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The Radiative Equilibrium of a Rotating System of Gaseous Masses.

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1. *Introduction and Summary.*—A rotating system of gaseous masses is said to be in mechanical equilibrium if the force of gravity, resulting from the force of attraction and the centrifugal force, is exactly balanced by the force of the total pressure, composed of the gas pressure and the radiation pressure. A system of gaseous bodies of given masses, mean densities, and angular velocity may be in mechanical equilibrium in an infinite number of different ways. Every such equilibrium is characterised by a functional relation between density and gravity-potential or, if one prefer, of a functional connection,

$$(1) \quad . \quad . \quad . \quad . \quad . \quad . \quad p=f(\mathrm{T})$$

between the gas pressure p, and the temperature T. But the function f may be arbitrary.

Through Schwarzschild's * introduction of the idea of radiative equilibrium an important step towards the solution of the problem was taken.

Schwarzschild † derived the differential equation for radiative intensity in a gaseous mass. Schwarzschild's equation fully defines the intensity of radiation, and thence also the absorption, for a given density, temperature, and absorption-coefficient of the gas.

Eddington ‡ went further and derived the differential equation for radiative equilibrium. For the difference between emission and absorption, Eddington obtained an approximate expression, not applicable very near the boundary surface of the gaseous mass, taking into account density,

* Göttingen, *Nachr. v. d. K. Ges. d. Wiss.*, 1906, p. 41; *Sitz-berichte d. K. preuss. Akad. d. Wiss.*, 28, 1914, p. 1183.

† *Loc. cit.*

‡ *M.N.*, 77, 20 (1916); *Zeitschr. f. Physik*, 7, 351 (1921).

temperature, and absorption-coefficient and their derivatives. In the state of radiative equilibrium this difference or defect is compensated by the liberation of new energy. But the liberation of energy as well as the absorption-coefficient depends physically on the nature (composition) and condition or state (density and temperature) of the gas. In radiative equilibrium, therefore, density and temperature must satisfy a partial differential equation including the production of energy, and the coefficient of absorption, as functions of the nature and condition of the gas. But the former function is at present completely unknown. For the latter, different theories give different expressions. It may therefore be said that Eddington introduced a new differential equation, but also one or two unknown functions, in the problem.

In Eddington's theory for the radiative equilibrium of stars without rotation, it is assumed that the production of energy and the absorption-coefficient are constant within the star. On the basis of this hypothesis Eddington arrives at the adiabatic law

$$(2) \qquad p = \mathrm{A} \,.\, \mathrm{T}^4,$$

where A is constant within the star. Eddington shows that even a rather large change in the production of energy within the star has only little influence on the state of equilibrium.

A. Kohlschütter,* on the other hand, assumes that the production of energy is proportional to the temperature. In his theory the functional relation (1) is more complicated.

Eddington's theory has been generalised by Milne † to rotating stars. Milne starts, like Eddington, with the assumption that the production of energy is constant within the star. This assumption, which in Eddington's theory led to the very simple adiabatic law (2), in Milne's theory no longer has such simple consequences.

In the following theory for the radiative equilibrium of a rotating system of gaseous masses, it is assumed as a *fundamental hypothesis that the nature of the gas is constant over every level surface*, but may change from one level surface to another. It is proved that the state of the gas is then constant along a level surface. Then also the production of energy, the coefficient of absorption, and the mean molecular weight are constant on a level surface. It may be remarked that our fundamental hypothesis attributes to the gaseous mass a property which will continue during its development when once existing.

A consequence of our general fundamental hypothesis is that the production of energy $4\pi\epsilon$ per second and gram within the gaseous mass follows the simple law

$$(3) \qquad 4\pi\epsilon = \mathrm{B}\left(1 - \frac{\omega^2}{2\pi \mathrm{G}\rho}\right),$$

where ω is the angular velocity, ρ the density, and G the constant of attraction, while B is constant within the gaseous mass. If in radiative

* *Publ. d. Astrophys. Obs. zu Potsdam*, Bd. 25, No. 78 (1922).
† *M.N.*, 83, 118 (1923).

equilibrium the nature of the gas is constant on a level surface, then the gaseous mass must be so arranged that its production of energy follows the law (3).

When $\omega = 0$, ϵ is constant, as in Eddington's theory. On the other hand, ϵ cannot be constant within a rotating gaseous mass in radiative equilibrium unless the density is constant also. The proof of this statement depends only on the postulate that the nature of the gas is constant on a level surface.

As a consequence of the law (3) regulating the production of energy, a general expression for the total radiation of every partial body of the rotating system is deduced.

It is also shown that the net flux of energy per second and cm.^{2} through a level surface anywhere within the partial body is proportional to the gravity. Where the boundary surface of the body has an edge or a point through which the gas can escape, the intensity of radiation is zero. This is perhaps the explanation of the dark equatorial belt in certain lenticular nebulæ.

According to modern theories of emission, the coefficient of absorption follows at least approximately the law * $k = \mathrm{K}p\mathrm{T}^{-4}$, where K is a constant. We accept a more general law

$$(4) \qquad k = k(p\mathrm{T}^{-4}),$$

where k is an arbitrary function of $p\mathrm{T}^{-4}$.

The adiabatic law (2) is proved to be a consequence of this general law of absorption and of our fundamental hypothesis that the nature of the gas is constant on a level surface. It is also shown that radiation force and gravity (which are always opposite to each other and normal to the level surface) are in constant ratio within the partial body, when a law of the form (4) is valid.

Our results apply to all rotating systems of gaseous masses in radiative equilibrium, whether they consist of one or of several distinct bodies. They may also be shell- or ring-shaped, etc. The angular velocity and the density of the gas are immaterial.

In a recently published paper † of great interest, Eddington has studied the connection between the absolute magnitudes of stars and their masses. Eddington shows that the law of perfect gases is valid in the interior of dwarf stars also. His reason for this is that the extension of the atoms is greatly decreased by ionisation. It would thus be unnecessary any longer to base the study of radiative equilibrium in ordinary dwarf stars on v. d. Waals' equation. I have, however, retained v. d. Waals' equation, in order that the theory may also be used for the study of the interior of planets and exceptionally dense dwarf stars.

2. *Intensity of Radiation.*—Let us consider an arbitrary gaseous mass not necessarily in equilibrium. An arbitrary element of the gas round the point x, y, z has energy radiating through it in all directions. Let us consider specially a vector whose axis has direction-cosines α, β, γ, with the axis of x, y, z. Through an element of surface $d\mathrm{S}$

* *M.N.*, **84**, 309 (1924). † *Ibid.*, 308 (1924).

normal to this vector, and within the angle of space $d\Sigma$, there passes per second the amount of energy

$$\text{(5)} \quad . \quad . \quad . \quad . \quad \mathrm{I}(x, y, z; \alpha, \beta, \gamma) d\mathrm{S} d\Sigma.$$

I is the intensity of radiation at the point x, y, z and in the direction α, β, γ.

The density of the gas ρ, its absolute temperature T, its coefficient of absorption k, and the intensity I are all functions of the length of path s, reckoned along the direction of the given vector.

Let us consider a small cylinder with base dS and height ds, whose axis is parallel to the vector. Through the base within a solid angle $d\Sigma$ passes an amount (5) of energy per second. Of this energy the quantity absorbed within the cylinder is

$$\text{(6)} \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{I} d\mathrm{S} d\Sigma \,.\, k\rho ds.$$

The cylinder, whose mass is $\rho d\mathrm{S} ds$, emits within the angle of space $d\Sigma$, *i.e.* through the top of the cylinder, an energy per second

$$\text{(7)} \quad . \quad . \quad . \quad . \quad . \quad \rho d\mathrm{S} ds \,.\, \mu \mathrm{T}^4 \,.\, k \,.\, d\Sigma,$$

where μ is a physical constant

$$\text{(8)} \quad . \quad . \quad . \quad . \quad . \quad \mu = 1 \cdot 685 \,.\, 10^{-5} \text{ (c, g, s)}.$$

Thus through the top of the cylinder passes a radiation per second of

$$(5) - (6) + (7).$$

But the radiation through the top of the cylinder is also

$$\left(\mathrm{I} + \frac{d\mathrm{I}}{ds} ds\right) d\mathrm{S} d\Sigma.$$

By a comparison of the last two expressions we get the differential equation for the intensity of radiation:

$$\text{(9)} \quad . \quad . \quad . \quad . \quad . \quad \frac{d\mathrm{I}}{ds} = -\mathrm{I} \,.\, k\rho + \mu \mathrm{T}^4 \,.\, k\rho.$$

This equation was first deduced by Schwarzschild.*

It is easy to integrate this equation. Introducing the so-called optical mass of the gas along the vector,

$$\text{(10)} \quad . \quad . \quad . \quad . \quad . \quad \sigma = \int_0^s k(s_1) \,.\, \rho(s_1) \,.\, ds_1$$

as a variable instead of the length of path s, we may write (9),

$$\text{(11)} \quad . \quad . \quad . \quad . \quad . \quad \frac{d\mathrm{I}}{d\sigma} + \mathrm{I} = \mu \mathrm{T}^4.$$

* Göttingen, *Nachr. v. d. K. Ges. d. Wiss.*, 1906, p. 41; *Sitz-berichte d. K. preuss. Akad. d. Wiss.*, **28**, 1914, p. 1183.

After integration we get for the intensity of radiation the integral expression

$$(12) \quad . \quad . \quad . \quad \mathrm{I}(\sigma)=\mathrm{I}_0 e^{-\sigma}+\int_0^\sigma e^{\sigma'-\sigma}\mu \mathrm{T}^4(\sigma')d\sigma'.$$

The intensity of radiation inwards at the surface (for $\sigma=0$, $s=0$) is denoted by I_0. If the system consists of a single gaseous mass limited by a single convex surface, I_0 is $=0$. But in other cases the intensity I_0 may be a function of the position of the point where the beam passes through the surface inwards as well as of the direction of the beam.

By successive partial integrations the following expression for $\mathrm{I}(\sigma)$ may be obtained from (12):

$$\mathrm{I}(\sigma)=\mu \mathrm{T}^4(\sigma)-\frac{d\mu \mathrm{T}^4(\sigma)}{d\sigma}+\frac{d^2\mu \mathrm{T}^4(\sigma)}{d\sigma^2}-\int_0^\sigma e^{\sigma'-\sigma}\frac{d^3\mu \mathrm{T}^4(\sigma')}{d\sigma'^3}d\sigma'$$
$$+e^{-\sigma}\left\{\mathrm{I}_0-\mu \mathrm{T}^4(\sigma')+\frac{d\mu \mathrm{T}^4(\sigma')}{d\sigma'}-\frac{d^2\mu \mathrm{T}^4(\sigma')}{d\sigma'^2}\right\}_{\sigma'=0}.$$

Within the mass, and for rays leaving it, the optical mass σ is in general very great. For these cases the expression in the last line is practically zero. Within the mass, except for the nearest vicinity of its surface, the first three derivatives of T^4 are generally falling very rapidly. Thus for the interior the formula of three terms

$$(13) \quad . \quad . \quad . \quad . \quad \mathrm{I}=\mu \mathrm{T}^4-\frac{d\mu \mathrm{T}^4}{d\sigma}+\frac{d^2\mu \mathrm{T}^4}{d\sigma^2}$$

is very approximately correct. It is independent of the ingoing radiation I_0. The formula (13) was first deduced by Milne.*

Finally (13) may be put into a form in which the dependence of the terms on x, y, z; α, β, γ, can be plainly seen. It follows from (10) that

$$(14) \quad . \quad . \quad . \quad \frac{d}{d\sigma}=\frac{d}{k\rho ds}=\frac{1}{k\rho}\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right).$$

3. *The Equation of Radiative Equilibrium.*—The energy which a gaseous mass in radiative equilibrium loses in radiation is replaced from within by a continuous production of energy. Only a small part of this production can be ascribed to the secular contraction of the mass. It is thought that the energy is mainly created partly by the transformation of elements, *e.g.* by the conversion of hydrogen into helium and partly by transmutation of mass into energy.

The production of energy per second and gram within the star is denoted by $4\pi\epsilon$. P represents the emission, Q the absorption per second and gram. The equation of radiative equilibrium is evidently

$$(15) \quad . \quad . \quad . \quad . \quad . \quad 4\pi\epsilon=\mathrm{P}-\mathrm{Q}.$$

Now, according to the laws of Kirchhoff and Stefan,

$$(16) \quad . \quad . \quad . \quad . \quad . \quad \mathrm{P}=4\pi k\mu \mathrm{T}^4.$$

* *M.N.*, 83, § 9, p. 122 (1923).

It is further clear that

$$(17) \quad . \quad . \quad . \quad . \quad . \quad Q = k\int I d\Sigma,$$

where $d\Sigma$ as before represents the surface element of the sphere of unit radius in the direction α, β, γ, and where the integration has to be carried out over the whole sphere. The expression (13) should be brought under the integral sign, taking into consideration the formula (14). We have

$$\int d\Sigma = 4\pi, \quad \int \alpha d\Sigma = 0, \ . \ . \ . \ ,$$

$$\int \beta\gamma d\Sigma = 0, \ . \ . \ . \ , \quad \int \alpha^2 d\Sigma = \frac{4\pi}{3}, \ . \ . \ .$$

Thus we get from (15), (16), and (17) the equation of radiative equilibrium

$$(18) \quad . \quad . \quad . \quad . \quad \epsilon = -\frac{1}{3\rho}\sum_3 \frac{d}{dx}\left(\frac{d\mu T^4}{k\rho dx}\right).$$

The sum $\sum_3$ contains three terms corresponding to x, y, z. The equation (18) was given by Eddington * in the case of spherical symmetry and by Milne † in the general case.

4. *The Net Flow of Energy.*—Let us consider a cm.^{2} whose positive normal lies in the direction of a vector s'; the direction-cosines of s' are α', β', γ'. Through this cm.^{2}, within the solid angle $d\Sigma$ (with direction-cosines α, β, γ), flows an amount of energy per second

$$I . \,|\, \alpha\alpha' + \beta\beta' + \gamma\gamma' \,|\, . \, d\Sigma.$$

If this expression is integrated over the positive half of the sphere we get the amount of energy going towards the positive side. Integration over the negative half of the sphere gives us the quantity of energy going to the negative side. The net flow $F_{s'}$ in the direction of the vector s' is now defined as the difference between the amount of energy going towards the positive side and that going towards the negative side. Thus we get

$$F_{s'} = \int I . \sum_3 \alpha\alpha' . \, d\Sigma,$$

where the integration is carried out over the whole surface of the sphere. After the expression (13) has been brought under the integral sign we get, with close approximation in the interior of the gaseous mass,

$$F_{s'} = -\frac{4\pi}{3k\rho}\sum_3 \alpha' \frac{d\mu T^4}{dx},$$

i.e.

$$(19) \quad . \quad . \quad . \quad . \quad . \quad F_{s'} = -\frac{4\pi}{3k\rho}\frac{d\mu T^4}{ds'}.$$

* *Zeitschr. f. Physik*, 7, 356, 357 (1921).
† *M.N.*, 83, 123 (1923).

5. *Radiation-Force and Radiation-Pressure.*—Let us consider an element of unit mass through which passes a beam of radiation within the solid angle $d\Sigma$ and with direction-cosines α, β, γ. Of this radiation is absorbed an amount of energy per second

$$\delta Q = I k d\Sigma.$$

The electro-magnetic theory teaches that the element of mass is then affected by a force of radiation in the direction α, β, γ, of the size

$$\frac{1}{c}\delta Q,$$

where c is the velocity of light

$$(20) \quad c = 3 \cdot 10^{10}.$$

This theoretical result has been verified experimentally by Lebedew.

Our object is to calculate the projection $S_{s'}$ in the direction of a vector s' (with direction-cosines α', β', γ') of the total radiation-force which affects the element of unit mass. Clearly

$$S_{s'} = \frac{k}{c}\int I \cdot \sum_3 \alpha\alpha' \cdot d\Sigma,$$

i.e.

$$(21) \quad S_{s'} = \frac{k}{c} F_{s'}.$$

From (19) follows the approximate expression

$$(22) \quad S_{s'} = -\frac{1}{\rho}\frac{d\frac{a}{3}T^4}{ds'}$$

where

$$(23) \quad a = \frac{4\pi\mu}{c} = 7{\cdot}060 \cdot 10^{-15}.$$

Now the radiation-pressure ϖ is defined by means of the formula

$$(24) \quad \rho S_{s'} = -\frac{d\varpi}{ds'}.$$

We get therefore approximately

$$(25) \quad \varpi = \tfrac{1}{3}aT^4.$$

The formulæ (19), (22), (24), and (25) are quite general also when the gaseous mass is not in equilibrium.

6. *The Equation of Mechanical Equilibrium.*—We now assume that the gaseous mass is in mechanical equilibrium rotating about a fixed axis with constant angular velocity. The components of the gravity are then the partial derivatives of the gravity-potential Φ, which is the sum of the attractional potential and the centrifugal potential.

Let us consider a cylindrical element of height ds in the direction of a vector s and with base dS. The cylinder is affected by the force of gravity. Its projection along the axis of the cylinder is

$$\frac{d\Phi}{ds}\rho dSds.$$

The cylinder is also affected by the force of radiation. Its projection on the axis of the cylinder is

$$S_s \rho dSds = -\frac{d\varpi}{ds}dSds.$$

On the surface of the cylinder the gas pressure p acts everywhere at right angles inwards. Only the pressure on the base and top of the cylinder gives rise to a component of force in the direction of the axis. This component is

$$-\frac{dp}{ds}dsdS.$$

In mechanical equilibrium the sum of the three components mentioned is zero. Hence the equation of mechanical equilibrium is

$$(26) \quad . \quad . \quad . \quad . \qquad d(p+\varpi)=\rho d\Phi.$$

The quantities p, ϖ, ρ, Φ are functions of the co-ordinates.

By a level surface is meant a surface along which

$$\Phi = \text{constant}.$$

It follows from (26) that $p+\varpi$ is constant on a level surface. Thus there exists a functional relation

$$(27) \quad . \quad . \quad . \quad . \qquad p+\varpi = \mathrm{P}(\Phi).$$

The equation (26) then gives

$$\rho = \frac{d(p+\varpi)}{d\Phi} = \mathrm{P}'(\Phi).$$

Thus ρ is constant on a level surface.

According to v. d. Waals' formula the gas pressure is obtained by means of the relation

$$(28) \quad . \quad . \quad . \quad . \qquad p = \frac{\mathsf{R}}{m}\frac{\rho}{1-b\rho}\mathrm{T}$$

where

$$(29) \quad . \quad . \quad . \quad . \qquad \mathsf{R} = 8{\cdot}2962 \,.\, 10^7$$

is the gas-constant and m denotes the mean molecular weight of the gas mixture. The density of the gas at greatest compression is $1/b$.

Further the relation (25) is applicable within the mass.

The gas must be considered as a mixture of free electrons and more or less ionised atoms of different kinds. In calculating the mean molecular weight the free electrons are included, and their mass may practically be neglected beside that of the atoms. In completely ionised hydrogen $m=\frac{1}{2}$; in completely ionised helium $m=\frac{4}{3}$. For other elements, on complete ionisation, m is in general near the value 2. Even when ionisation is fairly incomplete, m remains under 4. According to our fundamental hypothesis (p. 666) m and b are constant on a level surface.

The relations (27), (28), and (25) then show that T is constant on a level surface.

7. *The Production of Energy within the Gaseous Mass.*—I proceed to transform the second member of equation (18). In accordance with our fundamental hypothesis (p. 666) the coefficient of absorption k is constant on a level surface, *i.e.* k is a function of Φ.

Thus we have

$$\frac{1}{k\rho}\frac{dT^4}{dx}=\frac{1}{k\rho}\frac{dT^4}{d\Phi}\frac{d\Phi}{dx},$$

$$\frac{d}{dx}\left(\frac{1}{k\rho}\frac{dT^4}{dx}\right)=\frac{1}{k\rho}\frac{dT^4}{d\Phi}\frac{d^2\Phi}{dx^2}+\frac{d}{d\Phi}\left(\frac{1}{k\rho}\frac{dT^4}{d\Phi}\right)\left(\frac{d\Phi}{dx}\right)^2.$$

The equation of radiative equilibrium (18) now becomes

$$(30) \qquad \epsilon=-\frac{\mu}{3\rho}\left\{\frac{1}{k\rho}\frac{dT^4}{d\Phi}\Delta\Phi+\frac{d}{d\Phi}\left(\frac{1}{k\rho}\frac{dT^4}{d\Phi}\right)\left(\frac{d\Phi}{dn}\right)^2\right\},$$

since

$$\left(\frac{d\Phi}{dn}\right)^2=\sum_3\left(\frac{d\Phi}{dx}\right)^2.$$

From Poisson's equation it follows that

$$(31) \qquad \Delta\Phi=-4\pi G\rho+2\omega^2.$$

It will be shown later, in § 11, that

(32) . . . . . . $\frac{d\Phi}{dn}$ varies on the level surface within a *rotating* body in mechanical equilibrium.

The right member of (30) consists of two terms. The first is constant on the level surface. The second is variable, when it is not zero. In accordance with our fundamental hypothesis (p. 666) ϵ is constant on a level surface. Thus the second term in (30) must vanish, and our fundamental hypothesis leads to the consequence

$$\frac{d}{d\Phi}\left(\frac{1}{k\rho}\frac{dT^4}{d\Phi}\right)=0,$$

i.e. to the relation

$$(33) \qquad \rho d\Phi=C\frac{a}{3}\frac{dT^4}{k},$$

where C is constant within each partial body of the system.

By means of (30), (31), (33), and (23), we get the expression

$$(34) \quad \epsilon = \frac{cG}{C}\left(1 - \frac{\omega^2}{2\pi G\rho}\right).$$

Our general fundamental hypothesis thus leads to a very special relation between ϵ and ρ within the gaseous mass and also to the simpler equation (33) of radiative equilibrium.

In deducing the law (34) no special assumption has been made as to the quantities m, b, and k. They may be entirely arbitrary functions of the composition and state of the gas. The formula (34) is correct even if the laws of Kirchhoff, Stefan, Boyle, and v. d. Waals were not applicable within the stars.

The law (34) is reduced, if $\omega^2 = 0$, to Eddington's hypothesis—that ϵ = constant within the gaseous mass. By passing from the general case ($\omega^2 \neq 0$) to the special one ($\omega^2 = 0$) we have succeeded in finding firmer grounds for Eddington's hypothesis than was previously possible. It is to be noted that Eddington's hypothesis does not follow from our fundamental hypothesis if we are only dealing with a star without rotation. For in that case $\frac{d\Phi}{dn}$ is constant on every level surface. We cannot then proceed to the relation (33).

I will now show, assuming only that m, k, and b are constant over the level surface, *that ϵ cannot be constant within a rotating gaseous mass in radiative equilibrium unless the density is constant also.*

For let us suppose that ϵ is constant and ρ variable within the rotating gaseous mass. Then a relation of the form (34) cannot exist; and therefore a relation of the form (33) cannot exist either. It then follows from (30) and from (32), which will be proved later, that ϵ is not constant even along a level surface.

Thus Milne's hypothesis, according to which ϵ is constant within a rotating star, cannot be correct for non-homogeneous stars of constant nature along level surfaces. For such rotating gaseous masses the law (34) holds good.

The generation of energy is positive within the region where $\rho > \frac{\omega^2}{2\pi G}$. Close to the surface the formula (34) perhaps ceases to apply. In an intermediate region, where $\rho < \frac{\omega^2}{2\pi G}$, the production of energy is negative. This implies that radiative energy is transformed into potential energy or into matter. This region reaches a good way inwards even for nearly spherical stars. In a following paper it will be shown that the depth of this region with the radius as unity, is $= \cdot 0903\left(\frac{\omega^2}{2\pi G\rho_m}\right)^{1/3} = \cdot 0985$ (oblateness)$^{1/3}$. *E.g.* for the sun, considered as a perfect gas star with constant rotation, the density has the value $\left(\frac{\omega^2}{2\pi G}\right)$ at a depth of 2 per cent. of the radius.

8. *Net Flux, Surface Brightness, Effective Temperature, and Total Radiation.*—With net flux is meant the net flow of energy per second

and cm.[2] through the level surface. In accordance with (19), (23), and (33) the net flux has the expression

$$(35) \quad . \quad . \quad . \quad F = F_n = -\frac{4\pi\mu}{3k\rho}\frac{dT^4}{d\Phi}\frac{d\Phi}{dn} = \frac{c}{C}\frac{d\Phi}{dn_i}.$$

Here dn_i is positive in the direction of the inner normal of the level surface.

The formulæ collected or deduced above are very nearly true for the interior of the gaseous mass, but not too close to its limiting surface. We must assume that the production of energy in the neighbourhood of the surface is relatively insignificant compared with the total radiation. The net flux close beneath the limiting surface gives thus a good approximation to the surface brightness, *i.e.* the net flux through the limiting surface. Again the formula (35) applied to the limiting surface

$$(36) \quad . \quad . \quad . \quad . \quad . \quad \tilde{F} = \frac{c}{C}\frac{d\tilde{\Phi}}{dn_i}$$

gives, in consequence of the continuity of the gravity, an approximate expression for the net flux close under the limiting surface. Thus (36) gives an approximately correct expression of the surface brightness.

The outer layers of gas tend by their absorption, emission, and diffusion to deprive the outgoing radiation to some extent of the character of black body radiation, which it actually possesses at greater depths below the surface. Nevertheless, as a first approximation, the outgoing radiation may be said to follow Stefan's law.

T_e represents the effective temperature, *i.e.* the temperature of the black body with the surface brightness $\tilde{F}$. Then

$$(37) \quad . \quad . \quad . \quad T_e{}^4 = \frac{\tilde{F}}{\pi\mu} = \frac{4}{ac}\tilde{F} = \frac{4}{aC}\frac{d\tilde{\Phi}}{dn_i}.$$

We have not yet made any assumption as to the number of partial bodies composing the rotating system, nor of the density of the gas, nor of the angular velocity. We have only supposed that the nature of the gas is constant over the level surface. Our results have therefore a very wide extent. They apply to ordinary giant or dwarf stars. But they also apply to transition forms between single and double stars, to double and multiple stars, and to shell- and ring-formed figures which a rotating gaseous mass in radiative equilibrium can take.

The level surfaces along which Φ is constant may be divided into ordinary and singular level surfaces. On the former $\frac{d\Phi}{dn} \neq 0$; on the latter $\frac{d\Phi}{dn} = 0$, at least along certain curves or at single points. An ordinary level surface has throughout a definite tangent plane. On a singular level surface, on the other hand, edges or points appear at which $\frac{d\Phi}{dn} = 0$. Singular level surfaces are transition forms, in which the general character of the ordinary surfaces is changing.

Let us, for example, consider a singular level surface in which a series of closed surfaces is undergoing transition into other series of open surfaces. In a state of equilibrium the gas cannot then extend through the singular surface—in other words, the density must be zero, either before or at latest on the singular surface. Otherwise the gas would escape through an opening.

For another example we choose the case in which two series of closed level surfaces, each including its maximum value of Φ, coalesce in a single series of closed level surfaces, which are then at first hour-glass-shaped. The singular surface is then in one place pinched into a double point. If the gas extends past the singular surface the star has two nuclei surrounded by a common gaseous envelope. But if the gas does not reach the pinched singular surface the star is a double star.

If the limiting surface is singular the edges or points must appear dark. This is perhaps the explanation of the dark equatorial belt of certain lenticular nebulæ.

Let us return to the formula (34). After integration of $4\pi\epsilon\rho dxdydz$ over the partial body we get for its total radiation the expression

$$(38) \quad . \quad . \quad . \qquad L=\frac{4\pi c G}{C}M\left(1-\frac{\omega^2}{2\pi G\rho_m}\right),$$

in which the mean density of the partial body is denoted by ρ_m and its mass by M.

Poincaré,* neglecting radiation-force, has shown that

$$\frac{\omega^2}{2\pi G\rho_m}<1,$$

for otherwise gravity would at some place on the surface be directed outwards. Evidently this formula holds good also when radiation-force is taken into consideration, because $L>0$.

9. *Relation between Gas Pressure and Temperature. Different Assumptions as to the Coefficient of Absorption.*—In accordance with (26) and (33) and taking into account (25) we obtain the following differential equation between p and T,

$$(39) \quad . \quad . \quad . \quad . \qquad dp=\frac{a}{3}\left\{\frac{C}{k}-1\right\}dT^4.$$

We may regard k as a function of p and T. Different assumptions concerning this function give rise to different theories of radiative equilibrium.

Eddington has successively proposed several hypotheses. At first he supposed by way of trial that †

$$(40) \quad . \quad . \quad . \quad . \quad . \qquad k=\text{constant}.$$

With the notation

$$(41) \quad . \quad . \quad . \quad . \quad . \qquad C=\frac{k}{1-\beta},$$

* *Figures d'equilibre d'une masse fluide*, Paris, 1902, p. 11.
† *M.N.*, 77, 24 (1916).

where β is a constant between 0 and +1, he arrived at the functional connection

$$(42) \qquad p=\frac{a}{3}\frac{\beta}{1-\beta}\mathrm{T}^4.$$

This particular solution of (39) is characterised by the condition that $\mathrm{T}=0$ on the surface. The general solution, where $\mathrm{T}=\mathring{\mathrm{T}}$ for $p=0$, leads through (33) to the absurd consequence that $\Phi=\infty$ when $\mathrm{T}=\mathring{\mathrm{T}}$, *i.e.* on the limiting surface.

Later Eddington * supposed more in accordance with modern emission theories that $k\propto \rho m^{-1}\mathrm{T}^{-3}$ for a perfect gas. We are writing this hypothesis in the form

$$(43) \qquad k=\mathrm{K}p\mathrm{T}^{-4},$$

where K is a constant. In (39) we introduce instead of C a constant β by substituting

$$(44) \qquad \mathrm{C}=\frac{a}{3}\mathrm{K}\frac{\beta}{(1-\beta)^2}.$$

Even by the hypothesis (43) the particular solution (42) holds good. The general solution, where $\mathrm{T}\neq 0$ for $p=0$, can easily be formed; it is to be rejected because Φ must be finite on the surface. Evidently the relation (41) is still to be applied as in the first hypothesis.

From the relations (25) and (42) it follows that the gas-pressure and the radiation pressure are in constant relation to each other within the partial body.

V. d. Waals' equation (28) compared with (42) gives the relation

$$(45) \qquad \frac{\mathsf{R}}{m}\frac{\rho}{1-b\rho}=\frac{a}{3}\frac{\beta}{1-\beta}\mathrm{T}^3,$$

which may also be written

$$(45^1) \qquad \frac{1}{\rho}=b+\frac{\mathsf{R}}{m}\frac{3}{a}\frac{1-\beta}{\beta}\mathrm{T}^{-3}.$$

From (33), after integration from the limiting surface, where $\Phi=\tilde{\Phi}$, $\mathrm{T}=0$, we get

$$(46) \qquad \Phi-\tilde{\Phi}=\frac{a}{3}\frac{b}{1-\beta}\mathrm{T}^4+\frac{\mathsf{R}}{m}\frac{4}{\beta}\mathrm{T}.$$

If T is eliminated between (45) and (46) we find a relation between ρ and Φ of the form

$$(47) \qquad b\rho\frac{(1-\frac{3}{4}b\rho)^3}{(1-b\rho)^4}=\frac{ab}{192}\left(\frac{m}{\mathsf{R}}\right)^4\frac{\beta^4}{1-\beta}(\Phi-\tilde{\Phi})^3.$$

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* *Zeitschr. f. Physik*, 7, 389 (1921); *M.N.*, **84**, 309 (1924).

The above formulæ are simplified if $b\rho$ can be neglected in comparison with unity. Thus for a perfect gas

$$(48) \qquad \rho = \frac{m}{\mathsf{R}} \frac{a}{3} \frac{\beta}{1-\beta} \mathrm{T}^3,$$

$$(49) \qquad \Phi - \tilde{\Phi} = \frac{\mathsf{R}}{m} \frac{4}{\beta} \mathrm{T},$$

$$(50) \qquad \rho = \frac{a}{192} \left(\frac{m}{\mathsf{R}}\right)^4 \frac{\beta^4}{1-\beta} (\Phi - \tilde{\Phi})^3.$$

The expression (38) for the total radiation of the partial body takes different forms in the two cases (40) and (43). In the former case

$$(51) \qquad \mathrm{L} = \frac{4\pi c \mathrm{G}}{k} (1-\beta) \mathrm{M} \left(1 - \frac{\omega^2}{2\pi \mathrm{G} \rho_m}\right),$$

but in the second case we have

$$(52) \qquad \mathrm{L} = \frac{12\pi c \mathrm{G}}{a \mathrm{K}} \frac{(1-\beta)^2}{\beta} \mathrm{M} \left(1 - \frac{\omega^2}{2\pi \mathrm{G} \rho_m}\right).$$

The constant β has in either case the same meaning

$$(53) \qquad \beta = \frac{p}{p + \varpi}.$$

But β has also another meaning. Radiation force, whose direction coincides with the outer normal of the level surface, has, according to (24), (25), and (33), the expression

$$(54) \qquad \mathrm{S}_n = -\frac{1}{\rho} \frac{d\varpi}{dn} = -\frac{1}{\rho} \frac{a}{3} \frac{d\mathrm{T}^4}{d\Phi} \frac{d\Phi}{dn} = -\frac{k}{\mathrm{C}} \frac{d\Phi}{dn}.$$

Thus the radiation force is a fractional part of the gravity and acts in opposite direction. In the two cases (40) and (43) the formula (54), with regard to (41), reduces to

$$(54^1) \qquad \mathrm{S}_n = -(1-\beta) \frac{d\Phi}{dn}.$$

Hence $1 - \beta$ is the constant ratio between radiation force and gravity.

Evidently the two assumptions (40) and (43) are special cases of a more general formula

$$(55) \qquad k = k(p\mathrm{T}^{-4}),$$

where $k(x)$ is an arbitrary function. Introducing instead of C the constant β by the substitution

$$(56) \qquad \mathrm{C} = \frac{1}{1-\beta} k\left(\frac{a}{3} \frac{\beta}{1-\beta}\right)$$

we find again the relations (42), (41), (45), . . . , (50). But the total radiation then becomes because of (38) and (56)

$$(57) \quad . \quad . \quad L = 4\pi c G \frac{1-\beta}{k\left(\frac{a}{3}\frac{\beta}{1-\beta}\right)} \cdot M \cdot \left(1 - \frac{\omega^2}{2\pi G \rho_m}\right).$$

The constant β has the same meaning as before.

10. *Determination of the Gravity-Potential.*—Assuming that the coefficient of absorption k is a known function of the gas pressure p and the temperature T we get, after integrating (39), the relation between p and T. Further, the gravity-potential Φ is fixed as a function of the temperature after (33) has been integrated. Thus ρ, T and p can be expressed as functions of Φ.

If the coefficient of absorption k follows the law (40) or the law (43) or even the more general law (55) the functional relation between ρ and Φ is of the form (47) or (50) respectively.

It remains lastly to determine the gravity-potential Φ as a function of the co-ordinates. Φ is the sum of the attractional and centrifugal potentials. The latter is of the form $\frac{\omega^2}{2}s^2$, where s is the distance from the axis of rotation. Thus from the equations of Poisson and Laplace, Φ satisfies the differential equation

$$(58) \quad . \quad . \quad . \quad . \quad \Delta\Phi + 4\pi G\rho - 2\omega^2 = 0,$$

where

$$\Delta = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$$

is Laplace's differential symbol. The system may consist of several partial bodies rotating together as a rigid body about a fixed axis. Outside the partial bodies $\rho = 0$. Inside them, on the other hand, (47) or (50) are valid, the latter in the case of perfect gases. Thus we get as parameters in the first place the angular velocity ω, and in addition different values of β and $\tilde{\Phi}$ for every partial body. The differential equations of the problem thus contain $2n+1$ parameters, if the number of partial bodies is n. The required solution must be such that the function $\Phi - \frac{\omega^2}{2}s^2$ possesses the characteristic properties of the attraction potentials: this function and its derivatives of the first order must be finite and continuous through the whole of space, even at the surfaces of the partial bodies.

At present it would appear that the problem can only be solved in cases when the partial bodies are nearly spherical.

In two later papers I propose, taking as starting-point the functional relation (47), to deal with equilibrium first in a slightly oblate rotating single star or planet, and secondly for a double star system with bound rotation, in which the distance between the components is great in proportion to the mean radii of the stars. It will be seen that the gravity-potential depends in the first case ($n=1$) on three parameters, and in the second ($n=2$) on five. As fundamental parameters we may choose the masses, the mean densities, and the angular velocity of the system. The parameters β and $\tilde{\Phi}$ can be expressed as functions of these fundamental parameters. The form and extent of the system and its radiation are determined in all respects by the fundamental parameters.

11. *On the Non-existence of Rotating Equilibrium Figures with Parallel Level Surfaces.*—I will now proceed to show that the level surfaces of a rotating liquid or gaseous mass in mechanical equilibrium cannot be parallel.

For suppose that a series of such equilibrium figures exists for arbitrary values of the angular velocity ω. Let us first consider small values of ω, and let us assume that the functions and constants that appear can be expanded in series of ascending powers of ω^2.

By Φ and Ψ are denoted the inner and outer gravity-potentials of the mass. Between Φ and the density ρ there is a certain functional relation

$$(59) \quad \ldots \quad -4\pi G\rho = f(\Phi).$$

The independent variables are the radius-vector r, $u=\frac{z}{r}$, and the longitude v. Φ and Ψ satisfy the differential equations

$$(60) \quad \ldots \quad \Delta\Phi + 4\pi G\rho - 2\omega^2 = 0, \quad \Delta\Psi - 2\omega^2 = 0,$$

where

$$\Delta = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d}{du}\left[(1-u^2)\frac{d}{du}\right] + \frac{1}{r^2(1-u^2)}\frac{d^2}{dv^2}$$

is the symbol of Laplace in polar co-ordinates. The functions Φ and Ψ possess the characteristic properties of the gravity-potential. Thus Φ is finite inside, and $\Psi - \frac{\omega^2}{2}r^2(1-u^2)$ is finite outside the surface of the body. At this surface whose equation is

$$(61) \quad \ldots \quad \Phi = C$$

the conditions of continuity

$$(62) \quad \ldots \quad \Phi - \Psi = 0, \quad \dot{\Phi} - \dot{\Psi} = 0 \text{ (when } \Phi = C)$$

hold good. (Derivation with regard to r is indicated by a point.)

The parallel condition may be formulated

$$(63) \quad \ldots \quad \frac{d\Phi}{dn} = P(\Phi),$$

for if $\frac{d\Phi}{dn}$ is constant along the level surface, then $\frac{d\Phi}{dn}$ is a function of Φ.

The above functions are expanded according to powers of ω^2 in the form

$$(64) \quad \begin{cases} \Phi = \Phi_0 + \Phi_1 + \dots, \\ \Psi = \Psi_0 + \Psi_1 + \dots, \\ f(\Phi) = f_0(\Phi) + f_1(\Phi) + \dots, \\ P(\Phi) = P_0(\Phi) + P_1(\Phi) + \dots, \\ C = C_0 + C_1 + \dots \end{cases}$$

where the index shows the degree of the term in regard to ω^2.

Φ_0 and Ψ_0 are functions of r alone satisfying the ordinary differential equations

$$(65) \quad \ddot{\Phi}_0 + \frac{2}{r}\dot{\Phi}_0 = f_0(\Phi_0), \quad \ddot{\Psi}_0 + \frac{2}{r}\dot{\Psi}_0 = 0.$$

We have

$$(66) \quad \Psi_0 = \frac{GM}{r}$$

where M is the mass of the body.

The radius of the spherical equilibrium figure is denoted by $\bar{r}$. A straight line over a function signifies the value or expression of the function for $r = \bar{r}$.

We have, because of (59), $f_0(\bar{\Phi}_0) = 0$ for gases and $f_0(\bar{\Phi}_0) \neq 0$ for liquids.

According to (62)

$$(67) \quad \bar{\Phi}_0 - \bar{\Psi}_0 = 0, \quad \bar{\dot{\Phi}}_0 - \bar{\dot{\Psi}}_0 = 0,$$

and in accordance with (65) we also get

$$(68) \quad \bar{\ddot{\Phi}}_0 - \bar{\ddot{\Psi}}_0 = f_0(\bar{\Phi}_0).$$

The functions Φ_1 and Ψ_1 satisfy the differential equations

$$(69) \quad \Delta\Phi_1 = 2\omega^2 + f'_0(\Phi_0)\,.\,\Phi_1 + f_1(\Phi_0), \quad \Delta\Psi_1 = 2\omega^2.$$

(Derivation with regard to ϕ is indicated by an accent.)

From the conditions (62) we can derive certain conditions which Φ_1 and Ψ_1 must satisfy when $r = \bar{r}$. The equation (61) of the limiting surface may be written

$$\Phi_0 + \Phi_1 + \dots = C_0 + C_1 + \dots.$$

We have

$$C_0 = \bar{\Phi}_0 = \bar{\Psi}_0 = \frac{GM}{\bar{r}}.$$

From the equation of the limiting surface we get, after solution

$$(70) \quad r - \bar{r} = \frac{\bar{\Phi}_1 - C_1}{-\bar{\dot{\Phi}}_0} = \frac{\bar{r}^2}{GM}(\bar{\Phi}_1 - C_1).$$

The conditions (62) are expanded according to powers of $r-\bar{r}$. Introducing (70) we get, as a consequence of (67_2) and (68), if terms in ω^4, ω^6, . . . are omitted, the exact conditions

$$(71) \quad . \quad . \quad \bar{\Phi}_1 - \bar{\Psi}_1 = 0, \quad \bar{\dot{\Phi}}_1 - \bar{\dot{\Psi}}_1 = -f_0(\Phi_0)\frac{\bar{r}^2}{GM}(\bar{\Phi}_1 - C_1).$$

These are the looked for conditions for Φ_1 and Ψ_1.

I pass to the parallel condition (63). If ω^4 is neglected, then

$$\frac{d\Phi}{dn} = \dot{\Phi} + \ . \ . \ . \ .$$

With the same approximation (63) can therefore be written

$$\dot{\Phi}_0 + \dot{\Phi}_1 + \ . \ . \ . = P_0(\Phi_0) + P'_0(\Phi_0)\Phi_1 + P_1(\Phi_0) + \ . \ . \ . \ .$$

This equation splits up into the following exact relations

$$\begin{aligned} \dot{\Phi}_0 &= P_0(\Phi_0), \\ \dot{\Phi}_1 &= P'_0(\Phi_0)\Phi_1 + P_1(\Phi_0), \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

From the first, after derivation with regard to r, we get

$$\ddot{\Phi}_0 = P'_0(\Phi_0)\dot{\Phi}_0.$$

Using this, we get from the second, when divided by $\dot{\Phi}_0$,

$$\frac{d}{dr}\left(\frac{\Phi_1}{\dot{\Phi}_0}\right) = \frac{P_1(\Phi_0)}{\dot{\Phi}_0}.$$

After integration we get

$$(72) \quad . \quad . \quad . \quad \Phi_1 = \chi_1(r) + \dot{\Phi}_0(r) \, . \, g(u, v)$$

where

$$\chi_1(r) = \dot{\Phi}_0 \int \frac{P_1(\Phi_0)}{\dot{\Phi}_0} dr$$

is a function of r alone, while $g(u, v)$ is a function of u and v, independent of r, and at present unknown.

The function $g(u, v)$ can be expanded into spherical harmonics. Conserving only the spherical harmonic $X_2(u) = \frac{3u^2 - 1}{2}$, we get

$$\Phi_1 = A\dot{\Phi}_0 X_2(u) + \ . \ . \ .$$

where A is a constant.

This expression for Φ_1 is now introduced in the first equation (69). According to the theory of spherical harmonics we have

$$\Delta(\dot{\Phi}_0 X_2) = \left(\dddot{\Phi}_0 + \frac{2}{r}\ddot{\Phi}_0 - \frac{6}{r^2}\dot{\Phi}_0\right) X_2.$$

Further, the first of the equations (65) gives, after derivation,

$$\dddot{\Phi}_0+\frac{2}{r}\ddot{\Phi}_0-\frac{2}{r^2}\dot{\Phi}_0=f'_0(\Phi_0)\dot{\Phi}_0.$$

Thus

$$\Delta\Phi_1=\left[f_0(\Phi_0)-\frac{4}{r^2}\right]\dot{\Phi}_0 A X_2(u)+\ .\ .\ .\ .$$

Now $X_2(u)$ (as well as all the other spherical harmonics) must disappear out of (69_1). Hence it follows that

$$-\frac{4}{r^2}\dot{\Phi}_0 A=0,$$

i.e.

$$A=0.$$

Legendre's function $X_2(u)$ is thus lacking in the expansion of Φ_1 into spherical harmonics.

On the other hand, the function

$$\Psi_1-\frac{\omega^2}{3}r^2[1-X_2(u)]$$

is finite ($=0$) for $r=\infty$. Thus, regarding (69_2), the coefficient of $X_2(u)$ in the expansion of Ψ_1 in spherical harmonics is of the form

$$\left(-\frac{r^2}{3}+\frac{B}{r^3}\right)\omega^2,$$

where B is a constant.

When $r=\bar{r}$ Φ_1 and Ψ_1 must satisfy the conditions (71) for all values of u and v. The different spherical harmonics must all disappear out of these equations. If $X_2(u)$ is to disappear the constant B must satisfy the two conditions

$$\left(\frac{\bar{r}^2}{3}-\frac{B}{\bar{r}^3}\right)\omega^2=0,$$

$$\left(\frac{2\bar{r}}{3}+\frac{3B}{\bar{r}^4}\right)\omega^2=0.$$

They are inconsistent if $\omega^2\neq 0$.

Hence it follows that the gravity-potential of a slowly rotating liquid or gaseous mass in mechanical equilibrium cannot fulfil the parallel condition.

We now let the angular velocity assume higher and higher values. It is clear that the level surfaces must remain non-parallel, at least so long as the character of the equilibrium does not change essentially, *e.g.* by passing through some bifurcation form in which new forms of a different character attach themselves.

In this way the theorem (32) is proved under fairly general postulates.

The Radiative Equilibrium of a Slightly Oblate Rotating Star.
By H. v. Zeipel, Assoc.R.A.S.

1. *Introduction and Summary.*—In the previous paper (p. 665), here denoted by I, the general features of radiative equilibrium in a rotating system of gaseous masses were studied. As a fundamental hypothesis we assumed that the nature (composition) of the gas is constant along each level surface. From the further assumption that the absorption-coefficient is a function of the product pT^{-4}, where p is the gas pressure and T the temperature, a simple relation between the density and the gravity-potential was obtained [I, (47), (50)]. In that paper, as in this one, it was assumed that the gas follows v. d. Waals' equation. True, Eddington * has recently shown that the more special law of perfect gases is generally applicable, even to dwarf stars, since the size of the atoms is reduced in a high degree by ionisation. But gaseous planets, such as Jupiter, Saturn, and perhaps also the Earth (in its interior), have a relatively low temperature, and the ionisation is consequently incomplete. In the case of planets and certain dwarf stars of very great density it would therefore seem necessary to have recourse to v. d. Waals' equation.

If the system has a single boundary surface enclosing the centre of gravity the *form* of its equilibrium figure depends in general on two parameters, mean density ρ_m and angular velocity ω. But in the case of a perfect gas, the form is determined by the single parameter $\omega^2/2\pi G\rho_m$, where G is the attraction constant. It has not yet been possible to make a complete mathematical study of these equilibrium forms and their evolution series. A very important group of equilibrium forms can, however, even now be treated in detail with the help of the spherical harmonics. The characteristic of these equilibria is that the oblateness is slight.

In this paper the radiative equilibrium of a slightly oblate star of arbitrary central density is studied. The unknown quantities are expanded according to powers of the oblateness. The first two terms of the series are derived. The relation between mean density, angular velocity, and oblateness is deduced. The variation of gravity, surface brightness, and effective temperature with latitude is studied. For perfectly gaseous stars the theory is also worked out numerically.

2. *The Differential Equations of the Problem.*—Φ and Ψ represent the internal and external gravity-potentials respectively, *i.e.* the sum of the centrifugal potential and the internal or external attraction potential of the body. The differential equations

$$(1) \qquad \Delta\Phi + 4\pi G\rho - 2\omega^2 = 0, \quad \Delta\Psi - 2\omega^2 = 0,$$

follow directly from the equations of Poisson and Laplace.

Between the density ρ of the gas and the gravity-potential Φ exists a relation of the form [I, (47)]

$$(2) \qquad b\rho\frac{(1-\frac{3}{4}b\rho)^3}{(1-b\rho)^4} = \frac{ab}{3 \cdot 4^3}\left(\frac{m}{R}\right)^4 \frac{\beta^4}{1-\beta}(\Phi - \tilde{\Phi})^3.$$

* *Monthly Notices of R.A.S.*, **84**, 308.