1. The Physics of Nuclear Reaction Rates

1.1. The Four Fundamental Forces in Physics

There are four fundamental types of forces in physics. Two act on large scales, gravity and electro-magnetic forces. Gravity is not relevant to nuclear reactions and we ignore it here. The other two act on the scale of the size of atomic nuclei, and have $F \propto (1/r^2)exp(-r/r_0)$, where $r_0 \sim r_{nuc}$. The characteristic value of the constant S, where $F \sim S/r^2$ in the appropriate distance range is approximately $1/137$ for electro-magnetic reactions, 10^{-21} for weak nuclear interactions, and 10 for strong nuclear interactions.

Typical reaction products for the 3 relevant interactions: e-m forces typically produce photons, reaction is fast, $D(p,\gamma)^3He$. Weak interactions have very small cross sections and occur very slowly, they typically produce ν, for example $H(p,e^+\nu)D$. Strong interactions occur rapidly and produce $p,\ \alpha,\ n,\ ...$, an example being ${}^3He({}^3He,2p)^4He$.

The change in luminosity (energy flux) within the star is

$$\frac{dL_r}{dM_r} = \epsilon(nuc) + \epsilon(grav) - \epsilon(\nu) \ = \ \epsilon_{total}$$

$\epsilon(\nu)$ is negative as neutrinos escape from normal stars without interaction with matter, i.e. without thermalizing; this is not true of very dense compact stars. (Neutrinos will be discussed later, we ignore this term for the moment.) Our goal in this section is to evaluate the terms on the right side of the above equation.

In order to have a static star, the above equation must hold at all times. Otherwise,

$$\frac{\partial L_r}{\partial M_r} - \epsilon_{nuc} = \frac{\partial Q}{\partial t} = \frac{\partial E}{\partial t} + P\frac{\partial(1/\rho)}{\partial t}]$$

Note that $\epsilon(grav) = 0$ for an adiabatic process, by definition.

2. Reaction Mechanics

$A + B \rightarrow C + D$ is actually a two stage process, $A + B \rightarrow Z^* \rightarrow C + D$, where Z is a compound nucleus, and the $*$ denotes an excited state.

Once Z^* is formed, many exit channels are possible, including the entrance channel, $A + B$. As long as they obey all conservation laws (charge, mass/energy, momentum, angular momentum), they can occur, although the probability for the possible output channels varies enormously from channel to channel.

Analogy with cooking - Oven too hot - food burned, too cold - food raw, old yeast - flat,...for a given mixture of ingredients there are many possible outcomes.

Each valid exit channel has a mean time for decay, τ_i, hence a mean energy width Γ_i, such that $\Gamma_i \tau_i = \hbar$, as required by the Heisenberg uncertainty principle. The probability for a particular output channel i is:

$$P_i = \frac{1/\tau_i}{\sum_j 1/\tau_j}$$

and $\Gamma_i = 2\gamma_i^2 P_l$, where γ_i is the reduced width (the probability that Z^* is composed of the charged constitutents of the ith exit channel in a common nuclear potential; the value for this comes out of nuclear physics). P_l is the Coulomb penetration factor, which depends on the angular momentum quantum number l.

Nuclear reactions can alter the chemical composition, the mean weight per particle, and the thermal properties of a gas.

3. Nuclear Reaction Rates

Rate of nuclear reaction $A(B,C)D \equiv A + B \rightarrow C + D$ is a function of the relative velocity of A and B, v_{AB},

$$R_{AB}(v_{AB})(\text{reactions/cm}^3/\text{sec}) = \frac{n_A n_B \sigma_{AB}(v_{AB}) v_{AB}}{1 + \delta_{AB}}$$

where δ_{AB} is 1 if $A + B$ and 0 otherwise; this factor is needed so that identical pairs are not counted twice. v_{AB} is the relative velocity of A with respect to B, σ_{AB} is the cross section, which is a function of v_{AB}.

We must view the reaction in the center of mass frame, and the kinetic energy E is the energy of $A + B$ measured in that frame, the relative velocity v_{AB} is measured in that frame as well, but is the same as in the frame in which one of the 2 nuclei is at rest, and is $2E/m$, where m is the reduced mass, $m = (m_A m_B)/(m_A + m_B)$ (in gm).

The energy generation rate (ergs/sec/cm^3), is $Q_{AB}\ R_{AB}(v_{AB})$, where Q_{AB} is the average energy released/(reaction $A + B$), $Q_{AB} \equiv \Delta m/c^2$, where Δm is the mass deficit of the reaction (excluding neutrinos).

The energy generation rate ϵ (erg/sec/gm) is then $Q_{AB}\ R_{AB}(v_{AB})/\rho$.

$n_A = [X_A/(A_A m_H)]\rho$, where X_A is the fractional abundance by mass of the atom (actually a specific isotope of an atom) and A_A is the atomic weight of atom A.Note that initially X_H is ∼0.70. So

$$\epsilon_n = \frac{\rho X_A X_B v_{AB} \sigma_{AB} Q_{AB}}{m_H^2 A_A A_B (1 + \delta_{AB})}$$

Now we must average over all relative velocities v_{AB} to get $< v_{AB}\sigma_{AB} >$.

3.1. The Coulomb Barrier

The mean thermal energy per particle = $(3/2)kT = 2.1 \times 10^{-16}T$ ergs = 1.5 keV for $T = 10^7$ K. The rest energy mc^2 for 1 proton is 938 MeV, and typical Q for a nuclear reaction $\sim 1 - 10$ MeV, so $kT << Q$, and hence Q_{AB} is approximately independent of v_{AB}. We will assume Q_{AB} is constant.

The distribution of the speed of a particle is given by the Maxwellian velocity distribution, illustrated in the figure. Here μ_{AB} is the reduced mass = $m_A m_B/(m_A + m_B)$.

$$P(v_{AB})dv_{AB} = 4\pi[\frac{\mu_{AB}}{2\pi kT}]^{3/2}\ e^{-(m_{AB}v^2_{AB}(/(2kT)}\ v^2_{AB}\ dv_{AB} \qquad \propto v^2_{AB}\ e^{-m_{AB}v^2_{AB}/(2kT)}.$$

Neutrons have no charge, and there is thus no Coulomb barrier for them. There are no free n in stellar interiors. A free n, if created, reacts rapidly with and becomes part of a nucleus of an atom and is no longer free.

Thus A and B are both positively charged nuclei, and they repel each other.

Nuclear radius $\sim$ a few fermis (1 fermi = 10^{-13} cm). But the work to move 2 like charged particles together is $q_1q_2/r = Z_AZ_Be^2/r$, where Z_A is the charge of atom A (1 if singly ionized, 2 if doubly ionized, etc).

To get r_{min} for a thermal classical particle, we set $r_{min} = Z_AZ_Be^2/kT = 1.44Z_AZ_B/(0.8 \times 10^{-3}T_7)$ fermis = 2000 Z_AZ_B/T_7 fermis, where $T_7 = T/10^7$ K, a common notation we will use often in this document. Thus $r_{min} >> r_{nuc}$ for $T \sim 10^7$ K.

For $r_{min} \approx r_{nuc}$ we need $T \sim 10^{10}$ K, but only during the Big Bang can this be achieved. Inside stars, $r_{min} >> r_{nuc}$. In order to get nuclear reactions to occur, we need a miracle, i.e. quantum mechanical tunneling.

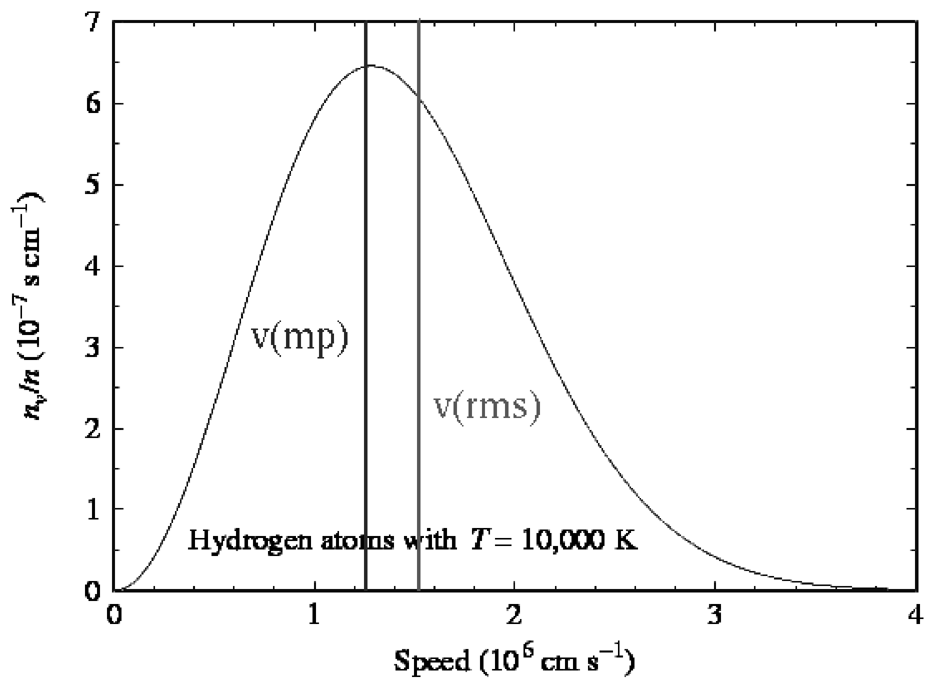

Fig. 1.— The distribution in speed (amplitude of the vector velocity) for H atoms with $T = 10,000$ K. The most probable speed is $\sim$12 km/sec. Source: website of Chris Mihos

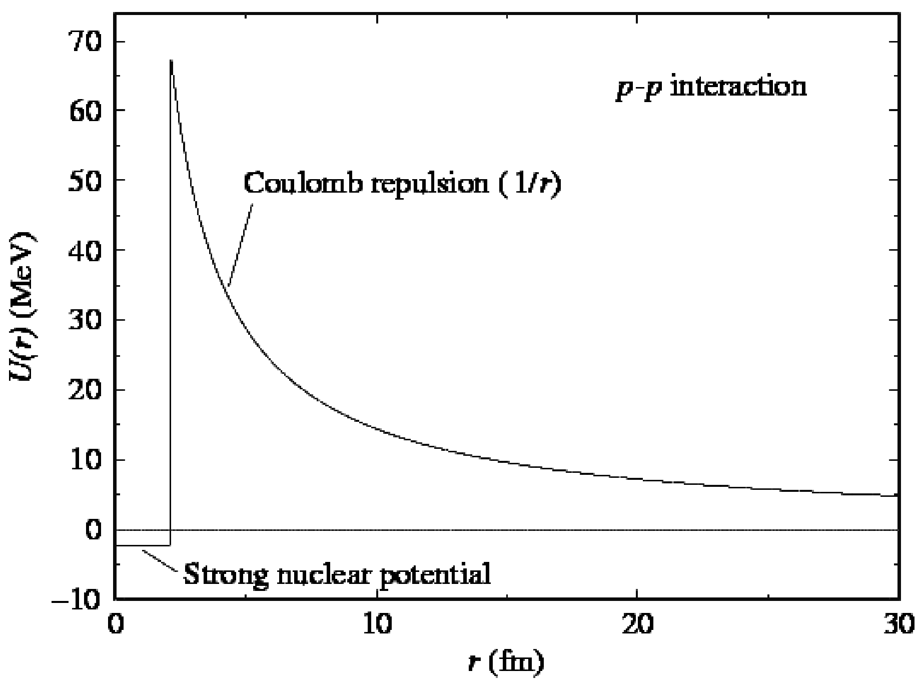

Fig. 2.— A sketch of the potential energy as a function of radius around a (positively) charged atomic nucleus. Source: website of Chris Mihos

3.2. Quantum Mechanical Tunneling

Nuclear reaction cross sections depend on tunneling through the Coulomb barrier. P (probability of reaction) rises as v_{AB} rises, but there are many fewer particles once $v_{AB} >> \sqrt{3kT/(2m_{AB})}$, the mean thermal velocity.

The incoming particle needs to get to the radius within which the nuclear forces take over and the reaction can occur versus the much larger r_{min}, determined from the particle speeds and the Coulomb barrier. This, as discussed above, requires QM tunneling. The relevant QM cross section is:

$$\sigma(E) \ = \ \frac{S(E)}{E} exp[-2\pi Z_A Z_B e^2/(\hbar v_{AB})] \ \propto \ exp[(-E_G/E)^{1/2}]$$

where E_G is the Gamow energy.

The term $S(E)/E$ occurs because $\sigma \sim \pi r^2 f(E)$, and $f(E) \approx$ constant with E. From the Heisenberg uncertainty principle, $\lambda \propto 1/p$ from QM, where p is the linear momentum, so $r^2 \propto 1/E$.

$S(E)$ is a slowly varying function of $E = m_{AB} v_{AB}^2/2$, unless there is a resonance. This is illustrated in the Fig.4-5 of Clayton.

Laboratory measurements of $S(E)$ are in general available over a range in E much higher than those prevalent in stellar interiors, where the values of $S(E)$ are too small for accurate lab measurements. Existing lab measurements thus often require extrapolation over many orders of magnitude to the smaller E values relevant to the stellar interior, which leads to concerns about the accuracy of such extrapolations.

A more detailed examination of nuclear potentials shows there can be several energy levels in the compound nucleus.

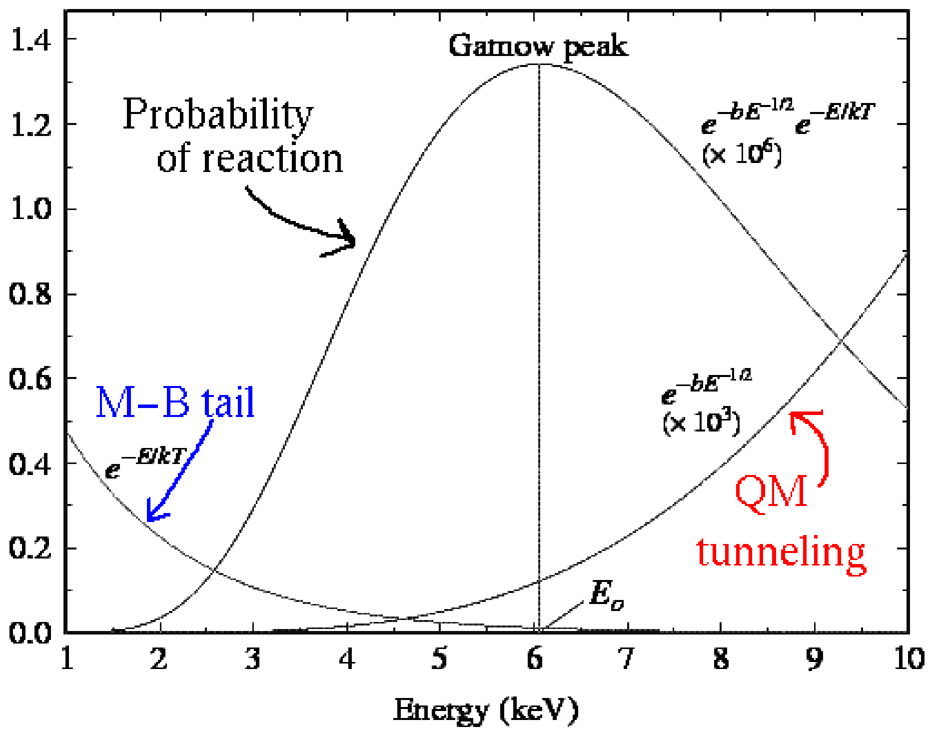

Fig. 3.— The tail of the velocity distribution and the QM tunneling are shown as a function of energy for $p + p$, the first reaction in the $p - p$ chain. Their product is the probability of the reaction. Source: website of Chris Mihos

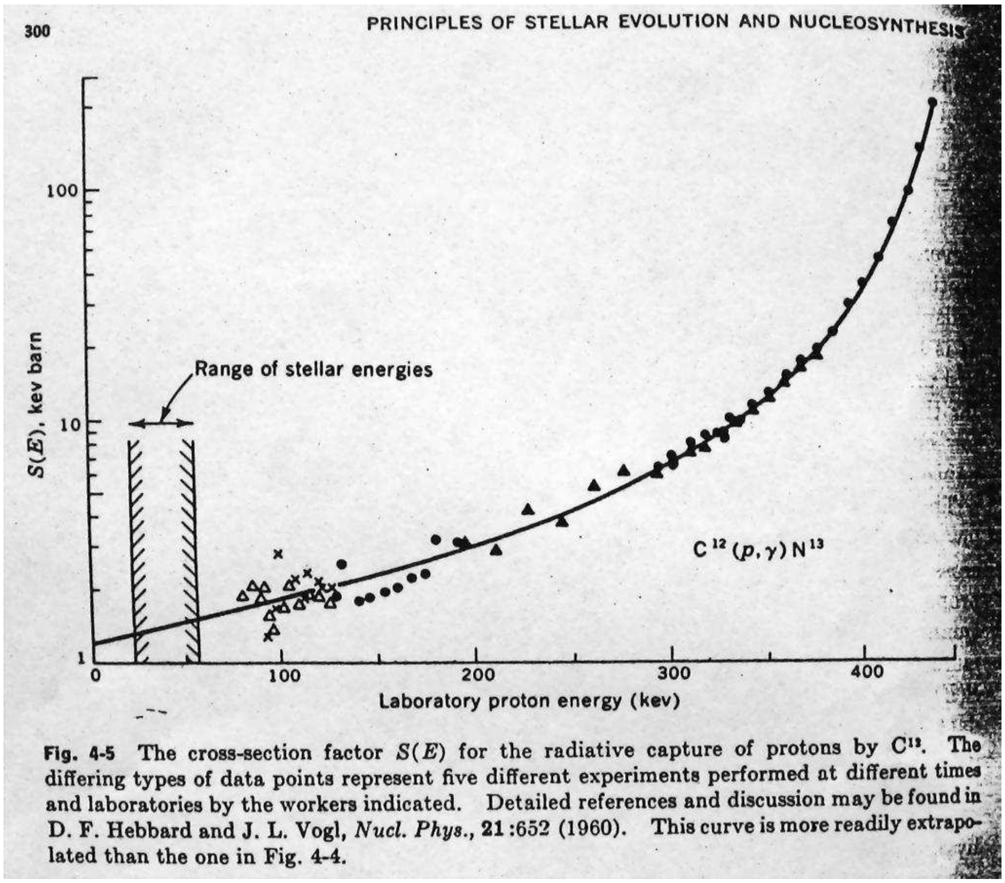

Fig. 4.— The cross section S for the reaction $^{13}C + p$ as a function of energy of the incoming proton. The lab measurements are indicated. A large extrapolation to lower E is required to reach the energy regime of interest for normal stars. Source: Fig.4-5 of Clayton

3.3. Quantum Mechanical Tunneling Probability

The Hamiltonian equation

$$(\frac{\hbar^2}{2m_{AB}})\frac{d^2\chi}{dr^2} = H = (V-E)\chi,$$

where $V(r) = Z_A Z_B e^2/r$ and $\chi(r)$ is the wavefunction.

We try to determine an approximation for the wavefunction. Define r_{min} as the closest distance possible for an incoming nucleus based on classical physics. Imagine $V - E$ constant in the above equation so as to simplify solving the equation. If $E > V$, the particle is classically allowed to be at that r, and in this region χ has an oscillatory behavior. In the classically forbidden region, corresponding to $E < V$, χ varies exponentially with r. At $r = r_{min}$, $V = E$ by definition, so $\frac{d^2\chi}{dr^2} = 0$. At $r = r_{min}/2$, $V = 2E$ and $H = E\chi$.

Assume that for $r < r_{min}$, $\chi(r)$ falls off exponentially, $\chi(r) \propto exp[(r_{min} - r)/l]$, where l is a characteristic length. Then evaluating H at $r_{min}/2$ we get $H \sim \chi(r_{min}/2)\hbar^2/(2m_{AB}l^2) \approx E\chi(r_{min}/2)$.

This determines the characteristic length l to be $\hbar/(\sqrt{2m_{AB}E}) = \hbar/(m_{AB}v_{AB}) = 1.3 \times 10^{-11}/\sqrt{T_7}$ cm.

At the nucleus, where $r \sim r_{min}/100$, we find $\chi \propto exp(-r_{min}/l) \propto exp(-17)$ for $T = 10^7$ K, and $Z_A = Z_B = 1$. The probability we need $\propto \chi^2$,

$$\chi^2 \propto exp(-2r_{min}/l) \propto exp[\frac{-2Z_A Z_B e^2}{1/2m_{AB}v_{AB}^2}\frac{m_{AB}v_{AB}}{\hbar}] = exp[-\frac{4Z_A Z_B e^2}{\hbar v_{AB}}]$$

Our rough derivation of the Gamow factor (the above negative exponential) is only slightly off from the correct value, $exp[-2\pi Z_A Z_B e^2/(\hbar v_{AB})]$.

3.4. Averaging Over Velocity

We now need to average over velocity to find $<\sigma v>$. We assume a Maxwellian (thermal) distribution for $p(v)$. We are going to omit the AB subscripts from σ, v and m for clarity.

$$< \sigma v > \;=\; \int_0^\infty \sigma v p(v) 4\pi v^2 dv \;=\; 4\pi\, [\frac{m}{2\pi kT}]^{3/2} \int_0^\infty v^3\, \frac{S(E)}{E}\, exp[-\frac{mv^2}{2kT}]\, exp[-\frac{-2\pi Z_A Z_B e^2}{\hbar v}]\, dv$$

We assume $S(E) = S_0$, a constant, so the above becomes

$$\sqrt{\frac{8}{\pi m}}\, S_0/(kT)^{3/2} \int_0^\infty exp[-E/(kT) - b/\sqrt{E}]\, dE$$

where b is a collection of constants, $b = (2\pi Z_A Z_B e^2/\hbar)(m/2)^{1/2}$. The term to be integrated represents the competition between the the Maxwellian velocity distribution, $\propto exp(-E/kt)$, and Coulomb barrier QM tunneling probability, $\propto exp(-b/\sqrt{E})$, as a product of two exponentials (see the previous figure).

The product of the two exponentials is shown in the figure. It is a function that cannot be more simply expressed nor integrated. To proceed further, we approximate it by a Gaussian centered at E_0, with a width Δ and an amplitude C, $g(E) = C\ exp[(E - E_0)^2/(\Delta/2)^2]$. Doing the integral will be easy once the Gaussian is defined, so we next need to solve for these three characteristic parameters E_0, Δ and C.

We define E_0 as the energy at which the product $exp[-E/(kT) - b/\sqrt{E}]$ has its maximum. We solve for the energy E_0 at which the derivative of the product is 0, and find

$$E_0 \;=\; (bkT/2)^{2/3}.$$

We solve for C by requiring that the maximum value of the Gaussian (C) is that of the product at E_0, $g(E_0) = C = exp[-E_0/(kT) - b/\sqrt{(E_0)}]$. This yields

$$C \; = \; e^{-\tau}, \quad \tau \; = \; \frac{3E_0}{kT}.$$

We find Δ from matching at E_0 the second derivative of $g(E)$ with that of the actual function $exp[-E/kT - b/\sqrt{E}]$; the first derivitive of $g(E)$ is 0 at E_0 by definition, since E_0 is at the maximum of $g(E)$. We end up with

$$\Delta = 4\sqrt{E_0 kT/3}$$

Now we can do the integral, i.e.

$$\int_0^\infty exp[-E/(kT) - b/\sqrt{E}] \; dE \; \approx \; \int_0^\infty g(E) \; dE \; = \; e^{-\tau} \; \frac{\Delta}{2} \int_0^\infty e^{-x^2} \; dx \; = \sqrt{\pi} \; e^{-\tau} \; \frac{\Delta}{2}$$

So the mean for σv, after some simplification of terms, is then:

$$< \sigma v > \;\; = \;\; \frac{8}{3^{5/2}} \; \frac{\hbar}{\pi e^2} \; \frac{1}{Z_A Z_B m} \; S_0 \; \tau^2 \; e^{-\tau}$$

where $\tau = 3E_0/(kT) = 42.5 \; (Z_A^2 Z_B^2 \mathcal{A})^{1/3} \; T_6^{-1/3}$

$\mathcal{A}$ is the reduced mass m in atomic mass units. Evaluating the constants, and assuming S_0 in keV-barns (1 barn = 10^{-24} cm^2), we get:

$$< \sigma v > \;\; = \;\; 7.2 \times 10^{-19} \; (Z_A Z_B \mathcal{A})^{-1} \; S_0 \; \tau^2 \; e^{-\tau} \mathrm{cm}^3 \mathrm{sec}^{-1}, \quad E_0 \; = \; 1.2 \; (Z_A^2 Z_B^2 \mathcal{A})^{1/3} \; T_6^{2/3} \; \mathrm{keV}$$

Because of the exponential $exp(-\tau)$, reactions involving charged particles with $Z_A > 1$ or $Z_B > 1$ become much less probable. Reactions involving incoming protons are much more likely.

Electron screening – a small correction factor for screening of positively charged nuclear particles by electrons (free electrons, if the atoms are fully ionized), so the effective Coulomb barrier becomes smaller, at least over a certain range of r. We ignore this factor, which produces a correction $< \sigma v > \ \propto\ S_0\, \tau^2\ exp(-\tau + \delta)$, where δ is small.

3.5. Power Law Form for Reaction Rates

We often want to express energy generation rates as powers of $T,\ P,\ \rho$. This makes it easier to derive approximations. The reaction rate $\propto \tau^2 exp(-\tau)$. It is this last term which we need to re-express. The power law variable is clearly T. Consider an expansion around an initial value of T_0, with $[T/T_0\ -\ 1]$ being small. Recall that $\tau = 3E_0/(kT) \propto T^{-1/3}$.

$$\frac{e^{-\tau(T)}}{e^{-\tau(T_0)}}\ =\ exp(-\tau(T_0)\ [(\frac{T}{T_0})^{-1/3}\ -\ 1])\ \approx\ [\frac{T}{T_0}]^{\tau(T_0)/3}$$

Thus if we want a reaction rate in the form of $\epsilon\ \propto \tau^2 exp(-\tau)$ in the form $\epsilon \propto T^{\eta}$, then $\eta = \tau(T_0)/3\ -\ 2/3\ \approx \tau(T_0)/3$, since $\tau(T_0) >> 1$ almost always.

3.6. Resonant Nuclear Reactions

The above formulation is correct ignoring possible resonances in the compound nucleus. If E is such that the compound nucleus is produced in a quasi bound state, then the interaction timescale becomes much longer, and the reaction rate increases dramatically. This is a local effect, occuring only near specific values of E. In order to predict this, we need to find the energy levels of the compound nucleus.

Fig.4-4 of Clayton shows the cross section of a resonant reaction, with an obvious peak near the marked resonant energy. These cross sections are harder to calculate. They have the general form near the resonance of $S(E) \sim \Gamma^2/[(E-E_r)^2 + (\Gamma/2)^2]$.

4. Burning H to He

At low temperatures, only the $p-p$ chain operates to burn H to He. At higher T, the CNO cycle begins to operate, and it dominates at high T, assuming some C exists to begin the CNO cycle.

4.1. The $p-p$ Chain

This is the only way to make He at low T, $T \leq 1.4 \times 10^7$ K.

$p + p \rightarrow D + e^+ + \nu_e$ (output: deuteron, positron and neutrino) $Q = 0.42$ MeV

$e^+ + e^- \rightarrow 2\,\gamma$ (output of annihilation is photons) $Q = 1.02$ MeV

The net effect of these two reactions is that a proton becomes a neutron while close to another proton.

The 0.26 MeV energy from ν is lost, so $Q_{eff} = 1.18$ MeV for these two reactions.

The cross section for the first reaction is tiny as it occurs via the weak interaction. $S_0 = 4 \times 10^{-22}$ keV barn, which is too small to measure in the lab.

The lifetime of a proton near the center of the Sun to the reaction $p + p$ is $2/(n_H < \sigma v >) \approx 2/[(1.2 \times 10^{26})\ (3.4 \times 10^{-44})] \approx 4.9 \times 10^{17}$ sec $\approx 1.5 \times 10^{10}$ yr. It is a good thing the Sun has LOTS of protons.

$D + p \rightarrow\ ^3He + \gamma$ Q=5.49 MeV, fast, $t_0 = 6$ sec, all D in a star (primordial, from the Big Bang) is destroyed rapidly at moderate T. This reaction is fast as it proceeds via the strong nuclear interaction, not the weak one. We just need the two participants $(p+n)\ +\ p$ to become associated into a single nucleus.

Since this is a fast reaction, the ratio n_D/n_p must reach equilibrium in a star such that

$$\frac{\partial n_D}{\partial t} = (n_p^2/2)\ < \sigma v >_{pp}\ -\ n_d n_p < \sigma v >_{Dp}\ \approx\ 0.$$

At $T = 2 \times 10^7$ K, we find $n_D/n_p \sim 10^{-18}$, while the Big Bang production ratio is $n_D/n_p \sim 2 \times 10^{-5}$ (the usually quoted ratio is a ratio by mass, this is by number). So D is rapidly destroyed and n_D/n_p drops to about 10^{-18} in the interior of the Sun.

The final reaction is $^3He\ +\ ^3He \rightarrow\ ^4He\ +\ 2\ p$, where $Q = 12.86$ MeV and the timescale is 10^6 yr.

Thus the total energy liberated per ^4He produced (which requires the $p+p$ reaction to occur twice) is 26.2 MeV, excluding the 0.26 MeV carried away by each of the 2 neutrinos produced.

There are many variations on this chain which occur with varying probabilities. The next most likely which can occur once some ^{4}He exists, and with $2.3 \times 10^7 \gtrsim T \gtrsim 1.4 \times 10^7$ K is:

$^3He +^4 He \rightarrow \ ^7Be + \gamma$ with $Q = 1.59$ MeV

$^7Be + e^- \rightarrow \ ^7Li + \nu_e$ with $Q = 0.86$ MeV, but 0.80 of that is carried away by the neutrino. This reaction rate is independent of T, as it involves oppositely charged particles.

$^7Li + p \rightarrow 2\ ^4He$ with $Q = 17.35$ MeV.

The total energy per ^4He produced, excluding ν losses, is 25.67 MeV.

At $T > 2.3 \times 10^7$ K, another varient occurs:

$^7Be + p \rightarrow \ ^8B + \gamma$ $Q = 0.13$ MeV

$^8B \rightarrow c^+ + \nu_e + \ ^8Be^*$ ^8B unstable, decays, $Q = 17.98$ MeV, $Q_{eff} = 10.78$ MeV excluding the neutrino energy

$^8Be^* \rightarrow \ 2\ ^4He$ $Q = 0.09$ MeV.

The total energy generated by converting 4 protons into ^{4}He is the same no matter which chain is followed. But the neutrino losses are somewhat different, so the effective energy generation per reaction (in MeV) is slightly different. Also the reaction rates differ, and the slowest reaction in each chain may have quite different t_0, so the energy generated/sec/gm (i.e. the values of ϵ for each of the $p-p$ chains) will be different. For example, in the main $p-p$ chain, the first and slowest reaction must occur twice to produce two deuterons, while in some of the side chains, only one deuteron is required.

The $p-p$ chain energy generation rate is

$$\epsilon_{pp} \approx 10^6 \rho X_H^2 T_6^{-2/3} \; exp(-33.81 T_6^{-1/3}) \text{ erg/sec/gm}$$

4.2. Numerical Examples

We have shown above that the cross section is determined by the QM tunneling, and is a function of v (or equivalently of energy E), both evaluated in the center of mass frame of the reacting particles A and B. Thus it must be averaged over the Maxwellian velocity distribution.

$$< \sigma v > \; = 4\pi \; [\frac{m}{2\pi kT}]^{3/2} \int_0^\infty v \; \frac{S(E)}{E} \; exp[-\frac{mv^2}{2kT}] \; exp[-\frac{-2\pi Z_A Z_B e^2}{\hbar v}] \; v^2 dv$$

Let us write $\sigma \; \propto \; exp(-b/\sqrt{E})$, where

$$b \; = \; \frac{\sqrt{2}\pi Z_A Z_B e^2 \sqrt{m_{AB}}}{\hbar} \; = \; 0.99 \; Z_A Z_B \sqrt{m_{AB}} \; (MeV)^{1/2}.$$

The minimum r achieved for a classical particle is set by $Z_A Z_B e^2 / r_{min} \; = \; kT$. Evaluating the constants, we find $r_{min} \; \approx \; 2000 Z_A Z_B / T_7$ fermis, where 1 fermi $\; = \; 10^{-13}$ cm, so $r_{min} \; \approx \; 1000$ fermis. On the other hand, the radius of an atomic nucleus is about 1 fermi.

The standard case we discuss here is for $p + p$ ($Z_A = Z_B = 1$) and with $T \sim 2 \times 10^7$ K. Adopting these values we find:

$$r_{min}/r_{nuc} \; \approx \; 1000$$

Recall that our approximation for the wave function is $\chi(r) \propto exp[-(r_{min} - r)/l]$, valid for $r < r_{min}$, and $l \; = \; \hbar/(\sqrt{2 m_{AB} E})$. We assume a particle needs to reach to within a few r_{nuc} in order for the nuclear forces to take over and a nuclear reaction to occur. When $r << r_{min}$, $\chi \ \propto \ exp[-r_{min}/l]$.

$l \ = \ 1.3 \times 10^{-11}/\sqrt{T_7}$ cm, so for our standard case $l \ \approx \ 10^{-11}$ cm.

The probability that the particle can reach this through QM tunneling is $\propto \chi^2 \ \propto \ exp(-20)$, which is 2×10^{-9}. Even QM tunneling is not very probable for such small radial separations as a few r_{nuc}.

Note that when T increases, $exp[-b/\sqrt{E}]$ also increases as the power becomes smaller and it is a negative exponential, so it cannot exceed e^0, which is 1.

It was shown that when we use a Gaussian approximation to the integral of $exp(-b/\sqrt{E})exp(-E/kT)$, we obtain $< \sigma v > \ \propto \ \tau^2 \ e^{-\tau}$, where $\tau \ = \ 3E_0/(kT)$, and $E_0 \ = \ (bkT/2)^{2/3}$.

So $\tau \ \propto \ T^{-1/3}$, and $< \sigma v > \ \propto \ T^{-2/3} exp(-aT^{-1/3})$.

$T^{-1/3}$ decreases when T increases, while $exp(-at^{-1/3})$ increases towards e^0 (i.e. 1) as T increases. We expect $< \sigma v >$ to increase as T increases, as higher T means higher mean velocity, a bigger QM tunneling probability, and thus a higher reaction rate. Thus the exponential must dominate.

We now demonstrate that this is the case.

$E_0 \ = \ 1.2(Z_A^2 Z_B^2 \mathcal{A})^{1/3} T_6^{2/3}$ keV. For our standard case, with $T = 20 \times 10^6$ K, $E_0 \sim 7$ keV. $\tau \ = \ 3E_0/(kT) \ \approx 7$. We evaluate $\tau^2 \ e^{-\tau}$ to find a value of 0.04 for our standard reaction and T.

For T which is 2 times larger than our standard value, $T = 40 \times 10^6$ K, $E_0 \ \approx \ 5.8$ keV. Again evaluating $\tau^2 \ e^{-\tau}$ we get 0.10, which is 2.5 times larger than at our standard T.

Because τ is significantly greater than 1, and we have a term $exp(-\tau)$, the exponential term overpowers the τ^2, and $< \sigma v >$ increases with T, as we expected.

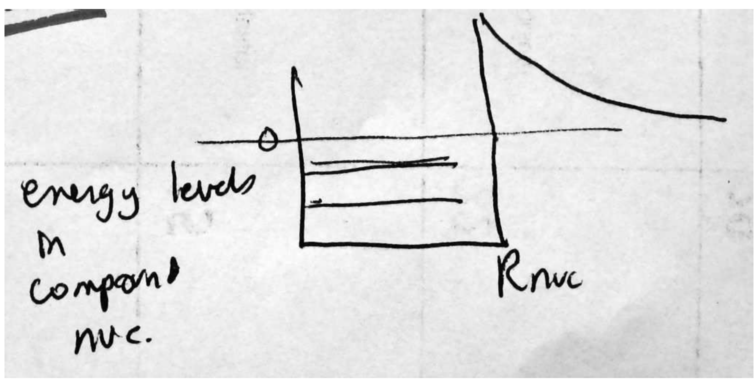

Fig. 5.— A sketch of energy levels (quasi-bound states) in the compound nucleus formed during a nuclear reaction.

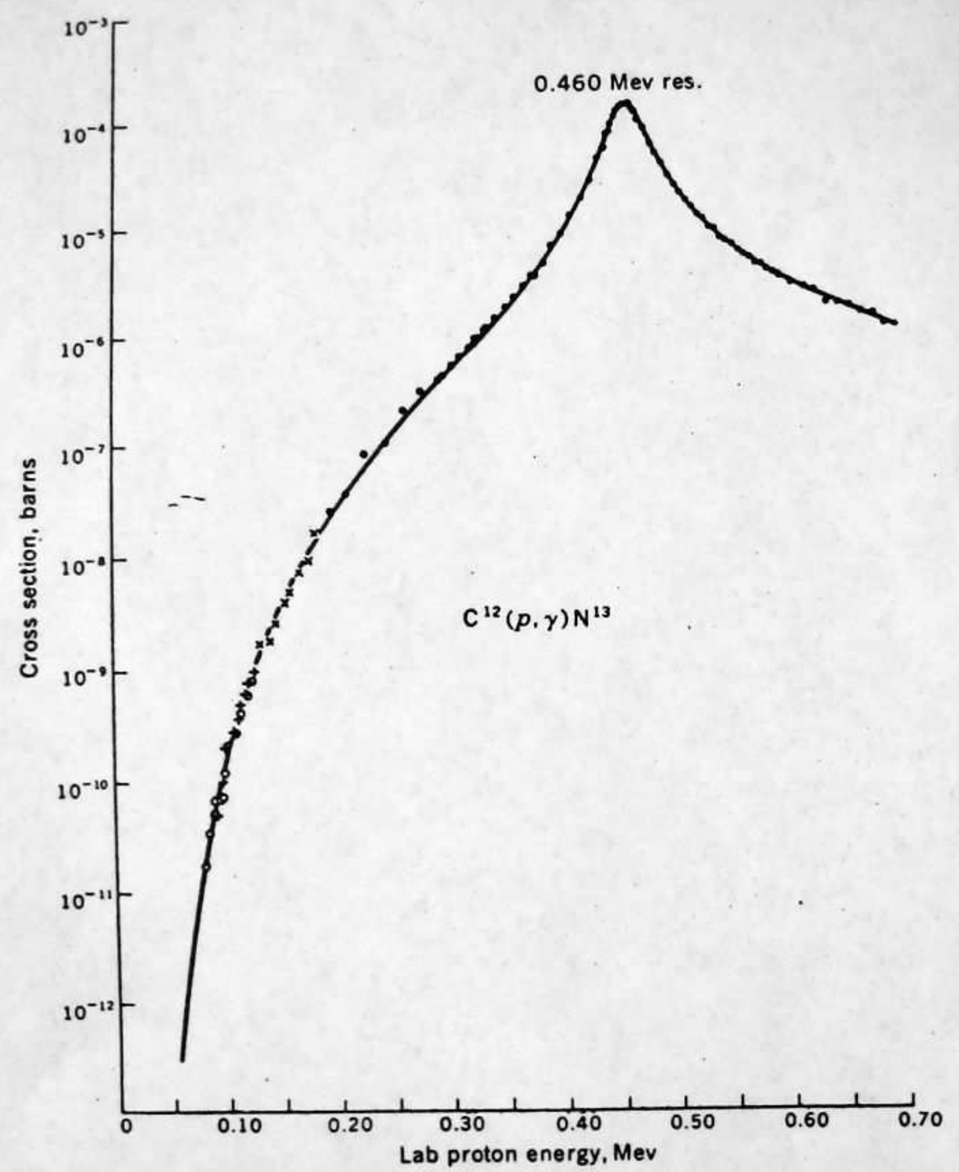

Fig. 4-4 **The measured cross section for the reaction $C^{12}(p,\gamma)N^{13}$ as a function of laboratory proton energy. A four-parameter theoretical curve has been fitted to the experimental points. An extrapolation to $E_p = 0.025$ Mev, which is an interesting energy for this reaction in astrophysics, appears treacherous. (*Courtesy of W. A. Fowler and J. L. Vogl.*)**

Fig. 6.— A sketch of the cross section for the reaction $^{12}C(p,\gamma)^{13}N$ as a function of energy of the incoming proton. Note the resonance. The energy of interest for normal stellar interiors is about 0.03 MeV, so a big extrapolation of the lab measurements to lower E is required. Source: Fig. 4-4 of Clayton

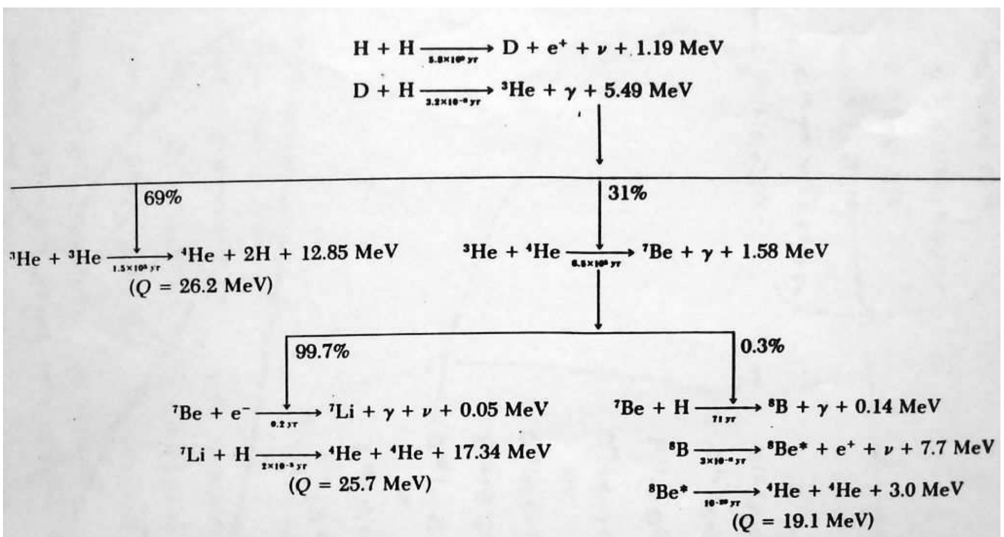

Figure 1. Three competing reaction chains with the net result, $4H \rightarrow {}^4He + Q$, where the locally contained heat, Q, varies from 19.1 to 26.2 MeV because of different neutrino-energy losses. The individual exoergic-reaction energies shown include positron annihilation but do not include neutrino energy. The partial life expectancy for the first particle written in each reaction is given under the arrow, for a typical solar-core ambient density and temperature, 150 g cm^{-3} and 15×10^6 K, respectively, and equal ^1H and ^4He mass fractions. The branch through $^7Be + {}^1H$, although quite rare, is nevertheless the dominant source of detectable neutrinos in the ^{37}Cl experiment because of the high energy released in the decay of ^8B.

Fig. 7.— The main $p - p$ chain and various alternative side chains are shown.

4.3. Nuclear Reaction Networks

Imagine we begin with a gas containing only atoms/particles of types A and B. Let R_{AB} be the number of reactions/sec of the type desired, $A + B \rightarrow C + D$. An expression for R_{AB} was given earlier in these notes. We can define a mean life time of nucleus A with respect to the desired reaction:

$$\frac{\partial N_A}{\partial t}|_{A+B} \quad = \quad - N_A/\tau_{A,B} \quad = \quad - R_{AB}$$

$$\tau_{A,B} = \frac{1 + \delta_{AB}}{N_B < \sigma v >}$$

Lets apply this to the main $p - p$ chain. Since the reaction rates are given per gm, we have

$$\frac{\partial n_H}{\partial t} = \rho[-3R(pp) + 2R(^3He\ ^3He) - R(^3He\ ^4He)].$$

But $n_i = (X_i\rho)/(m_i m_H)$, where m_i the the mass of isotope i in units of m_H. Then these equations become:

$$\frac{\partial X(^1H)}{\partial t} = m_H[-3R(pp) + 2R(^3He\ ^3He) - R(^3He\ ^4He)].$$

$$\frac{\partial X(^3He)}{\partial t} = 3m_H[R(pp) - 2R(^3He\ ^3He) - R(^3He\ ^4He)]$$

$$\frac{\partial X(^4He)}{\partial t} = 4m_H[R(^3He\ ^3He) + R(^3He\ ^4He)]$$

where the last term in each case comes from a side chain.

When nuclear equilibrium is reached, the amount of an intermediate product, such as ^{3}He, produced per unit time is the same as that destroyed in a subsequent reaction, i.e.

$$\frac{\partial X(^3He)}{\partial t} = 0 \quad \rightarrow \quad R(pp) - 2R(^3He\ ^3He) - R(^3He\ ^4He) = 0$$

In order for this to hold, we need certain relations among the abundances of the various isotopes to hold, i.e. $X(^1H)^2 - AX(^3He)^2 - BX(^3He)X(^4He) = 0$. A and B are constants related to the ratios of the various reaction rates for a given T. Specific ratios of $^3He/^4He$ (which are functions of T), for example, are required for nuclear equilibrium to prevail.

As time goes on, we have converted A and B into C and D. We have a set of interlocked time dependent equations, each expressing the time evolution of a particular atomic nucleus (an isotope of an atom). We must consider all the source terms and sink terms for a particular isotope in such an equation, i.e. all the possible reactions involving producing or destroying A. The set of such equations is called a nuclear reaction network. The solution is some initial ramp up. Then, after a time much longer than any of the individual reaction times, a steady flow through the network develops which is independent of time and persists as long as the initial nuclei required are not (too) depleted. This results in equal production and destruction rates for the intermediate products, with a conversion of the lightest particles at the beginning of the network into the heaviest ones at the end of the network. In a modern code, there may be hundreds of individual nuclear reactions considered to follow the change in abundances of the various isotopes with time.

4.4. The CNO Cycle

This cycle operates at higher T than that at which the $p-p$ chain turns on. The metals are used at catylists – they are not consumed. However, some ^{12}C is required for the CNO cycle to start. If the star truly has no metals, the CNO cycle cannot operate. The reactions in the main chain are listed, as are the energy released per reaction and the timescale of each reaction. The timescales for the captures given below exceed those of the β-decays at the temperature at which the CNO cycle operate, so the decays always have time to occur, and the reaction chains consequently do not have a multitude of possible branches.

$^{12}C + p \rightarrow\ ^{13}N + \gamma \qquad Q = 7.54$ MeV, $t_0 = 1.3 \times 10^7$ yr

$^{13}N \rightarrow\ ^{13}C + e^+ + \nu_e \qquad Q = 0.22\ MeV\ \ t_0 = 7$ min, ν_e carries off 0.72 MeV.

$^{13}C + p \rightarrow\ ^{14}N + \gamma \qquad Q = 7.54$ MeV, $t_0 = 2.7 \times 10^6$ yr

$^{14}N + p \rightarrow\ ^{15}O + \gamma \qquad Q = 7.29$ MeV, $t_0 = 3.2 \times 10^8$ yr

$^{15}O\ \rightarrow\ ^{15}N\ +\ e^+\ +\ \nu_e \qquad Q = 2.94$ MeV, $t_0 = 82$ sec, ν_e carries off 0.98 MeV

$^{15}N + p \rightarrow\ ^{12}C +\ ^4He \qquad Q = 4.96$ MeV, $t_0 = 1.1 \times 10^5$ yr

In this chain of reactions, ^{12}C is used as a catylist to drive the cycle. No change in the ^{12}C abundance results, but the ^{13}C abundance, and that of all isotopes of N and O, may be altered.

The total energy produced is 25.02 MeV/^{4}He produced, 1.70 MeV is lost via neutrinos.

The slowest reaction is the fourth, $^{14}N(p,\gamma)^{15}O$. This reaction drives the equilibrium, with ^{12}C converted into ^{14}N at any instant, and this reaction determines the nuclear energy generation rate.

$$\epsilon_{CNO}\ \approx\ 10^{28}\rho\ X_H X_{CN}\ T_6^{-2/3}\ exp[-152.3\ T_6^{-1/3}]\ \text{erg/gm/sec.}$$

For $T \sim 1.8 \times 10^7$ K, $\epsilon_{CNO} = \epsilon_{pp}$ for X_{CN} having the Solar value. At higher T, the energy generation rate from the CNO cycle exceeds that from the $p - p$ chain.

In the Sun, the CNO cycle contributes 3% of the energy generated. At $M = 1.9 M_\odot$, the CNO cycle and the $p - p$ chain contribute equally to the energy generation.

There are many alternate chains which operate with lower probability within the CNO cycle. Some of them are sketched below.

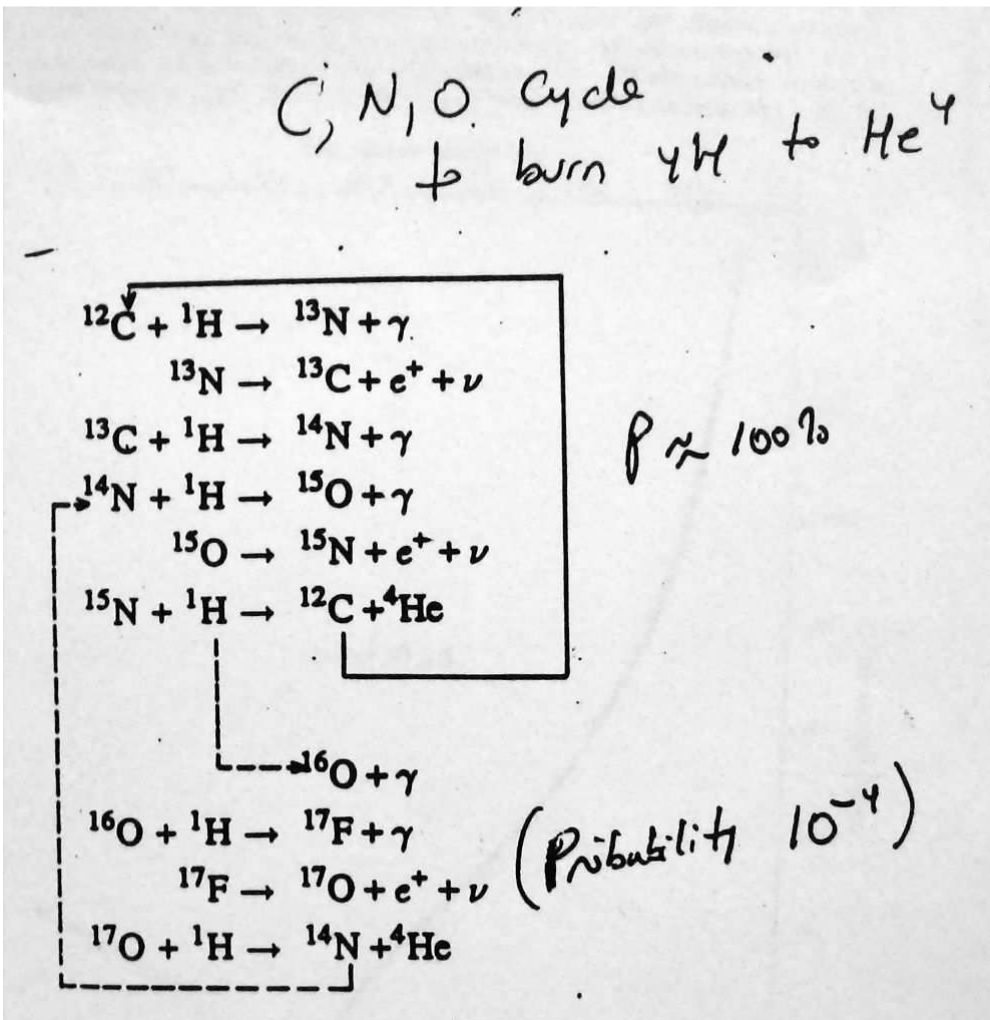

Fig. 8.— Example of a low probability alternate chain to the main CNO cycle.

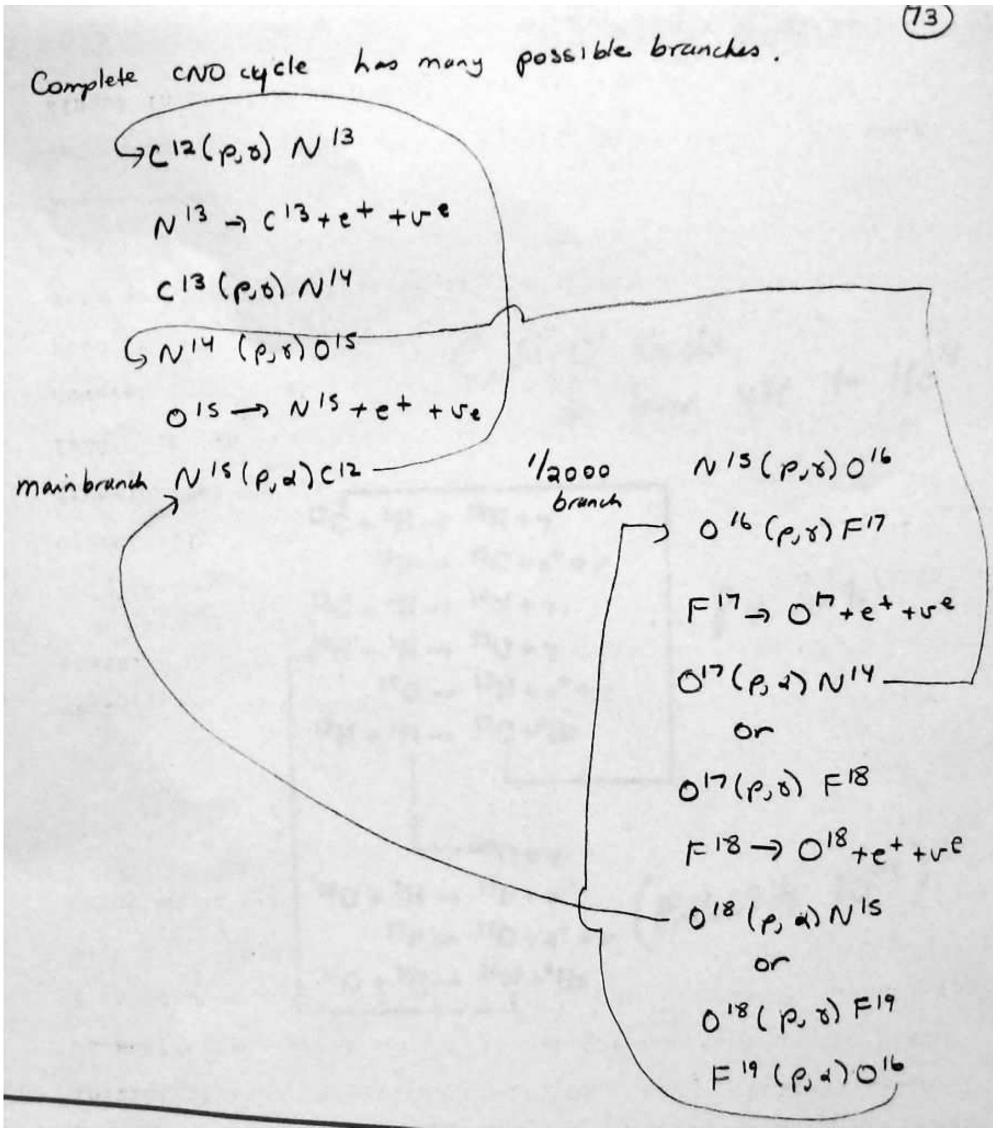

Fig. 9.— Examples of some side chains to the main CNO cycle. See also Table 6.2 of Hansen, Kawaler & Trimble.

5. Beyond He

We know we can build up to ^{4}He starting with just protons. But we can't build up heavier elements this way as there is no stable isotope of atomic mass 5 and of atomic mass 8. The list of stable isotopes is:

H 1, 2 He 3, 4 Li 6, 7 Be 9 B 10, 11 C 12, 13 N 14, 15 O 16, 17, 18

Also the Coulomb barriers are becoming higher and the probability for the reaction to occur is dropping rapidly as Z (the nuclear charge) increases.

We can't add neutrons, as they are gobbled up so fast that there are essentially none of them in a stellar interior except under very special circumstances. Since adding protons won't work, we need to add α-particles (^4He nuclei).

URLs for compilations of nuclear reaction rates include: Caughlan & Fowler (1988), nuclear reaction rate fits, http://www.phy.ornl.gov/astrophysics/data/data.html, and also, the NACRE Collaboration (in Belgium) gives nuclear reaction rates for charged particle reactions, see http://pntpm.ulb.ac.be/nacre.htm. Note the very recent compilation *Solar Fusion Cross Sectoins II: The pp Chain and CNO Cycles*, by Adelberger, Garcia, Harnish et al (2010, Reviews of Modern Physics, see arXiv:1004.2318).

6. The Triple α Process: Burning Helium Into Carbon

Burn He to C with triple-α process. The details of this process were worked out by Hans Bethe. Complicated as no stable nuclei with atomic mass 5 or 8. $T \sim 10^8$ K is required for this to occur.

$^4He + \ ^4He \rightarrow ^8 Be + \gamma$ $\quad Q = -0.095$ MeV. The lifetime of the ^8Be is only 10^{-16} sec.

$^8Be + {}^4He \to {}^{12}C^*$ (an excited state of ^{12}C.

$^{12}C^* \to {}^{12}C + \gamma$ $Q = 7.37$ MeV.

Since ^8Be is unstable in its ground state, it decays rapidly back to 2 α particles, $\tau = 2.6 \times 10^{-16}$ sec. We can calculate the equilibrium fraction of $^8Be/^4He$ by viewing this as similar to a Saha ionization equilibrium. The Gamow peak (the energy difference) between the ^8Be nucleus and that of two ^4He nuclei is 91 keV.

We find $E_0 = 3.9\, T_6^{2/3}$ keV (since $Z_\alpha = 2,\ \mu = 2$), so we need $T \sim 1.2 \times 10^8$ K.

If we ignore electron screening, we get an equilibrium equation:

$$\frac{n_\alpha^2}{n(^8Be)} = [\frac{\pi m_\alpha\, kT}{h^2}]^{3/2}\ exp[\chi/(kT)] = 1.69 \times 10^{34}\ T_9^{3/2}\ exp[-1.07/T_9]$$

where $\chi = 91.78$ keV, and the units of n are cm^{-3}.

We then get a small equilibrium abundance of ^8Be, $n(^8Be)/n(\alpha) \approx 7 \times 10^{-9}$ for pure He with $T = 10^8$ K and $\rho = 10^6$ gm/cm^3.

The only reason such a small fraction of ^8Be can lead to production of an interesting amount of ^{12}C is that there is a resonance in the reaction $^8Be + {}^4He \to {}^{12}C^*$ to an excited state of the ^{12}C nucleus. If there were no resonance, the rate would be much smaller. The figure displays the levels of the states in the ^{12}C nucleus compared to that of $^8Be + \alpha$. Once the excited state of ^{12}C is formed, it rapidly decays either back to $^8Be + {}^4He$ or to the ground state of ^{12}C, emitting 2 photons. Most of the time $^{12}C^*$ decays back to $^8Be +^4 He$, but once in about 2500 times it decays to $^{12}C + 2\ \gamma$.

We then get a small fraction of $^{12}C^*$, namely:

$$\frac{n(^{12}C^*)}{n_4 n_8} \;=\; (3/2)^{2/3}\;[\frac{h^2}{2\pi m_4 kT}]^{3/2}\; exp[-0.29 \text{ MeV}/kT]$$

$$\frac{dn(^{12}C)}{dt} \;=\; n(^{12}C^*)/\tau(^{12}C^* \rightarrow^{12} C) \;=$$

$$\frac{n(^4He)^3}{\tau(^{12}C^* \rightarrow^{12} C)}\; 3^{3/2}\; [\frac{h^2}{2\pi m_4\; kT}]^{3/2}\; exp[-(m(^{12}C^*) - 3m(^4He)/kT]$$

where this mass difference is 0.3795 keV (see the figure). The energy generation rate is:

$$\epsilon_{3\alpha} \;=\; [3m(^4He) - m(^{12}C)]\; c^2\; \frac{dn(^{12}C)}{dt} \;=\; \frac{5.1\times 10^8 \rho^2\; Y^3}{T_9^3}\; exp(-4.40/T_9) \text{ ergs/gm/sec}$$

For the triple-α process, $\epsilon_{3\alpha} \propto T^\nu,\; \nu \approx (4.4/T_9) - 3$. For $T \sim 10^8$ K, $\nu \;=\; 41$.

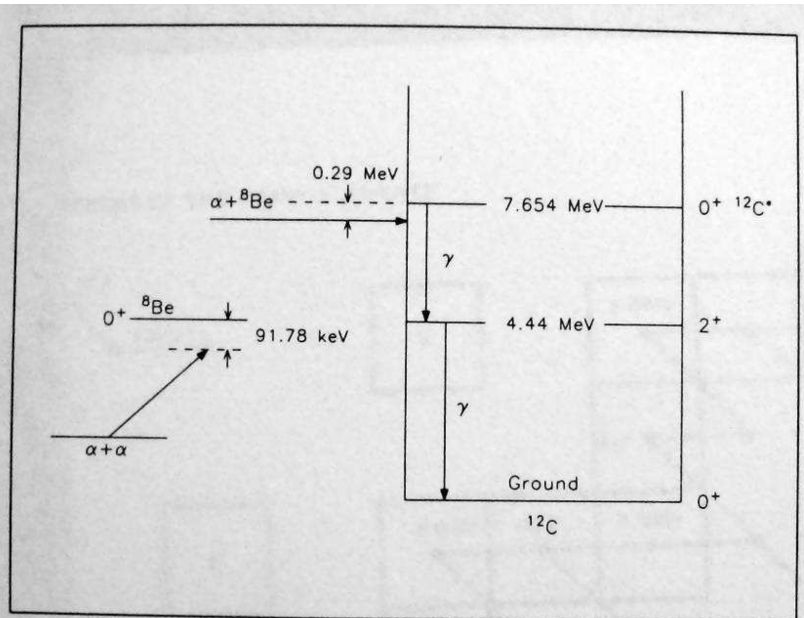

Fig. 6.11. The level diagrams and energetics of the two reactions composing the triple-α reaction (not to scale). The final result is the nucleus ^{12}C.

Fig. 10.— Energy levels of ^{8}Be versus an ^{12}C - note the presence of an excited state which produces a resonance in the reaction rate. (Fig. 6-11 of Hansen, Kawaler & Trimble)

7. He Burning After Carbon Is Formed

At this point there are no protons left; they have all been burned into He. Once ${}^{12}C$ is formed, many nuclear reactions of the general form $\alpha + {}^{N}X \rightarrow {}^{N+4}Y + \gamma$ become possible, where the atomic number of Y is $X + 2$ and its atomic weight is that of $X + 4$. Specific examples of such a a reaction chain beginning with ${}^{12}C$ are:

$\alpha + {}^{12}C \rightarrow {}^{16}O + \gamma$

$\alpha + {}^{16}O \rightarrow {}^{20}Ne + \gamma$

$\alpha + {}^{20}Ne \rightarrow {}^{24}Mg + \gamma$

etc.

7.1. Carbon Detonation

Ignition of a new nuclear energy source in a degenerate stellar core leads to a thermal runaway, as the nuclear reaction rate is highly temperature dependent, and the core does not expand when heated. This is the origin of the He flash (the ignition of He in a degenerate core) for low mass stars which ends the red giant branch phase.

For stars with $M > 10M_\odot$ the C-O core is non-degenerate when ignited, so there is no equivalent of the He-flash. Stars with $M < 10M_\odot$ have a degenerate C,O core when C,O are ignited ($T_c \sim 6 \times 10^8$ K), so a thermal runaway results.

The “carbon detonation” is a postulated process in which a “detonation wave” results which is sufficient to process all elements to ${}^{56}Fe$, releasing 2×10^{51} ergs.

7.2. Very Massive Stars

In very massive stars, the gas becomes hot enough in the core so that C-burning can occur in a non-degenerate environment via the reactions given below (plus similar additional ones). This requires $T_c \sim 8 \times 10^8$ K, $\rho \; = \; 3 \times 10^6 \;$ gm/cm^3.

$$^{12}C + \; ^{12}C \rightarrow \; ^{20}Ne + \alpha \qquad Q \; = \; 4.616 \text{ MeV}$$

$$^{12}C + \; ^{12}C \rightarrow \; ^{23}Na + p \qquad Q \; = \; 2.228 \text{ MeV}$$

$$^{12}C + \; ^{12}C \rightarrow \; ^{23}Mg + n \qquad Q \; = \; -\, 2.623 \text{ MeV}$$

The resulting composition in the $8 M_\odot$ core of a massive star is given in the table.

The table gives fractional abundances of various isotopes in a stellar core composed intially of ^{12}C, ^{16}O. After C detonation, this is transformed into mostly ^{16}O, ^{20}Ne and ^{24}Mg.

Table 1. Result of C-Burning in a Very Massive Star

Isotope	Fractional Abund	Isotope	Fractional Abund
^{12}C	2×10^{-3}	^{25}Mg	5×10^{-3}
^{16}O	0.5	^{26}Mg	5×10^{-3}
^{20}Ne	0.36	^{27}Al	1×10^{-3}
^{21}Ne	7×10^{-4}	^{28}Si	1×10^{-3}
^{22}Ne	2×10^{-4}	^{29}Si	1×10^{-4}
^{23}Ne	0.022	^{30}Si	7×10^{-6}
^{24}Mg	0.091		

8. The Iron Catastrophe

Stars can burn successively heavier elements, fusing them with α or p particles, and producing energy, until reaching ^{56}Fe. As each fuel is exhausted, the star can try to heat up its core to begin to burn the next fuel. If it fails to do so prior to Fe formation, the star can no longer maintain its luminosity, and it will slowly fade and die.

Once Fe is formed, fusion reactions no longer release energy. Only fission reactions release energy for nuclei heavier than the Fe-peak. Thus no matter how much the star attempts to contract and heat up its core to burn the next (heavier) fuel, it cannot generate energy that way, and a runaway collapse ensues (a SN).

To understand why this happens, we need to look at the binding energy of nuclei, $B(A,Z)$. $B(A,Z) \ = \ Zm_p + (A-Z)m_n - M(A,Z)$. For $A > 10, \ 7.4 \leq B/A \leq 8.8$ MeV.

If each nucleon interacted through an attractive force, $B \propto \ n(pairs) \propto A(A-1) \approx A^2, \ B/A \propto A$. This, while approximately true for the lightest elements, is not true for most elements.

So nuclear forces must saturate such that each nucleon effectively only interacts with a few other neighboring nucleons rather than all those in the nucleus itself.

r_n is the effective radius of the nuclear force and r_0 is the radius of typical nucleon. Both of these are constant. R is the nuclear radius, and $R = r_0 \ A^{1/3}$. If $R > r_n$, each particle interacts only with the fraction $(r_n/R)^3$ of the total number of nucleons.

E_V is the total potential energy of the nucleus due to nuclear interactions. $E_V \propto \ -(r_n/R)^3 \ A^2$. We need to minimize this by making R small (collapsing the nucleus), but this is opposed by the kinetic energy rising as R decreases due to the uncertainty principle.

In evaluating E_V, we must correct for points within r_n of the surface, so we get:

$$E_V \;\; = \;\; -\,(r_n/R)^3 \;A^2 \;[1 - \frac{9}{16}\frac{r_n}{R} + \frac{1}{32}(\frac{r_n}{R})^3 + ...]$$

We ignore all but the first correction term for the surface as the other terms are very small, so have:

$$E_V \;\; = \;\; -\,k_v \;[A - (9/16)(r_n/r_0) \;A^{2/3}]$$

We consider the thermal energy next. This is a Fermi gas of particles with spin 1/2, the nucleons behave like a fully degenerate gas. For a total mass of the nucleus of M,

$$E_T \;\; = \;\; \frac{9\pi}{40}(\frac{3}{\pi})^{1/3} \;\frac{\hbar^2}{MR^2} \;A^{5/3} \;\; = \;\; k_T \;A$$

since $R \propto A^{1/3}$.

The Coulomb force has a potential $E_C = k_c \; Z^2/A^{1/3} \;\propto Z^2/R$.

These terms have to be summed to get $B(A, Z)$. Then $B/A \;=\; [-E_T + E_V + E_C]/A = k_v - k_T - (9/16)(r_n/r_0) \; k_V \; A^{-1/3} - k_c \; Z^2 \; A^{-4/3}$. We set $Z \;=\; A/2$. This approximation is OK for heavier nuclei. The binding energy per nucleon then becomes:

$$B/A = k_V - k_T - (9/16)(r_n/r_0) k_V \; A^{-1/3} - (k_C/4) \; A^{2/3}$$

$$B/A \;\; = \;\; \alpha_V \;\; - \;\; \alpha_s \; A^{-1/3} \;\; - \;\; \alpha_C \; A^{2/3}$$

where the three different αs used above are combinations of constants. B/A begins at a small number and increases as A increases until $R > r_n$. If B/A decreases as A increases, then fusion reactions will not release energy. To find the maximum of B/A, we set its derivitive to 0.

$$\frac{\partial(B/A)}{\partial dt} = -\frac{\alpha_S\ A^{-4/3}}{3} - (2/3)\alpha_C A^{-1/3} = 0$$

Then the maximum of B/A occurs when $A = \alpha_s/(2\alpha_C)$. $\alpha_C \approx 0.19\ m_H$ and $\alpha_s \approx 19.7\ m_H$. The above formula givens $A_{max} \approx 51$, close to the Fe peak at $A = 56$. This simple model reproduces the observed behavior of B/A fairly well.

To determine which nuclei are stable we recall that we need to maximize the binding energy. The mass of a nucleus is $m(Z,A) = Zm_p + (A-Z)m_n - BE$, so maximizing the binding energy is equivalent to minimizing $m(Z,A)$. To do this, we need to add two additional terms to the above discussion on the binding energy. The first is a term which gives preference to nuclei with equal numbers of neutrons and protons ($A/2 = Z$), called the assymetry term. It is $-a_A(Z-A/2)^2/A$, where a_A is the assymetry constant.

The second is a pairing term favoring the case when both the number of neutrons and of protons is even, and penalizing the case when they are both odd. Set $\delta = 1$ for the even-even case, 0 for the even-odd case, and -1 for the odd, odd case. The this additional term is $a_p\delta A^{-1/2}$, where a_p is the pairing constant.

Adding in these two terms, the binding energy in the liquid drop model for nuclei becomes

$$BE = a_V A - a_s A^{2/3} - a_c Z^2 A^{-1/3} - a_A(Z-A/2)^2/A + a_p A^{-1/2}.$$

The resulting nuclear mass relation $m(Z,A)$ is known as the semi-empirical mass formula, first developed by Weizsacker. The solution for maximizing the binding energy, equivalent to minimizing the mass for a given A, combined with the possible modes of

β-decay define the stable isotopes of each element. The constants for each of the 5 terms are best determined empircally, by attempting to reproduce the actual masses of stable nuclei, and are approximately $a_V = 15.85$ MeV/c^2, $a_S = 18.34$ MeV/c^2, $a_C = 0.71$ MeV/c^2, $a_A = 92.86$ MeV/c^2, and $a_p = 11.46$ MeV/c^2.

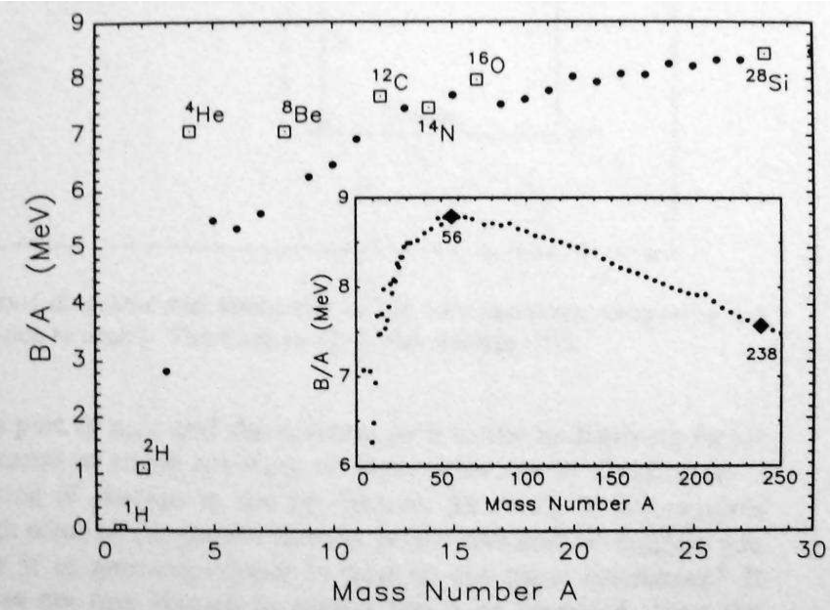

Fig. 6.1. Plotted is the binding energy per nucleon, $B\varepsilon/A$, as a function of atomic mass number A for the most stable isobar of A. The main area is for the lighter nuclei, whereas the insert is for most of the rest. Some nuclei discussed in the text are given special emphasis. Data from Wapstra et al. (1988).

Fig. 11.— The binding energy per nucleon B/A is shown as a function of atomic number A. (Fig. 6.1 of Hansen, Kawaler & Trimble)

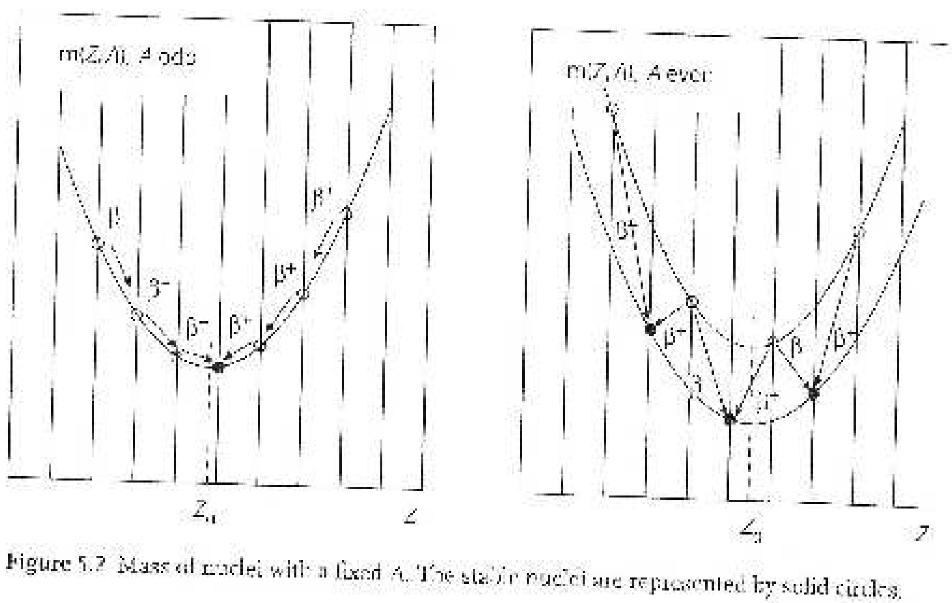

Fig. 12.— Predicted mass of nuclei with a fixed A (a fixed number of nucleons), from the liquid drop model of nuclei. The stable nuclei are represented by solid circles. The case of A odd is shown in the left panel; the right panel shows the case of even A. β^- decays are shown as arrows towards the lower right, β^+ decays are arrows towards the lower left. The dashed vertical line in each panel represents Z_0, the value of Z for which $m(Z, A)$ is a minimum. (This is the solution of an equation and may not be an integer.) (Fig. 5.2 of Bertulani's book *Nuclear Physics in a Nutshell*.)

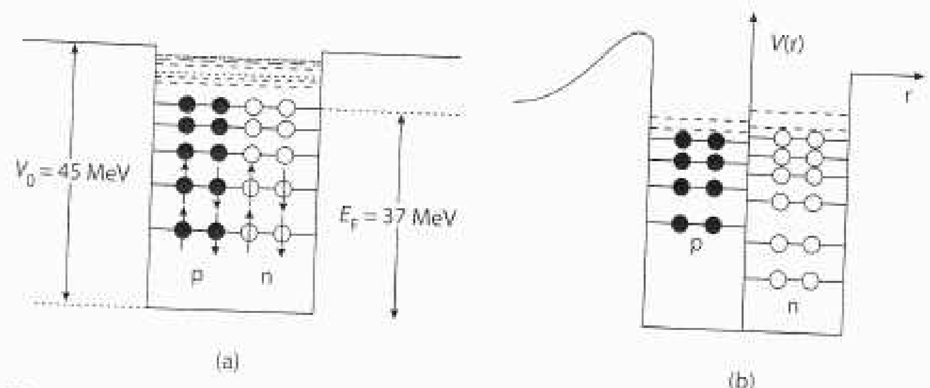

Figure 5.8 a) Potential well and states of a Fermi gas for $T = 0$. b) When one takes into account the Coulomb force. The potentials for protons and neutrons are different and we can imagine a well for each nucleon type. In two of the levels we show the spins associated to each nucleon.

Nuclear physics in a nutshell, by C. Bertoloni

Fig. 13.— Illustrative drawing of nuclear levels. The protons and neutrons each have a spin, and by quantum mechanics there can be only 2 of each in a given level. So if you have a fixed number of protons, and try to add more neutrons, you will start filling up the neutron levels and have to put the next ones into a higher energy nuclear level. Eventually the levels will have such high energy that the neutrons will not be bound and the nucleus will be unstable. (Fig. 5.8 of Bertulani's book *Nuclear Physics in a Nutshell.*)

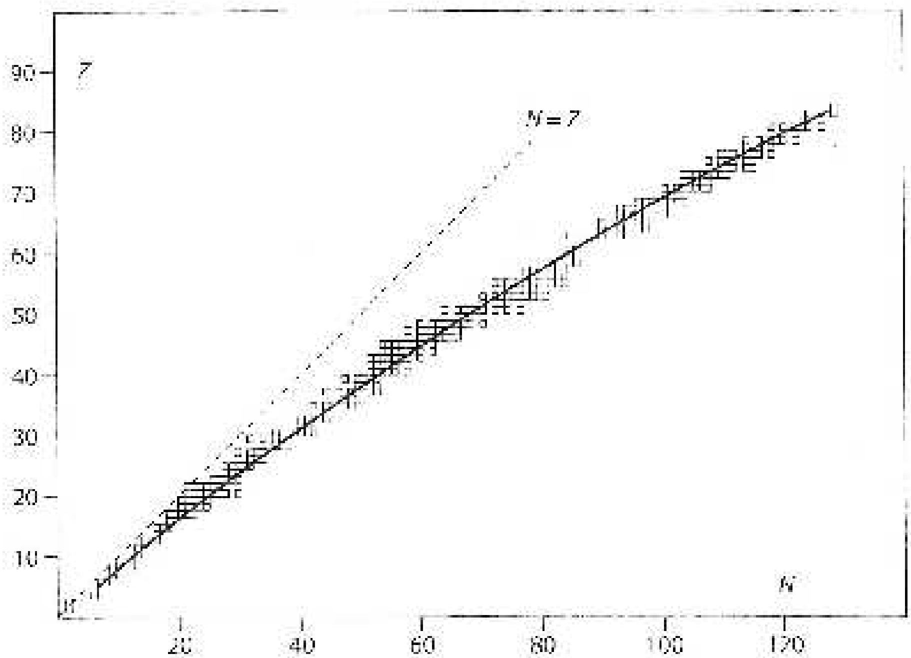

Figure 5.3 Location of the stable nuclei in the N, Z-plane. The solid line is the curve of Z_0 against $N = A - Z_0$, obtained from (5.11).

Fig. 14.— The location of the stable nuclei in the N, Z plane, often called the valley of stability. Here N is the number of neutrons, which is $A - Z_0$. The solid line is the prediction for Z_0 against N of the liquid drop model for the binding energy of nuclei. The dashed line is $N = Z$, which does not fit the data. Stability for heavy nuclei occurs when the number of protons is considerably smaller than the number of neutrons. (Fig. 5.3 of Bertulani's book *Nuclear Physics in a Nutshell*.)

9. Silicon Burning, Nuclear Statistical Equilibrium

After ^{23}C is burned to ^{16}O in the core of a massive star, the O is burned into Ne and eventually to silicon. The core contracts still further to ignite the next fuel, achieving $\rho \sim 10^8$ gm/cm^3 and $T \sim 3 \times 10^9$ K. At such a high temperature, ^{28}Si undergoes a photonuclear reaction,

$$^{28}Si \; + \; \gamma \; \rightarrow^{24} Mg \; + \; \alpha$$

$$^{24}Mg \; + \; \gamma \; \rightarrow^{20} Ne \; + \; \alpha.$$

Ultimately 7 α-particles can be produced from each of the ^{28}Si nuclei that began the above reactions. The resulting α-particles can be captured on other ^{28}Si nuclei to produce ^{32}S and higher mass nuclei. This leads to a complex reaction network involving many nuclei, n, p, and α-particles as well. The nuclei are pushed towards more massive, more tightly bound nuclei since the relevant rate equations all contain a factor $e^{(B/kT)}$, where B is the binding energy of the product of the reaction. Eventually the Fe-Ni peak is reached. There catastrophe awaits due to the fact that nuclei heavier than Fe-Ni are all less tightly bound, as shown above. Nuclear statistical equilibrium equations eventualy determine the final abundances of the various nuclei just prior to collapse; this process is sometimes called the α-rich freezeout.

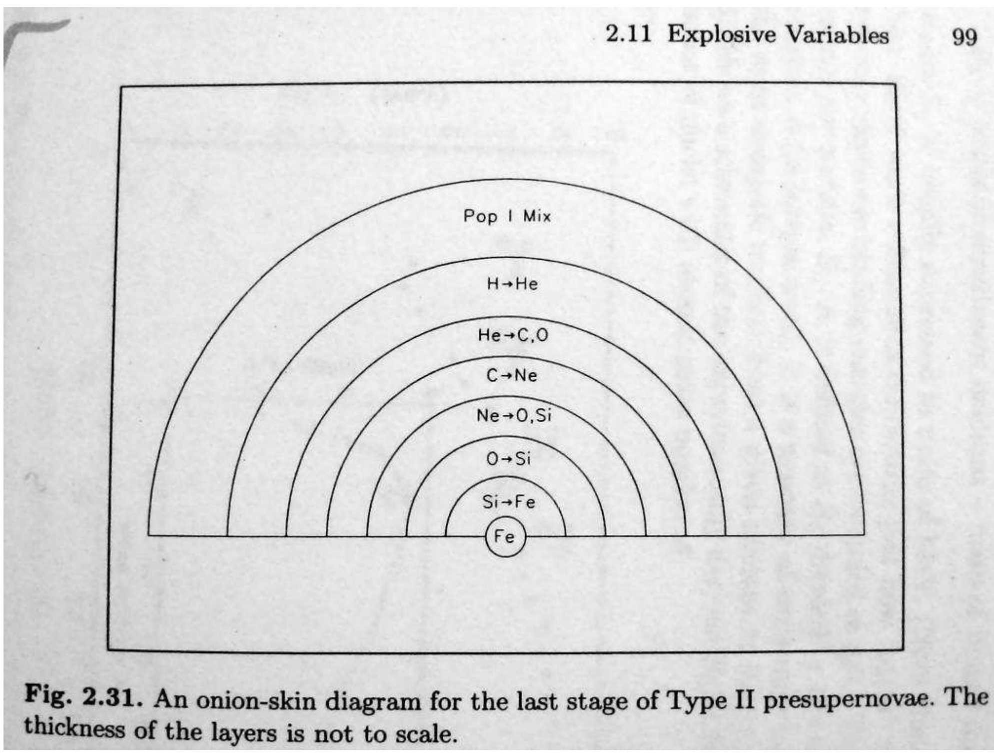

Fig. 2.31. An onion-skin diagram for the last stage of Type II presupernovae. The thickness of the layers is not to scale.

Fig. 15.— The composition of a very massive star at the end of its normal nuclear reactions after it produces a Fe core. It is like an onion, with shells of nuclear ashes of various burning processes. (Fig. 15.5 of Hansen, Kawaler & Trimble)

10. Core Collapse

After Fe and Ni dominate the core, no further compression can achieve turning on a new nuclear fuel, which would stop the compression. So the core continues collapsing until it achieves a density close to the nuclear density. All the pre-existing nuclei are reduced to α-particles and neutrons, and eventually to just p and n, a process called neutronization. The very high temperature means that the energy contained in the core is enormous, $\sim 10^{53}$ ergs. The core cools rapidly within a few seconds by emimtting neutrinos which do not interact with the stellar material before streaming out from the star. Some of the relevant reactions are:

$$p + e^- \rightarrow n + \nu_e$$

$$n + e^+ \rightarrow p + \overline{\nu_e}$$

$$\gamma \rightarrow e^- + e^+$$

$$e^- + e^+ \rightarrow \gamma + \gamma.$$

$$e^- + e^+ \rightarrow \nu_i + \overline{\nu_i}$$

The first two reactions listed above can operate as a cycle that produces neutrinos that can escape from the star and hence cool it. This is called the “URCA” process. The two photon electron, positron annihilation (the fourth reaction above) is much more likely (by more than a factor of 10^{10}) than the annihilation producing neutrinos (the fifthe reaction), but the neutrions can escape the star, and the photons can't, so the star will rapidly cool via neutrino losses. The outer layers are ejected by the collapse and bounce in a SN explosion, while the core is left as a cooling neutron star.

11. Neutron Capture Processes

Because of the decline in B/A for the elements heavier than the Fe-peak, neutron capture reactions must be used to produce them. Such reactions have no Coulomb barrier.

For neutron capture reactions, $\sigma \approx S/E$, where E represents the cross section r^2. $\sigma \propto 1/v$ for thermal energies and $\sigma \propto 1/v^2$ for keV energies. The basic reason is that lower relative velocities of the nucleus and the neutron give more time for an interaction to occur, and there is no Coulomb barrier. (See Clayton, sec 7-3, for details.) So in the stellar case, $\sigma v \approx$ constant.

Since free neutrons are very rare in stars, the abundances of elements beyond the Fe-peak are very low. In the Sun they are typically 10^{-5} times that of Fe itself, 10^{-10} times that of H.

11.1. Source of Neutrons in Stars

AGB stars are burning H in a shell. If ^{12}C from the core can be mixed back into a zone where H is not yet exhausted, then the following chain will produce neutrons:

$$^{12}C(p,\gamma)^{13}N(\beta^+,\nu)^{13}C(\alpha,n)^{16}O$$

Similarly if ^{14}N from a H burning zone where the CN cycle dominates is mixed into a He burning zone, neutrons can be produced:

$$^{14}N(\alpha,\gamma)^{18}F(\beta^+,\nu)^{18}O(\alpha,\gamma)^{22}Ne(\alpha,n)^{25}Mg$$

Thus there are two neutron sources available if contortions about moving isotopes from regions where they are produced to different parts of the star can occur. Presumably convection zones move the nuclear ashes around outside the stellar core, and this leads to production of neutrons through either ^{13}C or ^{22}Ne. The latter reaction requires a higher T, and so the former is believed to dominate in AGB stars.

If the star is too massive, core C burning occurs, and no neutrons are formed. Thus neutron production is confined to intermediate mass ($\approx$2 to 6 $M_\odot$) AGB stars.

There is also presumably a large flux of neutrons in SN, but these are produced on a very short timescale and at much higher rates (neutrons produced/cm^3/sec) than in AGB stars.

11.2. The s and r Processes

Without having to overcome the Coulomb barrier, one can synthesize heavy elements at relatively modest temperatures by exposing seed nuclei (normally Fe-peak nuclei, primarily Fe) to a flux of neutrons. Stable heavy nuclei have $N_p \approx N_n$. Nuclei have magic numbera ($N = $ 50, 82 or 126 neutrons or protons). This corresponds to filled shells of states of nuclear structure, just as the very stable noble gases correspond to filled shells of electrons in atoms.

Very heavy nuclei are stable when $N_p = N_n$. If they deviate strongly from this, the isotopes become unstable and decay. The last element with any stable isotope is Bi (immediately after Pb (lead)); all heavier elements known have no stable isotope.

The general form of the reactions is $(Z, A) + n \to (Z, A + 1) + \gamma$. Eventually, perhaps after adding several neutrons, this will produce an unstable isotope which will β-decay on timescales which range widely. Let τ_n be the timescale over which n captures occur. If $\tau_n >>$ a few hours, the s-process (slow neutron addition) prevails. If $\tau_n <<$ a few hours, the r-process prevails.

Neutron capture capture cross sections are fairly large, 0.1 to 1 barn. For $\sigma = 0.1$ barn, $T \approx 10^7$ K, then $\sigma v = 3 \times 10^{-17}$ cm^3/sec. If you want the mean time between neutron capture reactions τ_n to be 10^4 yr, then for a neutron density $n_n = 10^5$ cm^{-3}, $\sigma\ v\ n_n = 1/\tau_n = (3 \times 10^{11}\ \mathrm{sec}^{-1})$.

For the r(rapid) neutron capture process, $\tau_n < 1$ sec, $n_n \sim 10^{20}/\mathrm{cm}^3$. It is hard to get such a large flux of neutrons except in a SN.

Because the decay times for various isotopes vary widely (from fractions of a second to many years), the isotopes produced by the r and the s process differ. The s-process isotopes are the most tightly bound ones, those closest to the valley of nuclear stability which has $N_n = N_p$. (See the appended figure of the r and s process isotopic abundances in the Sun.)

Stellar spectroscopy almost always gives abundances of elements, which are the sum of those of all stable isotopes of that Z. The atomic abundances are a mixture of r and s process contributions of the various isotopes. Isotopic abundances are in general known only for the Solar wind, the Moon and cosmic rays (but the latter must be extrapolated back to the source), where direct sampling of the material enables the use of mass spectrometers.

The abundances as a function of time are those of a reaction chain in equilibrium, with seed nuclei feeding into the beginning of the chain, and very heavy nuclei emerging at the end of the chain. The neutron exposure e_n is defined as $d(e_n) = v_T n_n(t)\ dt$. The final abundance distribution of the elements produced beyond the Fe-peak will involve an integration over the reacton cross sections times the neutron exposure as a function of time.

$$\frac{dN_A}{dt} = -(\sigma v_A) N_A(t) + (\sigma v_{A-1}) N_{A-1}(t)$$

We take $< \sigma\ v >$ to be $< \sigma v(thermal) >$. Then

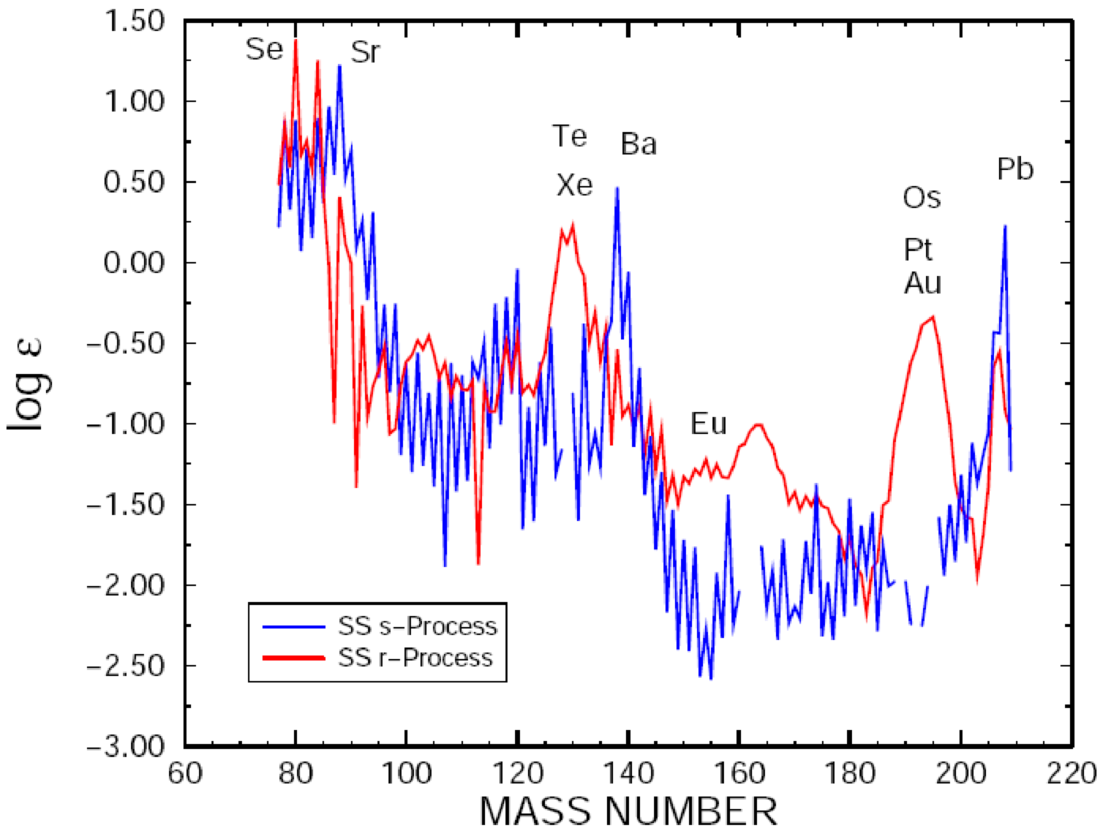

Fig. 16.— The *s* and *r* process isotopic abundances in the Sun, based on Kappeler et al (1989). These are on a scale such that $\epsilon(H) = 10^{12}/\mathrm{cm}^3$. Note the distinctive *s*-process peaks at $A \approx 88$, 138 and 208, and the *r*-process peaks at $A \approx 130$ and 195.

$$\frac{dN_A}{dt} \;=\; v_T n_n(t)[-\sigma_A(kT)\; N_A(t) + \sigma_{A-1}(kT)\; N_{A-1}(t)].$$

$$\frac{dN_A}{d(e_n)} \;=\; -\,\sigma_A(kT)\; N_A(t) + \sigma_{A-1}(kT)\; N_{A-1}(t).$$

This is a system of coupled linear equations. It presumably starts with $N_A(t = 0) = 0$ for $A \;>\; 56$, and with the ^{56}Fe (+ other Fe-peak elements, i.e. Ni etc) seed nuclei. To establish the equilibrium abundances of all isotopes inside the chain (i.e. not the seed and not the heaviest isotopes produced), we set :

$$\frac{dN_A}{d(e_n)} \;=\; 0,\text{ so that } \sigma_A N_A \;=\; \sigma_{A-1} N_{A-1} \quad \text{(in equilibrium)}$$

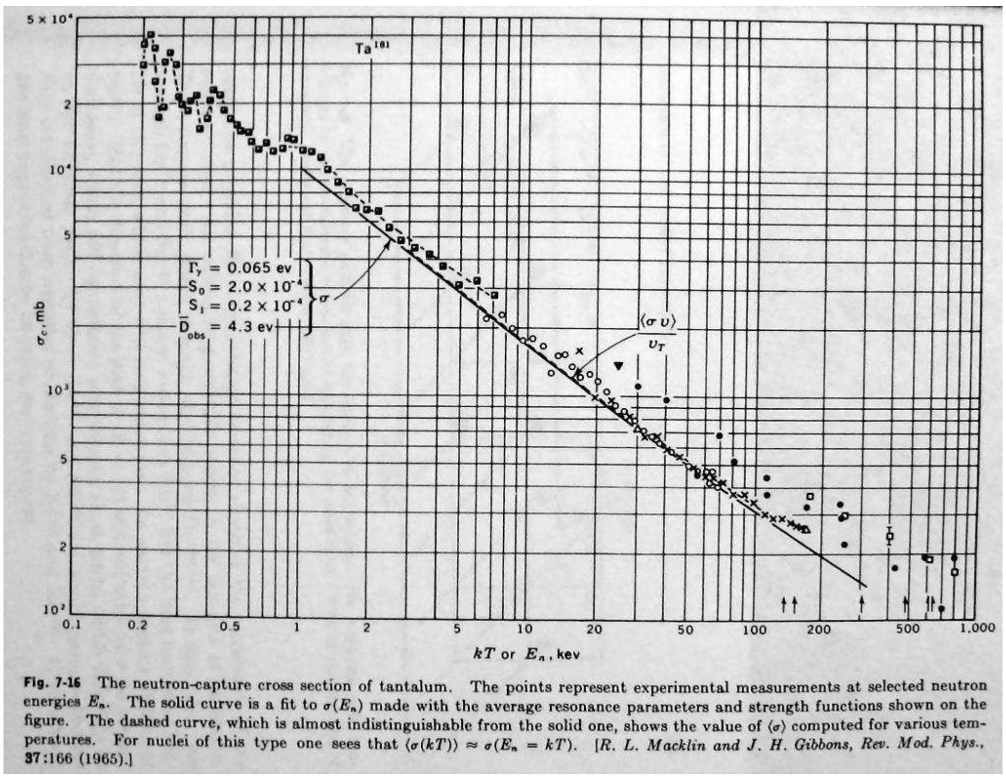

Fig. 17.— The neutron capture cross section as a function of energy for tantalum. (Figure from chapter 7 of Clayton)

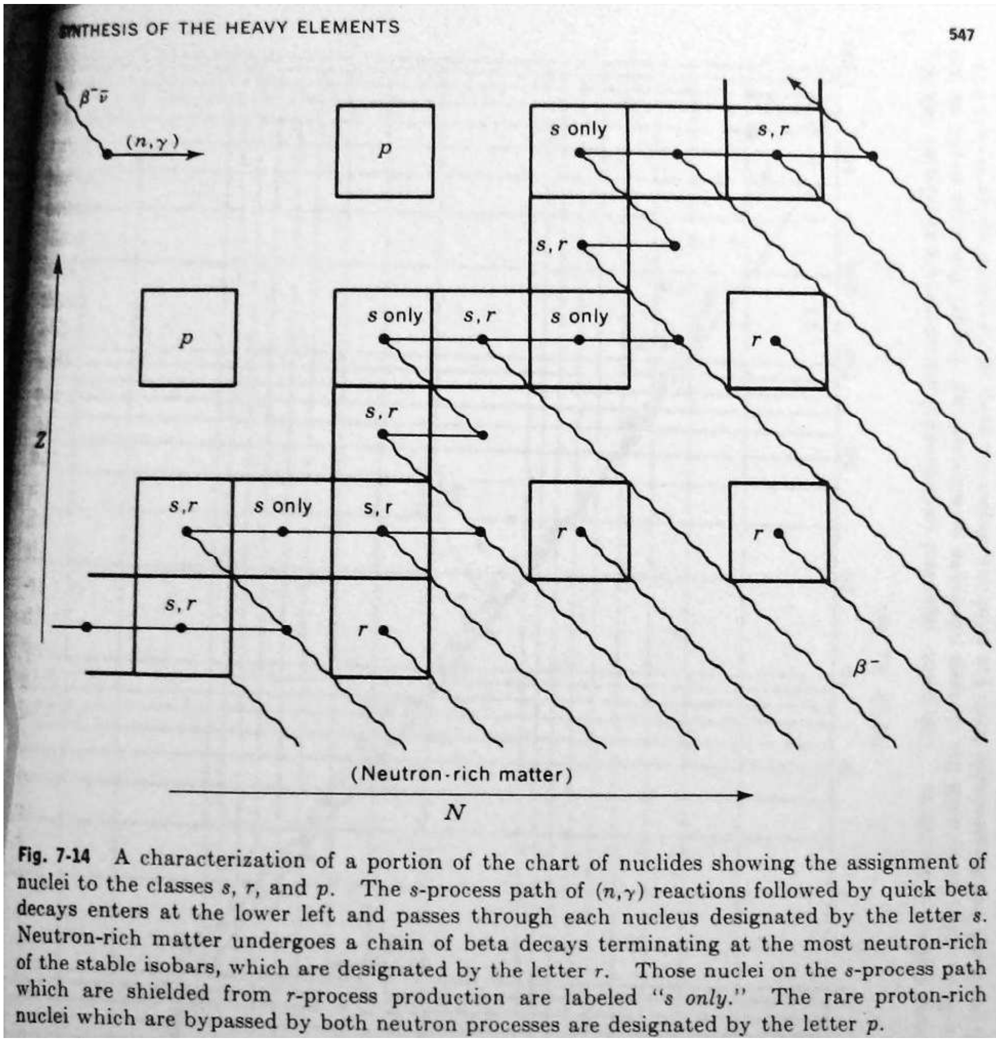

Fig. 18.— A schematic diagram of neutron number (X) versus atomic number (Y) showing the path of the s and the r process through the various isotopes. Wavy lines denote β decays. (Figure from chapter 7 of Clayton)

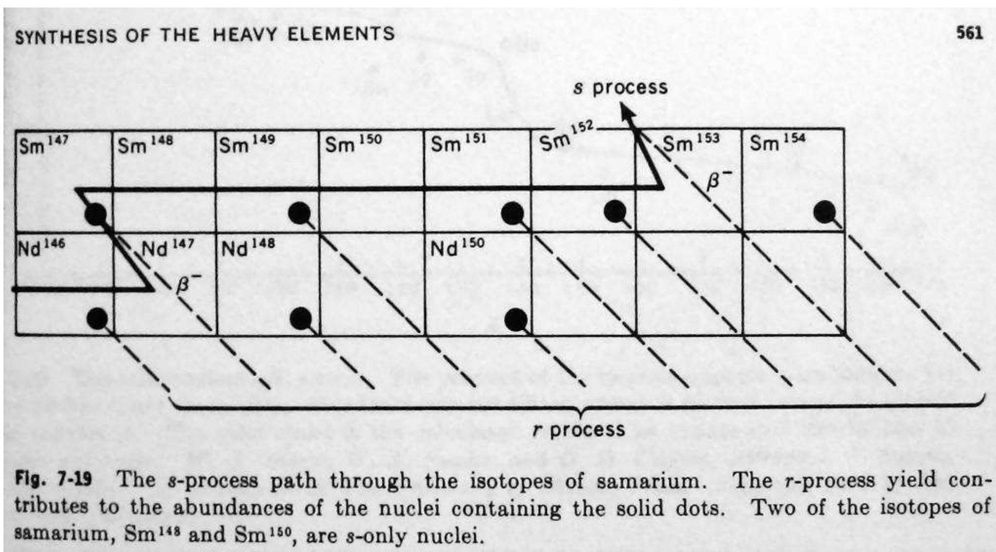

Fig. 7-19 The *s*-process path through the isotopes of samarium. The *r*-process yield contributes to the abundances of the nuclei containing the solid dots. Two of the isotopes of samarium, Sm^{148} and Sm^{150}, are *s*-only nuclei.

Fig. 19.— The s-process path through the isotopes of the rare earth samarium, with some contribution from the r-process in the case of certain specific isotopes. (Figure from chapter 7 of Clayton)

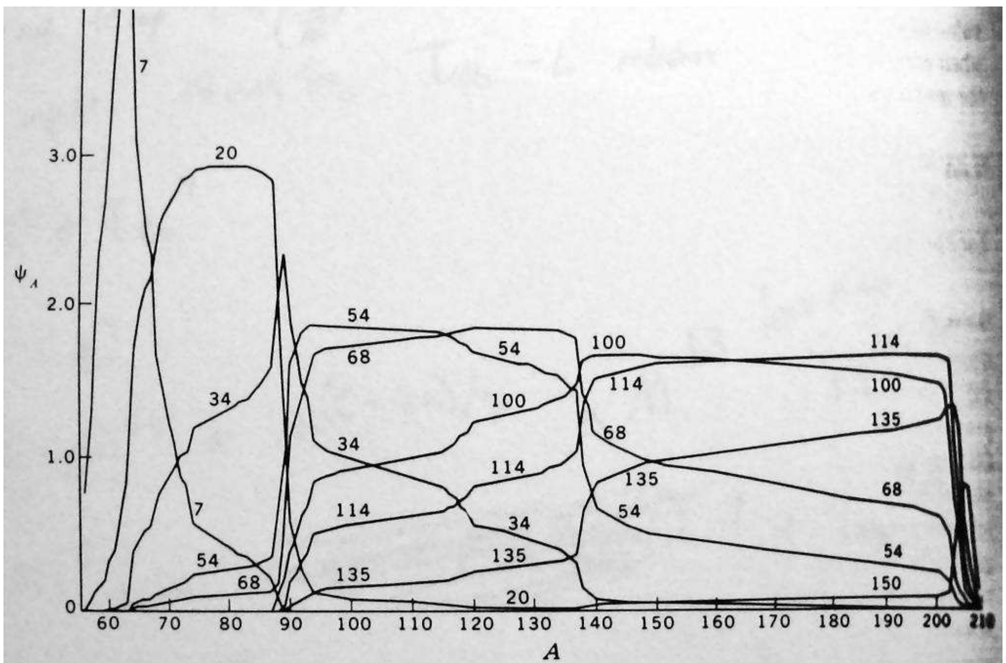

Fig. 7-22 The distributions $\psi_A = \sigma_A N_A$ for differing levels of neutron irradiation. Each curve is labeled by the parameter n_c, which is the average number of neutrons captured per initial iron seed nucleus. [*D. D. Clayton, W. A. Fowler, T. E. Hull, and B. A. Zimmerman, Ann. Phys.*, **12**:331 (1961).]

Fig. 20.— The distribution of heavy isotopes as a function of the neutron exposure. (Figure from chapter 7 of Clayton)

12. Isotopes

Nuclear reastions produce specific isotopes. Stellar measurements generally yield abundance of a particular elements, i.e. the sum of all stable isotopes.

Isotope ratios are easiest to measure among the very light elements where $\Delta(m)/m = m + 1/m$ is larger) and for very heavy elements (quirk of atomic spectroscopy).

Key isotopic ratios:

D/H 1.5 × 10−4 for the Earth, 3×10^{-5} for the Solar atmosphere, 0.8×10^{-5} for the ISM (from FUSE), 10^{-17} predicted for the center of the Sun

$^{12}C/^{13}C$ 90 (Sun, photosphere), 60 (ISM), 4 (CN cycle equilibrium)

Carbon stars, AGB stars etc. show $^{12}C/^{13}C$ 4 – 10.

Since isotopic ratios cannot be measured in stars for most elements, we must therefore look for specific element ratios that strongly indicate the nuclear process. We need to use both abundance and isotopic ratios to disintagle the r and s process contributions to the heavy elements. Examples of key abundance ratio (most common used s/r-process neutron capture is Ba/Eu or La/Eu):

Ba (mostly s)/Eu (mostly r) = 10 (pure r process), = 45 (Sun), = ∼500 (pure s)

The Solar Ba/Eu shows a mixture of r and s process neutron captures.

Isotopic ratios for Eu (2 stable isotopes, 151 and 153) can be measured (with some difficulty), but those for Ba (with several stable isotopes including 134, 135, 136, 137, 138) are almost impossible to measure.