Research Statement
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Diagrammatics and representations of algebras related to Howe and Schur–Weyl dualities

General context

The concept of **Howe duality** originates from the influential work of Howe [How89]. In its most classical inception, it relates the representations of a dual pair G, G' that are mutually centralising subgroups of the double cover of a symplectic group. The methods of Howe have been applied over the last 30 years in a vast array of cases [CW12].

The original **Schur–Weyl duality** [Sch01] states that the actions of GL_m and the symmetric group S_n on the tensor product of a fundamental representation $(\mathbb{C}^m)^{\otimes n}$ are each other centraliser. So we can decompose $(\mathbb{C}^m)^{\otimes n}$ into a direct sum of simple S_n module tensor simple GL_n-modules. It also fits under the theme of Howe duality.

My doctoral research tackled an algebra related to the Howe dual pair $9(\mathrm{Pin}(d), \mathfrak{osp}(1|2))$ in a deformed version of Weyl–Clifford algebras, where deformations occurred by mean of a reflection group. Previous a class of diagrammatic algebras, most notably the Temperley–Lieb algebra [TL71], related to problems inspired by physics.

In fact, many algebras related to these type of dualities admit a type of graphical calculus. This enables the statement of complex conditions by simple topological rules. Figure 1 below presents a famed example of knot theory and recoupling theory in the Temperley–Lieb algebra [LS94].

$$\mathsf{P}_n = \mathsf{P}_n^2,$$

$$e_i \mathsf{P}_n = 0 = \mathsf{P}_n e_i,$$

$$\mathsf{P}_n = \mathsf{P}_{n-1} - \frac{[n-1]_q}{[n]_q} \mathsf{P}_{n-1} e_1 \mathsf{P}_{n-1},$$

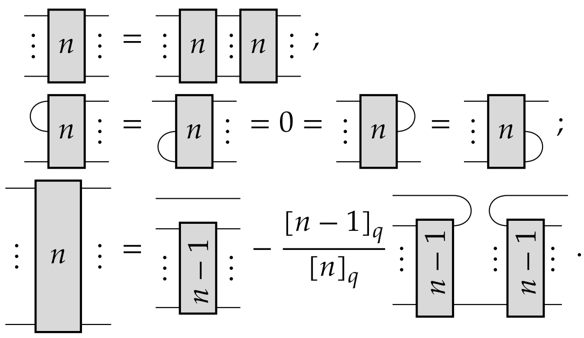


Figure 1: On the left, the algebraic properties of the elements P_n [Jon83; Wen87] in the Temperley–Lieb calculus, and their diagrammatic counterparts.

Statement of research problems

Main research aim

Construct representations and new diagrammatic calculus of algebras related to Howe and Schur–Weyl dualities.

Many algebras can be studied via these methods; I will focus first on three concrete problems to initiate the research. The overarching goal would be to then express links between the project, let it be by using similar methods, or directly by finding functorial relations between the families.

First concrete projects

1. Study the representation theory of the total angular momentum algebra for specific groups and define a diagrammatic calculus for general ones.

2. Study the untangled affine Temperley–Lieb algebras in characteristic 0 and p and define appropriate Jones–Wenzl elements.

3. Define an infinite symmetric webs calculus to study LKB representations.

Proposed research plan

Total angular momentum algebra The first problem is a continuation of my PhD thesis. The total angular momentum algebra studied here can be defined abstractly by generators and relations [DOV18a] or as the supercentraliser of an $\mathfrak{osp}(1|2)$ realisation inside the tensor product of a rational Cherednik algebra [EG02] and a Clifford algebra. This means that the algebra depends on a reflection group W and a weight function κ invariant on W-orbits. This instances is the algebra coming from the Howe dual pair $(\mathrm{Pin}(d), \mathfrak{osp}(1|2))$ [ØSS09] present in the product, but other Howe dualities have also been studied [CD20; Ciu+20]. Relatively little was known over the representation theory of this algebra, only the groups $W = \mathbb{Z}_2^N$ [DGV16] and $W = S_3$ [DOV18b] had been studied. In my doctoral thesis, I presented the representation theory of $W = D_{2m} \times \mathbb{Z}_2$ [De +22a] and $W = D_{2m} \times D_{2n}$, and I gave a realisation as polynomial solutions to the Dunkl–Dirac equation for any group W [De +22b].

At the moment, an ongoing collaboration with Marcelo De Martino and Roy Oste aims to extend the two first results to a stack of dihedral groups, and to consider the representation theory at "exotic" values of κ. Our preliminary computations hint that the general case will divide into 4-dimensional "slices" and, for odd dimension, with an extra 3-dimensional "slice"; the two cases we already studied, leaving only the question on how to coordinate the slices. Furthermore, in most previous works, we avoided values of κ that do not permit unitarity. In the low-dimensional cases, the values we avoided did not result in interesting behaviour, but we expect that having many values simultaneously conflicting could allow for remarkable types of representation, as is the case for representations of rational Cherednik algebras.

Once this first project is done, I propose to investigate the following directions, focussing on general W, and therefore much more difficult.

1. We know from [De +22b] that generalised symmetries can be used to create a basis for a realisation of an important representation: the polynomial null-solutions of a Dirac operators in which the derivatives are changed to Dunkl derivatives [Dun89]. This is only one representation, but we know the monogenic polynomials are one of, if note the, most important representation of the total angular momentum algebra, often encoding the behaviour of the representation outside exotic values of κ. An interesting avenue seems thus to study an extended algebra instead: a deformation of the conformal algebra defined in [CD15] and use a reduction to the total angular momentum algebra to obtain concrete information on the admissible representations.

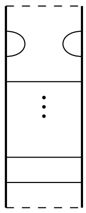
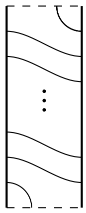
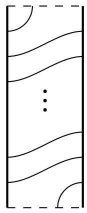
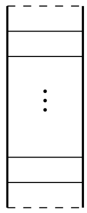
2. Create diagrammatics for this algebra by combining Webster's diagrammatics for rational Cherednik algebras [Web17] with a modification of Brundan's, Comes's and Kujawa's diagrammatics for Brauer–Clifford supercategory [BCK19]. A hint that these algebras encode interesting diagrammatics was already pointed out in [FH15] where crossing relations that could be represented via Temperley–Lieb algebras elements were found.

The untangled affine Temperley–Lieb algebras The affine Temperley–Lieb algebra is an algebra of very high relevance for physicists and algebraist. It is an infinite-dimensional algebra that appears in conformal field theory and is linked to Virasoro algebras. If the normal Temperley–Lieb algebra is naturally understood via Schur–Weyl duality, the duality breaks for the affine Temperley–Lieb algebra.

Since the influential work of Graham and Lehrer [GL98], its representation theory has been a central object of interest, mainly via the study of its monoidal category. It is presented via periodic planar diagrams, or diagrams on the cylinder. Another pair of generators is also added: the twist Ω and its inverse Ω^{-1}. Its presentation by generators and relations is given below, with $i, j \in \{0, \dots, n-1\}$ being periodic and id, the identity:

$$\begin{aligned} e_i^2 &= \beta e_i, & e_i e_{i\pm 1} e_i &= e_i, \\ e_i e_j &= e_j e_i, \quad |i-j| > 1, & \Omega e_j \Omega^{-1} &= e_{j-1}, \\ \Omega^2 e_1 &= e_{n-1} \dots e_2 e_1, & \Omega\Omega^{-1} &= \Omega^{-1}\Omega = \mathrm{id}. \end{aligned}$$

Diagrammatically, it is given by:

$e_1 =$  , $\Omega =$  , $\Omega^{-1} =$  and $\mathrm{id} =$  .

Recently, Martin and Spencer proved a modular version of the famed Jones–Wenzl projectors [MS22], extending on the work of Burrull, Libedinsky and Sentinelli [BLS19]. It enabled Spencer to generalise our work [LS20] and the work of Flores and Peltola [FP18] on the boundary seam algebra [MRR15] to the modular case [Spe21].

The goal of this project would be to approach the affine Temperley–Lieb algebra via the following quotient making it finite-dimensional:

$$\Omega^N = \gamma \mathrm{id}.$$

Diagrammatically, this amounts to unwinding full turns of the strands on the cylinder. It was motivated by a question of Tubbenhauer coming from their recent work with Khovanov and Sitaraman [KST22] where they used representation theory of specific algebras to make cryptographic protocols.

At the moment, we have defined the algebra, the untangled affine Temperley–Lieb algebra $uATL_n(\beta, \gamma)$, and proved it is sandwich cellular [TV22], a generalisation of cellularity [GL96] extending upon the notion of affine cellularity of König and Xi [KX12]. This gives ways to a study of its representation theory via its cell modules.

The algebra $uATL_n(\beta, \gamma)$ has n one-dimensional modules. In normal Temperley–Lieb algebras, the Jones–Wenzl projector is the idempotent linked to the only one-dimensional module. In the untangled affine version, we are able to compute the n Jones–Wenzl-like projectors $Q_{n,r}$ for characteristic 0 and we have linear recurrence formulas that uniquely determine their coefficients. The next step of the project, which should be completed before the start of the research stay, is to give closed forms for the coefficients and use the projector to study the representation theory à la [GL98] in characteristic 0 for roots of unity. This is part of an ongoing collaboration with Alexi Morin-Duchesne and Robert Spencer.

During my stay, I propose to investigate the following directions extending this project.

1. Define the projectors we find in the modular case, doing work similar to [MS22; BLS19]. In our case, the technical difficulties will be greater as the algebra has two parameters we need to tune. The tour de force of Martin and Spencer will need to be reproduced with care.

2. The quotient we use is, somehow, the simplest of a tower of algebras. We can define a family of imbricates untangled Temperley–Lieb algebra $uATL_n^k(\beta, \gamma)$ for which the quo-

tient is changed to $\Omega^{kn} = \gamma\mathrm{id}$. A comment of Théo Pinet suggests considering the limit of the process of successive quotients. Then the inductive limit is conjectured to be $aTL_n(\beta)$ and we could study the representation theory of $aTL_n(\beta)$ by lifting the projective modules in one untangled algebra.

3. Define its fusion rules and see how do they interact with the inductive limit. There are multiple fusion rules proposed for the affine Temperley–Lieb algebra, each with their own physical meanings, advantages and disadvantages. We would follow the recent definition of [IM22] adapted to the quotients.

4. Interestingly, similar work to define a Jones–Wenzl-type projector has been carried on another quotient of the affine Temperley–Lieb algebra by Queffelec and Wedrich [QW18] where they obtained a categorification of the skein algebra on the annulus. To define preceisely what our projector categorify would be of interest.

5. Lastly, I wish to investigate the physical meaning of the algebra $Q_{n,r}uTL_n(\beta,\gamma)Q_{n,r}$ and see if it gives rise to the same type of conformal theory with boundaries as what is explored by Flores and Peltola [FP18; FP20]. On the latter, we could also expect that an inductive process on the sequence of quotients could lead back to spin chains as module on the affine Temperley–Lieb [PS22], but we know the Schur–Weyl duality of Temperley–Lieb breaks down in the affine case, so how and when precisely it breaks in the limit is a key point to understand the affine Temperley–Lieb algebra.

LKB representations and infinite web calculus. In a recent preprint, Lacabanne, Tubbenhauer and Vaz gave a formulation of Verma Howe duality [LTV22] with the pair $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_n)$. As such, it gives a double centraliser formulation with the action of both quantum enveloping algebras on a tensor product of quantum Verma modules. In it, they found that it realises the Lawrence–Krammer–Bigelow (LKB) representations [JK11].

The last problem stems from a question of Tubbenhauer: *is it possible to find a diagrammatic calculus mimicking symmetric webs to replace $U_q(\mathfrak{sl}_2)$ in the duality?* A motivation to investigate lies in the fact that it is the case in the finite-dimensional case, in the quantum Howe duality outside Verma [RT16]. Symmetric webs offer a diagrammatic calculus for the category of the representations of $U_q(\mathfrak{sl}_n)$ and its presentation by generators and relation was proven in [CKM14]. Furthermore, it is of interest to note that this is somehow an extension of the Temperley–Lieb algebra calculus linked to $U_q(\mathfrak{sl}_2)$.

The goal of this project would be to extend this calculus outside finite-dimensional modules.

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