Representations and Wenzl–Jones elements of quotients of the affine Temperley–Lieb algebra

The untangled affine Temperley–Lieb algebra

Alexis Langlois-Rémillard

Work in progress with: Alexi Morin-Duchesne and Robert A. Spencer

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GHENT UNIVERSITY

FACULTY OF SCIENCES

EOS THE EXCELLENCE OF SCIENCE

Outline

1. Temperley–Lieb algebras and cellularity
2. Affine Temperley–Lieb algebras and the untangled version
3. A few words on a Wenzl–Jones type elements
4. Another thing to ask me at lunch

One-page description

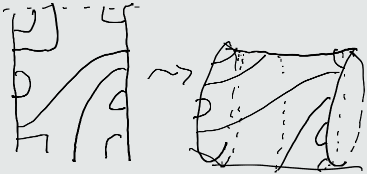
Goal

Study finite-dimensional quotients of the affine Temperley–Lieb algebras and construct a “Wenzl–Jones projector” in these quotients.

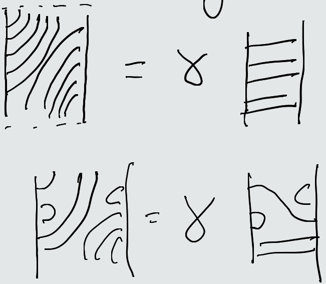
- Idempotent
- link to 1D module(s)

- has a diagrammatic construction

Affine Temperley–Lieb



Quotients Untangle!



Temperley–Lieb algebras and cellularity

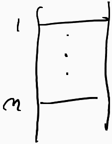
Primer: Temperley–Lieb algebras

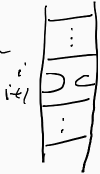
Generators and relations

Let $\beta = -q - q^{-1}$, $n \in \mathbb{N}$ and $q \in \mathbb{C}^{\times}$. $\mathrm{TL}_m(\beta)$ is generated by $I, e_1, \ldots, e_{n-1}$, subject to relations

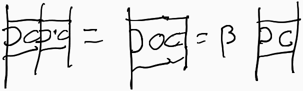
$$\begin{aligned} Ie_i &= e_i I = e_i, \\ e_i^2 &= \beta e_i, \\ e_i e_{i\pm 1} e_i &= e_i, \\ e_i e_j &= e_j e_i, \quad |i-j| > 1. \end{aligned}$$

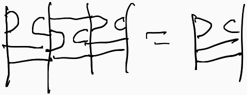
Diagrams

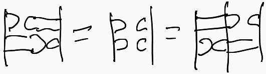
$I =$ 

$e_i \sim$ 

rel $(n=4)$

$e_2^2 = \beta e_2$, 

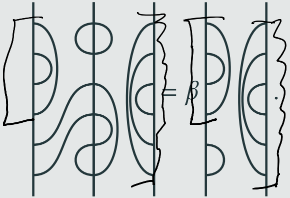
$e_1 e_2 e_1 = e_1$ 

$e_1 e_3 = e_3 e_1$, 

Properties of Temperley–Lieb

1. $\dim \mathrm{TL}_n = C(n)$, the Catalan number.
2. The vertical reflection is an anti-involution noted $*$.
3. The diagrams can be ordered by the number of through lines.
4. Through lines can only be closed, never created.
5. Left- and rightmost arcs of a multiplication are left untouched.
6. Semisimple for generic q

Multiplication



One peculiar element

$[k]_q$: q-number / Chebychev of second type $(n \neq 1)$

$q^k - q^{-k} / q - q^{-1}$

Wenzl–Jones projector

Defined by Jones (83) and studied by Wenzl (88).

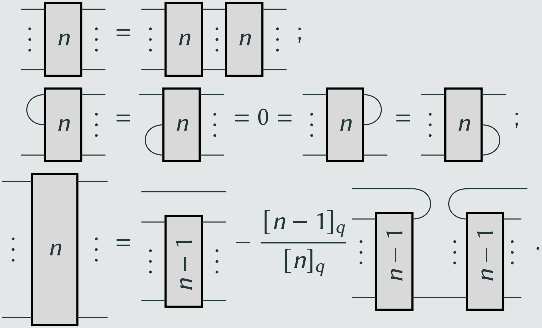
$$
\begin{aligned}
\mathrm{P}_n &= \mathrm{P}_n^2, \\
e_i \mathrm{P}_n &= 0 = \mathrm{P}_n e_i, \\
\mathrm{P}_n &= \mathrm{P}_{n-1} - \frac{[n-1]_q}{[n]_q} \mathrm{P}_{n-1} e_1 \mathrm{P}_{n-1},
\end{aligned}
$$

$P_1 = id$

$P_2 = id - \frac{[1]_q}{[2]_q} e_1$

Wenzl–Jones projector

Defined by Jones (83) and studied by Wenzl (88).



Cellular algebras

- Introduced in 1996 by Graham and Lehrer.
- Generalize the “specialness” of Kazhdan-Lusztig basis for Iwahori-Hecke algebras.

Take-home message

A cellular algebra is an associative algebras with a special basis, a filtration-like structure and an anti-involution.

Second message

A lot of generalisations of cellularity have been given the last 25 years.

Cellular algebras with an example

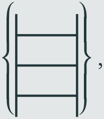
Definition (Graham. Lehrer. 96)

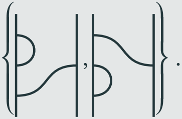
A R-algebra A is **cellular** with **cellular datum** $(\Lambda, M, C, *)$ if

1. Λ is a poset and $\{M(\lambda)\}_{\lambda \in \Lambda}$ a collection of finite sets;

TL_3

- $\Lambda = \{1 < 3\}$
- $M(\lambda)$ set of monic $3 \leftarrow \lambda$ diagrams:

$M(3) =$ 

$M(1) =$ 

Cellular algebras with an example

Definition (Graham. Lehrer. 96)

A R-algebra A is **cellular** with **cellular datum** $(\Lambda, M, C, *)$ if

2. $C : \bigcup_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \to A$ is an injective map; its image, a basis of A

$$\{C^{\lambda}_{s,t} := C(s,t) \mid \lambda \in \Lambda; s, t \in M(\lambda)\}.$$

TL_3

For $v, w \in M(\lambda)$

$$C(v, w) = vw^*$$

where w^* is the reflection of w.

TL_3 cellular basis

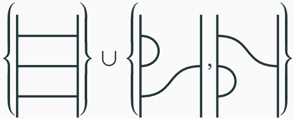
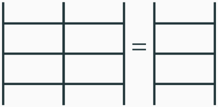
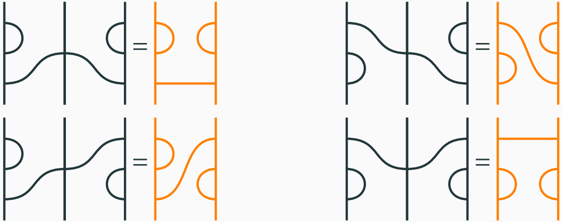
$M(3) \cup M(1) =$ 

Image of C given by





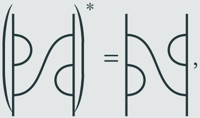
Definition (Graham and Lehrer, 1996)

A R-algebra A is **cellular** with **cellular datum** $(\Lambda, M, C, *)$ if

3. $*$ is an anti-involution such that $(C^\lambda_{s,t})^* = C^\lambda_{t,s}$;

TL_3

$*$ is the vertical reflection. As



we have $C(v, w)^* = (vw^*)^* = wv^* = C(w, v)$.

Cellular algebras,

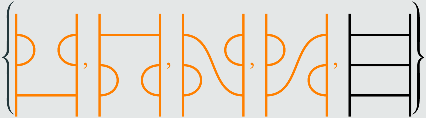
Definition (Graham and Lehrer, 1996)

A R-algebra A is **cellular** with **cellular datum** $(\Lambda, M, C, *)$ if

4. for all $a \in A$,
$$aC^{\lambda}_{s,t} \equiv \sum_{s' \in M(\lambda)} r_a(s', s) C^{\lambda}_{s',t} \mod A^{<\lambda}$$
where $A^{<\lambda} = \left\langle C^{\mu}_{s,t} \mid \mu < \lambda \right\rangle$.

TL_3

$\mathfrak{B}_3 =$



Here it means that multiplying two elements can only reduce the number λ of defects. This is verified as you can only close defect.

Cellular algebras, complete definition

Definition (Graham and Lehrer, 1996)

A R-algebra A is **cellular** with **cellular datum** $(\Lambda, M, C, *)$ if

1. Λ is a poset and $\{M(\lambda)\}_{\lambda\in\Lambda}$ a collection of finite sets;
2. $C : \bigcup_{\lambda\in\Lambda} M(\lambda) \times M(\lambda) \longrightarrow A$ is an injective map with a basis of A as image
$$\{C^{\lambda}_{s,t} := C(s,t) \mid \lambda \in \Lambda; s, t \in M(\lambda)\}$$
3. $*$ is an anti-involution such that $(C^{\lambda}_{s,t})^* = C^{\lambda}_{t,s}$;
4. for all $a \in A$,
$$aC^{\lambda}_{s,t} \equiv \sum_{s'\in M(\lambda)} r_a(s',s)C^{\lambda}_{s',t} \mod A^{<\lambda}$$
where $A^{<\lambda} = \left\langle C^{\mu}_{s,t} \mid \mu < \lambda \right\rangle$.

Drawing the definition

0) An algebra A

1) A Poset Λ

$\lambda_1 > \lambda_2 > \lambda_3 > \dots$

2) set

$M(\lambda) = \{ \boxed{t} \}_t$

3) Combining elements

$C\left(\boxed{t_\lambda}, \boxed{t'} \right) \rightsquigarrow \boxed{t \,|\, t'}$

$\to$ basis of A

4) Anti-involution

$\boxed{t \,|\, t'}^{*} = \boxed{t' \,|\, t}$

basis filtration:

λ_1

✓

λ_2

✓

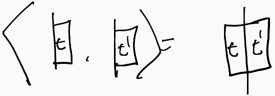
λ_3

8) Relation

$$\boxed{a}\,\boxed{t \,|\, t'} = \sum_{s \in M(\lambda)} r_{s,t} \boxed{s \,|\, t'} + \underbrace{\sum_{\mu < \lambda} c_{u,u'} \boxed{u \,|\, u'}}_{\in A^{<\lambda}}$$

$a \in A$, $\boxed{t \,|\, t'} \in C(M(\lambda), M(\lambda))$

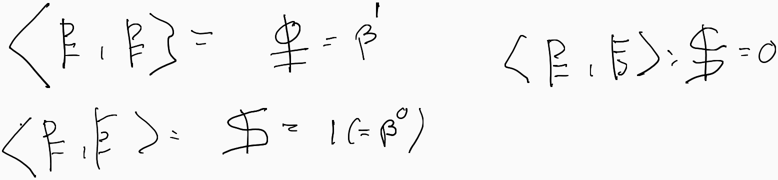
Bili:



$\in \mathbb{C}$

1. Construct a family of modules (cell modules)
2. Study semisimplicity via a bilinear form
3. Relation between indecomposable and simple modules

TL:



Affine Temperley–Lieb algebras and the untangled version

Generators and relations

Let $\beta = -q - q^{-1}$, $n \in \mathbb{N}$ and $q \in \mathbb{C}^\times$. $\mathrm{TL}_m(\beta)$ is generated by $I, e_0, e_1, \dots, e_{n-1}$, Ω and Ω^{-1} subject to relations

$\mathrm{TL}_m + e_0$

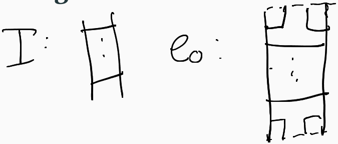
$$Ie_i = e_i I = e_i,$$
$$e_i^2 = \beta e_i,$$
$$e_i e_{i\pm1} e_i = e_i,$$
$$e_i e_j = e_j e_i, \quad |i-j| > 1.$$

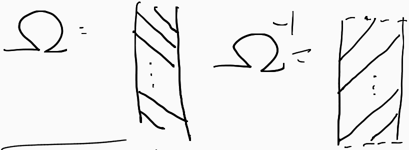
Ω

$$\Omega e_j \Omega^{-1} = e_{j-1}$$
$$e_{n-1} \dots e_2 e_1 = \Omega^2 e_1$$

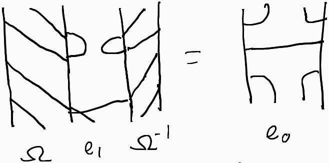
$e_2 e_1 = \Omega^2 e_1$

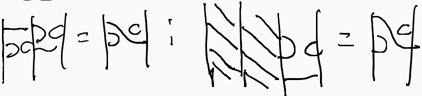
Diagrams





$(n=3)$ relation



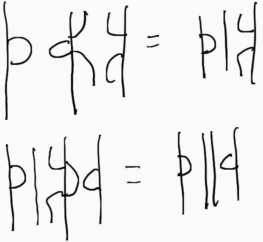


Cellularity?

Problem

The algebra is infinite-dimensional

$n=2$



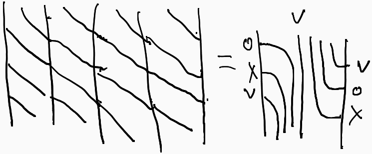
$\rightsquigarrow e_1 e_0 e_1 \overset{?}{=} e_1$

extra rel

(even)

$n=3$

$e^4 \rightsquigarrow$



(odd and even)

Cellularity?

Problem

The algebra is infinite-dimensional

Modification of cellularity

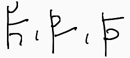
Affine cellular (König and Xi 2012)

1. Finite number in the poset filtration
2. Each level of has a commutative algebra associated to it.
3. For affine TL: $K[\Omega, \Omega^{-1}]$. $\to$ / $\Omega^{-1}\Omega = \Omega^{-1}\Omega = \mathrm{Id}$

Cell modules

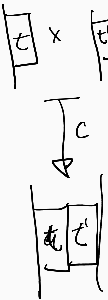
$$\{W_{n,d,z} \mid z \in \mathbb{C}^{\times}, 0 \leq d \leq n, d \equiv_2 n\}.$$

$\mathrm{Dim}\, W_{n,d,z} = \binom{n}{\frac{n-d}{2}}$

Basis:
$W_{3,1,z} = \{$  $\}$

Intuition with drawing

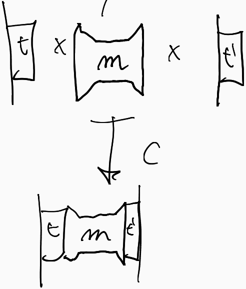
Cellular



(+) rel

Affine cellular

∈ commutative alg.



ATL$_n$: m is Ω^{l} or Ω^{-l}.

A subalgebra of the affine Temperley–Lieb

Periodic Temperley–Lieb algebra

$$\mathrm{PTL}_n(\beta) = \langle I, e_0, e_1, \ldots, e_{n-1} \rangle / R$$

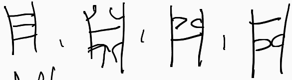
where R is the ideal generated by

$$Ie_i = e_i I = e_i,$$
$$e_i^2 = \beta e_i,$$
$$e_i e_{i\pm 1} e_i = e_i,$$
$$e_i e_j = e_j e_i, \quad |i - j| > 1.$$

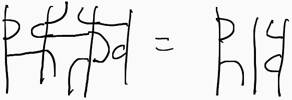
Properties

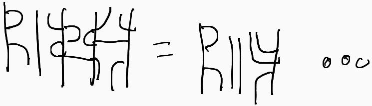
- $\mathrm{PTL}_n(\beta) \subset \mathrm{ATL}_n(\beta)$
- Diagrams with even crossing.

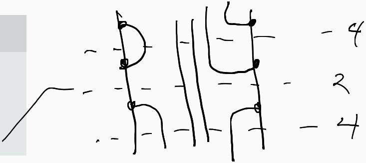
Diagrams



still infinite.







As much as the affine Temperley–Lieb as possible without going infinite.

Motivation

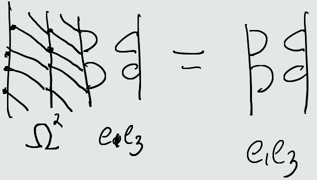
A question of Tubbenhauer on algebras with big representation gap; monoid theory, etc.

→ Step 1 of a tower of quotients

$$u_k ATL_n \rightsquigarrow \Omega^{kN} = \delta\, id \qquad ATL_n \stackrel{?}{=} \varprojlim_k u_k ATL_n$$

A distinction in n even and n odd

Particularities of n even



$\rightsquigarrow$ restriction on γ

We consider n odd for this talk

n even is similar, but with more care needed at the quotients and values.

Quotients, n odd

Quotient of $\mathrm{ATL}_n(\beta)$

Define the *untangled* affine Temperley–Lieb algebra $\mathrm{uATL}_n(\beta,\gamma)$ as $\mathrm{ATL}_n(\beta)$ quotiented by

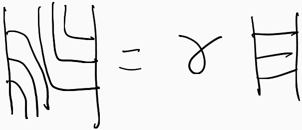
$$\Omega^n = \gamma I$$

Quotient of $\mathrm{PTL}_n(\beta)$

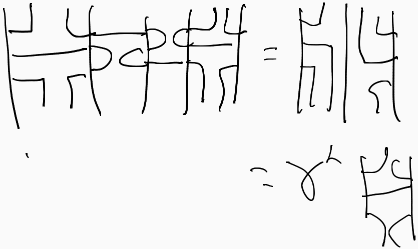
Define $\mathrm{uPTL}_n(\beta,\gamma)$ as $\mathrm{PTL}_n(\beta)$ quotiented by

$$e_0(e_{n-1}\dots e_1 e_0)^{n-2} = \gamma^2 e_0$$

Diagrams $(n=3)$



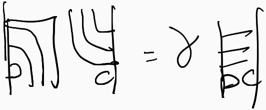
$e_0 \ (e_2 \ e_1 \ e_0)^1$

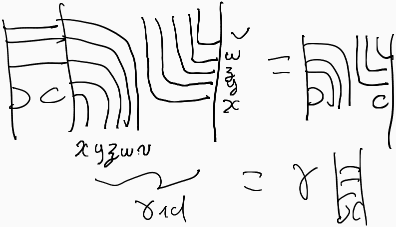


Computations

① Proposition

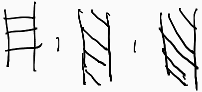
The relation $\Omega^n = \gamma\, id$ implies that there are maximally d crossings on diagrams with d through lines.

ex: 

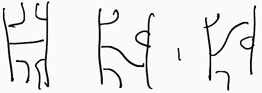


So dim will be d times the basis of diag with d through lines.

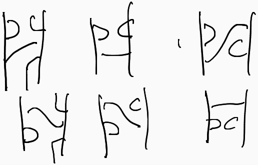
$n=3$



3×1^2



1×3^2



$\dim uAR_3 = 12$

First properties

$3 \times \binom{2}{1}^2 = 12$ ✓

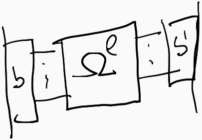
$5 \times \binom{4}{2}^2 = 120$

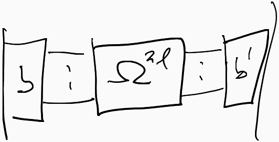
The dimension is

$$\dim \mathrm{uATL}_n(\beta,\gamma) = \sum_{d=1,3\ldots,n} d(\dim B_{n,d})^2 = n\binom{n-1}{\frac{n-1}{2}}^2$$

$$\dim \mathrm{uPTL}_n(\beta,\gamma) = n\binom{n-1}{\frac{n-1}{2}}^2 - (n-1)$$

All elements are of this form:





$b, b' \in B_{n,d}$. $0 \le l \le d$

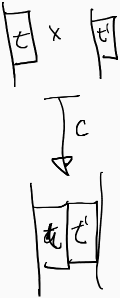
$\rightarrow$ basis $W_{n,d,z}$

uATL

uPTL_n $\quad 0 \le l < d$

Sandwich cellularity

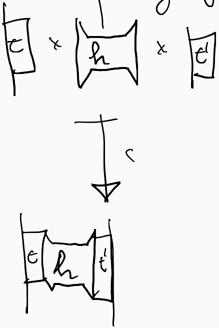
Cellular



$\oplus$ rel

Sandwich cellular

$\to$ any algebra H_d



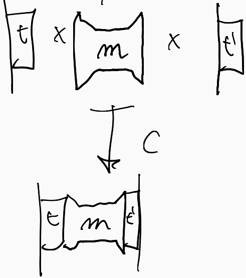
in $uATL_n$

$h \in H_d \cong \mathbb{Z}/_d$

$\cong \langle \Omega^0, \dots, \Omega^{d-1} \rangle$

Affine cellular

$\in$ commutative alg



ATL_n: m is Ω^l or Ω^{-l}.

Sandwich cellular (involutive)

Sandwich cellular (Tubbenhauer 22—Tubbenhauer–Vaz 21)

An algebra A is **sandwich cellular** if it has a sandwich cell datum $(\mathcal{P}, (\mathcal{L}, \mathcal{R}), (\mathscr{H}_\lambda, B_\lambda), C, *)$

1. $\mathcal{P}$ poset with order $<_{\mathcal{P}}$
2. $\mathcal{L}, \mathcal{R}$ collections of finite sets
3. for $\lambda \in \mathcal{P}$ algebras $\mathscr{H}_\lambda$ with bases B_λ
4. $C_\lambda : \mathcal{L}(\lambda) \times B_\lambda \times \mathcal{R}(\lambda) \to A; (l, b, r) \mapsto c^\lambda_{l,b,r}$
5. $*$ anti-involution such that $(c^\lambda_{l,b,r})^* \equiv c^\lambda_{r,b,l} \mod A^{<\lambda}$

Recap

We have two quotients and we know:

1. Dimensions
2. Sandwich cellular
3. Cell modules
4. n 1D modules for $\mathrm{uATL}_n(\beta)$; one for $\mathrm{uPTL}_n(\beta)$

1D modules

Condition on $W_{n,d,z}$ of $\mathrm{ATL}_n(\beta)$ to not be killed by the quotient: $\Omega^n = \gamma$ and $\Omega|_{W_{n,d,z}} = z$ so $z = \gamma^{1/n} e^{2\pi r/d}$

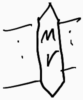
So $W_{n,d,r} := W_{n,d,\gamma^{1/n} e^{2\pi r/d}}$.

and many drawings!

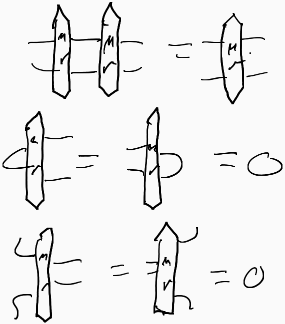
A few words on a Wenzl–Jones type elements

Properties wanted

$$Q_{n,r}^2 = Q_{n,r} \quad e_i Q_{n,r} = 0 = Q_{n,r} e_i \quad Q_{n,r} \mid_{W_{n,d,s}} = \begin{cases} 1 & d = n, r = s \\ 0 & \text{else} \end{cases}$$



such that



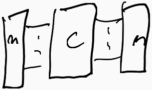
Subalgebras first

On uPTL

Only one one-dimensional $W_{n,n,r} = W_{n,n,0}$

$$Q_n^2 = Q_n \quad e_i Q_n = 0 = Q_n e_i \quad Q_n \mid_{W_{n,d,r}} = \begin{cases} 1 & d = n \\ 0 & \text{else} \end{cases}$$

idea: recall $\boxed{n}$: JW projector on $TL_n \subset uPTL_n$

Put $Q_n = \boxed{n} + \sum_c \lambda(c)$ 

So automatically $e_1 \cdots e_{n-1} \cdot Q_n = 0$

Properties of Q_n

Identity plus rest

Let c denote multi-index $c_1, \ldots, c_m$ and $e_c := e_{c_1} \ldots e_{c_m}$.

$$Q_n = I + \sum_c a_c e_c.$$

Unique

If Q_n' respects the properties, then $Q_n' = I + \sum_c a_c' e_c$ and $QQ' = Q = Q'$.

Commutation

As an element of ATL, Q_n commute with Ω since $Q_n' = \Omega Q_n \Omega^{-1}$ respect the properties and is in PTL, so

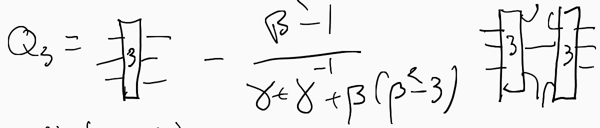
$$Q_n = \Omega Q_n \Omega^{-1}$$

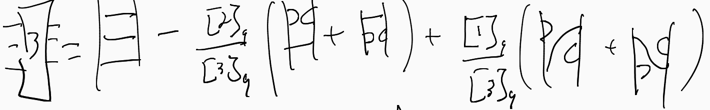
.

① Find the coefficients $\kappa(c)$ from:

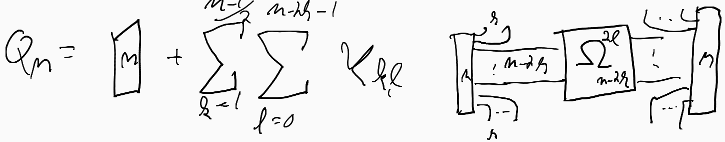
a) $e_0 Q_n = 0$ b) $Q_n e_0 = 0$ c) $Q_n^2 = Q_n$.

ex: $\underline{n=3}$





In general, we can further give the form



Lifting to $\mathrm{ATL}_n(\beta)$

Projector

Let $\omega_{n,r} = \gamma^{\frac{1}{n}} e^{2\pi i r/n}$. Then

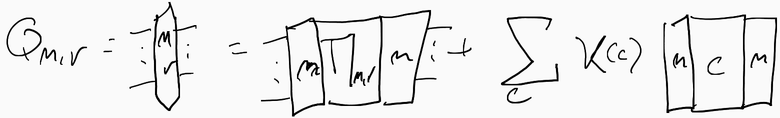
$$\Pi_{n,r} := 1/n \sum_{j=0}^{n-1} (\omega^{n,r})^{-j} \Omega^j.$$

- Squares to itself
- Sends 1D uPTL_n-module $W_{n,n,0}$ to 1D uATL_n-module $W_{n,n,\omega_{n,r}}$.

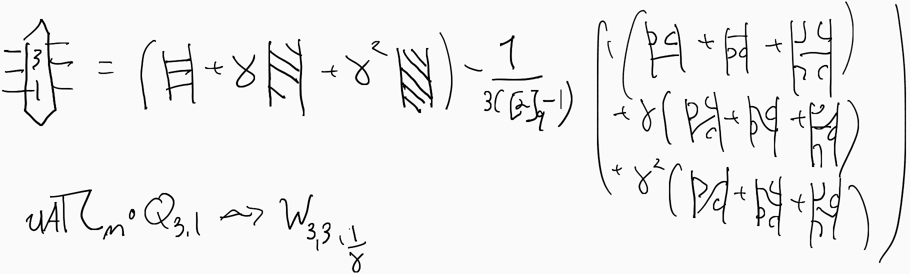
Wenzl–Jones projector $Q_{n,r}$

$$Q_{n,r} = \Pi_{n,r} Q_n$$

$Q_{n,r}$ in drawings



ex: $Q_{3,1}$



$uATL_n \circ Q_{3,1} \rightsquigarrow W_{3,3,\frac{1}{\gamma}}$

$\Omega\, Q_{3,1} = \frac{1}{\gamma} Q_{3,1}$

$\shortparallel$

γ^{2}

Proposition and recurrence form

Proposition

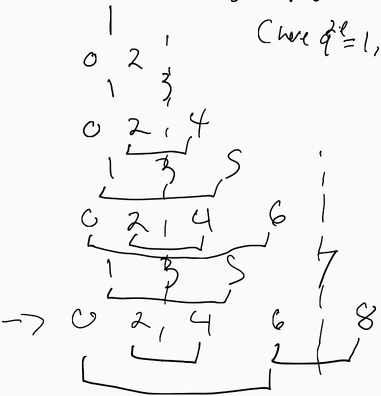
There are linear recurrence relations determining completely the coefficients $K(c)$ appearing in $Q_{n,r}$.

W.I.P. Find closed form for them

In TL_n $\to$ and ATL_n, Graham and Lehrer gave a way to study its representation theory via morphisms of cell modules:

Bratteli

(here $q^{2\ell}=1, \ell=4$)



Conjecture.

We have the same analysis for $uATL_n$

$\to$ and it is meaningful

Extra

More quotients.

Define $u_k ATL_n$ by $\Omega^{kn} = \gamma\, id$.

Question (Théo Pinet)

What is the inductive limit of $u_k ATL_n$ over k?

$\to ATL_n$ or something bigger?

Prop: All finite-dimensional $ATL_n^{(\beta)}$ appear as modules of $uATL_n(\beta, \gamma)$ for a certain γ

Extra

Future

Precise coefficients

We can now compute the coefficients and want to find "nice" formulas for them

Modular world

Spencer and Martin (2022), and Burrull, Libedinsky and Sentinelli (2019) described modular Wenzl–Jones elements. We would try this here.

Application: Cryptography (Khovanov, Sitaraman and Tubbenhauer 2022) and study of p-canonical bases

Use the quotients to study the affine Temprley–Lieb algebra

Can we say something non-trivial on ATL with these quotients?

Another thing to ask me at lunch

Chess domination on polycubes

Joint work with Érika Roldán and Christoph Müßig

Problem

What is the minimal and maximally-independent number of queens or rooks you can use to dominate a polycube?

Try it!

https://www.erikaroldan.net/queensrooksdomination

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Main results

It is a NP-complete problem to solve.

Questions?

References i