Diagrammatic algebra in representation theory

Selected topics in representation theory V5A6

Lecture notes
Universität Bonn, Summer Semester 2025

Alexis Langlois-Rémillard
Hausdorff Center for Mathematics, Bonn

Last update: July 22, 2025

Contents

Introduction

Warning

This is a work in progress. You can let me know of typos and mistake by mail at langlois@uni-bonn.de.
The last version will be available at `https://alexisl-r.github.io/teaching/S2025_Bonn_Diagrammatic/`.
This version: July 22, 2025.

These notes complement the course "Diagrammatic algebra in representation theory" given at the University of Bonn during the Summer Semester 2025.

Mostly, the course follows the book "Diagrammatic algebra" of Chris Bowman [Bow25].

Exceptionally, we followed

- D. Ridout and Y. Saint-Aubin. "Standard modules, induction and the structure of the Temperley-Lieb algebra". *Adv. Theor. Math. Phys.* 18 (2014), pp. 957–1041. arXiv: `1204.4505`

for dealing with Temperley–Lieb algebras and supplemented some general theory of finite-dimensional algebra from the references

- A. Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group*. Vol. 15. American Mathematical Soc., 1999
- C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Vol. 356. American Mathematical Soc., 1966
- I Assem, D Simson, and A Skowroński. *Elements of Representation Theory of Associative Algebras. Volume 1. Techniques of Representation Theory*. Vol. 65. London Mathematical Society Student Texts. New York: Cambridge University Press, 1997. DOI: `10.1017/CBO9780511614309`

When the lectures followed Chris' book, no notes are included, but the chapters are given. Problem sheets are also available on the webpage (and included at the end as extra).

Alexis Langlois-Rémillard,
July 22, 2025

Lecture 1

07/04/2025

`Introduction -- symmetric group -- presentation by generators and relations -- Coxeter groups -- other diagrammatic constructions`

This lecture followed roughly Chapters 1 and 2 (Sections 1.4, 2.1, 2.2, 2.4, 2.5, 2.6)

Lecture 2

14/04/2025

`Temperley--Lieb algebra -- dimension -- Proof of the equivalence between the diagrammatic and the generators and relations presentations.`

The lecture followed, up to a 90° shift for the diagrams, the reference was Ridout–Saint-Aubin (first sections).

D. Ridout and Y. Saint-Aubin. "Standard modules, induction and the structure of the Temperley-Lieb algebra". *Adv. Theor. Math. Phys.* 18 (2014), pp. 957–1041. arXiv: 1204.4505.

The relevant part of the book were Sections 5.1, 5.2, 6.1.

Lecture 3

28/04/2025

`End of the equivalence proof for TL -- Motivation from physics -- generalisations of TL`

This lecture gave some physical motivation to consider Temperley–Lieb algebras and other diagrammatics.

2025-05-17: My manuscript notes are online at the page course, they will be TEXed here soon.

Lecture 4

05/05/2025

`Refresher on finite-dimensional algebra representation theory -- Cellular algebra -- Cell modules -- Simple modules in cellular algebras`

This lecture presented cellular theory in a bit more details than the book and using a more standard definition of cellularity. I reserved the version of the book for weighted cellular algebra (Chapter 5). In particular, this allows us to treat Temperley–Lieb algebras $\mathsf{TL}_n(\beta)$ at $\beta = 0$, which I find is an interesting example.

The relevant parts of the book were Sections 6.1, 6.2, 6.3

Extra material and preliminaries of representation theory of algebra

This is a small compendium of results for finite-dimensional algebras. In general, we will not need that much so they will only be stated here without proof. References for this are the appendix of the book of Mathas [Mat99] and the classic of Curtis–Reiner [CR66].

Let A be a finite-dimensional algebra over a field $\mathbb{F}$. We will assume everything is over $\mathbb{C}$ to not trouble ourselves. Let M be a finite-dimensional A-module. We say M is a *simple* (or *irreducible*) if M is a proper and has no non-trivial proposer submodule. A *filtration* of M is a sequence of A-submodules of A

$$0 = M_0 \subset M_1 \subset M_1 \subset \cdots \subset M_k \subset M_{k+1} = M.$$

A *composition series* of M is a filtration of M where each *composition factor* M_i/M_{i-1} is a simple A-module.

Lemma 1. *Every A-module M has a composition series.*

Proof. Induction on the dimension of M. □

In particular, A viewed as an A-module on itself also has a composition series.

Lemma 2. *Suppose L is a simple A-module. Then $L \simeq A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$ and L is a composition factor of the A-module A.*

One of the key results in the representation theory of finite-dimensional algebras is that those composition series are well-defined and that, even though you will be able to filter a module in many ways, its composition factors are unique up to reordering.

Theorem 3 (Jordan–Hölder)**.** *Suppose that M is an A-module and that*

$$0 = M_1 \subset \cdots \subset M_k \subset M_{k+1} = M, \qquad 0 = N_1 \subset \cdots \subset N_l \subset N_{l+1} = M$$

are two composition series of M. Then $k = l$ and for each simple module A-module L

$$|\{i \mid L \simeq M_{i+1}/M_i\}| = |\{i \mid L \simeq N_{i+1}/N_i\}|.$$

Remark 4. *The Jordan–Hölder Theorem does not hold for $\mathbb{Z}$-modules.*

In particular, the Jordan–Hölder Theorem lets us define the *composition multiplicities* $[M : L]$ of L in M as the number of composition factors of M that are isomorphic to L.

Extra material on the course

Since I departed slightly from Chris's convention for cellularity (and also used left-module to keep in line with the general notion), I will provide some course notes. Feel free to follow Chris' convention instead of mine. They are slightly less general, but most cellular algebras of relevance fall in Chris' convention. I follow here mostly the original paper of Graham and Lehrer [GL96] with some proofs taken from the book of Mathas [Mat99]. Whatch out if you are to look into the book of Mathas: he considers right-modules and his order is the reverse of the other sources (and mine in particular).

We begin with the original definition of cellular algebras due to Graham–Lehrer. Here, we put R a commutative ring with unit and our algebra are associative and unital.

Definition 5 ([GL96, Definition 1.1])**.** *A* cellular algebra *is an R-algebra together with a* cell datum $(\Lambda, P, C, *)$ *where*

- *$\Lambda = (\Lambda, \leq)$ is a poset;*
- *for each $\lambda \in \Lambda$, $P(\lambda)$ is a finite set;*
- *$C : \bigsqcup_{\lambda\in\Lambda} P(\lambda) \times P(\lambda) \to A$ is an injective map whose image is an R-basis of A. We write $C(B, T) = C^\lambda_{BT}$ for $B, T \in P(\lambda)$. This basis is called the* cellular basis *of A;*
- *$* : A \to A$ is an anti-involution,*

and the following relations are respected

$$aC^\lambda_{BT} \equiv \sum_{S\in P(\lambda)} r^a_{SB} C^\lambda_{ST} \mod A^{<\lambda} \tag{4.1}$$

$$(C^\lambda_{BT})^* = C^\lambda_{TB}, \tag{4.2}$$

where $A^{<\lambda}$ is generated by $\{C^\mu_{B'T'} \mid \mu < \lambda, B', T' \in P(\mu)\}$.

It will be useful to combine (4.1) and (4.2):

$$C^\lambda_{BT} a \equiv \sum_{U\in P(\lambda)} r^a_{BU} C^\lambda_{BU} \mod A^{<\lambda} \tag{4.3}$$

The whole point of the definition is that this gives us, on the nose, a family of modules coming from the $P(\lambda)$. We fix for the next part, a cellular algebra over a field $R(= \mathbb{C})$ with cell datum $(\Lambda, P, C, *)$.

Definition 6. *We define the (left)* cell module V^λ *for* $\lambda \in \Lambda$ *to be the free vector space with basis* $\{v_B \mid B \in P(\lambda)\}$ *and action given by*

$$av_B = \sum_{S \in P(\lambda)} r^a_{BB'} v_{B'}$$

where $r^a_{BB'}$ *is determined by* aC^λ_{BB}.

More precisely, we define V^λ_T as the R-submodule of $A^{\leq\lambda}/A^{<\lambda}$ with basis $\{C^\lambda_{BT} + A^{<\lambda} \mid B \in P(\lambda)\}$. It is a left A-module by (4.1), and furthermore it is independent of T so we can identify it with V^λ.

Observe that we can use the anti-involution to define right A-modules.

We now want to define the bilinear form. We will use a technical lemma to make sure it is well-defined.

Lemma 7. *Suppose* $B, T \in P(\lambda)$. *Then there exists an elements* $r_{BT} \in R$ *such that, for any* $S, U \in P(\lambda)$

$$C^\lambda_{ST} C^\lambda_{BU} \equiv r_{BT} C^\lambda_{SU} \mod A^{<\lambda}.$$

Proof. We simply simplify the product in two ways, first with (4.1) and then with (4.3); what will remain is simply one coefficients r_{BT}. □

This allows us to give a bilinear form $\langle -, - \rangle_\lambda$.

Definition 8. *Define a bilinear form* $\langle -, - \rangle_\lambda : V^\lambda \times V^\lambda \to R$ *on the basis of* V^λ *by*

$$\langle v_B, v_T \rangle_\lambda = r_{BT} \mod A^{<\lambda},$$

and extending linearly.

The form is symmetric and associative.

Proposition 9. *Let* $\lambda \in \Lambda$, $B, T \in P(\lambda)$, $a \in A$ *and* $v, w \in V^\lambda$.

1. $\langle v, w \rangle_\lambda = \langle w, v \rangle_\lambda$.
2. $\langle av, w \rangle_\lambda = \langle v, a^* w \rangle_\lambda$.
3. $C^\lambda_{BT} v = \langle v, v_T \rangle_\lambda v_B$.

Proof. Since everything can be done on the basis $v = v_S, w = v_U$ and extended linearly, i) follows easily by application of $*$; ii) follows from the definition of the bilinear form by choosing the parentheses: $\langle av_S, v_U \rangle_\lambda C^\lambda_{UV} \equiv aC^\lambda_{SS} C^\lambda_{UU} \equiv (C^\lambda_{SS} a^*) C^\lambda_{UU} \equiv C^\lambda_{SS}(a^* C^\lambda_{UU}) \equiv \langle v_S, v_U \rangle_\lambda C^\lambda_{SU}$. Finally, the last is precisely the definition of the bilinear form. □

Lemma 10. *Suppose* $v \in V^\lambda$ *and* $a \in A^{\leq\mu}$. *Then* $\lambda \prec \mu \Rightarrow av = 0$.

Proof. Apply iii) of the previous proposition. □

Definition 11. *Let* $\mathrm{Rad}_\lambda = \{v \in V^\lambda \mid \langle v, w \rangle_\lambda = 0, \forall w \in V^\lambda\}$ *be the radical of the bilinear form. Denote also* $L^\lambda := V^\lambda / \mathrm{Rad}_\lambda$.

We denote $\Lambda^0 = \{\lambda \in \Lambda \mid \mathrm{Rad}_\lambda \neq V^\lambda\}$.

Proposition 12. *Let R be a field*[1]*; then Rad_λ is the unique maximal submodule of V^λ and L^λ is simple.*

Proof. In class. $\square$

Proposition 13. *Let R be a field and let $\lambda, \mu \in \Lambda^0$. Let M be a proposer submodule of V^μ and suppose that $\sigma : V^\lambda \to V^\mu/M$ is an A-modules morphism.*

1. *If $\sigma \neq 0$ then $\mu \leq \lambda$.*
2. *If $\mu = \lambda$ then $\sigma(v) = M + r_\sigma v$ for all $v \in V^\lambda$.*

Proof. Use the strategy of the proof of Proposition 12 to use that there exists elements $v, w \in V^\lambda$ such that $\langle v, w\rangle = 1$ and then use v to generate all $v_B \in V^\lambda$. So for v_B we get $\sigma(v_B) = \sigma(a_B v) = a_B\sigma(v) + M$ and then $\lambda \prec \mu$ implies $a_B\sigma(v) = 0$ by Lemma 10. If $\lambda = \mu$, then express $a_B = \sum_{U\in P(\lambda)} r_U C_{BT}$ and thus Proposition 12 iii) implies $\sigma(v_B) = \sigma(v)a_B + M = \sum_{U\in P(\lambda)} r_U \sigma(v) C_{BU}\sigma(v) = \langle\sigma(v), y\rangle v_B$ for $y = \sum_{U\in P(\lambda)} r_U v_U$, and $r_\sigma = \langle\sigma(v), y\rangle_\lambda$. $\square$

Corollary 14. *The L^λ are pairwise non-isomorphic.*

Proof. In class. $\square$

We now prove a technical lemma giving a filtration of the cellular algebra.

Lemma 15. *Suppose that Λ is finite with $\varnothing = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Lambda$ is a maximal chains of ideal of Λ. Then we can find a total ordering $\lambda_1, \dots, \lambda_k$ such that $\Gamma_i = \{\lambda_1, \dots, \lambda_i\}$. Then*

$$0 = A(\Gamma_0) \hookrightarrow A(\Gamma_1) \hookrightarrow \cdots \hookrightarrow A(\Gamma_k) = A$$

is a filtration of A with composition factors $A(\Gamma_i)/A(\Gamma_{i-1}) \simeq V^{\lambda_i *} \otimes_R V^{\lambda_i}$.

Proof. Since the chain of ideal is maximal, that means $\Gamma_i \setminus \Gamma_{i-1} = \{\mu\}$ for a certain μ. So we can find a total ordering. As a consequence, $A^{<\lambda_i} \subset A(\Gamma_{i-1})$ and the basis $\{C^{\lambda_i}_{BT} + A(\Gamma_{i+1}) \mid B, T \in P(\lambda_i)\}$ is a basis of the two-sided ideal $A(\Gamma_i)/A(\Gamma_{i-1})$. The isomorphisms of A-bimodule is then simple sending $C^{\lambda_i}_{BT} + A(\Gamma_{i-1}) \mapsto v_B \otimes v_T + A^{<\lambda_i}$ where $v_B \otimes v_T \simeq C^{\lambda_i}_{BT}$. $\square$

Then, the factors at level $\lambda$of the filtration are isomorphic to direct sum of $|P(\lambda)|$ left-module V^λ. This means that, when extended to composition series, the composition factors of A are composition factors of V^λ.

With this, we can already give one of the simple modules of A.

Lemma 16. *When λ is maximal, then $V^\lambda \simeq L^\lambda$.*

Proof. We need to prove that $\mathrm{Rad}_\lambda = 0$. Suppose $v \in \mathrm{Rad}_\lambda$. We write it $v = \sum_{B\in P(\lambda)} r_B v_B$. Fix $T \in P(\lambda)$ and write $a(vT) = \sum_{B\in P(\lambda)} r_B c^\lambda_{BT} \in A$. In particular, $a(vT) \in A^{\leq\Lambda}$, and it is in $A^{<\lambda}$ if and only if $v = 0$. Since v is in the radical, we have $\langle v, w\rangle_\lambda$ for all $w \in V^\lambda$. Therefore by the definition of the bilinear form we have, for $U, S \in P(\lambda)$

$$C^\lambda_{US} a(vT) = \sum_{B\in P(\lambda)} r_B C^\lambda_{US} C^\lambda_{BT} \equiv \sum_{B\in P(\lambda)} r_B \langle v_S, v_B\rangle_\lambda C_{UT} = \langle v_S, v\rangle_\lambda C^\lambda_{UT} \overset{v\in\mathrm{Rad}_\lambda}{=} 0 \mod A^{<\lambda}$$

Then $a(vT)a \in A^{<\lambda}$ for all $a \in A^{\leq\lambda}$ and since λ is maximal, that means $a(vT)\cdot 1 \in A^{<\lambda}$ so $x = 0$. $\square$

[1] Just a reminder.

Theorem 17 (Graham–Lehrer)**.** *Suppose that R is a field and that Λ is finite. Then $\{L^\lambda \mid \lambda \in \Lambda^0\}$ is a complete set of pairwise inequivalent simple modules.*

Proof. Done in class. See either [Mat99, Theorem 2.16] or [Bow25, Theorem 6.2.20] for proof. In Chris' proof, the chain of ideals comes from Lemma 15. □

This seems all dandy and fine, but sometimes it will be hard to find Λ^0. Still, we have reduced a difficult problem of abstract algebra into a much more manageable linear algebra problem.

Next lecture (13-05-2025) we will see that there is even more to those cellular algebras, and that they will also let us state meaningful results on the, much more elusive, *indecomposable modules*.

Lecture 5

12/05/2025

`Cellular algebra -- indecomposable modules -- composition series -- Jordan--Hölder Theorem -- decomposition number -- Krull--Schmidt Theorem -- idempotents -- radical`

Let $\mathcal{A}$ be a finite-dimensional algebra. When seen as a left-module on itself, it has a composition series, that is, a filtration where the factor are simple:

$$0 = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_k \subset \mathcal{A}_{k+1} = \mathcal{A}, \tag{5.1}$$

wehere $\mathcal{A}_i/\mathcal{A}_{i-1}$ is simple.

Lemma 18. *Any simple $\mathcal{A}$-module M appears a as a composition factor of $\mathcal{A}$.*

By the Jordan–Hölder Theorem 3, we can speak of the Jordan–Hölder composition series of a module.

Definition 19. *The decomposition number of a simple module L inside a module M is the number of time L appears as a composition factor inside the composition series of M. We denote it $[M : L]$.*

In cellular algebra, we define the decomposition matrix as

$$\mathbf{D} = ([V^\lambda : L^\mu])_{\lambda\in\Lambda,\mu\in\Lambda_0}. \tag{5.2}$$

As a consequence of Graham Lehrer Theorem 17, we know how $\mathbf{D}$ looks like.

Corollary 20. *The matrix $\mathbf{D}$ is unitriangular.*

In fact, we have even more, as cellular algebra also gives us information on the more elusive indecomposable modules.

Definition 21. *A module M is indecomposable if it does not decompose into the direct sum of two non-trivial submodule.*

In particular, simple modules are indecomposable, but the converse is not always true.

From a course of representation theory of finite groups over $\mathbb{C}$, one might get this impression, but on algebra, it is easy to find example where it does not work.

We will be interested in a class of special indecomposable modules.

Theorem 22 (Krull–Schmidt)**.** *Suppose M is an A-module and that*

$$M_1 \oplus \cdots \oplus M_k = M = N_1 \oplus \cdots \oplus N_l$$

are two decomposition of M into a direct sum of indecomposable modules. Then $k = l$ and we can rearrange the N_i via a permutation σ such that $N_{\sigma i} \simeq M_i$.

We will call the indecomposable module P_i appearing in the Krull–Schmidt decomposition of $\mathcal{A}$ viewed as a module on itself, the principal indecomposable.

The radical $\mathrm{Rad}(\mathcal{A})$ of a finite-dimensional $\mathbb{F}$-algebra $\mathcal{A}$ was defined as the sum of all nilpotent ideals (so ideal I for which there exists an $n \in \mathbb{N}$ such that $I^n = 0$). Then we define the radical of a submodule $M \subset \mathcal{A}$ as $\mathrm{Rad}(M) := \mathrm{Rad}(\mathcal{A}) \cap M$.

There is a one-to-one correspondence between principal indecomposable module and simple module given by sending $P \mapsto P/\operatorname{Rad} P$.

In cellular algebra, we can then speak of the matrix given by the decomposition number $[P^\lambda : L^\mu]$ for $\lambda, \mu \in \Lambda_0$. By denoting P^λ the principal indecomposable whose head is isomorphic to L^λ ($P^\lambda/\operatorname{Rad} P^\lambda \simeq L^\lambda$). We denote $\mathbf{C} = ([P^\lambda : L^\mu])_{\lambda,\mu\in\Lambda_0}$

A wonderful result of cellular theory is that we can access these decomposition number via the decomposition matrix.

Theorem 23 (Graham–Lehrer)**.** *Let A be a cellular algebra over a field with Λ finite. Then*

$$\mathbf{C} = \mathbf{D}^t\mathbf{D}.$$

Lecture 6

19/05/2025

$\mathbb{Z}$-gradings -- weighted cellular algebras -- $\mathbb{Z}$--graded (weighted) cellular algebras -- The Idempotent Trick -- The Grading Trick -- binary Schur algebra

This lecture covered material from the books. Relevant sections are Sections 5.3, 5.6, 5.7, 6.3 (gradings and binary Schur algebras); Sections 6.6, 6.7, 6.9 (weighted cellular algebras and $\mathbb{Z}$–graded cellular algebra)

Lecture 7

26/05/2025

Open class for problem sheets.

Lecture 8

02/06/2025

`Parabolic Coxeter systems -- Kazhdan--Lusztig polynomials --` q`-combinatorics -- poset particition`

This lecture followed Chapter 7, Sections 7.1, 7.2, 7.3 and 7.4.

Lecture 9

23/06/2025

`Oriented Temperley--Lieb algebra -- Correspondence with other presentation`

This lecture followed Chapter 7, Section 7.5.

Lecture 10

30/06/2025 (Recording)
`General Kazhdan--Lusztig theory -- Independence on the reduced path -- Kazhdan--Lusztig's positivity Conjecture`

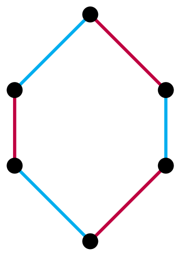
This lecture did a quick survey of Chapter 8, extracting bits and pieces from Sections 8.1 and 8.2

Goal of the lecture

Up to this lecture, we defined Kazhdan–Lusztig polynomials only on the symmetric group $\mathfrak{S}_{m+n}$ with maximal parabolic $\mathfrak{S}_m \times \mathfrak{S}_n$. This is a very particular case. We can, however, define Kazhdan–Lusztig theory for any (parabolic) Coxeter system. There will be one subtlety that we will explore on one example.

We construct its Bruhat graph in a similar fashion. We first put $\varnothing$ at the bottom and we construct recursively the graph by adding an edge coloured by the Coxeter generator σ if applying σ give a new element not previously in the graph.

Let's construct the Bruhat graph for the case $\mathfrak{S}_3$ with no parabolic. It has two Coxeter generators s_1, s_2 respecting Coxeter relation ${s_1}^2 = {s_2}^2 = (s_1 s_2)^3 = 1$.



As in the maximal parabolic case, we define certain quantities using paths on the graph following certain colours schemes. Let us fix, for μ, a reduced path T_μ and λ be a sequence $\varnothing \to \lambda^{(1)} \to \dots \to \lambda^{(\ell)} = \lambda$ following the colours (that is, the sequence of generators of) T_μ composed of the four following potential paths

(U^1_σ) The *up* move $\nu \xrightarrow{\sigma} \eta$ of degree 0.

(U^0_σ) The *failed up* move $\nu \xrightarrow{\sigma} \nu$ of degree 1.

(D^1_σ) The *down* move $\eta \xrightarrow{\sigma} \nu$ of degree 0.

(D^0_σ) The *failed down* move $\eta \xrightarrow{\sigma} \eta$ of degree -1.

When we look at the paths on this graph, we can see that we get a problem if we try to replicate the procedure we used in the maximal parabolic case: namely we will not have a procedure that is independent of the choice of reduced path!

That is because the true definition of Kazhdan–Lusztig polynomials make intervene a decomposition into a product of two uni-triangular matrix N, P of the matrix Δ constructed by the paths on the Bruhat graph: see Definition 8.2.8 of the book.

Well, not that the previous definition was not true, it's just a specific case

More precisely, we have $\Delta^{(W,P)} = N^{(W,P)}B^{(W,P)}$ for $N^{(W,P)} = (n_{\lambda,\nu}(q))_{\lambda,\nu\in {}^PW}$ and $B^{(W,P)} = (b_{\nu,\mu}(q))_{\nu,\mu\in {}^PW}$ where $n_{\lambda,\nu}(q) \in q\mathbb{Z}[q]$ for $\lambda \neq \nu$ and $b_{\nu,\mu}(q) \in \mathbb{Z}[q+q^{-1}]$. It is the element $n_{\lambda,\nu}(q)$ that are the Kazhdan–Lusztig polynomials.

In the maximal case: $B = I$.

Lecture 11

07/07/2025 (Recording)
`Diagrammatic Hecke algebra -- one-colour rules -- idea of the categorification`

This lecture introduced the ideas of Chapter 9 in Sections 9.1 and 9.2.

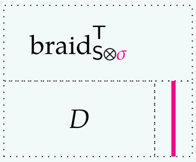
There was an infortunate mishap in the version of the book I shared: the diagrams for D^1_σ and D^0_σ are inverted in Definition 9.1.4. Here is a corrected version

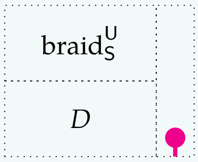
Alas my drawing do not pay justice to Chris' masterpieces.

Definition 9.1.4—Corrected

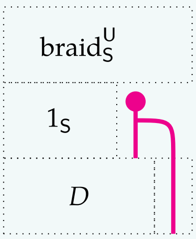
We define up and down operators on diagrams as follows:

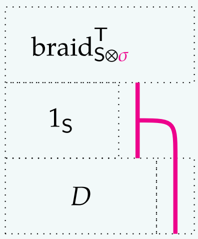
- Suppose that D has northern colour sequence $\mathsf{S} \in \mathrm{Path}(\lambda)$ and that $\sigma \in \mathrm{Add}(\lambda)$. Fix a preferred choice of $\mathsf{T} \in \mathrm{Path}(\lambda+\sigma)$ and $\mathsf{U} \in \mathrm{Path}(\lambda)$. We define

$\mathsf{U}^1_\sigma(D) =$ 

$\mathsf{U}^0_\sigma(D) =$ 

- Suppose that $\sigma \in \mathrm{Rem}(\lambda)$ and that D has northern colour sequence $\mathsf{S}\otimes\sigma \in \mathrm{Path}(\lambda)$. Fix a preferred choice of $\mathsf{T} \in \mathrm{Path}(\lambda)$ and $\mathsf{U} \in \mathrm{Path}(\lambda-\sigma)$. We define

$\mathsf{D}^1_\sigma(D) =$ 

$\mathsf{D}^0_\sigma(D) =$ 

The degree of the diagrammatic elements is computed by assigning a +1 for each spot  and a −1 to each fork . This way we can indeed verify that U^1_σ has degree 0, U^0_σ has degree 1, D^1_σ has degree $1-1=0$ and D^0_σ has degree -1.

Lecture 12

14/07/2025

```
Diagrammatic Hecke algebra -- multi-colour rules -- weighted graded cellular basis
```

We covered Sections 9.3, 9.4, 9.6 and 9.7. We had to skip the proof of Theorem 9.6.1, but I encourage you to have a look at it.

For each $\mu \in \mathcal{P}_{m,n}$, we fix a choice of reduced path T_μ.

Theorem 24. *Let $(W, P) = (\mathfrak{S}_{m+n}, \mathfrak{S}_m \times \mathfrak{S}_n)$. The algebra $\mathcal{H}_{(W,P)}$ is a graded weighted cellular algebra with idempotent $1_\mu := 1_{T_\mu}$, the anti-involution given by flipping through the horizontal axis and the cellular basis*

$$\{d_S 1_\lambda d_T^* \mid S \in \mathrm{Path}(\lambda, T_\mu), T \in \mathrm{Path}(\lambda, T_\nu), \lambda, \mu, \nu \in \mathcal{P}_{m,n}\}. \tag{12.1}$$

Lecture 13

21/07/2025

`General Coxeter system -- diagrammatic Hecke algebra -- p-Kazhdan--Lusztig polynomial -- graded weighted cellular algebra basis`

In this lecture, we went over Sections 10.1, 10.2, 10.3, 10.4 and 10.5. We could not do the example of Sections 10.5, but I encourage you to try them.

Extra lecture: Chris Bowman

22/07/2025

```
Counterexample of the Lusztig's Conjecture
```

Problem sheets

`Problem sheets for the course`

The next pages regroup the problems sheets given through the lecture.

1. 07-04-2025 Problem sheet 1, about Coxeter groups, the 15-puzzle and the first diagrammatical rules.
2. 14-04-2025 Problem sheet 2, about Temperley–Lieb algebras
3. 28-04-2025 Problem sheet 3, about tensor product and some Temperley–Lieb algebras.
4. 05-05-2025 Problem sheet 4, about cellularity (with material)
5. 12-05-2025 Problem sheet 5, about q-numbers and some (oriented) Temperley–Lieb.
6. 19-05-2025 Problem sheet 6, about idempotent trick and zig-zag algebra.
7. 02-06-2025 Problem sheet 7, about some Bruhat graph paths and KL polynomials
8. 23-06-2025 Problem sheet 8, bout some Bruhat graph paths and KL polynomials bis
9. 21-07-2025 Preparation for the oral examination: five problems similar in nature to what is asked at the oral examination.

Problem sheet 1, 07-04-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de

0. (Drill) Write the (24) elements of $\mathfrak{S}_4$ in diagram form and compute a few example.

1. Symmetric group Write a proof that the diagrammatic presentation $\mathfrak{S}_n$ shown in class is isomorphic to the symmetric group $S_n = \langle s_1, \ldots, s_{n-1} \mid s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = 1, |i-j| > 1 \rangle$ for all n.

2. 14-15 puzzle The *15-puzzle* is a sliding puzzle made out of 15 numbered blocks in a 4×4 grid with one empty cell. The 14-15 problem asks to reach Configuration B from Configuration A (see Figure 1). Prove that it is impossible, and that furthermore exactly half of the total number of configurations can be attained from Configuration A, and the other half, from Configuration B. [Bow25, Thm 2.3.1].

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Configuration A

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

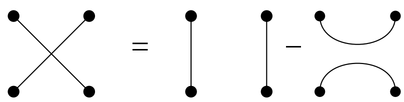
Configuration B

Figure 1: Two configurations of the 15-puzzle

3. [Bow25, Exercise 2.4.4, 2.4.5] Construct an isomorphisms $\phi : D_8 \to H_2$ for H_2 the hyper-octahedral group. Compute the order of H_n.

4. Coxeter graphs We can encode the Coxeter presentation in a graph; see [Bow25, Sect 2.6]. Give the Coxeter graph of the hyper-octahedral group H_n (So prove [Bow25, Proposition 2.6.3]).

5. Preparing the Temperley–Lieb algebra To prepare for the 14·04·2025 lecture, consider the following additional diagrammatic rules for $\mathfrak{S}_n^*$:



(1)

Compute the rule on all the 6 elements of $\mathfrak{S}_3$. What are your conclusions?

References

[Bow25] C. Bowman. *Diagrammatic algebra*. In press. Springer, 2025.

Problem sheet 2, 14-04-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de

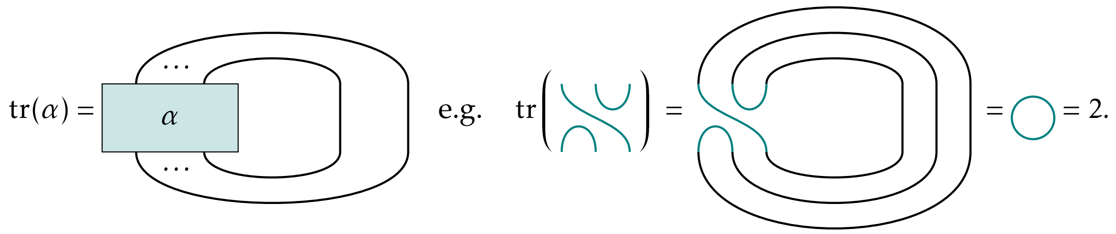
0. (Drill)

1. Write the (14) elements of $\mathsf{TL}_4(2)$ in diagram form and compute a few examples of multiplication.
2. Write the (14) walks on $\mathbb{Z}_2$ from $(0,0)$ to $(4,4)$ that do not cross the diagonal and relate them to the Temperley–Lieb diagram.
3. Choose a (big, say at least $n \geq 7$) Temperley–Lieb diagram and express it via a product of simple arcs (as we did in the proof that Ψ was surjective).

1. Catalan combinatorics Give a proof that the number of walks on $\mathbb{Z}^2$ from $(0,0)$ to (n,n) that does not cross the diagonal is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. (Hint: you might find it easier to count walks from $(0,0)$ to (n,p) that do not cross the diagonal and then specialise.)

2. Complete the proofs In the lecture, we did some of the proofs only for examples. Go back to your notes and add the necessary "$\cdots$" to make them work for all n.

3. A trace on the algebra We define a "trace" on the Temperley–Lieb algebra $\mathrm{tr} : \mathsf{TL}_n(2) \to \mathbb{C}$ by doing the following diagrammatic construction: given a Temperley–Lieb diagram α, embed it into a bigger space and connect the top and bottom strands with loops and compute the trace via the diagrammatic rule:



What is the trace of the identity? Relate this to the comment about Schur–Weyl duality in the course that stated that the Temperley–Lieb algebra $\mathsf{TL}_n(2)$ was the endomorphism algebra $\mathrm{End}_{U(\mathfrak{sl}_2)}((\mathbb{C}^2)^{\otimes n})$. Does this make sense to you?

4. Maps on the Temperley–Lieb algebras Flipping the diagram with respect to the horizontal axis gives an anti-involution (that is, $\iota^2 = \mathrm{id}$ and $\iota(ab) = \iota(b)\iota(a)$) on the (diagrammatic) Temperley–Lieb algebra. Define this anti-involution by its action on the generators.

5. Preparing the next course The diagrammatic rules we have is

$$\bigcirc = 2$$

Can we change it by

$$\bigcirc = \beta,$$

for a $\beta \in \mathbb{C}$?

We saw that we needed to have contractible loops close to 2 to be coherent with the symmetric group diagrammatic presentation. What this question asks is: “does this make sense diagrammatically” and then it begs “is there a dual structure for this new diagrammatic algebra where we have a similar Schur–Weyl duality?” [Those questions go beyond the scope of the course, hence why they get hidden here on the second page.]

Problem sheet 3, 28-04-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de

0. (Drill)

1. Compute the tensor product of the matrix $\sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with itself.

2. Do the computation of the Hamiltonian H_{XXZ} we saw in class to get expression for the summands E_i

$$H_{XXZ} = \frac{-1}{2} \sum_{i=1}^{n-1} \sigma_i^X \sigma_{i+1}^X + \sigma_i^Y \sigma_{i+1}^Y + \Delta \sigma_i^Z \sigma_{i+1}^Z + \delta(\sigma_i^Z - \sigma_{i+1}^Z) - \Delta(\mathrm{id}_i \otimes \mathrm{id}_{i+1})$$

   where

$$A_i = \mathrm{id}_2 \otimes \mathrm{id}_2 \otimes \cdots \otimes \underset{i}{a} \otimes \mathrm{id}_2 \otimes \cdots \otimes \mathrm{id}_2,$$

   and

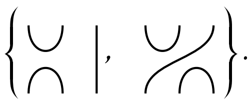
$$\sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3. Verify the relations the E_i you found respect.

1. Chapter 6.1 Read Chapter 6.1 of Chris' book.

2. Modules of Temperley–Lieb Consider the Temperley–Lieb algebra $\mathsf{TL}_3(\beta)$ as a left-module on itself. Can you decompose it into irreducible components? (Block-diagonalise simultaneously the matrices of the identity, s_1 and s_2.)

3. More on one Temperley–Lieb module Consider the left $\mathsf{TL}_3(\beta)$-module $\mathsf{TL}_3(\beta)s_1$ with basis [verify]

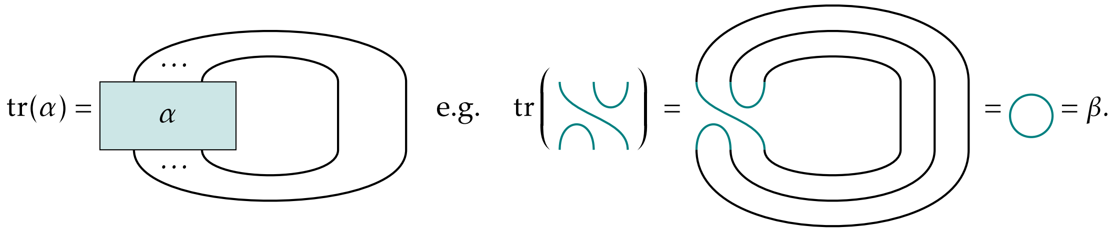


Compute the action of $\mathsf{TL}_3(\beta)$ on these module with (left) action given by (under) concatenation and evaluation. Are there value of β for which the structure of this representation changes?

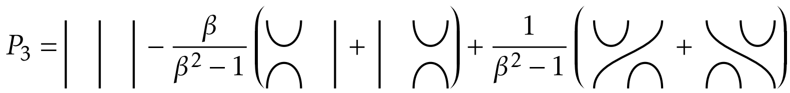
4. Preparing cellularity Write down the basis of TL_5. Give a statement on the number of through-strands acts with respect to the multiplication. Then do some numerology with respect to the number of diagrams with a given number of through strands. Compare this with the dimension of the Temperley–Lieb algebra[1].

[1]This should really tickle your (semisimple algebraically closed characteristic 0 field) representation theoretists sense. If it does not, please go contemplate Maschke's Theorem or some character—if it still is not clear, maybe check if you missed some diagrams!

5. A trace on the algebra We define a "trace" on the Temperley–Lieb algebra $\mathrm{tr} : \mathsf{TL}_n(\beta) \to \mathbb{C}$ by doing the following diagrammatic construction: given a Temperley–Lieb diagram α, embed it into a bigger space and connect the top and bottom strands with loops and compute the trace via the diagrammatic rule:



Compute the trace of the following element of $\mathsf{TL}_3(\beta)$



6. More on the pesky element Compute the actions of the two generators of $\mathsf{Tl}_3(\beta)$ on P_3.

Lecture notes and Problem sheet 4, 05-05-2025

Extra material and preliminaries of representation theory of algebra

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de

Some refresher of finite-dimensional algebras over fields

This is a small compendium of results for finite-dimensional algebras. In general, we will not need that much so they will only be stated here without proof. References for this are the appendix of the book of Mathas [Mat99] and the classic of Curtis–Reiner [CR66].

Let A be a finite-dimensional algebra over a field $\mathbb{F}$. We will assume everything is over $\mathbb{C}$ to not trouble ourselves. Let M be a finite-dimensional A-module. We say M is a *simple* (or *irreducible*) if M is a proper and has no non-trivial proposer submodule. A *filtration* of M is a sequence of A-submodules of A

$$0 = M_0 \subset M_1 \subset M_1 \subset \cdots \subset M_k \subset M_{k+1} = M.$$

A *composition series* of M is a filtration of M where each *composition factor* M_i/M_{i-1} is a simple A-module.

Lemma 1. *Every A-module M has a composition series.*

Proof. Induction on the dimension of M. □

In particular, A viewed as an A-module on itself also has a composition series.

Lemma 2. *Suppose L is a simple A-module. Then $L \simeq A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$ and L is a composition factor of the A-module A.*

One of the key results in the representation theory of finite-dimensional algebras is that those composition series are well-defined and that, even though you will be able to filter a module in many ways, its composition factors are unique up to reordering.

Theorem 3 (Jordan–Hölder)**.** *Suppose that M is an A-module and that*

$$0 = M_1 \subset \cdots \subset M_k \subset M_{k+1} = M, \qquad\qquad 0 = N_1 \subset \cdots \subset N_l \subset N_{l+1} = M$$

are two composition series of M. Then $k = l$ and for each simple module A-module L

$$|\{i \mid L \simeq M_{i+1}/M_i\}| = |\{i \mid L \simeq N_{i+1}/N_i\}|.$$

Remark 4. *The Jordan–Hölder Theorem does not hold for $\mathbb{Z}$-modules.*

In particular, the Jordan–Hölder Theorem lets us define the *composition multiplicities* $[M : L]$ of L in M as the number of composition factors of M that are isomorphic to L.

Extra material on the course

Since I departed slightly from Chris's convention for cellularity (and also used left-module to keep in line with the general notion), I will provide some course notes. Feel free to follow Chris' convention instead of mine. They are slightly less general, but most cellular algebras of relevance fall in Chris' convention. I follow here mostly the original paper of Graham and Lehrer [GL96] with some proofs taken from the book of Mathas [Mat99]. Whatchout if you are to look into the book of Mathas: he considers right-modules and his order is the reverse of the other sources (and mine in particular).

We begin with the original definition of cellular algebras due to Graham–Lehrer. Here, we put R a commutative ring with unit and our algebra are associative and unital.

Definition 5 ([GL96, Definition 1.1]). *A* cellular algebra *is an R-algebra together with a* cell datum $(\Lambda, P, C, *)$ *where*

- $\Lambda = (\Lambda, \leq)$ *is a poset;*
- *for each $\lambda \in \Lambda$, $P(\lambda)$ is a finite set;*
- $C : \bigsqcup_{\lambda\in\Lambda} P(\lambda)\times P(\lambda) \longrightarrow A$ *is an injective map whose image is an R-basis of A. We write $C(B,T) = C^{\lambda}_{BT}$ for $B,T \in P(\lambda)$. This basis is called the* cellular basis *of A;*
- $* : A \longrightarrow A$ *is an anti-involution,*

and the following relations are respected

$$aC^{\lambda}_{BT} \equiv \sum_{S\in P(\lambda)} r^{a}_{SB} C^{\lambda}_{ST} \mod A^{<\lambda} \tag{1}$$

$$(C^{\lambda}_{BT})^{*} = C^{\lambda}_{TB}, \tag{2}$$

where $A^{<\lambda}$ is generated by $\{C^{\mu}_{B'T'} \mid \mu < \lambda, B', T' \in P(\mu)\}$.

It will be useful to combine (1) and (2):

$$C^{\lambda}_{BT} a \equiv \sum_{U\in P(\lambda)} r^{a}_{BU} C^{\lambda}_{BU} \mod A^{<\lambda} \tag{3}$$

The whole point of the definition is that this gives us, on the nose, a family of modules coming from the $P(\lambda)$. We fix for the next part, a cellular algebra over a field $R(=\mathbb{C})$ with cell datum $(\Lambda, P, C, *)$.

Definition 6. *We define the (left)* cell module V^{λ} *for $\lambda \in \Lambda$ to be the free vector space with basis $\{v_B \mid B \in P(\lambda)\}$ and action given by*

$$av_B = \sum_{S\in P(\lambda)} r^{a}_{BB'} v_{B'}$$

where $r^{a}_{BB'}$ is determined by aC^{λ}_{BB}.

More precisely, we define V^{λ}_{T} as the R-submodule of $A^{\leq\lambda}/A^{<\lambda}$ with basis $\{C^{\lambda}_{BT} + A^{<\lambda} \mid B \in P(\lambda)\}$. It is a left A-module by (1), and furthermore it is independent of T so we can identify it with V^{λ}.

Observe that we can use the anti-involution to define right A-modules.

We now want to define the bilinear form. We will use a technical lemma to make sure it is well-defined.

Lemma 7. *Suppose $B, T \in P(\lambda)$. Then there exists an elements $r_{BT} \in R$ such that, for any $S, U \in P(\lambda)$*

$$C^{\lambda}_{ST} C^{\lambda}_{BU} \equiv r_{BT} C^{\lambda}_{SU} \mod A^{<\lambda}.$$

Proof. We simply simplify the product in two ways, first with (1) and then with (3); what will remain is simply one coefficients r_{BT}. □

This allows us to give a bilinear form $\langle -,- \rangle_{\lambda}$.

Definition 8. *Define a bilinear form $\langle -,- \rangle_{\lambda} : V^{\lambda} \times V^{\lambda} \longrightarrow R$ on the basis of V^{λ} by*

$$\langle v_B, v_T \rangle_{\lambda} = r_{BT} \mod A^{<\lambda},$$

and extending linearly.

The form is symmetric and associative.

Proposition 9. *Let $\lambda \in \Lambda$, $B, T \in P(\lambda)$, $a \in A$ and $v, w \in V^\lambda$.*

1. $\langle v, w\rangle_\lambda = \langle w, v\rangle_\lambda$.
2. $\langle av, w\rangle_\lambda = \langle v, a^* w\rangle_\lambda$.
3. $C^\lambda_{BT} v = \langle v, v_T\rangle_\lambda v_B$.

Proof. Since everything can be done on the basis $v = v_S, w = v_U$ and extended linearly, i) follows easily by application of $*$; ii) follows from the definition of the bilinear form by choosing the parentheses: $\langle av_S, v_U\rangle_\lambda C^\lambda_{UV} \equiv aC^\lambda_{SS}C^\lambda_{UU} \equiv (C^\lambda_{SS}a^*)C^\lambda_{UU} \equiv C^\lambda_{SS}(a^*C^\lambda_{UU}) \equiv \langle v_S, v_U\rangle_\lambda C^\lambda_{SU}$. Finally, the last is precisely the definition of the bilinear form. □

Lemma 10. *Suppose $v \in V^\lambda$ and $a \in A^{\preceq\mu}$. Then $\lambda \prec \mu \Rightarrow av = 0$.*

Proof. Apply iii) of the previous proposition. □

Definition 11. *Let $\mathrm{Rad}_\lambda = \{v \in V^\lambda \mid \langle v, w\rangle_\lambda = 0, \forall w \in V^\lambda\}$ be the radical of the bilinear form. Denote also $L^\lambda := V^\lambda/\mathrm{Rad}_\lambda$.*

We denote $\Lambda^0 = \{\lambda \in \Lambda \mid \mathrm{Rad}_\lambda \neq V^\lambda\}$.

Proposition 12. *Let R be a field*[1]*; then Rad_λ is the unique maximal submodule of V^λ and L^λ is simple.*

Proof. In class. □

Proposition 13. *Let R be a field and let $\lambda, \mu \in \Lambda^0$. Let M be a proposer submodule of V^μ and suppose that $\sigma : V^\lambda \to V^\mu/M$ is an A-modules morphism.*

1. *If $\sigma \neq 0$ then $\mu \preceq \lambda$.*
2. *If $\mu = \lambda$ then $\sigma(v) = M + r_\sigma v$ for all $v \in V^\lambda$.*

Proof. Use the strategy of the proof of Proposition 12 to use that there exists elements $v, w \in V^\lambda$ such that $\langle v, w\rangle = 1$ and then use v to generate all $v_B \in V^\lambda$. So for v_B we get $\sigma(v_B) = \sigma(a_B v) = a_B\sigma(v) + M$ and then $\lambda \prec \mu$ implies $a_B\sigma(v) = 0$ by Lemma 10. If $\lambda = \mu$, then express $a_B = \sum_{U\in P(\lambda)} r_U C_{BT}$ and thus Proposition 12 iii) implies $\sigma(v_B) = \sigma(v)a_B + M = \sum_{U\in P(\lambda)} r_U \sigma(v) C_{BU}\sigma(v) = \langle\sigma(v), y\rangle v_B$ for $y = \sum_{U\in P(\lambda)} r_U v_U$, and $r_\sigma = \langle \sigma(v), y\rangle_\lambda$. □

Corollary 14. *The L^λ are pairwise non-isomorphic.*

Proof. In class. □

We now prove a technical lemma giving a filtration of the cellular algebra.

Lemma 15. *Suppose that Λ is finite with $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Lambda$ is a maximal chains of ideal of Λ. Then we can find a total ordering $\lambda_1, \ldots, \lambda_k$ such that $\Gamma_i = \{\lambda_1, \ldots, \lambda_i\}$. Then*

$$0 = A(\Gamma_0) \hookrightarrow A(\Gamma_1) \hookrightarrow \cdots \hookrightarrow A(\Gamma_k) = A$$

is a filtration of A with composition factors $A(\Gamma_i)/A(\Gamma_{i-1}) \simeq V^{\lambda_i *} \otimes_R V^{\lambda_i}$.

Proof. Since the chain of ideal is maximal, that means $\Gamma_i \setminus \Gamma_{i-1} = \{\mu\}$ for a certain μ. So we can find a total ordering. As a consequence, $A^{<\lambda_i} \subset A(\Gamma_{i-1})$ and the basis $\{C^{\lambda_i}_{BT} + A(\Gamma_{i+1}) \mid B, T \in P(\lambda_i)\}$ is a basis of the two-sided ideal $A(\Gamma_i)/A(\Gamma_{i-1})$. The isomorphisms of A-bimodule is then simpl sending $C^{\lambda_i}_{BT} + A(\Gamma_{i-1}) \mapsto v_B \otimes v_T + A^{<\lambda_i}$ where $v_B \otimes v_T \simeq C^{\lambda_i}_{BT}$. □

[1] Just a reminder.

Then, the factors at level $\lambda$of the filtration are isomorphic to direct sum of $|P(\lambda)|$ left-module V^λ. This means that, when extended to composition series, the composition factors of A are composition factors of V^λ.

With this, we can already give one of the simple modules of A.

Lemma 16. *When λ is maximal, then $V^\lambda \simeq L^\lambda$.*

Proof. We need to prove that $\mathrm{Rad}_\lambda = 0$. Suppose $v \in \mathrm{Rad}_\lambda$. We write it $v = \sum_{B\in P(\lambda)} r_B v_B$. Fix $T \in P(\lambda)$ and write $a(vT) = \sum_{B\in P(\lambda)} r_B c^\lambda_{BT} \in A$. In particular, $a(vT) \in A^{\leq\Lambda}$, and it is in $A^{<\lambda}$ if and only if $v = 0$. Since v is in the radical, we have $\langle v, w\rangle_\lambda$ for all $w \in V^\lambda$. Therefore by the definition of the bilinear form we have, for $U, S \in P(\lambda)$

$$C^\lambda_{US} a(vT) = \sum_{B\in P(\lambda)} r_B C^\lambda_{US} C^\lambda_{BT} \equiv \sum_{B\in P(\lambda)} r_B \langle v_S, v_B\rangle_\lambda C_{UT} = \langle v_S, v\rangle_\lambda C^\lambda_{UT} \overset{v\in\mathrm{Rad}_\lambda}{=} 0 \mod A^{<\lambda}$$

Then $a(vT)a \in A^{<\lambda}$ for all $a \in A^{\leq\lambda}$ and since λ is maximal, that means $a(vT)\cdot 1 \in A^{<\lambda}$ so $x = 0$. □

Theorem 17 (Graham–Lehrer)**.** *Suppose that R is a field and that Λ is finite. Then $\{L^\lambda \mid \lambda \in \Lambda^0\}$ is a complexe set of pairwise inequivalent simple modules.*

Proof. Done in class. See either [Mat99, Theorem 2.16] or [Bow25, Theorem 6.2.20] for proof. In Chris' proof, the chain of ideals comes from Lemma 15. □

This seems all dandy and fine, but sometimes it will be hard to find Λ^0. Still, we have reduced a dfficult problem of abstract algebra into a much more manageable linear algebra problem.

Next lecture (13-05-2025) we will see that there is even more to those cellular algebras, and that they will also let us state meaningful results on the, much more elusive, *indecomposable modules*.

Problem sheet 4

0. (Drill)

1. Prove that the algebra of $n\times n$ matrices is cellular with respect to the cellular datum $\Lambda = \{n\}$, $P(n) = \{1, \dots, n\}$ $C_{ij} = E_{ij}$, the elementary matrices. (it amounts to checking that $AE_{ij} = \sum_{i'} r^a_{ii'} E_{i'j}$)

2. Define $V^\lambda{*}$, the right A-cell module.

3. Fill the details of Lemma 7.

1. Chapter 6.3 Read Chapter 6.3 of Chris' book to get an example of a different kind of cellular algebra.

2. Chapter 6.4 Read Chapter 6.4 of Chris' book. This gives a cell structure on $\mathbb{F}\mathfrak{S}_3$. In particular, it does not suppose the field has characteristic 0.

3. Some fun with TL We call the matrices $G_\lambda = (\langle v, w\rangle_\lambda)_{v,w\in V^\lambda}$ the *Gram* matrices of the bilinear form.

Compute all the Gram matrices for $\mathrm{TL}_n(\beta)$ for $n = 2, 3, 4, 5$. Exhibit the values of β where the algebra is not semisimple.

4. Gram determinant (Difficult) Find a recursion formula for the Gram determinant of Temperley–Lieb algebra.

References

[Bow25] C. Bowman. *Diagrammatic algebra*. In press. Springer, 2025.

[CR66] C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Vol. 356. American Mathematical Soc., 1966.

[GL96] J. Graham and G. Lehrer. "Cellular Algebras". In: *Invent. Math.* 123 (1996), pp. 1–34. DOI: 10.1007/BF01232365.

[Mat99] A. Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group*. Vol. 15. American Mathematical Soc., 1999.

Lecture notes and Problem sheet 5, 12-05-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de

Extra material and preliminaries of representation theory of algebra

The extra material on the representation theory of finite-dimensional algebras was taken from the books [Mat99; CR66], and the extra material on cellularity comes from [Mat99].

This extra material will be found on the course notes (along with a small recap of what we did lecture by lecture).

Still, let me just precise one notion from lecture. The radical $\mathrm{Rad}(\mathcal{A})$ of a finite-dimensional $\mathbb{F}$-algebra $\mathcal{A}$ was defined as the sum of all nilpotent ideals (so ideal I for which there exists an $n \in \mathbb{N}$ such that $I^n = 0$). Then we define the radical of a submodule $M \subset \mathcal{A}$ as $\mathrm{Rad}(M) := \mathrm{Rad}(\mathcal{A}) \cap M$.

Problem sheet 5

0. (Drill)

1. Write back the Gram matrices of $\mathsf{TL}_4(\beta)$ and identify the values where they have a radical (we did it in class).

1.Temperley–Lieb non-semisimple In class, we studied the representation theory of $\mathsf{TL}_4(\beta)$ for special $\beta = 0, 1$. Give the decomposition matric $D = ([V^d : L^{d'}])_{d,d'=0,2,4}$ for the last case we did not work out: $\beta = \sqrt{2}$.

2. Oriented Temperley–Lieb algebras [Chris' 5.5] Read the definition of the oriented Temperley–Lieb algebra. Pay attention, q is not an element of the field, it is an abstract element, and the reason this algebra is infinite dimensional.

***q*-numbers** Read Chapter 7.1 of Chris' book, but skip the definition of quantum number. There is a small typo in the version (the n of the middle member should be $n-1$):

$$[n]_q := q^{n-1} + q^{n-2} + \cdots + q + 1 = \frac{1-q^n}{1-q}$$

Often, especially in quantum group, it makes more sense to use a different notion of quantum number:

$$[\![n]\!]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$$

In particular, $[\![2]\!]_q = q + q^{-1}$ which is a convenient parametrisation of the parameter β in Temperley–Lieb algebras.

The exercise is then to give the relation between the two notions $[n]_q$ and $[\![n]\!]_q$. (Note that both of them define a generalisation of numbers and, indeed, retrieve n when $q \to 1$).

References

[CR66] C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Vol. 356. American Mathematical Soc., 1966.

[Mat99] A. Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group*. Vol. 15. American Mathematical Soc., 1999.

Problem sheet 6, 19-05-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de or by accessing online the book.

0. (Drill)

1. The algebra of polynomial of n variables $A = \mathbb{Z}[x_1, \dots, x_n]$ is a $\mathbb{Z}$–graded algebra (in fact, a $\mathbb{N}$–graded algebra). What are the level A_i in the decomposition $A = \bigoplus_{i\in\mathbb{Z}} A_i$?
2. Compute some q-binomials (pay attention to the typo in the definition of the q-numbers). They are defined as
$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}$$
where $[n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q!$ with $[0]_q := 1$.

1. The zig-zag algebra Read Section 5.4 on the zig-zag algebra and complete the exercises.

2. Idempotent and grading tricks in action Read Section 6.8 regarding the simple modules of the zig-zag algebra to appreciate the use of the grading and idempotent tricks. Then rewrite the section in as much details as you want for a small zig-zag algebra to convince yourself.

3. Relating the binary Schur and the zig-zag algebra We have seen in Lecture 5 how some idempotent truncation algebra can still be cellular. In particular, the zig-zag algebra ZZ_2 for example is an idempotent truncation of the binary Schur algebra, that is, $\mathsf{ZZ}_2 = e_\Pi S_{\mathbb{F}_2}(2)e_\Pi$ for some idempotent fixed the anti-involution e_Π. Give more details than Example 6.12.2, and extend this observation to ZZ_3 (pay attention as the zig-zag algebra field changes).

4. Understanding some poset construction Read Section 7.2 and convince yourself of the combinatorics you see there. This is best done by doing everything for the parabolic (if you forgive me for starting earlier with name) $\mathfrak{S}_2 \times \mathfrak{S}_2 \leq \mathfrak{S}_4$ and, in particular, Figure 7.2.

Problem sheet 7, 02-06-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de or by accessing online the book.

0. (Drill)

1. Compute $\begin{bmatrix} n+m \\ m \end{bmatrix}_q$ for $(m,n) = (2,2), (3,2)$ and $(3,3)$
2. Construct the Bruhat graph for (m,n) tile-partitions for the (m,n) above and compare with the value of the coefficients you found previously. (Note that Figs 7.2, 7.9 give you some indication.)

1. Paths in the Bruhat graph Compute all the potential paths in $\mathcal{P}_{2,3}$ of fixed colour sequence $s_2 s_3 s_4 s_1 s_2 s_3$ and draw the path in the graph $\mathcal{P}_{2,3}$. (We did two of them in class.) Then, associate the Kazdhan–Lusztig polynomial to them and their degree. Remember, the moves for $\lambda \xrightarrow{s_i} \mu$ are

$$\begin{aligned}
&U_i^1 : \lambda \to \mu, && \deg U_i^1 = 0; \\
&U_i^0 : \lambda \to \lambda, && \deg U_i^0 = 1; \\
&D_i^1 : \mu \to \lambda, && \deg D_i^1 = 0; \\
&D_i^0 : \mu \to \mu, && \deg D_i^0 = -1.
\end{aligned}$$

2. Kazhdan–Lusztig polynomials for $\mathfrak{S}_2 \times \mathfrak{S}_2 \leq \mathfrak{S}_4$ Compute all the the Kazhdan–Lusztig polynomials for $\mathfrak{S}_2 \times \mathfrak{S}_2 \leq \mathfrak{S}_4$ and draw their path on the Bruhat graph. (See Fig 7.5 in the book for the answer.)

3. (Exo 7.3.6) Kazhdan–Lusztig polynomials for $\mathfrak{S}_2 \times \mathfrak{S}_3 \leq \mathfrak{S}_5$ Compute all the the Kazhdan–Lusztig polynomials for $\mathfrak{S}_2 \times \mathfrak{S}_3 \leq \mathfrak{S}_5$ and draw their path on the Bruhat graph. (See Fig 7.2 in the book for the graph and Exercise 1 of this sheet, or Example 7.3.2 of the book for the first column (the $n(\lambda, \mu)$ for maximal colour sequence).) This is a 10×10 matrix.

4. Complete the isomorphism We swept under the carpet that the two ways of seeing the poset (with weight diagrams and tile-partitions) were the same. The beginning is done in Proposition 7.4.2 of the book; complete the details.

Problem sheet 8, 23-06-2025

Problems coming from Chris Bowman's book *Diagrammatic algebra* are referenced as the preliminary January version of the book available to the participants of the course by sending out an email to me: langlois@uni-bonn.de or by accessing the book online.

0. (Drill)

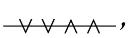
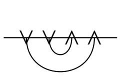
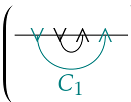
1. Write down the elements T_S of $\mathrm{TL}^{\uparrow\downarrow}_{2+2}(q)$ for the four paths $S = U_2^1U_3^1U_1^1U_2^1$; $U_2^1U_3^1U_1^1U_2^0$; $U_2^1U_3^0U_1^0D_2^0$; $U_2^1U_3^0U_1^0D_2^1$ for $\mathfrak{S}_2 \times \mathfrak{S}_2 \leq \mathfrak{S}_4$ and compute their degree (see Example 7.3.5).
2. Draw the bottom edge of the oriented Temperley–Lieb diagrams of Figure 7.12 (for the parabolic $\mathfrak{S}_2 \times \mathfrak{S}_3 \leq \mathfrak{S}_5$)

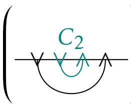
1. Proof of Proposition 7.5.7 Read the proof of Proposition 7.5.7 stating that for any reduced path T_μ, the Temperley–Lieb element $E_{T_\mu} = e_\mu$. This follows from the (alternative) proof of the correspondence between the diagrammatic and its generators and relations presentations of Temperley–Lieb algebras done in Theorem 5.2.3.

2. Complete the proof of Theorem 7.5.10 Follow the proof of Theorem 7.5.10 and complete the other diagrammatic verification that the degree is preserved by the map.

3. More on Proposition 7.5.18 This exercise gives the details of a combinatorial property of the Kazhdan–Lusztig polynomials. Let μ be a m, n diagram and $\overline{\mu}$ be the cup diagram generated by closing each $\vee$ with its closed left $\wedge$ neighbour. Then for each cup C, define the width of the cup $w(C)$ as twice the number of cups inside C, understanding that C is inside itself.

For example,

$\mu =$ , $\overline{\mu} =$ , $w\Big($ 

$\Big) = 2 \times 2 = 4$, $w\Big($ 

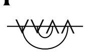
$\Big) = 1 \times 2 = 2$.

Proposition 7.5.18 *Given μ a (m, n)-diagram, the following column-sum of the Kazhdan–Lusztig polynomial matrix is palindromic and unimodal and equal*

$$\sum_{\lambda \subseteq \mu} q^{\ell(\mu)-\ell(\lambda)} n_{\lambda,\mu}(q) = \prod_{C \text{ a cup in } \overline{\mu}} (1 + q^{w(C)}).$$

For example, let us do the first and second-to-last columns of Figure 7.13. We have:

$$\begin{aligned}\sum_{\lambda \subseteq \mu} q^{\ell(\mu)-\ell(\lambda)} n_{\lambda,\mu}(q) &= q^{4-4}q^0 + q^{4-3}q + q^{4-1}q + q^{4-0}q^2 \\ &= 1 + q^2 + q^4 + q^6 \\ &= (1 + q^4)(1 + q^2) \\ &= \prod_{C \text{ a cup in }} (1 + q^{w(C)}).\end{aligned}$$



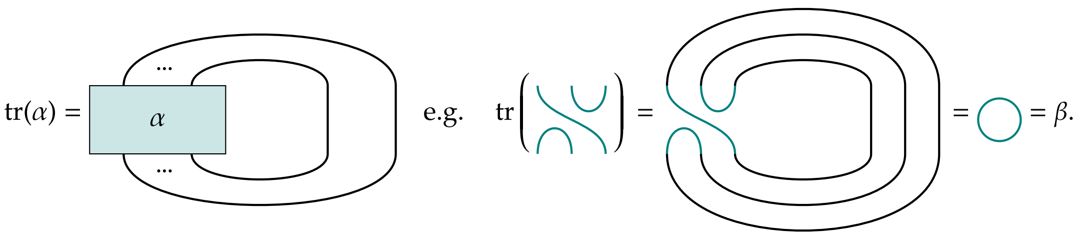
4.

Oral examination preparation, 21-07-2025

This sheet gives you an idea of the type of questions that will be asked during your oral examination. These will **not** be the questions asked at the oral examination. In addition, they do not include the follow-up questions I will ask. That being said, they are similar to the questions and should help you have a clear idea of what can be asked. The examination should take 30 minutes. I have added in parenthesis the time in minute I expect you to spend on each questions.

You do not need to give the "correct" answer to obtain full grade. You do need to explain your thought process.

1. A diagrammatic computation (2) We define a "trace" on the Temperley–Lieb algebra $\mathsf{TL}_n(\beta)$ that we defined in the course. It is a function $\mathrm{tr} : \mathsf{TL}_n(\beta) \to \mathbb{C}$ given by the following diagrammatic construction: given a Temperley–Lieb diagram α, embed it into a bigger space and connect the top and bottom strands with loops via the diagrammatic rule:



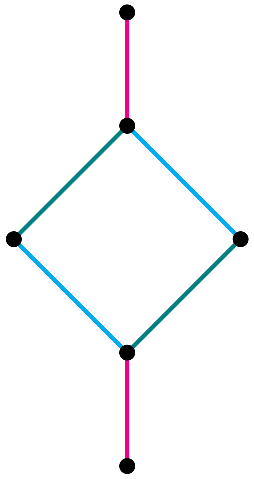
Compute the trace of the following element of $\mathsf{TL}_2(\beta)$

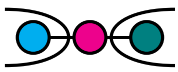
$$P_2 = \Big| \;\; \Big| - \frac{1}{\beta} \genfrac{}{}{0pt}{}{\cup}{\cap} \tag{A.1}$$

2. Cellular algebras (5) How do you define the simple modules of a given cellular algebra?

Can you give one example from the course?

3. Paths on the Bruhat graph for the symmetric group with maximal parabolic (5) Consider the following Bruhat graph of the maximal parabolic system $(\mathfrak{S}_4, \mathfrak{S}_2 \times \mathfrak{S}_2)$ with Coxeter generators $S^W = \{\tau, \sigma, \rho\}$ and parabolic $P = \langle \tau, \rho \rangle$.





1. Write down the following paths of colour pattern $\sigma\rho\tau\sigma$

$$\begin{aligned} S_1 &= U_2^1 U_3^1 U_1^1 U_2^1, \\ S_2 &= U_2^1 U_3^1 U_1^1 U_2^0, \\ S_3 &= U_2^1 U_3^0 U_1^0 D_2^0, \\ S_4 &= U_2^1 U_3^0 U_1^0 D_2^1, \end{aligned}$$

and compute their degree.

2. Can you draw the Soergel diagram d_{S_3} of the algebra $\mathcal{H}_{\mathfrak{S}_4,\mathfrak{S}_2\times\mathfrak{S}_2}$? If you do not remember the procedure, describe the shape of the diagram (what would go on top, what on the bottom, and what is inside), then go check your course notes and draw it.

4. Correspondence between diagrammatic and representation theory (8) Consider the subalgebra $\mathsf{TL}_2(\beta) \subset \mathsf{TL}_3(\beta)$. We look back at the elements $P_2 \in \mathsf{TL}_2(\beta)$ defined in (A.1) and its inclusion in $\mathsf{TL}_2(\beta)$.

1. Compute the action of the generator of $\mathsf{TL}_2(\beta)$ on $P_2 \in \mathsf{TL}_2(\beta)$ defined in (A.1).

   Do you have an interpretation of this computation in representation-theoretical terms?

2. Compute the actions of the generators of $\mathsf{TL}_3(\beta)$ on $P_2 \in \mathsf{TL}_2(\beta)$ under its inclusion in $\mathsf{TL}_3(\beta)$.

   Can you say something about the results of this computation?

5. A new algebra (hard) (10) Consider the tower of inclusion $\mathsf{TL}_2(\beta) \subset \mathsf{TL}_3(\beta) \subset \cdots \subset \mathsf{TL}_{n+2}(\beta)$ for $n \geq 0$. We denote $P_2^n \in \mathsf{TL}_{n+2}(\beta)$, the image of P_2 defined in (A.1) under these inclusion. We consider the idempotent truncation algebra $B_{n,2}(\beta) := P_2^n \mathsf{TL}_{n+2}(\beta) P_2^n$. We call it the boundary seam algebra[1]

1. Do you think this algebra is well defined?
2. Do you have a proposal for a diagrammatic calculus?
3. Do you think this algebra is semisimple?
4. If you were to characterise its simple modules, how would you go about it?

Now is a good time to reiterate that I do not expect you to blitz out all the correct answers, nor the complete steps of a solution right away to obtain full grade. The goal is to tell me how you would do and ask what you need. For example, a full answer to question 3.1 can be: "Here are

[1] The name comes from physics [MRR15].

the answers for S_1, S_2, but I don't remember if D_i^0 is going down or failing to go down, so the two last paths are uncertain. For their degrees, I remember reduced paths should be of degree zero so the U_i^1's need to be degree zero, but I'm not sure of the others; I will assume this and then the full degree is the sum of the degrees of the U and D."

Bibliography

[Bow25] C. Bowman. *Diagrammatic algebra*. Springer, 2025. DOI: 10.1007/978-3-031-88801-4 (cit. on pp. 3, 11).

[CR66] C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Vol. 356. American Mathematical Soc., 1966 (cit. on pp. 3, 7).

[GL96] J. Graham and G. Lehrer. "Cellular Algebras". *Invent. Math.* 123 (1996), pp. 1–34. DOI: 10.1007/BF01232365 (cit. on p. 8).

[IDA97] I Assem, D Simson, and A Skowroński. *Elements of Representation Theory of Associative Algebras. Volume 1. Techniques of Representation Theory*. Vol. 65. London Mathematical Society Student Texts. New York: Cambridge University Press, 1997. DOI: 10.1017/CBO9780511614309 (cit. on p. 3).

[Mat99] A. Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group*. Vol. 15. American Mathematical Soc., 1999 (cit. on pp. 3, 7, 8, 11).

[MRR15] A. Morin-Duchesne, J. Rasmussen, and D. Ridout. "Boundary algebras and Kac modules for logarithmic minimal models". *Nuclear Physics B* 899 (2015), pp. 677–769. arXiv: 1503.07584 (cit. on p. 40).

[RS14] D. Ridout and Y. Saint-Aubin. "Standard modules, induction and the structure of the Temperley-Lieb algebra". *Adv. Theor. Math. Phys.* 18 (2014), pp. 957–1041. arXiv: 1204.4505 (cit. on pp. 3, 5).