A Simple Commuting Model of Chicago

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Hog Butcher for the World,
Tool Maker, Stacker of Wheat,
Player with Railroads and the Nation's Freight Handler;
Stormy, husky, brawling,
City of the Big Shoulders...

Come and show me another city with lifted head singing
so proud to be alive and coarse and strong and cunning.

— Carl Sandburg

Abstract

This repo is intended to demonstrate the basics of conducting economics research with quantitative spatial models. I calibrate a simple quantitative spatial model of Chicago. I conduct various counterfactual exercises.

1. Introduction

2. Model

I begin with a simple model of commuting to demonstrate the basic mechanics of a common form of quantitative spatial model. I then extend this model to include other relevant features of the economy.

2.1. A Simple Model (Model A)

Chicago is comprised of discrete neighborhoods $i, n, k \in \mathcal{L}$. Each location i has a mass of H_i of residents.

2.1.1. Workers

Each agent inelastically supplies one unit of labor. An agent ω residing in location i and working in location n receives indirect[1] utility $\mathcal{U}_{ni}$, where

$$\mathcal{U}_{in\omega} = \left[\frac{w_n}{\kappa_{in}}\right]\varepsilon_{in\omega}. \tag{1}$$

w_n is the wage paid in location n. κ_{in} is a commuting cost of the iceberg form in the units of utility. $\varepsilon_{in\omega}$ is an idiosyncratic preference shock with a Fréchet distribution. The cumulative distribution function of $\varepsilon_{in\omega}$ is given by $F(\varepsilon_{in\omega}) = \exp\left(\varepsilon_{in\omega}^{-\theta}\right)$. θ governs the dispersion of this preference shock.

A worker ω in location i chooses the workplace that maximizes their indirect utility:

$$n^*_{i\omega} \overset{\text{def}}{=} \arg\max_{n \in \mathcal{L}} \mathcal{U}_{in\omega}. \tag{2}$$

[1] I omit the consumer's maximization problem for parsimony.

Since workers differ only in their draws of $\{\varepsilon_{in\omega}\}_{i,n\in\mathcal{L}}$ of preference shocks, we can drop the ω subscript in what follows. The Fréchet distributed preference shock implies

$$\begin{aligned} \pi_{in} &\overset{\text{def}}{=} \mathbb{P}\{n_i^* = n\} = \left[\frac{w_n}{\kappa_{in}}\right]^\theta \Phi_i^{-1}, \\ \text{where } \Phi_i &\overset{\text{def}}{=} \sum_{k\in\mathcal{L}} \left[\frac{w_k}{\kappa_{ik}}\right]^\theta. \end{aligned} \tag{3}$$

2.1.2. Firms

A unit mass of firms in each neighborhood produce a numeraire with the technology

$$Y_n = A_n L_n^\alpha \tag{4}$$

and pay workers their marginal product.[2] Accordingly, the wage in neighborhood n is given by

$$w_n = \alpha A_n L_n^{\alpha-1}. \tag{5}$$

2.1.3. Commuting Equilibrium

For the commuting market to clear, labor demand in location n must equal labor supply to location n across all residential locations i:

$$L_n = \sum_{i\in\mathcal{L}} \pi_{in} H_i. \tag{6}$$

We can substitute Equation 3 and Equation 5 into this expression to obtain the equilibrium characterization:

$$\underbrace{\left[\frac{\alpha A_n}{w_n}\right]^{\frac{1}{1-\alpha}}}_{\text{Labor Demand}} = \underbrace{\sum_{i\in\mathcal{L}} \left[\frac{w_n}{\kappa_{in}}\right]^\theta \Phi_i^{-1} H_i.}_{\text{Labor Supply}} \tag{7}$$

2.1.4. Counterfactual Equilibria

We consider a baseline equilibrium $\{\boldsymbol{w}^0, \boldsymbol{\pi}^0\}$ for parameters $\{\boldsymbol{A}^0, \boldsymbol{\kappa}^0, \boldsymbol{H}^0\}$ and a counterfactual equilibrium $\{\boldsymbol{w}', \boldsymbol{\pi}'\}$ for parameters $\{\boldsymbol{A}', \boldsymbol{\kappa}', \boldsymbol{H}'\}$. We denote proportional changes with hats, e.g.,

$$\hat{w}_n = \frac{w_n'}{w_n^0} \Longrightarrow w_n^0 \hat{w}_n = w_n'. \tag{8}$$

We start by expressing the market clearing condition for the counterfactual equilibrium and then substitute in Equation 5:

$$\begin{aligned} L_n^0 \hat{L}_n &= \left(\sum_{i\in\mathcal{L}} \left(\pi_{in}^0 H_i^0\right)\left(\hat{\pi}_{in} \widehat{H}_i\right) \right) \\ \Longrightarrow \left[\frac{\alpha \hat{A}_n}{\hat{w}_n}\right]^{\frac{1}{1-\alpha}} &= \frac{\sum_{i\in\mathcal{L}} \left(\pi_{in}^0 H_i^0\right)\left(\hat{\pi}_{in} \widehat{H}_i\right)}{L_n^0}. \end{aligned} \tag{9}$$

[2] Again, I omit details of market structure for parsimony. I do not model trade in goods.

We can use Equation 3 to write

$$\begin{aligned}\hat{\pi}_{in} &= \left[\frac{\hat{w}_n}{\hat{\kappa}_{in}}\right]^{\theta} \hat{\Phi}_i^{-1}, \\ \text{where } \hat{\Phi}_i &= \sum_{k\in\mathcal{L}} \pi_{ik}^0 \left[\frac{\hat{w}_k}{\hat{\kappa}_{ik}}\right]^{\theta}.\end{aligned} \tag{10}$$

The substantive piece of this expression is $\hat{\Phi}_i$. We derive it below:

$$\hat{\Phi}_i = \frac{\sum_{k\in\mathcal{L}} \left[\frac{w_k^0}{\kappa_{ik}^0}\right]^{\theta} \left[\frac{\hat{w}_k}{\hat{\kappa}_{ik}}\right]^{\theta}}{\sum_{l\in\mathcal{L}} \left[\frac{w_l^0}{\kappa_{il}^0}\right]^{\theta}} = \sum_{k\in\mathcal{L}} \pi_{ik}^0 \left[\frac{\hat{w}_k}{\hat{\kappa}_{ik}}\right]^{\theta}, \tag{11}$$

where we have used Equation 3 to substitute in for π_{ik}^0. We now combine Equation 9 and Equation 10 to obtain

$$\left[\frac{\alpha \hat{A}_n}{\hat{w}_n}\right]^{\frac{1}{1-\alpha}} = \left[\sum_{i\in\mathcal{L}} \frac{\pi_{in}^0 H_i^0 \widehat{H}_i \left[\frac{\hat{w}_n}{\hat{\kappa}_{in}}\right]^{\theta}}{\sum_{k\in\mathcal{L}} \pi_{ik}^0 \left[\frac{\hat{w}_k}{\hat{\kappa}_{ik}}\right]^{\theta}}\right] \frac{1}{L_n^0}. \tag{12}$$

What does this representation get us? If we express a counterfactual as a set of proportional changes to the parameter values $\left\{\widehat{\boldsymbol{A}}, \hat{\boldsymbol{\kappa}}, \widehat{\boldsymbol{H}}\right\}$, then we only need data on commuting probabilities $\boldsymbol{\pi}^0$, workplace population $\boldsymbol{L}^0$, and residential population $\boldsymbol{H}^0$ in the initial equilibrium to solve for the proportional changes in wages $\hat{\boldsymbol{w}}$ (using Equation 12) and commuting probabilities $\hat{\boldsymbol{\pi}}$ (using Equation 10).

Inspired by this representation, we define

$$\mathcal{Z}_n(\tilde{\boldsymbol{w}}) \stackrel{\text{def}}{=} \left[\frac{\alpha \hat{A}_n}{\tilde{w}_n}\right]^{\frac{1}{1-\alpha}} - \left[\sum_{i\in\mathcal{L}} \frac{\pi_{in}^0 H_i^0 \widehat{H}_i \left[\frac{\tilde{w}_n}{\hat{\kappa}_{in}}\right]^{\theta}}{\sum_{k\in\mathcal{L}} \pi_{ik}^0 \left[\frac{\tilde{w}_k}{\hat{\kappa}_{ik}}\right]^{\theta}}\right] \frac{1}{L_n^0}. \tag{13}$$

2.2. A Richer Model (Model B)

`In progress.`

3. Data and Calibration

`In progress.`

4. Counterfactual Exercises

`In progress.`