1 Introduction

With-profit insurance contracts are to this day one of the most popular life insurance contracts. They arose as a natural way to distribute the systematic surplus that emerges due to the prudent assumptions on which the contract is made. In recent years, sensible questions accompanied by a lot of attention have been aimed at the surplus, to name a few; is it distributed fairly? how should it be invested? How is it affected by the financial market? To answer these questions we need to understand the dynamics of the surplus in a model of practical relevance. The study of surplus and the interplay it has with other elements of an insurance contract, is not new. Norberg (1999) introduces the notion of individual surplus as well as the mean portfolio surplus. In Steffensen (2000) and Steffensen (2001), partial differential equations are used to describe the prospective second order reserve for various forms of bonus, when the surplus is invested in a Black-Scholes market. In this paper we pay little regard to the prospective reserve, and instead focus on the surplus and the retrospective second order reserve, also called the savings account.

The expected future value of the savings account is of particular interest, as it embodies the accumulation of dividends as well as the use of these dividends to increase benefits or decrease premiums. Due to the retrospective nature of the savings account and surplus, we may also take other retrospective considerations into account. In the existing literature, very little attention is paid to a very significant retrospective element of the with-profit insurance contract: the human element.

Insurance companies are governed by humans, and the decisions they make have an influence on the portfolio of policies - in particular concerning surplus and dividends. In a with-profit insurance contract many quantities are fixed at initialisation of the policy, but the rate at which dividends are paid out is not. The insurance company has a certain degree of freedom when it comes to the distribution of surplus, and the actions that have an influence on the insurance contracts are the so-called Management Actions. From a mathematical point of view, they pose a problem as they depend on the entire history of the portfolio of policies in a possibly non-linear fashion, making it difficult to calculate prospective reserves. If we want to take a glance into the crystal ball of liabilities, taking Future Management Actions (FMA's) into account, we need to embrace it's retrospective nature. In this paper, we do not incorporate FMA's to their full extent, but rather lay the retrospective groundwork on which models including FMA's can be built.

We derive a retrospective differential equation for the expected savings account and surplus, in a general model with affine dynamics. We devote special attention to a realistic model with affine dynamics, where dividends are used to increase future benefits.

1.1 Set-up

We consider the classic multi-state life insurance set-up, comprised of a state process Z denoting the state of the policy in a finite state space $\mathcal{J} = \{0, 1, ..., J\}$. By a permutation argument, we can without loss of generality assume that $Z(0) = 0$. The filtration generated by $Z(t)$ is denoted by $\mathcal{F}_t$. The counting process N^k defined by $N^k(t) = \#\{s; Z(s-) = i, Z(s) = k, s \in (0, t]\}$ describes the number of transitions into state k. The state process Z is assumed to be a continuous time Markov chain, with transition probabilities denoted by

$$p_{ij}(s, t) = \mathrm{P}(Z(t) = j | Z(s) = i)$$

for $s \leq t$. The corresponding transition intensities are denoted by

$$\mu_{ij}(t) = \lim_{h \searrow 0} p_{ij}(t, t + h)/h$$

for $i \neq j$. The predictable process $\mathbb{1}_{\{Z(t-) \neq k\}} \mu_{Z(t-)k}(t)$ is the intensity process for $N^k(t)$, i.e

$$M^k(t) := N^k(t) - \int_0^t \mathbb{1}_{\{Z(s-) \neq k\}} \mu_{Z(s-)k}(s) ds,$$

forms a martingale. The state process Z encapsulates the biometric risks involved with the insurance contract. Apart from the biometric risk, there is a financial risk connected to with-profit insurance contracts through the return on investment of the surplus. We make assumptions regarding the financial risk, by specifying the expected return on investment, r. Together, the transition intensities and expected return on investment form the second order basis, which describes the best guess on future development of the insurance portfolio. We take this second order basis as exogenously given. Note that a Monte-Carlo method can be used as a proxy for evaluation under the second order basis; perform evaluation under n simulated second order basis and take the mean. The Monte-Carlo approach for evaluation allows for great model flexibility, which is particularly appealing regarding the expected return on investment.

While the second order basis forms the best guess on future developments of the relevant technical elements, it would be far too risky for an insurance company to use these assumptions when signing contracts. What if a cure for cancer is invented in 10 years, or if the stock market crashes? To allow for events that make it difficult to meet the obligations to the insured, a much less risky set of assumptions are used when guarantees are given. These prudent assumptions form the first order (technical) basis. Using the standard notation, a " $*$ " symbolises first-order basis elements. It is precisely due to the difference between the first order basis and the realised (third order) basis that a surplus emerges. We have no way of knowing what the future is going to bring, so we cannot know how the surplus is going to evolve. We can however make an estimate by using the second order basis as a stand-in for the third order basis.
In order to define an insurance contract we introduce the payment process B, which depends on the dynamics of Z. The payment process is an $\mathcal{F}_t$-adapted process with dynamics given by

$$dB(t) = b^{Z(t)}(t)dt + \sum_{k \neq Z(t-)} b^{Z(t-)k}(t)dN^k(t).$$

The deterministic payment functions $b_i^j(t)$ and $b_i^{jk}(t)$ specify payments during sojourns in state j and on transition from state j to state k, respectively. Even though single payments during sojourns in states pose no mathematical difficulty, we assume that payments during sojourns in states are continuous for notational simplicity. Given the payment process B, we can define the prospective technical reserve as

$$V^{j*}(t) = \mathrm{E}\left[\int_t^n e^{-\int_t^s r*} dB(s) | Z(t) = j\right].$$

The dynamics of the technical reserves are found using Itô's lemma for FV-functions to be

$$\begin{aligned} dV^{Z(t)*}(t) =& r^*(t)V^{Z(t)*}(t)dt - b^{Z(t)}(t) - \sum_{k \neq Z(t-)} b^{Z(t-)k}(t)dN^k(t) \\ &- \sum_{k \neq Z(t-)} \rho^{Z(t-)k}(t)dt + \sum_{k \neq Z(t-)} R^{Z(t-)k}(t)(dN^k(t) - \mu_{Z(t-)k}(t)dt), \end{aligned} \tag{1}$$

where ρ^{jk} is the so-called risk premium for a transition from state j to state k, and R^{jk} is the so-called sum-at-risk for a transition from j to k. The sum-at-risk R^{jk} describes the required injection of capital on a transition from j to k in order to meet the future liabilities of the contract in state k, evaluated under the first-order basis. The sum-at-risk is given by

$$R^{jk}(t) = b^{jk}(t) + V^{k*}(t) - V^{j*}(t).$$

As the name suggests, the risk premium is the premium paid by the policyholder to cover the risk carried by the insurer that can not be diversified, such as medical advancements. Naturally the risk premium is the sum-at-risk multiplied by the difference in intensity for a transition from j to k between the first-order basis and the second-order basis, i.e

$$\rho^{jk}(t) = R^{jk}(t)(\mu^*_{jk}(t) - \mu_{jk}(t)).$$

Retrospective Reserve Without Bonus

One of the main contributions of Norberg (1991) is a definition of the retrospective reserve, as a conditional expected value of a past payments, in much the same manner as the prospective reserve is a conditional expected value of future payments. Formally Norberg (1991) defines the retrospective first order reserve, as

$$V^*_{\mathbb{E}}(t) = \mathrm{E}^* \left[\int_0^t e^{\int_s^t r^*} dB(s) \big| \mathcal{E}_t \right]$$

for some family of sigmaalgrebras $\mathbb{E} = \{\mathcal{E}_t\}_{0\leq t}$, where $\mathcal{E}_t$ represents the information available at time t. It is for an actuary very natural to assume that $\mathcal{E}_t = \sigma\{Z(s), 0 \leq s \leq t\}$, implying that all information about the past is accounted for. As noted by Norberg (1991) the family of sigmaalgebras may be increasing, i.e $\mathcal{E}_s \subseteq \mathcal{E}_t$ for $s < t$, but it is not required. So with this very general definition of the retrospective reserve, we may discard information, for instance by defining $\mathcal{E}_t = \sigma\{Z(0), Z(t)\}$. But why should we ever choose to discard information that is available to us? Because it is intractable to use $\mathcal{E}_t = \mathcal{F}_t$ when we want to calculate the expected value of $V^*_{\mathbb{E}}(t)$ and $\{Z(s)\}_{s\leq t}$ has not yet been realised. Computationally it is simply too demanding to take the expectation over $\mathcal{F}_t$ - all possible paths and all possible transition times have to be considered. Instead we therefore let $\mathcal{E}_t = \sigma\{Z(0), Z(t)\}$, implying that we only use the state at initialization and time t to evaluate the retrospective reserve. Using this formulation of $\mathcal{E}_t$, the retrospective reserve can be interpreted as the average reserve of a group of policies that all start in $Z(0)$ and end in $Z(t)$. In order to actually calculate this retrospective reserve, we note by the markov property that

$$P(Z(s) = j | Z(0) = 0, Z(t) = i) = \frac{p_{0j}(0,s)p_{ji}(0,t)}{p_{0i}(s,t)}, \tag{2}$$

and that the predictable compensator for $N^{jk}(s)|(Z(0) = 0, Z(t) = i)$ has intensity given by

$$\mathbb{1}_{\{Z(s-)=j\}}\mu_{jk|0i}(s|0,t) = \mathbb{1}_{\{Z(s-)=j\}}\mu_{jk}(s)\frac{p_{ki}(s,t)}{p_{ji}(s,t)}. \tag{3}$$

Define

$$V_i^*(t) = \mathrm{E}^*\left[\int_0^t e^{\int_s^t r*} dB(s)\big|Z(0)=0, Z(t)=i\right],$$

which by (2) and (3) is equal to

$$\begin{aligned}V_i^*(t) =& \int_0^t e^{\int_s^t r*} \sum_{j\in\mathcal{J}} \frac{p_{0j}(0,s)p_{ji}(s,t)}{p_{0i}(0,t)} \left(b^j(s) + \sum_{k\neq j} \mu_{jk}(s) b^{jk}(s) \frac{p_{ki}(s,t)}{p_{ji}(s,t)}\right) ds \\ =& \frac{1}{p_{0i}(0,t)} \int_0^t e^{\int_s^t r*} \sum_{j\in\mathcal{J}} p_{0j}(0,s) \left(p_{ji}(s,t)b^j(s) + \sum_{k\neq j} \mu_{jk}(s) b^{jk}(s) p_{ki}(s,t)\right) ds \qquad (4)\end{aligned}$$

as also derived by Norberg (1991). In itself (4) provides an interpretation of the retrospective reserve; it is the accumulated payments on transition and sojourn payments at all times prior to t, weighted by the corresponding probability of transition between states and sojourns in states, given the initial and terminal state of the policy.For sufficiently nice intensities and payment functions, analytical solutions for $V_i^*(t)$ can be derived. In general, we cannot provide a closed form expression for $V_i^*(t)$, and instead we have to rely on numerical methods, for instance by a numerical solution to the differential equation solved by $V_i^*(t)$. As it is a nuisance to directly derive a differential equation for V_i^*, due to the division by the probability of entering state i at time t, we define

$$\tilde{V}_i^*(t) = \mathrm{E}^*\left[\mathbb{1}_{\{Z(t)=i\}} \int_0^t e^{\int_s^t r^*} dB(s)\big|Z(0)=0\right] = V_i^*(t)p_{0i}(0,t).$$

Using the Kolmogorov differential equations, $p_{0i}(0,t)$ can be calculated for all i and t, and thus $V_i^*(t)$ can easily be calculated once $\tilde{V}_i^*(t)$ is available. Differentiating $\tilde{V}_i^*(t)$ with respect to t

gives

$$
\begin{aligned}
\frac{d}{dt}\tilde{V}_i^*(t) =& \frac{d}{dt}\int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(p_{ji}(s,t)b^j(s) + \sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{ki}(s,t)\right) ds \\
=& \sum_{j\in\mathcal{J}} p_{0j}(0,t)\left(\mathbb{1}_{\{j=i\}}b^j(t) + \sum_{k\neq j}\mu_{jk}(t)b^{jk}(t)\mathbb{1}_{\{k=i\}}\right) \\
&+ \int_0^t \frac{d}{dt}e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(p_{ji}(s,t)b^j(s) + \sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{ki}(s,t)\right) ds \\
=& p_{0i}(0,t)b^i(t) + \sum_{j\neq i} p_{0j}(0,t)\mu_{ji}(t)b^{ji}(t) + r^*(t)\tilde{V}_i^*(t) \\
&+ \int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(\frac{d}{dt}p_{ji}(s,t)b^j(s) + \sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)\frac{d}{dt}p_{ki}(s,t)\right) ds.
\end{aligned}
$$

The Kolmogorov forward differential equations state that

$$
\frac{d}{dt}p_{ji}(s,t) = \sum_{g\neq i} p_{jg}(s,t)\mu_{gi}(t) - \mu_{ig}(t)p_{ji}(s,t)
$$

which imply that

$$
\begin{aligned}
\frac{d}{dt}\tilde{V}_i^*(t) =& p_{0i}(0,t)b^i(t) + \sum_{j\neq i} p_{0j}(0,t)\mu_{ji}(t)b^{ji}(t) + r^*(t)\tilde{V}_i^*(t) \\
&+ \sum_{g\neq i}\mu_{gi}(t)\int_0^t e^{\int_s^t r^*}\sum_{j\in\mathcal{J}} p_{0j}(0,s)p_{jg}(s,t)b^j(s)ds \\
&- \sum_{g\neq i}\mu_{ig}(t)\int_0^t e^{\int_s^t r^*}\sum_{j\in\mathcal{J}} p_{0j}(0,s)p_{ji}(s,t)b^j(s)ds \\
&+ \sum_{g\neq j}\mu_{gi}(t)\int_0^t e^{\int_s^t r^*}\sum_{j\in\mathcal{J}} p_{0j}(0,s)\sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{kg}(s,t)ds \\
&- \sum_{g\neq i}\mu_{ig}(t)\int_0^t e^{\int_s^t r^*}\sum_{j\in\mathcal{J}} p_{0j}(0,s)\sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{kj}(s,t)ds
\end{aligned}
$$

$$
\begin{aligned}
=& p_{0i}(0,t)b^i(t) + \sum_{j\neq i} p_{0j}(0,t)\mu_{ji}(t)b^{ji}(t) + r^*(t)\tilde{V}_i^*(t) \\
&+ \sum_{g\neq i}\mu_{gi}(t) \underbrace{\int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(p_{jg}(s,t)b^j(s)\sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{kg}(s,t)\right) ds}_{\tilde{V}_g^*(t)} \\
&- \sum_{g\neq i}\mu_{ig}(t) \underbrace{\int_0^t e^{\int_s^t r^*} \sum_{j\in\mathcal{J}} p_{0j}(0,s)\left(p_{ji}(s,t)b^j(s)\sum_{k\neq j}\mu_{jk}(s)b^{jk}(s)p_{ki}(s,t)\right) ds}_{\tilde{V}_i^*(t)} \\
=& p_{0i}(0,t)b^i(t) + \sum_{j\neq i} p_{0j}(0,t)\mu_{ji}(t)b^{ji}(t) + r^*(t)\tilde{V}_i^*(t) \qquad (5) \\
&+ \sum_{g\neq i}\mu_{gi}(t)\tilde{V}_g^*(t) - \mu_{ig}(t)\tilde{V}_i^*(t), \qquad (6)
\end{aligned}
$$

and along with the initial condition

$$
\tilde{V}_i^*(0) = 0 \quad \text{for all } i,
$$

we have a system of differential equations describing $\tilde{V}_i^*(t)$. These differential equations have certain similarities with the classical prospective Thiele differential equations. The retrospective probability weighted reserve $\tilde{V}_i^*(t)$ develops in accordance with the probability weighted payments, the first order interest, and a diffusion between the reserves. Interestingly these differential equations are generalisations of the Kolmogorov forward differential equations. This can be seen by letting $r^*(t) = 0$ and defining the payment process

$$
dB(t) = \mathbb{1}_{\{t=s\}}\mathbb{1}_{\{Z(t)=j\}},
$$

that has a payout of one unit at time s, if $Z(t) = j$. Then

$$
\tilde{V}_i(t) = \mathrm{E}\left[\int_0^t \mathbb{1}_{\{Z(\tau)=j\}} d\delta_s(\tau) | Z(0)=0, Z(t)=i\right] p_{0i}(0,t) = p_{0j}(0,s)p_{ji}(s,t),
$$

and $\tilde{V}_i(s) = p_{0j}(0,s)\mathbb{1}_{\{i=j\}}$. Using the differential equation for the retrospective reserve, we see that for $s < t$

$$p_{0j}(0,s)\frac{d}{dt}p_{ji}(s,t) = \sum_{g \neq i} \mu_{gi}(t)p_{0j}(0,s)p_{jg}(s,t) - \mu_{ig}(t)p_{0j}(0,s)p_{ji}(s,t),$$

$$\Leftrightarrow$$

$$\frac{d}{dt}p_{ji}(s,t) = \sum_{g \neq i} \mu_{gi}(t)p_{jg}(s,t) - \mu_{ig}(t)p_{ji}(s,t),$$

which constitute the Kolmogorov forward differential equations. Norberg (1991) defined the retrospective reserve, and derived some of its important properties. At the time, the retrospective reserve was perhaps more of a mathematical curiosity than an actuarial tool, as the prospective reserves at the time provided all the information you could ask for. However, the retrospective reserve definitely deserves recognition when surplus and dividends are introduced. Norberg (1999) defines the individual surplus as a retrospective reserve, and derives a differential equation hereof in a simple model where no dividends are allotted.

1.1.1 Set-Up Including Surplus and Dividends

In this section we expand our set-up such that we can accurately describe the benefits and balances in a model where surplus and dividends are included. The first order basis on which insurance contracts are signed, are a set of prudent assumptions regarding interest and transition intensities. Knowing that the assumptions are prudent, the insurer and insured agree that when surplus has emerged as a consequence of the realized interest and transition intensities, this surplus should be given back to the insured. The surplus is returned to the insured through a dividend payment stream. What the insured chooses to do with his dividend can vary, but a very standard product design is to use the dividends to buy more insurance. In a sense, the dividend payment stream becomes a premium for a bonus benefit payment stream. We introduce two payment streams B_1 and B_2 with dynamics

$$dB_i(t) = b_i^{Z(t)}(t)dt + \sum_{k \neq Z(t-)} b_i^{Z(t-)k}(t)dN^k(t).$$

The payments specified by B_1 are the benefits which are fixed, and part of the original contract. The payments of B_2 specify the profile of the benefit stream that the dividend is converted into. When the contract is signed, both B_1 and B_2 are agreed upon, and while there is practically no restriction on their design, we assume that B_2 contains benefits only, implying that dividends are used to increase benefits, and not to decrease premiums. The payment streams B_1 and B_2, have corresponding technical reserves given by

$$V_i^{j*}(t) = \mathrm{E}\left[\int_t^n e^{-\int_t^s r^*} dB_i(s) \big| Z(t) = j\right].$$

We assume that the benefits of B_2 are triggered by events that have non-zero probability i.e. $V_2^{j*}(t) > 0$. In order to keep track of how much dividend has been materialized into the B_2 payment stream, we introduce the process $Q(t)$ which denotes the quantity of B_2 payment stream purchased at time t. We do not yet specify the dynamics of $Q(t)$. The payment process experienced by the policyholder, B, consists of one unit B_1 payment stream and Q units of B_2 payment stream, thus having dynamics

$$dB(t) = dB_1(t) + Q(t-)dB_2(t).$$

Where the left-limit version of Q is used in order to ensure that B is adapted to $\mathcal{F}_t$. We now define the savings account as the technical value of future guaranteed payments, given a certain quantity of B_2 payment stream,

$$\begin{aligned}
X(t) =& \mathrm{E}^*\left[\int_t^n e^{\int_t^s r*} d\left(B_1(s) + Q(t)B_2(s)\right)\right] \\
=& V_1^{Z(t)*}(t) + Q(t)V_2^{Z(t)*}(t).
\end{aligned}$$

Noting that

$$Q(t) = \frac{X(t) - V_1^{Z(t)*}(t)}{V_2^{Z(t)*}(t)}$$

we see that the payment stream experienced by the policyholder has dynamics

$$\begin{aligned}
dB(t) =& dB_1(t) + \frac{X(t-) - V_1^{Z(t-)*}(t-)}{V_2^{Z(t-)*}(t-)} dB_2(t) \\
=& b^{Z(t)}(t, X(t))dt + \sum_{k \neq Z(t-)} b^{Z(t-)k}(t, X(t-))dN^k(t),
\end{aligned}$$

for deterministic functions b^j and b^{jk} corresponding to sojourn payments and payments on transition, given by

$$\begin{aligned}
b^j(t, x) &= b_1^j(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^j(t) \\
b^{jk}(t, x) &= b_1^{jk}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^{jk}(t).
\end{aligned}$$

By introducing subscripts, the dynamics of $V_1^{Z(t)*}$ and $V_2^{Z(t)*}$ are given by (1). Using integration by parts for FV-functions we find the dynamics of X to be

$$\begin{aligned}dX(t) =& dV_1^{Z(t)*}(t) + Q(t-)dV_2^{Z(t)*}(t) + V_2^{Z(t)*}(t)dQ(t)\\ =& r^*(t)X(t)dt + V_2^{Z(t)*}(t)dQ(t) - b^{Z(t)}(t, X(t))dt - \sum_{k\neq Z(t-)} b^{Z(t-)k}(t, X(t-))dN^k(t)\\ &- \sum_{k\neq Z(t-)} \rho^{Z(t-)k}(t, X(t-))dt\\ &+ \sum_{k\neq Z(t-)} R^{Z(t-)k}(t, X(t-))(dN^k(t) - \mu_{Z(t-)k}(t)dt),\end{aligned} \tag{7}$$

where

$$\begin{aligned}\rho^{jk}(t, X(t-)) =& \rho_1^{jk}(t) + Q(t-)\rho_2^{jk}(t) = \rho_1^{jk}(t) + \frac{X(t-) - V_1^{j*}(t-)}{V_2^{j*}(t-)}\rho_2^{jk}(t),\\ R^{jk}(t, X(t-)) =& R_1^{jk}(t) + Q(t-)R_2^{jk}(t) = R_1^{jk}(t) + \frac{X(t-) - V_1^{j*}(t-)}{V_2^{j*}(t-)}R_2^{jk}(t),\end{aligned}$$

respectively can be interpreted as the risk premium and sum-at-risk for the savings account. The savings account plays a crucial role in the understanding of the with-profit insurance contract, just as the first order reserve plays a crucial role in the model without dividends. Given the savings account, we can readily define the surplus as

$$Y(t) = -\int_0^t e^{\int_s^t r} dB(s) - X(t),$$

corresponding to the accumulated premiums less benefits excess over the savings account. It is clear from the definition of the surplus that the savings account has an influence on Y, but the surplus also has an influence on the savings account though the dividends. The dividends flow from the surplus to the savings account, according to some dividend strategy determined by the insurer. These dividends are instantaneously used to increase benefits, by buying more of the B_2 payment stream. These additional benefits are, like the fixed benefits, priced under the first order basis, which means that one unit of B_2 has a value of $V_2^{Z(t)*}(t)$. The total amount of accrued dividends at time t are denoted by $D(t)$, and as the dividends are used to buy B_2, we must have that

$$dD(t) = V_2^{Z(t)*}(t)dQ(t). \tag{8}$$

By the principle of equivalence

$$0 = X(0) = V_1^{0*}(0) + Q(0-)V_2^{0*}(0)$$
$$\Leftrightarrow$$
$$Q(0-) = -\frac{V_1^{0*}(0)}{V_2^{0*}(0)}$$

providing us with the initial condition for Q, which along with (8) fully specifies Q. Note that the principle of equivalence puts no restrictions on the form of B_1 and B_2.

We assume the dynamics of the dividend process is given by

$$dD(t) = \delta^{Z(t)}(t, X(t), Y(t))dt,$$

for some deterministic function δ^j on the form

$$\delta^j(t, x, y) = \delta_1^j(t) + \delta_2^j(t)x + \delta_3^j(t)y.$$

In this paper, the form of δ is probably the assumption most eligible for criticism. In practice, the dividend is determined by an actuary who takes much more information into account than simply the value of the savings and surplus. Furthermore the dividend-deciding actuary is most likely going to take past development of the savings and surplus into account. The specification of the dynamics of D is at the heart of what a future management action is, and, as stated earlier, we do not fully incorporate these FMA's in all their generality and glory, but suffice with crude surrogates. Some of these crude surrogates can actually perform a decent job at describing real world dividend strategies, for instance by defining the dividend as some linear function of the contribution.
The dynamics of Y can easily be derived to be

$$dY(t) = rY(t)dt + dC(t) - dD(t) - \sum_{k \neq Z(t-)} R^{Z(t-)k}(t, X(t-))dM^k(t), \tag{9}$$

for

$$dC(t) = (r(t) - r^*(t))X(t)dt + \sum_{k \neq Z(t)} \rho^{Z(t)k}(t, X(t))dt,$$

which we call the contribution process, as it represents the contributions from the savings account to the surplus. As the dynamics of X and Y are affine, we can, for suitable affine functions g and h, write the dynamics of X and Y as

$$dX(t) = g_x^{Z(t)}(t, X(t), Y(t))dt + \sum_{k \neq Z(t-)} h_x^{Z(t-)k}(t, X(t-), Y(t-))dN^k(t) \tag{10}$$

$$dY(t) = g_y^{Z(t)}(t, X(t), Y(t))dt + \sum_{k \neq Z(t-)} h_y^{Z(t-)k}(t, X(t-), Y(t-))dN^k(t). \tag{11}$$

We refer to section C of the appendix for the specification of g and h leading to the dynamics given in (7) and (9). Apart from notational ease, the use of arbitrary g and h functions serve to generalise the results of the paper to any FV-process with affine dynamics of the form given by (10) and (11). Even though we work with the dynamics given by (10) and (11), we think of the g and h functions as the ones required to achieve the dynamics of (7) and (9). As we are interested in the interconnected dynamics of X and Y, we introduce the two-dimensional process

$$W(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}.$$

with dynamics given by

$$dW(t) = g^{Z(t)}(t, W(t))ds + \sum_{k \neq Z(t-)} h^{Z(t-)k}(t, W(t-))dN^k(t),$$

for g and h functions that are affine in all entries of W, and determined by the dynamics of X and Y. In practice, the surplus account is shared among, say N, policyholders. In that case, W should instead be an $N+1$-dimensional process - one dimension for each policyholder, and one dimension for the common surplus. This implies that we need a system of $\#\{\mathcal{J}\}^N + 1$ differential equations; one for each combination of all policy states, and one for the common surplus. There are several ways to reduce the dimensionality of the problem, making it computationally tractable. One way to diminish the problem of dependency between policyholders, is to discretize the dividend function and comprise it as lump-sum payments, which conforms to real-world practise. When there are no dividends, the savings are independent of surplus, and thus also the other policies. In between lump-sum payments of dividend, the state-wise contributions including investment gains from each policy is accumulated. When a lump-sum time is reached, the probability weighted sum of contributions are contributed to a single state-independent surplus where after the dividend is allocated. This method requires

$\#\{\mathcal{J}\} \times 2 \times N + 1$ differential equations be solved. One for the surplus, and one for the savings account and contribution for each state of each policy. Furthermore note that in between lump-sum payments of dividend, the policies are independent, allowing for parallelization across N cores. Even though $\#\{\mathcal{J}\} \times 2 \times N + 1$ independent differential equations is a vast improvement from $\#\{\mathcal{J}\}^N + 1$ dependent differential equations, it is still a computationally difficult problem, and a further digression on the subject is outside the scope of this paper.

It is important to realize the extent of applicable models that have affine dynamics, see Christiansen et al. (2014) for several relevant payment functions that are linear in the reserve, which corresponds to the savings when $D(t) = 0$ for all t. While the reach of models with affine dynamics is extensive, there are limitations to consider. It is not uncommon to have dynamics that include some min or max function, for instance in the case of guarantees, and these non-linear functions in savings cannot be described by affine dynamics.
As stated in the introduction, management actions are one of the main motivators of this paper, but they are hidden in mainly two terms; the second order interest and the dividend. This is because the management decides how to invest the surplus, and how it should be distributed to the customers. Due to the very human and abstract nature of management actions, we do not incorporate them directly in the dynamics of the savings and surplus, but instead let them work in the shadows.

2 State-Wise Probability Weighted Reserve

In this section we present the main result, which generalizes the result from Norberg (1991) by allowing for processes whose dynamics are affine functions of the process itself. To illustrate the central idea, consider the case where W has dynamics

$$dW(s) = g^{Z(s)}(s)W(s)ds,$$

and say we want to calculate

$$\tilde{W}^i(t) := \mathrm{E}_0[W(t)\mathbb{1}_{\{Z(t)=i\}}] = \mathrm{E}_0[W(t)|Z(t) = i]p_{Z(0)i}(0,t),$$

where we by the subscript 0 on the expectation denote the conditional expectation given $Z(0)$ and $W(0)$. That is

$$\mathrm{E}_0[\mathcal{A}] = \mathrm{E}[\mathcal{A}|Z(0) = 0, W(0)].$$

We can write $W(t)$ as an integral from 0 to t over the dynamics of W,

$$W(t) = W(0) + \int_0^t W(s) g^{Z(s)}(s) ds.$$

By the tower property and Fubinis theorem,

$$\begin{aligned}
\tilde{W}^i(t) =& p_{0i}(0,t)W(0) + \int_0^t \mathrm{E}[\mathbb{1}_{\{Z(t)=i\}} W(s) g^{Z(s)}(s)] ds \\
=& p_{0i}(0,t)W(0) + \int_0^t \mathrm{E}_0 \left[\sum_{j \in \mathcal{J}} \mathbb{1}_{\{Z(s)=j\}} \mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}} W(s) g^{Z(s)}(s) | Z(s) = j] \right] ds \\
=& p_{0i}(0,t)W(0) + \int_0^t \sum_{j \in \mathcal{J}} p_{Z(0)j}(0,s) g^j(s) \mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}} W(s) | Z(s) = j]] ds
\end{aligned}$$

By the Markov property $W(s) \perp\!\!\!\perp Z(t)|Z(s)$ for $s < t$, as $W(s)$ is $\mathcal{F}_s$-measurable, and therefore

$$\tilde{W}^i(t) = p_{0i}(0,t)W(0) + \int_0^t \sum_{j \in \mathcal{J}} g^j(s) \tilde{W}^j(t) p_{ji}(s,t) ds.$$

Differentiating with respect to t, and using Kolmogorov's forward differential equations yields the following system of differential equations

$$\begin{aligned}
\frac{d}{dt}\tilde{W}^i(t) =& \tilde{W}^i(t) g^i(t) + \sum_{j \neq i} \mu_{ji}(t) \tilde{W}^j(t) - \mu_{ij}(t) \tilde{W}^i(t) \\
\tilde{W}^i(0) =& \mathbb{1}_{\{Z(0)=i\}} W(0).
\end{aligned}$$

It is crucial to note that this differential equation relies on the affine structure of the dynamics of W, as it allows us to write $\tilde{W}^i(t)$ as an integral over $\tilde{W}^j(s)$. The result is generalized by considering a general FV process with dynamics that are affine in the process itself. By using the tower property and the fact that $W(s-) \perp\!\!\!\perp Z(t)|Z(s-)$, we get the following theorem.

Theorem 2.1.

Let $Z(t)$ be a Markov process on the state space $\mathcal{J}$, and let $W(t)$ be a $\mathcal{F}_t$-measurable process with dynamics

$$dW(s) = g^{Z(s)}(s, W(s)) ds + \sum_{k \neq Z(s-)} h^{Z(s-)k}(s, W(s-)) dN^k(s)$$

for g and h of the form

$$\begin{aligned} g^{Z(s)}(s, W(s)) =& g_1^{Z(s)}(s)W(s) + g_2^{Z(s)}(s) \\ h^{Z(s-)k}(s, W(s-)) =& h_1^{Z(s-)k}(s)W(s-) + h_2^{Z(s-)k}(s). \end{aligned}$$

Then $\tilde{W}^i(t) = E_0[\mathbb{1}_{\{Z(t)=i\}}W(t)]$ is described by the differential equation

$$\frac{d}{dt}\tilde{W}^i(t) = \sum_{j\neq i} \mu_{ji}(t)\tilde{W}^j(t) - \mu_{ij}(t)\tilde{W}^i(t) \tag{12}$$

$$+ \tilde{W}^i(t)g_1^i(t) + p_{Z(0)i}(0,t)g_2^i(t) \tag{13}$$

$$+ \sum_{j\neq i} \mu_{ji}(t)\left(\tilde{W}^j(t)h_1^{ji}(t) + p_{Z(0)j}(0,t)h_2^{ji}(t)\right) \tag{14}$$

$$\tilde{W}^i(0) = \mathbb{1}_{\{Z(0)=i\}}W(0) \tag{15}$$

The differential equations given by (12)-(15) bear close resemblance to the differential equation for the retrospective reserve given by (5)-(6). In fact, for

$$\begin{aligned} g_1 = h_1 = 0, &\qquad W(0) = 0, \\ g_2(t,i) = b^i(t), &\qquad h_2(t,j,i) = b^{ji}(t), \end{aligned}$$

we arrive at the differential equation derived by Norberg (1991). Norberg (1991) allows for dynamics that depend on the expected value of W, while we allow for dynamics of the process to depend on the process itself, which is a necessity in a model including bonus. The terms (12)-(14) in the differential equation can be intuitively explained.

If the policy is in state i at time t, it will develop with the continuous dynamics of that state, given by $W(t)g_1(t,i) + g_2(t,i)$. Due to the uncertainty involved pertaining to the state of the policy and the value of W, we have to weigh these dynamics with the probability of $Z(t) = i$, as well as the expected value of W, thus arriving at (13) as

$$\mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}}\left(W(t)g_1(t,i) + g_2(t,i)\right)] = \tilde{W}^i(t)g_1(t,i) + p_{Z(0)i}(0,t)g_2(t,i).$$

Similarly, we have to account for any transitions into the current state i, over the small interval $t + dt$. The infinitesimal probability of transition from j to i over an interval from t to $t + dt$ is given by $\mu_{ji}(t)$, and if such a transition was made, the savings and surplus are bumped by $W(t)h_1(t,j,i) + h_2(t,j,i)$. In order for a transition from j to i to be possible over the interval $t + dt$, the policy has to be in state j at time t, thus arriving at (14) as

$$\mathrm{E}_0[\mathbb{1}_{\{Z(t)=j\}}\left(W(t)h_1(t,j,i) + h_2(t,j,i)\right)] = \tilde{W}^j(t)h_1(t,j,i) + p_{Z(0)j}(0,t)h_2(t,j,i).$$

Furthermore, when a transition from j to i is made, the savings and surplus from state j (after the bump) are transferred to the savings and surplus of state i, amounting to the term given in (12). A myriad of models fit into the framework of Theorem 2.1, and even if only the savings account and surplus are projected, several other quantities of interest can be derived from these, for instance the present value of guaranteed future benefits

$$\begin{aligned}
\mathrm{GY}_i(t) =& \mathrm{E}\left[\int_t^n e^{-\int_t^s r} d\left(B_1(s) + \frac{X(t) - V_1^{Z(t)*}(t)}{V_2^{Z(t)*}(t)} B_2(s)\right) \big| Z(t) = i\right] \\
=& \mathrm{E}\left[\int_t^n e^{-\int_t^s r} dB_1(s) \big| Z(t) = i\right] \\
&+ \frac{\mathrm{E}[X(t)|Z(t) = i] - V_1^{i*}(t)}{V_2^{i*}(t)} \mathrm{E}\left[\int_t^n e^{-\int_t^s r} dB_2(s) \big| Z(t) = i\right] \\
=& V_1^i(t) + \frac{\tilde{X}^i(t) p_{0i}(0,t) - V_1^{i*}(t)}{V_2^{i*}(t)} V_2^i(t).
\end{aligned}$$

Where we in the second equality have used that $X(t) \perp\!\!\!\perp B_2(s)|Z(t)$ for $s > t$. We are interested in $\tilde{W}^i(t)$ for $i \in \mathcal{J}$, noting that the relation between $\tilde{W}^i$ and $\mathrm{E}_0[W(t)]$ is given by

$$\begin{aligned}
\mathrm{E}_0[W(t)] =& \mathrm{E}_0\left[\sum_{i \in \mathcal{J}} \mathbb{1}_{\{Z(t)=i\}} \frac{\mathrm{E}_0[W(t) \mathbb{1}_{\{Z(t)=i\}}]}{p_{0i}(0,t)}\right] \\
=& \sum_{i \in \mathcal{J}} \tilde{W}^i(t).
\end{aligned}$$

We can even calculate the present value of all future benefits including bonus

$$\begin{aligned}
\mathrm{G}_i(t) =& \mathrm{E}\left[\int_t^n e^{-\int_t^s r} d\left(B_1(s) + \frac{X(s) - V_1^{Z(s)*}(s)}{V_2^{Z(s)*}(s)} B_2(s)\right) \big| Z(t) = i\right] \\
=& \int_t^n e^{-\int_t^s r} \sum_{j \in \mathcal{J}} p_{ij}(t,s) \left(b_1^j(s) + \sum_{k \neq j} \mu^{jk}(s) b^{jk}(s)\right) ds \\
&+ \int_t^n e^{-\int_t^s r} \sum_{j \in \mathcal{J}} p_{ij}(t,s) \frac{\mathrm{E}[X(s)|Z(s) = j, Z(t) = i] - V_1^{j*}(s)}{V_2^{j*}(s)} \left(b_1^j(s) + \sum_{k \neq j} \mu^{jk}(s) b^{jk}(s)\right) ds
\end{aligned}$$

by using that $\mathrm{E}[X(s)|Z(s) = j, Z(t) = i] = \frac{\tilde{X}^j(s)}{p_{ij}(t,s)}$.

A Proof of Theorem 2.1

Proof of theorem 2.1. The proof consists of two steps. First, we derive an integral equation for $\tilde{W}^i(t)$. Second, we differentiate this integral equation.
Assume that $p_{Z(0)i}(0, s) > 0$ for all $s > 0$. The general case where some states cannot be reached by time s is considered at the end of the proof. By the tower property

$$\begin{aligned}
\tilde{W}^i(t) :=& \mathrm{E}_0[W(t)\mathbb{1}_{\{Z(t)=i\}}] \\
=& \mathrm{E}_0\left[\int_0^t \mathbb{1}_{\{Z(t)=i\}} dW(s)\right] \\
=& \mathrm{E}_0\left[\int_0^t \mathbb{1}_{\{Z(t)=i\}} g(s, Z(s), W(s))ds\right] \\
&+ \mathrm{E}_0\left[\int_0^t \sum_{k \neq Z(s-)} \mathbb{1}_{\{Z(t)=i\}} h(s, Z(s-), k, W(s-))dN^k(s)\right].
\end{aligned}$$

Based on the calculations in section C of Norberg (1991), note that the intensity process of the predictable compensator for $N^{jk}(s)|Z(s-) = j, Z(t) = i$ is given by

$$\mu_{jk}(s)\frac{p_{ki}(s, t)}{p_{ji}(s, t)}.$$

As $h(s, Z(s-), k, W(s-)$ is predictable, we may replace the integrator $dN^k(s)$ with its predictable compensator. Using the tower property once more,

$$
\begin{aligned}
\tilde{W}^i(t) &= \int_0^t \mathrm{E}_0\left[\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g(s, Z(s), W(s)) | Z(s)\right]\right] ds \\
&\quad + \mathrm{E}_0\left[\mathrm{E}_0\left[\int_0^t \sum_{k \neq Z(s-)} \mathbb{1}_{\{Z(t)=i\}} h(s, Z(s-), k, W(s-)) dN^k(s) | Z(s-)\right]\right] \\
&= \int_0^t \sum_{j \in \mathcal{J}} p_{Z(0)j}(0, s) \mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g(s, Z(s), W(s)) | Z(s) = j\right] ds \\
&\quad + \sum_{j \in \mathcal{J}} p_{Z(0)j}(0, s) \mathrm{E}_0\left[\int_0^t \sum_{k \neq j} \mathbb{1}_{\{Z(t)=i\}} h(s, j, k, W(s-)) dN^k(s) | Z(s-) = j\right] \\
&= \int_0^t \sum_{j \in \mathcal{J}} p_{Z(0)j}(0, s) \mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g(s, Z(s), W(s)) | Z(s) = j\right] ds \\
&\quad + \sum_{j \in \mathcal{J}} p_{Z(0)j}(0, s) p_{ji}(s, t) \mathrm{E}_0\left[\int_0^t \sum_{k \neq j} h(s, j, k, W(s-)) dN^k(s) | Z(s-) = j, Z(t) = i\right] \\
&= \int_0^t \sum_{j \in \mathcal{J}} p_{Z(0)j}(0, s) \mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g(s, Z(s), W(s)) | Z(s) = j\right] ds \\
&\quad + \sum_{j \in \mathcal{J}} p_{Z(0)j}(0, s) \int_0^t \sum_{k \neq j} \mathrm{E}_0\left[h(s, j, k, W(s-)) | Z(s-) = j, Z(t) = i\right] \mu_{jk}(s) p_{ki}(s, t) ds.
\end{aligned}
$$

Since $W(s)$ is $\mathcal{F}_s$-measurable, the Markov property gives us

$$
\mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}} W(s) | Z(s) = j] = \frac{\tilde{W}^j(s)}{p_{Z(0)j}(0, s)} p_{ji}(s, t),
$$

and by the continuity of $\tilde{W}^i(t)$ we get

$$\begin{aligned}
\tilde{W}^i(t) = &\int_0^t \sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s) \left(\frac{\tilde{W}^j(s)}{p_{Z(0)j}(0,s)} g_1(j,s) + g_2(j,s) \right) p_{ji}(s,t)ds \\
&+ \int_0^t \sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s) \left(\sum_{k\neq j} \mu_{jk}(t) p_{ki}(s,t) \left(\frac{\tilde{W}^j(s)}{p_{Z(0)j}(0,s)} h_1(s,j,k) + h_2(s,j,k) \right) \right) ds \\
= &\int_0^t \sum_{j\in\mathcal{J}} p_{ji}(s,t) \tilde{W}^j(s) g_1(j,s)ds \\
&+ \int_0^t \sum_{j\in\mathcal{J}} \sum_{k\neq j} \mu_{jk}(t) p_{ki}(s,t) \tilde{W}^j(s) h_1(s,j,k)ds \\
&+ \int_0^t \sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s) g_2(j,s) p_{ji}(s,t)ds \\
&+ \int_0^t \sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s) \sum_{k\neq j} \mu_{jk}(t) p_{ki}(s,t) h_2(s,j,k)ds.
\end{aligned}$$

Differentiating with respect to t gives

$$\begin{aligned}
\frac{d}{dt}\tilde{W}^i(t) = &\tilde{W}^i(t) g_1(i,t) + p_{Z(0)i}(0,t) g_2(i,t) \\
&+ \sum_{k\neq i} \mu_{ki}(t) \left(\tilde{W}^k(t) h_1(t,k,i) + p_{Z(0)k}(0,t) h_2(t,k,i) \right) \\
&+ \int_0^t \frac{\partial}{\partial t} \sum_{j\in\mathcal{J}} p_{ji}(s,t) \tilde{W}^j(s) g_1(j,s)ds \\
&+ \int_0^t \frac{\partial}{\partial t} \sum_{j\in\mathcal{J}} \sum_{k\neq j} \mu_{jk}(t) p_{ki}(s,t) \tilde{W}^j(s) h_1(s,j,k)ds \\
&+ \int_0^t \frac{\partial}{\partial t} \sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s) g_2(j,s) p_{ji}(s,t)ds \\
&+ \int_0^t \frac{\partial}{\partial t} \sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s) \sum_{k\neq j} \mu_{jk}(t) p_{ki}(s,t) h_2(s,j,k)ds.
\end{aligned}$$

By the Kolmogorov forward differential equations we arrive at

$$\begin{aligned}\frac{d}{dt}\tilde{W}^i(t) =& \tilde{W}^i(t)g_1(i,t) + p_{Z(0)i}(0,t)g_2(i,t) \\ &+ \sum_{k\neq i} \mu_{ki}(t) \left(\tilde{W}^k(t)h_1(t,k,i) + p_{Z(0)k}(0,t)h_2(t,k,i)\right) \\ &+ \sum_{k\neq i} \mu_{ki}(t)\tilde{W}^k(t) - \mu_{ik}(t)\tilde{W}^i(t).\end{aligned}$$

Combined with the initial condition

$$\tilde{W}^i(0) = \mathrm{E}_0[\mathbb{1}_{\{Z(0)=i\}}W(0)] = \mathbb{1}_{\{Z(0)=i\}}W(0),$$

we arrive at the differential equations given by (12)-(15). For the case where some state, q, cannot be reached before time s for $s > 0$, the product of intensities for all paths from $Z(0)$ into that state must be zero for all τ when $\tau \leq s$, whereby $\tilde{W}^q(s) = 0$ and therefore the differential equations still hold. Thus the proof is complete. □

B One Active State

We consider a simple model where the expected future savings are described by an easily derived differential equation. The model consists of n inactive states where there are no payments, and one active state with continuous dynamics g which, in this setting, may be non-linear. Denote by 0 the active state. On transition to any one of the inactive states, the surplus and savings are nullified. We need not specify what happens to the surplus and savings on a transition - they may be paid out to the customer or the insurance company, or any combination of the two - the only important requirement is that they are zero in all inactive states. The eradication of surplus and savings on transition corresponds to the relation $h_x(t,0,j,x,y) + h_y(t,0,j,x,y) = -x - y$, for $j = 1, ..., n$. For notational ease, we assume that

$$\begin{aligned}h_x(t,0,j,x,y) &= -x \\ h_y(t,0,j,x,y) &= -y,\end{aligned}$$

for all j. The survival model with and without surrender options are special cases of this model. The dynamics of X and Y are

$$\begin{aligned} dX(s) =& \mathbb{1}_{\{Z(s-)=0\}} g_x(s, 0, X(s), Y(s))ds - \sum_{h=1}^{n} X(s-)dN^h(s) \\ dY(s) =& \mathbb{1}_{\{Z(s-)=0\}} g_y(s, 0, X(s), Y(s))ds - \sum_{h=1}^{n} Y(s-)dN^h(s). \end{aligned}$$

Let $W(s) = (X(s), Y(s))^T$, and denote by T_1 the time of the first jump. For the deterministic function W_a that solves

$$W_a(t) = \int_0^t g(s, 0, W_a(s))ds,$$

we see that

$$\hat{W}(t) := \mathrm{E}[W(t)|Z(0) = 0] = \mathrm{E}[\mathbb{1}_{\{t<T_1\}} W_a(t)|Z(0) = 0] = p_{00}(0, t)W_a(t),$$

which comes at no surprise. In this case we know the past and present values of W given the current state of Z, so the only stochastic element pertains to the state of the policy at time t. By differentiating w.r.t. t, and applying Kolmogorov's forward differential equation, we get the following forward differential equation for $\hat{W}$,

$$\begin{aligned} \hat{W}(0) =& \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix}, \\ \frac{d}{dt}\hat{W}(t) =& p_{00}(0, t) g\left(s, 0, \frac{\hat{W}(t)}{p_{00}(0, t)}\right) - \frac{\hat{W}(t)}{p_{00}(0, t)} \sum_{k=1}^{n} \mu_{0k}(t). \end{aligned}$$

Even though it may seem very simple and perhaps even trivial, the model with one active state has great applicability.

B.0.1 Example With One Active State

If the benefits are identical after age 65, the states 0,1,3 and 4 can be lumped, as well as 2,5 and 6, thus creating a survival model. If the dynamics in two states are identical, they can be viewed as one. Life annuity at age 65.

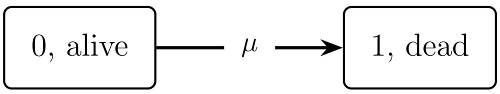

Figure 1: Life-Death model

C Dynamics of X and Y

The amount by which the savings surpass the first order reserve, is spent on B_2. where

$$b^j(t,x) = b_1^j(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^j(t), \qquad b^{jg}(t,x) = b_1^{jg}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^{jg}(t).$$

Dynamics of X

$$\begin{aligned} dX(t) =& r^*(t)X(t)dt + \delta^{Z(t)}(t, X(t), Y(t))dt - \sum_{g \neq Z(t-)} \rho^{Z(t-)g}(t, X(t-))dt \\ & - b^{Z(t)}(t, X(t))dt \\ & - \sum_{g \neq Z(t-)} \Big(b^{Z(t-)g}(t, X(t-)) + \chi^{Z(t-)g}(t, X(t-)) - X(t-) \Big) \mu^{Z(t)g}(t)dt \\ & + \sum_{g \neq Z(t-)} \Big(\chi^{Z(t-)g}(t, X(t-)) - X(t-) \Big) dN^g(t), \end{aligned}$$

and

$$dY(t) = Y(t)\frac{dS(t)}{S(t)} - \delta^{Z(t)}(t, X(t), Y(t)) + (r(t) - r^*(t))X(t) + \sum_{g \neq Z(t-)} \rho^{Z(t)g}(t, X(t)),$$

where

$$\begin{aligned} \rho^{jg}(t,x) =& (b^{jg}(t,x) + \chi^{jg}(t,x) - x)(\mu^{*jg}(t) - \mu^{jg}(t)) \\ \chi^{jg}(t,x) =& V_1^{g*}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{g*}(t), \end{aligned}$$

$$\delta^j(t,x,y) = \delta_1^j(t) + \delta_2^j(t)x + \delta_3^j(t)y + \delta_4^j(t)xy. \tag{16}$$

$$\tilde{W}^j(t) := \begin{pmatrix} \tilde{X}^j(t) \\ \tilde{Y}^j(t) \end{pmatrix} = \begin{pmatrix} \mathrm{E}[X(t)\mathbb{1}_{\{Z(t)=j\}}] \\ \mathrm{E}[Y(t)\mathbb{1}_{\{Z(t)=j\}}] \end{pmatrix}$$

With differential equation

$$
\begin{aligned}
\frac{d}{dt}\tilde{W}^j(t) =& \sum_{g\neq j} \mu^{gj}(t)\tilde{W}^g(t) - \mu^{jg}(t)\tilde{W}^j(t) \\
&+ \tilde{W}^j(t) \circ g_1(t,j,x,y) + p_{0j}(0,t) g_2(t,j) \\
&+ \sum_{g\neq j} \mu^{gj}(t) \left(\tilde{W}^g(t) \circ h_1(t,g,j,x,y) + p_{0g}(0,t) h_2(t,g,j) \right), \\
\tilde{W}^j(0) =& \mathbb{1}_{\{Z(0)=j\}} \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix},
\end{aligned}
$$

where $\circ$ denotes the Hadamard product (element-wise multiplication) and

$$
\begin{aligned}
g_1(t,j,x,y) &= \begin{pmatrix} g_{x1}(t,j,y) \\ g_{y1}(t,j,x) \end{pmatrix}, & h_1(t,j,g,x,y) &= \begin{pmatrix} h_{x1}(t,j,g,y) \\ h_{y1}(t,j,g,x) \end{pmatrix}, \\
g_2(t,j) &= \begin{pmatrix} g_{x2}(t,j,y) \\ g_{y2}(t,j,x) \end{pmatrix}, & h_2(t,j,g) &= \begin{pmatrix} h_{x2}(t,j,g,y) \\ h_{y2}(t,j,g,x) \end{pmatrix}.
\end{aligned}
$$

For

$$
\begin{aligned}
g_{x1}(t,j,y) =& r^*(t) + \delta_2^j(t) + \delta_4^j(t) y \\
&+ \frac{b_2^j(t)}{V_2^{j*}(t)} - \sum_{g\neq j} \rho_1^{jg}(t) - \sum_{g\neq j} \left(\frac{b_2^{jg}(t)}{V_2^{j*}(t)} + \frac{V_2^{g*}(t)}{V_2^{j*}(t)} - 1 \right) \mu^{jg}(t) \\
g_{x2}(t,j,y) =& \delta_1^j(t) + \delta_3^j(t) y - b_1^j(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} b_2^j(t) - \sum_{g\neq j} \rho_2^{jg}(t) \\
&- \sum_{g\neq j} \left(b_1^{jg}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} b_2^{jg}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{g*}(t) + V_1^{g*}(t) \right) \mu^{jg}(t).
\end{aligned}
$$

$$
\begin{aligned}
h_x(t,j,g,x,y) =& \chi^{jg}(t,x) - x \\
=& V_1^{g*}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{g*}(t) - x \\
=& \underbrace{x \left(\frac{V_2^{g*}(t)}{V_2^{j*}(t)} - 1 \right)}_{h_{x1}(t,j,g,y)} + \underbrace{V_1^{g*}(t) - \frac{V_1^{j*}(t) V_2^{g*}(t)}{V_2^{j*}(t)}}_{h_{x2}(t,j,g,y)}.
\end{aligned}
$$

For g_y we get

$$\begin{aligned}
g_y(t,j,x,y) =& y\frac{dS(t)}{S(t)} - \delta^j(t,x,y) + (r(t)-r^*(t))x + \sum_{g\neq j}\rho^{jg}(t,x) \\
=& y\frac{dS(t)}{S(t)} - \delta_1^j(t) - \delta_2^j(t)x - \delta_3^j(t)y - \delta_4^j(t)xy \\
& + (r(t)-r^*(t))x + \sum_{g\neq j}\rho^{jg}(t,x) \\
=& y\underbrace{\left(\frac{dS(t)}{S(t)} - \delta_3^j(t) - \delta_4^j(t)x\right)}_{g_{y1}(t,j,x)} \\
& \underbrace{+\sum_{g\neq j}\rho^{jg}(t,x) - \delta_1^j(t) - \delta_2^j(t)x + (r(t)-r^*(t))x}_{g_{y2}(t,j,x)}
\end{aligned}$$

Finally, as $h_y = 0$ we have $h_{y1} = h_{y2} = 0$.

We write out the differential equation for $\tilde{W}^i$ when $W(t) = (X(t), Y(t))^T$ and the dynamics of X and Y are given by (7)-(9). Note that ρ can be written as

$$\begin{aligned}
\rho^{jg}(t,x) =& (b^{jg}(t,x) + \chi^{jg}(t,x) - x)(\mu^{*jg}(t) - \mu^{jg}(t)) \\
=& \Bigg(x\frac{b_2^{jg}(t)}{V_2^{j*}(t)} + b_1^{jg}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)}b_2^{jg}(t) \\
& + V_1^{g*}(t) + x\frac{V_2^{g*}(t)}{V_2^{j*}(t)} - \frac{V_1^{j*}(t)}{V_2^{j*}(t)}V_2^{g*}(t) - x\Bigg)(\mu^{*jg}(t) - \mu^{jg}(t)) \\
=& x\underbrace{\left(\frac{b_2^{jg}(t)}{V_2^{j*}(t)} + \frac{V_2^{g*}(t)}{V_2^{j*}(t)} - 1\right)(\mu^{*jg}(t) - \mu^{jg}(t))}_{\rho_1^{jg}(t)} \\
& + \underbrace{\left(b_1^{jg}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)}b_2^{jg}(t) + V_1^{g*}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)}V_2^{g*}(t)\right)(\mu^{*jg}(t) - \mu^{jg}(t))}_{\rho_2^{jg}(t)} \\
=& \rho_1^{jg}(t)\cdot x + \rho_2^{jg}(t).
\end{aligned}$$

The differential equation for the conditional state-wise values of X are given by

$$
\begin{aligned}
\frac{d}{dt}\tilde{X}^i(t) = &\sum_{j\neq i} \mu_{ji}(t)\tilde{X}^j(t) - \mu_{ij}(t)\tilde{X}^i(t) \\
&+ \tilde{X}^i(t)\Bigg\{ r^*(t) + \delta_2^i(t) + \delta_4^i(t)\tilde{Y}^i(t) \\
&+ \frac{b_2^i(t)}{V_2^{i*}(t)} - \sum_{j\neq i} \rho_1^{ij}(t) - \sum_{j\neq i} \left(\frac{b_2^{ij}(t)}{V_2^{i*}(t)} + \frac{V_2^{j*}(t)}{V_2^{i*}(t)} - 1 \right) \mu_{ij}(t) \Bigg\} \\
&+ p_{Z(0)i}(0,t)\Bigg\{ \delta_1^i(t) + \delta_3^i(t)\tilde{Y}^i(t) - b_1^i(t) - \frac{V_1^{i*}(t)}{V_2^{i*}(t)} b_2^i(t) - \sum_{j\neq i} \rho_2^{ij}(t) \\
&- \sum_{j\neq i} \left(b_1^{ij}(t) - \frac{V_1^{i*}(t)}{V_2^{i*}(t)} b_2^{ij}(t) - \frac{V_1^{i*}(t)}{V_2^{i*}(t)} V_2^{j*}(t) + V_1^{j*}(t) \right) \mu_{ij}(t) \Bigg\} \\
&+ \sum_{j\neq i} \mu_{ji}(t) \Bigg\{ \tilde{X}^j(t) \left(\frac{V_2^{j*}(t)}{V_2^{i*}(t)} - 1 \right) + p_{Z(0)j}(0,t) \left(V_1^{j*}(t) - \frac{V_1^{i*}(t)V_2^{j*}(t)}{V_2^{i*}(t)} \right) \Bigg\}
\end{aligned}
$$

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