

Categorification of $U_q(\mathfrak{sl}_2)$: Part III

Let $\chi = \text{set of invertible elements in } k$, $\beta_m (m \in \mathbb{Z})$, $c_n^+ (n \in \mathbb{Z}^+)$, $c_n^- (n \in \mathbb{Z}^-)$

Def.: 2-Category \mathcal{U}_χ

- Objects: $n \in \mathbb{Z}$
- 1-morphisms: $\mathcal{E}_{(\varepsilon)} 1_{n-t} \{t\} \oplus \dots \oplus \mathcal{E}_{(\varepsilon_N)} 1_{n-t_N} \{t_N\} \in \text{1Morph}(n, m)$
 $\hookrightarrow (\varepsilon) = (\varepsilon_1, \dots, \varepsilon_m), \varepsilon_i \in \{+, -\}, \mathcal{E}_{(\varepsilon)} = \mathcal{E}_{\varepsilon_1} \dots \mathcal{E}_{\varepsilon_m}$
 $\mathcal{E}_+ = \mathcal{E}, \mathcal{E}_- = \mathcal{F}, m = n + 2 \sum_i \varepsilon_i$
- Composition: $\mathcal{E}_{(\varepsilon')} 1_{m-t'} \{t'\} \circ \mathcal{E}_{(\varepsilon)} 1_{n-t} \{t\} = \overbrace{\mathcal{E}_{(\varepsilon' \cup \varepsilon)} 1_{n-t+t'} \{t+t'\}}$
- 2-Morphs $(\mathcal{E}_{(\varepsilon)} 1_{n-t}, \mathcal{E}_{(\varepsilon')} 1_{n-t'}) \in \mathcal{U}_\chi$ = k -vector space of bi-coars of diagrams \rightsquigarrow horizontal/vertical composition of generating 2-morphisms of degree $t-t'$



diagram	from \rightarrow to	degree	diagram	from \rightarrow to	degree
$\begin{array}{c} n+2 \\ \downarrow \\ n \end{array}$	$\mathcal{E} 1_n \rightarrow \mathcal{E} 1_n$	0	$\begin{array}{c} n-2 \\ \downarrow \\ n \end{array}$	$\mathcal{F} 1_n \rightarrow \mathcal{F} 1_n$	0
$\begin{array}{c} n+2 \\ \downarrow \\ n \end{array}$	$\mathcal{E} 1_n \rightarrow \mathcal{E} 1_n$	2	$\begin{array}{c} n-2 \\ \downarrow \\ n \end{array}$	$\mathcal{F} 1_n \rightarrow \mathcal{F} 1_n$	2
$\begin{array}{c} \nearrow \\ \nwarrow \\ n \end{array}$	$\mathcal{E}^2 1_n \rightarrow \mathcal{E}^2 1_n$	-2	$\begin{array}{c} \searrow \\ \swarrow \\ n \end{array}$	$\mathcal{F}^2 1_n \rightarrow \mathcal{F}^2 1_n$	-2
$\begin{array}{c} \curvearrowleft \\ \curvearrowright \\ n \end{array}$	$\mathcal{F} \mathcal{E} 1_n \rightarrow 1_n$	$n+1$	$\begin{array}{c} \curvearrowright \\ \curvearrowleft \\ n \end{array}$	$1_n \rightarrow \mathcal{E} \mathcal{F} 1_n$	$-n+1$
$\begin{array}{c} \curvearrowleft \\ \curvearrowright \\ n \end{array}$	$1_n \rightarrow \mathcal{F} \mathcal{E} 1_n$	$n+1$	$\begin{array}{c} \curvearrowright \\ \curvearrowleft \\ n \end{array}$	$\mathcal{E} \mathcal{F} 1_n \rightarrow \mathcal{E} 1_n$	$-n+1$
$\begin{array}{c} \nearrow \\ \nwarrow \\ n \end{array}$	$\mathcal{E} \mathcal{F} 1_n \rightarrow \mathcal{F} \mathcal{E} 1_n$	0	$\begin{array}{c} \searrow \\ \swarrow \\ n \end{array}$	$\mathcal{F} \mathcal{E} 1_n \rightarrow \mathcal{E} \mathcal{F} 1_n$	0

which satisfy relations:

(1) Nil-Hecke relations:

$$(3) \quad \text{Diagram} = 0$$

$$(2) \quad \begin{array}{c} \text{Diagram showing two vertical lines with arrows pointing up, and a third line connecting them at their midpoints.} \\ = \end{array} \quad \begin{array}{c} \text{Diagram showing a single vertical line with a curved loop around it, and an arrow pointing up to its right.} \end{array}$$

$$(5) \quad \text{Diagram showing the cancellation of two terms: } - \text{Diagram 1} = \text{Diagram 2} - \text{Diagram 3} = \uparrow \uparrow$$

↓ which, upon applying repeatedly, becomes:

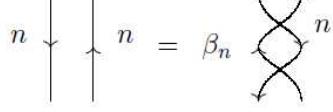
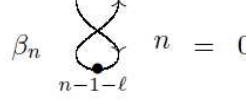
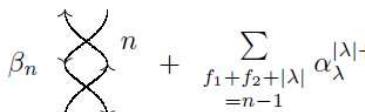
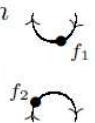
$$h \overbrace{X}^{\uparrow} - \overbrace{X}^{h\uparrow} = h \overbrace{X}^{\uparrow} - \overbrace{X}^{h\uparrow} = \sum_{\substack{l_1, l_2 \in \mathbb{N} \\ l_1 + l_2 = h-1}} l_1 \uparrow \quad \uparrow l_2$$

(2) The rel. that guarantee that

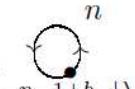
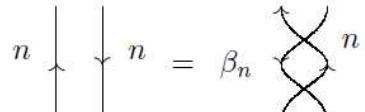
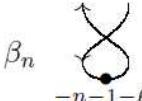
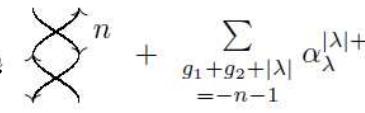
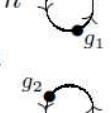
$$G_+ := \bigoplus_{n=1}^{\infty} \mathbb{C}^{1,n} \oplus \bigoplus_{n=1}^{\infty} \mathbb{C}^{1,n} \oplus \dots \oplus \bigoplus_{n=1}^{\infty} \mathbb{C}^{1,n} : \mathcal{F}E_1 \oplus E_1 \oplus \dots \oplus E_1 \rightarrow \mathcal{F}E_1$$

is isomorphism, with inverse

$$\text{is isomorphism, with inverse} \\ \sum_+ := \underbrace{\beta_m}_{\lambda \vdash m} \xrightarrow{\sim} \bigoplus_{l=0}^{n-1} \bigoplus_{\substack{\lambda \text{ partition} \\ |\lambda| \leq l}} \alpha_\lambda^l (n) \quad \begin{array}{c} \text{Diagram showing } n \text{ nodes in } \lambda \\ \text{with edges } \lambda_1, \dots, \lambda_m \end{array} \\ \lambda = (\lambda_1, \dots, \lambda_m)$$

Relations for $n \geq 0$	
(A1)	$\delta_{b,0} = \sum_{\lambda: \lambda \leq b} \alpha_\lambda^\ell(n) e_{\lambda,n}$ 
(A2)	$n \downarrow \quad \downarrow n = \beta_n$ 
(A3)	β_n  $n = 0$
(A4)	$\sum_{\lambda} \alpha_\lambda^\ell(n) e_{\lambda,n}$  $n = 0$
(A5)	$n \downarrow \quad \downarrow n = \beta_n$  $+ \sum_{f_1+f_2+ \lambda =n-1} \alpha_\lambda^{ \lambda +f_2}(n) e_{\lambda,n}$ 

Note that relations **A1**, **A3**, and **A4** are only valid for $n > 0$.

Relations for $n \geq 0$	
(B1)	$\delta_{b,0} = \sum_{\lambda: \lambda \leq b} \alpha_\lambda^\ell(n) e_{\lambda,n}$ 
(B2)	$n \downarrow \quad \downarrow n = \beta_n$ 
(B3)	β_n  $n = 0$
(B4)	$\sum_{\lambda} \alpha_\lambda^\ell(n) e_{\lambda,n}$  $n = 0$
(B5)	$n \downarrow \quad \downarrow n = \beta_n$  $+ \sum_{g_1+g_2+ \lambda =n-1} \alpha_\lambda^{ \lambda +g_2}(n) e_{\lambda,n}$ 

Note that relations **B1**, **B3**, and **B4** are only valid for $n < 0$.

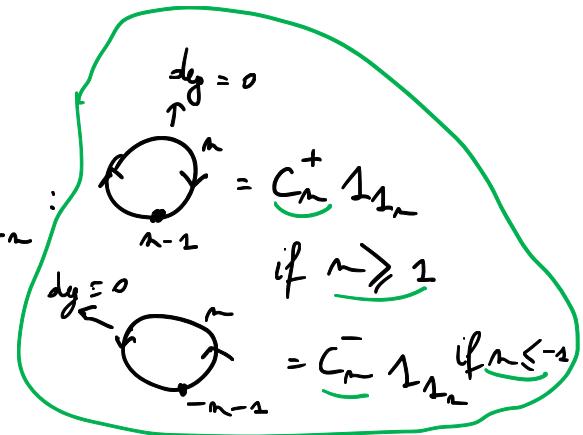
The map $\lambda \mapsto e_{\lambda, n}$ is injective $\Rightarrow \alpha_\lambda^l(n)$ are completely determined by (A1) - (B1)

$$\sum_{\substack{\lambda \text{ partition} \\ |\lambda| \leq b}} \alpha_\lambda^l(n) e_{\lambda, n} c_{(b-|\lambda|), n} = \delta_{b, 0} \quad \forall b \in \mathbb{N} \quad (*)$$

↪ Remark:

Dotted bubbles of degree 0 = multiple of 1_{1_n} :

$$\Rightarrow \alpha_\lambda^l(n) = \frac{\alpha_\lambda^l(n)}{(c_n^\pm)^{m+1}} \xrightarrow{\lambda_1, \dots, \lambda_m}$$



Coeff that expresses a complete sym. function in terms of elementary sym. functions

$$\hookrightarrow h_{s, n} := (-1)^s \sum_{|\lambda|=s} \alpha_\lambda^l(n) e_{\lambda, n}$$

$$(*) \Leftrightarrow \sum_{\substack{s, n \in \mathbb{N} \\ s+n=b}} (-1)^s h_{s, n} c_{n, n} = \delta_{b, 0} \quad \forall b \in \mathbb{N}$$

Equiv. relations for 2-morphisms:

(2'a): Biadjointness:

$$\begin{array}{l} \text{Diagram: } \text{Left} = \text{Right} = \uparrow, \quad \text{Left} = \text{Right} = \uparrow - \downarrow \end{array}$$

(2'b): Cyclicity w.r.t. biadjointness structure:

$$\begin{array}{l} \text{Diagram: } \text{Left} = \text{Right} = \uparrow, \quad \text{Left} = \text{Right} = \uparrow \end{array}$$

$$\begin{array}{l} \text{Diagram: } \text{Left} = \chi = \text{Right} \\ + \text{some other} \end{array}$$

\Rightarrow Isotopic diagrams
= same 2-morphisms

(2'c): • Bubbles of degree < 0 should vanish:

$$\begin{array}{c} \text{Diagram of a bubble with degree } k \\ \text{with boundary points } n-1+k \text{ and } -n+1+k \\ \text{and arrows pointing clockwise.} \end{array} = 0 \quad \text{if } k < 0$$

degree = k

• Bubbles of degree $0 = \text{cte. } 1_{1^n}$

• False bubbles: positive degree BUT carry a negative number of dots

$$\rightarrow \begin{array}{c} \text{Diagram of a bubble with degree } k \\ \text{with boundary points } n-1+k \text{ and } -1 \\ \text{and arrows pointing clockwise.} \end{array} \quad \begin{array}{l} n < 0 \\ 0 \leq k \leq -n \end{array} \Rightarrow \begin{array}{l} \text{not} \\ \text{allowed by graphical calculus} \end{array}$$

degree = $k > 0$

\Rightarrow Interpret false bubbles as formal symbols:

$$(A) \quad \begin{array}{c} \text{Diagram of a bubble with degree } 0 \\ \text{with boundary point } -1 \end{array} = C_0^+ 1_{1^0}, \quad \begin{array}{c} \text{Diagram of a bubble with degree } 0 \\ \text{with boundary point } -1 \end{array} = C_0^- 1_{1^0}$$

$$(B) \quad \sum_{\substack{f_1, f_2 \in \mathbb{N} \\ f_1 + f_2 = -n}} f_1 \uparrow \quad \begin{array}{c} \text{Diagram of a bubble with degree } n \\ \text{with boundary point } n-1+f_2 \end{array} = - \begin{array}{c} \text{Diagram of a bubble with degree } n \\ \text{with boundary point } -1-f_2 \end{array} \quad \forall n \in \mathbb{Z}$$

Similar with $\begin{array}{c} \text{Diagram of a bubble with degree } n \\ \text{with boundary point } -1 \end{array} \uparrow$

$$(C) \quad \begin{array}{c} \text{Diagram of a bubble with degree } n \\ \text{with boundary points } f_1, f_2, f_3 \in \mathbb{N} \\ \text{and } f_1 + f_2 + f_3 = n-1 \end{array} - \beta_n \sum_{\substack{f_1, f_2, f_3 \in \mathbb{N} \\ f_1 + f_2 + f_3 = n-1}} \begin{array}{c} \text{Diagram of a bubble with degree } n \\ \text{with boundary points } -n+1+f_2, f_3 \end{array} = \begin{array}{c} \uparrow \downarrow \\ m \end{array}$$

+ Similar

Prop.: Set of coeff X is completely determined by $\beta_m, (m \in \mathbb{Z})$ and C_0^+, C_1^+ :

$$\bullet C_0^+ C_0^- \beta_0 = -1$$

$$\bullet C_1^+ C_{-1}^- = 1$$

$$\bullet C_{n-2}^+ = -\beta_n C_n^+ \text{ if } n > 0$$

$$\bullet C_{n-2}^- = -\beta_n C_n^- \text{ if } n \leq 0$$

One can always chose: $\beta_n = -1, C_n^+ = 1, C_n^- = 1 \quad \forall n$

Def.: $\mathcal{U} := \mathcal{U}_X$ with $X = \{\beta_n = -1, C_n^\pm = 1\}_{n \in \mathbb{Z}}$

Theorem: Let χ be any set of invertible nodes that satisfy relations of Prop.

M : natural transformation $\mathcal{U} \rightarrow \mathcal{U}_\chi$:

$$n \in \text{Ob}(\mathcal{U}) \mapsto n$$

$$\mathcal{E}_{(e)} 1_n[t] \mapsto \mathcal{E}_{(e)} 1_{n'}[t]$$

$$M \left(\begin{array}{c|c} n+2 & n \\ \hline \bullet & \end{array} \right) = n+2 \quad \begin{array}{c} \downarrow \\ \bullet \end{array}$$

$$M \left(\begin{array}{c|c} n & n+2 \\ \hline \bullet & \end{array} \right) = n \quad \begin{array}{c} \downarrow \\ \bullet \end{array}$$

$$M \left(\begin{array}{c} \cancel{\times} \\ \cancel{\times} \end{array} \right) = \cancel{\times}$$

$$M \left(\begin{array}{c} \cancel{\times} \\ \cancel{\times} \end{array} \right) = \cancel{\times}$$

$$M \left(\begin{array}{c} n \\ \curvearrowleft \end{array} \right) = \begin{cases} \frac{1}{c_n^+} & n \\ c_{n-2}^- & n \\ & n \end{cases}$$

$$n = 2\ell, \text{ or } n = 2\ell + 1, \text{ for } \ell \in 2\mathbb{Z}_{\geq 0}$$

$$n = -2\ell, \text{ or } n = -(2\ell + 3), \text{ for } \ell \in 2\mathbb{Z}_{\geq 0}$$

otherwise

$$M \left(\begin{array}{c} n \\ \curvearrowright \end{array} \right) = \begin{cases} c_{n+2}^+ & n \\ \frac{1}{c_n^-} & n \\ & n \end{cases}$$

$$n = 2\ell, \text{ or } n = 2\ell + 1, \text{ for } \ell \in 2\mathbb{Z}_{\geq 0}$$

$$n = -2\ell, \text{ or } n = -(2\ell + 3), \text{ for } \ell \in 2\mathbb{Z}_{\geq 0}$$

otherwise

$$M \left(\begin{array}{c} n \\ \curvearrowleft \curvearrowright \end{array} \right) = \begin{cases} \frac{1}{c_n^+} & n \\ c_{n-2}^- & n \\ & n \end{cases}$$

$$n = 2\ell + 2, \text{ or } n = 2\ell + 3, \text{ for } \ell \in 2\mathbb{Z}_{\geq 0}$$

$$n = -(2\ell + 1), \text{ or } n = -(2\ell + 2), \text{ for } \ell \in 2\mathbb{Z}_{\geq 0}$$

otherwise

$$M \left(\begin{array}{c} n \\ \curvearrowright \curvearrowleft \end{array} \right) = \begin{cases} c_{n+2}^+ & n \\ \frac{1}{c_n^-} & n \\ & n \end{cases}$$

$$n = 2\ell + 2, \text{ or } n = 2\ell + 3, \text{ for } \ell \in 2\mathbb{Z}_{\geq 0}$$

$$n = -(2\ell + 1), \text{ or } n = -(2\ell + 2), \text{ for } \ell \in 2\mathbb{Z}_{\geq 0}$$

otherwise

M is isomorphism.

Proof: $M \left(\begin{array}{c} \cancel{\times} \\ \cancel{\times} \end{array} \right) = -\beta_n \cancel{\times}$

$$M \left(\begin{array}{c} n \\ \circlearrowleft \circlearrowright \end{array} \right) = \begin{cases} \frac{1}{c_n^+} & \text{if bubble is real} \\ -\beta_n c_n^- & \text{if bubble is fake} \end{cases}$$

$\Rightarrow M$ is 2-functor + invertible (bc. C_n^\pm, β_m are invertible). □

Recall. There only exists a categorification for \mathbb{A}^U (integral form of U)

$\mathbb{Z}[q, q^{-1}]$ -Subalgebra of U generated by "divided powers"

$$\frac{E^a}{[a]!}, \frac{F^b}{[b]!} \quad a, b \in \mathbb{N}, \quad [a]! = [a][a-1] \dots [1], \\ [a] = \frac{q^a - q^{-a}}{q - q^{-1}}$$

Why? • q -Shur pol in $U_q(g)$

have only $\mathbb{Z}[q, q^{-1}]$ wif we work in divided powers

• $E_{(\varepsilon_1)} 1_n \otimes t_1 \oplus \dots$

only accnts for the $\mathbb{Z}[q, q^{-1}]$ -module structure of $1_m \cup 1_n$,
NOT for the $\mathbb{Q}(q)$ -module structure of $1_m \cup 1_n$.



If shurz have 1-morphisms: $E_{(\varepsilon_1)} 1_n \{t_1\} \oplus \dots \oplus E_{(\varepsilon_N)} 1_n \{t_N\}$,

where $(\varepsilon) = (\varepsilon_1^{a_1}, \dots, \varepsilon_N^{a_N})$, $a_i \in \mathbb{N}$, $\varepsilon_i \in \{+, -\}$

$$E_{(\varepsilon)} = E_{\varepsilon_1}^{(a_1)} \dots E_{\varepsilon_N}^{(a_N)} \rightarrow \varepsilon_+^{(a)} = \varepsilon^{(a)}, \quad \varepsilon_-^{(a)} = F^{(a)}$$

$$E_{1_n}^{(a)} \cong \bigoplus_{[a]!} \varepsilon^{(a)} 1_n$$

$$\frac{E^a}{[a]!} \quad \frac{F^a}{[a]!}$$

$$\hookrightarrow \text{e.g. } a=2: [2]! = q + q^{-1} \\ \Rightarrow \varepsilon^2 1_n \cong \varepsilon^{(2)} 1_n \{1\} \oplus \varepsilon^{(2)} 1_n \{-1\}$$

$$\bullet a=3: [3]! = q^3 + 2q + 2q^{-1} + q^{-3}$$

$$\Rightarrow \varepsilon^3 1_n \cong \varepsilon^{(3)} 1_n \{3\} \oplus \varepsilon^{(3)} 1_n \{-1\} \oplus$$

$$\text{Solution: } \underset{n}{\text{1Morph}}(m, n) = \text{Ker}(\underset{n}{\text{1Morph}}(m, n)) \quad \varepsilon^{(1)} 1_n \{1\} \oplus \dots$$

The Karoubi envelope

- idempotent: morphism $e: b \rightarrow b$ s.t. $e \circ e = e$
 - split idempotent: $f, g: b \rightarrow b'$, $R: b' \rightarrow b$: $e = R \circ g$, $g \circ f = 1_b$
 \hookrightarrow If C is additive category: $b' := \text{im } e \rightsquigarrow g = p|_{\text{im } e}$
 $b = \text{im}(e) \oplus \text{im}(1-e)$ $R = \text{inj}_{\text{im}(e)} \hookrightarrow b$
 - $\text{Kar}(C)$:
 - Objects: (b, e) $b \in \text{Ob}(C)$, e idempotent in C
 - Morphisms: (e, f, e') $f \in \text{Morph}_C(b, b')$
 $e' \circ f = f \circ e = f$ $e: b \rightarrow b$, $e': b' \rightarrow b'$
- \hookrightarrow If C is additive category $\Rightarrow \text{Kar}(C)$ is additive
 $\text{im } e := (b, e) \in \text{Ob}(\text{Kar}(C)) \Rightarrow f \cong \text{im}(e) \oplus \text{im}(1-e)$
 $\downarrow \in \text{Ob}(C)$ $\uparrow \in \text{Ob}(\text{Kar}(C))$

\mathcal{U} = additive 2-category

- Objects: $n \in \mathbb{Z}$
- 1-Morph (n, m) : $(\mathcal{E}_{(\varepsilon_n)} 1_{\mathcal{U}}(t_1), e_1) \oplus \dots \oplus (\mathcal{E}_{(\varepsilon_n)} 1_{\mathcal{U}}(t_N), e_N)$
 $e_i \in \text{2Morph}_{\mathcal{U}}(\mathcal{E}_{(\varepsilon_i)} 1_{\mathcal{U}}(t_i), -)$
- 2-Morph $((\mathcal{E}_{(\varepsilon)} 1_{\mathcal{U}}(t), e), (\mathcal{E}_{(\varepsilon')} 1_{\mathcal{U}}(t'), e')) = (e, f, e')$
 $f \in \text{2Morph}_{\mathcal{U}}(\mathcal{E}_{(\varepsilon)} 1_{\mathcal{U}}(t), \mathcal{E}_{(\varepsilon')} 1_{\mathcal{U}}(t'))$

Prf: Description with $\mathcal{E}^{(2)} 1_n$

- minimal idempotent: idempotent e which cannot be written as $e' \oplus e''$,
 s.t. $e''e' = e'e'' = 0$

Df.: e a minimal idempotent $\in \text{2Morph}_{\mathcal{U}}(\mathcal{E}^{\alpha} 1_n, \mathcal{E}^{\alpha} 1_n)$ (upto grading shift)

\hookrightarrow Let D_2 = longest braid on 2 strands

e.g.: $a=4$:

Def.: $e_a := \begin{smallmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ a-1 & a-2 & \cdots & 0 \end{smallmatrix} \circ D_a$

↳ Example: $a=2$:

↳ Idempotent?

$$\Leftrightarrow \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \Leftrightarrow e_2^2 = e_2$$

$$\text{im } e_a \cong \text{im } (1 - e_a)$$

(up to grading shift)

Def.: $\forall a \in \mathbb{N}: \mathcal{E}^{(a)} 1_n := (\mathcal{E}^a 1_n \left\{ \frac{-a(a-1)}{2} \right\}, e_a)$
 \downarrow
 $\in \text{Ker}(\text{1Morph}_{\text{in}}(m, n))$
 $= \text{1Morph}_{\text{in}}(m, n)$

$$\text{Prop.: } \mathcal{E}^a 1_m \cong \bigoplus_{[a]!} \mathcal{E}^{(a)} 1_n.$$

Remark: Graphical calculus doesn't really work anymore for the
 Categorification theorem:

- $f: U \rightarrow K_0(\mathcal{U}): E_{(\mathcal{E})} 1_n \mapsto [E_{(\mathcal{E})} 1_n]$
 $\hookrightarrow \mathcal{E} = (\mathcal{E}_1^{a_1}, \dots, \mathcal{E}_N^{a_N}): E_{(\mathcal{E})}^{(a)} = \frac{E^a}{[a]!}$
 is isomorphism of $\mathbb{Z}[q, q^{-1}]$ -algebras

- The indecomposable 1-morphisms in \mathcal{U} (up to grading shift) give basis in $K_0(\mathcal{U})$ with structure coeffs $\in \mathbb{N}[q, q^{-1}]$, s.t.
 $\gamma^{-1}(\text{basis}) = \text{Lusztig's canonical basis of } \mathcal{U}$.

"Proof" / Motivation:

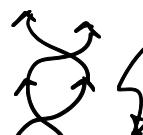
1. The spaces of 2-morphisms are not "too large"

Each 2Hom : lin. combi of "clear diagrams"

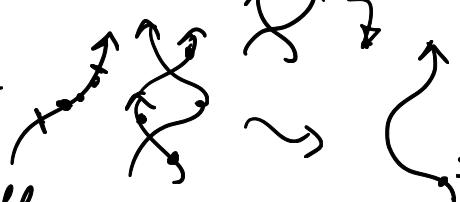
↳ no self-intersections of strands:



↳ no strands intersect more than once:



↳ all dots \in small interval



↳ closed diagrams:

products of dotted bubbles,

↳ non-nested

↳ same orientation

↳ far right corner

Clear diagrams = basis $\sim \exists$ analog of Reidemeister moves

$$\Rightarrow \text{gdim}_{\mathcal{U}}(\text{Hom}_{\mathcal{U}}(E_{(\varepsilon)}1_n, E_{(\varepsilon')}1_n)) = \langle E_{(\varepsilon)}1_n, E_{(\varepsilon')}1_n \rangle.$$

$$\frac{1}{\prod_{a=1}^{\infty} 1 - q^{2a}}$$

2. Relations are "not too strong"

1-Morph $\neq \{0\}$

3. $\gamma = \text{homomorphism: } \begin{cases} \exists \mathcal{U}^\pm \text{ isomorphisms} \\ \text{Composition of 1-morph} \Leftrightarrow \text{multiplication in } \mathcal{U} \end{cases}$

$$\text{gdim}_{\mathcal{U}}(\text{Hom}_{\mathcal{U}}(1_n, 1_n))$$