Stuff to fix

- Jeg kommer med flere påstande om industrien som jeg ikke er sikker på har hold i virkeligheden.
- Der er i princippet ikke nogen grund til at vi regner retrospektivt, når vi alligevel ikke bruger historikken... Skal vi udvidde, så X kan afhænge lineært af tidligere værdier? Vi kræver blot at

$$\mathrm{E}[g(t, Z(t), \{X(\tau)\}_{\tau\leq t})|Z(t)=i] = g(t, i, \{\mathrm{E}[X(\tau)|Z(t)=i]\}_{\tau\leq t})$$

fx ved

$$g(t, i, \{X(\tau)\}_{\tau\leq t}) = \int_0^t f_i^1(\tau)X(\tau)d\nu_1(\tau,t) + \int_0^t f_i^2(\tau)X(\tau)d\nu_2(\tau,t)$$

for ét eller andet sigma-additivt mål ν_1 (og ν_2). Fx kunne ν_1 være lebesque målet fra $t-1$ til t, mens ν_2 kunne være punktmålet i t hvilket er specialtilfældet som vi i øjeblikket kigger på. Vi kan også lade g afhænge af tidligere værdier af Z, fx på følgende måde

$$g(t, \{Z(\tau)\}_{\tau\leq t}, \{X(\tau)\}_{\tau\leq t}) = \int_0^t f(Z(\tau), \tau, t)X(\tau)d\nu(\tau),$$

hvorved

$$\begin{aligned}
&\mathrm{E}_{Z(0)}[\mathbb{1}_{\{Z(r)=i\}}g(t, \{Z(\tau)\}_{\tau\leq t}, \{X(\tau)\}_{\tau\leq t})|Z(t-)=g] \\
=&p_{gi}(t,r)\int_0^t \mathrm{E}[f(Z(\tau),\tau,t)X(\tau)|Z(t-)=g]d\nu(\tau,t) \\
=&\int_0^t p_{gi}(t,r)\mathrm{E}\left[\sum_{j\in\mathcal{J}}\mathbb{1}_{\{Z(\tau)=j\}}f(j,\tau,t)\mathrm{E}[X(\tau)|Z(\tau)=j, Z(t-)=g]\middle| Z(t-)=g\right]d\nu(\tau,t) \\
=&\int_0^t p_{gi}(t,r)\mathrm{E}\left[\sum_{j\in\mathcal{J}}\mathbb{1}_{\{Z(\tau)=j\}}f(j,\tau,t)\frac{\mathrm{E}[X(\tau)\mathbb{1}_{\{Z(\tau)=j\}}]}{p_{0j}(0,\tau)}\middle| Z(t-)=g\right]d\nu(\tau,t) \\
=&\int_0^t p_{gi}(t,r)\sum_{j\in\mathcal{J}}P(Z(\tau)=j|Z(0)=0, Z(t)=g)f(j,\tau,t)\frac{\tilde{X}^j(\tau)}{p_{0j}(0,\tau)}d\nu(\tau,t) \\
=&\int_0^t \sum_{j\in\mathcal{J}}\frac{p_{jg}(\tau,t)}{p_{0g}(0,t)}p_{gi}(t,r)f(j,\tau,t)\tilde{X}^j(\tau)d\nu(\tau,t).
\end{aligned}$$

hvor vi har brugt at $X(\tau)|Z(\tau)$ er uafhængig af $Z(t)|Z(\tau)$ for $\tau \leq t$. På denne måde kunne man fx. lade dividenden være en klumpbetaling svarende til det gennemsnitlige forventede bidrag over det sidste år, altså $f(j,\tau,t)X(\tau) = X(\tau)(r(\tau) - r^*(\tau))\sum_{k\neq j}\rho_1^{jk}(t) + \sum_{k\neq j}\rho_2^{jk}(t)$ for $\nu(\tau,t)$ værende lebesque målet for $(t-1,t]$ hvis t er et heltal, og ellers 0. Da ν ikke nødvendigvis er absolut kontinuert, vil g ikke svare til den kontinuerte udvikling af X - vi tillader klump-betalinger på deterministiske tidspunkter.

- Tilsvarende for h

$$h(t, \{Z(\tau)\}_{\tau\leq t}, \{X(\tau)\}_{\tau\leq t}) = \int_{(0,t]} \sum_{k\neq Z(\tau-)} \phi(\tau,t,Z(\tau-),k)X(\tau-)dN^k \otimes \nu(\tau,t)$$

Taking the expectation and conditioning on $Z(t-) = g$

$$\begin{aligned}
&\mathrm{E}[\mathbb{1}_{\{Z(r)=j\}} h(t, \{Z(\tau)\}_{\tau\leq t}, \{X(\tau-)\}_{\tau\leq t})|Z(t-) = g] \\
&= \int_{(0,t]} \mathrm{E}\left[\sum_{k\neq Z(\tau-)} \phi(\tau,t,Z(\tau-),k)\mathbb{1}_{\{Z(r)=j\}}X(\tau-)dN^k \otimes \nu(\tau,t)\middle| Z(t-) = g\right] \\
&= \int_{(0,t]} \mathrm{E}\left[\mathrm{E}\left[\sum_{k\neq Z(\tau-)} \phi(\tau,t,Z(\tau-),k)\mathbb{1}_{\{Z(r)=j\}}X(\tau-)dN^k \otimes \nu(\tau,t)|Z(\tau-), Z(t-) = g\right]\middle| Z(t-) = g\right] \\
&= \int_{(0,t]} \mathrm{E}\left[\sum_{i\in\mathcal{J}} \mathbb{1}_{\{Z(\tau)=i\}}\mathrm{E}\left[\sum_{k\neq i} \phi(\tau,t,i,k)\mathbb{1}_{\{Z(r)=j\}}X(\tau-)dN^k \otimes \nu(\tau,t)|Z(\tau-) = i, Z(t-) = g\right]\middle| Z(t-) \right. \\
&= \int_{(0,t]} \mathrm{E}\left[\sum_{i\in\mathcal{J}} \mathbb{1}_{\{Z(\tau)=i\}}\sum_{k\neq i} \phi(\tau,t,i,k)\mathrm{E}\left[\mathbb{1}_{\{Z(r)=j\}}X(\tau-)dN^k \otimes \nu(\tau,t)|Z(\tau-) = i, Z(t-) = g\right]\middle| Z(t-) \right. \\
&= \int_{(0,t]} \mathrm{E}\left[\sum_{i\in\mathcal{J}} \mathbb{1}_{\{Z(\tau)=i\}}\sum_{k\neq i} \phi(\tau,t,i,k)\mathrm{E}\left[\mathbb{1}_{\{Z(r)=j\}}X(\tau-)dN^k(\tau)|Z(\tau-) = i\right]\nu(\tau,t)\middle| Z(t-) = g\right]
\end{aligned}$$

og da $X(\tau-)|Z(\tau-)$ er uafhænig af $\mathbb{1}_{\{Z(r)=j\}}dN^h(\tau)|Z(\tau-)$

$$
\begin{aligned}
&= \int_{(0,t]} \mathrm{E}\left[\sum_{i\in\mathcal{J}} \mathbb{1}_{\{Z(\tau)=i\}} \sum_{k\neq i} \phi(\tau,t,i,k)\mathrm{E}\left[X(\tau-)|Z(\tau-)=i\right]\mathrm{E}[\mathbb{1}_{\{Z(r)=j\}}dN^k(\tau)|Z(\tau-)=i]\nu(\tau,t)\middle| Z(t-\right.\\
&= \int_{(0,t]} \mathrm{E}\left[\sum_{i\in\mathcal{J}} \mathbb{1}_{\{Z(\tau)=i\}} \sum_{k\neq i} \phi(\tau,t,i,k)\frac{\tilde{X}^i(\tau)}{p_{0i}(0,\tau)}\mathrm{E}[dN^k(\tau)|Z(\tau-)=i,Z(r)=j]p_{ij}(\tau,r)\nu(\tau,t)\middle| Z(t-)=\right.\\
&= \int_{(0,t]} \sum_{i\in\mathcal{J}} \mathrm{E}\left[\mathbb{1}_{\{Z(\tau)=i\}}|Z(t-)=g\right] \sum_{k\neq i} \phi(\tau,t,i,k)\frac{\tilde{X}^i(\tau)}{p_{0i}(0,\tau)}\mu_{ik|ij}(\tau|\tau,r)p_{ij}(\tau,r)d\nu(\tau,t)\\
&= \int_{(0,t]} \sum_{i\in\mathcal{J}} \frac{p_{0i}(0,\tau)p_{ig}(\tau,t)}{p_{0g}(0,t)} \sum_{k\neq i} \phi(\tau,t,i,k)\frac{\tilde{X}^i(\tau)}{p_{0i}(0,\tau)}\mu_{ik|ij}(\tau|\tau,r)p_{ij}(\tau,r)d\nu(\tau,t)\\
&= \int_{(0,t]} \sum_{i\in\mathcal{J}} \frac{p_{0i}(0,\tau)p_{ig}(\tau,t)}{p_{0g}(0,t)} \sum_{k\neq i} \phi(\tau,t,i,k)\frac{\tilde{X}^i(\tau)}{p_{0i}(0,\tau)}\mu_{ik}(\tau)\frac{p_{kj}(\tau,r)}{p_{ij}(\tau,r)}p_{ij}(\tau,r)d\nu(\tau,t)\\
&= \int_{(0,t]} \sum_{i\in\mathcal{J}} \frac{p_{ig}(\tau,t)}{p_{0g}(0,t)} \sum_{k\neq i} \phi(\tau,t,i,k)\tilde{X}^i(\tau)\mu_{ik}(\tau)p_{kj}(\tau,r)d\nu(\tau,t)
\end{aligned}
$$

- Meget i beviserne skal slettes. De er for lange
- Flere steder skriver jeg at dividende også kan bruges til at nedskrive præmier. Det skaber mere forvirring end nytte at holde styr på begge muligheder. Omskriv dette så vi kun kan bruge dividende til at opskrive ydelser.

1 Introduction

With-profit insurance contracts are to this day one of the most popular life insurance contracts. They arose as a natural way to distribute the systematic surplus that emerges due to the prudent assumptions on which the contract is made. In recent years, sensible questions accompanied by a lot of attention have been aimed at the surplus, to name a few; is it distributed fairly? how should it be invested? How is it affected by the financial market? To answer these questions we need to understand the dynamics of the surplus in a model of practical relevance. The study of surplus and the interplay it has with other elements of an insurance contract, is not new. Norberg (1999) introduces the notion of individual surplus as well as the mean portfolio surplus. In Steffensen (2000) and Steffensen (2001), partial differential equations are used to describe the prospective second order reserve for various forms of bonus, when the surplus is invested in a Black-Scholes market. In this paper we pay little regard to the prospective reserve, and instead focus on the surplus and the retrospective second order reserve, also called the savings account.

The expected future value of the savings account is of particular interest, as it embodies the accumulation of dividends as well as the use of these dividends to increase benefits or decrease premiums. Due to the retrospective nature of the savings account and surplus, we may also take other retrospective considerations into account. In the existing literature, very little attention is paid to a very significant retrospective element of the with-profit insurance contract: the human element.

Insurance companies are governed by humans, and the decisions they make have an influence on the portfolio of policies - in particular concerning surplus and dividends. In a with-profit insurance contract many quantities are fixed at initialisation of the policy, but the rate at which dividends are paid out is not. The insurance company has a certain degree of freedom when it comes to the distribution of surplus, and the actions that have an influence on the insurance contracts are the so-called Management Actions. From a mathematical point of view, they pose a problem as they depend on the entire history of the portfolio of policies in a possibly non-linear fashion, making it difficult to calculate prospective reserves. If we want to take a glance into the crystal ball of liabilities, taking Future Management Actions (FMA's) into account, we need to embrace it's retrospective nature. In this paper, we do not incorporate FMA's to their full extent, but rather lay the retrospective groundwork on which models including FMA's can be built.

We derive a retrospective differential equation for the expected savings account and surplus, in a general model with affine dynamics. We devote special attention to a realistic model with affine dynamics, where dividends are used to increase future benefits.

1.1 Set-up

We consider the classic multi-state life insurance set-up, comprised of a state process Z denoting the state of the policy in a finite state space $\mathcal{J} = \{0, 1, ..., J\}$. The counting process N^k defined by $N^k(t) = \#\{s; Z(s-) = i, Z(s) = k, s \in (0, t]\}$ describes the number of transitions into state k. The state process Z is assumed to be a continuous time Markov chain, with transition probabilities denoted by

$$p_{ij}(s,t) = \mathrm{P}(Z(t) = j | Z(s) = i)$$

for $s \leq t$. The corresponding transition intensities are denoted by

$$\mu_{ij}(t) = \lim_{h \searrow 0} p_{ij}(t, t+h)/h$$

for $i \neq j$. The predictable process $\mathbb{1}_{\{Z(t-)\neq k\}}\mu_{Z(t-)k}(t)$ is the intensity process for $N^k(t)$, i.e

$$M^k(t) := N^k(t) - \int_0^t \mathbb{1}_{\{Z(s-)\neq k\}}\mu_{Z(s-)k}(s)ds,$$

forms a martingale. The state process Z encapsulates the biometric risks involved with the insurance contract. Apart from the biometric risk, there is a financial risk connected to with-profit insurance contracts through the return on investment of the surplus. We make assumptions regarding the financial risk, by specifying the expected return on investment, r. Together, the transition intensities and expected return on investment form the second order basis, which describes the best guess on future development of the insurance portfolio. We take this second order basis as exogenously given. Note that a Monte-Carlo method can be used as a proxy for evaluation under the second order basis; perform evaluation under n simulated second order basis and take the mean. The Monte-Carlo approach for evaluation allows for great model flexibility, which is particularly appealing regarding the expected return on investment.

While the second order basis forms the best guess on future developments of the relevant technical elements, it would be far too risky for an insurance company to use these assumptions when signing contracts. What if a cure for cancer is invented in 10 years, or if the stock market crashes? To allow for events that make it difficult to meet the obligations to the insured, a much less risky set of assumptions are used when guarantees are given. These prudent assumptions form the first order (technical) basis. Using the standard notation, a " $*$ " symbolises first-order basis elements. It is precisely due to the difference between the first order basis and the realised (third order) basis that a surplus emerges. We have no way of knowing what the future is going to bring, so we cannot know how the surplus is going to evolve. We can however make an estimate by using the second order basis as a stand-in for the third order basis.

1.1.1 Savings and Surplus

The savings and surplus denoted by X and Y respectively, are the main quantities of interest. Three payment streams affect the savings and surplus, namely the benefits, dividends and contributions. The benefits are the contractual payments between the insurer and the insured. That is, the benefits are the state dependent premiums and benefits agreed upon on initialization of the policy. In order to incorporate the possibility of increasing benefits or decreasing premiums by spending dividends, we introduce two benefit processes B_1 and B_2 with dynamics

$$dB_i(t) = b_i^{Z(t)}(t)dt + \sum_{k \neq Z(t-)} b_i^{Z(t-)k}(t)dN^k(t).$$

The deterministic payment functions $b_i^j(t)$ and $b_i^{jk}(t)$ specify payments during sojourns in state j and on transition from state j to state k, respectively. Even though single payments during sojourns in states pose no mathematical difficulty, we assume that payments during sojourns in states are continuous for notational simplicity. The benefits specified by B_1 are the benefits which are fixed, and B_2 specifies the benefits which are increased when dividend is paid out. Accompanying the payment streams B_1 and B_2, are the technical reserves given by

$$V_i^{*j}(t) = \mathrm{E}\left[\int_t^n e^{-\int_t^s r^*} dB_i(s) \big| Z(t) = j\right].$$

The savings account is used to buy one unit of the payment stream B_1, and whatever remains, is spent on B_2. By the equivalence principle, the quantity Q, of B_2 cashflows that can be bought, is equal to

$$Q(t) = \frac{X(t-) - V_1^{Z(t-)*}(t-)}{V_2^{Z(t-)*}(t-)},$$

as the savings account has to meet the obligations of the B_1 cashflow, before any B_2 cashflow is bought and both B_1 and B_2 are priced under the first order basis. The payment stream experienced by the policyholder therefore has dynamics

$$\begin{aligned}dB(t) =&dB_1(t) + \frac{X(t-) - V_1^{Z(t-)*}(t-)}{V_2^{Z(t-)*}(t-)}dB_2(t)\\ =&b^{Z(t)}(t, X(t))dt + \sum_{k\neq Z(t-)} b^{Z(t-)k}(t, X(t-))dN^k(t),\end{aligned}$$

for

$$\begin{aligned}b^j(t,x) &= b_1^j(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)}b_2^j(t)\\ b^{jk}(t,x) &= b_1^{jk}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)}b_2^{jk}(t).\end{aligned}$$

Note at initialisation of the policy where no dividends are accrued and $X(0) = V_1^{Z(0)*}(0)$, then $b^j(0,x) = b_1(0)$. In general, if the savings account never receives any dividends, then $X(t) = V_1^{Z(t)*}(t)$ which is a special case of particular interest as it falls into the framework of Norberg (1991). To ease notation, we define the risk premium, ρ, as the sum at risk for a transition from j to k, R^{jk}, multiplied by the difference in intensity of the transition on the first- and second order basis

$$\begin{aligned}\rho^{jk}(t,x) &= R^{jk}(t,x)(\mu^*_{jk}(t) - \mu_{jk}(t)),\\ R^{jk}(t,x) &= b^{jk}(t,x) - x + \chi^{jk}(t,x).\end{aligned}$$

The three terms in the sum at risk describe what happens to the savings account on transition from j to k. A benefit $b^{jk}(t,x)$ is paid out, and the entire savings account are given to the insurance company as they no longer have any obligations for the policy in the state j. However, now the insurance company have other obligations for the policy in state k, and to be fair, the amount of extra benefits the policy could afford in state j are now carried over to state k. Therefore there is a deposit of

$$\chi^{jk}(t,x) = V_1^{k*}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)}V_2^{k*}(t),$$

to the savings account.

The dividend payment stream flows from the surplus to the savings account, and we assume the dynamics are given by

$$dD(t) = \delta^{Z(t)}(t, X(t), Y(t))dt,$$

for some deterministic function δ^j on the form

$$\delta^j(t, x, y) = \delta_1^j(t) + \delta_2^j(t)x + \delta_3^j(t)y + \delta_4^j(t)xy.$$

In this paper, the form of δ is probably the assumption most eligible for criticism. In practice, the dividend is determined by an actuary who takes much more information into account than simply the value of the savings and surplus. Furthermore the dividend-deciding actuary is most likely going to take past development of the savings and surplus into account. The specification of the dynamics of D is at the heart of what a future management action is, and, as stated earlier, we do not fully incorporate these FMA's in all their generality and glory, but suffice with crude surrogates. Some of these crude surrogates can actually perform a decent job at describing real world dividend strategies, for instance by defining the dividend as some linear function of the contribution.

The contributions are the safety contributions which ensure that the insurer can meet their liabilities. They represent the difference between the first and second order basis, and the artificial premium paid by the insured to cover the indiversifiable risk carried by the insurer. The contract is signed on the prudent first order basis, and as such, the contract is expected to generate a systematic surplus. This systematic surplus arises precisely because C is positive. The dynamics of C are defined to be

$$dC(t) = (r(t) - r^*(t))X(t)dt + \sum_{k \neq Z(t)} \rho^{Z(t)k}(t, X(t))dt.$$

By the principle of equivalence, the amount of B_2 cashflow that can be bought for the dividend implies that

$$dD(t) = V_2^{Z(t)*}(t)dQ(t).$$

Using Itô's lemma for FV-functions and integration by parts for FV-functions, we find the dynamics of X as

$$\begin{aligned}
dX(t) =& r^*(t)X(t)dt + \delta^{Z(t)}(t, X(t), Y(t))dt - b^{Z(t)}(t, X(t))dt - \sum_{k \neq Z(t-)} b^{Z(t-)k}(t, X(t-))dN^k(t) \\
&+ \sum_{k \neq Z(t-)} R^{Z(t-)k}(t, X(t-))(dN^k(t) - \mu^*_{Z(t-)k}(t)dt) \\
=& r^*(t)X(t)dt + \delta^{Z(t)}(t, X(t), Y(t))dt - b^{Z(t)}(t, X(t))dt - \sum_{k \neq Z(t-)} b^{Z(t-)k}(t, X(t-))dN^k(t) \\
&- \sum_{k \neq Z(t-)} \rho^{Z(t-)k}(t, X(t-))dt \\
&+ \sum_{k \neq Z(t-)} R^{Z(t-)k}(t, X(t-))(dN^k(t) - \mu_{Z(t-)k}(t)dt).
\end{aligned}$$

Note that the predictable compensator for dN^k when $k \neq Z(t-)$ is $\mu_{Z(t-)k}(t)$, under the second order basis, while $\mu^*_{Z(t-)k}(t)$ is the predictable compensator under the first order basis. We can now define the surplus of the accumulated contract as

$$Y(t) = -\int_0^t e^{\int_s^t r} dB(s) - X(t),$$

and derive the dynamics to be

$$dY(t) = rY(t)dt + dC(t) - dD(t) - \sum_{k \neq Z(t-)} R^{Z(t-)k}(t, X(t-))dM^k(t).$$

This concludes the formal set-up needed to work with the savings and surplus. However, as the dynamics of X and Y are imperative in order to understand the real-world we take a moment to reflect over these dynamics

For the sake of generality and notational ease, we use the following notation working with the explicit dynamics of X and Y, we choose to use the more general affine dynamics given by

$$\begin{aligned}
dX(t) =& X(t)g_{x1}(t, Z(t), Y(t))dt + \sum_{k \neq Z(t-)} X(t-)h_{x1}(t, Z(t-), k, Y(t-))dN^k(t) \\
&+ g_{x2}(t, Z(t), Y(t))dt + \sum_{k \neq Z(t-)} h_{x2}(t, Z(t-), k, Y(t-))dN^k(t), \\
dY(t) =& Y(t)g_{y1}(t, Z(t), X(t))dt + \sum_{k \neq Z(t-)} Y(t-)h_{y1}(t, Z(t-), k, X(t-))dN^k(t) \\
&+ g_{y2}(t, Z(t), X(t))dt + \sum_{k \neq Z(t-)} h_{y2}(t, Z(t-), k, X(t-))dN^k(t).
\end{aligned}$$

While it may not seem so, we work with g and h functions mainly for notational reasons. We refer to section B for a practically relevant choice of the functions g and h - the notational advantage of using g and h is also apparent there.

It is important to realize the extent of applicable models that have affine dynamics, see Christiansen et al. (2014) for several relevant payment functions that are linear in the reserve, which corresponds to the savings when $D(t) = 0$ for all t. While the reach of models with affine dynamics is extensive, there are limitations to consider. It is not uncommon to have dynamics that include some min or max function, for instance in the case of guarantees, and these non-linear functions in savings cannot be described by affine dynamics.
As stated in the introduction, management actions are one of the main motivators of this paper, but where are they evident in the dynamics of the savings and surplus? The answer is, the management actions are not evident. They are hidden in mainly two terms, the second order interest and the dividend. This is because the management decides how to invest the surplus, and how it should be distributed to the customers. Due to the very human and abstract nature of management actions, we do not incorporate them directly in the dynamics of the savings and surplus, but instead let them work in the shadows. It is important to

- Even though FMA's are one of the main reasons for considering the savings account, they are hidden in the dividend and surplus investment strategy.

Let W be some possibly multidimensional process with Z-dependent dynamics

$$dW(s) = g(s, Z(s), W(s))ds + \sum_{k \neq Z(s-)} h(s, Z(s-), k, W(s-))dN^k(s),$$

for g and h functions that are linear in all elements of W. This multidemsional process can for instance represent the savings and surplus

$$W(s) = \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix}.$$

In practice, the surplus account is shared among policyholders, corresponding to $W \in \mathbb{R}^{N+1}$ for N policies with a state process Z on a state space of size $\#\{\mathcal{J}\}^N$; one for each combination of all policy states. There are several ways to reduce the dimensionality of the problem, making it computationally tractable.

1.2 One Active State

We consider a simple model where the expected future savings are described by an easily derived differential equation. The model consists of n inactive states where there are no payments, and one active state with continuous dynamics g which, in this setting, may be non-linear. Denote by 0 the active state. On transition to any one of the inactive states, the surplus and savings are nullified. We need not specify what happens to the surplus and savings on a transition - they may be paid out to the customer or the insurance company, or any combination of the two - the only important requirement is that they are zero in all inactive states. The eradication of surplus and savings on transition corresponds to the relation $h_x(t,0,j,x,y)+h_y(t,0,j,x,y) = -x-y$, for $j = 1,...,n$. The survival model with and without surrender options are special cases of this model. The dynamics of X and Y are

$$dX(s) = \mathbb{1}_{\{Z(s-)=0\}} g_x(s,0,X(s),Y(s))ds - \sum_{h=1}^{n} X(s-)dN^h(s)$$
$$dY(s) = \mathbb{1}_{\{Z(s-)=0\}} g_y(s,0,X(s),Y(s))ds - \sum_{h=1}^{n} Y(s-)dN^h(s).$$

Let $W(s) = (X(s), Y(s))^T$, and denote by T_1 the time of the first jump. For the deterministic function W_a that solves

$$W_a(t) = \int_0^t g(s,0,W_a(s))ds,$$

we see that

$$\hat{W}(t) := \mathrm{E}[W(t)|Z(0)=0] = \mathrm{E}[\mathbb{1}_{\{t<T_1\}} W_a(t)|Z(0)=0] = p_{00}(0,t)W_a(t),$$

which comes at no surprise. In this case we know the past and present values of W given the current state of Z, so the only stochastic element pertains to the state of the policy at time t. By differentiating w.r.t. t, and applying Kolmogorov's forward differential equation, we get the following forward differential equation for $\hat{W}$,

$$\hat{W}(0) = \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix},$$
$$\frac{d}{dt}\hat{W}(t) = p_{00}(0,t) g\left(s, 0, \frac{\hat{W}(t)}{p_{00}(0,t)}\right) - \frac{\hat{W}(t)}{p_{00}(0,t)} \sum_{k=1}^{n} \mu_{0k}(t).$$

Even though it may seem very simple and perhaps even trivial, the model with one active state has great applicability.

1.3 Prospective vs. Retrospective

The expected value of the prospective reserve is of monumental interest for insurance companies, and for good reason - they describe their liabilities. So why should we pay any concern to the savings account, that only pertains to the past? The past and future are connected through the principle of equivalence, stating that under the first order basis the expected premiums and benefits should be equal. However, under the first order basis, surplus only grows due to financial returns and not due to the difference between The savings account stands on the verge between the surplus dependent past, and a future where no benefits are increased.

. So the savings account

While it may seem obvious that the prospective reserve must depend solely on information available today, it may not be evident that the retrospective

When incorporating human decisions into the projection of balances and benefits in life insurance, we need to embrace the fact that these decisions are based on the past. The fact that the savings account is retrospective, in the sense that it is $\mathcal{F}_t$ measurable, is the manifestation of the retrospective nature of the FMA's. If one imagines a world where the FMA's are prospective,

The definition and differential equation for the retrospective reserve without bonus, can be found in Norberg (1991).

There is an important distinction between the savings account, and the market value of future guaranteed benefits. The savings account is $\mathcal{F}_{[0,t]}$-measurable whereas the market value of future guaranteed benefits is $\mathcal{F}_{[t,n]}$-measurable.

Even though FMA's are one of the main reasons for considering retrospective quantities, they are not going to be evident in this paper, but are instead hidden from view under a veil of notation.

- Something about Monte-carlo method
- Something about FMA's - perhaps an example?
- Deterministic second order basis, and discussion regarding simulation.

1.3.1 Example With One Active State

If the benefits are identical after age 65, the states 0,1,3 and 4 can be lumped, as well as 2,5 and 6, thus creating a survival model. If the dynamics in two states are identical, they can be viewed as one. Life annuity at age 65.

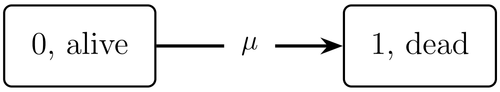


Figure 1: Life-Death model

1.4 Two Active States

When expanding to a model where there are two active states, and n inactive states, we need to use a different method to calculate $\hat{W}$, *if the active states are transient*. There is an important difference between the hierarchical model with two active states, and the transient model with two active states. In the model with one active state, we know the entire history of the policy, given that the policy is in the active state. When we introduce a second active state in the hierarchical model, we also know where the policy has been given the active state, but we do not know when it transitioned from one active state to the other. In order to calculate the expectation of the savings and surplus, we simply have to integrate over all possible transition times. If there are two transient states, there is an infinite amount of paths to any of the transient states, and for each possible path there is an infinite amount of possible jump times. To illustrate the naïve method of calculating expected savings and surplus in a hierarchical model, consider the model depicted in figure 2

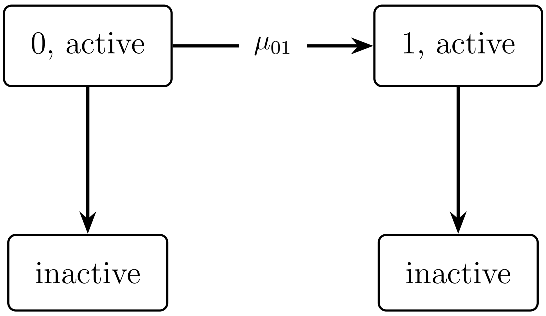


Figure 2: Two active state hierarchical model

In this model, there are two states for which the savings and surplus are non-zero; $Z(t) \in \{0, 1\}$. As in the case with one active state, we know the value of $W(t)$ for $Z(t) = 0$, but for $Z(t) = 1$

we need to consider all possible transition times. Let T_1 be the time of the transition from 0 to 1. If W_0 solves

$$W_0(t) = \int_0^t g(s, 0, W_0(s))ds,$$

then it characterizes the expected value of $W(t)$, given that $Z(t) = 0$. Similarly, W_1 characterizes the expected value of $W(t)$ given $Z(t) = 1$ and $dN_{01}(T_1) = 1$, if it solves

$$\begin{aligned} W_1(T_1, T_1) =& W_0(T_1) + h(T_1, 0, 1, W_0(T_1)), \\ W_1(T_1, t) =& \int_{T_1}^t g(s, 1, W_1(T_1, s))ds. \end{aligned}$$

The density of T_1, given that $Z(t) = 1$ is

$$q(s, t) = \frac{p_{00}(0, s)p_{11}(s, t)}{p_{01}(0, t)} \mu_{01}(s).$$

Let T_1 be the time of the first jump, then

$$W(t) = \mathbb{1}_{\{Z(t)=0\}} W_0(t) + \mathbb{1}_{\{Z(t)=1\}} W_1(T_1, t),$$

implying that

$$\mathrm{E}[W(t)] = p_{00}(0, t)\mathrm{E}[W_0(t)|Z(t) = 0] + p_{01}(0, t)\mathrm{E}[W_1(T_1, t)|Z(t) = 1].$$

When $Z(t) = 0$ all information about the history of the policy is known, and the value of W is deterministic. Conditioning on $Z(t) = 1$ does not provide full information about the history of the policy, as we do not know the time at which the transition from state 0 to state 1 was made. Therefore, to calculate $\mathrm{E}[W(t)|Z(t) = 1]$ we have to integrate over all possible transition times, weighted by the transition intensity given that a jump happened prior to t. Thus

$$\mathrm{E}[W(t)] = p_{00}(0, t)W_0(t) + p_{01}(0, t) \int_0^t q(s, t)W_1(s, t)ds.$$

We could apply this method of calculating $\mathrm{E}[W(t)]$ to any model. The basic principle is simple: given all information about the past of Z, we can calculate the value of $W(t)$, and the expected past can be calculated for each possible path of the policy. In general $\hat{W}$ can be calculated as

$$\mathrm{E}[W(t)] = \sum_{i \in \mathcal{P}} P(\text{path } i) \int_{(0,t]^{L_i}} W_i(t, \Theta_{L_i})dP_i(\Theta_{L_i}), \tag{1}$$

where $\mathcal{P}$ is the set of possible policy paths, L_i is the length of path i, Θ_{L_i} is an L_i-dimensional vector of jump-times, dP_i is the density of transition times for path i and $W_i(t, \Theta_{L_i})$ is the value of $W(t)$ given the path and transition times.

When the model is small and hierarchical, (1) provides a tractable method to calculate the expected savings and surplus, as there are few possible paths and they are short. When the model is transient the problem explodes, as there are infinitely many paths for the policy to take. Fortunately, there are some very large corners to cut, under the simple assumption that g and h are affine in W. Consider the case where W has dynamics

$$dW(s) = g(s, Z(s))W(s)ds,$$

then

$$W(t) = \int_0^t W(s)g(s, Z(s))ds.$$

Say we want to calculate

$$\tilde{W}^i(t) := \mathrm{E}_{Z(0)}[W(t)\mathbb{1}_{\{Z(t)=i\}}]$$

as we can use it to calculate $\mathrm{E}[W(t)] = \tilde{W}^0(t) + \tilde{W}^1(t)$. By the tower property and Fubinis theorem,

$$\begin{aligned}
\mathrm{E}_{Z(0)}[W(t)\mathbb{1}_{\{Z(t)=1\}}] &= \int_0^t \mathrm{E}[\mathbb{1}_{\{Z(t)=1\}}W(s)g(s, Z(s))]ds \\
&= \int_0^t \mathrm{E}_{Z(0)}[\mathbb{1}_{\{Z(s)=1\}}\mathrm{E}_{Z(0)}[\mathbb{1}_{\{Z(t)=1\}}W(s)g(s, Z(s))|Z(s) = 1]]ds \\
&\quad + \int_0^t \mathrm{E}_{Z(0)}[\mathbb{1}_{\{Z(s)=0\}}\mathrm{E}_{Z(0)}[\mathbb{1}_{\{Z(t)=1\}}W(s)g(s, Z(s))|Z(s) = 0]]ds \\
&= \int_0^t p_{Z(0)1}(0, s)\mathrm{E}_{Z(0)}[\mathbb{1}_{\{Z(t)=1\}}W(s)|Z(s) = 1]g(s, 1)ds \\
&\quad + \int_0^t p_{Z(0)0}(0, s)\mathrm{E}_{Z(0)}[\mathbb{1}_{\{Z(t)=1\}}W(s)|Z(s) = 0]g(s, 0)ds.
\end{aligned}$$

By the Markov property $W(s) \perp\!\!\!\perp Z(t)|Z(s)$, as $W(s)$ is $\mathcal{F}_s$-measurable.

$$\begin{aligned}
&= \int_0^t p_{11}(s, t)\tilde{W}^1(s)g(s, 1)ds \\
&\quad + \int_0^t p_{01}(s, t)\tilde{W}^0(s)g(s, 0)ds.
\end{aligned}$$

Differentiating with respect to t, and using Kolmogorov's forward differential equations yields the following system of differential equations

$$\begin{aligned}
\frac{d}{dt}\tilde{W}^1(t) =& \mu_{01}(t)\tilde{W}^0(t)(t) - \mu_{10}(t)\tilde{W}^1(t) + \tilde{W}^1(t)g(t,1) \\
\frac{d}{dt}\tilde{W}^0(t) =& \mu_{10}(t)\tilde{W}^1(t)(t) - \mu_{01}(t)\tilde{W}^0(t) + \tilde{W}^0(t)g(t,0) \\
\tilde{W}^i(0) =& \mathbb{1}_{\{Z(0)=i\}}W(0).
\end{aligned}$$

It is crucial to note that this differential equation is invariant to whether or not the model is transient, in contrast to the naive approach where all possible paths need to be considered individually. In the following section we generalise the result,

2 State-Wise Probability Weighted Reserve

In the previous section we presented a differential equation for a simple two state model, without any payments on transition. The same methodology can be applied for a general Markov model with affine dynamics. We are interested in $\tilde{W}^i(t)$ for $i \in \mathcal{J}$, noting by the tower property that the relation between $\tilde{W}^i$ and $\mathrm{E}_{Z(0)}[W(t)]$ is given by

$$\begin{aligned}
\mathrm{E}_{Z(0)}[W(t)] =& \mathrm{E}_{Z(0)}\left[\sum_{i\in\mathcal{J}} \mathbb{1}_{\{Z(t)=i\}} \frac{\mathrm{E}_{Z(0)}[W(t)\mathbb{1}_{\{Z(t)=i\}}]}{p_{0i}(0,t)}\right] \\
=& \sum_{i\in\mathcal{J}} \tilde{W}^i(t).
\end{aligned}$$

By using the tower property and the fact that $W(s-) \perp\!\!\!\perp Z(t)|Z(s-)$, we get the following theorem

Theorem 2.1.
Let $Z(t)$ be a Markov process on the state space $\mathcal{J}$, and let $W(t)$ be a $\mathcal{F}_t$ measurable process with dynamics

$$dW(s) = g(s, Z(s), W(s))ds + \sum_{k\neq Z(s-)} h(s, Z(s-), k, W(s-))dN^k(s)$$

for g and h of the form

$$\begin{aligned}
g(s, Z(s), W(s)) =& g_1(s, Z(s))W(s) + g_2(s, Z(s)) \\
h(s, Z(s-), k, W(s-)) =& h_1(s, Z(s-), k)W(s-) + h_2(s, Z(s), k).
\end{aligned}$$

Then

$$\frac{d}{dt}\tilde{W}^i(t) = \sum_{j\neq i} \mu_{ji}(t)\tilde{W}^j(t) - \mu_{ij}(t)\tilde{W}^i(t)$$
$$+ \tilde{W}^i(t)g_1(t,i) + p_{Z(0)i}(0,t)g_2(s,Z(s))$$
$$+ \sum_{j\neq i} \mu_{ji}(t)\left(\tilde{W}^j(t)h_1(t,j,i) + p_{Z(0)j}(0,t)h_2(t,j,i)\right)$$

The functions g and h respectively describe the continuous and discrete dynamics of $W(s)$. Skriv noget om antagelser og signifikans. Eksempel.

2.0.1 Free Policy

2.1 General Path dependent dynamics

Even though theorem 2.1 provides a powerful too for calculating future values of the savings and surplus, it is restricted to dynamics that only depends on the current value of Z and W. It is not unreasonable to assume that the dividend strategy depends on the history of Z and W. It turns out, that for certain dynamics that, in some sense, are linearly dependent on the past we can

Theorem 2.2.
Let $Z(t)$ be a Markov process on the state space $\mathcal{J}$, and let $W(t)$ be a $\mathcal{F}_t$ measurable process with dynamics

$$dW(s) = dg(s, \{Z(\tau)\}_{\tau\leq s}, \{W(\tau)\}_{\tau\leq s}) + dh(s, \{Z(\tau)\}_{\tau\leq s}, \{W(\tau)\}_{\tau\leq s})$$

for g and h of the form

$$g(s, \{Z(\tau)\}_{\tau\leq s}, \{W(\tau)\}_{\tau\leq s}) = \int_{(0,s]} \varphi_1(s,\tau,Z(\tau))W(\tau)d\nu_g(\tau,t)$$
$$+ \int_{(0,s]} \varphi_2(s,\tau,Z(\tau))d\eta_g(\tau,t)$$
$$h(s, \{Z(\tau)\}_{\tau\leq s}, \{W(\tau)\}_{\tau\leq s}) = \int_{(0,s]} \sum_{k\neq Z(\tau-)} \psi_1(s,\tau,Z(\tau-),k)W(\tau-)dN^k \otimes \nu_h(\tau,t)$$
$$+ \int_{(0,s]} \sum_{k\neq Z(\tau-)} \psi_2(s,\tau,Z(\tau-),k)dN^k \otimes \eta_h(\tau,t),$$

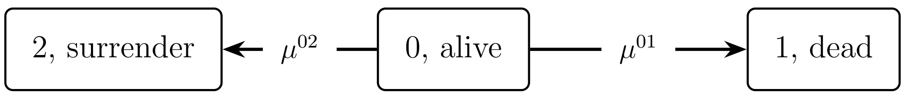
for some measures ν_g, ν_h, η_g and η_h. Then

$$\begin{aligned}
\frac{d}{dt}\tilde{W}^i(t) =& \sum_{j\neq i} \mu_{ji}(t)\tilde{W}^j(t) - \mu_{ij}(t)\tilde{W}^i(t) \\
&+ \int_{(0,t]} \sum_{k\in\mathcal{J}} p_{ki}(\tau,t)\varphi_1(t,\tau,k)\tilde{W}^k(\tau)d\nu_g(\tau,t)ds \\
&+ p_{Z(0)i}(0,t)\int_{(0,t]} \sum_{k\in\mathcal{J}} p_{ki}(\tau,t)\varphi_2(t,\tau,k)d\eta_g(\tau,t)ds \\
\tilde{W}^i(0) =& \mathbb{1}_{\{Z(0)=i\}}W(0)
\end{aligned}$$

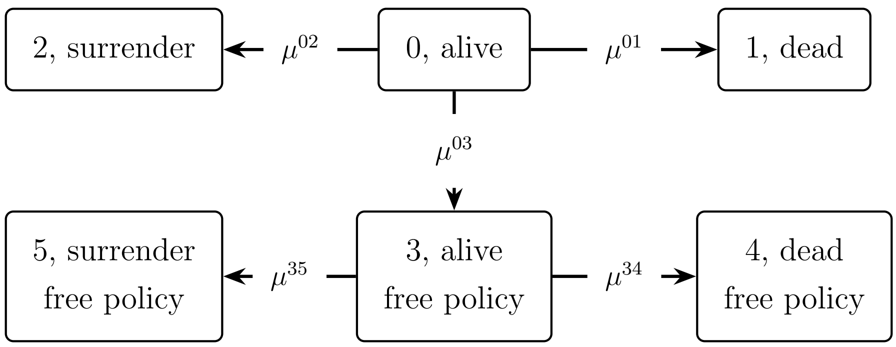
We see that theorem 2.1 is a special case of theorem 2.2 with ν_g, ν_h, η_g and η_h being the Dirac measures in t, i.e

$$\nu_g(\tau,t) = \nu_h(\tau,t) = \eta_g(\tau,t) = \eta_h(\tau,t) = \mathbb{1}_{\{\tau=t\}}$$

2.2 Life-Death-Surrender



2.3 Life-Death-Surrender With Free Policy



2.4 Use of Savings account

2.5 Thoughts

- With-profit insurance! Expected reserve including accumulation of dividends.

- Refer to Norberg (1991)
  - Introduction and motivation - stochastic reserve, Monte Carlo method. A little comment on the fact that the problem is still hard to solve.
  - Life-death (simple analytic solution).
  - Life-death free policy (how to deal with extra states).
  - General model without duration.
  - Life-death-surrender free policy, including discussion of free policy factor.
  - Lost all trick works.
  - General model with duration dependence.
  - Inclusion of surplus. Use independence when dividend is assigned on discrete points in time.
- Deterministic intensities.
- General Hierarchical models do not need linearity. In general the variance increases as the number of states increase as the variance of the sum of transition times increases.
- Market dependent intensities - allowed when directly dependent on the market, making them deterministic. Or intensities that depend on the expected reserve - in a sense corresponding to intensities that depend on the group of similar policies.
- We are only concerned with the reserve.
- Maybe we should use a different wording? **Savings**/stash/backlog/accumulation/hoard/reservoir instead of reserve, to distinguish between the Danish words for "reserve" and "depot"
- One could imagine that information about the jump time could be partially deduced from the intensities, thus almost allowing for non-linearity. Consider case where $\mu_{01}(t) = \kappa \mathbb{1}_{\{t \in (c_1, c_2]\}}$ for very small $|c_2 - c_1|$ and very large κ, providing almost perfect information about the jump time, whereby non-linearity in $g(s, 1, W(s))$ would be allowed for.

A Proof of Theorem 2.1

Proof of theorem 2.1. The proof consists of two steps. First, we derive an integral equation for $\tilde{W}^i(t)$. Second, we differentiate this integral equation.
By the tower property

$$
\begin{aligned}
\tilde{W}^i(t) :=& \mathrm{E}_{Z(0)}[W(t)\mathbb{1}_{\{Z(t)=i\}}] \\
=& \mathrm{E}_{Z(0)}\left[\int_0^t \mathbb{1}_{\{Z(t)=i\}} dW(s)\right] \\
=& \mathrm{E}_{Z(0)}\left[\int_0^t \mathbb{1}_{\{Z(t)=i\}} g(s, Z(s), W(s))ds\right] \\
&+ \mathrm{E}_{Z(0)}\left[\int_0^t \sum_{k \neq Z(s-)} \mathbb{1}_{\{Z(t)=i\}} h(s, Z(s-), k, W(s-))dN^k(s)\right].
\end{aligned}
$$

Based on the calculations in section C of Norberg (1991), note that the intensity process of the predictable compensator for $N^{jk}(s)|Z(s-) = j, Z(t) = i$ is given by

$$\mu_{jk}(s)\frac{p_{ki}(s,t)}{p_{ji}(s,t)}.$$

As $h(s, Z(s-), k, W(s-)$ is predictable, we may replace the integrator $dN^k(s)$ with its predictable compensator. Using the tower property once more,

$$
\begin{aligned}
\tilde{W}^i(t) &= \int_0^t \mathrm{E}_{Z(0)} \left[\mathrm{E}_{Z(0)} \left[\mathbb{1}_{\{Z(t)=i\}} g(s, Z(s), W(s))|Z(s)\right]\right] ds \\
&\quad + \mathrm{E}_{Z(0)} \left[\mathrm{E}_{Z(0)} \left[\int_0^t \sum_{k \neq Z(s-)} \mathbb{1}_{\{Z(t)=i\}} h(s, Z(s-), k, W(s-)) dN^k(s)|Z(s-)\right]\right] \\
&= \int_0^t \sum_{j \in \mathcal{J}} p_{Z(0)j}(0,s) \mathrm{E}_{Z(0)} \left[\mathbb{1}_{\{Z(t)=i\}} g(s, Z(s), W(s))|Z(s)=j\right] ds \\
&\quad + \sum_{j \in \mathcal{J}} p_{Z(0)j}(0,s) \mathrm{E}_{Z(0)} \left[\int_0^t \sum_{k \neq j} \mathbb{1}_{\{Z(t)=i\}} h(s, j, k, W(s-)) dN^k(s)|Z(s-)=j\right] \\
&= \int_0^t \sum_{j \in \mathcal{J}} p_{Z(0)j}(0,s) \mathrm{E}_{Z(0)} \left[\mathbb{1}_{\{Z(t)=i\}} g(s, Z(s), W(s))|Z(s)=j\right] ds \\
&\quad + \sum_{j \in \mathcal{J}} p_{Z(0)j}(0,s) p_{ji}(s,t) \mathrm{E}_{Z(0)} \left[\int_0^t \sum_{k \neq j} h(s, j, k, W(s-)) dN^k(s)|Z(s-)=j, Z(t)=i\right] \\
&= \int_0^t \sum_{j \in \mathcal{J}} p_{Z(0)j}(0,s) \mathrm{E}_{Z(0)} \left[\mathbb{1}_{\{Z(t)=i\}} g(s, Z(s), W(s))|Z(s)=j\right] ds \\
&\quad + \sum_{j \in \mathcal{J}} p_{Z(0)j}(0,s) \int_0^t \sum_{k \neq j} \mathrm{E}_{Z(0)} \left[h(s, j, k, W(s-))|Z(s-)=j, Z(t)=i\right] \mu_{jk}(s) p_{ki}(s,t) ds.
\end{aligned}
$$

Since $W(s)$ is $\mathcal{F}_s$-measurable, the Markov property gives us

$$
\mathrm{E}_{Z(0)}[\mathbb{1}_{\{Z(t)=i\}} W(s)|Z(s)=j] = \frac{\tilde{W}^j(s)}{p_{Z(0)j}(0,s)} p_{ji}(s,t),
$$

and by the continuity of $\tilde{W}^i(t)$ we get

$$\begin{aligned}
\tilde{W}^i(t) = &\int_0^t \sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s)\left(\frac{\tilde{W}^j(s)}{p_{Z(0)j}(0,s)}g_1(j,s)+g_2(j,s)\right)p_{ji}(s,t)ds \\
&+\int_0^t \sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s)\left(\sum_{k\neq j}\mu_{jk}(t)p_{ki}(s,t)\left(\frac{\tilde{W}^j(s)}{p_{Z(0)j}(0,s)}h_1(s,j,k)+h_2(s,j,k)\right)\right)ds \\
=&\int_0^t \sum_{j\in\mathcal{J}} p_{ji}(s,t)\tilde{W}^j(s)g_1(j,s)ds \\
&+\int_0^t \sum_{j\in\mathcal{J}}\sum_{k\neq j}\mu_{jk}(t)p_{ki}(s,t)\tilde{W}^j(s)h_1(s,j,k)ds \\
&+\int_0^t \sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s)g_2(j,s)p_{ji}(s,t)ds \\
&+\int_0^t \sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s)\sum_{k\neq j}\mu_{jk}(t)p_{ki}(s,t)h_2(s,j,k)ds.
\end{aligned}$$

Differentiating with respect to t gives

$$\begin{aligned}
\frac{d}{dt}\tilde{W}^i(t) =&\tilde{W}^i(t)g_1(i,t)+p_{Z(0)i}(0,t)g_2(i,t) \\
&+\sum_{k\neq i}\mu_{ki}(t)\left(\tilde{W}^k(t)h_1(t,k,i)+p_{Z(0)k}(0,t)h_2(t,k,i)\right) \\
&+\int_0^t \frac{\partial}{\partial t}\sum_{j\in\mathcal{J}} p_{ji}(s,t)\tilde{W}^j(s)g_1(j,s)ds \\
&+\int_0^t \frac{\partial}{\partial t}\sum_{j\in\mathcal{J}}\sum_{k\neq j}\mu_{jk}(t)p_{ki}(s,t)\tilde{W}^j(s)h_1(s,j,k)ds \\
&+\int_0^t \frac{\partial}{\partial t}\sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s)g_2(j,s)p_{ji}(s,t)ds \\
&+\int_0^t \frac{\partial}{\partial t}\sum_{j\in\mathcal{J}} p_{Z(0)j}(0,s)\sum_{k\neq j}\mu_{jk}(t)p_{ki}(s,t)h_2(s,j,k)ds.
\end{aligned}$$

By the Kolmogorov forward differential equations we arrive at

$$\begin{aligned}\frac{d}{dt}\tilde{W}^i(t) =& \tilde{W}^i(t)g_1(i,t) + p_{Z(0)i}(0,t)g_2(i,t) \\ &+ \sum_{k\neq i} \mu_{ki}(t)\left(\tilde{W}^k(t)h_1(t,k,i) + p_{Z(0)k}(0,t)h_2(t,k,i)\right) \\ &+ \sum_{k\neq i} \mu_{ki}(t)\tilde{W}^k(t) - \mu_{ik}(t)\tilde{W}^i(t).\end{aligned}$$

Combined with the initial condition

$$\tilde{W}^i(0) = \mathrm{E}_{Z(0)}[\mathbb{1}_{\{Z(0)=i\}}W(0)] = \mathbb{1}_{\{Z(0)=i\}}W(0),$$

the proof is complete. □

B Dynamics of X and Y

The amount by which the savings surpass the first order reserve, is spent on B_2. where

$$b^j(t,x) = b_1^j(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^j(t), \qquad b^{jg}(t,x) = b_1^{jg}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^{jg}(t).$$

Dynamics of X

$$\begin{aligned}dX(t) =& r^*(t)X(t)dt + \delta^{Z(t)}(t,X(t),Y(t))dt - \sum_{g\neq Z(t-)} \rho^{Z(t-)g}(t,X(t-))dt \\ &- b^{Z(t)}(t,X(t))dt \\ &- \sum_{g\neq Z(t-)} \left(b^{Z(t-)g}(t,X(t-)) + \chi^{Z(t-)g}(t,X(t-)) - X(t-)\right)\mu^{Z(t)g}(t)dt \\ &+ \sum_{g\neq Z(t-)} \left(\chi^{Z(t-)g}(t,X(t-)) - X(t-)\right)dN^g(t),\end{aligned}$$

and

$$dY(t) = Y(t)\frac{dS(t)}{S(t)} - \delta^{Z(t)}(t,X(t),Y(t)) + (r(t) - r^*(t))X(t) + \sum_{g\neq Z(t-)} \rho^{Z(t)g}(t,X(t)),$$

where

$$
\begin{aligned}
\rho^{jg}(t,x) =& (b^{jg}(t,x) + \chi^{jg}(t,x) - x)(\mu^{*jg}(t) - \mu^{jg}(t)) \\
\chi^{jg}(t,x) =& V_1^{g*}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{g*}(t),
\end{aligned}
$$

$$
\delta^j(t,x,y) = \delta_1^j(t) + \delta_2^j(t)x + \delta_3^j(t)y + \delta_4^j(t)xy. \tag{2}
$$

$$
\tilde{W}^j(t) := \begin{pmatrix} \tilde{X}^j(t) \\ \tilde{Y}^j(t) \end{pmatrix} = \begin{pmatrix} \mathrm{E}[X(t)\mathbb{1}_{\{Z(t)=j\}}] \\ \mathrm{E}[Y(t)\mathbb{1}_{\{Z(t)=j\}}] \end{pmatrix}
$$

With differential equation

$$
\begin{aligned}
\frac{d}{dt}\tilde{W}^j(t) =& \sum_{g \neq j} \mu^{gj}(t)\tilde{W}^g(t) - \mu^{jg}(t)\tilde{W}^j(t) \\
&+ \tilde{W}^j(t) \circ g_1(t,j,x,y) + p_{0j}(0,t) g_2(t,j) \\
&+ \sum_{g \neq j} \mu^{gj}(t) \left(\tilde{W}^g(t) \circ h_1(t,g,j,x,y) + p_{0g}(0,t) h_2(t,g,j) \right), \\
\tilde{W}^j(0) =& \mathbb{1}_{\{Z(0)=j\}} \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix},
\end{aligned}
$$

where $\circ$ denotes the Hadamard product (element-wise multiplication) and

$$
\begin{aligned}
g_1(t,j,x,y) &= \begin{pmatrix} g_{x1}(t,j,y) \\ g_{y1}(t,j,x) \end{pmatrix}, \qquad & h_1(t,j,g,x,y) &= \begin{pmatrix} h_{x1}(t,j,g,y) \\ h_{y1}(t,j,g,x) \end{pmatrix}, \\
g_2(t,j) &= \begin{pmatrix} g_{x2}(t,j,y) \\ g_{y2}(t,j,x) \end{pmatrix}, \qquad & h_2(t,j,g) &= \begin{pmatrix} h_{x2}(t,j,g,y) \\ h_{y2}(t,j,g,x) \end{pmatrix}.
\end{aligned}
$$

For

$$
\begin{aligned}
g_{x1}(t,j,y) =& r^*(t) + \delta_2^j(t) + \delta_4^j(t)y \\
&+ \frac{b_2^j(t)}{V_2^{j*}(t)} - \sum_{g \neq j} \rho_1^{jg}(t) - \sum_{g \neq j} \left(\frac{b_2^{jg}(t)}{V_2^{j*}(t)} + \frac{V_2^{g*}(t)}{V_2^{j*}(t)} - 1 \right) \mu^{jg}(t) \\
g_{x2}(t,j,y) =& \delta_1^j(t) + \delta_3^j(t)y - b_1^j(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} b_2^j(t) - \sum_{g \neq j} \rho_2^{jg}(t) \\
&- \sum_{g \neq j} \left(b_1^{jg}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} b_2^{jg}(t) - \frac{V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{g*}(t) + V_1^{g*}(t) \right) \mu^{jg}(t).
\end{aligned}
$$

$$
\begin{aligned}
h_x(t,j,g,x,y) =& \chi^{jg}(t,x) - x \\
=& V_1^{g*}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} V_2^{g*}(t) - x \\
=& \underbrace{x \left(\frac{V_2^{g*}(t)}{V_2^{j*}(t)} - 1 \right)}_{h_{x1}(t,j,g,y)} + \underbrace{V_1^{g*}(t) - \frac{V_1^{j*}(t) V_2^{g*}(t)}{V_2^{j*}(t)}}_{h_{x2}(t,j,g,y)} .
\end{aligned}
$$

For g_y we get

$$
\begin{aligned}
g_y(t,j,x,y) =& y \frac{dS(t)}{S(t)} - \delta^j(t,x,y) + (r(t) - r^*(t))x + \sum_{g \neq j} \rho^{jg}(t,x) \\
=& y \frac{dS(t)}{S(t)} - \delta_1^j(t) - \delta_2^j(t)x - \delta_3^j(t)y - \delta_4^j(t)xy \\
& + (r(t) - r^*(t))x + \sum_{g \neq j} \rho^{jg}(t,x) \\
=& \underbrace{y \left(\frac{dS(t)}{S(t)} - \delta_3^j(t) - \delta_4^j(t)x \right)}_{g_{y1}(t,j,x)} \\
& \underbrace{+ \sum_{g \neq j} \rho^{jg}(t,x) - \delta_1^j(t) - \delta_2^j(t)x + (r(t) - r^*(t))x}_{g_{y2}(t,j,x)}
\end{aligned}
$$

Finally, as $h_y = 0$ we have $h_{y1} = h_{y2} = 0$.

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