Stuff to fix

- Jeg kommer med flere påstande som jeg ikke er sikker på har hold i virkeligheden.

Introduction

With-profit insurance contracts are to this day one of the most popular life insurance contracts. They arose as a natural way to distribute the systematic surplus that develops due to the prudent assumptions on which the contract is made. In recent years, sensible questions accompanied by a lot of attention have been aimed at the surplus, to name a few; is it distributed fairly? what is the risk carried by the equity? how should it be invested? One might look for answers in the existing literature e.g. Møller and Steffensen (2007), Norberg (1999), Steffensen (2000) or Steffensen (2001), where partial differential equations are used to describe the prospective second order reserve and its interplay with surplus. While these PDE's provide a conceptually powerful tool, they are limited to simple market dynamics, and they do not provide realistic models for long term financial markets. Norberg (1999) considers the development of the surplus in a financial Markov chain environment, allowing for great model flexibility but even still, an important element is completely neglected: the human element.

Insurance companies are governed by humans, and the decisions they make have an influence on the portfolio of policies - in particular concerning surplus and dividends. In a with-profit insurance contract many quantities are fixed at initialisation of the policy, but the rate at which dividends are accrued is not. The insurance company has a certain degree of freedom when it comes to the distribution of surplus, and the actions that have an influence on the insurance contracts are the so-called Management Actions. From a mathematical point of view, they pose a problem as they depend on the entire history of the portfolio of policies, making it difficult to calculate prospective reserves. If we want to take a glance into the crystal ball of liabilities, taking Future Management Actions (FMA's) into account, we need to embrace it's retrospective nature.

In this paper we derive a retrospective differential equation for the expected savings account and surplus, in a model where dividends are spent on increasing benefits.

Prospective vs. Retrospective

When incorporating human decisions into the projection of balances and benefits in life insurance, we need to embrace the fact that these decisions are based on the past. How to embrace the retrospective nature, is the main contribution of this paper.

- Something about Monte-carlo method
- Something about FMA's - perhaps an example?

Setup

We consider the classic multi-state life insurance setup, comprised of a state process Z denoting the state of the policy in a finite state space $\mathcal{J} = \{0, 1, ..., J\}$. As all dividends are spent on increasing benefits B_2, the savings account at time t is the technical value of all future payments guaranteed at time t... Måske skal vi motivere dynamikken af X mere grundigt. The amount by which the savings surpass the first order reserve, is spent on B_2. The payment process $B(t)$ is thus given by

$$dB(t) = b_1^{Z(t)}(t, X(t))dt + \sum_{k \neq Z(t-)} b^{Z(t-)k}(t, X(t))dN^k(t)$$

where

$$b^j(t, x) = b_1^j(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^j(t), \qquad b^{jg}(t, x) = b_1^{jg}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^{jg}(t).$$

Dynamics of X

$$\begin{aligned} dX(t) =& r^*(t)X(t)dt + \delta^{Z(t)}(t, X(t), Y(t))dt - \sum_{g \neq Z(t-)} \rho^{Z(t-)g}(t, X(t-))dt \\ &- b^{Z(t)}(t, X(t))dt \\ &- \sum_{g \neq Z(t-)} \Big(b^{Z(t-)g}(t, X(t-)) + \chi^{Z(t-)g}(t, X(t-)) - X(t-) \Big) \mu^{Z(t)g}(t)dt \\ &+ \sum_{g \neq Z(t-)} \Big(\chi^{Z(t-)g}(t, X(t-)) - X(t-) \Big) dN^g(t), \end{aligned}$$

and

$$dY(t) = Y(t)\frac{dS(t)}{S(t)} - \delta^{Z(t)}(t, X(t), Y(t)) + (r(t) - r^*(t))X(t) + \sum_{g \neq Z(t-)} \rho^{Z(t)g}(t, X(t)),$$

where

$$\begin{aligned}
\rho^{jg}(t,x) =& (b^{jg}(t,x) + \chi^{jg}(t,x) - x)(\mu^{*jg}(t) - \mu^{jg}(t)) \\
\chi^{jg}(t,x) =& V_1^{g*}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{g*}(t),
\end{aligned}$$

$$\delta^j(t,x,y) = \delta_1^j(t) + \delta_2^j(t)x + \delta_3^j(t)y + \delta_4^j(t)xy. \tag{1}$$

- Definition of X and Y
- Deterministic second order basis, and discussion regarding simulation.
- remark on continuity of $b^i(t)$
- model limitations.
- Even though FMA's are one of the main reasons for considering the savings account, they are hidden in the dividend and surplus investment strategy.

Let W be some possibly multidimensional process with Z-dependent dynamics

$$dW(s) = g(s, Z(s), W(s))ds + \sum_{k \neq Z(s-)} h(s, Z(s-), k, W(s-))dN^k(s),$$

for g and h functions that are linear in all elements of W. This multidemsional process can for instance represent the savings and surplus

$$W(s) = \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix}.$$

In practice, the surplus account is shared among policyholders, corresponding to $W \in \mathbb{R}^{N+1}$ for N policies with a state process Z on a state space of size $\#\{\mathcal{J}\}^N$; one for each combination of all policy states. There are several ways to reduce the dimensionality of the problem, making it computationally tractable.

One Active State

We consider a simple model where the expected future savings are described by an easily derived differential equation. The model consists of n inactive states where there are no payments, and one active state with continuous dynamics g which, in this setting, may be non-linear. Denote by 0 the active state. On transition to any one of the inactive states, the surplus and savings are nullified. We need not specify what happens to the surplus and savings on a transition - they may be paid out to the customer or the insurance company, or any combination of the two - the only important requirement is that they are zero in all inactive states. The eradication of surplus and savings on transition corresponds to the relation $h_x(t,0,j,x,y)+h_y(t,0,j,x,y) = -x-y$, for $j=1,...,n$. The survival model with and without surrender options are special cases of this model. The dynamics of X and Y are

$$\begin{aligned} dX(s) =& \mathbb{1}_{\{Z(s-)=0\}} g_x(s,0,X(s),Y(s))ds - \sum_{h=1}^{n} X(s-)dN^h(s) \\ dY(s) =& \mathbb{1}_{\{Z(s-)=0\}} g_y(s,0,X(s),Y(s))ds - \sum_{h=1}^{n} Y(s-)dN^h(s). \end{aligned}$$

Let $W(s) = (X(s),Y(s))^T$, and denote by T_1 the time of the first jump. For the deterministic function W_a that solves

$$W_a(t) = \int_0^t g(s,0,W_a(s))ds,$$

we see that

$$\hat{W}(t) := \mathrm{E}[W(t)|Z(0)=0] = \mathrm{E}[\mathbb{1}_{\{t<T_1\}} W_a(t)|Z(0)=0] = p_{00}(0,t)W_a(t),$$

which comes at no surprise. In this case we know the past and present values of W given the current state of Z, so the only stochastic element pertains to the state of the policy at time t. By differentiating w.r.t. t, and applying Kolmogorov's forward differential equation, we get the following forward differential equation for $\hat{W}$,

$$\begin{aligned} \hat{W}(0) =& \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix}, \\ \frac{d}{dt}\hat{W}(t) =& p_{00}(0,t) g\left(s, 0, \frac{\hat{W}(t)}{p_{00}(0,t)}\right) - \frac{\hat{W}(t)}{p_{00}(0,t)} \sum_{k=1}^{n} \mu_{0k}(t). \end{aligned}$$

Even though it may seem very simple and perhaps even trivial, the model with one active state has great applicability.

Example With One Active State

If the benefits are identical after age 65, the states 0,1,3 and 4 can be lumped, as well as 2,5 and 6, thus creating a survival model. If the dynamics in two states are identical, they can be viewed as one. Life annuity at age 65.

$$\boxed{\text{0, alive}} \xrightarrow{\mu} \boxed{\text{1, dead}}$$

Figure 1: Life-Death model

Two Active States

When expanding to a model where there are two active states, and n inactive states, we need to use a different method to calculate $\hat{W}$, *if the active states are transient*. There is an important difference between the hierarchical model with two active states, and the transient model with two active states. In the model with one active state, we know the entire history of the policy, given that the policy is in the active state. When we introduce a second active state in the hierarchical model, we also know where the policy has been given the active state, but we do not know when it transitioned from one active state to the other. In order to calculate the expectation of the savings and surplus, we simply have to integrate over all possible transition times. If there are two transient states, there is an infinite amount of paths to any of the transient states, and for each possible path there is an infinite amount of possible jump times. To illustrate the naïve method of calculating expected savings and surplus in a hierarchical model, consider the model depicted in figure 2

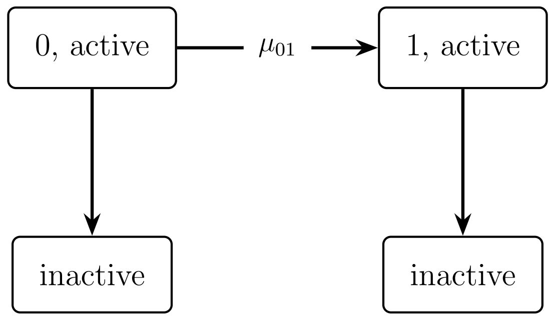


Figure 2: Two active state hierarchical model

In this model, there are two states for which the savings and surplus are non-zero; $Z(t) \in \{0,1\}$. As in the case with one active state, we know the value of $W(t)$ for $Z(t) = 0$, but for $Z(t) = 1$ we need to consider all possible transition times. Let T_1 be the time of the transition from 0 to 1. Define the quantity

$$W_0(t) = \int_0^t g(s, 0, W_0(s))ds,$$

characterizing the expected value of $W(t)$, given that $Z(t) = 0$. Similarly, W_1 characterizes the expected value of $W(t)$ given $Z(t) = 1$ and $dN_{01}(T_1) = 1$, if it solves

$$\begin{aligned} W_1(T_1, T_1) =& W_0(T_1) + h(T_1, 0, 1, W_0(T_1)), \\ W_1(T_1, t) =& \int_{T_1}^t g(s, 1, W_1(T_1, s))ds \end{aligned}$$

Lastly we define the density of T_1, given that $Z(t) = 1$ as

$$q(s,t) = \frac{p_{00}(0,s)p_{11}(s,t)}{p_{01}(0,t)}\mu_{01}(s).$$

Let T_1 be the time of the first jump, then

$$W(t) = \mathbb{1}_{\{Z(t)=0\}}W_0(t) + \mathbb{1}_{\{Z(t)=1\}}W_1(T_1, t)$$

Implying that

$$\mathrm{E}[W(t)] = p_{00}(0,t)\mathrm{E}[W(t)|Z(t) = 0] + p_{01}(0,t)\mathrm{E}[W(t)|Z(t) = 1].$$

Now let

$$\hat{W}_i(t) := \mathrm{E}[W(t)|Z(t) = i] = \begin{cases} W_0(t) & \text{for } Z(t) = 0 \\ \displaystyle\int_0^t q(s,t)W_1(s,t)ds & \text{for } Z(t) = 1 \\ 0 & \text{otherwise ,} \end{cases}$$

denote the expectation of W given the current state of the policy. When $Z(t) = 0$ all information about the history of the policy is known, and the value of W is deterministic. Conditioning on $Z(t) = 1$ does not provide full information about the history of the policy, as we do not know the time at which the transition from state 0 to state 1 was made. Therefore, to calculate $\hat{W}_1(t)$ we have to integrate over all possible transition times, weighted by the transition intensity given that a jump happened prior to t. Thus

$$\mathrm{E}[W(t)] = p_{00}(0,t)W_0(t) + p_{01}(0,t)\int_0^t q(s,t)W_1(s,t)ds.$$

We could extend this method to hierarchical models of arbitrary size. The basic principle is the same: given all information about the past, we can calculate the value of $W(t)$, and the expected past can be calculated for each possible state. These calculations very quickly become very extensive, as there are many high-dimensional integrals to calculate when the policy has made more than one transition. In general $\hat{W}$ can be calculated as

$$\mathrm{E}[W(t)] = \sum_{i\in\mathcal{P}} P(\text{path } i)\int_{(0,t]^{L_i}} W_i(t,\Theta_{L_i})dP_i(\Theta_{L_i}) \tag{2}$$

where $\mathcal{P}$ is the set of possible policy paths, L_i is the length of path i, Θ_{L_i} is an L_i-dimensional vector of jump-times, dP_i is the density of transition times for path i and $W_i(t,\Theta_{L_i})$ is the value of $W(t)$ given the path and transition times.

When the model is small and hierarchical, (**??**) provides a tractable method to calculate the expected savings and surplus, as there are few possible paths and they are relatively short. When the model is transient the problem explodes, as there are infinitely many paths for the policy to take.

n-state model

State-Wise Probability Weighted Reserve

Define

$$\tilde{X}^j(t) := \mathrm{E}_{Z(0)}[X(t)\mathbb{1}_{\{Z(t)=j\}}]$$

and note that

$$\mathrm{E}_{Z(0)}[X(t)\mathbb{1}_{\{Z(t)=j\}}] = \mathrm{E}_{Z(0)}[X(t)|Z(t)=j]p_{Z(0),j}(0,t), \tag{3}$$

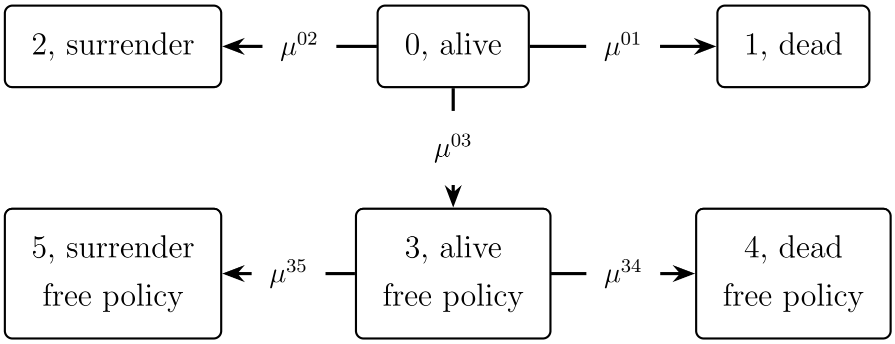
by the definition of conditional expectation. We can think of $\tilde{X}^j$ as the probability weighted state-wise reserves. The relation between $\tilde{X}^j$ and $\mathrm{E}[X(t)]$ is

$$\begin{aligned}
\mathrm{E}_{Z(0)}[X(t)] =& \mathrm{E}_{Z(0)}[\mathrm{E}_{Z(0)}[X(t)|Z(t)]] \\
=& \mathrm{E}_{Z(0)}\left[\sum_{j\in\mathcal{J}} \mathbb{1}_{\{Z(t)=j\}} \mathrm{E}_{Z(0)}[X(t)|Z(t)=j]\right] \\
=& \mathrm{E}_{Z(0)}\left[\sum_{j\in\mathcal{J}} \mathbb{1}_{\{Z(t)=j\}} \frac{\mathrm{E}_{Z(0)}[X(t)\mathbb{1}_{\{Z(t)=j\}}]}{p_{0j}(0,t)}\right] \\
=& \sum_{j\in\mathcal{J}} p_{0j}(0,t)\frac{\tilde{X}^j}{p_{0j}(0,t)} \\
=& \sum_{j\in\mathcal{J}} \tilde{X}^j.
\end{aligned}$$

Life-Death-Surrender

$$\begin{tikzcd}
\boxed{\text{2, surrender}} & \boxed{\text{0, alive}} \arrow[l, "\mu^{02}"'] \arrow[r, "\mu^{01}"] & \boxed{\text{1, dead}}
\end{tikzcd}$$

Life-Death-Surrender With Free Policy



Use of Savings account

Thoughts

- With-profit insurance! Expected reserve including accumulation of dividends.
- Refer to Norberg (1991)
  - Introduction and motivation - stochastic reserve, Monte Carlo method. A little comment on the fact that the problem is still hard to solve.
  - Life-death (simple analytic solution).
  - Life-death free policy (how to deal with extra states).
  - General model without duration.
  - Life-death-surrender free policy, including discussion of free policy factor.
  - Lost all trick works.
  - General model with duration dependence.
  - Inclusion of surplus. Use independence when dividend is assigned on discrete points in time.
- Deterministic intensities.
- General Hierarchical models do not need linearity. In general the variance increases as the number of states increase as the variance of the sum of transition times increases.

- Market dependent intensities - allowed when directly dependent on the market, making them deterministic. Or intensities that depend on the expected reserve - in a sense corresponding to intensities that depend on the group of similar policies.
- We are only concerned with the reserve.
- Maybe we should use a different wording? **Savings**/stash/backlog/accumulation/hoard/reservoir instead of reserve, to distinguish between the Danish words for "reserve" and "depot"
- One could imagine that information about the jump time could be partially deduced from the intensities, thus almost allowing for non-linearity. Consider case where $\mu_{01}(t) = \kappa \mathbb{1}_{\{t\in(c_1,c_2]\}}$ for very small $|c_2 - c_1|$ and very large κ, providing almost perfect information about the jump time, whereby non-linearity in $g(s, 1, W(s))$ would be allowed for.

Proof of q

References

Kristian Buchardt, Thomas Møller, and Kristian Schmidt. Cash flows and policyholder behaviour in the semi-markov life insurance setup. *Scandinavian Actuarial Journal*, 2015(8), 2015. ISSN 03461238. URL `http://search.proquest.com/docview/1718994569/`.

Thomas Møller and Mogens Steffensen. *Market-valuation methods in life and pension insurance*. International Series on Actuarial Science. Cambridge University Press, Cambridge, 2007. ISBN 0521868777.

Ragnar Norberg. Reserves in life and pension insurance. *Scand. Actuar. J. 1*, pages 3–24, 1991.

Ragnar Norberg. A theory of bonus in life insurance. *Finance and Stochastics*, 3(4):373–390, 1999. ISSN 0949-2984.

Mogens Steffensen. A no arbitrage approach to thiele's differential equation. *Insurance, Mathematics and Economics*, 27(2):201–214, 2000. ISSN 01676687. URL `http://search.proquest.com/docview/208166412/`.

Mogens Steffensen. *On valuation and control in life and pension insurance*. Laboratory of Actuarial Mathematics, University of Copenhagen, Copenhagen, 2001. ISBN 8778344492.