of $W = \mathrm{D}_{2m} \otimes \mathrm{D}_{2n}$, using the formulation of the current chapter.

Theorem 2.2.23 becomes the following.

Theorem 6.3.4 ([Ost22, Prop. 4.9]). *As an associative subalgebra of $H_\kappa \otimes Cl$, the centralizer $\mathcal{O}_\kappa$ is generated by*

$$\{O(v) \mid v \in \bigwedge^2(V) \text{ or } v \in \bigwedge^3(V)\} \cup \{\rho(\tilde{w}) \mid \tilde{w} \in \widetilde{W}\}.$$

For completeness, we give the explicit expressions of the elements of the form (6.18). For $x \in V$, we have

$$O(x) = \left(\sum_{p=1}^{m} \kappa_{\alpha_p} \left\langle x, \alpha_p \right\rangle \tilde{\sigma}_p + \sum_{q=1}^{n} \kappa_{\beta_q} \left\langle x, \beta_q \right\rangle \tilde{\tau}_q \right) \in \rho(\mathbb{C}\widetilde{W}). \tag{6.20}$$

In particular, if $x \in \operatorname{span}\{x_1, x_2\}$, then $O(x)$ can be identified with an element of $\rho(\widetilde{\mathsf{D}}_{2m})$, and if $x \in \operatorname{span}\{x_3, x_4\}$, with an element of $\rho(\widetilde{\mathsf{D}}_{2n})$.

Given vectors $x, y, z \in V$, we have

$$O(x \wedge y) = (x\mathrm{D}_y - y\mathrm{D}_x) + (O(x)\gamma(y) - O(y)\gamma(x)) + \tfrac{1}{2}\gamma(x \wedge y), \tag{6.21}$$

which reduces to a total angular momentum operator when $\kappa = 0$. For three-index symmetries it is instead

$$\begin{aligned} O(x \wedge y \wedge z) = {} & O(x \wedge y)\gamma(z) - O(x \wedge z)\gamma(y) + O(y \wedge z)\gamma(x) \\ & - O(x)\gamma(y \wedge z) + O(y)\gamma(x \wedge z) - O(z)\gamma(x \wedge y) \\ & - \tfrac{1}{2}\gamma(x \wedge y \wedge z). \end{aligned} \tag{6.22}$$

The space $O(\bigwedge^4(V))$ is one-dimensional, we will get to it in (6.30).

Notation 6.3.5. *For elements of the linear basis $\{x_A \mid A \subseteq \{1, \ldots, 4\}\}$ of $\bigwedge(V)$, we shall use the notations $O_A = O(x_A)$, or even $O_{a_1 a_2 \cdots a_p}$, if $A = \{a_1, a_2, \ldots, a_p\} \subseteq \{1, \ldots, 4\}$.*

We will often call, as in the previous chapters, symmetries O_j, O_{ij}, O_{ijk} and O_{ijkl}, one-, two-, three- and four-index symmetry, respectively. The one- and three-index symmetries are odd elements, and the two- and four-index symmetries are even elements.

The commutation relations of Theorem 2.2.23 are now rewritten and simplified for the four dimensional context. Recall that $[\![-,-]\!]$ is the supercommutator for $\mathbb{Z}_2$-graded algebras introduced in Section 2.2.2.

Theorem 6.3.6 ([Ost22])**.** *Let $a,b,c,p,q \in \{1,2,3,4\}$.*

The generators satisfy the following commutation relations:

$$\begin{aligned}\llbracket O_{ab}, O_{pq} \rrbracket = \delta_{bp}A_{aq} - \delta_{bq}A_{ap} - \delta_{ap}A_{bq} + \delta_{aq}A_{bp} \\ + \{O_a, O_{bpq}\} - \{O_b, O_{apq}\},\end{aligned} \tag{6.23}$$

where $A_{ab} = O_{ab} + O_aO_b - O_bO_a$ is an anti-symmetric expression on the indices. Moreover,

$$\llbracket O_{ab}, O_{apq} \rrbracket = -(O_{bpq} + \{O_b, O_{pq}\} + [O_a, O_{abpq}]), \tag{6.24}$$

$$\llbracket O_{ab}, O_{abq} \rrbracket = -(\{O_a, O_{aq}\} + \{O_b, O_{bq}\}), \tag{6.25}$$

and also

$$\llbracket O_{abp}, O_{abq} \rrbracket = \{O_p, O_q\} + \{O_{ap}, O_{aq}\} + \{O_{bp}, O_{bq}\}, \tag{6.26}$$

$$\llbracket O_{abc}, O_{abc} \rrbracket = 2(O_a^2 + O_b^2 + O_c^2 + O_{ab}^2 + O_{ac}^2 + O_{bc}^2 - \tfrac{1}{4}). \tag{6.27}$$

It will be useful to consider also an isotropic basic for the complexified space V with respect to $\langle \cdot, \cdot \rangle$ besides the orthonormal basis $\{x_1, x_2, x_3, x_4\}$ of $V_{\mathbb{R}}$. We decompose $V = V_1 \oplus V_2$, with V_a the span of $\{x_{2a-1}, x_{2a}\}$ for $a \in \{1,2\}$. The space V_1 is fixed under the action of the dihedral group D_{2n}, and the space V_2 is fixed under the action of D_{2m}. For each two-dimensional space V_a, we let

$$z_a^+ := z_a = x_{2a-1} + ix_{2a}, \qquad z_a^- := \bar{z}_a = x_{2a-1} - ix_{2a}. \tag{6.28}$$

Notation 6.3.7. *It will also be convenient to express the symmetry operators with respect to the isotropic basis $\{z_1^+, z_1^-, z_2^+, z_2^-\}$. We shall write, accordingly, with $\delta, \epsilon, \nu \in \{-1,+1\}$ and $a,b,c \in \{1,2\}$,*

$$O_a^\delta := O(z_a^\delta), \quad O_{ab}^{\delta\epsilon} := O(z_a^\delta \wedge z_b^\epsilon) \quad \text{and} \quad O_{abc}^{\delta\epsilon\nu} := O(z_a^\delta \wedge z_b^\epsilon \wedge z_c^\nu), \tag{6.29}$$

with $\delta, \epsilon, \nu \in \{-1,+1\}$ and $a,b,c \in \{1,2\}$. In proofs, we will also often abbreviate $T_a^\varepsilon := O_{abb}^{\varepsilon+-}$.

Denote by Ω the Casimir and by $\mathcal{S}$ the Scasimir of the $\mathfrak{osp}(1|2)$ realisation (2.55). The Casimir is an element of $\mathcal{O}_\kappa$, but the Scasimir is not. However, it is closely related to the symmetry corresponding to the volume form $x_1 \wedge x_2 \wedge x_3 \wedge x_4 \in \bigwedge^4(V)$. Let us denote

$$Z := O_{1234} = -O(z_1 \wedge \bar{z}_1 \wedge z_2 \wedge \bar{z}_2)/4, \tag{6.30}$$

which has the following properties.

Proposition 6.3.8 ([Ost22]). *One has $Z = \mathcal{S}e_{1234} = e_{1234}\mathcal{S}$. For $v \in \bigwedge^k(V)$, we have*

$$Z\,O(v) = (-1)^k O(v) Z\,. \tag{6.31}$$

The element Z can be expressed in terms of the other symmetries as

$$\begin{aligned} Z = {} & \{O_{12}, O_{34}\} - \{O_{13}, O_{24}\} + \{O_{14}, O_{23}\} \\ & - 2(O_{123}O_4 + O_{124}O_3 + O_{134}O_2 + O_{234}O_1). \end{aligned} \tag{6.32}$$

or

$$\begin{aligned} -4Z = {} & \{O_{11}^{+-}, O_{22}^{+-}\} + \{O_{12}^{+-}, O_{12}^{-+}\} + \{O_{12}^{++}, O_{12}^{--}\} \\ & - 2(O_{112}^{+-+}O_2^{-} + O_{112}^{+-+}O_2^{-} + O_{112}^{+-+}O_2^{-} + O_{112}^{+-+}O_2^{-}). \end{aligned} \tag{6.33}$$

Furthermore, we have

$$Z^2 = \Omega + \tfrac{1}{4} = \tfrac{3}{4} - 2\sum_{j=1}^{4} O_j^2 - \sum_{1\le j<k\le 4} O_{jk}^2. \tag{6.34}$$

We note that $O(z_1 \wedge \bar{z}_1 \wedge z_2 \wedge \bar{z}_2) = -4Z$, and that we can rewrite (6.34) as

$$\begin{aligned} Z^2 = {} & \frac{3}{4} - \{O_1^+, O_1^-\} - \{O_2^+, O_2^-\} + \tfrac{1}{4}(O_{11}^{+-})^2 + \tfrac{1}{4}(O_{22}^{+-})^2 \\ & - \tfrac{1}{4}\{O_{12}^{+-}, O_{12}^{-+}\} - \tfrac{1}{4}\{O_{12}^{++}, O_{12}^{--}\}. \end{aligned} \tag{6.35}$$

The (graded) centre of $\mathfrak{O}_\kappa$ was computed in [CDO22] for a general reflection group W. It is always a univariate polynomial ring, but its generator is different depending on whether the longest element w_0 of W is a scalar multiple of the identity $(-1)_V$ in $GL(V)$. Note that for $W = \mathsf{D}_{2m} \times \mathsf{D}_{2n}$, this is the case if and only if both parameters of the dihedral groups are even.

Denote by Ω the Casimir and by $\mathcal{S}$ the Scasimir of the $\mathfrak{osp}(1|2,\mathbb{C})$ realisation of Theorem 2.2.16.

Theorem 6.3.9 ([CDO22]). *The graded centre of $\mathfrak{O}_\kappa$ is the polynomial ring $\mathbb{C}[\mathbb{S}]$ with generator*

$$\mathbb{S} := \begin{cases} \mathcal{S}w_0, & \text{if } w_0 = (-1)_V, \\ \Omega, & \text{if } w_0 \neq (-1)_V. \end{cases}$$

6.4 Ladder operators and the triangular subalgebra

This section introduces the main tools of the study of the representation theory of $\mathcal{O}_\kappa$. It is specific to the double dihedral case.

6.4.1 Definitions and notations

We start by describing an abelian sub-Lie superalgebra of $\mathcal{O}_\kappa$.

Definition 6.4.1. *For $a \in \{1,2\}$, define*

$$H_a := \tfrac{1}{2}O(z_a \wedge \bar{z}_a)/2 = -iO(x_{2a-1} \wedge x_{2a}) \tag{6.36}$$

and let $\mathfrak{h} := \operatorname{span}\{H_1, H_2\}$. Furthermore, define $\mathfrak{a} := \operatorname{span}\{H_1, H_2, Z\}$ and define $\mathfrak{t}_0 := \mathfrak{a} \oplus \operatorname{span}\{\tilde{r}_1, \tilde{r}_2\}$.

Remark 6.4.2. *We will construct below a subalgebra of $\mathcal{O}_\kappa$ containing $\mathfrak{t}_0$ that will play an important role in the representation theory of $\mathcal{O}_\kappa$. The weights with respect to the abelian subalgebra $\mathfrak{t}_0$ will be vital to our arguments. The role of the algebra $\mathfrak{a}$ will be analogous to that of the maximal toral subalgebra of a reductive Lie algebra, and $\mathfrak{h}$ will be analogous to the toral subalgebra of the semisimple part.*

Proposition 6.4.3. *We have $[\![H_1, H_2]\!] = 0$. Furthermore, the element Z commutes with $\mathfrak{h}$.*

Proof. Using (6.23) we get

$$[\![H_1, H_2]\!] = -[\![O_{12}, O_{34}]\!] = -\{O_1, O_{234}\} + \{O_2, O_{134}\} = 0,$$

since $\rho(\tilde{\sigma})O(x \wedge x_3 \wedge x_4) = -O(x \wedge x_3 \wedge x_4)\rho(\tilde{\sigma})$, whenever $x \in V_1$ and $\pi(\widetilde{\sigma}) \in D_{2n}$, in view of relations (6.19) and (6.20). The last claim follows from (6.31). ∎

Let $\varpi_1, \varpi_2 \in \mathfrak{h}^*$ denote the dual functionals defined by $\varpi_a(H_b) = \delta_{ab}$, for $a, b \in \{1,2\}$. Consider the subset $\Upsilon = \Upsilon_{\bar{0}} \sqcup \Upsilon_{\bar{1}}$ of $\mathfrak{h}^*$ defined by

$$\Upsilon_{\bar{0}} := \{\pm(\varpi_1 - \varpi_2), \pm(\varpi_1 + \varpi_2)\}, \qquad \Upsilon_{\bar{1}} := \{\pm\varpi_1, \pm\varpi_2\}. \tag{6.37}$$

Whenever $\alpha \in \Upsilon_{\bar{0}}$, we shall write $\alpha = \delta\varpi_1 + \epsilon\varpi_2$, with $\delta, \epsilon \in \{-1, +1\}$. Similarly, we write $\beta = \epsilon\varpi_a \in \Upsilon_{\bar{1}}$.

Definition 6.4.4. *Define the ladder elements* $\{L_\alpha \mid \alpha \in \Upsilon\}$ *via the following: if* $\alpha = \delta\varpi_1 + \epsilon\varpi_2 \in \Upsilon_{\bar{0}}$, *with* $\delta, \epsilon \in \{\pm 1\}$ *then*

$$L_\alpha := \Big\{H_1, \Big\{H_2, O(z_1^\delta \wedge z_2^\epsilon)\Big\}\Big\}, \tag{6.38}$$

and if $\beta = \epsilon\varpi_b \in \Upsilon_{\bar{1}}$ *with* $\epsilon \in \{\pm 1\}$, *then*

$$L_\beta := \Big\{H_b, O(z_b^\epsilon \wedge z_a \wedge \bar{z}_a)\Big\}, \tag{6.39}$$

where $a \in \{1,2\} \setminus \{b\}$. *For computations, it will be convenient to also denote the ladder elements as*

$$L_{12}^{\delta\epsilon} := L_{\delta\varpi_1 + \epsilon\varpi_2}, \qquad L_b^\epsilon := L_{\epsilon\varpi_b}. \tag{6.40}$$

Recall that the diagonal homomorphism $\rho : \mathbb{C}\widetilde{W} \to \mathcal{O}_\kappa$ is not injective. In fact, its kernel is $\mathbb{C}\widetilde{W}^+ \cong \mathbb{C}W$ and $\rho(\mathbb{C}\widetilde{W}) \cong \mathbb{C}\widetilde{W}^-$. However, restricting ρ to $\widetilde{W}$ induces an injective group homomorphism into the units of $\mathcal{O}_\kappa$. We denote the conjugation action of $\widetilde{W}$ on $\mathcal{O}_\kappa$ by

$$\mathrm{Ad}(\tilde{w})(X) := \rho(\tilde{w})X\rho(\tilde{w})^{-1}, \tag{6.41}$$

for any $\tilde{w} \in \widetilde{W}$ and $X \in \mathcal{O}_\kappa$. Clearly, $\mathrm{Ad}(z)$ acts trivially on $\mathcal{O}_\kappa$. Also, note that since the volume element $x_1 \wedge x_2 \wedge x_3 \wedge x_4 \in \bigwedge(V)$ is invariant for any orthogonal transformation and has even parity, the element $Z \in \mathcal{O}_\kappa$ is invariant for the conjugation action of $\widetilde{W}$.

The ladder elements $\{L_\alpha, \alpha \in \Upsilon\}$ and the abelian algebra $\mathfrak{a}$ determine an 11-dimensional subspace

$$\mathfrak{l} := \mathfrak{a} \oplus \mathrm{span}\{L_\alpha \mid \alpha \in \Upsilon\} \subset \mathcal{O}_\kappa. \tag{6.42}$$

This subspace is a $\widetilde{W}$-module for the conjugation action (6.41). Let $\zeta := e^{i\pi/m}$, $\eta := e^{i\pi/n}$. We have the following description of the $\widetilde{W}$-action.

Lemma 6.4.5. *The* $\widetilde{W}$*-conjugation action on* $\mathfrak{l}$ *is given as follows. First* $\mathrm{Ad}(z)L = L$ *for any* $L \in \mathfrak{l}$ *and* $\mathrm{Ad}(\tilde{w})Z = Z$ *for all* $\tilde{w} \in \widetilde{W}$. *For* $1 \le p \le m$ *and* $1 \le q \le n$, *we have:*

$$\begin{aligned} &\mathrm{Ad}(\tilde{\sigma}_p)H_1 = -H_1, &&\mathrm{Ad}(\tilde{\tau}_q)H_1 = H_1,\\ &\mathrm{Ad}(\tilde{\sigma}_p)H_2 = H_2, &&\mathrm{Ad}(\tilde{\tau}_q)H_2 = -H_2. \end{aligned} \tag{6.43}$$

Furthermore, we have

$$\begin{aligned}
\mathrm{Ad}(\tilde{\sigma}_p)L_1^{\pm} &= \zeta^{\pm 2p}L_1^{\mp}, & \mathrm{Ad}(\tilde{\tau}_q)L_1^{\pm} &= L_1^{\pm}, \\
\mathrm{Ad}(\tilde{\sigma}_p)L_2^{\pm} &= L_2^{\pm}, & \mathrm{Ad}(\tilde{\tau}_q)L_2^{\pm} &= \eta^{\pm 2q}L_2^{\mp},
\end{aligned} \tag{6.44}$$

and

$$\mathrm{Ad}(\tilde{\sigma}_p)L_{12}^{\pm\epsilon} = -\zeta^{\pm 2p}L_{12}^{\mp\epsilon}, \qquad \mathrm{Ad}(\tilde{\tau}_q)L_{12}^{\delta\pm} = -\eta^{\pm 2q}L_{12}^{\delta\mp}. \tag{6.45}$$

for all $\delta, \epsilon \in \{-1, +1\}$.

In particular, for $\tilde{r}_1 = z\tilde{\sigma}_1\tilde{\sigma}_m$ *and* $\tilde{r}_2 = z\tilde{\tau}_1\tilde{\tau}_n$ *we have*

$$\begin{aligned}
\mathrm{Ad}(\tilde{r}_1)L_1^{\pm} &= \zeta^{\mp 2}L_1^{\pm}, & \mathrm{Ad}(\tilde{r}_2)L_1^{\pm} &= L_1^{\pm}, \\
\mathrm{Ad}(\tilde{r}_1)L_2^{\pm} &= L_2^{\pm}, & \mathrm{Ad}(\tilde{r}_2)L_2^{\pm} &= \eta^{\mp 2}L_2^{\pm}, \\
\mathrm{Ad}(\tilde{r}_1)L_{12}^{\pm\epsilon} &= \zeta^{\mp 2}L_{12}^{\pm\epsilon}, & \mathrm{Ad}(\tilde{r}_2)L_{12}^{\delta\pm} &= \eta^{\mp 2}L_{12}^{\delta\pm}.
\end{aligned} \tag{6.46}$$

Proof. Straightforward computations using (6.19). ∎

Remark 6.4.6. *Since* $\mathrm{Ad}(z)$ *acts trivially, the decomposition of* $\mathfrak{l}$ *is with respect to representations in* $\mathrm{Irr}(W)$*. Specifically, in terms of* $\mathsf{D}_{2m} \times \mathsf{D}_{2n}$*-modules, we have*

$$\mathfrak{l} = (\mathrm{trv}\otimes\mathrm{trv})\oplus(\mathrm{sgn}\otimes\mathrm{trv})\oplus(\mathrm{trv}\otimes\mathrm{sgn})\oplus(\mathrm{ref}\otimes\mathrm{trv})\oplus(\mathrm{trv}\otimes\mathrm{ref})\oplus(\mathrm{ref}'\otimes\mathrm{ref}'),$$

where trv, sgn *and* ref *are, respectively, the trivial, the sign and the reflection representation of the corresponding dihedral group, and* ref′ *is the twist of the reflection by the sign representation. In particular,* $\mathrm{ref}'\otimes\mathrm{ref}' \simeq \mathrm{ref}\otimes\mathrm{ref}$.

Proposition 6.4.7. *Let* $\widetilde{W}^{\mathfrak{h}} := \{\tilde{w} \in \widetilde{W} \mid \mathrm{Ad}(\widetilde{w})H = H \text{ for all } H \in \mathfrak{h}\}$*. We have*

$$\widetilde{W}^{\mathfrak{h}} = \langle \tilde{r}_1, \tilde{r}_2 \rangle = W_{(\bar{0},\bar{0})}$$

Proof. It is straightforward to check that $\mathrm{Ad}(\tilde{\sigma}_p)H_1 = -H_1$ and that $\mathrm{Ad}(\tilde{\sigma}_p)H_2 = H_2$, for $p = 1, \ldots, m$, while $\mathrm{Ad}(\tilde{\tau}_q)H_1 = H_1$ and $\mathrm{Ad}(\tilde{\tau}_q)H_2 = -H_2$, for $q = 1, \ldots, n$. Hence, from the descriptions of $\widetilde{G}_{\bar{\imath}}$, $\widetilde{H}_{\bar{\jmath}}$ in (6.10), it follows that $\widetilde{W}^{\mathfrak{h}} = \widetilde{W}_{(\bar{0},\bar{0})}$, which finishes the proof. ∎

6.4.2 Properties of the ladder operators

Lemma 6.4.8. *In the associative algebra $H_\kappa \otimes Cl$, for $\delta, \epsilon \in \{-1, +1\}$, the following holds:*

$$\begin{aligned}
[H_1, O_{12}^{\delta\epsilon}] &= \delta(O_{12}^{\delta\epsilon} + [O_1^\delta, O_2^\epsilon]) - \tfrac{1}{2}\{O_1^\delta, O_{112}^{+-\epsilon}\},\\
[H_2, O_{12}^{\delta\epsilon}] &= \epsilon(O_{12}^{\delta\epsilon} + [O_1^\delta, O_2^\epsilon]) + \tfrac{1}{2}\{O_2^\epsilon, O_{122}^{\delta+-}\}.
\end{aligned}$$

Proof. Using (6.23) and the linearity of $O : \bigwedge(V) \to \mathfrak{O}_\kappa$, it is straightforward to compute

$$\begin{aligned}
[H_1, O_{12}^{\delta\epsilon}] = \tfrac{1}{2}(&\left\langle z_1^-, z_1^\delta\right\rangle (O_{12}^{+\epsilon} + O_1^+ O_2^\epsilon - O_2^\epsilon O_1^+)\\
&- \left\langle z_1^+, z_1^\delta\right\rangle (O_{12}^{-\epsilon} + O_1^- O_2^\epsilon - O_2^\epsilon O_1^-) + \{O_1^+, O_{112}^{-\delta\epsilon}\} - \{O_1^-, O_{112}^{+\delta\epsilon}\}),
\end{aligned}$$

where we recall that $H_1 = \frac{1}{2} O_{11}^{+-}$. When $\delta = +1$, we get $\left\langle z_1^-, z_1^\delta\right\rangle = 2$, $\left\langle z_1^+, z_1^\delta\right\rangle = 0$ and $O_{112}^{+\delta\epsilon} = 0$. Hence

$$[H_1, O_{12}^{+\epsilon}] = (O_{12}^{+\epsilon} + O_1^+ O_2^\epsilon - O_2^\epsilon O_1^+) + \tfrac{1}{2}\{O_1^+, O_{112}^{-+\epsilon}\}.$$

When $\delta = -1$, we get $\left\langle z_1^-, z_1^\delta\right\rangle = 0$, $\left\langle z_1^+, z_1^\delta\right\rangle = 2$ and $O_{112}^{-\delta\epsilon} = 0$. Thus

$$[H_1, O_{12}^{-\epsilon}] = -(O_{12}^{-\epsilon} + O_1^- O_2^\epsilon - O_2^\epsilon O_1^-) - \tfrac{1}{2}\{O_1^-, O_{112}^{+-\epsilon}\}.$$

In any case, the identity

$$[H_1, O_{12}^{\delta\epsilon}] = \delta(O_{12}^{\delta\epsilon} + O_1^\delta O_2^\epsilon - O_2^\epsilon O_1^\delta) - \tfrac{1}{2}\{O_1^\delta, O_{112}^{+-\epsilon}\}$$

holds. The computation for $[H_2, O_{12}^{\delta\epsilon}]$ is similar. ■

Lemma 6.4.9. *For $a \neq b \in \{1, 2\}$ and $\epsilon \in \{-1, +1\}$, we have*

$$\begin{aligned}
[H_a, O_{abb}^{\epsilon+-}] &= \epsilon(O_{abb}^{\epsilon+-} + 2\{O_a^\epsilon, H_b\}) + 2[O_a^\epsilon, Z];\\
[H_b, O_{abb}^{\epsilon+-}] &= 0.
\end{aligned}$$

Proof. For the first equation, we compute directly using the linearity of $O : \bigwedge(V) \to \mathfrak{O}_\kappa$.

$$[\![H_a, O_{abb}^{\epsilon+-}]\!] = [\![O_{2a-1,2a}, (-2)O_{2a-1,2b-1,2b} - 2i\epsilon O_{2a,2b-1,2b}]\!]. \tag{6.47}$$

We can then use (6.24), and the first identity easily follows using linearity. For the second equation, one could similarly compute

(6.47) for H_b instead of H_a and apply (6.25). We shall, however, proceed in a different way. First, note that if $S, T \in H_\kappa \otimes Cl$, then $P(P(S)T) = P(S)P(T)$. It follows from this that if $[\![H_b, \gamma(x)]\!] = 0$ then $[\![H_b, O(x)]\!] = 0$, for any $x \in \bigwedge(V)$. We apply this to $\gamma(z_a^\epsilon \wedge z_b \wedge \bar{z}_b) = \gamma(z_a^\epsilon)\gamma(z_b \wedge \bar{z}_b)$. We have

$$[\![H_b, \gamma(z_a^\epsilon \wedge z_b \wedge \bar{z}_b)]\!] = [\![H_b, \gamma(z_a^\epsilon)]\!]\gamma(z_b \wedge \bar{z}_b) + \gamma(z_a^\epsilon)[\![H_b, \gamma(z_b \wedge \bar{z}_b)]\!] = 0,$$

since, using (6.21), it is easy to see that $[\![O(z_b \wedge \bar{z}_b), \gamma(z_a^\epsilon)]\!] = 0 = [\![O(z_b \wedge \bar{z}_b), \gamma(z_b \wedge \bar{z}_b)]\!]$. This finishes the proof. ■

The next proposition give the ladder operators in this context. If we compare with Proposition 5.5.10, we see that in four dimensions we have two distinct families of operators: the even ones who were introduced in the three-dimensional case, and the new odd ones.

Proposition 6.4.10. *For any $H \in \mathfrak{h}$ and $\alpha \in \Upsilon$, we have*

$$[\![H, L_\alpha]\!] = \alpha(H)L_\alpha. \tag{6.48}$$

Proof. Let $H = cH_1 + dH_2$ with $c, d \in \mathbb{C}$. Suppose first that $\alpha = \delta\varpi_1 + \epsilon\varpi_2 \in \Upsilon_{\bar{0}}$ with $\delta, \epsilon \in \{-1, +1\}$. Note that $\alpha(H) = \delta c + \epsilon d$. Using Lemma 6.4.8 and the definition of L_α (6.38), we compute

$$\begin{aligned}
[\![H, L_\alpha]\!] &= [H, \{H_1, \{H_2, O_{12}^{\delta\epsilon}\}\}] \\
&= \{H_1, \{H_2, [H, O_{12}^{\delta\epsilon}]\}\} \\
&= c\{H_1, \{H_2, e(O_{12}^{\delta\epsilon} + [O_1^\delta, O_2^\epsilon]) - \{O_1^\delta, O_{112}^{+-\epsilon}\}\}\} \\
&\quad + d\{H_1, \{H_2, f(O_{12}^{\delta\epsilon} - [O_1^\delta, O_2^\epsilon]) + \{O_2^\epsilon, O_{122}^{\delta+-}\}\}\} \\
&= (\delta c + \epsilon d)L_\alpha = \alpha(H)L_\alpha,
\end{aligned}$$

where we used the following relations: $\{H_1, \{H_2, X\}\} = \{H_2, \{H_1, X\}\}$, for any $X \in H_\kappa \otimes Cl$; $[O_1^\delta, O_2^\epsilon] = 2O_1^\delta O_2^\epsilon$; $\{O_1^\delta, O_{112}^{+-\epsilon}\} = 2O_1^\delta O_{112}^{+-\epsilon}$; $\{O_2^\epsilon, O_{122}^{\delta+-}\} = 2O_2^\epsilon O_{122}^{\delta+-}$; $\{H_a, O_1^\delta O_2^\epsilon\} = 0$, and Lemma 6.4.9. Then

$$\begin{aligned}
\{H_2, \{H_1, O_1^\delta O_{112}^{+-\epsilon}\}\} &= -\{H_2, O_1^\delta [H_1, O_{112}^{+-\epsilon}]\} = 0, \\
\{H_1, \{H_2, O_2^\epsilon O_{122}^{\delta+-}\}\} &= -\{H_1, O_2^\epsilon [H_2, O_{122}^{\delta+-}]\} = 0.
\end{aligned}$$

Next, suppose that $\beta = \epsilon\varpi_a \in \Upsilon_{\bar{1}}$. Then, using Lemma 6.4.9 and the definition of L_β (6.39), we get

$$[H, L_\beta] = [H, \{H_a, O_{abb}^{\epsilon+-}\}]$$

$$
\begin{aligned}
&= \{H_a, [H, O_{ab\bar{b}}^{\epsilon+-}]\} \\
&= \{H_a, \epsilon\varpi_a(H)(O_{ab\bar{b}}^{\epsilon+-} + 2\{O_a^\epsilon, H_b\}) + 2[O_a^\epsilon, O_{1234}]\} \\
&= \beta(H) L_\beta,
\end{aligned}
$$

since $\{O_a^\epsilon, H_b\} = 2O_a^\epsilon H_b$, $[O_a^\epsilon, O_{1234}] = 2O_a^\epsilon O_{1234}$ and, moreover, H_a anti-commutes with O_a^ϵ and commutes with H_b, O_{1234}. This finishes the proof. ■

Remark 6.4.11. *If we consider* $\dim(V) = 2N$ *in the classical case, that is, when* $\kappa = 0$ *and* W *is the trivial group, the 2-index symmetries* $O(x_j \wedge x_k)$ *span the Lie algebra* $\mathfrak{so}(2N)$ *of type* D_N*. The set* $\Upsilon_{\bar{0}}$ *treated here is its corresponding root system when* $N = 2$*. Furthermore, in this situation, if we consider the ladder elements* L_α*, for* $\alpha \in \Upsilon_{\bar{0}}$*, as a product in* $\mathrm{End}(V)$ *using the root-space decomposition of the orthogonal Lie algebra [KV16], then* L_α *is proportional to the root vector in the direction of* α*.*

6.4.3 The triangular subalgebra

Despite Proposition 6.4.10, the ladder elements do not behave as nicely as root vectors in a Lie (super)algebra. In general, they do not satisfy the property $[\![\mathbb{C}L_\alpha, \mathbb{C}L_\beta]\!] \subseteq \mathbb{C}L_{\alpha+\beta}$. However, the products of any two ladder elements have useful factorisations, which we describe in Propositions 6.4.13–6.4.15.

We first rewrite the ladder operators in a way similar to (5.92).

Lemma 6.4.12. *The ladder elements can be written alternatively as*

$$
L_a^\delta = O_{ab\bar{b}}^{\delta+-}(2H_a + \delta) + 4O_a^\delta(Z + \delta H_b) \tag{6.49}
$$

$$
\begin{aligned}
L_{12}^{\delta\epsilon} = {} & O_{12}^{\delta\epsilon}(2H_1 + \delta)(2H_2 + \epsilon) - O_1^\delta O_{2\bar{1}\bar{1}}^{\epsilon+-}(2H_2 + \epsilon); \\
& + O_{1\bar{2}\bar{2}}^{\delta+-} O_2^\epsilon (2H_1 + \delta) + 2O_1^\delta O_2^\epsilon(\epsilon\delta - 2Z).
\end{aligned} \tag{6.50}
$$

Proof. For (6.49), we compute directly using Lemma 6.4.9:

$$
L_a^\delta = \{H_a, O_{ab\bar{b}}^{\delta+-}\} = 2O_{ab\bar{b}}^{\delta+-} H_a + [H_a, O_{ab\bar{b}}^{\delta+-}] = O_{ab\bar{b}}^{\delta+-}(2H_a + \delta) + 4O_a^\delta(Z + \delta H_b).
$$

The expression (6.50) is slightly more complicated. From the definition of $L_{12}^{\delta\epsilon}$ we compute

$$
\begin{aligned}
L_{12}^{\delta\epsilon} &= \{H_1, \{H_2, O_{12}^{\delta\epsilon}\}\} = 2\{H_1, O_{12}^{\delta\epsilon} H_2\} + \{H_1, [H_2, O_{12}^{\delta\epsilon}]\} \\
&= \{H_1, [H_2, O_{12}^{\delta\epsilon}]\} + 2[H_1, O_{12}^{\delta\epsilon}]H_2 + 4O_{12}^{\delta\epsilon} H_2 H_1.
\end{aligned} \tag{6.51}
$$

We use Lemma 6.4.8, $\{H_1, O_1^\delta O_2^\epsilon\} = 0$ and $\{H_1, O_2^\epsilon O_{122}^{\delta+-}\} = O_2^\epsilon L_1^\delta$ to get the two identities

$$\begin{aligned}
\{H_1,[H_2,O_{12}^{\delta\epsilon}]\} &= O_{12}^{\delta\epsilon}(2\epsilon H_1+\epsilon\delta)+O_1^\delta O_2^\epsilon(2\epsilon\delta-4Z-4\delta H_2)\\
&\quad -\epsilon O_1^\delta O_2^\epsilon + O_{122}^{\delta+-}O_2^\epsilon(2H_1+\delta),\\
2[H_1,O_{12}^{\delta\epsilon}]H_2 &= 2\delta O_{12}^{\delta\epsilon}H_2+4\delta O_1^\delta O_2^\epsilon H_2-2O_1^\delta O_{122}^{\delta+-}H_2.
\end{aligned}$$

Then replacing in (6.51) and factorising finish the proof. ■

The following proposition, and Proposition 6.4.15 present the factorisations of odd ladder operators, similar to Proposition 5.5.10.

Proposition 6.4.13. *For each pair $(\beta,-\beta)$ with $\beta \in \Upsilon_{\bar{1}}$, the ladder elements admit the factorisations*

$$\begin{aligned}
L_1^+L_1^- &= 16((H_1-1/2)^2-O_1^+O_1^-)((Z-H_2)^2-(H_1-1/2)^2),\\
L_1^-L_1^+ &= 16((H_1+1/2)^2-O_1^-O_1^+)((Z+H_2)^2-(H_1+1/2)^2),\\
L_2^+L_2^- &= 16((H_2-1/2)^2-O_2^+O_2^-)((Z-H_1)^2-(H_2-1/2)^2),\\
L_2^-L_2^+ &= 16((H_2+1/2)^2-O_2^-O_2^+)((Z+H_1)^2-(H_2+1/2)^2).
\end{aligned}$$

Proof. We shall indicate the computation for $\beta = \varpi_1$, as the case $\beta = \varpi_2$ is analogous. Recall that we abbreviate $T_1^\pm := O_{122}^{\pm+-}$. From (6.49) we have

$$L_1^\pm = (2H_1 \mp 1)T_1^\pm + 4(Z \mp H_2)O_1^\pm. \tag{6.52}$$

Computing directly from (6.52) using Lemma 6.4.9 and $T_1^\pm O_1^\mp = O_1^\pm T_1^\mp$ we get

$$L_1^\pm L_1^\mp = 4(H_1 \mp \tfrac{1}{2})^2 T_1^\pm T_1^\mp - 16(Z \mp H_2)^2 O_1^\pm O_1^\mp. \tag{6.53}$$

Now, switching from the isotropic basis $\{z_1, \bar{z}_1, z_2, \bar{z}_2\}$ to the orthogonal basis $\{x_1, x_2, x_3, x_4\}$ and using the formula (6.34) for Z^2, together with (6.27), we obtain

$$T_1^\pm T_1^\mp = 4(Z^2 - H_1^2 + H_2^2 - \tfrac{1}{4}) + 2\{O_1^+, O_1^-\} \pm 4i[O_{134}, O_{234}]. \tag{6.54}$$

Applying $-\frac{1}{2}P$ to the expression $O_{134}e_{234} - O_{234}e_{134}$, which we rewrite using (6.21) and (6.22), we compute

$$\begin{aligned}
4i[O_{134},O_{234}] &= 4H_1 - 8H_2Z\\
&\quad + 4i([O_3,O_{123}]+[O_4,O_{124}]-[O_1,O_2]).
\end{aligned} \tag{6.55}$$

Further, we note that, by using $T_2^{\pm}O_2^{\mp} = O_2^{\pm}T_2^{\mp}$,

$$\begin{aligned}
-8i([O_3, O_{123}] + [O_4, O_{124}]) &= [O_2^+ + O_2^-, T_2^+ + T_2^-] - [O_2^+ - O_2^-, T_2^+ - T_2^-] \\
&= 2([O_2^+, T_2^-] + [O_2^-, T_2^+]) \\
&= 0,
\end{aligned}$$

so that (6.55) reads as $4i[O_{134}, O_{234}] = 4H_1 - 8H_2 Z - 4i[O_1, O_2]$. Finally, we combine (6.53), (6.54), (6.55) together with the equation

$$\{O_1^+, O_1^-\} = 2O_1^{\pm}O_1^{\mp} \pm 2i[O_1, O_2]$$

to obtain the desired claim. ■

Proposition 6.4.14. *The ladder operators products $L_{\beta_1}L_{\beta_2}$ with $\beta_1, \beta_2 \in \Upsilon_{\bar{1}}$, $\beta_1 \neq \beta_2$ admit the following factorisations:*

$$\begin{aligned}
L_1^+L_2^+ &= 4L_{12}^{++}(H_1 - H_2 + Z - 1/2), & L_2^+L_1^+ &= 4L_{12}^{++}(H_1 - H_2 - Z + 1/2),\\
L_1^+L_2^- &= -4L_{12}^{+-}(H_1 + H_2 - Z - 1/2), & L_2^-L_1^+ &= -4L_{12}^{+-}(H_1 + H_2 + Z + 1/2),\\
L_1^-L_2^+ &= 4L_{12}^{-+}(H_1 + H_2 + Z + 1/2), & L_2^+L_1^- &= 4L_{12}^{-+}(H_1 + H_2 - Z - 1/2),\\
L_1^-L_2^- &= -4L_{12}^{--}(H_1 - H_2 - Z + 1/2), & L_2^-L_1^- &= -4L_{12}^{--}(H_1 - H_2 + Z - 1/2).
\end{aligned}$$

Proof. We show that, for any $\delta, \epsilon \in \{\pm 1\}$ and $a \neq b \in \{1, 2\}$ the identity

$$L_a^{\delta}L_b^{\delta} = 4L_{ab}^{\delta\epsilon}(Z + \epsilon H_a - \delta H_b - \tfrac{1}{2}\delta\epsilon) \tag{6.56}$$

holds. In this proof, we shall again abbreviate $T_a^{\delta} := O_{ab\bar{b}}^{\delta+-}$. Using equation (6.49), the commutation relations of Lemma 6.4.9 and the action described in Lemma 6.4.5, we compute

$$\begin{aligned}
L_a^{\delta}L_b^{\epsilon} = {} & T_a^{\delta}T_b^{\epsilon}(2H_a + \delta)(2H_b + \epsilon) + 4T_a^{\delta}O_b^{\epsilon}(2H_a + \delta)(Z + \epsilon H_b) \\
& - 4O_a^{\delta}T_b^{\epsilon}(Z - \delta\epsilon - \delta H_b)(2H_b + \epsilon) - 16O_a^{\delta}O_b^{\epsilon}(Z - \delta\epsilon - \delta H_b)(Z + \epsilon H_a).
\end{aligned} \tag{6.57}$$

Further, using the expansion (6.22) of the three-index symmetry $T_a^{\delta} = O_{ab\bar{b}}^{\delta+-}$, together with the fact that $T_a^{\delta}T_b^{\epsilon} = -\frac{1}{2}P(T_a^{\delta}\gamma(z_b^{\epsilon} \wedge z_a \wedge \bar{z}_a))$ we obtain

$$T_a^{\delta}T_b^{\epsilon} = 4O_{ab}^{\delta\epsilon}(Z + \epsilon H_1 - \tfrac{1}{2}\delta\epsilon - \delta H_2) - 2\epsilon O_a^{\delta}T_b^{\epsilon} - 2\delta T_a^{\delta}O_b^{\epsilon} - 4\delta\epsilon O_a^{\delta}O_b^{\epsilon}.$$

Substituting this in (6.57) and using the expression (6.50) yield the desired (6.56). Noting that $L_{ba}^{\epsilon\delta} = -L_{ab}^{\delta\epsilon}$, we obtain all the factorisations claimed in the statement. ■

Proposition 6.4.15. *The ladder operators products $L_{\alpha_1}L_{\alpha_2}$ for $\alpha_1,\alpha_2 \in \Upsilon_{\bar{0}}$ admit the following factorisations*

$$\begin{aligned}
L_{12}^{++}L_{12}^{--} &= 16((H_1-1/2)^2 - O_1^+O_1^-)((H_2-1/2)^2 - O_2^+O_2^-) \\
&\quad \times ((H_1+H_2-1)^2 - (Z-1/2)^2), \\
L_{12}^{--}L_{12}^{++} &= 16((H_1+1/2)^2 - O_1^-O_1^+)((H_2+1/2)^2 - O_2^-O_2^+) \\
&\quad \times ((H_1+H_2+1)^2 - (Z-1/2)^2), \\
L_{12}^{+-}L_{12}^{-+} &= 16((H_1-1/2)^2 - O_1^+O_1^-)((H_2+1/2)^2 - O_2^-O_2^+) \\
&\quad \times ((H_1-H_2-1)^2 - (Z+1/2)^2), \\
L_{12}^{-+}L_{12}^{+-} &= 16((H_1+1/2)^2 - O_1^-O_1^+)((H_2-1/2)^2 - O_2^+O_2^-) \\
&\quad \times ((H_1-H_2+1)^2 - (Z+1/2)^2).
\end{aligned}$$

Proof. Given $a,b \in \{1,2\}$ with $a \neq b$ and $\delta,\epsilon \in \{-1,+1\}$, consider the following elements of $\mathfrak{O}_\kappa$:

$$\begin{aligned}
p_a^\delta &:= (H_a - \tfrac{1}{2}\delta)^2 - O_a^\delta O_a^{-\delta}, \\
q_a^\delta &:= (Z - \delta H_b)^2 - (H_a - \tfrac{1}{2}\delta)^2, \\
\Xi^{\delta,\epsilon} &:= 4\epsilon H_1 Z + 2\epsilon\delta Z + 2\delta H_1 + 1, \\
r = r^{\delta,\epsilon} &:= Z + \epsilon H_1 - \delta H_2 - \tfrac{1}{2}\delta\epsilon.
\end{aligned}$$

It is straightforward to check that the polynomial r divides both q_1^δ and $q_2^\epsilon + \Xi^{-\delta,\epsilon}$ and we have

$$\begin{aligned}
q_1^\delta &= (Z - \epsilon H_1 - \delta H_2 + \tfrac{1}{2}\delta\epsilon)r, \\
q_2^\epsilon + \Xi^{-\delta,\epsilon} &= (Z + \epsilon H_1 + \delta H_2 - \tfrac{3}{2}\delta\epsilon)r.
\end{aligned}$$

Furthermore, note that, on the one hand, we have

$$L_1^\delta L_2^\epsilon L_2^{-\epsilon} L_1^{-\delta} = -16 L_{12}^{\delta\epsilon} L_{12}^{-\delta-\epsilon} r^2,$$

as $[r, L_{12}^{-\delta-\epsilon}] = 0$, and on the other hand, using $[q_2^\epsilon, L_1^{-\delta}] = L_1^{-\delta}\Xi^{-\delta,\epsilon}$ and $[p_2^\epsilon, L_1^{-\delta}] = 0$, we have

$$\begin{aligned}
L_1^\delta L_2^\epsilon L_2^{-\epsilon} L_1^{-\delta} &= L_1^\delta (16 p_2^\epsilon q_2^\epsilon) L_1^{-\delta} \\
&= 16 L_1^\delta L_1^{-\delta} p_2^\epsilon q_2^\epsilon + 16 L_1^\delta [p_2^\epsilon q_2^\epsilon, L_1^{-\delta}] \\
&= 16 L_1^\delta L_1^{-\delta} p_2^\epsilon (q_2^\epsilon + \Xi^{-\delta,\epsilon}) \\
&= 16^2 p_1^\delta p_2^\epsilon q_1^\delta (q_2^\epsilon + \Xi^{-\delta,\epsilon}),
\end{aligned}$$

from which we obtain

$$L_{12}^{\delta\epsilon}L_{12}^{-\delta-\epsilon} = 16(p_1^{\delta}p_2^{\epsilon})((\delta H_1 + \epsilon H_2 - 1)^2 - (Z - \tfrac{1}{2}\delta\epsilon)^2),$$

where the identity

$$(Z-\epsilon H_1-\delta H_2+\tfrac{1}{2}\delta\epsilon)(Z+\epsilon H_1+\delta H_2-\tfrac{3}{2}\delta\epsilon)) = (Z-\tfrac{1}{2}\delta\epsilon)^2-(\delta H_1+\epsilon H_2-1)^2$$

is obtained by completing squares. This finishes the proof. ■

Now, fix a partition $\Upsilon = \Upsilon_+ \cup \Upsilon_-$ with

$$\Upsilon_+ = \{(\varpi_1 - \varpi_2), (\varpi_1 + \varpi_2), \varpi_1, \varpi_2\}$$

and $\Upsilon_- = -\Upsilon_+$. Let $\mathfrak{t}_\pm = \operatorname{span}\{L_\alpha \mid \alpha \in \Upsilon_\pm\}$ and recall the vector space $\mathfrak{t}_0$ of Definition 6.4.1. Finally, given any vector subspace $U \subset \mathfrak{O}_\kappa$, let $\mathcal{A}(U)$ denote the associative subalgebra of $\mathfrak{O}_\kappa$ generated by U.

Definition 6.4.16. *Let* $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_+ \oplus \mathfrak{t}_-$. *We define* $\mathfrak{T}_\pm := \mathcal{A}(\mathfrak{t}_\pm)$, $\mathfrak{T}_0 := \mathcal{A}(\mathfrak{t}_0)$, *and* $\mathfrak{T} := \mathcal{A}(\mathfrak{t})$. *This last subalgebra will be called the triangular subalgebra of* $\mathfrak{O}_\kappa$.

This nomenclature is justified by Proposition 6.4.17, proved below. The computations in Propositions 6.4.13, 6.4.14 and 6.4.15 indicate that the product of ladder elements is again a ladder element, modulo right-multiplication by a polynomial expression in $\mathfrak{t}_0$.

Proposition 6.4.17. *The triangular subagalgebra* $\mathfrak{T}$ *admits a decomposition*

$$\mathfrak{T} = \mathfrak{T}_-\mathfrak{T}_+\mathfrak{T}_0. \tag{6.58}$$

To prove this factorisation, we shall need a preliminary result. We note that the case $(\alpha,\beta) \in \Upsilon_{\bar{1}} \times \Upsilon_{\bar{1}}$ is Proposition 6.4.13

Lemma 6.4.18. *Let* (α,β) *be a pair of roots which are non-opposite to each other. Then, the following assertions hold:*

1. *if* $(\alpha,\beta) \in \Upsilon_{\bar{0}} \times \Upsilon_{\bar{1}} \cup \Upsilon_{\bar{1}} \times \Upsilon_{\bar{0}}$ *and the angle between these roots is obtuse, then* $L_\alpha L_\beta = (L_{\alpha+\beta})p$, *for some* $p \in \mathfrak{T}_0$;
2. *if* $(\alpha,\beta) \in \Upsilon_{\bar{0}} \times \Upsilon_{\bar{1}} \cup \Upsilon_{\bar{1}} \times \Upsilon_{\bar{0}}$ *and the angle between these roots is acute, then* $[L_\alpha, L_\beta] = 0$;

3. *if* $(\alpha,\beta) \in \Upsilon_{\bar{0}} \times \Upsilon_{\bar{0}}$*, then* $L_\alpha L_\beta = (L_{(\alpha+\beta)/2})^2 p$*, for some* $p \in \mathcal{T}_0$*.*

Proof. In 1., note that we have $\alpha + \beta \in \Upsilon_{\bar{1}}$. We can thus assume $\alpha = \delta\varpi_a + \epsilon\varpi_b$ and $\beta = -\epsilon\varpi_b$. Then, by Propositions 6.4.13 and 6.4.14, there exist polynomials $q_1, q_2 \in \mathcal{T}_0$ such that q_2 commutes with $L_b^{-\epsilon}$ and

$$L_a^\delta (L_b^\epsilon L_b^{-\epsilon}) = L_a^\delta q_1 = L_{ab}^{\delta\epsilon} L_b^{-\epsilon} q_2 = (L_a^\delta L_b^\epsilon) L_b^{-\epsilon}.$$

Moreover, from the precise expressions for q_1 and q_2, one can compute that q_2 divides q_1.

For 2., we can assume $\alpha = \delta\varpi_a + \epsilon\varpi_b$ and $\beta = \epsilon\varpi_b$ so that, using Proposition 6.4.14, we can find $q_1, q_2 \in \mathcal{T}_0$ such that

$$L_b^\epsilon (L_a^\delta L_b^\epsilon) = L_b^\epsilon L_{ab}^{\delta\epsilon} q_1 = L_{ab}^{\delta\epsilon} q_2 L_b^\epsilon = (L_b^\epsilon L_a^\delta) L_b^\epsilon.$$

Furthermore, one can compute that $q_2 L_b^\epsilon = L_b^\epsilon q_1$.

Finally, for 3., we can assume $\alpha = \delta\varpi_a + \epsilon\varpi_b$ and $\beta = \delta\varpi_a - \epsilon\varpi_b$. Using again Propositions 6.4.13 and 6.4.14, we find polynomials $q_1, q_2, q_3 \in \mathcal{T}_0$ such that

$$L_a^\delta (L_b^\epsilon L_b^{-\epsilon}) L_a^\delta = L_a^\delta L_a^\delta q_1 = L_{ab}^{\delta\epsilon} L_{ab}^{\delta-\epsilon} q_2 q_3 = (L_a^\delta L_b^\epsilon)(L_b^{-\epsilon} L_a^\delta)$$

with the property that q_2 and q_3 are coprime of degree one and each of them divides q_1; hence, $q_2 q_3$ divides q_1. ■

Proof of Proposition 6.4.17. Due to Proposition 6.4.10, Lemma 6.4.5 and $Z L_\alpha = (-1)^{|\alpha|} L_\alpha Z$, it is clear that any expression in $\mathcal{T}$ is written as sums of elements in $\mathcal{T}_\pm \mathcal{T}_0$. Now let $M = T_1 T_2 \cdots T_n A$ be any monomial expression in $\mathcal{T}$, with $T_i \in \mathcal{T}_\pm$ and $A \in \mathcal{T}_0$. We show that we can rearrange M and write it as an element in $\mathcal{T}_- \mathcal{T}_+ \mathcal{T}_0$, that is, a finite sum $M = \sum_j N_j P_j A_j$, with $N_j \in \mathcal{T}_-$, $P_j \in \mathcal{T}_+$ and $A_j \in \mathcal{T}_0$. Define the inversion set of a monomial M to be

$$\mathcal{I}(M) = \{(i,j) \mid 1 \le i,j \le n, i < j \text{ and } T_i \in \mathcal{T}_+, T_j \in \mathcal{T}_-\}.$$

If $|\mathcal{I}(M)| = 0$, we are done. Else, there exist ladder elements T_i, T_{i+1} with $T_i = L_\alpha \in \mathcal{T}_+$ and $T_{i+1} = L_\beta \in \mathcal{T}_-$. Moreover, we can assume that the index i is maximal for this property, which implies that the remaining factors $T_{i+2} \cdots T_n$ are correctly ordered. There are four cases we need to analyse: (i) $(\alpha,\beta) \in \Upsilon_{\bar{1}} \times \Upsilon_{\bar{1}}$, (ii) $(\alpha,\beta) \in \Upsilon_{\bar{0}} \times \Upsilon_{\bar{1}}$, (iii) $(\alpha,\beta) \in \Upsilon_{\bar{1}} \times \Upsilon_{\bar{0}}$ and (iv) $(\alpha,\beta) \in \Upsilon_{\bar{0}} \times \Upsilon_{\bar{0}}$. We claim that, in each case, when we evaluate T_iT_{i+1}, we get $M = \sum_j M_j$ with each $M_j \in \Upsilon_\pm \Upsilon_0$ monomials with $|\mathcal{I}(M_j)| < |\mathcal{I}(M)|$. So after finitely many steps, we shall write $M \in \Upsilon_- \Upsilon_+ \Upsilon_0$.

Note that we can assume that (α, β) are not opposite roots, since otherwise $L_\alpha L_{-\alpha} \in \Upsilon_0$ would trivially decrease the cardinality of the inversion set. In case (i), we then use Proposition 6.4.14, and in (ii) and (iii), we use items 1. and 2. of Lemma 6.4.18 . In all these cases, we can either commute $T_iT_{i+1} = T_{i+1}T_i$ or $T_iT_{i+1} = TA$ for some $T \in \Upsilon$ and $A \in \Upsilon_0$. After sending A to the right, we have a linear combination of monomials with smaller inversion sets.

Case (iv) is a bit more complicated, since, by Lemma 6.4.18-3., the product $T_iT_{i+1} = T^2A$, for some odd ladder element T. If $T \in \Upsilon_-$, after sending A to the right, this would produce correctly ordered monomials M' so that in each $T_1 \cdots T_{i-1}M'$, the maximal index where an inversion occurs becomes $i_1 < i$, and we can repeat the process for that i_1. So assume $T \in \Upsilon_+$. Note that this could increase the inversion sets of the produced monomials M' after we send A to the right. However, from the maximality assumption on i, we necessarily need to deal with the configuration $T^2L_1 \cdots L_m$ where each $L_j \in \Upsilon_-$. Since T is an odd ladder element, when we deal with this configuration, case (iv) will not occur in the first step, and hence we can, after finitely many steps, reorder $T(L_1 \cdots L_m) = \sum_j M_jP_jA_j$ in $\Upsilon_-\Upsilon_+\Upsilon_0$. We then obtain $T^2L_1 \cdots L_m = \sum_j TM_jP_jA_j$. But note that $|\mathcal{I}(TL_1 \cdots L_m)| = m$ and also $|\mathcal{I}(TM_j)| \leq m$ for each j, since the expressions $M_j \in \Upsilon_-$ are a product of negative ladder elements with at most m factors. After reordering each TM_j (using the same principle that allowed us to reorder $TL_1 \cdots L_m$), the original monomial is replaced by a linear combination

$$M = T_1T_2 \cdots T_nA = T_1T_2 \cdots T_{i-1}\left(\sum_k M_kP_kA_k\right).$$

with $|\mathcal{I}(T_1T_2 \cdots T_{i-1}M_k)| < |\mathcal{I}(M)|$, for each k. This finishes the proof by induction. ■

6.5 Finite representation theory

In this section, we present a coarse classification of the finite-dimensional irreducible representations of $\mathfrak{O}_\kappa$. We then focus in detail on one branch of representations that can be realised by the Dunkl polynomial monogenics and explore certain other possibilities from the coarse classification. The main tool is the triangular subalgebra $\mathfrak{T}$ of the previous section.

6.5.1 Restriction to the triangular subalgebra

In this section, we show that the restriction to the triangular subalgebra $\mathfrak{T}$ is enough to completely determine the finite-dimensional representation theory of $\mathfrak{O}_\kappa$.

Let M be an $\mathfrak{O}_\kappa$-module. If $\mu \in \mathfrak{a}^*$, we let $M_\mu = \{v \in M \mid Av = \mu(A)v$, for all $A \in \mathfrak{a}\}$ and we denote by $\mathrm{Wt}_{\mathfrak{a}}(M) = \{\mu \in \mathfrak{a}^* \mid M_\mu \neq 0\}$ the set of $\mathfrak{a}$-weights of M. We similarly define the set $\mathrm{Wt}_{\mathfrak{h}}(M)$ of $\mathfrak{h}$-weights. Recall that $\mathfrak{a} = \mathrm{span}\{H_1, H_2, Z\}$. We shall write $\mu = \mu_1\varpi_1 + \mu_2\varpi_2 + \mu_Z\varpi_Z$, or simply $\mu = (\mu_1, \mu_2, \mu_Z)$ where $\mu(H_a) = \mu_a$, for $a \in \{1,2\}$ and $\mu_Z = \mu(Z)$.

Lemma 6.5.1. *Let M be a finite-dimensional $\mathfrak{O}_\kappa$-module and assume $0 \neq u \in M_\mu$. If $\mu_a = -\frac{1}{2}\delta$ for some $\delta \in \{\pm 1\}$, then the following hold:*

1. $O^{\delta+-}_{abb}(u) \in M_\nu$, *with* $\nu = \frac{1}{2}\delta\varpi_a + \mu_b\varpi_b - \mu_Z\varpi_Z$;
2. $L^\delta_a(u) = 0$.

Proof. Assume $a = 1$, as the case $a = 2$ is similar. We have $\mu = (-\delta/2, \mu_2, \mu_Z)$. We write

$$O^{\delta+-}_{122}(u) = \sum_{\nu \in \mathrm{Wt}_{\mathfrak{a}}(M)} u_\nu,$$

for some $u_\nu \in M_\nu$. We claim that $u_\nu = 0$, unless $\nu = (\delta/2, \mu_2, -\mu_Z)$. Indeed, since $[H_2, O^{\delta+-}_{122}] = 0$ by Lemma 6.4.9 and $\{Z, O^{\delta+-}_{122}\} = 0$, then by linear independence, it follows that $u_\nu = 0$, unless $\nu = (\nu_1, \mu_2, -\mu_Z)$, for some ν_1. Using Lemma 6.4.9, we get

$$\begin{aligned} H_1 O^{\delta+-}_{122}(u) &= -\tfrac{1}{2}\delta O^{\delta+-}_{122}(u) + [H_1, O^{\delta+-}_{122}](u) \\ &= \tfrac{1}{2}\delta O^{\delta+-}_{122}(u) + 4O^{\delta}_{1}(\delta H_2 + Z)(u). \end{aligned}$$

Thus

$$(H_1 - \tfrac{1}{2}\delta)O^{\delta+-}_{122}(u) = \sum_\nu (\nu_1 - \tfrac{1}{2})u_\nu = 4(\mu_Z + \delta\mu_2)O^{\delta}_{1}(u) \in M_{(\delta/2, \mu_2, -\mu_Z)}.$$

It follows that $u_\nu = 0$ unless $\nu = (\delta/2, \mu_2, -\mu_Z)$. This finishes 1.

But, since 1. holds, it then follows that

$$L_1^\delta(u) = H_1 O_{122}^{\delta+-}(u) + O_{122}^{\delta+-} H_1(u) = 0.$$

The lemma is proved. ∎

In analogy with the theory of semisimple Lie algebras, given $\mu, \lambda \in \mathfrak{h}^*$, we say $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a linear combination of positive roots with positive integer coefficients. We say that $\mu \in \mathrm{Wt}_{\mathfrak{h}}(M)$ is a maximal weight of M if there is no $\lambda \in \mathrm{Wt}_{\mathfrak{h}}(M)$ such that $\mu \prec \lambda$. Furthermore, any nonzero element $\mathfrak{v} \in M_\mu$, with μ a maximal weight, will be called a highest weight vector.

Proposition 6.5.2. *If M is a finite-dimensional $\mathfrak{O}_\kappa$-module, then there exists $\mu \in \mathrm{Wt}_{\mathfrak{h}}(M)$ maximal.*

Proof. We write $\mathfrak{Re}(\mu)$ for the real part of μ. Since $|\mathrm{Wt}_{\mathfrak{h}}(M)| < \infty$, let $a = \max\{\mathfrak{Re}(\mu_1) \mid \mu = \mu_1\varpi_1 + \mu_2\varpi_2 \in \mathrm{Wt}_{\mathfrak{h}}(M)\}$. Amongst all elements $\mu = \mu_1\varpi_1 + \mu_2\varpi_2 \in \mathrm{Wt}_{\mathfrak{h}}(M)$ such that $a = \mathfrak{Re}(\mu_1)$, choose one for which $\mathfrak{Re}(\mu_2)$ is maximal. Since, for every $\alpha \in \Upsilon_+$, we have that $\mu + \alpha$ increases $\mathfrak{Re}(\mu_1)$ or $\mathfrak{Re}(\mu_2)$, the element $\mu \in \mathrm{Wt}_{\mathfrak{h}}(M)$ thus described is a maximal weight. ∎

Given a finite-dimensional $\mathfrak{O}_\kappa$-module M, let $\lambda \in \mathrm{Wt}_{\mathfrak{h}}(M)$ be the maximal weight and choose any highest-weight vector $\mathfrak{v} \in M_\lambda$. Define $U(\mathfrak{v}) = \rho(\mathbb{C}\widetilde{W})\mathfrak{T}\mathfrak{v}$ as the vector subspace of M spanned by all translates of the actions of $\mathbb{C}\widetilde{W}$ and $\mathfrak{T}$. That is

$$U(\mathfrak{v}) := \mathrm{span}\{\rho(\widetilde{w})T\mathfrak{v} \mid \widetilde{w} \in \widetilde{W}, T \in \mathfrak{T}\}.$$

In view of the classical case and of the monogenic representation, we consider the case where the highest-weight space is one-dimensional. Then all weight spaces are one-dimensional and we can construct the representations of $\mathfrak{O}_\kappa$ from that of $\mathfrak{T}$.

Proposition 6.5.3. *Suppose $\mathfrak{v}$ is a highest-weight vector of the irreducible $\mathfrak{O}_\kappa$-module M and that* $\mathrm{Wt}(\mathfrak{v})$ *is one-dimensional. Then:*

1. *all weight spaces of M are one-dimensional;*

2. *$U(\mathfrak{v})$ is a submodule of M.*

Furthermore, if $\mu(2H_1+\delta) \neq 0 \neq \mu(2H_2+\epsilon)$ for $\delta, \epsilon \in \{-1,+1\}$, we have explicit formulas for the actions.

Proof. We begin by the first statement. If the highest-weight vector is one-dimensional, then applying the triangular subalgebra $\mathfrak{T}$ on $\mathfrak{v}$ will divide into one-dimensional weight spaces since the actions are invertible. Then acting with $\widetilde{W}$ gives $U(\mathfrak{v})$. Now, we claim that the action of the full algebra $\mathfrak{O}_\kappa$ will always be a sum of multiples of the vectors of the $\mathfrak{T}$ weight spaces. Suppose it is not the case, then we have two linearly independent vectors of weight (μ_1, μ_2). Using reflections and ladder operators, we can navigate back to the highest weight space. Indeed, if it is impossible to navigate back to the highest-weight space, then there would be a $\mathfrak{O}_\kappa$-submodule of M that does not contain $\mathfrak{v}$, contradicting the irreducibility. So, we can navigate back to the highest-weight space. Then we would find now two linearly independent highest weight vectors, since all operation are invertible and preserve thus linear independence, but the highest-weight space is one-dimensional, a contradiction.

For the second point, we now know that all weight spaces are one-dimensional. Since $\mathfrak{O}_\kappa$ is generated by the 2- and 3-index symmetries and $\rho(\widetilde{W})$, it suffices to show that $O_{12}^{\delta\epsilon}(U(\mathfrak{v})) \subseteq U(\mathfrak{v})$ and $O_{abb}^{\delta+-}(U(\mathfrak{v})) \subseteq U(\mathfrak{v})$, for all $\delta, \epsilon \in \{\pm 1\}$ and $a, b \in \{1,2\}$. Note that

$$U(\mathfrak{v}) = \bigoplus_{\mu \in \mathrm{Wt}_{\mathfrak{h}}(M)} U(\mathfrak{v})_\mu,$$

with $U(\mathfrak{v})_\mu \subseteq M_\mu$, since the actions of $\widetilde{W}$ and $\mathfrak{T}$ preserves the $\mathfrak{h}$-decomposition. Given $0 \neq u \in U(\mathfrak{v})_\mu$.

Suppose now $\mu(2H_1+\delta) \neq 0 \neq \mu(2H_2+\epsilon)$ for $\delta, \epsilon \in \{-1,+1\}$ for all weight spaces. From (6.49), it follows that, up to a constant, $O_{abb}^{\delta+-}$ acts on u via $L_a^\delta - 4O_a^\delta(Z+\delta H_b)$. Then

$$O_{abb}^{\delta+-}(u) = \frac{(L_a^\delta - 4O_a^\delta(Z+\delta H_b))}{\mu(2H_a+\delta)} u \in U(\mathfrak{v}).$$

Likewise, using (6.50), we have

$$O_{12}^{\delta\epsilon}(u) = \frac{1}{(2\lambda_1^u+\delta)(2\lambda_2^u+\epsilon)} \Big(L_{12}^{\delta\epsilon} + O_1^\delta O_{211}^{\epsilon+-}(2\lambda_2^u+\epsilon) - O_{122}^{\delta+-} O_2^\epsilon (2\lambda_2^u+\delta$$

$$- 2O_1^\delta O_2^\epsilon(\epsilon\delta - 2\Lambda^u)\Big)u \in U(\mathfrak{v}).$$

Thus we have fully described the representation, proving that $U(\mathfrak{v})$ is a submodule of M, and thus that it is M by irreducibility.

However, even in the cases where there would be a division by 0 in the formula, we still know in which weight spaces three-index symmetries go by Lemma 6.5.1, and we can follow a similar process for two-index symmetries by Lemma 6.4.8, in both cases we see that the weight spaces visited are the same as $U(\mathfrak{v})$ and so $U(\mathfrak{v})$ is a submodule of M since all the weight spaces are one-dimensional. ■

Remark 6.5.4. *We will assume for the remaining of the chapter that the representations we are considering have a one-dimensional highest-weight space. This is always the case by the proof of the previous proposition outside the problematic case of highest-weight space of* (H_1, H_2)*-weight being* $(1/2, 1/2)$*. We leave the care of this particular problematic case for later inquiries.*

6.5.2 Actions of group algebra elements

A certain class of group elements will be relevant in the study of the representations. We can express their action with a simple case-by-case formula. For $a \in \{1, 2\}$, let m_a denote the dihedral parameter $m_a := m$ and $m_2 := n$.

Lemma 6.5.5. *For odd integers* $\ell_a \in \{1, \dots, 2m_a - 1\}$ $(a \in \{1, 2\})$*, let* $u(\ell_1, \ell_2)$ *be an irreducible module for the quotient group* $\widetilde{W}_{\bar{0},\bar{0}} = \langle \tilde{r}_1, \tilde{r}_2 \rangle (\cong C_{2m} \times C_{2n}/C_2)$ *with the action (6.11) on* $\mathfrak{u} \in u(\ell_1, \ell_2)$*. Then, for* $a \in \{1, 2\}$

$$O_a^\pm O_a^\mp \mathfrak{u} = Q_a(\ell_a \pm 1)^2 \mathfrak{u}\,, \tag{6.59}$$

where

$$Q_a(j) := \begin{cases} m_a \kappa_{2a-1}, & m_a \text{ odd, } j \equiv 0 \bmod m_a; \\ \frac{m_a}{2}(\kappa_{2a-1} + \kappa_{2a}), & m_a \text{ even, } j \equiv 0 \bmod m_a; \\ \frac{m_a}{2}|\kappa_{2a-1} - \kappa_{2a}|, & m_a \text{ even, } j \equiv m_a/2 \bmod m_a; \\ 0, & \text{otherwise.} \end{cases} \tag{6.60}$$

Proof. We will prove this for the first dihedral group D_{2m}, the second D_{2n} follows in the same fashion. Note that $\tilde{r}_1^p \tilde{f}_1 = z^p \tilde{\sigma}_p$, for $p = 1, \dots, m$.

From (6.20), using (6.3), (6.28) and $\rho(z) = -1$, we have, with $\zeta = e^{i\pi/m}$,

$$O(z_1^{\pm}) = \pm i \sum_{p=1}^{m} \kappa_{\alpha_p} \zeta^{\pm p} \rho(\tilde{\sigma}_p) = \pm i \sum_{p=1}^{m} \kappa_{\alpha_p} (-\zeta^{\pm 1} \rho(\tilde{r}_1))^p \rho(\tilde{f}_1). \tag{6.61}$$

Denoting $\omega_{\pm} := \zeta^{\pm 1}\rho(\tilde{r}_1)$, which satisfies $\omega_{\pm}^m = (\zeta^{\pm 1}\rho(\tilde{r}_1))^m = -\rho(z) = 1$, we thus find

$$\begin{aligned} O(z_1^{\pm})O(z_1^{\mp}) &= \sum_{p,q=1}^{m} \kappa_{\alpha_p}\kappa_{\alpha_q}\omega_{\pm}^{p-q} = \sum_{p=0}^{m-1}\sum_{q=1}^{m} \kappa_{\alpha_q}\kappa_{\alpha_{p+q}}\omega_{\pm}^{p} \\ &= \begin{cases} m\kappa_1^2 \sum_{p=0}^{m-1} \omega_{\pm}^p & \text{if } m \text{ is odd;} \\ \sum_{p=0}^{m/2-1} \left(\frac{m}{2}(\kappa_1^2+\kappa_2^2)\omega_{\pm}^{2p} + m\kappa_1\kappa_2\omega_{\pm}^{2p+1}\right) & \text{if } m \text{ is even,} \end{cases} \end{aligned}$$

where we used

$$\sum_{q=1}^{m} \kappa_{\alpha_q}\kappa_{\alpha_{p+q}} = \begin{cases} \frac{m}{2}(\kappa_1^2+\kappa_2^2) & \text{if } m \text{ is even and } p \text{ is even;} \\ m\kappa_1\kappa_2 & \text{if } m \text{ is even and } p \text{ is odd;} \\ m\kappa_1^2 & \text{if } m \text{ is odd.} \end{cases} \tag{6.62}$$

Acting on an eigenvector for $\tilde{r}_1$ gives the desired result. ∎

Note that $Q_a(j)$ is always non-negative.

It will be useful in what follows to have an expression for the action of products of the one-index symmetries on ladder operators.

Lemma 6.5.6. *Let $u(\ell,k)$ be an irreducible module for the group $\widetilde{W}_{(\bar{0},\bar{0})} \subset \widetilde{W}$. Let $\delta,\epsilon \in \{-,+\}$. We have, for $u \in u(\ell_1,\ell_2)$ and $a,b \in \{1,2\}$,*

$$O_1^{\pm}O_1^{\mp}(L_{12}^{\delta\epsilon})^K u = Q_1(\ell_1 \pm 1 + 2\delta K)^2 (L_{12}^{\delta\epsilon})^K u, \tag{6.63}$$

$$O_2^{\pm}O_2^{\mp}(L_{12}^{\delta\epsilon})^K u = Q_2(\ell_2 \pm 1 + 2\epsilon K)^2 (L_{12}^{\delta\epsilon})^K u, \tag{6.64}$$

$$O_a^{\pm}O_a^{\mp}(L_b^{\epsilon})^K u = Q_a(\ell_a \pm 1 + \delta_{ab}2\epsilon K)^2 (L_b^{\epsilon})^K u,\ a,b \in \{1,2\}, \tag{6.65}$$

where $Q_a(j)$ is defined in (6.60), and $\delta_{ab} := 1$ if $a = b$ and $\delta_{ab} := 0$ if $a \neq b$.

Proof. From Lemma 6.4.5, we have

$$\rho(\tilde{r}_1)(L_{12}^{\delta\epsilon})^K = \zeta^{-2\delta K}(L_{12}^{\delta\epsilon})^K \rho(\tilde{r}_1), \quad \rho(\tilde{r}_2)(L_{12}^{\delta\epsilon})^K = \eta^{-2\epsilon K}(L_{12}^{\delta\epsilon})^K \rho(\tilde{r}_2),$$

$$\rho(\tilde{r}_1)(L_1^\epsilon)^K = \zeta^{-2\epsilon K}(L_1^\epsilon)^K\rho(\tilde{r}_1), \qquad \rho(\tilde{r}_1)(L_2^\epsilon)^K = (L_2^\epsilon)^K\rho(\tilde{r}_1),$$
$$\rho(\tilde{r}_2)(L_1^\epsilon)^K = (L_1^\epsilon)^K\rho(\tilde{r}_2), \qquad \rho(\tilde{r}_2)(L_2^\epsilon)^K = \eta^{-2\epsilon K}(L_2^\epsilon)^K\rho(\tilde{r}_2),$$

and then, one application of Lemma 6.5.5 concludes the proof. ∎

The previous lemma will often be applied to products of the form $(L_a^-)^K\mathfrak{v}$, $(L_{12}^{-+})^K\mathfrak{v}$ and $(L_{12}^{--})^K\mathfrak{v}$; it will thus be useful to reserve a notation for the coefficients appearing.

Notation 6.5.7. *Let $a \in \{1,2\}$. We will denote*

$$\begin{aligned} E_a^\pm(K) &:= Q_a(\ell_a \pm 1 - 2K)^2; \\ F_a^\pm(K) &:= Q_a(\ell_a \pm 1 + (-1)^a 2K)^2; \\ G_a^\pm(K) &:= Q_a(\ell_a \pm 1 - 2K)^2. \end{aligned} \tag{6.66}$$

Then we have, if $\tilde{r}_1\mathfrak{v} = \zeta^{\ell_1}\mathfrak{v}$ and $\tilde{r}_2\mathfrak{v} = \eta^{\ell_2}\mathfrak{v}$, for $\mathfrak{v} \in \mathcal{V}$, $K \in \mathbb{N}$, odd $\ell_1 \in \{1,\dots,2m-1\}$ and odd $\ell_2 \in \{1,\dots,2n-1\}$,

$$\begin{aligned} O_a^\pm O_a^\mp (L_a^-)^K\mathfrak{v} &= E_a^\pm(K)(L_a^-)^K\mathfrak{v}, \\ O_a^\pm O_a^\mp (L_{12}^{-+})^K\mathfrak{v} &= F_a^\pm(K)(L_{12}^{-+})^K\mathfrak{v}, \\ O_a^\pm O_a^\mp (L_{12}^{--})^K\mathfrak{v} &= G_a^\pm(K)(L_{12}^{--})^K\mathfrak{v}. \end{aligned}$$

Remark 6.5.8. *In view of Theorem 6.3.6 and equation (2.41), the restriction of any $\mathfrak{O}_\kappa$-module $\mathcal{V}$ to the subalgebra $\rho(\mathbb{C}\widetilde{W}) \subset \mathfrak{O}_\kappa$ decomposes as a direct sum of spin representations in* $\mathrm{sIrr}(\widetilde{W})$*, by Maschke's Theorem.*

6.5.3 Coarse classification

In this section, we will use the triangular subalgebra $\mathfrak{T}$ to characterise the finite-dimensional irreducible representations of $\mathfrak{O}_\kappa$. We will use the weight theory coming from $\mathfrak{T}$. This will label the representation by a set of weights, and we will give a set of equations they must satisfy in Theorem 6.5.9. The next sections will extend this skeleton to a full $\mathfrak{O}_\kappa$-representation in certain cases.

We begin by defining a set of values that will play an important role in what follows. They come from the factorisations of Propositions 6.4.13 and 6.4.15.

$$A_1 := ((\lambda_1 + 1/2)^2 - F_1^-(0))((\Lambda + \lambda_2)^2 - (\lambda_1 + 1/2)^2); \tag{6.67}$$

$$
\begin{aligned}
A_2 &:= ((\lambda_2+1/2)^2 - F_2^-(0))((\Lambda+\lambda_1)^2 - (\lambda_2+1/2)^2); && (6.68)\\
B_1 &:= ((\lambda_1+1/2)^2 - F_1^-(0))((\lambda_2-1/2)^2 - F_2^+(0))\\
&\quad \times ((\lambda_1-\lambda_2+1)^2 - (\Lambda+1/2)^2); && (6.69)\\
B_2 &:= ((\lambda_1+1/2)^2 - F_1^-(0))((\lambda_2+1/2)^2 - F_2^-(0))\\
&\quad \times ((\lambda_1+\lambda_2+1)^2 - (\Lambda-1/2)^2); && (6.70)\\
C_{12}^{-+}(j) &:= ((\lambda_1-j-1/2)^2 - F_1^+(j))((\lambda_2+j+1/2)^2 - F_2^-(j))\\
&\quad \times ((\lambda_1-\lambda_2-2j-1)^2 - (\Lambda+1/2)^2); && (6.71)\\
C_{12}^{--}(j) &:= ((\lambda_1-j-1/2)^2 - G_1^+(j))((\lambda_2-j-1/2)^2 - G_2^+(j))\\
&\quad \times ((\lambda_1+\lambda_2-2j-1)^2 - (\Lambda-1/2)^2) && (6.72)\\
C_1(j) &:= ((\lambda_1-j-1/2)^2 - E_1^+(j))\\
&\quad \times (((-1)^j\Lambda-\lambda_2)^2 - (\lambda_1-j-1/2)^2); && (6.73)\\
C_2(j) &:= ((\lambda_2-j-1/2)^2 - E_2^+(j))\\
&\quad \times (((-1)^j\Lambda-\lambda_1)^2 - (\lambda_2-j-1/2)^2); && (6.74)
\end{aligned}
$$

where $E_a^\pm(j)$ and $F_a^\pm(j)$ are defined as in (6.66).

We can now state the main theorem of the section.

Theorem 6.5.9. *A finite-dimensional irreducible $\mathcal{V}$ admits a highest-weight vector $\mathfrak{v}$ with weight $(\mu_1,\mu_2,\mu_Z) = (\lambda_1,\lambda_2,\Lambda)$ inside a $\widetilde{W}$-representation $U(\ell,k)$ for certain integer ℓ and k. We call then the element of $(\Lambda,\lambda_1,\lambda_2,\zeta^\ell,\eta^k) \in \mathfrak{t}_0^*$, the label of $\mathcal{V}$. The elements of the label satisfy the following system of equations:*

$$
\begin{cases}
A_1 = 0,\ A_2 = 0;\\
B_1 = 0,\ B_2 = 0;\\
C_{12}^{-+}(N) = 0,\ C_{12}^{-+}(j) \neq 0, & 0 \le j < N;\\
C_{12}^{--}(N') = 0,\ C_{12}^{--}(j) \neq 0, & 0 \le j < N';\\
C_1(M_1) = 0,\ C_1(j) \neq 0, & 0 \le j < M_1;\\
C_2(M_2) = 0,\ C_2(j) \neq 0, & 0 \le j < M_2;
\end{cases} \tag{6.75}
$$

for certain integers $N, N', M_1, M_2 \in \mathbb{N}$.

Proof. Let $\mathcal{V}$ be a finite-dimensional irreducible representation of $\mathcal{O}_\kappa$. We can restrict it to a $\mathfrak{T}$-representation. By Proposition 6.4.17, we then know $\mathcal{V}$ admits a basis of common eigenvectors of Z, H_1 and

H_2. We take the highest-weight vector with respect to the order on the Cartan part $\mathfrak{T}_0$ and denote it by $\mathfrak{v}$. This highest-weight vector has thus three eigenvalues linked to Z, H_1 and H_2 and we denote them by:

$$Z\mathfrak{v} = \Lambda\mathfrak{v}, \qquad H_1\mathfrak{v} = \lambda_1\mathfrak{v}, \qquad H_2\mathfrak{v} = \lambda_2\mathfrak{v}. \tag{6.76}$$

Furthermore, if we act on $\mathfrak{v}$ by $\widetilde{W}$, we will find an irreducible spin $\widetilde{W}$-representation U such that $\widetilde{W} \circlearrowleft \mathfrak{v} \in U \subset \mathcal{V}$, and from Theorem 6.2.3 then there are odd positive integers ℓ, k such that $U \simeq U(\ell, k)$.

Finally, since $\mathcal{V}$ is finite-dimensional, it means that there exists minimal integers N, N' M_1 and M_2 such that the following chains stop:

$$\begin{aligned}
(L_{12}^{-+})^{N+1}\mathfrak{v} &= 0, &\quad \text{and} \quad && (L_{12}^{-+})^{N}\mathfrak{v} &\neq 0,\\
(L_{12}^{--})^{N'+1}\mathfrak{v} &= 0, &\quad \text{and} \quad && (L_{12}^{--})^{N'}\mathfrak{v} &\neq 0,\\
(L_{1}^{-})^{M_1+1}\mathfrak{v} &= 0, &\quad \text{and} \quad && (L_{1}^{-})^{M_1}\mathfrak{v} &\neq 0,\\
(L_{2}^{-})^{M_2+1}\mathfrak{v} &= 0, &\quad \text{and} \quad && (L_{2}^{-})^{M_2}\mathfrak{v} &\neq 0.
\end{aligned}$$

It remains to show that the relations (6.75) are respected. Since $\mathfrak{v}$ is a highest-weight vector then:

$$L_1^{+}\mathfrak{v} = 0 \quad \text{and} \quad L_2^{+}\mathfrak{v} = 0 \quad \text{and} \quad L_{12}^{-+}\mathfrak{v} = 0.$$

This translates, using the factorisations of Propositions 6.4.14 and 6.4.15, to the following three relations coming respectively from $L_1^{-}L_1^{+}\mathfrak{v} = 0$, $L_2^{-}L_2^{+}\mathfrak{v} = 0$ and $L_{12}^{-+}L_{12}^{+-}\mathfrak{v} = 0$:

$$\begin{aligned}
((\lambda_1 + 1/2)^2 - F_1^{-}(0))((\Lambda + \lambda_2)^2 - (\lambda_1 + 1/2)^2) &= 0,\\
((\lambda_2 + 1/2)^2 - F_2^{-}(0))((\Lambda + \lambda_1)^2 - (\lambda_2 + 1/2)^2) &= 0,\\
((\lambda_1 + 1/2)^2 - F_1^{-}(0))((\lambda_2 - 1/2)^2 - F_2^{+}(0))&\\
\times((\lambda_1 - \lambda_2 + 1)^2 - (\Lambda + 1/2)^2) &= 0.
\end{aligned}$$

Thus, we found $A_1 = A_2 = B_1 = 0$.

Since L_{12}^{-+}, L_{12}^{--}, L_1^{-} and L_2^{-} are ladder operators with respect to H_1 and H_2, the eigenvalues of $(L_{12}^{-+})^j\mathfrak{v}$, $(L_{12}^{--})^j\mathfrak{v}$, $(L_1^{-})^j\mathfrak{v}$ and $(L_2^{-})^j\mathfrak{v}$ are:

$$\begin{aligned}
H_1(L_{12}^{-+})^j\mathfrak{v} &= (\lambda_1 - j)(L_{12}^{-+})^j\mathfrak{v}, & \qquad H_2(L_{12}^{-+})^j\mathfrak{v} &= (\lambda_2 + j)(L_{12}^{-+})^j\mathfrak{v},\\
H_1(L_{12}^{--})^j\mathfrak{v} &= (\lambda_1 - j)(L_{12}^{--})^j\mathfrak{v}, & \qquad H_2(L_{12}^{--})^j\mathfrak{v} &= (\lambda_2 - j)(L_{12}^{--})^j\mathfrak{v},
\end{aligned}$$

$$H_1(L_1^-)^j\mathfrak{v} = (\lambda_1 - j)(L_1^-)^j\mathfrak{v}, \qquad H_2(L_1^-)^j\mathfrak{v} = (\lambda_2)(L_1^-)^j\mathfrak{v},$$
$$H_1(L_2^-)^j\mathfrak{v} = (\lambda_1)(L_2^-)^j\mathfrak{v}, \qquad H_2(L_2^-)^j\mathfrak{v} = (\lambda_2 - j)(L_2^-)^j\mathfrak{v}.$$

Also, L_{12}^{-+} commutes with Z, and L_1^-, L_2^- anticommute, so

$$Z(L_{12}^{-+})^j\mathfrak{v} = \Lambda(L_{12}^{-+})^j\mathfrak{v},\ Z(L_1^-)^j\mathfrak{v} = (-1)^j\Lambda(L_1^-)^j\mathfrak{v},\ Z(L_2^-)^j\mathfrak{v} = (-1)^j\Lambda(L_2^-)^j\mathfrak{v}.$$

We know that the chains of elements created by applying successively L_{12}^{-+}, L_{12}^{--}, L_1^- and L_2^- must end, and so there exist minimal N, N', M_1 and M_2 such that each chain becomes 0. This implies the following equations:

$$L_1^+L_1^-(L_1^-)^{M_1}\mathfrak{v} = 0, \qquad L_2^+L_2^-(L_2^-)^{M_2}\mathfrak{v} = 0,$$
$$L_{12}^{+-}L_{12}^{-+}(L_{12}^{-+})^N\mathfrak{v} = 0, \qquad L_{12}^{++}L_{12}^{--}(L_{12}^{--})^{N'}\mathfrak{v} = 0.$$

We use the factorisations of Propositions 6.4.13 and 6.4.15 to get that $C_{12}^{-+}(N) = C_{12}^{--}(N') = C_1(M_1) = C_2(M_2) = 0$. The minimality of M_1, M_2 N and N' then implies that $C_{12}^{-+}(j) \neq 0$, $C_{12}^{--}(j') \neq 0$ $C_1(j_1) \neq 0$, $C_2(j_2) \neq 0$ for $0 \leq j < N$, $0 \leq j' < N'$, $0 \leq j_1 < M_1$, and $0 \leq j_2 < M_2$.

We thus have found a label for the representation $\mathcal{V}$ respecting (6.75), concluding the proof. ∎

Remark 6.5.10. *In most cases, it will be possible to extract the values of N, N', M_1 and M_2 from the weights.*

We express pictorially in Figure 6.1 what Theorem 6.5.9 entails. Each representation of $\mathfrak{O}_\kappa$ has four chains of elements in $\mathfrak{T}$ of respective length N, N', M_1 and M_2 and a highest-weight vector $\mathfrak{v}$. This sketches how the corresponding irreducible $\mathfrak{T}$-representation looks like.

Theorem 6.5.9 gives necessary conditions on the set of weights. We close this chapter by investigating what certain specific choices of weights imply.

6.5.4 Monogenic-type representations

It is possible to refine the coarse classification of Theorem 6.5.9 by studying all candidate tuples of parameters respecting the equations (6.75), seeing if they indeed give rise to a finite-dimensional

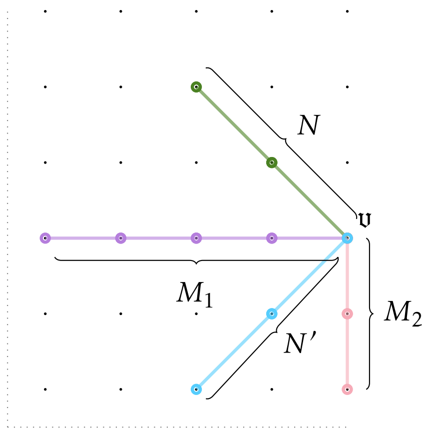


Figure 6.1: Four chains of vectors from Theorem 6.5.9 in any finite-dimensional $\mathcal{O}_\kappa$-representation. Diagonal North-West links mean moving with L_{12}^{-+}; diagonal South-West links mean moving with L_{12}^{--}; horizontal to the left, with L_1^-, and vertical above, with L_2^-.

representation. The classification for $W = D_{2m} \times \mathbb{Z}_2$ proceeded in the same way, see Section 5.6.

Here, we focus on one branch of the possibilities by fixing the H_1 and H_2-eigenvalue to have $M_1 = N$ and $M_2 = 0 = N'$, and with a restriction on ℓ and k; it will contain the motivating example of the monogenic representations and, for κ sufficiently small, will also be the only possibility; see Proposition 6.5.16. For general κ, however, it is by no mean the only possibility for representations and we also include in the following sections some cases with different weights, but a complete exploration of all possibilities would get us outside the scope of the thesis.

We will call $Q_a(j)$ of (6.60) *maximal* if $Q_a(j) = \max_i Q_a(i)$.

Definition 6.5.11. *Fix ℓ and k such that $Q_1(\ell+1-2N)$ and $Q_2(k+1)$ are maximal. Let $\lambda_1 = 1/2 + N + Q_1(\ell+1-2N)$, $\lambda_2 = 1/2 + Q_2(k+1)$ and $\Lambda = -1/2 - \lambda_1 - \lambda_2$. We call then the representation and such weights* of monogenic type.

The rationale behind this choice is the following consequence.

Lemma 6.5.12. *Let $\mathcal{V}$ be of monogenic type, and take $\mathfrak{v} \in \mathcal{V}$ to be the highest-weight vector of Theorem 6.5.9. Then $L_2^- \mathfrak{v} = 0 = L_{12}^{--}\mathfrak{v}$ and so $M_2 = 0 = N'$. Furthermore, $M_1 = N$.*

Proof. Having $\lambda_2 = 1/2 + Q_2(k+1)$ has the following consequence on the factorisations from Propositions 6.4.13 and 6.4.14: $L_2^+ L_2^- \mathfrak{v} = 0 = L_{12}^{++} L_{12}^{--}\mathfrak{v}$, so $M_2 = 0 = N'$.

From the value of λ_1, we know $C_1(N) = 0 = C_{12}^{-+}(N)$. We will check that $C_1(j) \neq 0$ and $C_{12}(j) \neq 0$ for $0 \leq j < N$. For $C_1(j)$, we only have to check the second factor of $C_1(j)$, since by maximality of $Q_1(\ell+1-2N)$ the first is never zero, and indeed looking at

$$((-1)^j \Lambda - \lambda_2)^2 - (\lambda_1 - j - \tfrac{1}{2}) = \begin{cases} (\frac{1}{2} + \lambda_1)^2 - (\lambda_1 - j - \frac{1}{2})^2 & j \text{ odd;} \\ (\frac{1}{2} + \lambda_1 + 2\lambda_2)^2 - (\lambda_1 - j - \frac{1}{2}) & j \text{ even;} \end{cases}$$

we see that it is not zero for $0 \leq j < N$. For $C_{12}^{-+}(j)$ the first and second factors are never zero by maximality of $Q_1(\ell+1-2N)$ and $Q_2(k+1)$ and the last one is never zero since it is only zero if

$$(\lambda_1 - \lambda_2 - 2j - 1)^2 = (\Lambda + 1/2)^2 \tag{6.77}$$

$$\Leftrightarrow (\lambda_1 - \lambda_2 - 2j - 1)^2 = (\lambda_1 + \lambda_2)^2 \quad \Leftrightarrow \begin{cases} \lambda_1 = j + 1/2; \\ \lambda_2 = -j - 1/2; \end{cases} \tag{6.78}$$

which is never the cases since $j < N$ and $\lambda_2 > 0$. ∎

We will show that the representation U generated from k and ℓ will never be two-dimensional for representation of monogenic type, that is, we show the condition forces $k \neq n$ and $\ell \neq m$.

Proposition 6.5.13. *Suppose $\mathcal{V}$ is an irreducible representation of monogenic type, then $\ell \neq m$ and $k \neq n$.*

Proof. Suppose $\ell = m$ or $k = n$. We will show by contradiction that it cannot happen. We have three cases to look at: $(\ell, k) = (m, n)$; $(\ell, k) = (m, k)$, and $(\ell, k) = (\ell, n)$.

We begin with $(\ell, k) = (m, k)$. Then the highest-weight vector $\mathfrak{v}$ of Theorem 6.5.9 enjoys the following action of elements of Υ:

$$Z\mathfrak{v} = \Lambda \mathfrak{v}, \qquad H_1 \mathfrak{v} = \lambda_1 \mathfrak{v}, \qquad H_2 \mathfrak{v} = (1/2 + Q_2(k+1))\mathfrak{v}. \tag{6.79}$$

The action of $\widetilde{W}$ is then given, from Theorem 6.2.3, by

$$\tilde{f}_1\mathfrak{v} = \mathfrak{v}, \qquad \tilde{f}_2\mathfrak{v} = \mathfrak{v}_2, \qquad \tilde{r}_1\mathfrak{v} = -\mathfrak{v}, \qquad \tilde{r}_2\mathfrak{v} = \eta^k\mathfrak{v}, \tag{6.80}$$

for a certain $\mathfrak{v}_2 \in \mathcal{V}$ in the same $\widetilde{W}$-representation as $\mathfrak{v}$. Then, the relation $\tilde{f}_1 H_1 = -H_1\tilde{f}_1$ implies that $\lambda_1 = 0$ since

$$\lambda_1\mathfrak{v} = H_1\mathfrak{v} = H_1\tilde{f}_1\mathfrak{v} = -\tilde{f}_1 H_1\mathfrak{v} = -\lambda_1\mathfrak{v}. \tag{6.81}$$

But this contradicts $\lambda_1 = 1/2 + N + Q_1(\ell + 1 - 2N)$.

Now the case $(\ell, k) = (\ell, n)$. Then we know $Q_2(n+1) = 0$ and so $\lambda_2 = 1/2$. However then $\tilde{f}_2\mathfrak{v} = \mathfrak{v}$ and $H_2\mathfrak{v} = \lambda_2\mathfrak{v}$ so, using $\tilde{f}_2 H_2 = -H_2\tilde{f}_2$, we have $\lambda_2 = -\lambda_2$, a contradiction. There are no representation of monogenic type for those values.

We conclude with the last case, $(\ell, k) = (m, n)$. In this, by the same argument as the first case, we have $\lambda_1 = 0$, a contradiction. ∎

We are now ready to reconstruct representation of monogenic type from a label.

Proposition 6.5.14. *Let $N \in \mathbb{N}$. Define $\lambda_1 := N + 1/2 + Q_1(\ell + 1 - 2N)$, $\lambda_2 := 1/2 + Q_2(k+1)$ and $\Lambda = -1/2 - \lambda_1 - \lambda_2$ with $1 \leq \ell < m$ and $1 \leq k < n$ making $Q_1(\ell + 1 - 2N)$ and $Q_2(k+1)$ maximal. A highest-weight element $\mathfrak{v}$ with action of Υ_0 given by*

$$Z\mathfrak{v} = \Lambda\mathfrak{v},\ H_1\mathfrak{v} = \lambda_1,\ \mathfrak{v}\ H_2\mathfrak{v} = \lambda_2\mathfrak{v}, \tag{6.82}$$

$$\tilde{r}_1\mathfrak{v} = -\zeta^\ell\mathfrak{v},\ \tilde{r}_2\mathfrak{v} = -\eta^k\mathfrak{v} \tag{6.83}$$

defines an irreducible $\mathcal{O}_\kappa$-representation of monogenic type of dimension $2(N+1)(N+2)$ with a basis given by

$$\mathcal{B} := \{\mathfrak{v}_{ij}^{\delta\epsilon} := \tilde{\tau}_n^{\epsilon}\tilde{\sigma}_m^{\delta}(L_2^+)^j(L_1^-)^i\mathfrak{v} \mid \delta, \epsilon \in \{0,1\}, j \leq i \in \{0, \ldots, N\}\}. \tag{6.84}$$

Proof. The element $\mathfrak{v}$ defined in the proposition is a highest-weight vector since the equations of Theorem 6.5.9 are respected and so $L_1^+\mathfrak{v} = L_2^+\mathfrak{v} = 0 = L_{12}^{++}\mathfrak{v} = L_{12}^{+-}\mathfrak{v}$. The value of λ_1 makes it so that $C_{12}^{-+}(N) = 0$. From the maximality of $Q_1(\ell + 1 - 2N)$ and $Q_2(k+1)$, we have that $C_{12}^{-+}(j) \neq 0$ for $0 \leq j < N$.

Then the action of Υ on the set (6.84) is retrieved from Propositions 6.4.13–6.4.15. The full action of $\mathcal{O}_\kappa$ on the $\mathfrak{v}_{ij}^{\delta\epsilon}$ is retrieved from

Lemma 6.4.12. Indeed, on a $\mathfrak{u} \in \mathcal{B}$ with $H_a\mathfrak{u} = \lambda_a^{\mathfrak{u}}\mathfrak{u}$ and $Z\mathfrak{u} = \Lambda^{\mathfrak{u}}\mathfrak{u}$, we have:

$$O_{abb}^{\delta+-}(\mathfrak{u}) = \frac{(L_a^{\delta} - 4O_a^{\delta}(\Lambda^{\mathfrak{u}} + \delta\lambda_b^{\mathfrak{u}}))}{(2\lambda_a^{\mathfrak{u}} + \delta)}\mathfrak{u}, \tag{6.85}$$

$$\begin{aligned} O_{12}^{\delta\epsilon}(\mathfrak{u}) = \frac{1}{(2\lambda_1^{\mathfrak{u}} + \delta)(2\lambda_2^{\mathfrak{u}} + \epsilon)} \Big(& L_{12}^{\delta\epsilon} + O_1^{\delta} O_{211}^{\epsilon+-}(2\lambda_2^{\mathfrak{u}} + \epsilon) \\ & - O_{122}^{\delta+-} O_2^{\epsilon}(2\lambda_2^{\mathfrak{u}} + \delta) - 2O_1^{\delta} O_2^{\epsilon}(\epsilon\delta - 2\Lambda^{\mathfrak{u}}) \Big)\mathfrak{u}, \end{aligned} \tag{6.86}$$

and the two actions are well defined since the H_1- and H_2-weights of elements of $\mathcal{B}$ are given by:

$$H_1\mathfrak{v}_{ij}^{\delta\varepsilon} = (-1)^{\delta}(\lambda_1 - i)\mathfrak{v}_{ij}^{\delta\varepsilon}, \qquad H_2\mathfrak{v}_{ij}^{\delta\varepsilon} = (-1)^{\varepsilon}(\lambda_2 + i)\mathfrak{v}_{ij}^{\delta\varepsilon}, \tag{6.87}$$

which implies in particular that $(2\lambda_1^{\mathfrak{u}} + \delta) \neq 0$ and $(2\lambda_2^{\mathfrak{u}} + \epsilon) \neq 0$ for all $u \in \mathcal{B}$ since $|(\lambda_1 - i)| > 1/2$ and $|(\lambda_2 + i)| > 1/2$ for all $0 \leq i \leq N$.

Finally, all the vectors of $\mathcal{B}$ have distinct pairs of H_1- and H_2-eigenvalues, so they are all linearly independent. It is thus a basis.

Hence, we have defined an irreducible representation of monogenic type of dimension $2(N+1)(N+2)$. ∎

The algebra $\mathcal{O}_\kappa$ contains a deformation of the Lie algebra $\mathfrak{so}(2d)$ by the parameter function κ and the reflection group. When κ is in a neighbourhood of 0, we will see that the representation theory is restricted. The following definition makes precise what is meant by neighbourhood of 0.

Definition 6.5.15. *We call the parameter function κ* ***small*** *if $Q_a(j) \leq 1/2$ for all $j \in \mathbb{N}$ and $a \in \{1,2\}$.*

When κ is small or zero, the only possible weights are the ones of monogenic type, or close to those of monogenic type.

Proposition 6.5.16. *When κ is small, then $\lambda_1 = N + 1/2 \pm Q_1(\ell + 1 - 2N)$, $\lambda_2 = 1/2 \pm Q_2(k+1)$ and $\Lambda = -1/2 - \lambda_1 - \lambda_2$. Furthermore, $M_1 = N$ and $M_2 = 0 = N'$*

Proof. Since $0 \leq Q_1(\ell - 1), Q_2(k-1) \leq 1/2$, the only way to satisfy the equations $A_1 = 0$, $A_2 = 0$ and $B_2 = 0$ of Proposition 6.5.9 is to have the following set of quadratic equations:

$$\begin{gathered}(\Lambda+\lambda_2)^2=(\lambda_1+1/2)^2, \quad (\Lambda+\lambda_1)^2=(\lambda_2+1/2)^2, \\ (\lambda_1+\lambda_2+1)^2-(\Lambda-1/2)^2=0. \end{gathered} \tag{6.88}$$

The only solution to (6.88) is $\Lambda=-1/2-\lambda_1-\lambda_2$. Then $B_1=0$ is satisfied either by $\lambda_2=1/2$ or $\lambda_2=1/2\pm Q_2(k+1)$. For both, $M_2=0=N'$.

Finally, with $\Lambda=-1/2-\lambda_1-\lambda_2$, the only way to have $C_{12}^{-+}(N)=0$ is by

$$\lambda_1-N-1/2=\pm Q_1(\ell+1-2N), \text{ so } \lambda_1=1/2+N\pm Q_1(\ell+1-2N),$$

since

$$\begin{gathered}(\lambda_2+N+1/2)^2-Q_2(k-1+2N)^2\neq 0, \\ (\lambda_1-\lambda_2-2N-1)^2-(\lambda_1+\lambda_2)^2\neq 0.\end{gathered}$$

Then we also have $C_1(N)=0$, and furthermore, $C_1(j)\neq 0$ for $0\leq j<N$ since

$$\begin{aligned}((N-j\pm Q_1(\ell+1-2N))^2-E_1^+(j))\neq 0 \\ ((-1)^j(N+1/2\mp Q_1(\ell+1-2N)\mp Q_2(k+1))+1/2\pm Q_2(k+1))^2\neq \\ (N-j\pm Q_1(\ell+1-2N))^2,\end{aligned}$$

which comes from the fact that $Q_a(j)\leq 1/2$. ■

Remark 6.5.17. *When $\kappa=0$, we find $\lambda_1=N+1/2$, $\lambda_2=1/2$, and $\Lambda=-1/2-\lambda_1-\lambda_2$, and $M_1=N$, $M_2=0=N'$.*

Remark 6.5.18. *For small κ, Proposition 6.5.14 will also work for any set of label of the form $(\Lambda=-1/2-\lambda_1-\lambda_2, \lambda_1=N+1/2\pm Q_1(\ell+1-2N), \lambda_2=1/2\pm Q_2(k+1), \zeta^\ell, \eta^k)$. Furthermore, all finite-dimensional irreducible representations for small κ are of this form by the proof of Proposition 6.5.16.*

6.5.5 Representations not of monogenic type

The representations of monogenic type falls in but one of the branches that Theorem 6.5.9 leaves to investigation. In this section, we explore some of the other possibilities.

We will relax the assumptions on λ_1, λ_2 and Λ. From now on, assume $\lambda_2 = 1/2 \pm Q_2(k+1)$, understanding that if $\lambda_2 = 1/2 - Q_2(k+1)$ then $Q_2(k+1) \leq 1/2$, and assume $\Lambda = -1/2 - \lambda_1 - \lambda_2$. Note that we do not assume anything about the maximality of $Q_2(k+1)$. Furthermore, we still have $L_2^- \mathfrak{v} = 0 = L_{12}^{--}\mathfrak{v}$ and so $M_2 = 0 = N'$. We summarise in the following.

Definition 6.5.19. *A finite-dimensional representation and its highest weight are in the* relaxed *case if*

$$\lambda_2 = 1/2 \pm Q_2(k+1), \quad \Lambda = -1/2 - \lambda_1 - \lambda_2. \tag{6.89}$$

We first show that λ_1 is restricted to a certain form from the relaxed data (6.89)

Proposition 6.5.20. *Let $\mathcal{V}$ be an irreducible finite-dimensional representation with condition (6.89). The values λ_1 can take are then restricted to $\lambda_1 = M + 1/2 \pm Q_1(\ell + 1 - 2M)$ for a certain M.*

Proof. When $\lambda_2 = 1/2 \pm Q_2(k+1)$ and $\Lambda = -1/2 - \lambda_1 - \lambda_2$, the relation $C_{12}^{-+}(N) = 0$ can be satisfied in the following ways:

1) $(\lambda_2 + \mathbf{N} + \mathbf{1/2})^2 - \mathbf{Q_2}(\mathbf{k} - \mathbf{1} + \mathbf{2N})^2 = \mathbf{0}.$ This can happen if $Q_2(k - 1 + 2N) = 1 + N + Q_2(k+1)$. Then from $C_1(M_1) = 0$ we get the equation

$$((\lambda_1 - M_1 - 1/2)^2 - Q_1(\ell + 1 - 2M_1)^2)(((-1)^{M_1}\Lambda - \lambda_2)^2 - (\lambda_1 - M_1 + 1/2) = 0,$$

that can be satisfied by $\lambda_1 = M_1 + 1/2 \pm Q_1(\ell + 1 - 2M_1)$, which satisfy the proposition, or else by solving for the second factor, which results in $\lambda_1 = -1/2 + M_1/2 \pm Q_2(k+1)$ for M_1 even or $\lambda_1 = 1/2 + (M_1 - 1)/2$ for M_1 odd.

2) $(\lambda_\mathbf{1} - \lambda_2 - \mathbf{2N} - \mathbf{1})^2 = (\lambda_\mathbf{1} + \lambda_2)^2.$ This only happens if $\lambda_1 = 1/2 + N$ or if $\lambda_2 = -(2N+1)/2$, the latter being a contradiction on the positivity of λ_2. So, $M = N$, and $\lambda_1 = N + 1/2$ with $Q_1(\ell + 1 - 2N) = 0$. The last conditions come from the fact that, by Lemma 6.5.1, $(L_1^-)^{N+1}\mathfrak{v} = 0$ since the H_1 weight of $(L_1^-)^N\mathfrak{v}$ is 1/2 and so $L_1^+ L_1^- (L_1^-)^N \mathfrak{v} = 0$ and

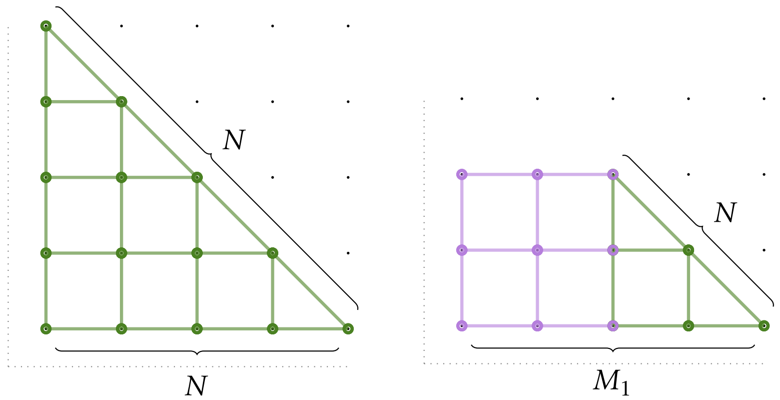
$$\begin{aligned}((\lambda_1 - N - 1/2)^2 - Q_1(\ell + 1 - 2N)^2)((\Lambda - \lambda_2)^2 - (\lambda_1 - N - 1/2)^2) = 0 \\ Q_1(\ell + 1 - 2N)^2(-N - 2 \mp 2Q_2(k+1))^2 = 0,\end{aligned}$$

which forces $Q_1(\ell+1-2N)^2 = 0$. Indeed, $Q_2(k+1) \geq 0$ and if $\lambda_2 = 1/2 - Q_2(k+1)$, then $Q_2(k+1) \leq 1/2$ so $-N-2 \mp 2Q_2(k+1) \neq 0$.

3) $(\boldsymbol{\lambda_1 - N - 1/2})^2 - \mathbf{Q_1}(\boldsymbol{\ell + 1 - 2N})^2 = \mathbf{0}$**.** This only happens if $\lambda_1 = (2N+1)/2 \pm Q_1(\ell+1-2N)$.

All cases have been covered, which concludes the proof. ■

The only shape possible for the Υ weight space of representations of monogenic type was a triangle. We see from Proposition 6.5.20 that more possibilities can happen if we relax the conditions. Two of the possibilities are presented in Figure 6.2.



(a) The triangle. **(b)** The trapezoid.

Figure 6.2: Two possibilities of weight space geometries for representations with $\lambda_2 = 1/2 \pm Q_2(k+1)$ and $\Lambda = -1/2 - \lambda_1 - \lambda_2$.

In this relaxed case (6.89), there are representations with $U := U(\ell, k)$ of dimension 2. In all instances, the $\mathcal{O}_\kappa$-representation obtained from the data will be shown to be also 2-dimensional.

Proposition 6.5.21. *Let $(\Lambda, \lambda_1, \lambda_2, \zeta^\ell, \eta^k)$ be the label of an irreducible representation $\mathcal{V}$ of $\mathcal{O}_\kappa$ respecting conditions of Theorem 6.5.9. If $\lambda_2 = 1/2 \pm Q_2(k+1)$, $\Lambda = -1/2 - \lambda_1 - \lambda_2$ and $\ell = m$ or $k = n$, then $\mathcal{V}$ is 2-dimensional with remaining weights*

$$\begin{gathered} \lambda_1 = 0,\ \Lambda = -1,\ \lambda_2 = 1/2,\ \ell = m,\ N = N' = 0 = M_1 = M_2, \\ \textit{with } k \textit{ and } \kappa \textit{ such that } Q_2(k+1) = 0 \textit{ and } Q_2(k-1) = 1. \end{gathered} \tag{6.90}$$

Proof. We will show that $\mathcal{V}$ must be characterised by the label and the action given in (6.90). We have three cases to look at: $(\ell, k) = (m, n)$; $(\ell, k) = (m, k)$, and $(\ell, k) = (\ell, n)$. The first two will be shown to contradict (6.89) and the remaining, to lead to a two-dimensional representation with label given in (6.90).

We begin with $(\ell, k) = (m, n)$. Then the highest-weight vector $\mathfrak{v}$ of Theorem 6.5.9 enjoys the following action of elements of Υ:

$$Z\mathfrak{v} = \Lambda\mathfrak{v}, \qquad H_1\mathfrak{v} = \lambda_1\mathfrak{v}, \qquad H_2\mathfrak{v} = (1/2 \pm Q_2(k+1))\mathfrak{v}. \tag{6.91}$$

The action of $\widetilde{W}$ is then given, from Theorem 6.2.3, by

$$\tilde{f}_1\mathfrak{v} = \mathfrak{v}, \qquad \tilde{f}_2\mathfrak{v} = \mathfrak{v}_2, \qquad \tilde{r}_1\mathfrak{v} = -\mathfrak{v}, \qquad \tilde{r}_2\mathfrak{v} = -\mathfrak{v}, \tag{6.92}$$

for a certain $\mathfrak{v}_2 \in \mathcal{V}$ in the same $\widetilde{W}$-representation as $\mathfrak{v}$. Then, the relation $\tilde{f}_1 H_1 = -H_1\tilde{f}_1$ implies that $\lambda_1 = 0$ since

$$\lambda_1 = H_1\mathfrak{v} = H_1\tilde{f}_1\mathfrak{v} = -\tilde{f}_1 H_1\mathfrak{v} = -\lambda_1\mathfrak{v}. \tag{6.93}$$

Since $\ell = m$ and $k = n$ and both $m, n \geq 2$, then $Q_1(m\pm1) = 0 = Q_2(n\pm1)$ and so $\lambda_2 = 1/2$.

We can also already know that $L_1^-\mathfrak{v} = 0$ since $\mathfrak{v}$ is a highest-weight vector and so it is annihilated by L_1^+. Indeed, applying Lemma 6.4.5 implies:

$$L_1^-\mathfrak{v} = L_1^-\tilde{\sigma}_m\mathfrak{v} = -\tilde{\sigma}_m L_1^+\mathfrak{v} = 0. \tag{6.94}$$

So $M_1 = 0$.

As $L_2^+\mathfrak{v} = 0 = L_1^-\mathfrak{v}$, then factorisations of Proposition 6.4.14 implies that $L_{12}^{-+}\mathfrak{v} = 0$ since $\lambda_1 + \lambda_2 + \Lambda + 1/2$ and $\lambda_1 + \lambda_2 - \Lambda - 1/2$ cannot both be 0 with $\lambda_1 = 0$ and $\lambda_2 = 1/2$. Hence $N = 0$. So relations (6.75) becomes (note that $B_1 = 0$ since $\mathcal{V}$ respects (6.89)):

$$(A_1 = 0) \qquad (\Lambda + 1/2)^2 - 1/4 = 0; \tag{6.95}$$
$$(A_2 = 0) \qquad \Lambda^2 - 1 = 0; \tag{6.96}$$
$$(B_2 = 0) \qquad (3/2)^2 - (\Lambda - 1/2)^2 = 0; \tag{6.97}$$
$$(C_{12}^{-+}(0) = 0) \qquad (3/2)^2 - (\Lambda + 1/2)^2 = 0; \tag{6.98}$$

and we can see from the two last equations that both cannot be true at the same time, so there are no representation with λ_2 and Λ given in (6.89) when $\ell = m$, $k = n$.

Now the case $(\ell, k) = (\ell, n)$. Then we know $Q_2(n+1) = 0$ and so $\lambda_2 = 1/2$. However then $\tilde{f}_2\mathfrak{v} = \mathfrak{v}$ and $H_2\mathfrak{v} = \lambda_2\mathfrak{v}$ so, using $\tilde{f}_2H_2 = -H_2\tilde{f}_2$, we have $\lambda_2 = -\lambda_2$, a contradiction. There are no representation with relaxed data (6.89) for those values.

We conclude with the last case, $(\ell, k) = (m, k)$. Recall that we still have $M_2 = 0 = N'$ by (6.89). Furthermore, by the same argument as the first case, we have $\lambda_1 = 0$ and $N = 0 = M_1$ since $L_1^-\mathfrak{v} = 0 = L_2^+\mathfrak{v}$. However, now the values of $Q_2(k \pm 1)$ are unknown. Relations (6.75) become

$$\begin{aligned}
(A_1 = 0) &\qquad (\Lambda + \lambda_2)^2 - 1/4 = 0;\\
(A_2 = 0) &\qquad ((\lambda_2 + 1/2)^2 - F_2^-(0))(\Lambda^2 - (\lambda_2 + 1/2)^2) = 0;\\
(B_2 = 0) &\qquad ((\lambda_2 + 1/2)^2 - F_2^-(0))((\lambda_2 + 1)^2 - (\Lambda - 1/2)^2) = 0;\\
(C_{12}^{-+}(0) = 0) &\qquad ((\lambda_2 + 1/2)^2 - F_2^-(0))((\lambda_2 + 1)^2 - (\Lambda + 1/2)^2) = 0.
\end{aligned}$$

From $A_1 = 0$ and $A_2 = 0$ we have $\Lambda = -1/2 - \lambda_2$. Then $B_2 = 0$ is also satisfied. For $C_{12}^{-+}(0) = 0$ then either

$$(\lambda_2 + 1)^2 - \lambda_2^2 = 0, \tag{6.99}$$

but then $\lambda_2 = -1/2$, a contradiction with the positivity, or

$$(\lambda_2 + 1/2)^2 - Q_2(k-1)^2 = 0 \qquad \Leftrightarrow \qquad (1 \pm Q_2(k+1))^2 = Q_2(k-1)^2,$$

which then means that $Q_2(k+1) = 0$ and $Q_2(k-1) = 1$ with $\lambda_2 = 1/2$.

The $\mathcal{O}_\kappa$-representation is 2-dimensional and given by $\mathfrak{v}$ and $\tilde{f}_2\mathfrak{v}$.

We thus retrieve the conditions described in (6.90). ■

Everything is set to do a construction similar to that of Proposition 6.5.14 with the values of λ_1 in Proposition 6.5.20. There, the representations depends on the values of κ: for some values, they become reducible and split in two smaller representations. A process similar to that of Chapter 5 would yield a complete classification of the finite-dimensional representations. One would then have conditions on the values of κ after long verifications. We leave the complete classification for the future and conclude here with a worked-out example of the representation of monogenic type.

Remark 6.5.22. *Of course, there are other possibilities outside the relaxed case* (6.89)*. The complete classification is far outside the scope of the thesis and is left for future work.*

6.5.6 An example of representations

The monogenic type Let $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{R}^4_{\geq 0}$. In the following example, we take $m = 3$, $n = 4$, and the label $\lambda_1 = 7/2 + 3\kappa_1$, $\lambda_2 = 1/2 + 2(\kappa_3 + \kappa_4)$, $\ell = 5$, $k = 7$ (for ease of computation, we took an equivalent index for the $\widetilde{W}$-representation). This gives us

$$\Lambda = -\lambda_1 - \lambda_2 = -9/2 - 3\kappa_1 - 2(\kappa_3 + \kappa_4).$$

We check if the first relations (6.75) are satisfied:

$$\begin{aligned}
A_1 &= (4+3\kappa_1)^2((-4-3\kappa_1-2(\kappa_3+\kappa_4)+2(\kappa_3+\kappa_4))^2-(4+3\kappa_1)^2) = 0;\\
A_2 &= (1+2(\kappa_3+\kappa_4))^2((-1-2(\kappa_3+\kappa_4))^2-(1+2(\kappa_3-\kappa_4))^2) = 0;\\
B_1 &= (4+3\kappa_1)^2((2(\kappa_3+\kappa_4))^2-4(\kappa_3+\kappa_4)^2)\times\\
&\qquad((4+3\kappa_1-2(\kappa_3+\kappa_4))^2-(-4-3\kappa_1-2(\kappa_3+\kappa_4))^2) = 0;\\
B_2 &= (4+3\kappa_1)^2(2(\kappa_3+\kappa_4))^2\times\\
&\qquad((5+3\kappa_1+2(\kappa_3+\kappa_4))^2-(-5-3\kappa_1-2(\kappa_3+\kappa_4))^2) = 0.
\end{aligned}$$

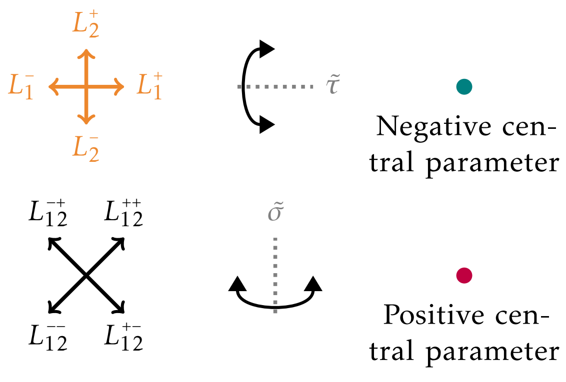


Figure 6.3: Navigation convention.

There are other relations to check, but we will already start and do so along the way. We begin by expliciting our graphical notation in Figure 6.3.

Since we have a highest weight vector, we act as much as we can with L_{12}^{-+} until we reach the point where acting further would be 0, following the relations between our parameters, it is illustrated in Figure 6.4. It gives us the value of the parameter N in the theorem. Here, $N = 3$. Indeed, we know $L_{12}^{-+}L_{12}^{+-}(L_{12}^{+-})^K\mathfrak{v} = C_{12}^{-+}(K)\mathfrak{v}$, so we only need to check that $C_{12}^{-+}(0) \neq 0, C_{12}^{-+}(1) \neq 0, C_{12}^{-+}(2) \neq 0$ and $C_{12}^{-+}(3) = 0$:

$$\begin{aligned}
C_{12}^{-+}(0) &= ((3+3\kappa_1)^2 - 9\kappa_1^2)(1+2(\kappa_3-\kappa_4))^2 \times \\
&\qquad ((2+3\kappa_1-2(\kappa_3+\kappa_4))^2 - (-4-3\kappa_1-2(\kappa_3+\kappa_4))^2) \neq 0 \\
C_{12}^{-+}(1) &= (2+3\kappa_1)^2((2+2(\kappa_3+\kappa_4))^2 - 4(\kappa_1+\kappa_4)^2) \times \\
&\qquad ((3\kappa_1-2(\kappa_3-\kappa_4))^2 - (-3-3\kappa_1-2(\kappa_3-\kappa_4))^2) \neq 0 \\
C_{12}^{-+}(2) &= (1+3\kappa_1)^2(3+2(\kappa_3+\kappa_4))^2 \times \\
&\qquad ((-2+3\kappa_1-2(\kappa_3-\kappa_4))^2 - (-3-3\kappa_1-2(\kappa_3-\kappa_4))^2) \neq 0 \\
C_{12}^{-+}(3) &= ((3\kappa_1)^2 - 9\kappa_1^2)((4+2(\kappa_3+\kappa_4))^2 - 4(\kappa_3+\kappa_4)^2) \times \\
&\qquad ((-4+3\kappa_1-2(\kappa_3+\kappa_4))^2 - (-3-3\kappa_1-2(\kappa_3+\kappa_4))^2) = 0
\end{aligned}$$

It was also possible to extract directly N from this specific value of λ_1 as the label uses maximal $Q_a(j)$.

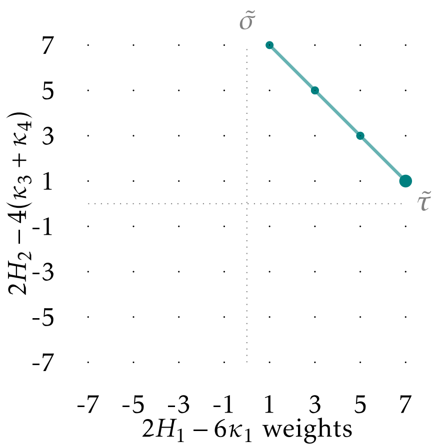


Figure 6.4: Finding the longest chain.

We also have to check that $C_1(3) = 0$, and $C_1(0) \neq 0$, $C_1(1) \neq 0$, $C_1(2) \neq 0$. We simply replace the values in the equations and find:

$$C_1(0) = ((3+3\kappa_1)^2 - 9\kappa_1^2)((-5-3\kappa_1-4(\kappa_3+\kappa_4))^2 - (3+3\kappa_1)^2) \neq 0;$$

$$C_1(1) = (2+3\kappa_1)^2((2+3\kappa_1)^2 - (2+3\kappa_1)^2) \neq 0;$$
$$C_1(2) = (1+3\kappa_1)^2((-5-3\kappa_1-4(\kappa_3+\kappa_4))^2 - (1+3\kappa_1)^2) \neq 0;$$
$$C_1(3) = ((0+3\kappa_1)^2 - 9\kappa_1^2)((2+3\kappa_1)^2 - (3\kappa_1)^2) = 0.$$

Note that $L_{\bar{1}\bar{2}}^{-}\mathfrak{v} = 0 = L_{\bar{2}}^{-}\mathfrak{v}$, and so $M_2 = 0 = N'$, since

$$((\lambda_2 - 1/2)^2 - F_2^+(0)) = ((1/2 + 2(\kappa_3+\kappa_4) - 1/2)^2 - 4(\kappa_3+\kappa_4)^2 = 0.$$

Once this is done, we can visit all the other possible weights of the grid in the quadrant by defining $\mathfrak{v}_{ij} := (L_2^+)^j (L_1^-)^i \mathfrak{v}$ for $j \leq i \in \{0,\ldots,N\}$. This is shown in Figure 6.5. That they are not zero can be verified by checking that the action of $L_a^{\pm} L_a^{\mp}$ is non-zero in the appropriate directions.

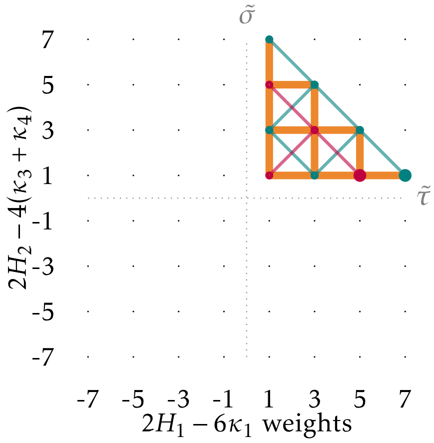


Figure 6.5: Visiting the whole quadrant with Υ.

The last step is to look at the action of $\mathcal{O}_\kappa$ on $\mathcal{V}$ with $\widetilde{W}$. An orbit of the $\widetilde{W}$-representation U is given by the four possible reflections along the axis (including the trivial one). Then all the weight spaces for the four distinct Υ-representations are visited. It is shown in Figure 6.6.

The basis of the 40 elements is given by

$$\mathcal{B} := \{\mathfrak{v}_{ij}^{\delta\epsilon} := \tilde{\sigma}_m^{\delta} \tilde{\tau}_n^{\epsilon} (L_2^-)^j (L_1^-)^i \mathfrak{v} \mid \delta, \epsilon \in \{0,1\}, 0 \leq j \leq i \leq 3\}. \tag{6.100}$$

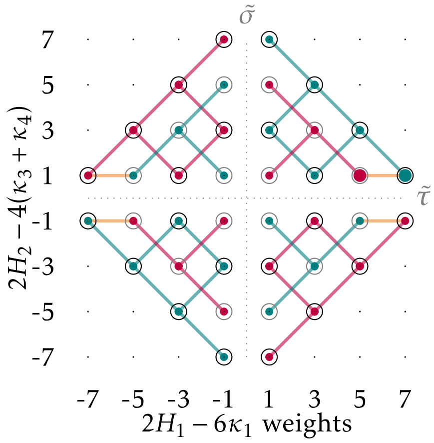


Figure 6.6: Induction from Υ to $\mathfrak{O}_\kappa$ by $\widetilde{W}$.

That it is indeed a $\mathfrak{O}_\kappa$-representation is observed by giving the actions of the other generating elements of $\mathfrak{O}_\kappa$ as in Proposition 6.5.14. Since the denominators of (6.85) and (6.86) are never zero in the monogenic case, the formulas are explicit. For example, we have $O^{-+}_{12}\mathfrak{v}^{00}_{00} = -\mathfrak{v}^{00}_{11}/(4(\lambda_1 + \lambda_2 - \Lambda - 1/2))$.

7

Conclusion and further research

We give a brief reminder of what was done in each chapter, present the main ideas of the thesis and end by reviewing avenues for future works.

7.1 Review of the thesis

In this thesis, we have studied the representations of the Dunkl total angular momentum algebra, and we have focussed specifically on two families of reflection groups. For the dihedral groups and the product of two dihedral groups, we gave the classifications of the finite-dimensional irreducible representations. For general groups, we gave a construction for the polynomial null solutions of the Dunkl–Dirac equation using generalised symmetries of the Dunkl–Dirac operator.

In Chapter 2, we reviewed the definitions and properties of the algebraic objects encountered in the rest of the thesis. We also presented some general properties of the total angular momentum algebra collected from the literature.

In Chapter 3 we presented a construction of bases of monogenic polynomials by using generalised symmetries. This way, for any reflection group, we can realise one irreducible representation of $\mathfrak{O}_\kappa$. The construction uses a Dunkl–Kelvin inverse transform and retrieves the formulas of [DGV16b] for the group $W = \mathbb{Z}_2^d$.

In Chapter 4, we presented the total angular momentum algebra for the group $W = \mathsf{D}_{12} \subset \mathcal{O}(3)$ linked with the root system G_2. We obtained structural results on the algebra and ladder operators.

In Chapter 5, we presented the irreducible finite-dimensional representations of $\mathfrak{O}_\kappa(W,V)$ for $V = \mathbb{R}^3$ and $W = \mathsf{D}_{2m} \times \mathbb{Z}_2 \subset \mathcal{O}(3)$. The main tools for that study were ladder operators and a weight theory reminiscent of that of $\mathfrak{sl}(2)$. The full classification was given along with restrictions on the parameter function κ needed in order for the representation to be unitary. A family of examples of irreducible representations was also given via spinor-valued monogenic polynomials, and their explicit form was given using orthogonal polynomials.

In Chapter 6, we studied the representations of $\mathfrak{O}_\kappa$ associated with the group $W = \mathsf{D}_{2m} \times \mathsf{D}_{2n} \subset \mathcal{O}(4)$. The main idea was to use a subalgebra $\Upsilon \subset \mathfrak{O}_\kappa$ with a triangular decomposition $\Upsilon = \Upsilon_- \Upsilon_+ \Upsilon_0$ to get a weight theory. Then we could use the weight theory to give a coarse classification of the possible finite-dimensional representations, and in some cases, we constructed the representation from the weights.

7.2 Main takeaways

We now highlight the main results of the thesis by revisiting them as takeaways. The first one comes from Theorem 3.4.4.

Takeaway A (Generalised symmetries monogenic bases)**.** *The generalised symmetries of the Dunkl–Dirac operator*

$$\mathfrak{z}_j := 2\varepsilon x_j H - \underline{x} \mathrm{D}_j \underline{x}$$

can be used to create a basis of the polynomial monogenics $P_n(\mathbb{R}^d, V)$ of degree n. Let ν be a basis of V and denote $\mathfrak{z}^\beta := \mathfrak{z}_1^{\beta_1} \dots \mathfrak{z}_d^{\beta_n}$ for $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$ the basis is given by:

$$\mathcal{B}_n = \{\mathfrak{z}^{\mathbf{j}} s \mid \mathbf{j} = (j-1, \dots, j_{d-1}, 0) \in \mathbb{N}^d, |j|_1 = n, s \in \nu\}.$$

The second one concerns Theorems 5.6.1 and 5.6.2.

Takeaway B (Finite-dimensional representations for $W = \mathsf{D}_{2m} \times \mathbb{Z}_2$)**.** *The finite-dimensional irreducible representations of $\mathfrak{O}_\kappa$ are classified for the group $W = \mathsf{D}_{2m} \times \mathbb{Z}_2 \subset \mathcal{O}(3)$.*

The following is an extension of the results presented in Lemma 5.6.3 and Proposition 6.4.17. We will say dihedral $\mathfrak{O}_\kappa$ for the cases where $W = \mathsf{D}_{2m_1} \times \cdots \times \mathsf{D}_{2m_d}$ and $V = \mathbb{R}^{2d}$ or $W = \mathsf{D}_{2m_1} \times \cdots \times \mathsf{D}_{2m_d} \times \mathbb{Z}_2$ and $V = \mathbb{R}^{2d+1}$.

Takeaway C (Triangular subalgebras for the dihedral cases)**.** *For dihedral $\mathfrak{O}_\kappa$, there exists a triangular subalgebra $\mathfrak{T} \subset \mathfrak{O}_\kappa$ with a weight structure that captures the representation theory of $\mathfrak{O}_\kappa$.*

The last takeaway is extracted from Theorems 5.6.1 and 5.6.2, and Proposition 6.5.16.

Takeaway D (Representation for small parameter function)**.** *When κ is small, the representations of the dihedral $\mathfrak{O}_\kappa$ share the same structure as those appearing when $\kappa = 0$.*

7.3 Ongoing and future work

We now briefly go over ongoing work and avenues for future investigations.

7.3.1 Algebra of generalised symmetries

In Chapter 3, we used a family of generalised symmetries of the Dunkl–Dirac operator to obtain bases for polynomial monogenics for any reflection group W. It invites the question: under a suitable generalisation of Dunkl–Dirac operator, what is the algebraic structure generated by the generalised symmetries of the Dirac operator? We know that it will generate an algebra containing the total angular momentum algebra and that the polynomial monogenics will be an infinite-dimensional irreducible representation. Since the generalised symmetries do not necessarily commute with the dual symbol, it means in particular that they do not preserve the degree of polynomials, hence the infinite-dimensionality of the representations.

The algebraic structure generated by those generalised symmetries was studied in the super-case by Coulembier and De Bie [CD15]. Similar work should be achievable in the Dunkl deformed case, with a fair refinement of the complexity due to the addition of the group algebra. An encouraging sign however is the presence of the Dunkl–Clifford–Kelvin inverse (see (3.24)) similar to the one that was used in [CD15].

Another avenue would be to use the anti-involution given by $x_i^* = \mathrm{D}_i$ and $\mathrm{D}_i^* = x_i$ to have the dual of the generalised symmetries of $\underline{x}$, and see what the interactions of the two families of generalised symmetries reveal.

7.3.2 Dihedral total angular momentum algebra

Chapters 4–6 were concerned with various families of total angular momentum algebras linked with dihedral groups. As we remarked in Takeaway C, the triangular subalgebra $\mathfrak{T} \subset \mathfrak{O}_\kappa$ and its properties hold, up to suitable generalisations, in all total angular momentum algebras linked with a product of p dihedral groups and possibly $\mathbb{Z}_2$. In this case, from preliminary computations and investigations with De Martino and Oste, we know that the weight space will be of dimension p and that it will be possible to navigate in the weight space using ladder-like operators with a triangular structure. From this, it should be relatively straightforward to get a coarse classification of the finite-dimensional representations. However, the precise locus of the algebraic variety generated by the intersection of the polynomial equations that will be obtained might be hard as we would have to find the proper factorisations, as in Propositions 5.5.10, 6.4.13–6.4.15.

The second direction is to consider in more detail the cases with specific parameters to better understand what happens. When there is only one dihedral group as in Chapter 5, the behaviour only separates the irreducible representations: they split into two smaller irreducible cases present at lower level. Our investigations into the double dihedral cases seems to indicate that something more interesting happens: the representation splits into non-deformed cases translated on the weight space and a case only appearing in the deformed case. Figure 7.1 illustrates this behaviour.

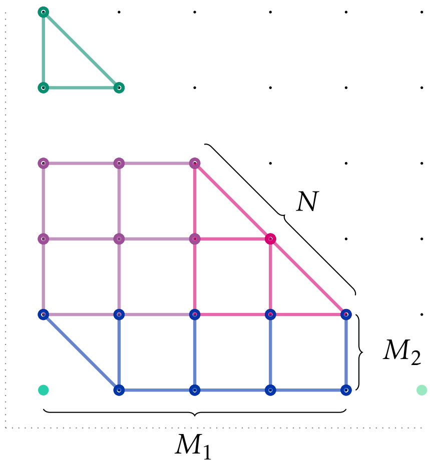


Figure 7.1: The conjectured main form of a $\mathfrak{T}$-weight space in $W = \mathsf{D}_{2m} \times \mathsf{D}_{2n}$. The green parts at the top, the right and the bottom are parts "cut" from the main representation. The undeformed representation would be the full triangle, but deformations split it in possibly four parts.

The question is then if adding more dihedral groups could lead to more exotic representation theory from the interaction between the different problematic values of the parameter function. Example of what could happen would be the split into indecomposable parts when the extra mobility in the weight space could lead to paths that go in only one direction. Naturally, this would have to happen for values where unitarity fails.

7.3.3 Other groups and other dual pairs

The total angular momentum algebra $\mathfrak{O}_\kappa$ is linked with the Howe dual pair $(\mathrm{Pin}(d), \mathfrak{osp}(1|2))$. Other dual pairs lead to different algebras. Recently, Ciubotaru and De Martino studied the Dunkl deformation of the dual pair $(\mathcal{O}(d), \mathfrak{spo}(2|2))$ [CD20], and Ciubotaru, De Bie, De Martino and Oste have studied the unitary dual pairs $(U(n), \mathfrak{u}(1,1))$ and $(U(n), \mathfrak{u}(2|1))$ [Ciu+20].

The algebras linked to these other dual pairs are similar to the algebra

$\mathcal{O}_\kappa$ studied in this thesis. It would be interesting to see if we could do a similar work at the level of the representation theory for other dual pairs.

One theme of the thesis was to consider what happens when we add another group, that is, to consider reducible root systems. It would be interesting to see if we can use the theory developed here to say something about the representations of the symmetry algebra associated with groups of the form $W = G \times H$, when the representation theory of the algebra associated to G and H are known in the smaller spaces. The double coverings could be extracted from Morris [Mor80], and already, the results of Chapter 3 indicate that on the monogenic representations, there seems to be a way of passing from monogenic polynomials of smaller rank to those of higher rank, generalising in a way the Cauchy–Kovalevskaya proven in the case $W = \mathbb{Z}_2^d$.

Finally, the last direction we are considering is, of course, to consider the representation theory of the algebra $\mathcal{O}_\kappa(W, V)$ for a general group W. The priority is for the general family A_n. The main difficulty is the irreducibility of the root system which means that there are no subalgebra that can play the role of $\mathfrak{T}$.

To study the family, the first step is to understand the spin representations of S_{n+1}, coming back to the seminal thesis of Schur [Sch07] and revisited in works of Józefiak [Józ89], Nazarov [Naz90], Bessenrodt [Bes94], Hoffman and Humphreys [HH92] or Brundan and Kleschev [BK03]; each with different flavours. The second step we propose is to study a chain of subalgebras by considering the inclusion $A_1 \subset A_2 \subset \cdots \subset A_n$. A way to proceed in this direction could pass by the series of papers by Okounkov, Sergeev and Vershik [OV96; VO05; Ver06; VS08] since they make strong use of the tower of subalgebra and use the specificity of the Jucy–Murphy elements present.. The projective representation of $\widetilde{S}_n$ were also described by Dirac cohomology [Cal19], and it seems useful to consider the interaction with the total angular momentum algebra there. It seems to us that the stepping stone would then be to obtain a description of the representation theory for the A_3 and the A_4 cases.

A full classification of the finite-dimensional representations seems out of reach, but a restriction to generic values of the parameter function should be attainable. By this we mean that focussing on values for which the representations can be unitary and taking small enough deformation to avoid problems might yield conditions strict enough to give a better classification of the finite-dimensional representations. However, we note that for our current methods to extend, we would need to pursue work in the direction of obtaining a proper definition of the relations of the algebra.

At the moment, it seems more realistic to focus on what can be learned from the structure of the rational Cherednik algebra. A first indication of the structure for the representations could be obtained by taking modules of rational Cherednik algebras, tensoring them with spinor spaces for the Clifford algebra and then restricting to $\mathfrak{O}_\kappa$.

A
Nederlandstalige samenvatting

Dunkl-operatoren zijn veralgemeningen van partiële afgeleiden waaraan elementen van de groepsalgebra van een reflectiegroep toegevoegd zijn. Samen met de variabelen (als vermenigvuldigingsoperatoren) en de groepsalgebra van de reflectiegroep, genereren Dunkl-operatoren een representatie van de rationale Cherednik algebra geassocieerd met die reflectiegroep.

Binnen de rationale Cherednik algebra kan men een $\mathfrak{sl}(2)$-tripel identificeren. In de Dunkl-representatie realiseren de Dunkl-veralgemeningen van de Laplace-operator, de Euler-operator en de norm in het kwadraat dit $\mathfrak{sl}(2)$-tripel.

In de klassieke setting kan de Dirac-operator als de vierkantswortel van de Laplace-operator gedefinieerd worden aan de hand van Clifford-algebra elementen. De algebraïsche structuur gegenereerd door de Dirac-operator en de vectorvariabele is de Lie-superalgebra $\mathfrak{osp}(1|2)$. We kunnen hetzelfde proces uitvoeren voor rationale Cherednik algebra's. Daar vinden we ook een realisatie van de Lie-superalgebra $\mathfrak{osp}(1|2)$ binnen het tensorproduct met een Clifford-algebra. Een geschikte Dunkl-generalisatie van de Dirac-operator en zijn duale geven de realisatie in de Dunkl-representatie.

In hoofdstuk 3 geven we een constructie voor de veeltermoplossingen van de Dirac-vergelijking, geldig voor elke reflectiegroep. Hiervoor gebruiken we veralgemeende symmetrieën van de Dunkl–Dirac-operator. Voor specifieke groepen kunnen we expliciete uitdrukkingen met Jacobi-polynomen van de veeltermen vinden. We presenteren dit voor het geval van een compleet reduceerbare abelse groep. Bovendien geven we een verband met een Cauchy–Kovalevskaya extensiestelling.

Onze focus voor de rest van de proefschrift gaat naar een subalgebra in het tensorproduct van een rationale Cherednik-algebra en een Clifford-algebra: de (Dunkl) totaalimpulsmomentalgebra. Het is de supercentraliser van de $\mathfrak{osp}(1|2)$ realisatie.

In hoofdstuk 5 classificeren we alle eindig-dimensionale irreducibele representaties van de totaalimpulsmomentalgebra geassocieerd met een dihedrale groep in een drie-dimensionale ruimte. We bepalen ook de voorwaarden voor de parameterfunctie opdat de representaties unitair zouden zijn en de representaties kunnen gesplitst worden. We geven een (spinor-waardige) veeltermrealisatie voor een specifieke familie van representaties aan de hand van Jacobi-polynomen. Dit is afkomstig van een Cauchy–Kovalevskaya extensiestelling.

In hoofdstuk 6 bestuderen we de totaalimpulsmomentalgebra geassocieerd met de vermenigvuldiging van twee dihedrale groepen in een vier-dimensionale ruimte. Hiervoor gebruiken we een specifieke subalgebra die een triangulaire decompositie toelaat. Door gebruik te maken van de geassocieerde gewichtstructuur en een verzameling van ladderoperatoren, wordt het mogelijk om de representaties van de dubbele dihedrale totaalimpulsmomentalgebra op te stellen. De methoden van de twee laatste hoofstukken kunnen uitgebreid worden naar de studie van de totaalimpulsmomentalgebra geassocieerd met een arbitrair aantal dihedrale groepen.

English summary

Dunkl operators are generalisations of partial derivatives adding terms coming from the group algebra of a reflection group. Given a reflection group and a parameter function invariant under the action of the group, the Dunkl operators with the variable multiplication and the group algebra of the reflection group generate a representation of the rational Cherednik algebra associated with this reflection group.

Inside the rational Cherednik algebra, there is an $\mathfrak{sl}(2)$ triple; in the Dunkl representation, this is realised as the Dunkl generalisation of the Laplace operator, the Euler operator and the squared norm.

In the classical setting, the Dirac operator is defined as the square root of the Laplace operator using Clifford algebra. The algebraic structure generated by the Dirac operator and its vector variable is the Lie superalgebra $\mathfrak{osp}(1|2)$. We can mimic the same process in the rational Cherednik algebra context and find a realisation of $\mathfrak{osp}(1|2)$ in the tensor product with a Clifford algebra. In the Dunkl realisation, it is given by the suitable Dunkl generalisation of the Dirac operator and its dual symbol.

In Chapter 3, we give a construction for the Dunkl polynomial monogenics, solutions of the Dunkl–Dirac equation, for any reflection group using generalised symmetries of the Dunkl–Dirac operator. For specific groups, it enables explicit expressions of the polynomial monogenics using special functions. We present it for the completely reducible abelian case and link partial generalised symmetries with a Cauchy–Kovalevskaya extension theorem.

Our focus for the rest of the thesis lies on a subalgebra in the tensor product of a rational Cherednik algebra and a Clifford algebra: the (Dunkl) total angular momentum algebra. It is the supercentraliser

of the $\mathfrak{osp}(1|2)$ realisation present inside.

We give in Chapter 5 the complete classification of the finite-dimensional irreducible representations of the total angular momentum algebra associated with a dihedral group in a three-dimensional space. We also give condition on the parameter function for unitarity and as to when the representations will split. For a specific family of representations, we give a (spinor-valued) polynomial realisation by expressing Dunkl polynomial monogenics explicitly using Jacobi polynomials. This came from a Cauchy–Kovalevskaya extension theorem.

In Chapter 6, we study the total angular momentum algebra in a four-dimensional space associated to the product of two dihedral groups. This is done by using a subalgebra specific to this reducible context admitting a triangular decomposition. With a weight structure on the subalgebra and using a set of ladder operators, it is possible to characterise the representation of the double dihedral total angular momentum algebra. It is our hope that the method of the two last chapters will transfer to the study of the total angular momentum algebra associated with an arbitrary number of dihedral groups.

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