Kleine seminar 20/01/2020

Based on introduction to categorification by Savage.

Goal To categorify the Fock representation of the Heisenberg algebra

$\mathfrak{h} = \langle x, y, z \rangle \qquad [x, y] = \langle x, y \rangle z$

central

$\mathfrak{h}_{2n+1} = \langle x_i, y_i, z \rangle \quad i \in \{1, \dots, n\}$

$[x_i, y_j] = \langle x_i, y_j \rangle z$

$[x_i, x_j] = [y_i, y_j] = 0$

$$\left\{ \begin{pmatrix} 0 & \vec{x} & z \\ & 0 & \vec{y} \\ & & 0 \end{pmatrix} \right\} \qquad \begin{array}{l} x_i = q_i \\ y_i = p_i \end{array}$$

Today infinite dimensional, unital, associative algebra over $\mathbb{Q}$ $\ \ \ \nwarrow H^{\mathbb{Q}}_{\infty}$

$p_i, q_j \quad i, j \in \mathbb{N}_+$

$$\begin{cases} [p_i, p_j] = [q_i, q_j] = 0 \\ [p_i, q_j] = i\, \delta_{ij}\, 1 \end{cases} \Rightarrow \text{not suited for categorification}$$

Consider an integer Heisenberg algebra $\mathfrak{h}$ over $\mathbb{Z}$

$e_n, a^*_n \quad n \in \mathbb{N}_+$

$$\begin{cases} [e_n, e_m] = [a^*_n, a^*_m] = 0 \\ [e_n, a^*_m] = e_{n-1}\, a^*_{m-1} \end{cases}$$

$H \otimes_{\mathbb{Z}} \mathbb{Q} = H^{\mathbb{Q}}_{\infty}$

Realisation using Sym

Outline

→ introduce Sym

  ⇝ Several Sym

  → $H \subseteq \mathrm{End}_{\mathbb{Z}}(\mathrm{Sym})$

    → Fock representation

→ Introduce category $\mathcal{A}$ symmetric group modules

  $K(\mathcal{A}) \cong \mathrm{Sym}$

→ For each $M \in \mathcal{A}$

  $\mathrm{Res}_M : \mathcal{A} \to \mathcal{A}$

  $\mathrm{Ind}_M \quad \mathcal{A} \to \mathcal{A}$

  → weak categorification

Sym : algebra of symmetric ~~functions~~ polynomials

  in countable many variables over $\mathbb{Z}$

In 2 variables $\quad f(x,y) = f(y,x)$

Ex $\quad x+y$

$\quad x^2+y^2$

$\quad xy$

homogeneous polynomial of order n

$$\mathrm{Sym} = \bigoplus_{n \in \mathbb{N}} \mathrm{Sym}_n$$

Ex $\mathrm{Sym}_0 = \mathbb{Z}\,\underline{1}$ $\quad m_{\emptyset}$

$\mathrm{Sym}_1 \quad \mathbb{Z}(\underbrace{x_1 + x_2 + x_3 + \dots + x_i + \dots}_{S(x_1)})$ $\quad m_{\square}$

$\mathrm{Sym}_2 : \quad \mathbb{Z}\, S(x_1^2) + \mathbb{Z} S(x_1 x_2)$ $\quad m_{\boxminus}$

$\mathrm{Sym}_3 . \quad \mathbb{Z}\, S(x_1^3) + \mathbb{Z}\, S(x_1^2 x_2) + \mathbb{Z} S(x_1 x_2 x_3)$

$m_{\square\square}$ $\quad m_{\square\square\square}$ $\quad m_{\text{⊞}}$ $\quad m_{\boxminus}$

Basis labelled by partitions

$\lambda \in \mathcal{P}$

$$m_\lambda = \sum_{\substack{\alpha \text{ is} \\ \text{a rearrangement} \\ \text{of } \lambda}} x^\alpha \qquad x_1^{\alpha_1} x_2^{\alpha_2} \dots \; x_i^{\alpha_i} \dots$$

monomial

$\lambda = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Complete symmetric function

$$h_n = \sum_{\lambda \in \mathcal{O}_n} m_\lambda \qquad \Rightarrow \quad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_q}$$

elementary symm function

$$e_n = m_{1^n} \qquad\qquad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_q}$$

power sum symm function

$$p_n = m_{(n)} \qquad\qquad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_q}$$

Ex

$$h_0 = e_0 = p_0 = m_\emptyset = 1$$

$$h_1 = e_1 = p_1 = m_{(1)} = x_1 + x_2 + \cdots$$

$$h_2 = S(x_1^2) + S(x_1 x_2) = h_{(2)}$$

$$e_2 = S(x_1 x_2) = e_{(2)}$$

$$p_2 = S(x_1^2) = p_{(2)}$$

$$h_{(1,1)} = h_1 h_1 = S(x_1) S(x_1) = S(x_1^2) + 2 S(x_1 x_2)$$

$$e_{(1,1)} = e_1 e_1 = h_{(1,1)} = p_{(1,1)}$$

$$\left\{ \begin{array}{l} h_3 = S(x_1^3) + S(x_1^2 x_2) + S(x_1 x_2 x_3) = h_{(3)} \\ h_{(2,1)} = h_2 h_1 = S(x_1^3) + 3 S(x_1^2 x_2) \\ h_{(1,1,1)} = h_1 h_1 h_1 = S(x_1^3) + 5 S(x_1^2 x_2) + 2 S(x_1 x_2 x_3) \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{lll} h_\lambda & \lambda \in \mathcal{O} & \mathbb{Z}\text{-basis} \\ e_\lambda & \lambda \in \mathcal{O} & \mathbb{Z}\text{-basis} \\ p_\lambda & & \text{not a } \mathbb{Z}\text{-basis but } \mathbb{Q}\text{-basis.} \end{array} \right.$$

$$\mathrm{Sym} \otimes_{\mathbb{Z}} \mathbb{Q}$$

$m_\lambda, \ h_\lambda, \ e_\lambda, \qquad , p_\lambda$

Schur functions

$\lambda \in \sigma$

$\boxed{h_{-n} = 0}$

$s_\lambda = \det (h_{\lambda_i - i + j})_{1 \le i, j \le n} \qquad n \gg$

$s_\emptyset = h_\emptyset = 1$

$$s_1 = \begin{vmatrix} h_1 & h_2 \\ h_{-1} & h_0 \end{vmatrix} = h_1$$

$$s_1 = \begin{vmatrix} h_1 & h_2 & h_3 \\ h_{-1} & h_0 & h_1 \\ h_{-2} & h_{-1} & h_0 \end{vmatrix} = \begin{vmatrix} h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$s_{\square\square} = \begin{vmatrix} h_2 & h_3 \\ 0 & h_0 \end{vmatrix} = h_2$$

$$s_{(n)} = \begin{vmatrix} h_n & h_{n+1} & \cdots & \cdots \\ 0 & h_0 & & \\ 0 & 0 & h_0 & \\ 0 & 0 & & h_0 \end{vmatrix} = h_n$$

$s_{(1^n)}$

$$s_{\substack{\square\\\square}} = \begin{vmatrix} h_1 & h_2 & h_3 \\ h_0 & h_1 & h_2 \\ 0 & 0 & h_0 \end{vmatrix} = h_1^2 - h_2 = S(x_1, x_2)$$
$$= e_2$$

$$s_{\substack{\square\\\square\\\square}} = \begin{vmatrix} h_1 & h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 & h_3 \\ 0 & 1 & h_1 & h_2 \\ 0 & 0 & 0 & h_0 \end{vmatrix} = e_3$$

$s_{(1^n)} = e_n$

Define inner product

$$\begin{cases} \langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} \\ \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu} \end{cases}$$

Heisenberg algebra

Let $f \in Sym$ $\Rightarrow f \in End_{\mathbb{Z}}(Sym)$

$f: a \mapsto f \cdot a$

$f^*(a)$: is so

$\langle b, f^*(a) \rangle = \langle f \cdot a, b \rangle$

well defined because $\langle \cdot, \cdot \rangle$ is non degy

algebra generated by f, f^* is isomorphic to Heisenberg algebra

$Sym \otimes Sym^* \cong H$

e_n, h_n^* satisfy the relations of Heisenberg alg.

$[e_n, e_m] = 0$, $[h_n^*, h_m^*] = 0$

$h_n^* e_m = e_m h_n^* + e_{m-1} h_{n-1}$

$$\begin{cases} H \subseteq End_{\mathbb{Z}}(Sym) \\ \text{representation of } H \text{ on } Sym \quad \text{Fock representation} \end{cases}$$

$A_n = \mathbb{C}[S_n]$ semisimple

irreps are labelled by partition of σ_n

S^λ Specht module

$E_n = S^{(1^n)}$ Sign module

$L_n = S^{(n)}$ trivial module

$\mathcal{A} = \bigoplus_{n \in \mathbb{N}} A_n\text{-mod}$

$G_{\mathcal{A}} = \bigoplus_{n \in \mathbb{N}} G_0(A_n) = \bigoplus_{\lambda \in \sigma} \mathbb{Z}[S^\lambda]$

Bilinear form on $G_{\mathcal{A}}$.

$\langle \cdot, \cdot \rangle : G_{\mathcal{A}} \times G_{\mathcal{A}} \to \mathbb{Z}$

$\langle [M], [N] \rangle = \dim_{\mathbb{C}} Hom_{\mathcal{A}}(M, N)$

Prop

$G_A \to Sym$

$\varphi_A: [S^\lambda] \mapsto s_\lambda \quad \lambda \in P$

is an isomorphism

$[E_n] \mapsto e_n$

$[L_n] \mapsto h_n$

Furthermore

$\langle a, b\rangle = \langle \varphi_A(a), \varphi_A(b)\rangle \quad a, b \in G_A$

Let $M \in \mathcal{A}$ $\quad M \in A_m$-mod

Then $\quad A_m \otimes A_n \hookrightarrow A_{m+n}$

$Ind_M : \mathcal{A} \to \mathcal{A}$

$N \mapsto Ind_{A_m \otimes A_n}^{A_{m+n}} M \otimes N \qquad A_{m+n} \otimes_{A_m \otimes A_n} M \otimes N$

A_n-mod $\qquad A_{m+n}$-mod

$Res_M : \mathcal{A} \to \mathcal{A}$

$m \le n \qquad N \mapsto Hom_{A_m}(M, Res^{A_n}_{A_{n-m} \otimes A_m} N)$

$(1 \otimes a)$

A_{n-m}-mod

If L is an $A_q \otimes A_\ell$-module, M is an A_ℓ-mod

then $Hom_{A_\ell}(M, L)$ is an A_q-mod.

$(a \cdot f)m = (a \otimes 1) f(m)$

A_q

$m > n \qquad N \mapsto 0$

$\Rightarrow$ Exact functors

Induce operations on G_A.

Prop

$$
\begin{tikzcd}
G_A \arrow[r, "{[\mathrm{Ind}_M]}"] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\circlearrowleft"] & G_A \arrow[d, "\varphi_A"] \\
\mathrm{Sym} \arrow[r, "{\varphi_A([M])}"'] & \mathrm{Sym}
\end{tikzcd}
\qquad
\begin{tikzcd}
G_A \arrow[r, "{[\mathrm{Res}_M]}"] \arrow[d, "\varphi_A"'] \arrow[dr, phantom, "\circlearrowleft"] & G_A \arrow[d, "\varphi_A"] \\
\mathrm{Sym} \arrow[r, "{\varphi_A([M])^*}"'] & \mathrm{Sym}
\end{tikzcd}
$$

$$
\begin{tikzcd}
G_A \arrow[r, "{[\mathrm{Ind}_{E_n}]}"] \arrow[d] & G_A \arrow[d] \\
\mathrm{Sym} \arrow[r, "e_n"'] & \mathrm{Sym}
\end{tikzcd}
\qquad
\begin{tikzcd}
G_A \arrow[r, "{[\mathrm{Res}_{L_n}]}"] \arrow[d] & G_A \arrow[d] \\
\mathrm{Sym} \arrow[r, "e_n^*"'] & \mathrm{Sym}
\end{tikzcd}
$$

$\rightarrow$ naive categorification

Weak categorification

Thm

$$
\begin{cases}
\mathrm{Ind}_{E_n} \circ \mathrm{Ind}_{E_m} \cong \mathrm{Ind}_{E_m} \circ \mathrm{Ind}_{E_n} \\
\mathrm{Res}_{L_n} \circ \mathrm{Res}_{L_m} \cong \mathrm{Res}_{L_m} \circ \mathrm{Res}_{L_n}. \\
\mathrm{Res}_{L_m} \circ \mathrm{Ind}_{E_n} \cong \mathrm{Ind}_{E_n} \circ \mathrm{Res}_{L_m} \oplus \mathrm{Ind}_{E_{n-1}} \circ \mathrm{Res}_{L_{m-1}}
\end{cases}
$$

$\Rightarrow$ weak categorification of the Fock rep of Heisenberg algebra.