J. Astrophys. Astr. (2005) **26,** 139–160

Theory of Asteroseismology

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Abstract. The increasing number of pulsating modes in a star leads to the increase of the amount of information derived from the pulsation. Probing the invisible interior of stars is a dream come true. In this review, I summarize the fundamental properties of stellar pulsations and the excitation mechanisms, emphasizing several cases of recent progress.

Key words. Asteroseismology—stars: pulsation—pulsation: excitation.

1. Introduction

We have recently witnessed a new development in observations and theories of stellar pulsations. This new development comes basically from the discovery of pulsations and oscillation-related phenomena in many stars, which were hitherto regarded as non-pulsating stars. They include our sun, white dwarfs, Ap stars, early-type O/B stars, and so on. The most important characteristics of oscillations in these stars are that their oscillations are usually multi-periodic with several modes of oscillations involved, while most of the classical pulsating variables, such as Cepheids and RR Lyrae stars, present only a single mode. The increasing number of pulsating modes in a star leads to the increase of the amount of information derived from the pulsation. Indeed, any single period of classical pulsating stars, which are thought to be pulsating with the radial fundamental mode or the first overtone, has provided us a measure of the mean density of the star, but in the case of the double-mode Cepheids, which are pulsating in two different modes, then the stellar mass can be determined as one more item of information from the pulsations in addition to the mean density. A more conspicuous case is our sun. The sun is oscillating in thousands of nonradial eigenmodes, and this richness in the number of identified modes leads to the great success of "helioseismology", by which we can probe the invisible internal structure of the sun in detail. The discovery of very small amplitude pulsations, especially those found from observations of the sun as a star, has raised a question concerning the concept of pulsating and non-pulsating stars. We now expect that pulsations, at least with small amplitudes, will be ubiquitous among stars in general. This opens the possibility of a seismological approach to stars, and the research field probing the internal structure of stars in general is now called "asteroseismology". We may not detect such large numbers of eigenmodes of individual distant stars even if they are oscillating, since we may not resolve the stellar image as in the case of the sun. However, we should be reminded that every one of the stars differs from all stars in its mass, and in its evolutionary stage. From the standpoint of the theory of stellar oscillation, stars are like musical instruments that are able to oscillate in modes which differ from one star to another and change in delicate ways as a star evolves. Probing the invisible interior of distant stars in detail like in the case of the sun is thus worth doing. Asteroseismology is at the present moment still in its infancy, but it has the potential to develop into a major field of stellar physics.

2. Basics of stellar oscillations

In this section, I summarize the general properties of pulsations of stars. For more complete reviews on this subject, text books such as those by Ledoux and Walraven (1958); by Cox (1980); by Unno *et al.* (1989); and by Gough (1993) should be consulted. Recent progress has also been concisely described by Gautschy and Saio (1995, 1996).

2.1 *Basic equations of linear oscillations*

Stars are like musical instruments which have various modes of oscillations and tones. Such oscillations of a star are described by equations of hydrodynamics. We consider a spherically symmetric star as an unperturbed state upon which small perturbations of oscillation are superimposed. We assume that no magnetic field and no motions exist in the unperturbed state. We also assume in the linear theory that all perturbations are sufficiently small so that only terms in the first-order in perturbations are retained while those higher than the second are neglected. There are two different ways to express perturbations: the Eulerian form and the Lagrangian form. The Eulerian perturbation is defined as the perturbation of a physical quantity at a given position, denoted by prime, while the Lagrangian perturbation is defined by that for a given fluid element, denoted by a symbol δ. The connection between the Eulerian and Lagrangian perturbations f' and δf of any physical quantity f, respectively, is, to first order in the displacement of the fluid element,

$$\boldsymbol{\xi} \equiv \boldsymbol{r} - \boldsymbol{r}_0, \tag{1}$$

given by

$$\delta f = f' + \boldsymbol{\xi} \cdot \nabla f_0, \tag{2}$$

where $\boldsymbol{r}$ in equation (1) denotes the Lagrangian position variable of a given fluid element which is at $\boldsymbol{r} = \boldsymbol{r}_0$ in the equilibrium state, and the subscript 0 in equation (2) denotes the unperturbed, equilibrium state of f. (In what follows, we omit the subscript 0 for equilibrium quantities unless there is confusion.)

The set of linearized basic equations of hydrodynamics is derived as follows:

(a) Equation of continuity:

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}') = 0. \tag{3}$$

(b) Equation of motion:

$$\frac{\partial \boldsymbol{v}'}{\partial t} = -\frac{1}{\rho} \nabla p' + \frac{\rho'}{\rho^2} \nabla p - \nabla \psi'. \tag{4}$$

(c) Poisson equation:

$$\nabla^2\psi' = 4\pi G\rho'. \tag{5}$$

(d) Equation of thermal energy conservation:

$$\begin{aligned}\rho T\frac{\partial\delta S}{\partial t} &\equiv \rho T\frac{c_p}{v_T}\left\{\frac{\partial(\delta\rho/\rho)}{\partial t} - \frac{1}{\Gamma_1}\frac{\partial(\delta p/p)}{\partial t}\right\}\\ &= \rho\varepsilon\left(\frac{\delta\rho}{\rho} + \frac{\delta\varepsilon}{\varepsilon}\right) - \delta(\nabla\cdot\boldsymbol{F}),\end{aligned} \tag{6}$$

where S denotes the specific entropy, ε the nuclear energy generation rate per unit mass, $\boldsymbol{F}$ the energy flux, $c_p \equiv T(\partial S/\partial T)_p$ the specific heat per unit mass at constant pressure, $v_T \equiv -(\partial \ln\rho/\partial\ln T)_p$, $\Gamma_1 \equiv (\partial\ln p/\partial\ln\rho)_S$, and the other symbols have their usual meanings (t: time, ρ: density, p: pressure, T: temperature, $\boldsymbol{v}$: velocity, ψ: gravitational potential, G: gravitational constant). Here we neglect viscosity since it is generally small in the stellar interior. The Lagrangian perturbation of the chemical composition is also neglected, since the periods of pulsations are much shorter than the time scales of nucleosynthesis and material diffusion. The Lagrangian variation of the velocity is the Stokes derivative of the displacement of the fluid element:

$$\delta\boldsymbol{v} = \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{\xi}. \tag{7}$$

Since we are assuming no velocity field in the unperturbed state, equations (2) and (7) lead to

$$\boldsymbol{v}' = \frac{\partial\boldsymbol{\xi}}{\partial t}. \tag{8}$$

2.2 *Linear adiabatic wave equation: Eigenvalue problem*

The temporal dependence of any perturbed quantities can be represented by $\exp(i\sigma t)$. The ratio of thermal to dynamical time scales is one of the most important parameters that govern the entropy perturbation of stellar pulsations. This fact is apparent in the dimensionless equation of energy conservation. By replacing $\partial/\partial t$ with $i\sigma$ after multiplying equation (6) by $4\pi r^3/L$, where L denotes the luminosity of the star and r, the distance from the stellar center, we obtain a dimensionless form of the linearized equation of thermal energy conservation,

$$i\omega\frac{\tau_{\rm th}}{\tau_{\rm dyn}}\frac{\delta S}{c_p} = \frac{4\pi r^3\rho\varepsilon}{L}\left[\rho\varepsilon\left(\frac{\delta\rho}{\rho} + \frac{\delta\varepsilon}{\varepsilon}\right) - \delta(\nabla\cdot\boldsymbol{F})\right]. \tag{9}$$

Here the thermal time scale $\tau_{\rm th}$ and the dynamical time scale $\tau_{\rm dyn}$ are defined as $\tau_{\rm th} \equiv 4\pi r^3\rho c_p T/L$, $\tau_{\rm dyn} \equiv (R^3/GM)^{1/2}$, respectively, where M and R denote the mass and the radius of the star, respectively, and ω denotes the dimensionless angular frequency normalized by $\tau_{\rm dyn}^{-1}$; $\omega \equiv \sigma/(R^3/GM)^{-1/2}$. Equation (9) indicates that for a given energy excess and for a given c_p the entropy perturbation is smaller for larger $\tau_{\rm th}/\tau_{\rm dyn}$. The ratio $\tau_{\rm th}/\tau_{\rm dyn}$ is very large in the interior of a star except near the stellar surface. In most of the stellar interior, pulsations are then almost adiabatic, and the adiabatic pulsation is, in many cases, a very good approximation for the actual stellar pulsation. In this case, $\delta S = 0$, and hence

$$c^2 \frac{\partial \rho'}{\partial t} - \frac{\partial p'}{\partial t} = -\boldsymbol{v}' \cdot (c^2 \nabla \rho - \nabla p), \tag{10}$$

where $c \equiv (\Gamma_1 p/\rho)^{1/2}$ denotes the sound speed.

By operating $\partial/\partial t$ to the equation of motion (4), and eliminating $\partial \rho'/\partial t$ and $\partial p'/\partial t$ with the help of equations (3) and (10), we deduce the basic equation in terms of an operator $\mathcal{L}$ with respect to the displacement vector $\boldsymbol{\xi}$

$$\begin{aligned}\mathcal{L}(\boldsymbol{\xi}) \equiv & \frac{1}{\rho^2} \nabla p \nabla \cdot (\rho \boldsymbol{\xi}) - \frac{1}{\rho} \nabla (\boldsymbol{\xi} \cdot \nabla p) - \frac{1}{\rho} \nabla (c^2 \rho \nabla \cdot \boldsymbol{\xi}) \\ & + \nabla \left\{ G \int \frac{\nabla \cdot [\rho \boldsymbol{\xi}] \mathrm{d}^3 \boldsymbol{x}}{|\boldsymbol{x} - \boldsymbol{r}|} \right\} = \sigma^2 \boldsymbol{\xi}. \end{aligned} \tag{11}$$

This equation, together with boundary conditions, forms an eigenvalue problem with the eigenvalue σ^2.

The operator $\mathcal{L}$ is Hermitian in the case of vanishing pressure at the stellar surface (the zero-boundary condition):

$$\int_0^M \boldsymbol{\eta}^* \mathcal{L}(\boldsymbol{\xi}) \mathrm{d}M_r = \int_0^M \boldsymbol{\xi}^* \mathcal{L}(\boldsymbol{\eta}) \mathrm{d}M_r, \tag{12}$$

where M_r denotes the mass inside the radius r; that is, $\mathrm{d}M_r = 4\pi \rho r^2 \mathrm{d}r$. Hence the eigenvalues, σ^2, are real; the eigenfunctions form a normalized orthogonal set:

$$\int_0^M \boldsymbol{\xi}_q^* \cdot \boldsymbol{\xi}_{q'} \mathrm{d}M_r = \delta_{qq'}, \tag{13}$$

where the subscripts attached to $\boldsymbol{\xi}$ label the eigenmode, and $\delta_{qq'}$ denotes the Kronecker delta.

The normal modes with non-zero eigenfrequencies σ in a spherically symmetric stars are characterized by the eigenfunctions that are proportional to the spherical harmonics, $Y_l^m(\theta, \phi) (l = 0, 1, 2, \ldots; m = 0, \pm 1, \ldots, \pm l)$, and the displacement eigenfunction is given by

$$\boldsymbol{\xi} = \left[\xi_r(r) \boldsymbol{e}_r + \xi_h(r) \left(\boldsymbol{e}_\theta \frac{\partial}{\partial \theta} + \boldsymbol{e}_\phi \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right) \right] Y_l^m(\theta, \phi) \exp(i \sigma t). \tag{14}$$

The degree l represents the total number of nodal lines which divide the stellar surface into zones of opposite phase, while the azimuthal order m indicates the number of nodal lines which are longitudinal. The normal modes belonging to a spherical harmonic are further distinguished by the number of nodes, n, in the radial component of displacement from the center to the surface of a star. The normal modes are then labelled by the three quantum numbers: the radial order n, the degree l, and the azimuthal order m. The eigenfrequencies, however, depend on the radial order n and the degree l but are degenerated by $(2l+1)$-folds in the azimuthal order m in the case of a non-rotating, non-magnetic spherical star. This degeneracy is lifted if rotation or the magnetic field of the star is taken into account.

There exists a trivial solution to the basic equation (11) of oscillations besides the spheroidal modes given by equation (14). In this solution, the eigenfrequency σ and all scalar variables are zero, but the displacement vector is non-zero and given by

$$\boldsymbol{\xi} = -\frac{1}{\sqrt{l(l+1)}}\left(-\boldsymbol{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \boldsymbol{e}_\phi \frac{\partial}{\partial\theta}\right) Y_l^m(\theta,\phi). \tag{15}$$

These toroidal modes are steady eddy motions and not of much interest in the case of a non-rotating, non-magnetic spherical star, but they become oscillatory modes in the case of a rotating and/or magnetic star.

It is known that the set of eigenfunctions which consist of the spheroidal modes and the toroidal modes is complete (Eisenfeld 1969), and hence any oscillation can be expressed in terms of a series expansion of this set.

Taking the scalar product of equation (11) with $\rho\boldsymbol{\xi}^*$ and integrating it throughout the stellar volume, we obtain an expression of the eigenvalue in terms of the eigenfunction:

$$\sigma^2 = \int_0^M \boldsymbol{\xi}^* \cdot \mathcal{L}(\boldsymbol{\xi}) \mathrm{d}M_r \tag{16}$$

From this expression, it can be shown that the eigenfrequencies obey a variational principle (Chandrasekhar 1964):

$$\delta\sigma^2 = \int_0^M \boldsymbol{\xi}^* \cdot \delta\mathcal{L}(\boldsymbol{\xi}) \mathrm{d}M_r \tag{17}$$

The variational principle indicates that eigenfrequencies are determined more accurately than the eigenfunctions, and this fact was utilized to improve eigenfrequencies before the modern computer age (Rayleigh–Ritz method). The variational principle recently came to light again as a basis of the inverse problem of the oscillations, in which the stellar structure is determined from the eigenfrequencies. In this case, $\delta\sigma^2$ in the left-hand-side of equation (17) is regarded as the difference between the observed frequency and the theoretical frequency of the same mode of a model of the star, and is a known quantity. On the other hand, the right-hand-side is dependent on the difference between the equilibrium quantities of the true star and those of the model, which is an unknown to be solved. From equation (17) for many observed modes in the case of the sun, we have well determined the sound speed profile and the density profile in the solar interior, from which we deduce other physical parameters in the solar interior.

2.3 *Linear adiabatic radial pulsation*

In the case of $l = 0$ (radial pulsation), the displacement vector has only the radial component

$$\boldsymbol{\xi} = \xi(r)\boldsymbol{e}_r \exp(i\sigma t), \tag{18}$$

and equation (11) is reduced to a Sturm-Liouville type equation:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Gamma_1 p r^4 \frac{\mathrm{d}\zeta}{\mathrm{d}r}\right) + r^3 \frac{\mathrm{d}}{\mathrm{d}r}\left[(3\Gamma_1 - 4)p\right]\zeta = \sigma^2 \rho r^4 \zeta, \tag{19}$$

where $\zeta \equiv \xi/r$. Hence, the eigenfrequencies are lower bounded:

$$\sigma_1^2 < \sigma_2^2 < \sigma_3^2 < \cdots . \tag{20}$$

The restoring force is the pressure force, and then the pulsation modes are acoustic waves.

2.4 *Linear adiabatic nonradial oscillations*

For non-radial pulsations ($l \neq 0$), a star pulsates in such a way as to deviate from a spherical shape. Two different kinds of restoring forces (the pressure force and the buoyancy force) operate, and there exist, therefore, two different kinds of modes: acoustic modes (p-modes) and gravity modes (g-modes). To discuss general properties of non-radial oscillations the Cowling (1941) approximation, in which the Eulerian perturbation to the gravitational potential is neglected ($\psi' = 0$), is appropriate to use, since it has a sufficient degree of accuracy and simplifies the treatment greatly. The accuracy of this approximation is quite good for modes with large values of n and l. Using the Cowling approximation allows us to ignore the last term in the middle of equation (11). Note that this equation is not of the Sturm–Liouville type.

It is instructive to consider the wave propagation in a plane isothermal atmosphere under a constant gravitational field g, in which the pressure p and density ρ decrease with height z as $\propto \exp(-z/H)$. The perturbation is described by

$$\boldsymbol{\xi} \propto \exp\left(\frac{z}{2H}\right) \exp[i(\sigma t + k_h x + k_z z)], \tag{21}$$

where x stands for the horizontal coordinates. The exponentially growing factor with height in equation (21) arises so as to conserve wave energy in the vertical direction since the density ρ in atmosphere decreases with height. The angular frequency σ and the horizontal and vertical wavenumbers k_h and k_z must then satisfy a dispersion relation which is given by

$$\sigma^4 - \sigma^2[(k_h^2 + k_z^2)c^2 + \sigma_{\mathrm{ac}}^2] + k_h^2 c^2 N^2 = 0, \tag{22}$$

where $N \equiv [g\,\mathrm{d}\ln(p^{1/\Gamma_1}/\rho)/\mathrm{d}z]^{1/2}$ is the Brunt–Väisälä frequency and $\sigma_{\mathrm{ac}} \equiv c/(2H)$ is the acoustic cut-off frequency. If σ and k_h are given, this dispersion relation determines k_z^2. If $k_z^2 > 0$, waves can propagate vertically. On the other hand, if $k_z^2 < 0$, the energy density perturbation decreases exponentially with height if no wave flux is coming from above. The situation can be seen if we plot the relation (22) in the (k_h, σ)-diagram, which is divided into three regions (see Fig. 1): (i) $k_z^2 > 0$, where acoustic waves can propagate (region 'P' of Fig. 1), (ii) $k_z^2 > 0$, where gravity waves can propagate (region 'G' of Fig. 1), and (iii) $k_z^2 < 0$, where waves are evanescent.

Actual stars are by no means isothermal or plane under constant gravity. However, the diagnostic diagram is still helpful in approximately representing the wave nature

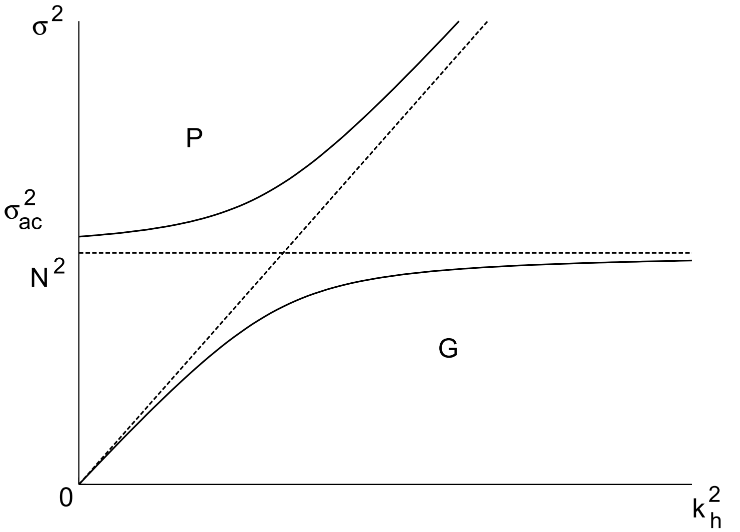

Figure 1. Diagnostic diagram for the plane isothermal atmosphere. The quantity k_z^2 is positive in the region 'P' and in the region 'G', while it is negative in the other regions.

of non-radial pulsations of stars. Let us substitute the form of solution (14) into the wave equation (11) under the Cowling approximation, and write the wave equation (11) symbolically

$$\frac{\mathrm{d}}{\mathrm{d}r}\begin{pmatrix}\xi_r\\ \xi_h\end{pmatrix} = \begin{pmatrix}\mathcal{A}\,\mathcal{B}\\ \mathcal{C}\;\mathcal{D}\end{pmatrix}\begin{pmatrix}\xi_r\\ \xi_h\end{pmatrix}. \tag{23}$$

With the transformation of variables ξ_r and ξ_h to new variables u and v defined by $u \equiv \mathcal{B}^{-1/2}\xi_r$ and $v \equiv \mathcal{C}^{-1/2}\xi_h$, the wave equation results in

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{c^2\sigma^2}[\sigma^2 - \Phi_p(r)][\sigma^2 - \Phi_g(r)]u = 0, \tag{24}$$

where

$$\Phi_p \simeq \frac{l(l+1)c^2}{r^2} + \sigma_{\mathrm{ac}}^2 \tag{25}$$

and

$$\Phi_g \simeq N^2 \tag{26}$$

are functions of r. If we assume $u \propto \exp(\int k_r \mathrm{d}r)$, the wave propagates radially as an acoustic wave if $\Phi_g, \Phi_p < \sigma^2$, and as a gravity wave if $\sigma^2 < \Phi_g, \Phi_p$. The wave is evanescent if $\Phi_g < \sigma^2 < \Phi_p$ or $\Phi_p < \sigma^2 < \Phi_g$. Note that equation (24) tends to a Sturm–Liouville type as $\sigma^2 \to \infty$ or $\sigma^2 \to 0$. The angular eigenfrequency σ and the radial dependence of the eigenfunction $u(r)$ and $v(r)$ are described by the radial order n and the degree l of the mode, but are independent of the azimuthal order m for the case of a spherical star in its unperturbed state. For a given l, the azimuthal

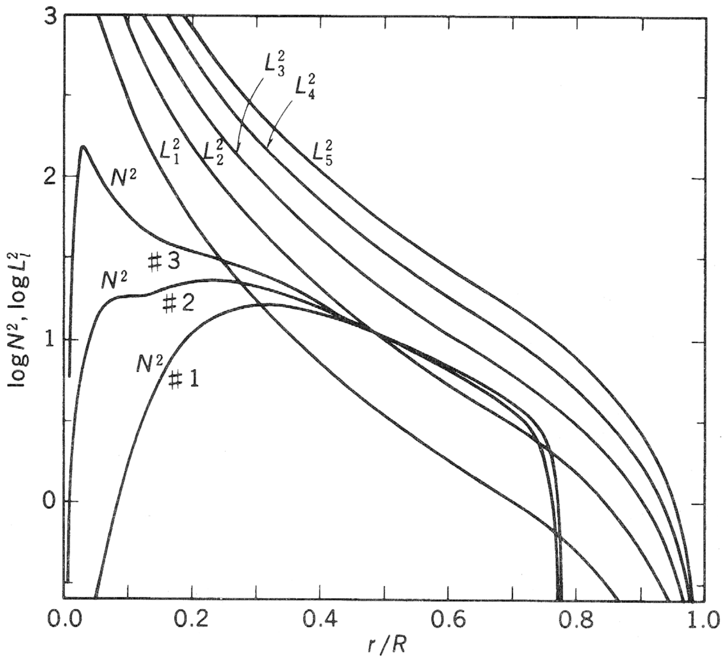

Figure 2. Propagation diagram for models of a $1\,\mathrm{M}_\odot$ star at the ZAMS stage (1) and at the advanced evolutionary stages (2 and 3). Model 2 has an internal structure close to that of the present sun. The quantities N^2 and $L_l^2 \equiv l(l+1)c^2/r^2$ are measured in units of GM/R^3. The L_l^2-curves are of model 2, but they do not differ much for different models.

order m takes integer values from $-l$ to l, and thus the nonradial modes are $(2l+1)$-fold degenerate. This degeneracy is lifted if rotation or the magnetic field of the star is taken into account.

The wave propagation feature is easily seen in the propagation diagram (Fig. 2), on which $\Phi_p(r)$ and $\Phi_g(r)$ are plotted as functions of r. Let us consider the propagation diagrams for the ZAMS model and later core-hydrogen-burning phase models of a star. Note that the squared Brunt-Väisäla frequency is negative in the convective zone and the peak frequency increases with evolution in the μ-gradient zone located just outside the nuclear reacting core, since N^2 includes a term of the gradient of the mean molecular weight $\nabla_\mu \equiv \mathrm{d}\ln\mu/\mathrm{d}\ln p$, which increases with evolution:

$$N^2 = g^2\frac{\rho}{p}\left[v_T(\nabla_{\mathrm{ad}} - \nabla) + \left(\frac{\partial \ln\rho}{\partial \ln\mu}\right)_{p,T}\nabla_\mu\right], \tag{27}$$

where $g \equiv GM_r/r^2$, $\nabla_{\mathrm{ad}} \equiv (\partial \ln T/\partial \ln p)_S$ and $\nabla \equiv \mathrm{d}\ln T/\mathrm{d}\ln p$. The μ-gradient zone acts like a potential well that traps gravity waves. In the case of a star of $M \geq 2\,\mathrm{M}_\odot$, with the increase of N^2 in the μ-gradient zone, there appears an evanescent zone, for modes in the frequency range between the two maxima of the N^2-curve, and these modes possess the dual character such that they behave like gravity waves in the μ-gradient zone, but like acoustic waves in the outer part of the star. The evolutionary development of the μ-gradient zone causes the star to have a structure similar to two potential wells separated by an evanescent zone. One of them is a potential well for a gravity-wave oscillation trapped in the interior and the other is a potential well for an acoustic-wave oscillation trapped in the envelope. The dichotomy is particularly distinguishable in the case of high degree l, but it is obscure for low degree modes such as $l = 1$ or 2. Hence, though there is no chance to detect the high degree modes well trapped in the μ-gradient zone, it may be possible to detect the low degree modes oscillating even in the μ-gradient zone. Detection of such modes will provide us with information about the evolutionary stage of the star and the structure of the deep interior of the star.

When a pair of the frequencies, one for the mode trapped in the interior and the other for the mode trapped in the envelope, approach each other, these modes interact with each other, and they exchange the oscillation properties without equalizing their frequencies. If we plot the evolutionary change of these frequencies as a function of evolutionary time, these frequencies approach once each other and then separate without crossing. This phenomenon is called "avoided crossing" in the community of stellar pulsation, while a similar phenomenon is called "nearly kissing" in the community of geophysics.

After the central hydrogen is exhausted and the core becomes almost isothermal, the squared Brunt–Väisälä frequency still becomes higher near the center, while farther out it becomes negative since the envelope becomes convectively unstable. As the star evolves further into the white dwarf stage, the core becomes degenerate. The Brunt–Väisälä frequency becomes zero in the completely degenerate and chemically homogeneous core, and N^2 is positive only near the stellar surface in the case of white dwarfs. In this case, the gravity modes are the oscillations trapped near the surface while the acoustic waves can penetrate into the deep interior, contrary to the case of non-degenerate stars. As the white dwarf cools and the luminosity becomes lower, the frequencies of gravity modes become lower.

3. Excitation mechanisms of stellar pulsations: Thermal overstability

As discussed in the previous section, adiabatic oscillations in a non-rotating and non-magnetic spherical star are strictly periodic. Oscillations in nature are, however, inevitably nonadiabatic. This corresponds to the fact that the eigenfrequencies and eigenfunctions are no longer purely real in the mathematical description of linear nonadiabatic oscillations. In order for a particular mode to be excited, some excitation mechanism must be present in a region where the amplitudes of eigenfunctions are large. The pulsational stability is a problem of the exchange from one type of energy to another in a system. Here we consider the linear stability problem to investigate the restoring forces and excitation mechanisms of oscillation. For more complete reviews on this subject, the text book by Unno *et al.* (1989) should be consulted.

3.1 *Energy equation and the work integral*

We arrange the basic equations (3), (4), and (6) in such a way as to give the partial time derivatives $\partial \boldsymbol{v}'/\partial t$, $\partial p'/\partial t$, and $\partial(p' - c^2\rho')/\partial t$; multiply these equations by $\rho\boldsymbol{v}'$, $p'/(\Gamma_1 p)$, and $g^2(p' - c^2\rho')/(\rho c^4 N^2)$, respectively; and add them. After somewhat lengthy manipulations (Unno *et al.* 1989), the result turns out to be

$$\frac{\partial}{\partial t}(\rho e_W) + \nabla \cdot \boldsymbol{F}_W = \rho \delta T \frac{\partial \delta S}{\partial t} - \Phi' \frac{\partial \rho'}{\partial t}, \tag{28}$$

where wave energy per unit mass, e_W, is defined as

$$e_W = \frac{1}{2}\left\{\boldsymbol{v}'^2 + \left(\frac{p'}{\rho c}\right)^2 + \left(\frac{g}{N}\right)^2 \left[\left(\frac{p'}{\Gamma_1 p} - \frac{\rho'}{\rho}\right)^2 - \frac{\nabla}{\nabla_{\rm ad}}\left(v_T \frac{\delta S}{c_p}\right)^2\right]\right\}, \tag{29}$$

and

$$\boldsymbol{F}_W = p'\boldsymbol{v}' + \rho \boldsymbol{v}'\Phi'. \tag{30}$$

We can replace $\partial/\partial t$ by the Stokes derivative $\mathrm{d}/\mathrm{d}t$ in equation (28) because the difference appears only in higher order terms. The second term in the right-hand side of equation (28) can be incorporated into the global wave energy E_W by integrating equation (28) over the whole volume of the star; *i.e.*,

$$\frac{\mathrm{d}E_W}{\mathrm{d}t} = \int_0^M \delta T \frac{\mathrm{d}\delta S}{\mathrm{d}t} \mathrm{d}M_r - \int_S \boldsymbol{F}_W \cdot \mathrm{d}\boldsymbol{S}, \tag{31}$$

where

$$E_W = \int_0^M \left(e_W + \frac{1}{2}\frac{\rho'}{\rho}\Phi'\right) \mathrm{d}M_r. \tag{32}$$

This equation states that change in the wave energy of a star is caused by non-adiabatic processes in the interior and by the outgoing wave flux $\boldsymbol{F}_W$ at the surface. The first term in e_W is the kinetic energy and the other terms represent the potential energies corresponding to the various restoring forces. If there is a negative potential energy, monotonically unstable modes could exist, because the kinetic energy can increase without changing the total energy. For example, a dynamically unstable convective mode arises if $N^2 < 0$ somewhere in the stellar interior, due to the exchange of energy between the first and the third term in the right-hand side of equation (29). In the case of an acoustic wave, the second term in e_W is dominant, and the wave energy averaged over one period is equi-partitioned into the kinetic energy $(1/2)\boldsymbol{v}'^2$ and the acoustic potential energy $(1/2)(p'/\rho c)^2$. When the third term is dominant, the internal gravity wave ($N^2 > 0$) or the convective mode ($N^2 < 0$) appears. The term proportional to $(\delta S)^2$ is related to nonadiabatic effect. Although this term cannot be discussed independently of the right-hand side of equation (28), equation (29) suggests that if this term is so large as to make the square bracket negative, the convective mode ($N^2 < 0$) could become oscillatory.

Integrating equation (31) over one period of oscillation, we obtain

$$\begin{aligned} W \equiv \oint \frac{\mathrm{d}E_W}{\mathrm{d}t}\mathrm{d}t &= \oint \mathrm{d}t \left[\int_0^M \delta T \frac{\mathrm{d}\delta S}{\mathrm{d}t} \mathrm{d}M_r - \int_S \boldsymbol{F}_W \cdot \mathrm{d}\boldsymbol{S}\right] \\ &= \oint \mathrm{d}t \left[\int_0^M \frac{\delta T}{T}\delta\left(\varepsilon - \frac{1}{\rho}\nabla \cdot \boldsymbol{F}\right) \mathrm{d}M_r\right] \\ &\quad - \oint \mathrm{d}t \left[\int_{r=R} (p' + \rho\Phi')\boldsymbol{v}' \cdot \mathrm{d}\boldsymbol{S}\right], \end{aligned} \tag{33}$$

where $\oint$ indicates the integration over one period of oscillation. This quantity is defined as the work integral W and it means the increase of the global wave energy over one period of oscillation. In the case of the zero-boundary condition, that is, if the pressure goes to zero at the surface, the work integral W turns to be the first term in the second line of equation (33) (Eddington 1926). If W is positive, the star is vibrationally unstable. The energy increase is provided from photon energy, which is originally produced by nuclear reactions in the core.

Equation (33) indicates that if a finite value can artificially be given to the surface pressure, a strictly periodic oscillation can be produced by equating the first term in the right-hand side to the second term. This corresponds to a strictly periodic imaginary oscillation, in which all the excitation effect is consumed by the work done on the artificially placed matter above the stellar surface. The work integral W can also be defined as the amount of energy which must be removed from the star in order for the star to oscillate strictly periodically.

The work integral W is related to the growth rate $\sigma_{\Im}$ of amplitude of oscillation, which is the imaginary part of σ ($\sigma = \sigma_{\Re} + i\sigma_{\Im}$):

$$\sigma_{\Im} = -\frac{1}{2}\frac{W/E_W}{\Pi}, \tag{34}$$

where Π denotes the period. The factor 1/2 comes from the fact that the energy is proportional to the square of amplitude. We consider that $|\sigma_{\Im}/\sigma_{\Re}| \ll 1$, since the ratio is of the order of magnitude of the dynamical to thermal time-scale ratio. Then, $\Pi = 2\pi\sigma_{\Re}^{-1}$. The total energy of oscillation E_W is twice the time average of the kinetic energy since there is equi-partition of the kinetic and potential energies of oscillation in the time average:

$$E_W = \frac{\sigma_{\Re}^2}{2}\int_0^M |\boldsymbol{\xi}|^2 \mathrm{d}M_r. \tag{35}$$

Then, we have

$$\sigma_{\Im} = -\frac{1}{2\pi\sigma_{\Re}} W \left[\int_0^M |\boldsymbol{\xi}|^2 \mathrm{d}M_r\right]^{-1}. \tag{36}$$

3.2 *κ-mechanism*

In quasi-adiabatic analysis the work integral W is estimated by using adiabatic eigenfunctions as well as adiabatic relations. In this case we should evaluate the work integral W by using the expression given in equation (33). For the sake of simplicity, in what follows we neglect convection and the surface energy loss. Let us decompose W into W_N and W_F, which are related to the perturbations of nuclear energy generation rate and radiative flux, respectively:

$$W = W_N + W_F \tag{37}$$

with

$$W_N = \frac{\pi}{\sigma_{\Re}}\int_0^M \frac{\delta T^*}{T}\delta\varepsilon \mathrm{d}M_r \tag{38}$$

and

$$W_F = \frac{\pi}{\sigma_{\Re}} \int_0^M \frac{\delta T^*}{T} \left[-\frac{\mathrm{d}\delta L_r}{\mathrm{d}M_r} + \frac{l(l+1)}{\mathrm{d}\ln T/\mathrm{d}\ln r} \frac{F}{\rho r} \frac{\delta T}{T} + l(l+1) \left(\frac{\xi_h}{r} \frac{\mathrm{d}L_r}{\mathrm{d}M_r} - \frac{\xi_r}{r} \frac{F}{\rho r} \right) \right] \mathrm{d}M_r, \tag{39}$$

where L_r and $F \equiv L_r/(4\pi r^2)$ denote the radiative luminosity at the radius r and the radiative flux, respectively. Here we neglected the surface integral $\int \boldsymbol{F}_W \cdot \mathrm{d}\boldsymbol{S}$ representing the energy loss by the outgoing wave which vanishes for the trapped oscillation.

The contribution of the first term in the square bracket in the right-hand-side of equation (39) is positive if a positive value of δT is associated with a decrease of δL_r towards the surface. This means that instability may occur if matter gains energy at the compression phase and loses energy during expansion. It is instructive to modify further the expression for W_F to a form in which all the terms are proportional to $(\delta T/T)^2$ or $[\mathrm{d}(\delta T/T)/\mathrm{d}r]^2$. After some manipulations (Unno *et al.* 1989), we obtain an approximate expression for W_F:

$$\begin{aligned} \frac{\sigma_{\Re}}{\pi} W_F \simeq & -\frac{1}{2} \left[\alpha_1 L_r \left(\frac{\delta T}{T} \right)^2 \right]_{r=R} - \frac{1}{2} \int_0^R \mathrm{d}r \left(\frac{\delta T}{T} \right)^2 \frac{\mathrm{d}}{\mathrm{d}r} (\alpha_1 L_r) \\ & + \int_0^R \mathrm{d}r \, L_r \frac{H_p}{\nabla_{\mathrm{ad}}} \left(\frac{\nabla - \nabla_{\mathrm{ad}}}{\nabla} - \frac{c_1\omega^2 - U}{\alpha_0} \right) \\ & \times \left\{ \left[\frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\delta T}{T} \right) \right]^2 + \frac{l(l+1)}{r^2} \left(\frac{\delta T}{T} \right)^2 \right\} \\ & + l(l+1) \int_0^R \mathrm{d}r \left(\frac{\delta T}{T} \right)^2 \frac{1}{\nabla_{\mathrm{ad}}} \left[\frac{4 - V}{\alpha_0 V} \left(\frac{1}{c_1\omega^2} \frac{\mathrm{d}L_r}{\mathrm{d}r} - \frac{L_r}{r} \right) \right. \\ & \left. + \frac{c_1\omega^2 - U}{V\alpha_0 c_1 \omega^2} \frac{\mathrm{d}L_r}{\mathrm{d}r} + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{c_1\omega^2 L_r + \mathrm{d}L_r/\mathrm{d}\ln r}{c_1\omega^2 \alpha_0 V} \right) \right]. \end{aligned} \tag{40}$$

Here α_0 and α_1 are defined as

$$\alpha_0 = 4 - U - \frac{l(l+1)}{c_1\omega^2} + c_1\omega^2 \tag{41}$$

and

$$\alpha_1 \equiv 4 - \frac{1}{\nabla_{\mathrm{ad}}} - \kappa_T - \frac{\kappa_\rho}{\Gamma_3 - 1} + \left(1 - \frac{l(l+1)c^2}{\Gamma_1 \sigma_R^2 r^2} \right) \left(\frac{c_1\omega^2 - U}{\alpha_0 \nabla_{\mathrm{ad}}} \right), \tag{42}$$

respectively, where

$$c_1 \equiv \left(\frac{r}{R}\right)^3 \Big/ \left(\frac{M_r}{M}\right),$$

$$\omega^2 \equiv \frac{\sigma_{\Re}^2}{(GM/R^3)},$$

$$U \equiv \frac{\mathrm{d}\ln M_r}{\mathrm{d}\ln r},$$

$$V \equiv -\frac{\mathrm{d}\ln p}{\mathrm{d}\ln r},$$

$$H_p \equiv -\frac{\mathrm{d}r}{\mathrm{d}\ln p},$$

$$\Gamma_3 - 1 \equiv \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_S,$$

$$\kappa_T \equiv \left(\frac{\partial \ln \kappa}{\partial \ln T}\right)_\rho,$$

$$\kappa_\rho \equiv \left(\frac{\partial \ln \kappa}{\partial \ln \rho}\right)_T,$$

and where κ denotes the opacity.

The first and the second terms in the right-hand side of equation (40) describe the κ*-mechanism* (Baker and Kippenhahn 1962). In the outer envelope in the radiative equilibrium, L_r is constant and the κ-mechanism works for driving an oscillation if

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\kappa_T + \frac{\kappa_\rho}{\Gamma_3 - 1}\right) > 0. \tag{43}$$

If a region in the stellar envelope satisfies this condition, radiative flux from the stellar interior is blocked by the effect of the temperature and density dependence of opacity. The blocked energy is converted to the energy of the oscillation. The value of κ_T increases in the inner part of an ionization zone and decreases in the outer part. Therefore, the excitation and damping zones due to the κ-mechanism are located in, respectively, the inner and outer parts of the ionization zone. In an ionization zone an adiabatic exponent $(\Gamma_3 - 1)$ (>0) is minimum. This spatial variation of $(\Gamma_3 - 1)$ enhances the effect of the κ*-mechanism*. This effect is sometimes called the γ*-mechanism* (e.g., Cox *et al.* 1966).

It should be noted here that we would expect that the strong damping of the outer portions of an ionization zone would largely cancel the strong driving of the inner portions. However, because of non-adiabatic effects, the non-adiabatic eigenfunctions can be quite different from the adiabatic ones in layers sufficiently far out in a star. It is this lack of equality between these two quantities that leads to the overall effect of the ionization zone on the stability of the star. The strongly non-adiabatic exterior is separated from the quasi-adiabatic interior by a transition region of which location in the stellar envelope is determined by

$$\tau_{\mathrm{th}}/\tau_{\mathrm{dyn}} \sim 1, \tag{44}$$

where τ_{th} and τ_{dyn} are defined in section 2.2. Since the decrease of τ_{th} with r is very rapid, the location of the transition region depends on the oscillation modes only weakly. Depending on the equilibrium models and the oscillation mode, the transition region can be located in the hydrogen ionization zone or in the helium ionization zone or in the ionization of carbon and oxygen.

It is well known after Zhevakin (1953) that the κ-mechanism in the hydrogen and helium ionization zones is responsible for the Cepheid instability strip of radial pulsations which extends from the classical Cepheids in the giant region to δ Scuti variables near the main sequence. Moreover, the κ-mechanism in the hydrogen ionization zone has been investigated as a possible excitation mechanism for the observed nonradial g-modes of the variable DA white dwarfs, namely DAV stars. Also, overstable g-modes of variable DB white dwarfs (helium envelope; DBV stars) are thought to be excited by the κ-mechanism in the second helium ionization zone. It should be noted, however, that an alternative convective mechanism has been proposed and the excitation mechanism for DAV and DBV stars is still controversial (Brickhill 1991; Goldreich & Wu 1999). The κ-mechanism in the hydrogen ionization zone in the atmosphere of Ap stars has been extensively investigated by Balmforth *et al.* (2001) and Cunha (2002) as a possible excitation mechanism for rapid oscillations in roAp stars [see also Saio (2005)].

The new opacity data which were recalculated for the OPAL project (Rogers & Iglesias 1992; Iglesias & Rogers 1993, 1996) and for the OP project (Seaton *et al.* 1994) show a significant enhancement over the old Los Alamos opacity library in the temperature range of $T \sim 100{,}000$–$300{,}000$ K. This feature called the "Z-bump" is caused by intra-M-shell transitions mainly in Fe and by fine-structure transitions in Mg, Cr, and Ni. Based on the new opacity data, one could eventually, after many years of futile attempts, identify driving mechanisms for pulsations in early-type stars, that is, β Cephei stars and slowly pulsating B-type (SPB) stars (Kiriakidis *et al.* 1992; Moskalik & Dziembowski 1992; Gautschy & Saio 1993; Dziembowski *et al.* 1993). It was also predicted that the κ-mechanism associated with a local enrichment of iron caused by diffusion in hot subdwarf B (sdB) stars works to drive pulsations in these stars. Independent discovery of real sdB pulsators was sensational (see review, Charpinet *et al.* 2001).

The κ-mechanism associated with the ionization of carbon and oxygen is also possible. In the case of very hot pre-white-dwarf stars, including planetary nebula nuclei, of which hydrogen-rich matter has been lost, carbon/oxygen matter produced by helium burning could be located in the temperature region of $T \sim 10^6$K. In such a case the κ-mechanism associated with the ionization of the K-shell electrons of carbon/oxygen can excite global oscillations. DOV stars showing nonradial pulsations are interpreted as such stars.

3.3 *δ-mechanism*

The second line of equation (40) arises from the radiative diffusions of the thermal energy of oscillating gaseous elements. For high order g-modes $(c_1\omega^2 - U)/\alpha_0$ is very small. This indicates a superadiabatic region ($\nabla > \nabla_{\rm ad}$) excites high order g-modes. This excitation effect is called the δ-*mechanism* in the text book by Unno *et al.* (1989). This excitation effect was first demonstrated by Cowling (1957), who showed the overstability of g-modes for the superadiabatic stratification of plasma stabilized by the existence of horizontal magnetic fields. If the superadiabatic region is stabilized by a spatial gradient of the mean molecular weight, this mechanism corresponds to Kato's (1966) mechanism. On the other hand, this mechanism does not exist for p-modes or radial pulsations, because in such cases $[(\nabla - \nabla_{\rm ad})\nabla^{-1} - (c_1\omega^2 - U)\alpha_0^{-1}]$ reduces to $\sim -\nabla_{\rm ad}/\nabla$. This term causes radiative damping for p-modes and radial pulsations irrespective of the sign of $(\nabla - \nabla_{\rm ad})$.

A better understanding of this mechanism may be obtained by a local analysis. To make the analysis simple we use the Boussinesq approximation, in which density is treated as a constant in the equation of continuity and in the equation of motion except the buoyancy term. Let us suppose a plane-parallel, gravitationally stratified layer of fluid in hydrostatic and radiative equilibrium. We consider the higher overtone of gravity waves whose wavelengths in the vertical direction are much shorter than the density and pressure scale heights. This implies that the density variations are very small, and this is the situation appropriate for the Boussinesq approximation, in which the density variations are taken into account only in the term of the buoyancy, and the pressure perturbation p' is neglected in the energy equation. If the temporal and spatial dependence of the perturbation of an arbitrary quantity f is taken as

$$f' \propto \exp[i(k_h x + k_z z) + st], \tag{45}$$

where the gravity is the z-direction, then, the basic equations of oscillations lead to the dispersion relation

$$s^3 + \frac{Kk^2}{c_p\rho}s^2 + \frac{k_h^2}{k^2}N^2 s + \frac{k_h^2}{k^2}\frac{g}{H_p}\nabla_\mu\frac{Kk^2}{c_p\rho} = 0, \tag{46}$$

where $k^2 \equiv k_h^2 + k_z^2$, $K \equiv (4acT^3)/(3\kappa\rho)$ is the radiative conductivity, and g denotes the gravitational acceleration. The necessary and sufficient conditions for stability is $\Re(s) < 0$. Disregarding a root for the secular mode, we find two roots in the first approximation of small non-adiabaticity

$$s_{1,2} \simeq \frac{Kk^2}{2c_p p}\upsilon_T g^2\frac{\nabla - \nabla_{\rm ad}}{N^2} \pm i\frac{k_h}{k}N, \tag{47}$$

where N means the Brunt–Väisälä frequency given by equation (27). The dynamical instability occurs if $N^2 < 0$, that is, if

$$\upsilon_T(\nabla - \nabla_{\rm ad}) - \nabla_\mu > 0, \tag{48}$$

which is often called the Ledoux (1947) criterion for the convective instability. In the dynamically stable case, the medium is stable against the ordinary convection since $N^2 > 0$, but perturbation grows oscillatory with increasing amplitude if

$$\nabla - \nabla_{\rm ad} > 0 \tag{49}$$

because the real part of s in equation (47) is positive. The physical cause of this overstability is that the radiative heat exchange brings about an asymmetry in the oscillatory motion in such a way that an oscillatory element overshoots its equilibrium position with an increasing velocity.

The favorable condition for this kind of overstability is realized in the μ-gradient zone around the convective core in massive stars, and it has been confirmed by a global stability analysis that some g-modes trapped in the μ-gradient zone are in fact overstable due to this mechanism (Shibahashi and Osaki 1976). In the presence of the uniform magnetic field, the condition for the dynamical stability is modified so that $k_h^2 N^2/k^2 + (\boldsymbol{B} \cdot \boldsymbol{k})^2/(4\pi\rho) > 0$, but the overstability condition (49) is unchanged. Shibahashi (1983) proposed this kind of overstability due to the superadiabatic layer in the magnetic stars' envelope as a possible excitation mechanism of roAp stars.

3.4 *DB gap and pulsation of white dwarfs*

In terms of surface composition, white dwarfs come in two major subgroups: those with essentially pure hydrogen atmospheres (DAs), which constitute about 80% of all white dwarfs, and those with essentially pure helium atmospheres (DBs). The ratio of DA to DB white dwarfs is temperature sensitive. There seem no DBs between 45,000 K and 30,000 K, and this exclusion zone is known as the "DB gap" (Fontaine and Wesemael 1991). Since the temperatures of the upper and the lower bounds of the DB gap coincide with the effective temperatures where the He II/III and He I/II convection zones show up, respectively, convective mixing is suspected of the cause of appearance of DBs outside the DB gap. Inversely, chemical separation due to gravitational settling in the convectively stable atmosphere is suspected of the cause of the presence of the DB gap. If one adopts this scenario, the DBs near the boundaries of the DB gap are expected to have a super-adiabatic layer which is nonetheless convectively stable due to the μ-gradient caused by chemical separation. Such DBs may be pulsationally unstable and then a new type of DB variables is expected (Shibahashi, in preparation).

4. Stochastic excitation of solar-like oscillations

Mode stability is governed not only by the perturbations in the radiative fluxes but also by the perturbations in the turbulent convective fluxes, which have been completely ignored in the previous section for the sake of simplicity. This is particularly true for the sun and solar-like stars, in which energy is mainly transported by convection in their outer envelopes. The principle source of excitations of the observed oscillations in the sun was controversial for many years. If solar p-modes were thermally overstable, some nonlinear mechanism must limit their amplitudes to the low values that are observed. But no such mechanism has been found. Alternatively, it is now widely believed that the observed low-amplitude oscillations in the sun are determined by the local balance between the stochastic driving due to the acoustic radiation by turbulent convection and the damping of the intrinsically stable p-modes. According to this view, the turbulent convection generates acoustic noise, and acoustic noise in the sun's resonant cavity results in the excitation of the cavity's normal modes. The kinetic energy of modes is stochastically supplied by the turbulence, and the radiation works to damp the modes. This view is extended to solar-like stars in general, and then detection of solar-like oscillations in late-type stars are vigorously attempted. In fact, detection of solar-like oscillations in these stars has been accomplished in the last few years, after decades of disappointing attempts (see Kurtz 2004). Probing the invisible interior of stars is a dream come true. In this section, following mainly Osaki (1990), I first outline analytically the stochastic excitation mechanism. For more details, papers such as those by Goldreich and Keeley (1977); by Osaki (1990); by Balmforth (1992); by Goldreich *et al.* (1994); by Houdek *et al.* (1999); by Samadi and Goupil (2001); and by Chaplin *et al.* (2005) should be consulted. Then I briefly discuss the recent results based on numerical simulations.

4.1 *Damping rate of oscillations and acoustic noise generation*

Let us write the energy of a particular eigenmode, its angular frequency, and its energy dissipation rate as E_q, σ_q, and Γ_q, where the subscript, q, signifies a particular eigenmode. The energy of the mode is then given by

$$E_q = 2 \times \left\langle \frac{1}{2} \int_0^M \boldsymbol{v}_q^2 \mathrm{d}M_r \right\rangle = \left\langle \int_0^M \boldsymbol{v}_q^2 \mathrm{d}M_r \right\rangle, \tag{50}$$

where $\boldsymbol{v}_q$ denotes the velocity amplitude of the oscillation mode and the integration is performed over the whole mass of the star. Here the angular bracket means time average and we note that the energy of oscillation is the sum of the kinetic energy plus the potential energy. The equation of energy of the mode is then given by

$$\frac{\mathrm{d}E_q}{\mathrm{d}t} = G - \Gamma E_q, \tag{51}$$

where G denotes the acoustic power by turbulent convection available to the particular mode in our interest.

The damping rate of oscillation may be divided into two parts:

$$\Gamma = \Gamma_{\mathrm{therm}} + \Gamma_{\mathrm{dyn}}, \tag{52}$$

where Γ_{therm} and Γ_{dyn} stand for damping rates of oscillation that are caused through the thermal equation and the momentum equation, respectively. Furthermore, each of them may be written by several contributions by different causes. In fact, the thermal damping due to the non-adiabatic effects of oscillation may be divided into the damping due to radiative flux variation and that due to convective flux variation, and then Γ_{therm} may be written into two parts:

$$\Gamma_{\mathrm{therm}} = \Gamma_{\mathrm{rad}} + \Gamma_{\mathrm{conv}}, \tag{53}$$

where Γ_{rad} and Γ_{conv} stand for the damping rates of oscillation due to radiative and convective flux variations. The thermal damping rate Γ_{therm} can be negative (*i.e.*, "negative" dissipation) as discussed in the previous section on thermal overstability. It is still hard to estimate theoretically the damping rate of a mode, particularly Γ_{conv}, because we need to treat "time-dependent" convection, for which theory has not yet been well established.

On the other hand, the dynamical damping rate consists of those due to the turbulent viscous dissipation, due to the leakage of waves from boundaries, and due to energy transfer to other modes by nonlinear interaction. That is,

$$\Gamma_{\rm dyn} = \Gamma_{\rm visc} + \Gamma_{\rm leak} + \Gamma_{\rm coupl}, \tag{54}$$

where the turbulent viscous damping and the damping due to the leakage of waves are supposed to be always positive but the term due to the mode coupling may be either positive or negative.

Since we assume in this picture the local balance between the stochastic driving by turbulent convection and the damping of the intrinsically stable p-modes, the energy of oscillation of the mode is then given by

$$E_q = G/\Gamma. \tag{55}$$

In the case of solar oscillation, the energy dissipation rate Γ is observationally determined by measuring the linewidth of the mode in the power spectrum, and the energy of oscillation of the mode E_q is also observationally determined by measuring the amplitude of the mode and by using theoretical eigenfunction $\boldsymbol{\xi}_q$. In order to make progress in asteroseismology, however, we have to deduce rather theoretically the expected amplitude of oscillation by estimating G and Γ.

4.2 *Estimate of acoustic power*

The generation of acoustic noise by turbulence is known as the Lighthill mechanism (Lighthill 1952). Let us consider the acoustic power generated by a turbulent eddy of size λ_T having velocity v_λ, density ρ, and Mach number $M = v_\lambda/c$. According to the Lighthill theory, the total acoustic power generated by a turbulent eddy is given by

$$P \sim \frac{\rho v_\lambda^3}{\lambda_T} M^{2n+1}, \tag{56}$$

where n is the multipole index. Since the turbulence is thought to be subsonic, the lowest order multipole is the most effective. The lowest three multipoles are:

- monopole radiation ($n = 0$), which occurs when a parcel of gas executes periodical expansion and contraction,
- dipole radiation ($n = 1$), which occurs when an external force acts on the gas,
- quadrupole radiation ($n = 2$), which is emitted when turbulent Reynolds stresses act as the source term.

In the original Lighthill mechanism, the quadrupole radiation is the lowest order multipole generated by turbulence. In the following, we estimate the acoustic power based on the quadrupole radiation.

The characteristic time of the eddy (or the turn-over time) may be given by $\tau_\lambda \sim \lambda_T/v_\lambda$, and then the characteristic frequency of the eddy is given by $\sigma_{\rm eddy} \sim v_\lambda/\lambda_T$. The characteristic frequency emitted by the quadrupole radiation is twice as high as this frequency, *i.e.*, $\sigma_Q \sim 2\sigma_{\rm eddy}$ because the product of turbulent velocity components is involved in the Reynolds stress tensor $T_{ij} = \rho v_i v_j$ (Osaki 1990). Since the quadrupole radiation depends strongly on the Mach number, most of the acoustic radiation in the convection zone is generated near the top of the convection zone where the turbulent convective velocity attains its maximum value. We approximate this zone by a layer having a thickness of H at $r = r_s$ and the convective velocity $v_{\max}$, and set $\lambda_T \sim H$ and $v_\lambda \sim v_{\max}$. Hence, $\sigma_Q \sim 2v_{\max}/H$, then the total acoustic power generated is given by

$$P(\sigma) \sim 4\pi R^2 H \epsilon(\sigma), \tag{57}$$

where R is the stellar radius and $\epsilon(\sigma)$ stands for the acoustic emissivity per unit volume and per unit frequency. If we assume the Kolmogoroff spectrum for the turbulent convective eddy, the acoustic emissivity is then given by (see, e.g., Goldreich and Kumar 1988)

$$\epsilon(\sigma) \sim \rho v_{\max}^2 M^5 \begin{cases} (\sigma/\sigma_Q)^2 & \text{for } \sigma \le \sigma_Q \\ (\sigma/\sigma_Q)^{-7/2} & \text{for } \sigma \ge \sigma_Q. \end{cases} \tag{58}$$

The acoustic power per mode at the source is given by

$$G_{\text{source}} \sim P(\sigma) \frac{\Delta k}{k} \Delta\sigma, \tag{59}$$

where $\Delta k/k$ and $\Delta\sigma$ denote the horizontal wave number band-width for a particular non-radial mode and the frequency band-width, respectively. The horizontal wave-number band-width is estimated by assuming isotropic radiation as

$$\frac{\Delta k}{k} \sim \frac{1}{2(kR)^2} = \frac{1}{\sigma^2} \frac{c^2}{2R^2}. \tag{60}$$

Substituting equations (57), (58) and (60) into equation (59), we estimate the acoustic power *at the source*

$$G_{\text{source}} \sim \rho H^3 v_{\max}^2 M^3 \Delta\sigma \begin{cases} (\sigma/\sigma_Q)^0 & \text{for } \sigma \le \sigma_Q \\ (\sigma/\sigma_Q)^{-5.5} & \text{for } \sigma \ge \sigma_Q. \end{cases} \tag{61}$$

It should be noted here that what we should estimate is the acoustic power *transmitted into the stellar acoustic cavity*. If the source layer is located within the cavity, the acoustic power in the cavity is G_{source} itself. On the other hand, if the source layer is located in the evanescent zone outside the acoustic cavity, the power is attenuated when it is transmitted into the cavity. Whether the source layer is inside the cavity or outside the cavity is dependent on the frequency under consideration. As shown in section 2.4, wave propagates as an acoustic wave only if $\sigma > \Phi_p(r)$ and $\sigma > \Phi_g(r)$. This condition is deduced to $\sigma > \sigma_{\text{ac}}(r)$ in the outer part of the star, since $\Phi_p(r) \simeq \sigma_{\text{ac}}(r)$ and $\Phi_g(r) \simeq \sigma_{\text{ac}}(r)$ there. Inversely, the wave is evanescent if $\sigma \le \sigma_{\text{ac}}(r)$. In order to estimate the attenuation effect, let us assume that the $\sigma_{\text{ac}}(r)$ is a monotonically increasing function with r near the surface. Then, there is no attenuation effect for $\sigma > \sigma_{\text{ac}}(r_s)$, while the attenuation factor is $\exp[\int_{r_t}^{r_s} c^{-1}(1-\sigma^2/\sigma_{\text{ac}}^2)^{1/2}\mathrm{d}r]$ for $\sigma < \sigma_{\text{ac}}(r_s)$, where r_t is the radius at which $\sigma_{\text{ac}}(r_t) = \sigma$. Hence, in the case of $\sigma_{\text{ac}}(r_s) < \sigma_Q$, that is, if $v_{\max}/c(r_s) > 1/4$, the acoustic power *transmitted into the acoustic cavity*, G, is estimated as

$$G \sim \rho H^3 v_{\max}^2 M^3 \Delta\sigma \begin{cases} (\sigma/\sigma_Q)^{-5.5} & \text{for } \sigma \geq \sigma_Q \\ 1 & \text{for } \sigma_Q \geq \sigma \geq \sigma_{\rm ac}(r_s) \\ \exp\left[-\int_{r_t}^{r_s} (1-\sigma^2/\sigma_{\rm ac}^2)^{1/2} \dfrac{\mathrm{d}r}{c}\right] & \text{for } \sigma_{\rm ac}(r_s) \geq \sigma. \end{cases} \tag{62}$$

On the other hand, if $\sigma_{\rm ac}(r_s) > \sigma_Q$, that is, if $v_{\max}/c(r_s) < 1/4$, G is estimated as

$$G \sim \rho H^3 v_{\max}^2 M^3 \Delta\sigma \begin{cases} (\sigma/\sigma_Q)^{-5.5} & \text{for } \sigma \geq \sigma_{\rm ac}(r_s) \\ \exp\left[-\int_{r_t}^{r_s} (1-\sigma^2/\sigma_{\rm ac}^2)^{1/2} \dfrac{\mathrm{d}r}{c}\right] & \text{for } \sigma_{\rm ac}(r_s) \geq \sigma. \end{cases} \tag{63}$$

It should be noted here that, for both the cases, for $\sigma > \sigma_{\rm ac}(r_s)$, the wave leaks outward as a running wave since there is no reflection, and then the damping rate $\Gamma_{\rm leak}$ in equation (54) becomes large.

4.3 *Numerical results*

Expressions for the excitation rate have been derived analytically as discussed in the previous subsections. Evaluation of these expressions, however, depends on knowing the properties of convection, which are derived under appropriate approximations. Unfortunately, the theories of convection are still far from being complete. Both the traditional mixing length theory and the recent alternatively developed theories do not adequately describe the dynamics of convection and in addition contain free parameters and so lack predictive capability. In this situation, various attempts of hydrodynamical simulations have been made. Such simulations have succeeded in reproducing fairly well the properties of solar granulation, and then they are highly prospective for investigation of the stochastic excitation mechanism of p-mode oscillations. Using results of these numerical simulations of the near surface layer of a star it becomes possible to calculate the excitation rate of p-modes for the sun and other stars. Recent results are well summarized by Stein *et al.* (2004). The excitation rate of radial oscillations of stars near the main sequence from K to F and a subgiant K IV star has been calculated from numerical simulations of their surface convection zones. According to Stein *et al.* (2004), p-mode excitation increases with increasing effective temperature until envelope convection ceases in the F stars and also increases with decreasing gravity.

4.4 *Comparison with observations*

Detection of solar-like oscillations in late-type stars are vigorously attempted. In fact, detection of solar-like oscillations in those stars has been accomplished in the last few years, after decades of disappointing attempts (see Kurtz 2004). The bright star Procyon (α CMi) has been considered one of the best candidates for asteroseismology, on the basis of models and previous reports of p-modes detected in ground-based spectroscopy. A search for p-modes by nearly continuous photometric satellite-based observations was made by MOST (Matthews *et al.* 2004). However, the peak amplitudes are much smaller than the theoretical expectation and are inconsistent with the ground-based observations of this star. More careful analyses are definitely needed to proceed with asteroseismology of solar-like stars.

Acknowledgements

I was honoured to have the occasion to attend the ARIES workshop which was held to celebrate the golden jubilee of the institution. I would like to express my sincere thanks to the SOC of the workshop for their kind invitation to the meeting and to all the members of ARIES for their warm hospitality during my stay in India.

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