Quantitative Spatial Models in Economics: A Simple Commuting Model of Chicago

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`This is a live document and subject to change.`

Hog Butcher for the World,
Tool Maker, Stacker of Wheat,
Player with Railroads and the Nation's Freight Handler;
Stormy, husky, brawling,
City of the Big Shoulders…

Come and show me another city with lifted head singing
so proud to be alive and coarse and strong and cunning.

— Carl Sandburg

Abstract

This repo is intended to demonstrate the basics of conducting economics research with quantitative spatial models. I derive and calibrate a simple quantitative spatial model of Chicago and conduct two counterfactual exercises. I then repeat this process for a richer model and compare the results.

1. Introduction

`In progress.`

The models presented below might strike you as restrictive and unrealistic. I have opted for an especially simple pair of models to demonstrate the basic mechanics of quantitative spatial models. This document will hopefully make the richer models of the literature more accessible. All quantitative spatial models (indeed, all economic models) necessarily abstract from certain features of reality. Results must always be interpreted in light of a researcher's modeling choices and the appropriateness of these choices for the research question.

2. Model

I begin with a simple model of commuting to demonstrate the basic mechanics of a common form of quantitative spatial model. I then extend this model to include other relevant features of the economy.

2.1. A Simple Model (Model A)

Chicago is comprised of discrete neighborhoods $i, n, k, l \in \mathcal{L}$. Each location i has a fixed mass R_i of residents.

2.1.1. Workers

Each agent inelastically supplies one unit of labor. An agent ω residing in location i and working in location n receives indirect[1] utility $\mathcal{U}_{ni}$, where

$$\mathcal{U}_{in\omega} = \left(\frac{w_n}{\kappa_{in}}\right) b_{in\omega}. \tag{1}$$

w_n is the wage paid in location n. κ_{in} is a commuting cost of the iceberg form in the units of utility. $b_{in\omega}$ is an idiosyncratic preference shock with a Fréchet distribution. The cumulative distribution function of $b_{in\omega}$ is given by $F_{in}(b_{in\omega}) = \exp\left(b_{in\omega}^{-\theta}\right)$. θ governs the dispersion of this preference shock.

A worker ω in location i chooses the workplace that maximizes their indirect utility:

$$n_{i\omega}^* \stackrel{\text{def}}{=} \arg\max_{n \in \mathcal{L}} \mathcal{U}_{in\omega}. \tag{2}$$

Since workers differ only in their draws of $\{b_{in\omega}\}_{i,n \in \mathcal{L}}$ of preference shocks, we can drop the ω subscript in what follows. The Fréchet-distributed preference shock implies

$$\begin{aligned} \pi_{in \mid i} &\stackrel{\text{def}}{=} \mathbb{P}\{n_i^* = n\} = \varphi_{in} \Phi_i^{-1}, \\ \text{where} \quad \varphi_{in} &\stackrel{\text{def}}{=} \left(\frac{w_n}{\kappa_{in}}\right)^{\theta} \\ \text{and } \Phi_i &\stackrel{\text{def}}{=} \sum_{k \in \mathcal{L}} \varphi_{ik}. \end{aligned} \tag{3}$$

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Pending a citation on discrete choice magic.
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2.1.2. Firms

A unit mass of perfectly competitive firms in each neighborhood produce a freely traded final good priced at 1 with the technology

$$Y_n = \tilde{A}_n L_n^{\beta} K_n^{1-\beta}, \tag{4}$$

where $\tilde{A}_n$ is a productivity parameter, L_n is labor input, and K_n is capital input. We assume that each neighborhood has a fixed stock of immobile capital $\overline{K}_n$, so it suffices to consider the production function

$$Y_n = A_n L_n^{\beta}, \tag{5}$$

where $A_n = \tilde{A}_n \overline{K}_n^{1-\beta}$. These perfectly competitive firms will pay workers their marginal product; wage and labor demand in neighborhood n are given by

$$\begin{aligned} w_n &= \beta A_n L_n^{\beta-1} \\ \Longrightarrow L_n &= \left(\frac{\beta A_n}{w_n}\right)^{\frac{1}{1-\beta}}. \end{aligned} \tag{6}$$

[1] I omit the subproblem of utility maximization given location choice for parsimony. Model B will explicitly discuss this subproblem, which nests the subproblem of utility maximization in this model.

We assume that capital owners spend all their rental income locally so that goods market clearing still holds. This model shuts down a potentially important margin of adjustment: changing capital stocks. Again, results must always be interpreted in light of such modeling choices and their appropriateness for the research question.

2.1.3. Commuting Equilibrium

For the commuting market to clear, labor demand in location n must equal labor supply to location n across all residential locations i:

$$L_n = \sum_{i \in \mathcal{L}} \pi_{in \mid i} R_i. \tag{7}$$

We can substitute Equation 3 and Equation 6 into this expression to obtain an equilibrium characterization:

$$\underbrace{\left(\frac{\beta A_n}{w_n}\right)^{\frac{1}{1-\beta}}}_{\text{Labor Demand}} = \underbrace{\sum_{i \in \mathcal{L}} \varphi_{in} \Phi_i^{-1} R_i.}_{\text{Labor Supply}} \tag{8}$$

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This section does not discuss the existence and uniqueness of the equilibrium, nor does
it discuss welfare. I will add sections on these topics in the future. As a note of
caution, we cannot compare welfare between the two models, given the different utility
functions.
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2.1.4. Counterfactual Equilibria

I will denote the vector-collection of a variable x_i over all locations with boldface: $\{x_i\}_{i \in \mathcal{L}} \stackrel{\text{def}}{=} \boldsymbol{x}$. We consider a baseline equilibrium $\{\boldsymbol{w}^0, \boldsymbol{\pi}^0\}$ for parameters $\{\boldsymbol{A}^0, \boldsymbol{\kappa}^0, \boldsymbol{R}^0\}$ and a counterfactual equilibrium $\{\boldsymbol{w}', \boldsymbol{\pi}'\}$ for parameters $\{\boldsymbol{A}', \boldsymbol{\kappa}', \boldsymbol{R}'\}$. We denote proportional changes with hats, e.g.,

$$\hat{w}_n = \frac{w_n'}{w_n^0} \Longrightarrow w_n^0 \hat{w}_n = w_n'. \tag{9}$$

This representation leads us to "exact hat algebra," a popular method to model and summarize counterfactual equilibria. We start by expressing the market clearing condition for the counterfactual equilibrium and then substitute in Equation 6:

$$\begin{aligned} L_n^0 \hat{L}_n &= \left(\sum_{i \in \mathcal{L}} \left(\pi_{in \mid i}^0 R_i^0\right)\left(\hat{\pi}_{in \mid i} \hat{R}_i\right)\right) \\ \Longrightarrow \left(\frac{\hat{A}_n}{\hat{w}_n}\right)^{\frac{1}{1-\beta}} &= \frac{\sum_{i \in \mathcal{L}} \left(\pi_{in \mid i}^0 R_i^0\right)\left(\hat{\pi}_{in \mid i} \hat{R}_i\right)}{L_n^0}. \end{aligned} \tag{10}$$

We can use Equation 3 to write

$$
\begin{aligned}
\hat{\pi}_{in\,|\,i} &= \hat{\varphi}_{in}\hat{\Phi}_i^{-1}, \\
\text{where} \quad \hat{\varphi}_{in} &\stackrel{\text{def}}{=} \left(\frac{\hat{w}_n}{\hat{\kappa}_{in}}\right)^{\theta} \\
\text{and } \hat{\Phi}_i &\stackrel{\text{def}}{=} \sum_{k\in\mathcal{L}} \pi^0_{ik\,|\,i}\hat{\varphi}_{ik}
\end{aligned}
\tag{11}
$$

The substantive piece of this expression is $\hat{\Phi}_i$. We derive it below:

$$
\hat{\Phi}_i = \frac{\sum_{k\in\mathcal{L}} \varphi^0_{ik}\hat{\varphi}_{ik}}{\sum_{l\in\mathcal{L}} \varphi^0_{il}} = \sum_{k\in\mathcal{L}} \pi^0_{ik\,|\,i}\hat{\varphi}_{ik}, \tag{12}
$$

where we have used Equation 3 to substitute in for $\pi^0_{ik\,|\,i}$ (see the portions colored red). We now combine Equation 10 and Equation 11 to obtain

$$
\left(\frac{\hat{A}_n}{\hat{w}_n}\right)^{\frac{1}{1-\beta}} = \left[\sum_{i\in\mathcal{L}} \frac{\pi^0_{in\,|\,i} R^0_i \hat{R}_i (\hat{w}_n/\hat{\kappa}_{in})^{\theta}}{\sum_{k\in\mathcal{L}} \pi^0_{ik\,|\,i}(\hat{w}_k/\hat{\kappa}_{ik})^{\theta}}\right]\frac{1}{L^0_n}. \tag{13}
$$

What does this characterization of a counterfactual equilibria buy us? If we express a counterfactual as a set of proportional changes to the parameter values $\left\{\widehat{\boldsymbol{A}}, \hat{\boldsymbol{\kappa}}, \widehat{\boldsymbol{R}}\right\}$, then we only need data on initial conditional commuting probabilities $\boldsymbol{\pi}^0$, workplace population $\boldsymbol{L}^0$, and residential population $\boldsymbol{R}^0$ to solve for the proportional changes in wages $\hat{\boldsymbol{w}}$ (using Equation 13) and conditional commuting probabilities $\hat{\boldsymbol{\pi}}$ (using Equation 11).

Inspired by this representation, we define

$$
\mathcal{Z}_n(\tilde{\boldsymbol{w}}) \stackrel{\text{def}}{=} \left(\frac{\hat{A}_n}{\tilde{w}_n}\right)^{\frac{1}{1-\beta}} - \left[\sum_{i\in\mathcal{L}} \frac{\pi^0_{in\,|\,i} R^0_i \hat{R}_i (\tilde{w}_n/\hat{\kappa}_{in})^{\theta}}{\sum_{k\in\mathcal{L}} \pi^0_{ik\,|\,i}(\tilde{w}_k/\hat{\kappa}_{ik})^{\theta}}\right]\frac{1}{L^0_n}. \tag{14}
$$

We can use this vector-valued function $\boldsymbol{\mathcal{Z}}(\tilde{\boldsymbol{w}})$ to compute the proportional changes in wages (and other equilibrium objects) in counterfactual equilibria. I provide pseudocode for this procedure below. I implement the algorithm in `analysis/model_A.ipynb`.

Model A Algorithm:

1. s = 0
2. $\varepsilon = \text{tolerance} + 1$
3. $\tilde{\boldsymbol{w}}^0 = \vec{1}$
4. **while** $\varepsilon > \text{tolerance}$ **do**
5. $\quad \tilde{\boldsymbol{w}}^{s+1} = \tilde{\boldsymbol{w}}^s + \kappa_w \boldsymbol{\mathcal{Z}}(\tilde{\boldsymbol{w}}^s)$
6. $\quad \varepsilon = \max\{|\boldsymbol{\mathcal{Z}}(\tilde{\boldsymbol{w}}^s)|\}$
7. $\quad$ s = s + 1
8. **end while**
9. **return** $\tilde{w}^s$

```
It would be appropriate to discuss zero flows here. I will do so in the future.
```

2.2. A Richer Model (Model B)

We now consider a model with a housing market and residential choice. The mass of agents is denoted $\overline{R}$. We no longer fix an agent's residential location.

2.2.1. Setup

Utility for an agent ω residing in location i and working in location n is given by

$$U_{in\omega} = \left(\frac{c_{in\omega}}{\alpha}\right)^{\alpha}\left(\frac{h_{in\omega}}{1-\alpha}\right)^{1-\alpha}\frac{b_{in\omega}}{\kappa_{in}} \tag{15}$$

where $c_{in\omega}$ is final good consumption, $h_{in\omega}$ is housing consumption, and $F_{in}(b_{in\omega}) = \exp\left(-B_{in}b_{in\omega}^{-\theta}\right)$. We've added a parameter B_{in} that governs average utility for agents that live in location i and work in location n. The Cobb-Douglas form of Equation 15 implies that agents spend a constant fraction α of their income on the final good and $(1-\alpha)$ on housing. The price of the final good is again 1, and we denote the price of housing in location i by q_i. Accordingly, indirect utility for an agent ω residing in location i and working in location n with wage w_n is given by

$$\mathcal{U}_{in\omega} = \left(\frac{\alpha w_n}{\alpha}\right)^{\alpha}\left(\frac{\alpha w_n}{q_i(1-\alpha)}\right)^{1-\alpha}\frac{b_{in\omega}}{\kappa_{in}} = \left(\frac{w_n q_i^{\alpha-1}}{\kappa_{in}}\right)b_{in\omega}. \tag{16}$$

A worker ω now chooses both a residence and workplace:

$$\{i,n\}_{\omega}^{*} \stackrel{\text{def}}{=} \arg\max_{i,n\in\mathcal{L}} \mathcal{U}_{in\omega}. \tag{17}$$

Similar to before, the Fréchet-distributed preference shock implies the following expression for the *unconditional* residential and commuting probability

$$\begin{aligned} \pi_{in} &\stackrel{\text{def}}{=} \mathbb{P}\{\{i,n\}^* = \{i,n\}\} = \varphi_{in}\Phi^{-1}, \\ \text{where} \quad \varphi_{in} &\stackrel{\text{def}}{=} B_{in}\left(\frac{w_n q_i^{\alpha-1}}{\kappa_{in}}\right)^{\theta} \\ \text{and} \quad \Phi &\stackrel{\text{def}}{=} \sum_{k\in\mathcal{L}}\sum_{l\in\mathcal{L}}\varphi_{kl}. \end{aligned} \tag{18}$$

In what follows, it will be useful to define the mass of residents in each location i

$$R_i \stackrel{\text{def}}{=} \sum_{n\in\mathcal{L}} \pi_{in}\overline{R}, \tag{19}$$

following the notation from Model A.

2.2.2. Housing Market

Each location i has a fixed stock of land available for rent H_i. Landlords face no costs and spend all of their rental income on the final good to ensure goods market clearing. Let $\overline{\nu}_i$ denote the average income of residents in location i. We can than express *aggregate* income for resident in location i

$$\overline{\nu}_i R_i = \sum_{n\in\mathcal{L}} \pi_{in} w_n \overline{R}. \tag{20}$$

Land market clearing implies that housing expenditure (given by utility maximization) must equal landlord income in neighborhood i:

$$\underbrace{(1-\alpha)\overline{\nu}_i R_i}_{\text{Housing Expenditure}} = \underbrace{H_i q_i}_{\text{Landlord Income}} \tag{21}$$

2.2.3. Firms

We maintain the same set of assumptions on the firm side as in Model A. This yields the wage equation and labor demand

$$\begin{aligned} w_n &= \beta A_n L_n^{\beta-1} \\ \implies L_n &= \left(\frac{\beta A_n}{w_n}\right)^{\frac{1}{1-\beta}}. \end{aligned} \tag{22}$$

2.2.4. Commuting Equilibrium

We now use the unconditional commuting probability in Equation 18 to define the commuting market clearing condition:

$$L_n = \sum_{i\in\mathcal{L}} \pi_{in}\overline{R}. \tag{23}$$

2.2.5. Counterfactual Equilibria

We proceed as in model A and derive the exact hat system.

$$\begin{aligned} \hat{\varphi}_{in} &= \hat{B}_{in}\left(\frac{\hat{w}_n \hat{q}_i^{\alpha-1}}{\hat{\kappa}_{in}}\right)^{\theta} \\ \hat{\pi}_{in} &= \frac{\hat{\varphi}_{in}}{\sum_{k\in\mathcal{L}}\sum_{l\in\mathcal{L}} \pi_{kl}^0 \hat{\varphi}_{kl}} \\ \hat{R}_i &= \frac{\overline{R}^0 \hat{\overline{R}}}{R_i^0} \sum_{n\in\mathcal{L}} \pi_{in}^0 \hat{\pi}_{in} \\ \left(\frac{\hat{A}_n}{\hat{w}_n}\right)^{\frac{1}{1-\beta}} &= \left(\frac{\overline{R}^0 \hat{\overline{R}}}{L_n^0}\right) \sum_{i\in\mathcal{L}} \pi_{in}^0 \hat{\pi}_{in} \\ \hat{\overline{\nu}}_i \hat{R}_i &= \hat{\overline{R}}\left(\frac{\sum_{n\in\mathcal{L}} \pi_{in}^0 w_n^0 \hat{\pi}_{in} \hat{w}_n}{\sum_{k\in\mathcal{L}} \pi_{ik}^0 w_k^0}\right) \\ \hat{q}_i &= \frac{\hat{\overline{\nu}}_i \widehat{R}_i}{\widehat{H}_i} \end{aligned} \tag{24}$$

We combine the expressions from above and define

$$
\begin{aligned}
\mathcal{Z}_n(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{q}}) &\stackrel{\text{def}}{=} \left(\frac{\hat{A}_n}{\hat{w}_n}\right)^{\frac{1}{1-\beta}} - \left(\frac{\overline{R}^0 \hat{\overline{R}}}{L_n^0}\right) \sum_{i \in \mathcal{L}} \frac{\pi_{in}^0 \hat{B}_{in} \left(\hat{w}_n \hat{q}_i^{\alpha-1} / \hat{\kappa}_{in}\right)^{\theta}}{\sum_{k \in \mathcal{L}} \sum_{l \in \mathcal{L}} \pi_{kl}^0 \hat{B}_{kl} \left(\hat{w}_l \hat{q}_k^{\alpha-1} / \hat{\kappa}_{kl}\right)^{\theta}} \\
\mathcal{Q}_i(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{q}}) &\stackrel{\text{def}}{=} \left(\frac{\hat{\overline{R}}}{\widehat{H}_i}\right) \left(\frac{\sum_{n \in \mathcal{L}} \pi_{in}^0 w_n^0 \hat{\pi}_{in} \hat{w}_n}{\sum_{k \in \mathcal{L}} \pi_{ik}^0 w_k^0}\right).
\end{aligned}
\tag{25}
$$

Model B Algorithm:

1. s = 0
2. $\varepsilon = \text{tolerance} + 1$
3. $\tilde{\boldsymbol{w}}^0 = \tilde{\boldsymbol{q}}^0 = \vec{1}$
4. **while** $\varepsilon >$ tolerance **do**
5. $\quad \tilde{\boldsymbol{q}}^{s+1} = \left(1 - \kappa_q\right) \tilde{\boldsymbol{q}}^s + \kappa_q \boldsymbol{\mathcal{Q}}(\tilde{\boldsymbol{w}}^s, \tilde{\boldsymbol{q}}^s)$
6. $\quad \tilde{\boldsymbol{w}}^{s+1} = \tilde{\boldsymbol{w}}^s + \kappa_w \boldsymbol{\mathcal{Z}}(\tilde{\boldsymbol{w}}^s, \tilde{\boldsymbol{q}}^s)$
7. $\quad \varepsilon = \max\{|\boldsymbol{\mathcal{Z}}(\tilde{\boldsymbol{w}}^s, \tilde{\boldsymbol{q}}^s)|\}$
8. $\quad$ s = s + 1
9. **end while**
10. **return** $\tilde{\boldsymbol{w}}^s, \tilde{\boldsymbol{q}}^s$

3. Data and Calibration

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I will discuss the data and parameter choices used to generate the results below. The
code to generate the data is available upon request and will be incorporated into a
future version of this repository.
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4. Counterfactual Exercises

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I plan to summarize equilibrium changes for a larger set of endogenous variables and a
discussion of the differences between the two models.
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I compare the equilibrium impact of two parameter shocks: a 5% increase in productivity in the Far Southeast Side and a 5% reduction in commuting costs from the Far Southeast to Chicago's employment core (the Loop, Near North Side, and Near West Side). For interpretation, it is important to note that the $\hat{\boldsymbol{w}}$ reports the changes in wages *paid* to agents working in a given location.

4.1. Local Productivity Shock

I plot the proportional changes in productivity in Figure 1. Productivity is unchanged outside of the Far Southeast.

Figure 1: Local Productivity Shock, $\widehat{\boldsymbol{A}}$

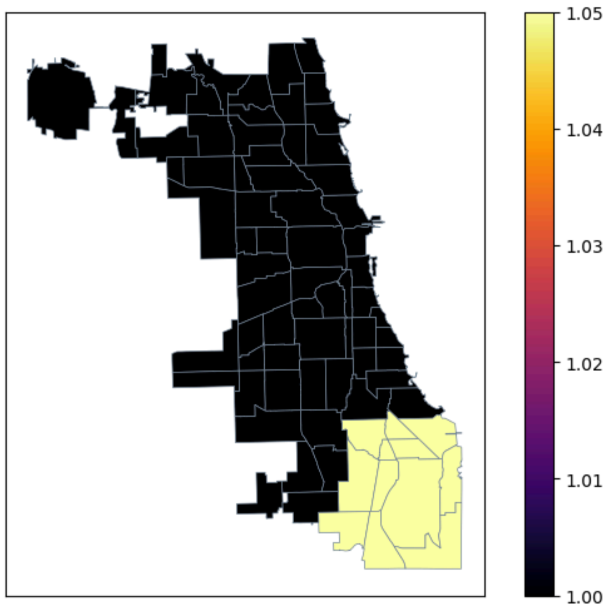


Figure 2: Local Productivity Shock, $\hat{\boldsymbol{w}}$

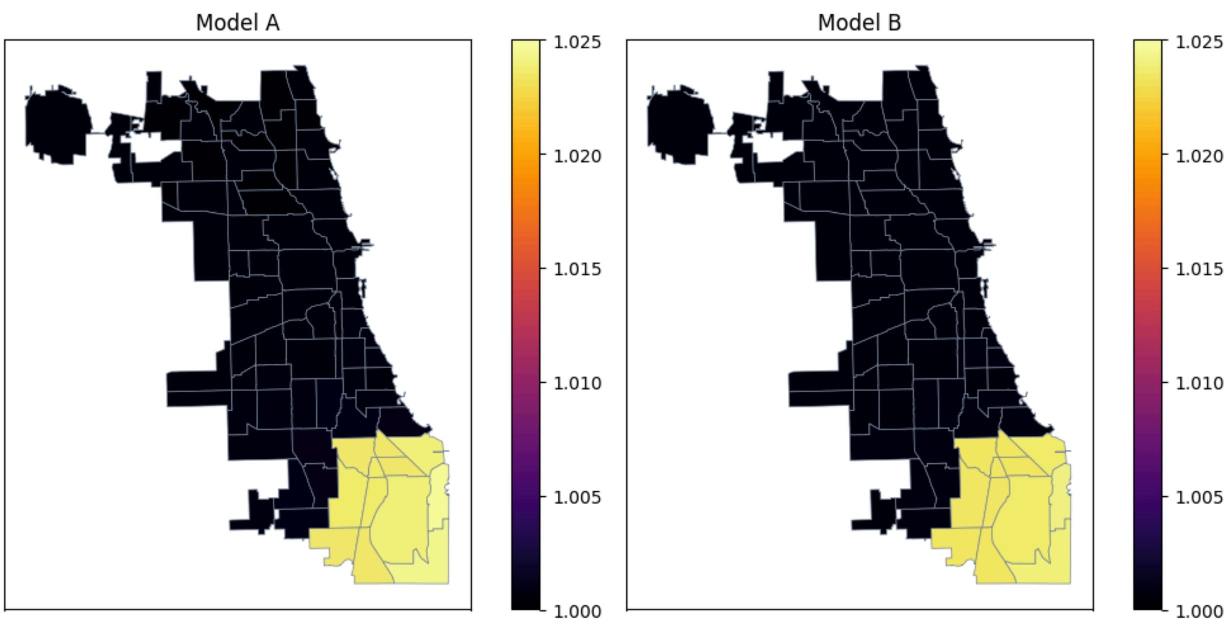


Figure 3: Local Productivity Shock, $\hat{\boldsymbol{w}}$

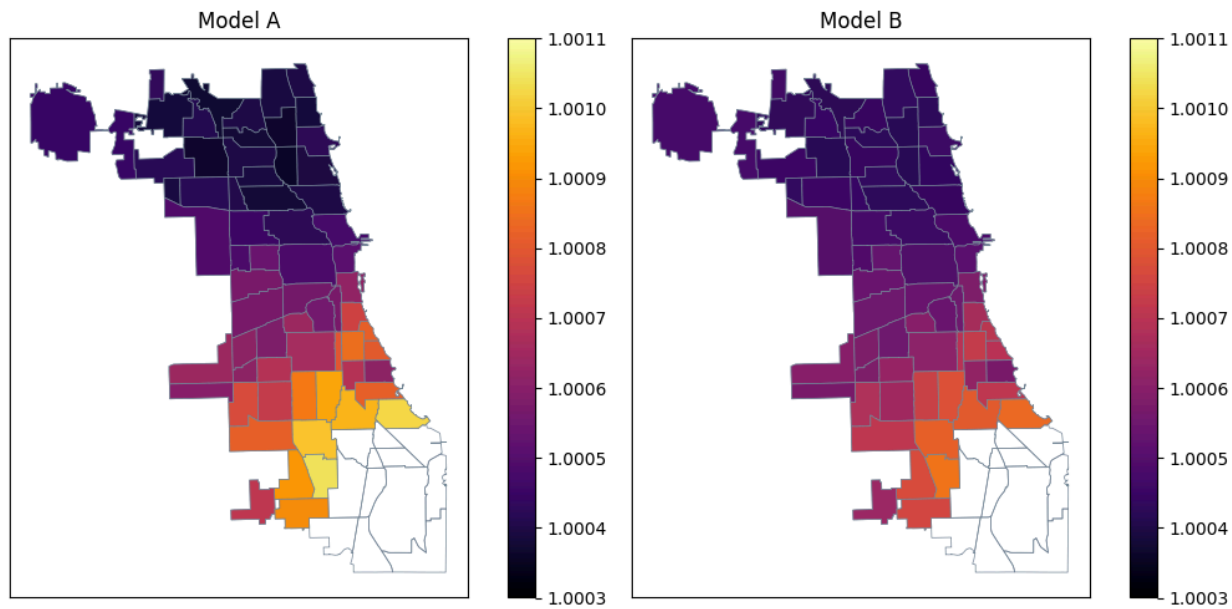
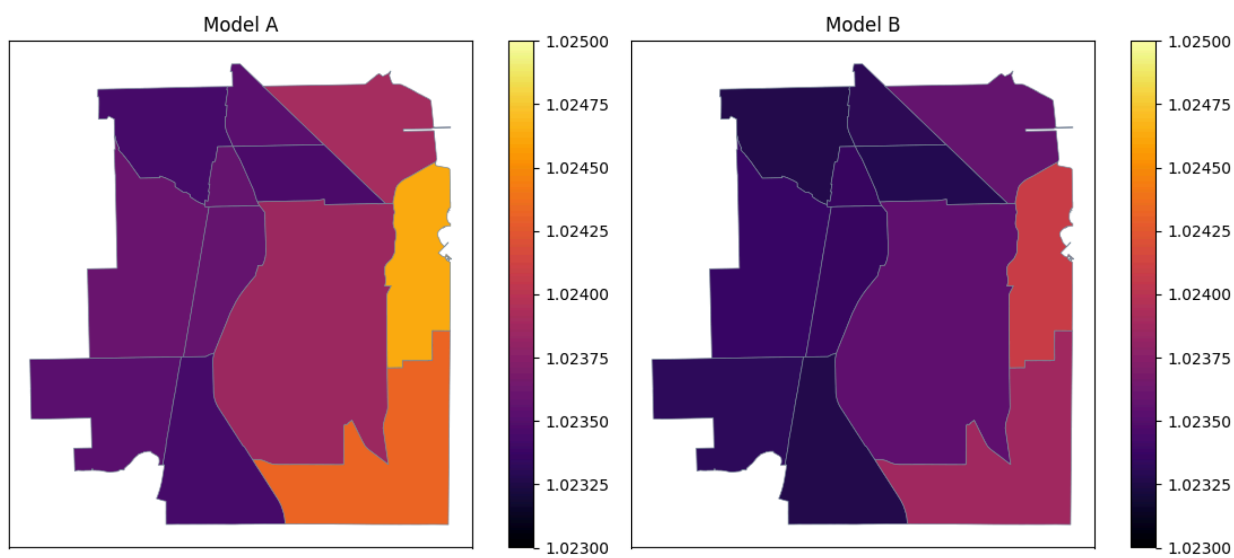


Figure 4: Local Productivity Shock, $\hat{\boldsymbol{w}}$



4.2. Local Productivity Shock

I plot the proportional changes in commuting costs from the Far Southeast in Figure 5. Commuting costs are unchanged outside of the Far Southeast.

Figure 5: Commuting Cost Shock, $\widehat{\boldsymbol{A}}$

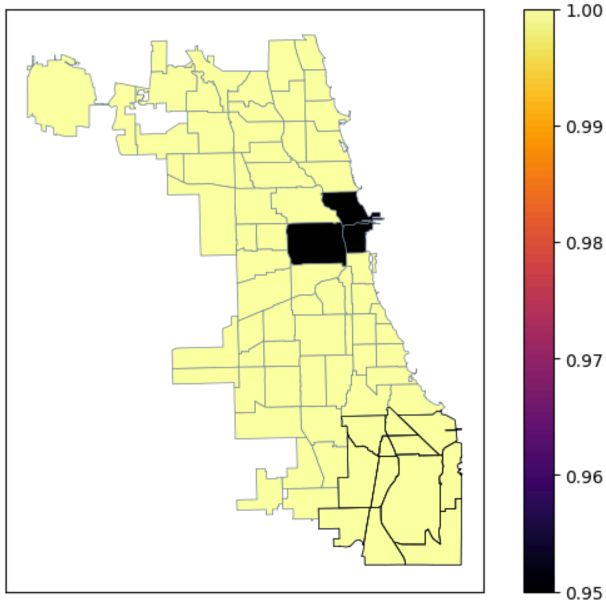


Figure 6: Commuting Cost Shock, $\hat{\boldsymbol{w}}$

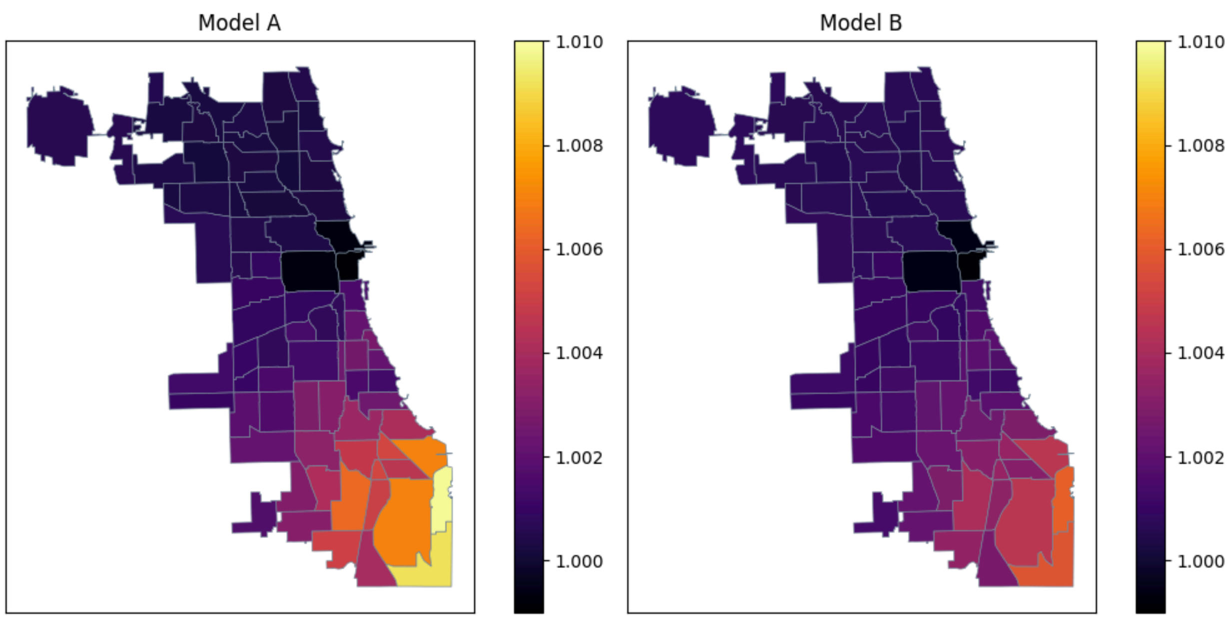


Figure 7: Commuting Cost Shock, $\hat{w}$

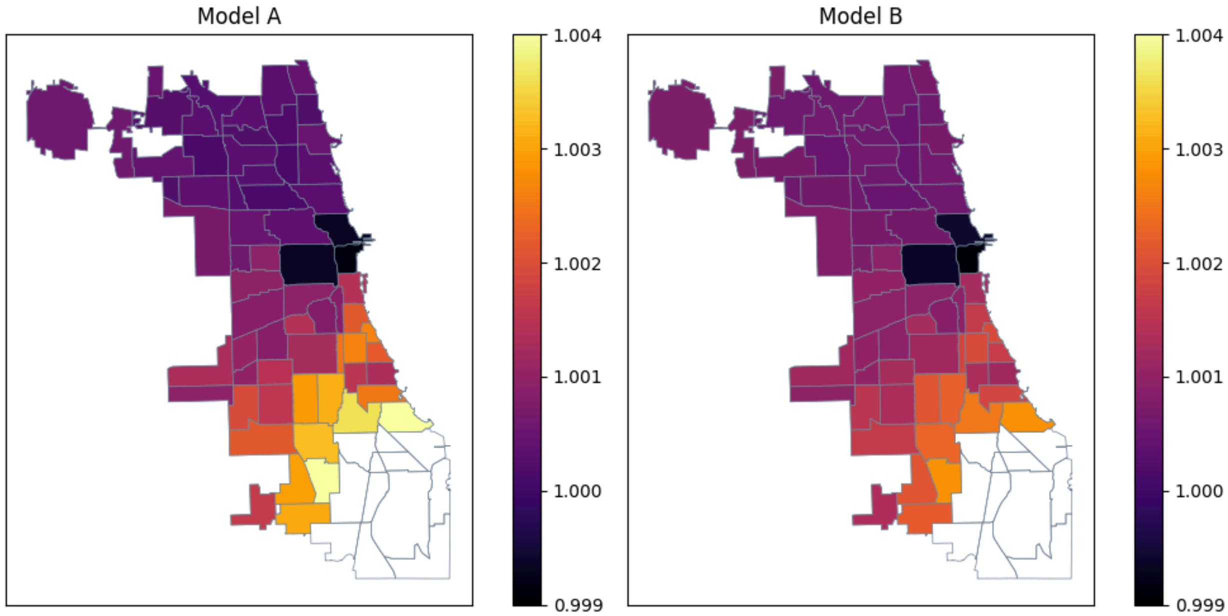


Figure 8: Commuting Cost Shock, $\hat{w}$

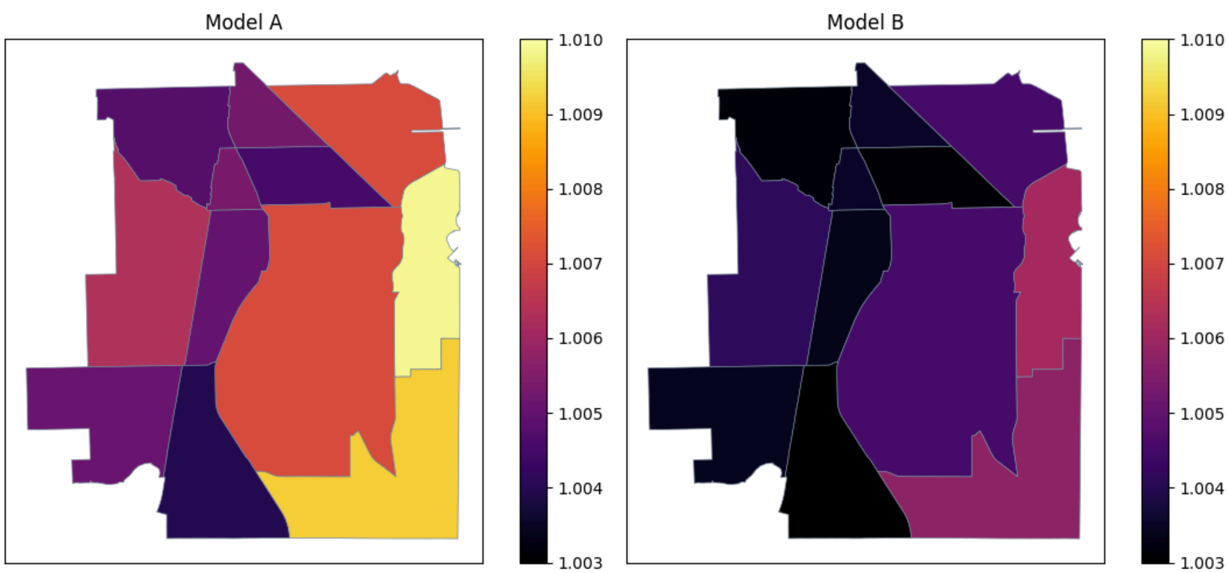


Figure 9: Commuting Cost Shock, $\hat{w}$