Lecture notes and Problem sheet 5, 12-05-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de

Extra material and preliminaries of representation theory of algebra

The extra material on the representation theory of finite-dimensional algebras was taken from the books [Mat99; CR66], and the extra material on cellularity comes from [Mat99].

This extra material will be found on the course notes (along with a small recap of what we did lecture by lecture).

Still, let me just precise one notion from lecture. The radical $\mathrm{Rad}(\mathcal{A})$ of a finite-dimensional $\mathbb{F}$-algebra $\mathcal{A}$ was defined as the sum of all nilpotent ideals (so ideal I for which there exists an $n \in \mathbb{N}$ such that $I^n = 0$). Then we define the radical of a submodule $M \subset \mathcal{A}$ as $\mathrm{Rad}(M) := \mathrm{Rad}(\mathcal{A}) \cap M$.

Problem sheet 5

0. (Drill)

1. Write back the Gram matrices of $\mathsf{TL}_4(\beta)$ and identify the values where they have a radical (we did it in class).

1.Temperley–Lieb non-semisimple In class, we studied the representation theory of $\mathsf{TL}_4(\beta)$ for special $\beta = 0, 1$. Give the decomposition matric $D = ([V^d : L^{d'}])_{d,d'=0,2,4}$ for the last case we did not work out: $\beta = \sqrt{2}$.

2. Oriented Temperley–Lieb algebras [Chris' 5.5] Read the definition of the oriented Temperley–Lieb algebra. Pay attention, q there is not an element of the field, it's an astract element, and the reason this algebra is infinite-dimensional.

q-numbers Read Chapter 7.1 of Chris' book, but skip the definition of quantum number. There is a small typo in the version (the n of the middle member should be $n - 1$):

$$[n]_q := q^{n-1} + q^{n-2} + \cdots + q + 1 = \frac{1 - q^n}{1 - q}$$

Often, especially in quantum group, it makes more sense to use a different notion of quantum number:

$$[\![n]\!]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$$

In particular, $[\![2]\!]_q = q + q^-1$ which is a convenient parametrisation of the parameter β in Temperley–Lieb algebras.

The exercise is then to give the relation between the two notions $[n]_q$ and $[\![n]\!]_q$. (Note that both of them define a generalisation of number and, indeed, retrieve n when $q \to 1$).

References

[CR66] C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Vol. 356. American Mathematical Soc., 1966.

[Mat99] A. Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group*. Vol. 15. American Mathematical Soc., 1999.