Chapter 6: Extensions and Resolutions

§ 6.1 Fix $\lambda \in \Lambda^+$. Want to realise

$$\operatorname{ch} L(\lambda) = \sum_{w} (-1)^{\ell(w)} \operatorname{ch} M(w\cdot\lambda) \qquad (2.4)$$

Def A **BGG resolution of $L(\lambda)$** is an exact sequence

$$(*) \quad 0 \to C_m \xrightarrow{\delta_m} C_{m-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\varepsilon} L(\lambda) \to 0$$

with $\displaystyle C_k := \bigoplus_{w \in W^{(k)}} M(w\cdot\lambda)$, $k = 0, 1, \ldots, m = |\Phi^+|$

$(C_m = M(w_0\cdot\lambda)$, $C_0 = M(\lambda))$.

Goals:
(1) BGG resolutions exist
(2) Uniqueness ?
(3) Applications (LA cohomology, Homology of $\mathcal{O}$).

Exercise: $(C_\bullet, \delta_\bullet) \to L(\lambda)$ BGG resol $\Rightarrow \delta|_{M(w\cdot\lambda)} \neq 0$

Sketch: $\lambda \in \Lambda^+ \Rightarrow w_0 \cdot \lambda = \mu$ is anti dom regular.

$M(w\cdot\lambda) = \langle v_+ \rangle$, $w \in W^{(k)}$

$$\begin{aligned}
\delta(v_+) = 0 \quad &\Rightarrow \exists\, v \in C_{k+1} \text{ s.t. } \delta(v) = v_+ \\
&\Rightarrow C_{k+1} \twoheadrightarrow M(w\cdot\lambda) \\
&\Rightarrow [M(u\cdot\lambda) : L(w\cdot\lambda)] \neq 0, \quad \exists\, u \in W^{(k+1)} \\
&\Rightarrow M(w\cdot\lambda) \hookrightarrow M(u\cdot\lambda) \\
&\Leftrightarrow w w_0 \cdot \mu \leq u w_0 \cdot \mu \\
&\Leftrightarrow w w_0 \leq u w_0 \\
&\Rightarrow k = \ell(w) \geq \ell(u) = k+1
\end{aligned}$$

$\square$

§ 6.2

Thm $\lambda \in \Lambda^+$. There is an exact seq. $(D_\bullet^\lambda, \partial_\bullet) \to L(\lambda)$ s.t. D_k^λ has std filt with $(D_k^\lambda : M(w \cdot \lambda)) = 1$, $\forall w \in W^{(k)}$.

Sketch: Assume $\lambda = 0$

$\mathfrak{b}$-cosets: $v_j = y_j + \mathfrak{b}$

(A) Let $\mathfrak{n}^- \cong V = \mathfrak{g}/\mathfrak{b} \supseteq \{v_1, \dots, v_m\}$ basis

Wts V : $-\alpha_i \leftrightarrow v_i$

Wts $\Lambda^k V$: $-\sum \alpha_{i_j} \leftrightarrow v_{i_1} \wedge \dots \wedge v_{i_k}$

(B) $\underline{D_k := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\Lambda^k V)}$, has std filt. (3.6)

$\Rightarrow D_0 = M(0) \qquad (\Lambda^0 V = \mathrm{triv})$

$D_m = M(w_0 \cdot 0) \qquad (h \cdot v_1 \wedge \dots \wedge v_m = -2\rho(h)\, v_1 \wedge \dots \wedge v_m)$

(C) Introduce $\partial_k : D_k \to D_{k-1}$, ε (general construction)

(D) $D_k^0 := D_k \cap \mathcal{O}^{\chi_0}$ (principal block)

(E) Apply T_0^λ to pass from $L(0) \to L(\lambda)$ $\square$

Details are in §6.3 – §6.5

Claim Each D_k has std filtration

Pf: $M = \Lambda^k V \supseteq \{z_1, \dots, z_N\}$

$\mathrm{Wts}(M) \supseteq \{\mu_1, \dots, \mu_N\}$

$\mu_i \leq \mu_j \Rightarrow i \leq j$

$$\rightsquigarrow \quad 0 \subseteq M_N \subseteq \cdots \subseteq M_2 \subseteq M_1 = D_k$$

$$M_j := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \langle z_j, \dots, z_N \rangle$$

$$\text{s.t.} \quad M_j / M_{j+1} \cong M(\mu_j)$$

$\square$

<u>Example</u> $\mathfrak{g} = \mathfrak{sl}(3)$, $\quad \Phi^+ = \{\alpha, \beta, \gamma = \alpha+\beta\}$

$$V \supseteq \{ v_\alpha, \quad v_\beta, \quad v_\gamma \}$$

wts: $\quad -\alpha \quad -\beta \quad -\alpha-\beta$

$$M := \Lambda^2 V \supseteq \{ v_\alpha \wedge v_\gamma, \; v_\beta \wedge v_\gamma, \; v_\alpha \wedge v_\beta \}$$

wts: $\quad -(2\alpha+\beta) \quad -(\alpha+2\beta) \xrightarrow{+\alpha} -(\alpha+\beta)$

Now: $\quad \mu_3 - \mu_1 = \alpha > 0$

$$x_\alpha (v_\alpha \wedge v_\gamma) = x_\alpha v_\alpha \wedge v_\gamma + v_\alpha \wedge x_\alpha v_\gamma \in \langle v_\alpha \wedge v_\beta \rangle$$

$$x_\beta (v_\alpha \wedge v_\gamma) = x_\beta v_\alpha \wedge v_\gamma + v_\alpha \wedge x_\beta v_\gamma = 0$$

$$x_\gamma (v_\alpha \wedge v_\gamma) = x_\gamma v_\alpha \wedge v_\gamma + v_\alpha \wedge x_\gamma v_\gamma = 0$$

$\therefore \quad v_\alpha \wedge v_\gamma$ is HWV modulo M_2.

Further $W^{(2)} \cdot 0 = \{ s_\alpha s_\beta \cdot 0 = -(2\alpha+\beta),$

$\qquad s_\beta s_\alpha \cdot 0 = -(\alpha+2\beta) \}$

$$\therefore \quad W^{(k)} \cdot 0 \neq \mathrm{Wts}(\Lambda^k V)$$

Recall $M = \bigoplus_{\chi} M^{\chi} \quad \forall M \in \mathcal{O}$

Claim: $M, N \in \mathcal{O}$, $\varphi \in \mathrm{Hom}_{\mathcal{O}}(M,N) \Rightarrow \varphi(M^{\chi}) \subseteq N^{\chi}$

Pf: $v \in M^{\chi}$, $z \in Z(\mathfrak{g}) \Rightarrow (z - \chi(z))^n v = 0, \ \exists n$

$$\Rightarrow 0 = (z-\chi(z))^n \varphi(v) \qquad \square$$

$\therefore$ $(D_\bullet, \partial_\bullet) \to L(\lambda)$ exact $\Rightarrow$ $(D_\bullet^{\chi_\lambda}, \partial_\bullet^{\chi_\lambda}) \longrightarrow L(\lambda)$ exact

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Def: $\Pi_w := \Phi^+ \cap w(\Phi^-)$

$\Gamma_w := \Phi^+ \cap w(\Phi^+)$

$$\begin{array}{ccc} D_k & \longrightarrow & D_{k-1} \\ \parallel & & \\ \bigoplus_{\chi} D_k^{\chi} & \longrightarrow & \bigoplus_{\chi} D_{k-1}^{\chi} \end{array}$$

Notation: $\Pi \subseteq \Phi^+ \Rightarrow \overline{\Pi} = \sum_{\alpha \in \Pi} \alpha \in \mathfrak{h}^*$.

Lemma: $\mu = w \cdot 0$ occurs in $\Lambda^{\ell(w)} V$

Pf: $\Phi^+ = \Phi^+ \cap (w\Phi^+ \cup w\Phi^-)$

$= \Gamma_w \cup \Pi_w$

$w\Phi^+ = w\Phi^{\overset{-}{+}} \cap (\Phi^+ \cup \Phi^{\overset{+}{-}})$

$= \Gamma_w \cup (-\Pi_w)$

$\Rightarrow w \cdot 0 = w\rho - \rho$

$= \frac{1}{2}(\overline{\Gamma_w} - \overline{\Pi_w}) - (\overline{\Gamma_w} + \overline{\Pi_w})$ $\square$

$= -\overline{\Pi_w}$

Lemma $\mu = w.0$ occurs only once in $\Lambda^\bullet V$

Pf: We show: $\Pi \subset \Phi^+$ s.t. $\overline{\Pi} = \overline{\Pi_w} \Rightarrow \Pi = \Pi_w$.

Clear $\ell(w) = 0$

Suppose $\ell(w) = k > 0$

$\Rightarrow \ell(s_\alpha w) = k-1$, $\exists \alpha \in \Delta$

(0.3) $\Rightarrow w^{-1}\alpha < 0$

$\Rightarrow \begin{cases} \alpha \in \Pi_w \\ (w')^{-1}\alpha > 0 \end{cases}$ $w' = s_\alpha w$

$\Rightarrow \underline{\alpha \notin \Pi_{w'}}$

Claim $\Pi_w = s_\alpha \Pi_{w'} \cup \{\alpha\}$ (*)

Pf: Have

$$s_\alpha \Pi_{w'} = s_\alpha(\Phi^+ \cap w'\Phi^-)$$

$$= (\Phi^+ \setminus \{\alpha\} \cup \{-\alpha\}) \cap \overset{s_\alpha w'}{w}\Phi^-$$

$(\alpha \in \Pi_w \subseteq w\Phi^-)$ $= (\Phi^+ \setminus \{\alpha\}) \cap w\Phi^-$

$$\Rightarrow \{\alpha\} \cup s_\alpha(\Pi_{w'}) = \Phi^+ \cap w\Phi^- = \Pi_w. \quad \square$$

Back to the Lemma:

$$\Pi \subseteq \Phi^+, \quad \overline{\Pi} = \overline{\Pi_w} = \rho - w\rho$$

$$\Rightarrow \underline{s_\alpha \overline{\Pi}} = (\rho - \alpha) - s_\alpha w\rho = (\rho - w'\rho) - \underline{\alpha}$$

$$= \underline{\overline{\Pi_{w'}} - \alpha}$$

$\alpha \notin \Pi \Rightarrow s_\alpha \Pi \subseteq \Phi^+$

$\Rightarrow s_\alpha \Pi \cup \{\alpha\} \subseteq \Phi^+$

$\Rightarrow \overline{s_\alpha \Pi \cup \{\alpha\}} = \overline{\Pi_{w'}}$

(IH) $\Rightarrow \Pi_{w'} = s_\alpha \Pi \cup \{\alpha\}$

$\Rightarrow \alpha \in \Pi_{w'}$ (contr.)

$\therefore \alpha \in \Pi$

Let $\Pi' = s_\alpha(\Pi \setminus \{\alpha\}) \subseteq \Phi^+$

$\Rightarrow \overline{\Pi}' = \overline{\Pi_{w'}}$ $\qquad \overline{\Pi} = \Pi_w$

(IH) $\Rightarrow \underline{\Pi'} = \underline{\Pi_{w'}}$

$\Rightarrow \Pi = s_\alpha(\Pi_{w'}) \cup \{\alpha\} = \Pi_w$ $\square$

So far we showed:

- $\mathrm{Wts}(D_k^0) = \{w \cdot 0, \; w \in W^{(k)}\}$
- D_k^0 has a std filtration.

Question: $\mathrm{Ext}_{\mathcal{O}}(M(w\cdot\lambda), M(w'\cdot\lambda)) = ?$

if $\lambda \in \Lambda^+$, $\ell(w) = \ell(w')$.

Thm (6.3) $\lambda \in \mathfrak{h}^*$

(a) $\mathrm{Ext}_{\mathcal{O}}(M(\mu), M(\lambda)) \neq 0 \Rightarrow \mu \uparrow \lambda, \ \mu \neq \lambda$

(b) $\lambda \in \Lambda^+$, $w, w' \in W$. Then

$$\mathrm{Ext}_{\mathcal{O}}(M(w\cdot\lambda), M(w'\cdot\lambda)) \neq 0 \Rightarrow w' < w$$
$$\Rightarrow \ell(w) < \ell(w')$$

Pf: (a) $(3.1\,a) \Rightarrow \mu \neq \lambda$. Given

$$\begin{tikzcd} & & & P(\mu) \arrow[dl, "\varphi"'] \arrow[d, "\pi"] & & \\ 0 \arrow[r] & M(\lambda) \arrow[r, "f"] & M \arrow[r, "g"] & M(\mu) \arrow[r] \arrow[l, bend left, dashed, "\sigma"] & 0 & (*) \end{tikzcd}$$

if $v \in M(\mu)$, $x, y \in P(\mu)$ s.t. $v = \pi x = \pi y$

$$0 = \pi(x-y) = g\varphi(x-y)$$
$$\Rightarrow \varphi(x-y) \in \mathrm{im}(f) \Leftrightarrow \varphi(x-y) \in M(\lambda)$$

<u>if:</u> $\varphi P(\mu) \cap \mathrm{im} f = 0 \Rightarrow \sigma(v) = \varphi(x), \ \exists x \in \pi^{-1} v$

$\Rightarrow (*)$ splits

(3.10) $\quad 0 \subseteq P_0 \subseteq P_1 \subset \cdots \subset P_n = P(\mu), \ P_i / P_{i-1} \cong M(\mu_i)$

(3.11) $\quad (P(\mu) : M(\mu_i)) = [M(\mu_i) : L(\mu)] > 0$

(5.1) $\quad \underline{\mu \uparrow \mu_i} \quad \forall i$

$f(M(\lambda)) \cap \varphi(P(\mu)) \neq 0 \Rightarrow \varphi P_i \cap f(M(\lambda))$, $\exists i$ min.

$\Rightarrow \varphi | P_i : P_i \to M(\lambda)$

$\quad \to P_i / P_{i-1} \cong M(\mu_i)$

$\Rightarrow [M(\lambda) : L(\mu_i)] \neq 0 \Rightarrow \underline{\exists \mu_i \uparrow \lambda}$. $\Rightarrow \mu \uparrow \lambda$

(b) $\lambda \in \Lambda^+ \Rightarrow \mu = w_0 \cdot \lambda$ is antidom regular

$\therefore \ \mathrm{Ext}_{\mathcal{O}}(M(w\cdot\lambda), M(w'\cdot\lambda)) \neq 0$

$(a) \implies ww_0 \cdot \mu \uparrow w'w_0 \cdot \mu$ & $w \neq w'$

$(5.2) \implies ww_0 < w'w_0$

$\implies w > w'$

$\square$