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A GENERALIZED TRIGONOMETRIC SOLUTION OF THE CUBIC EQUATION.

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The trigonometric solution of the cubic, as commonly given, is either limited to the "irreducible case," or treated by a special method for each case. The following solution treats all cases by the same device, and if tables of hyperbolic functions are at hand, is no less convenient for numerical calculation than other methods. Nor is it restricted to real coefficients, a fact which may sometimes make it convenient in applications of the theory of functions.

As a preliminary, consider the sine, $p+iq$, belonging to a complex angle $u+iv$, *i. e.*, let

$$\sin(u+iv)=p+iq=re^{ai}. \tag{1}$$

By expanding, and separating real and imaginary parts, we find that

$$\sin u \operatorname{Cosh} v=p, \quad \cos u \operatorname{Sinh} v=q. \tag{2}$$

Hence,

$$\frac{p}{\sin u}=\operatorname{Cosh} v, \quad \frac{q}{\cos u}=\operatorname{Sinh} v. \tag{3}$$

Squaring each equation in (2), and subtracting, we find that

$$\frac{p^2}{\sin^2 u}-\frac{q^2}{\cos^2 u}=\operatorname{Cosh}^2 v-\operatorname{Sinh}^2 v=1. \tag{4}$$

By substituting $\cos^2 u=1-\sin^2 u$ in (4), and solving the resulting equation as a quadratic in $\sin^2 u$, we get

$$\left.\begin{aligned} u&=\sin^{-1}[\pm\sqrt{\tfrac{1}{2}(p^2+q^2+1)-\tfrac{1}{2}\surd(p^2+q^2+1)^2-4p^2}] \\ &=\sin^{-1}[\pm\sqrt{\tfrac{1}{2}(r^2+1)-\tfrac{1}{2}\surd r^4-2r^2\cos\alpha+1}] \end{aligned}\right\} \tag{5}$$

By a similar process'

$$\left.\begin{aligned} v&=\mathrm{Sinh}^{-1}[\pm\sqrt{\tfrac{1}{2}(p^2+q^2-1)+\tfrac{1}{2}\surd(p^2+q^2-1)^2+4q^2}] \\ &=\mathrm{Sinh}^{-1}[\pm\sqrt{\tfrac{1}{2}(r^2-1)+\tfrac{1}{2}\surd r^4-2r^2\cos 2\alpha+1}] \end{aligned}\right\} \tag{6}$$

The signs before the inner radicals are apparently ambiguous; but by considering that $\sin^2 u$ and $\sin^2 v$ must be positive, and $\sin^2 u \leq 1$ (in order to give real values to u and v), we see that the signs of the inner radicals must be as written in (5) and (6). The signs before the outer radicals may be taken the same as those of p and q, respectively.

Suppose the cubic to have been deprived of its second term, and to be in the form

$$x^3-ax+b=0, \tag{7}$$

where a and b may be positive, negative, or complex. From trigonometry,

$$\sin^3\phi-\tfrac{3}{4}\sin\phi+\tfrac{1}{4}\sin 3\phi=0. \tag{8}$$

This suggests reducing (7) to the form

$$y^3-\tfrac{3}{4}y+c=0, \tag{9}$$

which is accomplished by the substitution

$$x=2\sqrt{\frac{a}{3}}\,y, \text{ giving } c=\frac{3b}{8a}\sqrt{\frac{3}{a}}. \tag{10}$$

If a be complex, either determination may be taken for $\sqrt{\frac{a}{3}}$, but $\sqrt{\frac{3}{a}}$ should then have an argument equal to the negative of the argument of $\sqrt{\frac{a}{3}}$.

If we assume $\sin\phi=y$, we find by comparing (8) and (9) that

$$\sin 3\phi=\frac{3b}{2a}\sqrt{\frac{3}{a}}. \tag{11}$$

To solve equation (7), we compute $\frac{3b}{2a}\sqrt{\frac{3}{a}}$, treat it as a sine, and compute the corresponding angle—which we call $3\phi_1$—by (5) and (6), taking the real part of 3ϕ between $-90°$ and $+90°$. Other possible values of 3ϕ are $3\phi_2=3\phi_1+360°$, $3\phi_3=3\phi_1+720°$. The roots are therefore, by (10),

$$\left.\begin{aligned} x_1&=2\sqrt{\frac{a}{3}}\sin\phi_1 \\ x_2&=2\sqrt{\frac{a}{3}}\sin\phi_2=2\sqrt{\frac{a}{3}}\sin(\phi_1+120°) \\ x_3&=2\sqrt{\frac{a}{3}}\sin\phi_3=2\sqrt{\frac{a}{3}}\sin(\phi_1+240°) \end{aligned}\right\} \tag{12}$$

When a and b are restricted to real values, we readily find the well-known conditions for the reality of all roots:

$$a>0 \text{ and } 27b^2<4a^3.$$

The condition for equal roots holds whether the coefficients are real or complex. By introducing the Gudermannian angle θ, defined by $\phi=\log_e\tan(45°+\frac{1}{2}\theta)$, from which follows that

$$\operatorname{Cosh}\phi=\sec\theta,\quad \operatorname{Sinh}\phi=\tan\theta,\quad \text{etc.},$$

we can reduce the general formulas given above to the special ones involving tangents of auxiliary angles sometimes used in the case of a single real root.

It is worth noting that a table of Mercator's parts—calculated for a spherical earth—is a table of inverse Gudermannians, and may be used to give the hyperbolic functions if a regular table of them is not at hand. A table of Mercator's parts carried to $\frac{1}{100}$ minute is given in Callet's "Tables Nautiques." The second alternative form under (5) and (6) is generally the more convenient for numerical work with complex coefficients. The inner radical represents the side of a triangle whose other sides are r^2 and 1, and their included angle α. There are small tables for the solution of this case, but in view of its frequent occurence either directly, or in an equivalent form, it seems rather remarkable that no extensive tables for it are in general use.

Example: Solve $x^3+ix+1+i=0$.

Here $a=-i$, $b=1+i$, $\sqrt{\frac{a}{3}}=\frac{1-i}{\sqrt{6}}$, $\sqrt{\frac{3}{a}}=(1+i)\sqrt{\frac{3}{2}}$, $\sin 3\phi=-\frac{3}{2}\sqrt{6}$.

From (5) and (6), since $p=-\frac{3}{2}\sqrt{6}$ and $q=0$, $u=\sin^{-1}(-1)=-90°$, $v=\mathrm{Sinh}^{-1}(\frac{5}{2}\sqrt{2})=1.97544$, $\mathrm{Sinh}\frac{v}{3}=0.70709$, $\mathrm{Cosh}\frac{v}{3}=1.22473$;

$$x_1=\frac{2(1-i)}{\sqrt{6}}(-\tfrac{1}{2}\times1.22473+i\frac{\sqrt{3}}{2}\times0.70709)=-0.00001+i\times0.99997,$$

$$x_2=\frac{2(1-i)}{\sqrt{6}}(1.22473)=0.99998(1-i),$$

$$x_3=\frac{2(1-i)}{\sqrt{6}}(-\tfrac{1}{2}\times1.22473-i\frac{\sqrt{3}}{2}\times0.70709)=-0.99997+i\times0.00001.$$

These are from five-figure tables; the exact roots are i, $1-i$, and -1.

ON THE EXPANSION OF DEVERTEBRATED THREE DIMENSIONAL DETERMINANTS AND THE EXTENSION OF CAYLEY'S EXPANSION THEOREM.

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The primary object of this paper is to extend to three dimensional or cubical determinants* the general law for the expansion of a two dimensional devertebrated determinant† in terms of the elements of the principal diagonal of the given determinant and their co-axial minors or, in other words, to develop a general law for the expansion of a devertebrated cubical determinant in terms of the elements of its principal diagonal plane and their co-planar minors.

The secondary object is to show how Cayley's Expansion Theorem may be extended to cubical determinants with but slight modifications. A formula for the number of terms of a cubical determinant which are independent of the elements of the principal diagonal plane is also given.

Let $\triangle$ denote a cubical determinant of order n and let $\triangle'_a$ denote a new cubical determinant formed by adding a different variable to each of the n^2 elements of the principal diagonal plane.

Expanding $\triangle'_a$ as a cubical determinent with binomial elements, the term independent of the variable will be the given cubical determinant $\triangle$. The next n^2 terms are the products of the respective variables and their corresponding cubical minors of order $n-1$; similarly, the next $\frac{n^2(n-1)^2}{2!}$ terms are products

*An exposition and bibliography of cubical determinants is given by E. R. Hedrick, *Annals of Mathematics*, Ser. 2, Vol. 1 (1900), pp. 49—67.

†Stetson, MONTHLY, Vol. XI (1904), pp. 166—168.