Quantitative Spatial Models in Economics: A Simple Commuting Model of Chicago

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This is a live document and subject to change.
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Hog Butcher for the World,
Tool Maker, Stacker of Wheat,
Player with Railroads and the Nation's Freight Handler;
Stormy, husky, brawling,
City of the Big Shoulders...

Come and show me another city with lifted head singing
so proud to be alive and coarse and strong and cunning.

— Carl Sandburg

Abstract

This repo is intended to demonstrate the basics of conducting economics research with quantitative spatial models. I derive and calibrate a simple quantitative spatial model of Chicago and conduct two counterfactual exercises. I then repeat this process for a richer model and compare the results.

1. Introduction

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In progress.
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2. Model

I begin with a simple model of commuting to demonstrate the basic mechanics of a common form of quantitative spatial model. I then extend this model to include other relevant features of the economy.

2.1. A Simple Model (Model A)

Chicago is comprised of discrete neighborhoods $i, n, k, l \in \mathcal{L}$. Each location i has a fixed mass R_i of residents.

2.1.1. Workers

Each agent inelastically supplies one unit of labor. An agent ω residing in location i and working in location n receives indirect[1] utility $\mathcal{U}_{ni}$, where

$$\mathcal{U}_{in\omega} = \left(\frac{w_n}{\kappa_{in}}\right) b_{in\omega}. \tag{1}$$

[1] I omit the subproblem of utility maximization given location choice for parsimony. Model B will explicitly discuss this subproblem, which nests the subproblem of utility maximization in this model.

w_n is the wage paid in location n. κ_{in} is a commuting cost of the iceberg form in the units of utility. $b_{in\omega}$ is an idiosyncratic preference shock with a Fréchet distribution. The cumulative distribution function of $b_{in\omega}$ is given by $F_{in}(b_{in\omega}) = \exp\left(b_{in\omega}^{-\theta}\right)$. θ governs the dispersion of this preference shock.

A worker ω in location i chooses the workplace that maximizes their indirect utility:

$$n_{i\omega}^* \overset{\text{def}}{=} \arg\max_{n\in\mathcal{L}} \mathcal{U}_{in\omega}. \tag{2}$$

Since workers differ only in their draws of $\{b_{in\omega}\}_{i,n\in\mathcal{L}}$ of preference shocks, we can drop the ω subscript in what follows. The Fréchet-distributed preference shock implies

$$\begin{aligned} \pi_{in \mid i} &\overset{\text{def}}{=} \mathbb{P}\{n_i^* = n\} = \varphi_{in}\Phi_i^{-1}, \\ \text{where} \quad \varphi_{in} &\overset{\text{def}}{=} \left(\frac{w_n}{\kappa_{in}}\right)^{\theta} \\ \text{and } \Phi_i &\overset{\text{def}}{=} \sum_{k\in\mathcal{L}} \varphi_{ik}. \end{aligned} \tag{3}$$

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Pending a citation on discrete choice magic.
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2.1.2. Firms

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This section currently omits some details concerning market structure. I will add a
discussion for the sake of completeness, but the equilibrium characterization will
not change.
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A unit mass of firms in each neighborhood produce a freely traded final good with the technology

$$Y_n = A_n L_n^{\beta} \tag{4}$$

and pay workers their marginal product.[2] The price of the final good is 1. Accordingly, the wage and labor demand in neighborhood n are given by

$$\begin{aligned} w_n &= \beta A_n L_n^{\beta-1} \\ \Longrightarrow L_n &= \left(\frac{\beta A_n}{w_n}\right)^{\frac{1}{1-\beta}}. \end{aligned} \tag{5}$$

2.1.3. Commuting Equilibrium

For the commuting market to clear, labor demand in location n must equal labor supply to location n across all residential locations i:

$$L_n = \sum_{i\in\mathcal{L}} \pi_{in \mid i} R_i. \tag{6}$$

We can substitute Equation 3 and Equation 5 into this expression to obtain an equilibrium characterization:

[2] Again, I omit details of market structure for parsimony. I do not explicitly model trade in goods.

$$\underbrace{\left(\frac{\beta A_n}{w_n}\right)^{\frac{1}{1-\beta}}}_{\text{Labor Demand}} = \underbrace{\sum_{i\in\mathcal{L}} \varphi_{in} \Phi_i^{-1} R_i}_{\text{Labor Supply}}. \tag{7}$$

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This section does not discuss the existence and uniqueness of the equilibrium, nor does
it discuss welfare. I will add sections on these topics in the future.
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2.1.4. Counterfactual Equilibria

I will denote the vector-collection of a variable x_i over all locations with boldface: $\{x_i\}_{i\in\mathcal{L}} \stackrel{\text{def}}{=} \boldsymbol{x}$. We consider a baseline equilibrium $\{\boldsymbol{w}^0, \boldsymbol{\pi}^0\}$ for parameters $\{\boldsymbol{A}^0, \boldsymbol{\kappa}^0, \boldsymbol{R}^0\}$ and a counterfactual equilibrium $\{\boldsymbol{w}', \boldsymbol{\pi}'\}$ for parameters $\{\boldsymbol{A}', \boldsymbol{\kappa}', \boldsymbol{R}'\}$. We denote proportional changes with hats, e.g.,

$$\hat{w}_n = \frac{w'_n}{w^0_n} \Longrightarrow w^0_n \hat{w}_n = w'_n. \tag{8}$$

This representation leads us to "exact hat algebra," a popular method to model and summarize counterfactual equilibria. We start by expressing the market clearing condition for the counterfactual equilibrium and then substitute in Equation 5:

$$\begin{aligned} L^0_n \hat{L}_n &= \left(\sum_{i\in\mathcal{L}} \left(\pi^0_{in} R^0_i\right)\left(\hat{\pi}_{in} \hat{R}_i\right)\right) \\ \Longrightarrow \left(\frac{\hat{A}_n}{\hat{w}_n}\right)^{\frac{1}{1-\beta}} &= \frac{\sum_{i\in\mathcal{L}} \left(\pi^0_{in} R^0_i\right)\left(\hat{\pi}_{in} \hat{R}_i\right)}{L^0_n}. \end{aligned} \tag{9}$$

We can use Equation 3 to write

$$\begin{aligned} \hat{\pi}_{in\,|\,i} &= \hat{\varphi}_{in} \hat{\Phi}_i^{-1}, \\ \text{where} \quad \hat{\varphi}_{in} &\stackrel{\text{def}}{=} \left(\frac{\hat{w}_n}{\hat{\kappa}_{in}}\right)^{\theta} \\ \text{and } \hat{\Phi}_i &\stackrel{\text{def}}{=} \sum_{k\in\mathcal{L}} \pi^0_{ik} \hat{\varphi}_{ik} \end{aligned} \tag{10}$$

The substantive piece of this expression is $\hat{\Phi}_i$. We derive it below:

$$\hat{\Phi}_i = \frac{\sum_{k\in\mathcal{L}} \varphi^0_{ik} \hat{\varphi}_{ik}}{\sum_{l\in\mathcal{L}} \varphi^0_{il}} = \sum_{k\in\mathcal{L}} \pi^0_{ik} \hat{\varphi}_{ik}, \tag{11}$$

where we have used Equation 3 to substitute in for π^0_{ik} (see the portions colored red). We now combine Equation 9 and Equation 10 to obtain

$$\left(\frac{\hat{A}_n}{\hat{w}_n}\right)^{\frac{1}{1-\beta}} = \left[\sum_{i\in\mathcal{L}} \frac{\pi^0_{in} R^0_i \hat{R}_i \left(\hat{w}_n/\hat{\kappa}_{in}\right)^{\theta}}{\sum_{k\in\mathcal{L}} \pi^0_{ik} \left(\hat{w}_k/\hat{\kappa}_{ik}\right)^{\theta}}\right] \frac{1}{L^0_n}. \tag{12}$$

What does this characterization of a counterfactual equilibria buy us? If we express a counterfactual as a set of proportional changes to the parameter values $\left\{\widehat{\boldsymbol{A}}, \hat{\boldsymbol{\kappa}}, \widehat{\boldsymbol{R}}\right\}$, then we only need data on initial conditional commuting probabilities $\boldsymbol{\pi}^0$, workplace population $\boldsymbol{L}^0$, and residential population $\boldsymbol{R}^0$ to solve for the proportional changes in wages $\hat{\boldsymbol{w}}$ (using Equation 12) and conditional commuting probabilities $\hat{\boldsymbol{\pi}}$ (using Equation 10).

Inspired by this representation, we define

$$\mathcal{Z}_n(\tilde{\boldsymbol{w}}) \stackrel{\text{def}}{=} \left(\frac{\hat{A}_n}{\tilde{w}_n}\right)^{\frac{1}{1-\alpha}} - \left[\sum_{i \in \mathcal{L}} \frac{\pi_{in}^0 R_i^0 \hat{R}_i (\tilde{w}_n / \hat{\kappa}_{in})^\theta}{\sum_{k \in \mathcal{L}} \pi_{ik}^0 (\tilde{w}_k / \hat{\kappa}_{ik})^\theta}\right] \frac{1}{L_n^0}. \tag{13}$$

We can use this vector-valued function $\boldsymbol{\mathcal{Z}}(\tilde{\boldsymbol{w}})$ to compute the proportional changes in wages (and other equilibrium objects) in counterfactual equilibria. I provide pseudocode for this procedure below. I implement the algorithm in `analysis/model_A.ipynb`.

Model A Algorithm:

1. s = 0
2. $\varepsilon = \text{tolerance} + 1$
3. $\tilde{\boldsymbol{w}}^0 = \vec{1}$
4. **while** $\varepsilon >$ tolerance **do**
5. $\quad \tilde{\boldsymbol{w}}^{s+1} = \tilde{\boldsymbol{w}}^s + \kappa_w \boldsymbol{\mathcal{Z}}(\tilde{\boldsymbol{w}}^s)$
6. $\quad \varepsilon = \max\{|\boldsymbol{\mathcal{Z}}(\tilde{\boldsymbol{w}}^s)|\}$
7. $\quad$ s = s + 1
8. **end while**
9. **return** $\tilde{\boldsymbol{w}}^s$

2.2. A Richer Model (Model B)

We now consider a model with a housing market and residential choice. The mass of agents is denoted $\overline{R}$. We no longer fix an agent's residential location.

2.2.1. Setup

Utility for an agent ω residing in location i and working in location n is given by

$$U_{in\omega} = \left(\frac{c_{in\omega}}{\alpha}\right)^\alpha \left(\frac{h_{in\omega}}{1-\alpha}\right)^{1-\alpha} \frac{b_{in\omega}}{\kappa_{in}} \tag{14}$$

where $c_{in\omega}$ is final good consumption, $h_{in\omega}$ is housing consumption, and $F_{in}(b_{in\omega}) = \exp(-B_{in} b_{in\omega}^{-\theta})$. We've added a parameter B_{in} that governs average utility for agents that live in location i and work in location n. The Cobb-Douglas form of Equation 14 implies that agents spend a constant fraction α of their income on the final good and $(1-\alpha)$ on housing. The price of the final good is again 1, and we denote the price of housing in location i by q_i. Accordingly, indirect utility for an agent ω residing in location i and working in location n with wage w_n is given by

$$\mathcal{U}_{in\omega} = \left(\frac{\alpha w_n}{\alpha}\right)^\alpha \left(\frac{\alpha w_n}{q_i(1-\alpha)}\right)^{1-\alpha} \frac{b_{in\omega}}{\kappa_{in}} = \left(\frac{w_n q_i^{\alpha-1}}{\kappa_{in}}\right) b_{in\omega}. \tag{15}$$

A worker ω now chooses both a residence and workplace:

$$\{i,n\}^*_\omega \overset{\text{def}}{=} \arg\max_{i,n\in\mathcal{L}} \mathcal{U}_{in\omega}. \tag{16}$$

Similar to before, the Fréchet-distributed preference shock implies the following expression for the *unconditional* residential and commuting probability

$$\begin{aligned} \pi_{in} &\overset{\text{def}}{=} \mathbb{P}\{\{i,n\}^* = \{i,n\}\} = \varphi_{in}\Phi^{-1}, \\ \text{where} \quad \varphi_{in} &\overset{\text{def}}{=} B_{in}\left(\frac{w_n q_i^{\alpha-1}}{\kappa_{in}}\right)^\theta \\ \text{and} \quad \Phi &\overset{\text{def}}{=} \sum_{k\in\mathcal{L}}\sum_{l\in\mathcal{L}} \varphi_{kl}. \end{aligned} \tag{17}$$

In what follows, it will be useful to define the mass of residents in each location i

$$R_i \overset{\text{def}}{=} \sum_{n\in\mathcal{L}} \pi_{in}\overline{R}, \tag{18}$$

following the notation from Model A.

2.2.2. Housing Market

Each location i has a fixed stock of land available for rent H_i. Landlords face no costs and spend all of their rental income on the final good to ensure goods market clearing. Let $\overline{\nu}_i$ denote the average income of residents in location i. We can than express *aggregate* income for resident in location i

$$\overline{\nu}_i R_i = \sum_{n\in\mathcal{L}} \pi_{in} w_n \overline{R}. \tag{19}$$

Land market clearing implies that housing expenditure (given by utility maximization) must equal landlord income in neighborhood i:

$$\underbrace{(1-\alpha)\overline{\nu}_i R_i}_{\text{Housing Expenditure}} = \underbrace{H_i q_i}_{\text{Landlord Income}} \tag{20}$$

2.2.3. Firms

We maintain the same set of assumptions on the firm side as in Model A. This yields the wage equation and labor demand

$$\begin{aligned} w_n &= \alpha A_n L_n^{\alpha-1} \\ \Longrightarrow L_n &= \left(\frac{\alpha A_n}{w_n}\right)^{\frac{1}{1-\alpha}}. \end{aligned} \tag{21}$$

2.2.4. Commuting Equilibrium

We now use the unconditional commuting probability in Equation 17 to define the commuting market clearing condition:

$$L_n = \sum_{i\in\mathcal{L}} \pi_{in}\overline{R}. \tag{22}$$

2.2.5. Counterfactual Equilibria

We proceed as in model A and derive the exact hat system.

$$
\begin{aligned}
\hat{\varphi}_{in} &= \hat{B}_{in}\left(\frac{\hat{w}_n \hat{q}_i^{\alpha-1}}{\hat{\kappa}_{in}}\right)^{\theta} \\
\hat{\pi}_{in} &= \frac{\hat{\varphi}_{in}}{\sum_{k\in\mathcal{L}}\sum_{l\in\mathcal{L}} \pi_{kl}^0 \hat{\varphi}_{kl}} \\
\hat{R}_i &= \frac{\overline{R}^0 \hat{\overline{R}}}{R_i^0} \sum_{n\in\mathcal{L}} \pi_{in}^0 \hat{\pi}_{in} \\
\left(\frac{\hat{A}_n}{\hat{w}_n}\right)^{\frac{1}{1-\beta}} &= \left(\frac{\overline{R}^0 \hat{\overline{R}}}{L_n^0}\right) \sum_{i\in\mathcal{L}} \pi_{in}^0 \hat{\pi}_{in} \\
\hat{\overline{\nu}}_i \hat{R}_i &= \hat{\overline{R}}\left(\frac{\sum_{n\in\mathcal{L}} \pi_{in}^0 w_n^0 \hat{\pi}_{in} \hat{w}_n}{\sum_{k\in\mathcal{L}} \pi_{ik}^0 w_k^0}\right) \\
\hat{q}_i &= \frac{\hat{\overline{\nu}}_i \widehat{R}_i}{\widehat{H}_i}
\end{aligned}
\tag{23}
$$

We combine the expressions from above and define

$$
\begin{aligned}
\mathcal{Z}_n(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{q}}) &\stackrel{\text{def}}{=} \left(\frac{\hat{A}_n}{\hat{w}_n}\right)^{\frac{1}{1-\beta}} - \left(\frac{\overline{R}^0 \hat{\overline{R}}}{L_n^0}\right) \sum_{i\in\mathcal{L}} \frac{\pi_{in}^0 \hat{B}_{in}\left(\hat{w}_n \hat{q}_i^{\alpha-1}/\hat{\kappa}_{in}\right)^{\theta}}{\sum_{k\in\mathcal{L}}\sum_{l\in\mathcal{L}} \pi_{kl}^0 \hat{B}_{kl}\left(\hat{w}_l \hat{q}_k^{\alpha-1}/\hat{\kappa}_{kl}\right)^{\theta}} \\
\mathcal{Q}_i(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{q}}) &\stackrel{\text{def}}{=} \left(\frac{\hat{\overline{R}}}{\widehat{H}_i}\right)\left(\frac{\sum_{n\in\mathcal{L}} \pi_{in}^0 w_n^0 \hat{\pi}_{in} \hat{w}_n}{\sum_{k\in\mathcal{L}} \pi_{ik}^0 w_k^0}\right).
\end{aligned}
\tag{24}
$$

Model B Algorithm:

1. s = 0
2. $\varepsilon = \text{tolerance} + 1$
3. $\tilde{\boldsymbol{w}}^0 = \tilde{\boldsymbol{q}}^0 = \vec{1}$
4. **while** $\varepsilon >$ tolerance **do**
5. $\quad \tilde{\boldsymbol{q}}^{s+1} = \left(1-\kappa_q\right)\tilde{\boldsymbol{q}}^s + \kappa_q \boldsymbol{\mathcal{Q}}(\tilde{\boldsymbol{w}}^s, \tilde{\boldsymbol{q}}^s)$
6. $\quad \tilde{\boldsymbol{w}}^{s+1} = \tilde{\boldsymbol{w}}^s + \kappa_w \boldsymbol{\mathcal{Z}}(\tilde{\boldsymbol{w}}^s, \tilde{\boldsymbol{q}}^s)$
7. $\quad \varepsilon = \max\{|\boldsymbol{\mathcal{Z}}(\tilde{\boldsymbol{w}}^s, \tilde{\boldsymbol{q}}^s)|\}$
8. $\quad$ s = s + 1
9. **end while**
10. **return** $\tilde{\boldsymbol{w}}^s, \tilde{\boldsymbol{q}}^s$

3. Data and Calibration

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In progress.
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4. Counterfactual Exercises

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4.1. Local Productivity Shock

Figure 1: Local Productivity Shock, Simple QSM, $\widehat{\boldsymbol{A}}$

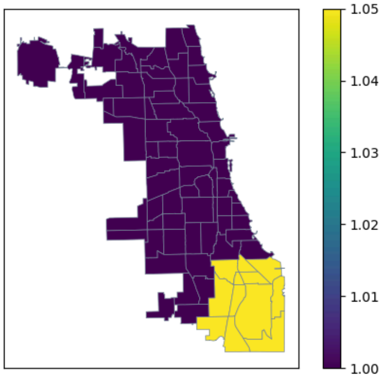

Figure 2: Local Productivity Shock, Simple QSM, $\hat{\boldsymbol{w}}$

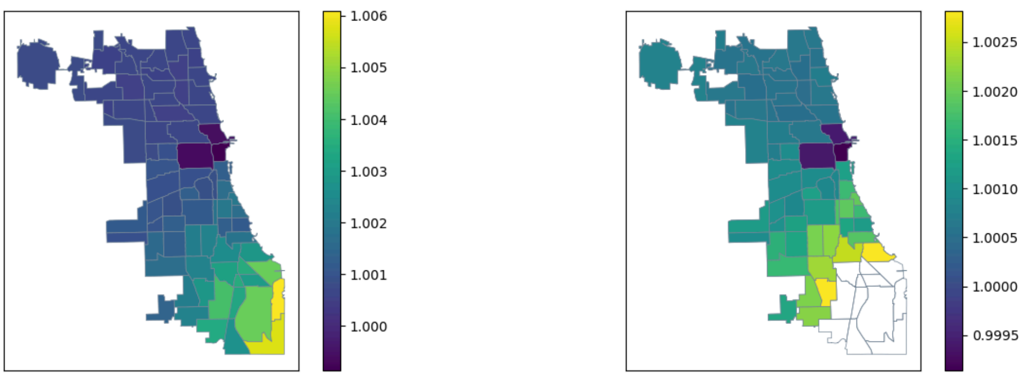

Figure 3: Local Productivity Shock, Simple QSM, $\hat{\boldsymbol{w}}$