# Review

$\rho$: half sum of weight

$$w \cdot \lambda = w(\lambda+\rho)-\rho$$

## Prop 1.4'

if $\alpha \in \Delta$ $\lambda \in \check{\mathfrak{h}}^*$ $\langle \lambda+\rho, \check{\alpha} \rangle \in \mathbb{Z}^{>0}$

There exists

$$M(s_\alpha \cdot \lambda) \longrightarrow N(\lambda) \subset M(\lambda)$$

## Theorem Harish-Chandra

a) $Z(\mathfrak{g}) \xrightarrow[\sim]{\Psi} S(\mathfrak{h})^W$

b) $\chi_\mu = \chi_\lambda$ if and only if $\mu = w \cdot \lambda$ for a $w \in W$

c) Every central character $\chi : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ is a

$\chi = \chi_\lambda$, $\lambda \in \check{\mathfrak{h}}^*$

# Kleine Seminar Chapter 1 part III

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## 1.11

**Theorem**

Category $\mathcal{O}$ is artinian.

1) $M \in \mathcal{O}$ is artinian

2) $\dim \mathrm{Hom}_{\mathcal{O}}(M, N) < \infty$ $M, N \in \mathcal{O}$

**Idea of Proof** – Prove it for Verma module.

take $V = \sum_{w \in W} M(\lambda)_{w \cdot \lambda}$

If $N' \subset N \subset M(\lambda)$ are submodules

1) $\dim V < \infty$ (weight module)

$N/N'$ has a maximal vector of weight $\mu \leq \lambda$ and $Z(\mathfrak{g})$ acts by $\chi_\lambda$ on $N/N'$ So $\chi_\mu = \chi_\lambda$

by Harish-Chandra Theorem, b) $\mu = w \cdot \lambda$

So $N \cap V \neq \emptyset$

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$N \cap V' \neq \emptyset$ implies

$\Rightarrow \dim \cancel{V} \cap N > \dim V \cap N'$

These dimension are finite

So a descending chain of submodule will end.

## Corollary

Each $M \in \mathcal{O}$ possesses a composition series of finite lenght with simple quotients isomorphic to some $L(\lambda)$.

## Definition

The Grothendieck group of $\mathcal{O}$ is $AB\langle [M] \mid M \in \mathcal{O} \rangle$ ~~/~~ $[B] = [A] + [C]$ if $0 \to A \to B \to C \to 0$ exact

## Composition series

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset M$$

with $M_i / M_{i-1}$ simple

By Jordan Hölder theorem, we can speak of "the composition series" because all simple quotient appears in any composition series.

We write $[M(\lambda), L(\mu)]$ for the number of simple quotient isomorphic to $L(\mu)$ in $M(\lambda)$.

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Socle $M$, $\operatorname{Soc} M$ : sum of simple submodule of $M$

radical: $\operatorname{Rad} M$ : intersection of maximal submodule

head : $\operatorname{Hd} M$ : quotient $M / \operatorname{Rad} M$.

**Definition** $\chi$ central character

$$M^{\chi} := \{ v \in M \mid (z - \chi(z))^{n} \cdot v = 0 \text{ for } n \gg 0,\ n = n(z),\ z \in Z(\mathfrak{g}) \}.$$

$\mathcal{O}_{\chi} \subset \mathcal{O}$: subcategory of objects $M = M^{\chi}$

**Proposition**

$$\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^{*}/\text{mod } W} \mathcal{O}_{\chi_\lambda}$$

$\to$ orbits of dot action

Indecomposable module lies in one $\mathcal{O}_{\chi_\lambda}$

**Comment**

$\chi$ is a central character that may be expressed as $\chi_\lambda$ one for each $W$-orbit.

# 1.13

**Definition**

If $M_1$ and $M_2$ are such that there is a $M$ with

$$0 \to M_1 \to M \to M_2 \to 0$$

$$0 \to M_2 \to M \to M_1 \to 0$$

short exact non-split sequences then $M_1$ and $M_2$ are in the same block.

**Prop** If $\lambda \in \Lambda$ (integral) then $\mathcal{O}_{\chi_\lambda}$ is a block

$M$ arbitrary

$M$ belongs to a block $\mathcal{O}$ if all its composition factors belong to it

$\exists$ block $\subset \mathcal{O}_{\chi_\lambda}$

* ↳ Try to see it

[That each indecomposable module belongs to a block is not obvious and follows from artinian condition.]

**Idea** We use Prop 1.4 to get that all $L(w\cdot\lambda)$ are in the same block.

$$L(w\cdot\lambda) = M(w\cdot\lambda)/N(w\cdot\lambda) \hookrightarrow N(\lambda)/\mathrm{rm}\, W(w\cdot\lambda)$$

The block $\mathcal{O}_{\chi_0}$ is called the principal block.

$\mathcal{O}$ is integral in $\mathfrak{h}^*$

The proposition fails for non-integral weights

$sl_2$ Verma module

when $\lambda$ was positive but not integral, we still had .

$M(\lambda)$ simple..

but $\lambda$ and $-\lambda-\alpha$ were ~~two~~ linked

So $L(\lambda)$ and $L(-\lambda-\alpha) \in \mathcal{O}_{\chi_\lambda}$

but they are different. So $\mathcal{O}_{\chi_\lambda}$ not a block

1.14

## Definition

Let $e(\lambda)$ be a symbol associated to $e(\lambda) \in \mathbb{Z}\Lambda$. The formal character of $M$ lying in $\mathbb{Z}\Lambda$ is

$$\operatorname{ch} M := \sum_{\lambda \in \Lambda} \dim M_\lambda \, e(\lambda)$$

$$\operatorname{ch}(M \otimes N) = \operatorname{ch} M \cdot \operatorname{ch} N$$

$M$ finite-dimensional

If $M \in \mathcal{O}$, $\operatorname{ch} M$ is well defined as a finite sum in $\mathbb{Z}\Lambda$

# 1.15

For the possibly infinite-dimensional cases we need:

$\mathrm{ch}\, M \longrightarrow \mathbb{Z}^+$ valued function on $\mathfrak{h}^*$

$e(\lambda) \longrightarrow$ characteristic function

$$e(\mu) = \begin{cases} 1 & \mu = \lambda \\ 0 & \mu \neq \lambda \end{cases}$$

Multiplication $\longrightarrow$ convolution

**Definition** The group of functions $f : \mathfrak{h}^* \longrightarrow \mathbb{Z}$ with support lies in a finite union of $\lambda - \Gamma$ (OS) is denoted $\mathcal{X}$

all $e(\lambda) \in \mathcal{X}$ and $e_0$ is identity under convolution.

$\mathcal{X}_0 \subset \mathcal{X}$ subgroup generated by $\mathrm{ch}\, M \mid M \in \mathcal{O}$.

$M \in \mathcal{O}$

Work as functions of $\mathfrak{h}^*$, not simply as elements of ring

$e(\lambda)$ a function.

$$(f * g)(\lambda) = \sum_{\mu + \nu} f(\mu) g(\nu)$$

$\mathcal{X}$ : group of functions

$f : \mathfrak{h}^* \longrightarrow \mathbb{Z}$,

$\rightarrow$ The set of weights is contained in a finitely many sets of the form $\lambda - \Gamma$, $\Gamma$ semigroup in $\Lambda_r$ generated by $\Phi^+$

$$e_0 * f(\lambda) = \sum_{\mu + \nu = \lambda} e_0(\mu) f(\nu) = f(\lambda)$$

## Proposition $\mathrm{ch}\,M$ respects:

a) $0 \to M' \to M \to M'' \to 0$ short exact sequence then

$$\mathrm{ch}\,M = \mathrm{ch}\,M' + \mathrm{ch}\,M''$$

b) $\mathcal{X}_0 \xrightarrow{\sim} K(\mathcal{O})$

$$\mathrm{ch}\,M \longmapsto [M]$$

c) if $M \in \mathcal{O}$, $L$ fin. dim.

$$\mathrm{ch}(L \otimes M) = \mathrm{ch}\,L * \mathrm{ch}\,M$$

## Comments

a) If we have such a short exact sequence, we will have

$$\dim M_\mu = \dim M'_\mu + \dim M''_\mu$$

($L \otimes M$ is in $\mathcal{O}$)

$$(L \otimes M)_\lambda = \sum_{\mu+\nu} L_\mu \otimes M_\nu$$

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$M(\lambda)$ Verma Module

**Definition** The Kostantfunction $\wp \in \mathcal{X}$ is

$$\wp(\gamma) = \#\text{ families } (c_\alpha)_{\alpha>0},\ c_\alpha \in \mathbb{Z}^+$$

$$\gamma = -\sum_{\alpha>0} c_\alpha \alpha$$

**Proposition** for $\lambda \in \tilde{\mathfrak{h}}^+$

$$\operatorname{ch} M(\lambda) = \wp * e(\lambda)$$

$$\operatorname{ch}(0) = \wp.$$

More info are needed on
$\mathrm{ch}\, L(\lambda)$.

Write
$$\mathrm{ch}\, M(\lambda) = \mathrm{ch}\, L(\lambda) + \sum_{\substack{\mu \leq \lambda \\ \mu = w\cdot\lambda}} [M(\lambda) : L(\mu)]\, \mathrm{ch}\, L(\mu)$$

Invert it (Triangular)

$$\mathrm{ch}\, L(\lambda) = \sum_{w\cdot\lambda \leq \lambda} b(\lambda, w)\, \mathrm{ch}\, M(w\cdot\lambda)$$

$b(\lambda, w) \in \mathbb{Z}$

$sl_2$

$\lambda < 0$, we know $M(\lambda) = L(\lambda)$

$\lambda \geq 0$ we know we have one linkage $M(-\lambda-2)$ so

$chL(\lambda) = chM(\lambda) - ch(M(-\lambda-2))$?

for $\lambda \in \Lambda^+$, chapter 2

$\lambda < 0$. $M(\lambda)$ is simple

if $\lambda \notin \mathbb{Z}^+$, then $M(\lambda)$ is simple also $M(\lambda) = L(\lambda)$

$\lambda > 0$ $-\lambda-2$ is linked by the $W$ action to $\lambda$

if $\lambda \in \mathbb{Z}^+$ the theorem in in chapter

it is finite-dimensional