1 Introduction

With-profit insurance contracts are to this day one of the most popular life insurance contracts. They arose as a natural way to distribute the systematic surplus that emerges due to the prudent assumptions on which the contract is made. In recent years, sensible questions accompanied by a lot of attention have been aimed at the surplus, to name a few; is it distributed fairly? how should it be invested? How is it affected by the financial market? To answer these questions we need to understand the dynamics of the surplus in a model of practical relevance. The study of surplus and the interplay it has with other elements of an insurance contract, is not new. Norberg (1999) introduces the notion of individual surplus as well as the mean portfolio surplus. In Steffensen (2000) and Steffensen (2001), partial differential equations are used to describe the prospective second order reserve for various forms of bonus, when the surplus is invested in a Black-Scholes market. In this paper we pay little regard to the prospective reserve, and instead focus on the surplus and the retrospective reserve including dividends, also called the savings account. Furthermore we do not restrict ourselves to the Black-Scholes market, but allow for arbitrary specification of the financial market.

In the existing literature, very little attention is paid to a very significant retrospective element of the with-profit insurance contract: the human element. Insurance companies are governed by humans, and the decisions they make have an influence on the portfolio of policies - in particular concerning surplus and dividends. In a with-profit insurance contract many quantities are fixed at initialisation of the policy, but the rate at which dividends are paid out is not. The insurance company has a certain degree of freedom when it comes to the distribution of surplus, and the actions that have an influence on the insurance contracts are the so-called Management Actions. From a mathematical point of view they pose a problem as they are retrospective in nature, and may depend on the entire history of the portfolio of policies in a possibly non-linear fashion, making it difficult to calculate prospective reserves. If we want to take a glance into the crystal ball of liabilities, taking Future Management Actions (FMA's) into account, we need to embrace it's retrospective nature. In this paper, we do not incorporate FMA's to their full extent, but rather lay the retrospective groundwork on which models including FMA's can be built.

In section 1.1 we present a standard model for life insurance contracts, that form the foundation for the results of section 2, which is a summary of the relevant results from Norberg (1991). In section 3 we extend the set-up of section 1.1 to allow for a model where surplus and dividends are considered. The main result is presented in 4, where we derive a retrospective differential equation for the expected savings account and surplus, in a general model with affine dynamics.

1.1 Set-up

We consider the classic multi-state life insurance set-up, comprised of a state process Z denoting the state of the policy in a finite state space $\mathcal{J} = \{0, 1, ..., J\}$. By a permutation argument, we can without loss of generality assume that $Z(0) = 0$. The filtration generated by $Z(t)$ is denoted by $\mathcal{F}_t$. The counting process N^k defined by $N^k(t) = \#\{s; Z(s-) = i, Z(s) = k, s \in (0, t]\}$ describes the number of transitions into state k. The state process Z is assumed to be a continuous time Markov chain, with transition probabilities denoted by

$$p_{ij}(s, t) = \mathrm{P}(Z(t) = j | Z(s) = i)$$

for $s \leq t$. The corresponding transition intensities are denoted by

$$\mu_{ij}(t) = \lim_{h \searrow 0} p_{ij}(t, t + h)/h$$

for $i \neq j$. The predictable process $\mathbb{1}_{\{Z(t-) \neq k\}} \mu_{Z(t-)k}(t)$ is the intensity process for $N^k(t)$, i.e

$$M^k(t) := N^k(t) - \int_0^t \mathbb{1}_{\{Z(s-) \neq k\}} \mu_{Z(s-)k}(s) ds,$$

forms a martingale. The state process Z encapsulates the biometric risks involved with the insurance contract. Apart from the biometric risk, there is a financial risk connected to with-profit insurance contracts through the return on investment of the surplus. We make assumptions regarding the financial risk, by specifying the expected return on investment, r. Together, the transition intensities and expected return on investment form the second order basis, which describes the best guess on future development of the insurance portfolio. We take this second order basis as exogenously given. Note that a Monte-Carlo method can be used as a proxy for evaluation under the second order basis; perform evaluation under n simulated second order basis and take the mean. The Monte-Carlo approach for evaluation allows for great model flexibility, which is particularly appealing regarding the expected return on investment.

While the second order basis forms the best guess on future developments of the relevant technical elements, it would be far too risky for an insurance company to use these assumptions when signing contracts. What if a cure for cancer is invented in 10 years, or if the stock market crashes? To allow for events that make it difficult to meet the obligations to the insured, a much less risky set of assumptions are used when guarantees are given. These prudent assumptions form the first order (technical) basis. Using the standard notation, a "$*$" symbolises first-order basis elements. It is precisely due to the difference between the first order basis and the realised (third order) basis that a surplus emerges. We have no way of knowing what the future is going to bring, so we cannot know how the surplus is going to evolve. We can however make an estimate by using the second order basis as a stand-in for the third order basis.
In order to define an insurance contract we introduce the payment process B, which depends on the dynamics of Z. The payment process is an $\mathcal{F}_t$-adapted process with dynamics given by

$$dB(t) = b^{Z(t)}(t)dt + \sum_{k \neq Z(t-)} b^{Z(t-)k}(t)dN^k(t),$$

for sufficiently nice $b^i(t)$ and $b^{jk}(t)$. The deterministic payment functions $b_i^j(t)$ and $b_i^{jk}(t)$ specify payments during sojourns in state j and on transition from state j to state k, respectively. Even though single payments during sojourns in states pose no mathematical difficulty, we assume that payments during sojourns in states are continuous for notational simplicity. Given the payment process B, we can define the prospective technical reserve as

$$V^{j*}(t) = \mathrm{E}\left[\int_t^n e^{-\int_t^s r*} dB(s) | Z(t) = j\right].$$

The dynamics of the technical reserves are found using Itô's lemma for FV-functions to be

$$\begin{aligned} dV^{Z(t)*}(t) =& r^*(t)V^{Z(t)*}(t)dt - b^{Z(t)}(t) - \sum_{k \neq Z(t-)} b^{Z(t-)k}(t)dN^k(t) \\ &- \sum_{k \neq Z(t-)} \rho^{Z(t-)k}(t)dt + \sum_{k \neq Z(t-)} R^{Z(t-)k}(t)(dN^k(t) - \mu_{Z(t-)k}(t)dt), \end{aligned} \tag{1}$$

where ρ^{jk} is the so-called risk premium for a transition from state j to state k, and R^{jk} is the so-called sum-at-risk for a transition from j to k. The sum-at-risk R^{jk} describes the required injection of capital on a transition from j to k in order to meet the future liabilities of the contract in state k, evaluated under the first-order basis. The sum-at-risk is given by

$$R^{jk}(t) = b^{jk}(t) + V^{k*}(t) - V^{j*}(t).$$

As the name suggests, the risk premium is the premium paid by the policyholder to cover the risk carried by the insurer that can not be diversified, such as medical advancements. Naturally the risk premium is the sum-at-risk multiplied by the difference in intensity for a transition from j to k between the first-order basis and the second-order basis, i.e

$$\rho^{jk}(t) = R^{jk}(t)(\mu^*_{jk}(t) - \mu_{jk}(t)).$$

2 Retrospective Reserve Without Bonus

One of the main contributions of Norberg (1991) is a definition of the retrospective reserve, as a conditional expected value of a past payments, in much the same manner as the prospective reserve is a conditional expected value of future payments. Formally Norberg (1991) defines the retrospective first order reserve, as

$$V^*_{\mathbb{E}}(t) = \mathrm{E}^* \left[\int_0^t e^{\int_s^t r^*} dB(s) \big| \mathcal{E}_t \right]$$

for some family of sigmaalgrebras $\mathbb{E} = \{\mathcal{E}_t\}_{0\leq t}$, where $\mathcal{E}_t$ represents the information available at time t. It is for an actuary very natural to assume that $\mathcal{E}_t = \sigma\{Z(s), 0 \leq s \leq t\}$, implying that all information about the past is accounted for. As noted by Norberg (1991) the family of sigmaalgebras may be increasing, i.e $\mathcal{E}_s \subseteq \mathcal{E}_t$ for $s < t$, but it is not required. So with this very general definition of the retrospective reserve, we may discard information, for instance by defining $\mathcal{E}_t = \sigma\{Z(0), Z(t)\}$. But why should we ever choose to discard information that is available to us? Because it is intractable to use $\mathcal{E}_t = \mathcal{F}_t$ when we want to calculate the expected value of $V^*_{\mathbb{E}}(t)$ and $\{Z(s)\}_{s\leq t}$ has not yet been realised. Computationally it is simply too demanding to take the expectation over $\mathcal{F}_t$ - all possible paths and all possible transition times have to be considered. Instead we therefore let $\mathcal{E}_t = \sigma\{Z(0), Z(t)\}$, implying that we only use the state at initialization and time t to evaluate the retrospective reserve. Using this formulation of $\mathcal{E}_t$, the retrospective reserve can be interpreted as the average reserve of a group of policies that all start in $Z(0)$ and end in $Z(t)$. In order to actually calculate this retrospective reserve, we note by the markov property that

$$P(Z(s) = j | Z(0) = 0, Z(t) = i) = \frac{p_{0j}(0, s)p_{ji}(0, t)}{p_{0i}(s, t)}, \tag{2}$$

and that the predictable compensator for $N^{jk}(s)|(Z(0) = 0, Z(t) = i)$ has intensity given by

$$\mathbb{1}_{\{Z(s-)=j\}}\mu_{jk|0i}(s|0, t) = \mathbb{1}_{\{Z(s-)=j\}}\mu_{jk}(s)\frac{p_{ki}(s, t)}{p_{ji}(s, t)}. \tag{3}$$

Define

$$V^{i*}(t) = \mathrm{E}^* \left[\int_0^t e^{\int_s^t r*} dB(s) \big| Z(0) = 0, Z(t) = i \right],$$

which by (2) and (3) is equal to

$$\begin{aligned} V^{i*}(t) &= \int_0^t e^{\int_s^t r*} \sum_{j \in \mathcal{J}} \frac{p_{0j}(0,s) p_{ji}(s,t)}{p_{0i}(0,t)} \left(b^j(s) + \sum_{k \neq j} \mu_{jk}(s) b^{jk}(s) \frac{p_{ki}(s,t)}{p_{ji}(s,t)} \right) ds \\ &= \frac{1}{p_{0i}(0,t)} \int_0^t e^{\int_s^t r*} \sum_{j \in \mathcal{J}} p_{0j}(0,s) \left(p_{ji}(s,t) b^j(s) + \sum_{k \neq j} \mu_{jk}(s) b^{jk}(s) p_{ki}(s,t) \right) ds \qquad (4) \end{aligned}$$

as also derived by Norberg (1991). In itself (4) provides an interpretation of the retrospective reserve; it is the accumulated payments on transition and sojourn payments at all times prior to t, weighted by the corresponding probability of transition between states and sojourns in states, given the initial and terminal state of the policy. For sufficiently nice intensities and payment functions, analytical solutions for $V^{i*}(t)$ can be derived. In general, we cannot provide a closed form expression for $V^{i*}(t)$, and instead we have to rely on numerical methods, for instance by a numerical solution to the differential equation solved by $V^{i*}(t)$. As it is a nuisance to directly derive a differential equation for V^{i*}, due to the division by the probability of entering state i at time t, we define

$$\tilde{V}^{i*}(t) = \mathrm{E}^* \left[\mathbb{1}_{\{Z(t)=i\}} \int_0^t e^{\int_s^t r^*} dB(s) \big| Z(0) = 0 \right] = V^{i*}(t) p_{0i}(0,t).$$

Using the Kolmogorov differential equations, $p_{0i}(0,t)$ can be calculated for all i and t, and thus $V^{i*}(t)$ can easily be calculated once $\tilde{V}^{i*}(t)$ is available. Differentiating $\tilde{V}^{i*}(t)$ with respect to t

gives

$$
\begin{aligned}
\frac{d}{dt}\tilde{V}^{i*}(t) =& \frac{d}{dt}\int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(p_{ji}(s,t)b^j(s) + \sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{ki}(s,t)\right) ds \\
=& \sum_{j\in\mathcal{J}} p_{0j}(0,t)\left(\mathbb{1}_{\{j=i\}}b^j(t) + \sum_{k\neq j}\mu_{jk}(t)b^{jk}(t)\mathbb{1}_{\{k=i\}}\right) \\
&+ \int_0^t \frac{d}{dt}e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(p_{ji}(s,t)b^j(s) + \sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{ki}(s,t)\right) ds \\
=& p_{0i}(0,t)b^i(t) + \sum_{j\neq i} p_{0j}(0,t)\mu_{ji}(t)b^{ji}(t) + r^*(t)\tilde{V}^{i*}(t) \\
&+ \int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(\frac{d}{dt}p_{ji}(s,t)b^j(s) + \sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)\frac{d}{dt}p_{ki}(s,t)\right) ds.
\end{aligned}
$$

The Kolmogorov forward differential equations state that

$$
\frac{d}{dt}p_{ji}(s,t) = \sum_{g\neq i} p_{jg}(s,t)\mu_{gi}(t) - \mu_{ig}(t)p_{ji}(s,t)
$$

which imply that

$$
\begin{aligned}
\frac{d}{dt}\tilde{V}^{i*}(t) =& p_{0i}(0,t)b^i(t) + \sum_{j\neq i} p_{0j}(0,t)\mu_{ji}(t)b^{ji}(t) + r^*(t)\tilde{V}^{i*}(t) \\
&+ \sum_{g\neq i}\mu_{gi}(t)\int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)p_{jg}(s,t)b^j(s)ds \\
&- \sum_{g\neq i}\mu_{ig}(t)\int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)p_{ji}(s,t)b^j(s)ds \\
&+ \sum_{g\neq j}\mu_{gi}(t)\int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{kg}(s,t)ds \\
&- \sum_{g\neq i}\mu_{ig}(t)\int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{kj}(s,t)ds
\end{aligned}
$$

$$
\begin{aligned}
=& p_{0i}(0,t)b^i(t) + \sum_{j\neq i} p_{0j}(0,t)\mu_{ji}(t)b^{ji}(t) + r^*(t)\tilde{V}^{i*}(t) \\
&+ \sum_{g\neq i}\mu_{gi}(t) \underbrace{\int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(p_{jg}(s,t)b^j(s)\sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{kg}(s,t)\right)ds}_{\tilde{V}_{g*}(t)} \\
&- \sum_{g\neq i}\mu_{ig}(t) \underbrace{\int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(p_{ji}(s,t)b^j(s)\sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{ki}(s,t)\right)ds}_{\tilde{V}^{i*}(t)} \\
=& p_{0i}(0,t)b^i(t) + \sum_{j\neq i} p_{0j}(0,t)\mu_{ji}(t)b^{ji}(t) + r^*(t)\tilde{V}^{i*}(t) \qquad (5)\\
&+ \sum_{g\neq i}\mu_{gi}(t)\tilde{V}_g^*(t) - \mu_{ig}(t)\tilde{V}^{i*}(t), \qquad (6)
\end{aligned}
$$

and along with the initial condition

$$\tilde{V}^{i*}(0) = 0 \quad \text{for all } i,$$

given by the principle of equivalence, we have a system of differential equations describing $\tilde{V}^{i*}(t)$. These differential equations have certain similarities with the classical prospective Thiele differential equations. The retrospective probability weighted reserve $\tilde{V}^{i*}(t)$ develops in accordance with the probability weighted payments, the first order interest, and a diffusion between the reserves. Interestingly these differential equations are generalisations of the Kolmogorov forward differential equations. This can be seen by letting $r^*(t) = 0$ and defining the payment process

$$dB(t) = \mathbb{1}_{\{t=s\}}\mathbb{1}_{\{Z(t)=j\}},$$

that has a payout of one unit at time s, if $Z(t) = j$. Then

$$\tilde{V}^{i*}(t) = \mathrm{E}^*\left[\int_0^t \mathbb{1}_{\{Z(\tau)=j\}} d\delta_s(\tau) | Z(0)=0, Z(t)=i\right] p_{0i}(0,t) = p_{0j}(0,s)p_{ji}(s,t),$$

and $\tilde{V}^{i*}(s) = p_{0j}(0,s)\mathbb{1}_{\{i=j\}}$ providing the initial condition for the Kolmogorov forward differential equations. The differential equation for the retrospective reserve for $s < t$ is given by

$$p_{0j}(0,s)\frac{d}{dt}p_{ji}(s,t) = \sum_{g\neq i}\mu_{gi}(t)p_{0j}(0,s)p_{jg}(s,t) - \mu_{ig}(t)p_{0j}(0,s)p_{ji}(s,t),$$

$$\Leftrightarrow$$

$$\frac{d}{dt}p_{ji}(s,t) = \sum_{g\neq i}\mu_{gi}(t)p_{jg}(s,t) - \mu_{ig}(t)p_{ji}(s,t),$$

which constitute the Kolmogorov forward differential equations.

Norberg (1991) defined the retrospective reserve, and derived some of its important properties. At the time, the retrospective reserve was perhaps more of a mathematical curiosity than an actuarial tool, as the prospective reserves at the time provided all the information you could ask for. Furthermore when $\mathcal{E}_t = \mathcal{F}_t$ the retrospective reserve is observable, and not something you need to calculate. However, the retrospective reserve with $\mathcal{E}_t = \sigma\{Z(0), Z(t)\}$ definitely deserves recognition when surplus and dividends are introduced. Norberg (1999) defines the individual surplus as a retrospective reserve, and derives a differential equation hereof in a simple model where no dividends are allotted.

3 Set-Up Including Surplus and Dividends

In this section we expand our set-up such that we can accurately describe the benefits and balances in a model where surplus and dividends are included. The first order basis on which insurance contracts are signed, are a set of prudent assumptions regarding interest and transition intensities. Knowing that the assumptions are prudent, the insurer and insured agree that when surplus has emerged as a consequence of the realized interest and transition intensities, this surplus should be given back to the insured. The surplus is returned to the insured through a dividend payment stream. What the insured chooses to do with his dividend can vary, but a very standard product design is to use the dividends to buy more insurance. In a sense, the dividend payment stream becomes a premium for a bonus benefit payment stream. We introduce two payment streams B_1 and B_2 with dynamics

$$dB_i(t) = b_i^{Z(t)}(t)dt + \sum_{k\neq Z(t-)} b_i^{Z(t-)k}(t)dN^k(t).$$

The payments specified by B_1 are the benefits which are fixed, and part of the original contract. The payments of B_2 specify the profile of the benefit stream that the dividend is converted into. When the contract is signed, both B_1 and B_2 are agreed upon, and while there is practically no restriction on their design, we assume that B_2 contains benefits only, implying that dividends are used to increase benefits, and not to decrease premiums. The payment streams B_1 and B_2, have corresponding technical reserves given by

$$V_i^{j*}(t) = \mathrm{E}\left[\int_t^n e^{-\int_t^s r^*} dB_i(s) \big| Z(t) = j\right].$$

We assume that the benefits of B_2 are triggered by events that have non-zero probability i.e. $V_2^{j*}(t) > 0$. In order to keep track of how much dividend has been materialized into the B_2 payment stream, we introduce the process $Q(t)$ which denotes the quantity of B_2 payment stream purchased at time t. We do not yet specify the dynamics of $Q(t)$. The payment process experienced by the policyholder, B, consists of one unit B_1 payment stream and Q units of B_2 payment stream, thus having dynamics

$$dB(t) = dB_1(t) + Q(t-)dB_2(t).$$

Where the left-limit version of Q is used in order to ensure that B is adapted to $\mathcal{F}_t$. We now define the savings account as the technical value of future guaranteed payments, given a certain quantity of B_2 payment stream,

$$\begin{aligned} X(t) =& \mathrm{E}^*\left[\int_t^n e^{\int_t^s r*} d\left(B_1(s) + Q(t)B_2(s)\right)\right] \\ =& V_1^{Z(t)*}(t) + Q(t)V_2^{Z(t)*}(t). \end{aligned}$$

Noting that

$$Q(t) = \frac{X(t) - V_1^{Z(t)*}(t)}{V_2^{Z(t)*}(t)}$$

we see that the payment stream experienced by the policyholder has dynamics

$$\begin{aligned} dB(t) =& dB_1(t) + \frac{X(t-) - V_1^{Z(t-)*}(t-)}{V_2^{Z(t-)*}(t-)} dB_2(t) \\ =& b^{Z(t)}(t, X(t))dt + \sum_{k \neq Z(t-)} b^{Z(t-)k}(t, X(t-))dN^k(t), \end{aligned}$$

for deterministic functions b^j and b^{jk} corresponding to sojourn payments and payments on transition, given by

$$
\begin{aligned}
b^j(t,x) &= b_1^j(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^j(t) \\
b^{jk}(t,x) &= b_1^{jk}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^{jk}(t).
\end{aligned}
$$

By introducing subscripts, the dynamics of $V_1^{Z(t)*}$ and $V_2^{Z(t)*}$ are given by (1). Using integration by parts for FV-functions we find the dynamics of X to be

$$
\begin{aligned}
dX(t) =& dV_1^{Z(t)*}(t) + Q(t-)dV_2^{Z(t)*}(t) + V_2^{Z(t)*}(t)dQ(t) \\
=& r^*(t)X(t)dt + V_2^{Z(t)*}(t)dQ(t) - b^{Z(t)}(t,X(t))dt - \sum_{k\neq Z(t-)} b^{Z(t-)k}(t,X(t-))dN^k(t) \\
&- \sum_{k\neq Z(t-)} \rho^{Z(t-)k}(t,X(t-))dt \\
&+ \sum_{k\neq Z(t-)} R^{Z(t-)k}(t,X(t-))(dN^k(t) - \mu_{Z(t-)k}(t)dt),
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
\rho^{jk}(t,X(t-)) =& \rho_1^{jk}(t) + Q(t-)\rho_2^{jk}(t) = \rho_1^{jk}(t) + \frac{X(t-) - V_1^{j*}(t-)}{V_2^{j*}(t-)}\rho_2^{jk}(t), \\
R^{jk}(t,X(t-)) =& R_1^{jk}(t) + Q(t-)R_2^{jk}(t) = R_1^{jk}(t) + \frac{X(t-) - V_1^{j*}(t-)}{V_2^{j*}(t-)}R_2^{jk}(t),
\end{aligned}
$$

respectively can be interpreted as the risk premium and sum-at-risk for the savings account. The dynamics of X are remarkably similar to the dynamics of the retrospective reserve as seen in (1), in fact if $Q(t-)dV_2^{Z(t)*}(t) = 0$ they are identical, and this is a special case that we return to.

The savings account plays a crucial role in the understanding of the with-profit insurance contract, just as the first order reserve plays a crucial role in the model without dividends. Given the savings account, we can readily define the surplus as

$$
Y(t) = -\int_0^t e^{\int_s^t r} dB(s) - X(t),
$$

corresponding to the accumulated premiums less benefits excess over the savings account. It is clear from the definition of the surplus that the savings account has an influence on Y, but the surplus also has an influence on the savings account though the dividends. The dividends flow from the surplus to the savings account, according to some dividend strategy determined by the insurer. These dividends are instantaneously used to increase benefits, by buying more of the B_2 payment stream. These additional benefits are, like the fixed benefits, priced under the first order basis, which means that one unit of B_2 has a value of $V_2^{Z(t)*}(t)$. The total amount of accrued dividends at time t are denoted by $D(t)$, and as the dividends are used to buy B_2, we must have that

$$dD(t) = V_2^{Z(t)*}(t)dQ(t). \tag{8}$$

By the principle of equivalence

$$0 = X(0) = V_1^{0*}(0) + Q(0-)V_2^{0*}(0)$$
$$\Leftrightarrow$$
$$Q(0-) = -\frac{V_1^{0*}(0)}{V_2^{0*}(0)}$$

providing us with the initial condition for Q, which along with (8) fully specifies Q. Note that the principle of equivalence puts no restrictions on the form of B_1 and B_2. If no dividends are ever allotted, i.e. $D(t) = Q(t-)dV_2^{Z(t)*}(t) = 0$, then the dynamics of X are identical to the dynamics of the technical reserve found in (1) for an X-independent payment process B_C given by

$$dB_C(t) = dB_1(t) - \frac{V_1^{0*}(0)}{V_2^{0*}(0)}dB_2(t).$$

If no dividends are allotted, then the savings account is simply a retrospective first order reserve, and therefore falls under the framework of section 2. The dividend process is the crucial term that separates the results of Norberg (1991) from the results of this paper, and it is precisely due to the dividend process that we need to extend the results of Norberg (1991).

We assume the dynamics of the dividend process is given by

$$dD(t) = \delta^{Z(t)}(t, X(t), Y(t))dt,$$

for some deterministic function δ^j on the form

$$\delta^j(t, x, y) = \delta_1^j(t) + \delta_2^j(t)x + \delta_3^j(t)y.$$

In this paper, the form of δ is probably the assumption most eligible for criticism. In practice, the dividend is determined by an actuary who takes much more information into account than simply the value of the savings and surplus. Furthermore the dividend-deciding actuary is most likely going to take past development of the savings and surplus into account. The specification of the dynamics of D is at the heart of what a future management action is, and, as stated earlier, we do not fully incorporate these FMA's in all their generality and glory, but suffice with crude surrogates. Some of these crude surrogates can actually perform a decent job at describing real world dividend strategies, for instance by defining the dividend as some linear function of the contribution, which is defined below.
The dynamics of Y can easily be derived to be

$$dY(t) = rY(t)dt + dC(t) - dD(t) - \sum_{k \neq Z(t-)} R^{Z(t-)k}(t, X(t-))dM^k(t), \tag{9}$$

for

$$dC(t) = (r(t) - r^*(t))X(t)dt + \sum_{k \neq Z(t)} \rho^{Z(t)k}(t, X(t))dt,$$

which we call the contribution process, as it represents the contributions from the savings account to the surplus. As the dynamics of X and Y are affine, we can, for suitable affine functions g and h, write the dynamics of X and Y as

$$dX(t) = g_x^{Z(t)}(t, X(t), Y(t))dt + \sum_{k \neq Z(t-)} h_x^{Z(t-)k}(t, X(t-), Y(t-))dN^k(t) \tag{10}$$

$$dY(t) = g_y^{Z(t)}(t, X(t), Y(t))dt + \sum_{k \neq Z(t-)} h_y^{Z(t-)k}(t, X(t-), Y(t-))dN^k(t). \tag{11}$$

We refer to section B of the appendix for the specification of g and h leading to the dynamics given in (7) and (9). Apart from notational ease, the use of arbitrary g and h functions serve to generalise the results of the paper to any FV-process with affine dynamics of the form given by (10) and (11). Even though we work with the dynamics given by (10) and (11), we think of the g and h functions as the ones required to achieve the dynamics of (7) and (9). As we are interested in the interconnected dynamics of X and Y, we introduce the two-dimensional process

$$W(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}.$$

with dynamics given by

$$dW(t) = g^{Z(t)}(t, W(t))ds + \sum_{k \neq Z(t-)} h^{Z(t-)k}(t, W(t-))dN^k(t),$$

for g and h functions that are affine in all entries of W, and determined by the dynamics of X and Y. Without loss of generality we assume that $W(0) = w_0$ for some deterministic but arbitrary w_0. A function f is affine in all entries of W if and only if

$$f(t, W(t)) = f_1(t) \circ W(t) + f_2(t)$$

where we by "$\circ$" denote the element wise product.

In practice, the surplus account is shared among, say N, policyholders. In that case, W should instead be an $N+1$-dimensional process - one dimension for each policyholder, and one dimension for the common surplus. This implies that we need a system of $\#\{\mathcal{J}\}^N + 1$ differential equations; one for each combination of all policy states, and one for the common surplus. There are several ways to reduce the dimensionality of the problem, making it computationally tractable. One way to diminish the problem of dependency between policyholders, is to discretize the dividend function and comprise it as lump-sum payments, which conforms to real-world practise. When there are no dividends, the savings are independent of surplus, and thus also the other policies. In between lump-sum payments of dividend, the state-wise contributions including investment gains from each policy is accumulated. When a lump-sum time is reached, the probability weighted sum of contributions are contributed to a single state-independent surplus where after the dividend is allocated. This method requires $\#\{\mathcal{J}\} \times 2 \times N + 1$ differential equations be solved. One for the surplus, and one for the savings account and contribution for each state of each policy. Furthermore note that in between lump-sum payments of dividend, the policies are independent, allowing for parallelization across N cores. Even though $\#\{\mathcal{J}\} \times 2 \times N + 1$ independent differential equations is a vast improvement from $\#\{\mathcal{J}\}^N + 1$ dependent differential equations, it is still a computationally difficult problem, and a further digression on the subject is outside the scope of this paper.

It is important to realize the extent of applicable models that have affine dynamics, see Christiansen et al. (2014) for several relevant payment functions that are linear in the reserve, which corresponds to the savings when $D(t) = 0$. While the reach of models with affine dynamics is extensive, there are limitations to consider. It is not uncommon to have dynamics that

include some min or max function, for instance in the case of guarantees, and these non-linear functions in savings cannot be described by affine dynamics.
As stated in the introduction, management actions are one of the main motivators of this paper, but they are hidden in mainly two terms; the second order interest and the dividend. This is because the management decides how to invest the surplus, and how it should be distributed to the customers. Due to the very human and abstract nature of management actions, we do not incorporate them directly in the dynamics of the savings and surplus, but instead let them work in the shadows.

4 State-Wise Probability Weighted Reserve

In this section we present the main result, which generalizes the result from Norberg (1991) by allowing for processes whose dynamics are affine functions of the process itself. To illustrate the central idea, consider the case where W has dynamics

$$dW(s) = g^{Z(s)}(s) \circ W(s)ds,$$

and say we want to calculate

$$\tilde{W}^i(t) := \mathrm{E}_0[W(t)\mathbb{1}_{\{Z(t)=i\}}] = \mathrm{E}_0[W(t)|Z(t)=i]p_{Z(0)i}(0,t),$$

where we by the subscript 0 on the expectation denote the conditional expectation given $Z(0)$ and $W(0)$. That is

$$\mathrm{E}_0[\mathcal{A}] = \mathrm{E}[\mathcal{A}|Z(0)=0, W(0)].$$

We can write $W(t)$ as an integral from 0 to t over the dynamics of W,

$$W(t) = W(0) + \int_0^t W(s) \circ g^{Z(s)}(s)ds.$$

By the tower property and Fubinis theorem,

$$\begin{aligned}
\tilde{W}^i(t) =& p_{0i}(0,t)W(0) + \int_0^t \mathrm{E}[\mathbb{1}_{\{Z(t)=i\}}W(s) \circ g^{Z(s)}(s)]ds \\
=& p_{0i}(0,t)W(0) + \int_0^t \mathrm{E}_0\left[\sum_{j\in\mathcal{J}} \mathbb{1}_{\{Z(s)=j\}}\mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}}W(s) \circ g^{Z(s)}(s)|Z(s)=j]\right]ds \\
=& p_{0i}(0,t)W(0) + \int_0^t \sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s)g^j(s) \circ \mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}}W(s)|Z(s)=j]]ds
\end{aligned}$$

By the Markov property $W(s) \perp\!\!\!\perp Z(t)|Z(s)$ for $s < t$, as $W(s)$ is $\mathcal{F}_s$-measurable, and therefore

$$\tilde{W}^i(t) = p_{0i}(0,t)W(0) + \int_0^t \sum_{j\in\mathcal{J}} g^j(s) \circ \tilde{W}^j(t) p_{ji}(s,t)ds.$$

Differentiating with respect to t, and using Kolmogorov's forward differential equations yields the following system of differential equations

$$\begin{aligned}\frac{d}{dt}\tilde{W}^i(t) =& \tilde{W}^i(t) \circ g^i(t) + \sum_{j\neq i} \mu_{ji}(t)\tilde{W}^j(t) - \mu_{ij}(t)\tilde{W}^i(t)\\ \tilde{W}^i(0) =& \mathbb{1}_{\{Z(0)=i\}}W(0).\end{aligned}$$

It is crucial to note that this differential equation relies on the affine structure of the dynamics of W, as it allows us to write $\tilde{W}^i(t)$ as an integral over $\tilde{W}^j(s)$ for $0 \leq s \leq t$. The result is generalized by considering a general FV process with dynamics that are affine in the process itself. By using the tower property and the fact that $W(s-) \perp\!\!\!\perp Z(t)|Z(s-)$, we get the following theorem.

Theorem 4.1.
Let $Z(t)$ be a Markov process on the state space $\mathcal{J}$, and let $W(t)$ be a q-dimensional, $\mathcal{F}_t$-measurable process with dynamics

$$dW(s) = g^{Z(s)}(s, W(s))ds + \sum_{k\neq Z(s-)} h^{Z(s-)k}(s, W(s-))dN^k(s)$$

for q-dimensional functions g and h of the form

$$\begin{aligned}g^{Z(s)}(s, W(s)) =& g_1^{Z(s)}(s) \circ W(s) + g_2^{Z(s)}(s)\\ h^{Z(s-)k}(s, W(s-)) =& h_1^{Z(s-)k}(s) \circ W(s-) + h_2^{Z(s-)k}(s).\end{aligned}$$

Then $\tilde{W}^i(t) = E_0[\mathbb{1}_{\{Z(t)=i\}}W(t)]$ is described by the differential equation

$$\frac{d}{dt}\tilde{W}^i(t) = \sum_{j\neq i} \mu_{ji}(t)\tilde{W}^j(t) - \mu_{ij}(t)\tilde{W}^i(t) \tag{12}$$

$$+ \tilde{W}^i(t) \circ g_1^i(t) + p_{Z(0)i}(0,t)g_2^i(t) \tag{13}$$

$$+ \sum_{j\neq i} \mu_{ji}(t) \left(\tilde{W}^j(t) \circ h_1^{ji}(t) + p_{Z(0)j}(0,t)h_2^{ji}(t)\right) \tag{14}$$

$$\tilde{W}^i(0) = \mathbb{1}_{\{Z(0)=i\}}W(0) \tag{15}$$

The differential equations given by (12)-(15) bear close resemblance to the differential equation for the retrospective reserve given by (5)-(6). In fact, for

$$\begin{aligned} g_1 = h_1 = 0, \qquad & W(0) = 0, \\ g_2(t,i) = b^i(t), \qquad & h_2(t,j,i) = b^{ji}(t), \end{aligned}$$

we arrive at the differential equation derived by Norberg (1991). Norberg (1991) allows for dynamics that depend on the expected value of W, while we allow for dynamics of the process to depend on the process itself, which is a necessity in a model including dividends. The terms (12)-(14) in the differential equation can be intuitively explained.

If the policy is in state i at time t, it will develop with the continuous dynamics of that state, given by $W(t)g_1^i(t) + g_2^i(t)$. Due to the uncertainty involved pertaining to the state of the policy and the value of W, we have to weigh these dynamics with the probability of $Z(t) = i$, as well as the expected value of W, thus arriving at (13) as

$$\mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}}\left(W(t)g_1^i(t) + g_2^i(t)\right)] = \tilde{W}^i(t)g_1^i(t) + p_{Z(0)i}(0,t)g_2^i(t).$$

Similarly, we have to account for any transitions into the current state i, over the small interval $t + dt$. The infinitesimal probability of transition from j to i over an interval from t to $t + dt$ is given by $\mu_{ji}(t)$, and if such a transition was made, the savings and surplus are bumped by $W(t)h_1^{ji}(t) + h_2^{ji}(t)$. In order for a transition from j to i to be possible over the interval $t + dt$, the policy has to be in state j at time t, thus arriving at (14) as

$$\mathrm{E}_0[\mathbb{1}_{\{Z(t)=j\}}\left(W(t)h_1^{ji}(t) + h_2^{ji}(t)\right)] = \tilde{W}^j(t)h_1^{ji}(t) + p_{Z(0)j}(0,t)h_2^{ji}(t).$$

Furthermore, when a transition from j to i is made, the savings and surplus from state j (after the bump) are transferred to the savings and surplus of state i, amounting to the term given in (12).

For dynamics of X and Y given by (7) and (9) it is important to note that if the dividend function δ is affine in X and Y, then the dynamics of X and Y are also affine in X and Y. The applicability of theorem 4.1 relies solely on the affinity of the dynamics of the dividends in savings and surplus. From a practical point of view, the projection of expected savings and surplus provide us with useful information. A practically important quantity that can be calculated based on $\tilde{X}$ and $\tilde{Y}$ is the present value of guaranteed future benefits, given by

$$\begin{aligned}
\mathrm{GY}_i(t) =& \mathrm{E}\left[\int_t^n e^{-\int_t^s r} d\left(B_1(s) + \frac{X(t) - V_1^{Z(t)*}(t)}{V_2^{Z(t)*}(t)} B_2(s)\right) \big| Z(t) = i\right] \\
=& \mathrm{E}\left[\int_t^n e^{-\int_t^s r} dB_1(s) \big| Z(t) = i\right] \\
&+ \frac{\mathrm{E}[X(t)|Z(t) = i] - V_1^{i*}(t)}{V_2^{i*}(t)} \mathrm{E}\left[\int_t^n e^{-\int_t^s r} dB_2(s) \big| Z(t) = i\right] \\
=& V_1^i(t) + \frac{\tilde{X}^i(t)/p_{0i}(0,t) - V_1^{i*}(t)}{V_2^{i*}(t)} V_2^i(t).
\end{aligned}$$

Where we in the second equality have used that $X(t) \perp\!\!\!\perp B_2(s)|Z(t)$ for $s > t$. Note that $\mathrm{E}[\mathrm{GY}_i(t)|X(t) = x]$ is affine in x, and therefore it can be used as an input to the dividend function δ - for instance by letting the dividend be some percentage of the guaranteed future benefits.

5 Dealing With Free Policy

Syvtilstandsmodel. Forklar hvad der sker ved overgang til fripolice. Bemærk $V^{3*}(t,u) = V^{0*+}(t)f(t-u)$ når intensiteterne er ens.

The free policy option is the option for the insured to cease all future premiums in exchange all future benefits being scaled accordingly. In the case where benefits are not scaled according to the savings account, there is a natural way to calculate the scaling factor, also called the free policy factor, as the function f that solves

$$\mathrm{E}^*\left[\int_t^n e^{-\int_t^s r*} dB(s)|Z(t)=0\right] = \mathrm{E}^*\left[\int_t^n e^{-\int_t^s r*} f(t)dB^+(s)|Z(t)=3\right]$$

$$\Leftrightarrow$$

$$f(t) = \frac{V^{0*}(t)}{V^{0*+}(t)}.$$

However, as the value of future guarantees under the first order basis is precisely X, no matter how the benefits are scaled, we cannot use this method to define the free policy factor as

$$\mathrm{E}^*\left[\int_t^n e^{-\int_t^s r*} d\left(B_1(s) + \frac{X(t) - V_1^{Z(t)*}(t)}{V_2^{Z(t)*}(t)} B_2(s)\right)|Z(t)=0\right] =$$

$$\mathrm{E}^*\left[\int_t^n e^{-\int_t^s r*} f(t) d\left(B_1^+(s) + \frac{X(t) - V_1^{Z(t)*}(t,0)}{V_2^{Z(t)*}(t,0)} B_2^+(s)\right)|Z(t)=3\right]$$

$$\Leftrightarrow$$

$$V_1^{0*}(t) + \frac{X(t) - V_1^{0*}(t)}{V_2^{0*}(t)} V_2^{0*}(t) = f(t)V_1^{0*+}(t) + \frac{X(t) - V_1^{0*+}(t)f(t)}{V_2^{0*+}(t)f(t)} V_2^{0*+}(t)f(t)$$

$$\Leftrightarrow$$

$$1 = 1$$

leaving us none the wiser. Instead we require that $Q(T_F-) = Q(T_F)$ for T_F being the time of transition from premium paying to free policy. That is, we require that the number of extra B_2 payment streams bought, does not change when transitioning from premium paying to free policy. Benefits of both B_1 and B_2 are scaled by the free policy factor on transition to free policy. Now the free policy factor must satisfy

$$\mathrm{E}^*\left[\int_t^n e^{-\int_t^s r*} d\left(B_1(s) + \frac{X(t) - V_1^{Z(t)*}(t)}{V_2^{Z(t)*}(t)} B_2(s)\right) |Z(t) = 0\right] =$$

$$\mathrm{E}^*\left[\int_t^n e^{-\int_t^s r*} f(t) d\left(B_1^+(s) + \frac{X(t) - V_1^{0*}(t)}{V_2^{0*}(t)} B_2^+(s)\right) |Z(t) = 3\right]$$

$$\Leftrightarrow$$

$$X(t) = f(t)V_1^{0*+}(t) + \frac{X(t) - V_1^{0*+}(t)}{V_2^{0*+}(t)} V_2^{0*+}(t) f(t)$$

$$\Leftrightarrow$$

$$f(t) = \frac{X(t)V_2^{0*}(t)}{V_1^{0*+}(t)V_2^{0*}(t) + (X(t) - V_1^{0*}(t))V_2^{0*+}(t)},$$

implying that f is now not a function, but a process. It is worth noting that this choice of free policy factor leads to $R^{03}(t, X(t)) = b^{03}(t, X(t)) = 0$. As we assume defined benefits i.e $dB_2^+ = dB_2$, we see that

$$f(t) = \frac{X(t)}{X(t) - V_1^{0*-}(t)}.$$

To our dismay this free policy factor is not affine in X, which causes some trouble as we shall see. Note that the only duration dependent part of the benefit pertains to the payments that are not scaled by the savings account,

$$\begin{aligned}
dB^i(t, x, u) =& f(t-u)dB_1^{i+}(t) + \frac{x - V_1^{i*}(t, u)}{V_2^{i*}(t, u)} dB_2^{i+}(t) f(t-u) \\
=& f(t-u)dB_1^{i+}(t) + \frac{x - V_1^{i*+}(t)f(t-u)}{V_2^{i*+}(t)f(t-u)} dB_2^{i+}(t) f(t-u) \\
=& f(t-u)\left(dB_1^{i+}(t) - \frac{V_1^{i*+}(t)}{V_2^{i*+}(t)} dB_2^{i+}(t)\right) + \frac{x}{V_2^{i*+}(t)} dB_2^{i+}(t),
\end{aligned}$$

implying that the dynamics of X are on the form

$$\begin{aligned}
dX(s) =& X(s)g_1(s, Z(s))ds + g_2(s, Z(s))ds + \mathbb{1}_{\{Z(s)\in\mathbb{F}\}} f(s - U(s))g_3(s, Z(s))ds \\
&+ \sum_{h \neq Z(s-)} \left(X(s-)h_1(s, Z(s-), h) + h_2(s, Z(s-), h) +\right) dN^h(s) \\
&+ \sum_{h \neq Z(s-)} \mathbb{1}_{\{Z(s)\in\mathbb{F}\}} f(s - U(s))h_3(s, Z(s-), h) dN^h(s),
\end{aligned} \tag{16}$$

where $\mathbb{F} \subseteq \mathcal{J}$ are the set of free policy states. Note the two very important special cases;

- $dB_1^{Z(t)+}(t) = dB_2^{Z(t)+}(t)$, corresponding to the assumption that whatever benefits the insured has already bought, are the same benefits he wants to buy using his dividend.
- $dB_1^{Z(t)+}(t) = 0$, which is the case when B_1 only relates to the premium of the policy.

In these cases, the dynamics of X are independent of the free-policy duration. This is because the otherwise duration dependent terms of the dynamics of X, dB and χ^{jk}, can be written as

$$\begin{aligned}
dB^i(t,x,u) =& \frac{x}{V_1^{i*+}(t)} dB_2^{i+}(t) \\
\chi^{jk}(t,x,u) =& V_1^{k*+}(t) f(t-u) + \frac{x - V_1^{j*+}(t) f(t-u)}{V_2^{j*+}(t) f(t-u)} V_2^{k*+}(t) f(t-u) \\
=& \frac{x}{V_2^{j*+}(t)} V_2^{k*+}(t).
\end{aligned}$$

Pleasing as these special cases may be, they are not general enough to encompass the real-world complexity we need. Instead we should try to derive a differential equation for the expectation of the savings account when it has dynamics given by (16).

For general B_1 and B_2, the extra terms when taking expectation of dynamics of X compared to the case without duration dependence, are

$$\mathrm{E}_0[\mathbb{1}_{\{Z(s)\in\mathbb{F}\}} \mathbb{1}_{\{Z(t)=j\}} g_3(s, Z(s-)) f(s-U(s)) | Z(s-) = g] \tag{17}$$

and

$$\sum_{h\neq g} \mathrm{E}_0[\mathbb{1}_{\{Z(s)\in\mathbb{F}\}} \mathbb{1}_{\{Z(t)=j\}} h_3(s, Z(s-), h) dN^h(s) f(s-U(s)) | Z(s-) = g] \tag{18}$$

Commencing with (17),

$$\begin{aligned}
&\mathrm{E}_0[\mathbb{1}_{\{Z(s)\in\mathbb{F}\}} \mathbb{1}_{\{Z(t)=j\}} g_3(s, Z(s-)) f(s-U(s)) | Z(s-) = g] \\
&= \mathrm{E}_0[\mathbb{1}_{\{Z(t)=j\}} f(s-U(s)) | Z(s-) = g] \mathbb{1}_{\{g\in\mathbb{F}\}} g_3(s,g).
\end{aligned}$$

By conditioning on the indicator function and multiplying with its probability we get

$$= \mathrm{E}_0[f(s-U(s)) | Z(s-) = g, Z(t) = j] p_{gj}(s,t) \mathbb{1}_{\{g\in\mathbb{F}\}} g_3(s,g).$$

Note that $f(s-U(s))$ is $\mathcal{F}_s$-measurable, and by the Markov property independent of $Z(t)$ given $Z(s)$, for $t > s$. Therefore

$$\begin{aligned}
&\mathrm{E}_0[f(s-U(s))|Z(s-)=g, Z(t)=j]p_{gj}(s,t)\mathbb{1}_{\{g\in\mathbb{F}\}}g_3(s,g) \\
&= \mathrm{E}_0[f(s-U(s))|Z(s-)=g]p_{gj}(s,t)\mathbb{1}_{\{g\in\mathbb{F}\}}g_3(s,g) \\
&= \mathrm{E}_0[\mathbb{1}_{\{Z(s-)=g\}}f(s-U(s))]\frac{p_{gj}(s,t)}{p_{Z(0)g}(0,s)}\mathbb{1}_{\{g\in\mathbb{F}\}}g_3(s,g) \\
&= g_3(s,g)\frac{p_{gj}(s,t)}{p_{Z(0)g}(0,s)}\mathbb{1}_{\{g\in\mathbb{F}\}}\int_0^s \mathrm{E}_0[f(\tau)\mathbb{1}_{\{Z(s-)=g\}}|s-U(s)=\tau]dP(s-U(s)\leq\tau|Z(0)).
\end{aligned}$$

Performing the same calculations as in section A.2 of Buchardt and Møller (2015), and noting that $f(\tau) \perp\!\!\!\perp \mathbb{1}_{\{Z(s-)=g\}}|s-U(s)$, we get

$$\begin{aligned}
&g_3(s,g)\frac{p_{gj}(s,t)}{p_{Z(0)g}(0,s)}\mathbb{1}_{\{g\in\mathbb{F}\}}\int_0^s \mathrm{E}[f(\tau)\mathbb{1}_{\{Z(s-)=g\}}|Z(0), s-U(s)=\tau]dP(s-U(s)\leq\tau|Z(0)) \\
&\qquad = g_3(s,g)\frac{p_{gj}(s,t)}{p_{Z(0)g}(0,s)}\mathbb{1}_{\{g\in\mathbb{F}\}}\int_0^s p_{Z(0)0}(0,\tau)\mu_{03}(\tau)\mathrm{E}_0[f(\tau)|Z(\tau)=0]p_{3g}(\tau,s)d\tau,
\end{aligned}$$

As $f(\tau)$ is not affine in $X(\tau)$, we cannot simply replace X with its expectation given $Z(\tau)$. We can however perform a Taylor expansion of f around $\tilde{X}^0(\tau)$, and get

$$\mathrm{E}_0[f(\tau)|Z(\tau)=0] = \mathrm{E}_0[\tilde{f}(\tau)|Z(\tau)=0] + \mathrm{E}_0\left[O\left(\left(X(\tau)-\frac{\tilde{X}^0(\tau)}{p_{Z(0)0}(0,\tau)}\right)\right)\middle|Z(\tau)=0\right]$$

where

$$\tilde{f}(\tau) = \frac{\frac{\tilde{X}^0(\tau)}{p_{Z(0)0}(0,\tau)}}{\frac{\tilde{X}^0(\tau)}{p_{Z(0)0}(0,\tau)} - V_1^{0*-}(\tau)},$$

which is independent of X. Thus,

$$\begin{aligned}
\mathrm{E}_0[f(\tau)|Z(\tau)=0] =& \frac{\frac{\tilde{X}^0(\tau)}{p_{Z(0)0}(0,\tau)}}{\frac{\tilde{X}^0(\tau)}{p_{Z(0)0}(0,\tau)} - V_1^{0*-}(\tau)} \\
&+ \mathrm{E}_0\left[O\left(\left(X(\tau)-\frac{\tilde{X}^0(\tau)}{p_{Z(0)0}(0,\tau)}\right)^2\right)\middle|Z(\tau)=0\right].
\end{aligned}$$

Now we perform an approximation by disregarding the expectation of the O-function, leaving us with a deterministic approximated free policy factor, and as the duration dependent benefits are independent of X, we may use the lost-all trick to conclude

$$\mathrm{E}_0[\mathbb{1}_{\{Z(s)\in\mathbb{F}\}}\mathbb{1}_{\{Z(t)=j\}}g_3(s,Z(s-))f(s-U(s))|Z(s-)=g] = \frac{p^{\text{lost}}_{Z(0)g}(0,s)}{p_{Z(0)g}(0,s)}\mathbb{1}_{\{g\in\mathbb{F}\}}p_{gj}(s,t)g_3(s,g).$$

Now, consider (18)

$$\mathrm{E}_0[\mathbb{1}_{\{Z(t)=j\}}dN^h(s)f(s-U(s))|Z(s-)=g]h_3(s,g,h)\mathbb{1}_{\{g\in\mathbb{F}\}}.$$

Note that $U(s)|Z(s-) \perp\!\!\!\perp \mathbb{1}_{\{Z(t)=j\}}dN^h(s)|Z(s-)$ **OBS!** By the same arguments used to prove (17).

$$\begin{aligned}
\mathrm{E}_0[f(s-U(s))|Z(s-)=g]\mathrm{E}[\mathbb{1}_{\{Z(t)=j\}}dN^h(s)|Z(s-)=g]h_3(s,g,h)\mathbb{1}_{\{g\in\mathbb{F}\}}.\\
=\frac{p^{\text{lost}}_{0g}(0,s)}{p_{0g}(0,s)}p_{hj}(s,t)\mu_{gh}(s)h_3(s,g,h)\mathbb{1}_{\{g\in\mathbb{F}\}}.
\end{aligned}$$

Performing the same procedure as in the case without duration dependence brings us to the differential given by

$$\boxed{\begin{aligned}
\frac{d}{dt}\tilde{X}^j(t) =& \sum_{g\neq j}\mu^{gj}(t)\tilde{X}^g(t)-\mu^{jg}(t)\tilde{X}^j(t)\\
&+\tilde{X}^j(t)g_1(t,j)+p_{Z(0)j}(0,t)g_2(t,j)+p^{\text{lost}}_{Z(0)j}(0,t)g_3(t,j)\mathbb{1}_{\{j\in\mathbb{F}\}}\\
&+\sum_{g\neq j}\mu^{gj}(t)\left(\tilde{X}^g(t)h_1(t,g,j)+p_{Z(0)g}(0,t)h_2(t,g,j)\right)\\
&+\sum_{g\neq j}\mu^{gj}(t)p^{\text{lost}}_{Z(0)g}(0,t)h_3(t,g,j)\mathbb{1}_{\{g\in\mathbb{F}\}}\\
\tilde{X}^j(0) =& \mathbb{1}_{\{Z(0)=j\}}X(0).
\end{aligned}}$$

A Proof of Theorem 4.1

Proof of theorem 4.1. The proof consists of two steps. First, we derive an integral equation for $\tilde{W}^i(t)$. Second, we differentiate this integral equation.
Assume that $p_{0i}(0,s)>0$ for all $s>0$. The general case where some states cannot be reached

by time s is considered at the end of the proof. By the tower property

$$\begin{aligned}\tilde{W}^i(t) :=&\mathrm{E}_0[W(t)\mathbb{1}_{\{Z(t)=i\}}]\\ =&p_{0i}(0,t)W(0) + \mathrm{E}_0\left[\int_0^t \mathbb{1}_{\{Z(t)=i\}} dW(s)\right]\\ =&p_{0i}(0,t)W(0) + \mathrm{E}_0\left[\int_0^t \mathbb{1}_{\{Z(t)=i\}} g^{Z(s)}(s,W(s))ds\right]\\ &+ \mathrm{E}_0\left[\int_0^t \sum_{k\neq Z(s-)} \mathbb{1}_{\{Z(t)=i\}} h^{Z(s-)k}(s,W(s-))dN^k(s)\right].\end{aligned}$$

Based on the calculations in section C of Norberg (1991), note that

$$\mathrm{E}\left[N^{jk}(s) - \int_0^s \mathbb{1}_{\{Z(\tau-)=j\}}\mu_{jk}(\tau)\frac{p_{ki}(\tau,t)}{p_{ji}(\tau,t)}d\tau \big| Z(s-)=j, Z(t)=i\right] = 0.$$

As $h^{Z(s-)k}(s,W(s-)$ is predictable, we may therefore replace the integrator $dN^k(s)$ with $\mu_{jk}(s)\frac{p_{ki}(s,t)}{p_{ji}(s,t)}ds$, when conditioning on $Z(s-)=j, Z(t)=i$. Using the tower property once more,

$$\begin{aligned}\tilde{W}^i(t) =&p_{0i}(0,t)W(0) + \int_0^t \mathrm{E}_0\left[\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g^{Z(s)}(s,W(s))|Z(s)\right]\right]ds\\ &+ \mathrm{E}_0\left[\mathrm{E}_0\left[\int_0^t \sum_{k\neq Z(s-)} \mathbb{1}_{\{Z(t)=i\}} h^{Z(s-)k}(s,W(s-))dN^k(s)|Z(s-)\right]\right]\\ =&p_{0i}(0,t)W(0) + \int_0^t \sum_{j\in\mathcal{J}} p_{0j}(0,s)\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g^{Z(s)}(s,W(s))|Z(s)=j\right]ds\\ &+ \sum_{j\in\mathcal{J}} p_{0j}(0,s)\mathrm{E}_0\left[\int_0^t \sum_{k\neq j} \mathbb{1}_{\{Z(t)=i\}} h^{jk}(s,W(s-))dN^k(s)|Z(s-)=j\right]\end{aligned}$$

$$
\begin{aligned}
&= p_{0i}(0,t)W(0) + \int_0^t \sum_{j\in\mathcal{J}} p_{0j}(0,s)\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g^{Z(s)}(s,W(s))|Z(s)=j\right] ds \\
&\quad + \sum_{j\in\mathcal{J}} p_{0j}(0,s)p_{ji}(s,t)\mathrm{E}_0\left[\int_0^t \sum_{k\neq j} h^{jk}(s,W(s-))dN^k(s)|Z(s-)=j, Z(t)=i\right] \\
&= p_{0i}(0,t)W(0) + \int_0^t \sum_{j\in\mathcal{J}} p_{0j}(0,s)\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g^{Z(s)}(s,W(s))|Z(s)=j\right] ds && (20) \\
&\quad + \sum_{j\in\mathcal{J}} p_{0j}(0,s)\int_0^t \sum_{k\neq j} \mathrm{E}_0\left[h^{jk}(s,W(s-))|Z(s-)=j, Z(t)=i\right]\mu_{jk}(s)p_{ki}(s,t)ds. && (21)
\end{aligned}
$$

Since $W(s)$ is $\mathcal{F}_s$-measurable, the Markov property gives us

$$
\mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}}W(s)|Z(s)=j] = \frac{\tilde{W}^j(s)}{p_{0j}(0,s)}p_{ji}(s,t).
$$

Plugging into (20)-(21), and using that g and h are affine in W gives

$$
\begin{aligned}
\tilde{W}^i(t) =& p_{0i}(0,t)W(0) + \int_0^t \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(\frac{\tilde{W}^j(s)}{p_{0j}(0,s)} \circ g_1^j(s) + g_2^j(s)\right)p_{ji}(s,t)ds \\
&+ \int_0^t \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(\sum_{k\neq j}\mu_{jk}(t)p_{ki}(s,t)\left(\frac{\tilde{W}^j(s)}{p_{0j}(0,s)} \circ h_1^{jk}(s) + h_2^{jk}(s)\right)\right)ds \\
=& p_{0i}(0,t)W(0) + \int_0^t \sum_{j\in\mathcal{J}} p_{ji}(s,t)\tilde{W}^j(s)\circ g_1^j(s)ds \\
&+ \int_0^t \sum_{j\in\mathcal{J}}\sum_{k\neq j}\mu_{jk}(t)p_{ki}(s,t)\tilde{W}^j(s)h_1^{jk}(s)ds \\
&+ \int_0^t \sum_{j\in\mathcal{J}} p_{0j}(0,s)g_2^j(s)p_{ji}(s,t)ds \\
&+ \int_0^t \sum_{j\in\mathcal{J}} p_{0j}(0,s)\sum_{k\neq j}\mu_{jk}(t)p_{ki}(s,t)h_2^{jk}(s)ds.
\end{aligned}
$$

Differentiating with respect to t gives

$$\begin{aligned}\frac{d}{dt}\tilde{W}^i(t) =& W(0)\left(\sum_{k\neq i} p_{0k}(0,t)\mu_{ki}(t) - \mu_{ik}(t)p_{0i}(0,t)\right)\\ &+ \tilde{W}^i(t)\circ g_1^i(t) + p_{0i}(0,t)g_2^i(t)\\ &+ \sum_{k\neq i}\mu_{ki}(t)\left(\tilde{W}^k(t)\circ h_1^{ki}(t) + p_{0k}(0,t)h_2^{ki}(t)\right)\\ &+ \int_0^t \frac{\partial}{\partial t}\sum_{j\in\mathcal{J}} p_{ji}(s,t)\tilde{W}^j(s)\circ g_1^j(s)ds\\ &+ \int_0^t \frac{\partial}{\partial t}\sum_{j\in\mathcal{J}}\sum_{k\neq j}\mu_{jk}(t)p_{ki}(s,t)\tilde{W}^j(s)\circ h_1^{jk}(s)ds\\ &+ \int_0^t \frac{\partial}{\partial t}\sum_{j\in\mathcal{J}} p_{0j}(0,s)g_2^j(s)p_{ji}(s,t)ds\\ &+ \int_0^t \frac{\partial}{\partial t}\sum_{j\in\mathcal{J}} p_{0j}(0,s)\sum_{k\neq j}\mu_{jk}(t)p_{ki}(s,t)h_2^{jk}(s)ds.\end{aligned}$$

Using the Kolmogorov forward differential equations and recognizing $\tilde{W}^k$ and $\tilde{W}^i$, we arrive at

$$\begin{aligned}\frac{d}{dt}\tilde{W}^i(t) =& \tilde{W}^i(t)\circ g_1^i(t) + p_{0i}(0,t)g_2^i(t)\\ &+ \sum_{k\neq i}\mu_{ki}(t)\left(\tilde{W}^k(t)\circ h_1^{ki}(t) + p_{0k}(0,t)h_2^{ki}(t)\right)\\ &+ \sum_{k\neq i}\mu_{ki}(t)\tilde{W}^k(t) - \mu_{ik}(t)\tilde{W}^i(t).\end{aligned}$$

Combined with the initial condition

$$\tilde{W}^i(0) = \mathrm{E}_0[\mathbb{1}_{\{Z(0)=i\}}W(0)] = \mathbb{1}_{\{Z(0)=i\}}W(0),$$

we get the differential equations given by (12)-(15). For the case where some state, q, cannot be reached before time s for $s > 0$, the product of intensities for all paths from $Z(0)$ into that state must be zero for all τ when $\tau \leq s$, whereby $\tilde{W}^q(s) = 0$ and therefore the differential equations still hold. Thus the proof is complete. $\square$

B Dynamics of X and Y

The dynamics of X are found in (7), and given by

$$
\begin{aligned}
dX(t) =& r^*(t)X(t)dt + \delta^{Z(t)}(t, X(t), Y(t))dt - \sum_{k\neq Z(t-)} \rho^{Z(t-)k}(t, X(t-))dt \\
&- b^{Z(t)}(t, X(t))dt \\
&- \sum_{k\neq Z(t-)} \Big(b^{Z(t-)g}(t, X(t-)) + \chi^{Z(t-)k}(t, X(t-)) - X(t-) \Big) \mu^{Z(t)k}(t)dt \\
&+ \sum_{k\neq Z(t-)} \Big(\chi^{Z(t-)k}(t, X(t-)) - X(t-) \Big) dN^k(t),
\end{aligned}
$$

and the dynamics of Y are found in (9), and given by

$$
dY(t) = Y(t)\frac{dS(t)}{S(t)} - \delta^{Z(t)}(t, X(t), Y(t)) + (r(t) - r^*(t))X(t) + \sum_{k\neq Z(t)} \rho^{Z(t)k}(t, X(t)),
$$

for

$$
\begin{aligned}
\rho^{jg}(t, x) =& (b^{jg}(t, x) + \chi^{jg}(t, x) - x)(\mu^{*jg}(t) - \mu^{jg}(t)) \\
\chi^{jg}(t, x) =& V_1^{g*}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{g*}(t),
\end{aligned}
$$

$$
\delta^j(t, x, y) = \delta_1^j(t) + \delta_2^j(t)x + \delta_3^j(t)y.
$$

Note that ρ can be written as

$$
\begin{aligned}
\rho^{jg}(t, x) =& \bigg(x\frac{b_2^{jg}(t)}{V_2^{j*}(t)} + b_1^{jg}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} b_2^{jg}(t) \\
&+ V_1^{g*}(t) + x\frac{V_2^{g*}(t)}{V_2^{j*}(t)} - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{g*}(t) - x \bigg) (\mu^{*jg}(t) - \mu^{jg}(t)) \\
=& x \underbrace{\left(\frac{b_2^{jg}(t)}{V_2^{j*}(t)} + \frac{V_2^{g*}(t)}{V_2^{j*}(t)} - 1 \right) (\mu^{*jg}(t) - \mu^{jg}(t))}_{\rho_1^{jg}(t)} \\
&+ \underbrace{\left(b_1^{jg}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} b_2^{jg}(t) + V_1^{g*}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{g*}(t) \right) (\mu^{*jg}(t) - \mu^{jg}(t))}_{\rho_2^{jg}(t)} \\
=& \rho_1^{jg}(t) \cdot x + \rho_2^{jg}(t).
\end{aligned}
$$

We are interested in the specification of g_1, g_2, h_1 and h_2 for which the differential equation

$$\begin{aligned}
\frac{d}{dt}\tilde{W}^i(t) =& \sum_{j\neq i} \mu_{ji}(t)\tilde{W}^j(t) - \mu_{ij}(t)\tilde{W}^i(t) \\
&+ \tilde{W}^i(t) \circ g_1^i(t) + p_{Z(0)i}(0,t) g_2^i(t) \\
&+ \sum_{j\neq i} \mu_{ji}(t) \left(\tilde{W}^j(t) \circ h_1^{ji}(t) + p_{Z(0)j}(0,t) h_2^{ji}(t)\right) \\
\tilde{W}^i(0) =& \mathbb{1}_{\{Z(0)=i\}} W(0)
\end{aligned}$$

determines

$$\tilde{W}^j(t) := \begin{pmatrix} \tilde{X}^j(t) \\ \tilde{Y}^j(t) \end{pmatrix} = \begin{pmatrix} \mathrm{E}[X(t)\mathbb{1}_{\{Z(t)=j\}}] \\ \mathrm{E}[Y(t)\mathbb{1}_{\{Z(t)=j\}}] \end{pmatrix}.$$

The 2-dimensional functions g_1, g_2, h_1 and h_2 are given by

$$\begin{aligned}
g_1^j(t,x,y) &= \begin{pmatrix} g_{x1}^j(t) \\ g_{y1}^j(t) \end{pmatrix}, & h_1^{jk}(t,x,y) &= \begin{pmatrix} h_{x1}^{jk}(t) \\ h_{y1}^{jk}(t) \end{pmatrix}, \\
g_2^j(t,x,y) &= \begin{pmatrix} g_{x2}^j(t,y) \\ g_{y2}^j(t,x) \end{pmatrix}, & h_2^{jk}(t) &= \begin{pmatrix} h_{x2}^{jk}(t,y) \\ h_{y2}^{jk}(t,x) \end{pmatrix}.
\end{aligned}$$

For the g and h functions that describe the dynamics of X and Y. We separate the dynamics of X into the terms that are linear in X and those that are not, such that the continuous dynamics of X are given by $g_{x1}^{Z(t)}(t)X(t) + g_{x2}^{Z(t)}(t)$. The functions g_{x1} and g_{x2} can easily be found to be

$$\begin{aligned}
g_{x1}^j(t) =& r^*(t) + \delta_2^j(t) \\
&+ \frac{b_2^j(t)}{V_2^{j*}(t)} - \sum_{g\neq j} \rho_1^{jg}(t) - \sum_{g\neq j} \left(\frac{b_2^{jg}(t)}{V_2^{j*}(t)} + \frac{V_2^{g*}(t)}{V_2^{j*}(t)} - 1 \right) \mu^{jg}(t) \\
g_{x2}^j(t,y) =& \delta_1^j(t) + \delta_3^j(t)y - b_1^j(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} b_2^j(t) - \sum_{g\neq j} \rho_2^{jg}(t) \\
&- \sum_{g\neq j} \left(b_1^{jg}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} b_2^{jg}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{g*}(t) + V_1^{g*}(t) \right) \mu^{jg}(t).
\end{aligned}$$

Similarly we can write the non-continuous dynamics of X as

$$\begin{aligned}
h_x^{jk}(t,x,y) =& \chi^{jk}(t,x) - x \\
=& V_1^{k*}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{k*}(t) - x \\
=& x \underbrace{\left(\frac{V_2^{k*}(t)}{V_2^{j*}(t)} - 1 \right)}_{h_{x1}^{jk}(t)} + \underbrace{V_1^{k*}(t) - \frac{V_1^{j*}(t) V_2^{k*}(t)}{V_2^{j*}(t)}}_{h_{x2}^{jk}(t,y)} .
\end{aligned}$$

Concerning the continuous dynamics of Y we see

$$\begin{aligned}
g_y^{jk}(t,x,y) =& y \frac{dS(t)}{S(t)} - \delta^j(t,x,y) + (r(t) - r^*(t))x + \sum_{g \neq j} \rho^{jg}(t,x) \\
=& y \frac{dS(t)}{S(t)} - \delta_1^j(t) - \delta_2^j(t)x - \delta_3^j(t)y \\
&+ (r(t) - r^*(t))x + \sum_{k \neq j} \rho^{jk}(t,x) \\
=& y \underbrace{\left(\frac{dS(t)}{S(t)} - \delta_3^j(t) \right)}_{g_{y1}^j(t)} \\
&\underbrace{+ \sum_{k \neq j} \rho^{jk}(t,x) - \delta_1^j(t) - \delta_2^j(t)x + (r(t) - r^*(t))x}_{g_{y2}^j(t,x)}
\end{aligned}$$

Finally, as $h_y = 0$ we have $h_{y1} = h_{y2} = 0$.

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