

THERMOHALINE CONVECTION IN STELLAR INTERIORS

ROGER K. ULRICH

Department of Astronomy, University of California, Los Angeles

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ABSTRACT

A quantitative theory of mixing induced by an inverted gradient of mean molecular weight is presented. This theory is applied to three stellar problems, with the following results: (1) during ${}^3\text{He}$ burning in a $2 M_{\odot}$ star the change of X_3 between the center and surface is 0.002; (2) the μ -mechanism proposed by Stothers and Simon is too short-lived to explain the β Cephei variables, and (3) after the initial ignition of ${}^4\text{He}$ burning in a degenerate shell flash, the ${}^4\text{He}$ core and the ${}^{12}\text{C}$ shell mix on a time scale greater than 10^6 years. The theory is checked by comparison with the laboratory experiment by Stommel and Faller quoted by Stern. The agreement is satisfactory. An important uncertainty in the theory is the ratio of length to width of a moving finger of matter.

I. INTRODUCTION

The term "thermohaline convection" has been used by Stern (1960) to describe a variety of hydrodynamic instability which occurs when $d\mu/dr > 0$ and $-(d\rho/dr)_{\text{actual}} > -(d\rho/dr)_{\text{adiabatic}}$ (μ is the mean molecular weight and r is measured positive outward). Under normal conditions with $d\mu/dr \leq 0$ no instability exists when the density gradient and the temperature gradient are both stable. However, in the presence of an inverted distribution of mean molecular weight, the adverse heat fluctuation associated with a displaced cell of matter in thermally stable surroundings may diffuse out of the cell and leave a residual, favorable density fluctuation resulting from the chemical composition. The displacement of the cell may then grow at a rate which is limited either by (a) the diffusion rate of the thermal energy, (b) the viscous drag of the surrounding matter, or (c) (in the case of very small cells in a medium with low viscosity) the maximum gravitational potential energy available from the chemical gradient. The favored geometry for these cells is that of a long thin finger. Spiegel (1969) has given a brief review of this instability. The discussion by Stern has emphasized case b.

A related instability discovered by Goldreich and Schubert (1967) and extended by Fricke (1968) has been discussed frequently in the recent literature (Kippenhahn 1969; James and Kahn 1970, 1971). The aim of these papers has been similar to that of the present discussion, namely, to derive a mixing rate. This problem is essentially nonlinear and is accordingly very difficult to solve. Kippenhahn has used a blob theory to derive a mixing rate for the Goldreich-Schubert instability. The present theory is similar, although some approximations made by Kippenhahn could not be made here: the neglect of time derivatives in the momentum and thermal energy equations limits application of his results to case (a) of thermally limited modes. One of the main difficulties in both the work by Kippenhahn and the present work is to determine a maximum distance of travel for a moving blob. The procedure used by Kippenhahn is to assume that the blob dissolves as soon as it is unstable against shear-flow turbulence. Caution must be exercised in using an approach such as this, since the entire region is already unstable and it must be demonstrated that the additional instability is effective in disrupting the flow. In § V it is pointed out that shear-flow turbulence in the case of thermohaline convection can disrupt the flow only after the displacement is some small multiple of the width of the blob. An example of an astrophysical situation in which a condition of simple shear-flow instability would yield the wrong answer is the solar granulation. These granules

are vastly unstable against shear-flow turbulence and yet maintain their identity sufficiently long that the product of their velocity and lifetime is roughly equal to their diameter.

Thermohaline convection has been called upon in several astrophysical situations to induce mixing. These are: (1) during ${}^3\text{He}$ burning when the ${}^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + 2{}^1\text{H}$ reaction increases the number of particles per gram (Abrams and Iben 1970; Ulrich 1971); (2) after mass transfer in a binary system when some ${}^4\text{He}$ -rich matter is deposited on the surface of the original secondary (Stothers and Simon 1969); and (3) after the initial ignition of ${}^4\text{He}$ burning in a degenerate shell (Thomas 1967, 1970). In order to check the accuracy of the theory it is also applied to the aquatic case where empirical evidence is available. Results identical to those given by Stern are obtained, and the application of the theory is extended. Both the size and velocity of the unstable flow are in agreement with the theory.

The theoretical development starts with the basic conservation equations for a parcel of matter which arises from rest out of the ambient medium. These equations are cast in a dimensionless form which facilitates the simultaneous treatment of the aquatic and stellar problems. The small-scale, most rapidly growing modes corresponding to case *b* or *c* above are examined first. From stability arguments it is concluded in § IVc that in stars these modes cannot assume the excessively long and narrow geometry characteristic of aquatic laboratory experiments. Larger-scale modes corresponding to case *a* are shown to be more effective in transporting matter in the stellar cases. Finally, a quantitative theory based upon an assumed ratio of maximum displacement to horizontal wavelength is developed in § V and applied to stellar cases 1 and 2 in § VI.

II. FORMULATION

In order to simplify the analysis I start from the assumption that the moving matter is in the form of a relatively narrow finger and that the velocity is primarily in a vertical direction. In addition, I assume that the effect of the pressure of the surrounding matter on the momentum of the moving finger can be approximated as a virtual mass. Thus the effective inertial mass per volume is $C\rho$ instead of ρ . The actual value of C depends on the flow in the surrounding matter and is uncertain in this approximate treatment. Fortunately this uncertainty is not significant, and I have used $C = 2$ in all applications. Because of the relative narrowness of the moving region, the Laplacian ∇^2 can be approximated as ∇_h^2 , the horizontal Laplacian. I denote the unperturbed quantity and the fluctuation in a quantity by the subscripts 0 and 1, respectively.

The equations of conservation of momentum, thermal energy, and chemical composition are

$$C \frac{dw}{dt} = - \frac{GM_r}{r^2} \frac{\rho_1}{\rho} + \nu \nabla_h^2 w, \quad (1)$$

$$\frac{dT_1}{dt} = w \left[\left(\frac{dT}{dr} \right)_{\text{ad}} - \frac{dT_0}{dr} \right] + K \nabla_h^2 T_1, \quad (2)$$

$$\frac{d\mu_1}{dt} = -w \frac{d\mu_0}{dr} + D \nabla_h^2 \mu_1; \quad (3)$$

and the equation of state is

$$\frac{\rho_1}{\rho} = \left(\frac{\partial \ln \rho}{\partial T} \right)_{P, \mu} T_1 + \left(\frac{\partial \ln \rho}{\partial \mu} \right)_{P, T} \mu_1. \quad (4)$$

In these equations w is the vertical velocity, ρ the density, P the pressure, GM_r/r^2 is the local acceleration of gravity, ν is the kinematical viscosity, K is the kinematical

thermal conductivity, and D is the diffusion coefficient. These last three quantities all have dimensions of $\text{cm}^2 \text{ s}^{-1}$. The primary difference between the aquatic and stellar cases comes in the equation of state. In the aquatic case $\ln \mu$ is the salinity, whereas in the stellar case μ has the usual interpretation as the average mass per particle relative to the mass of hydrogen. We also must include radiation pressure and partial electron degeneracy in the stellar equation of state. Another important difference between the aquatic and stellar cases is the interpretation of $(dT/dr)_{\text{ad}}$. This quantity is zero in water whereas it is given by

$$\left(\frac{dT}{dr}\right)_{\text{ad}} = -\left(\frac{\partial T}{\partial P}\right)_{\text{ad}} \frac{GM_r}{r^2} \rho \tag{5}$$

in the stellar cases. For compatibility with standard notation in the two cases I shall use the following additional definitions:

Stellar cases:

$$\nabla = d \ln T_0 / d \ln P, \quad \nabla_{\text{ad}} = (\partial \ln T / \partial \ln P)_{\text{ad}},$$

$$Q = -(\partial \ln \rho / \partial \ln T)_{P, \mu}, \quad K = \frac{4acT^3}{3\rho^2 \kappa C_P};$$

Aquatic case:

$$\alpha = -(\partial \ln \rho / \partial T)_{P, \mu}, \tag{6}$$

where κ is the Rosseland mean opacity, a is the radiation density constant, c is the speed of light, and C_P is the heat capacity at constant pressure.

The simultaneous treatment of the stellar and aquatic cases is aided by transforming to dimensionless variables and coefficients. The choice of a scaling length is not obvious in the stellar case. In keeping with tradition I use the pressure scale height $Pr^2/GM_r\rho$ as the scaling length l_s . In the aquatic cases l_s is simply the depth of the region where the temperature and salinity gradients exist. A convenient scaling velocity is then the velocity attained after a free fall over a distance l_s . The various scaling factors and dimensionless coefficients are defined in Table 1. In astrophysical notation the temperature gradient β_T is $Q(\nabla_{\text{ad}} - \nabla)$. It is worth noting that in the case of a solute in water

TABLE 1
DEFINITIONS OF SCALING FACTORS AND DIMENSIONLESS QUANTITIES

Quantity	Symbol	Definition
Scaling length.....	l_s	$Pr^2/GM_r\rho$ (stellar), depth of the salt gradient (aquatic)
Scaling velocity.....	v_s	$(l_s GM_r/r^2)^{1/2}$
Scaling time.....	t_s	l_s/v_s
Scaling temperature.....	T_s	$-(\partial T/\partial \ln \rho)_{P, \mu}$
Temperature gradient.....	β_T	$l_s[dT_0/dr - (dT/dr)_{\text{ad}}]/T_s$
Salinity gradient.....	β_S	$l_s(\partial \ln \rho/\partial \mu)_{T, P}(d\mu_0/dr)$
Viscosity.....	ϕ	$\nu/l_s v_s$
Thermal conductivity.....	K	$K/l_s v_s$
Diffusion coefficient.....	D	$D_{12}/l_s v_s$
Time.....	τ	t/l_s
Vertical velocity.....	W	w/v_s
Vertical displacement.....	ξ	$\Delta r/l_s$
Temperature fluctuation.....	θ	T_1/T_s
Salinity fluctuation.....	S	$\mu_1(\partial \ln \rho/\partial \mu)_{P, T}$
Square of the horizontal wavenumber...	k^2	$-l_s^2 \nabla^2$

$\mu_0 = 1$ and S is the usual salinity in grams of solute per gram of water (after setting $[\partial \ln \rho / \partial \ln \mu]_{P,T} = 1$). Also in this case β_T is simply α times the total temperature change, and β_S is simply the total salinity change between the two layers.

Since I have already assumed a geometry of the flow in representing the effect of pressure fluctuations and in representing ∇^2 , no further loss of generality entails if $d/d\tau$ is represented by $Wd/d\xi$. This latter derivative follows the motion of a part of the salt finger and must not be confused with $\partial/\partial\tau$ which refers to gradients within the salt finger at a particular instant of time. As a further simplification in notation, I denote $d/d\xi$ with a prime.

The equations of motion in dimensionless variables are now

$$CW' = \theta/W + S/W - k^2\phi, \quad (7)$$

$$\theta' = -\beta_T - k^2 K\theta/W, \quad (8)$$

$$S' = \beta_S - k^2 DS/W. \quad (9)$$

The solution of interest is the one for which $W = \theta = S = 0$ at $\xi = 0$. This boundary condition essentially defines the quantity Δr used in Table 1. As long as all the coefficients are constant, the boundary condition assures us that W' , θ' , and S' are constants. Hence, the ratios of quantities on the right are simply the ratios of the derivatives. For this particular solution W' and k^2 are related by

$$CW'^3 + k^2(CK + CD + \phi)W'^2 + [\beta_T - \beta_S + k^4(K\phi + KD + D\phi)]W' - k^2(\beta_S K - \beta_T D) + k^6\phi KD = 0. \quad (10)$$

The dynamical equations are written in a form closely akin to the mixing-length equations. As long as the assumed geometry remains a valid description of the parcel of matter and as long as the representation of the pressure fluctuation gradients as a virtual mass is sufficiently accurate, then the dynamical equations are fully nonlinear. Mathematically the dispersion relation, equation (10), is identical with the dispersion relation obtained from a linear analysis provided the growth rate is identified with W' . However, the physical interpretation of the variables is by no means the same as in a linear analysis. The zero boundary conditions ensure that only the solution which grows exponentially with time is excited initially. In fact, since there is only one growing solution to the dispersion relation, any other solution excited by nonzero boundary conditions or by slow variations in the coefficients must decay toward the solution with zero boundary conditions. Before applying equation (10) to the problem of mixing in stellar interiors, I first discuss the numerical values of the parameters.

III. CASES STUDIED

a) Stellar Cases

The dimensionless thermal conductivity is given by

$$K = \frac{4acT^3GM_r}{3\kappa C_p r^2 (P^3 \rho)^{1/2}}. \quad (11)$$

In many cases we are interested in a radiative zone where $L_r/4\pi r^2 = (4acT^3/3\rho\kappa)dT_0/dr$, the temperature and pressure are roughly related by a polytrope of index 3 so $\nabla \approx 0.25$, and the gas is nearly perfect so that $\nabla_{ad} = 0.4$, $P \approx \rho kT/\mu m_H$, and $C_p = 2.5 k/\mu m_H$. When these assumptions are used, a convenient formula for estimating K is

$$K = 3.5 \times 10^{-11} \frac{L_r}{L_\odot} \left(\frac{r_\odot}{r}\right)^2 \left(\frac{\mu}{T_6}\right)^{3/2} \frac{1}{\rho \nabla}, \quad (12)$$

where ρ is in g cm^{-3} and T_6 , L_\odot , and r_\odot are the temperature in 10^6 ° K and the solar luminosity and radius. The viscosity has been found from the formula given by Spitzer (1962)

$$\nu = 2.2 \frac{T_6^{5/2}}{\rho \ln \Lambda} \text{ cm}^2 \text{ s}^{-1}, \quad (13)$$

where Λ is the Debye length and $\ln \Lambda$ has been taken from Table 5.1 of Spitzer's (1962) work. The diffusion coefficient is, according to Aller and Chapman (1960),

$$D_{12} = 3.2 \frac{T_6^{5/2}}{A_r^{1/2} Z^2 X_1 \rho A_1(2)} \text{ cm}^2 \text{ s}^{-1}, \quad (14)$$

where

$$A_1(2) = \ln \left(1 + 453 \frac{T_6^3 \mu_e}{Z^2 \rho} \right) \quad (15)$$

and A_r is the reduced mass of the diffusing species relative to hydrogen, Z is the charge, and X_1 is the mass fraction of hydrogen.

In degenerate matter none of the above formulae are valid. The conductivity has been taken from Hubbard and Lampe (1969), and the viscosity from Hubbard's (1966) work via the formula given by van Horn (1969) and a short table of $H_T(K_F)$ kindly provided by Hubbard (1971). There is at present no theory of diffusion in degenerate matter; however, diffusion should be less important relative to conduction and viscosity under conditions of degeneracy since ion mobility is not enhanced as is electron mobility.

The basic properties of the three stellar cases are given in Table 2. Case 1 was taken from the ^3He burning calculations previously presented by the author. The mass of $2.0 M_\odot$ is in the middle of the range where ^3He burning is important. The choice of $15 M_\odot$ for case 2 was made to obtain a lower limit to the mixing rate and still have a model possibly relevant to the β Cephei variables. The radius, luminosity, and state variables were kindly provided by Simpson (1971a) for a zero-age main-sequence model. Effects of radiation pressure were included in the state derivatives. Also, the formula for

TABLE 2
PROPERTIES OF THE STELLAR CASES

Quantity	Case 1: Envelope of a Star Burning ^3He	Case 2: Binary Star Secondary after Mass Transfer	Case 3: He Core after a Degenerate Shell He Flash
M_*/M_\odot	2.0	15.0	1.3
M_r/M_*	0.5	0.5	0.20
L_r/L_\odot	9.5	1.8×10^4	...
r/R_\odot	0.50	1.5	0.019
$T(10^6 \text{ }^\circ\text{K})$	7.2	19	35
$\rho(\text{g cm}^{-3})$	5.0	2.0	10^5
$P(10^{15} \text{ dyn cm}^{-2})$	5.0	5.6	7.5×10^5
∇	+0.25	+0.31	-0.20
∇_{ad}	0.40	0.35	0.40
$\mu m_H C_P/k$	2.5	3.8	0.90
Q	1.0	1.2	0.10
μ	0.60	0.60	1.34
$K(\text{cm}^2 \text{ s}^{-1})$	8.1×10^6	5.1×10^9	3.7×10^2
$\nu(\text{cm}^2 \text{ s}^{-1})$	15	7.6×10^3	0.10
$D_{12}(\text{cm}^2 \text{ s}^{-1})$	3.9	75	...

radiative viscosity due to Thomas (1930) was used to calculate ν for case 2. This viscosity is related to K by $\nu = KC_P T/5c^2$. Case 3 was chosen from the work by Thomas (1967) to represent the ${}^4\text{He}$ core under a ${}^{12}\text{C}$ -rich shell. The contribution of the degenerate electrons to the state variables was calculated from asymptotic formulae given by Landau and Lifshitz (1958, § 57).

The gradient of mean molecular weight for the stellar cases is somewhat arbitrary since in fact the objective of this study is the determination of the degree of mixing. In all cases I have chosen the value of β_S in such a way as to strengthen the conclusions reached. Also we must always have β_S less than β_T to prevent the occurrence of ordinary mixing. The gradient required by Stothers and Simon (1969) probably violates this condition. For case 1, I anticipate the results of § IV and choose $\beta_S \approx 10^{-4}$. In case 2, I use a gradient somewhat less than that required by Stothers and Simon (1969) to explain the β Cephei phenomenon. For case 3, I have chosen a uniform gradient of ${}^{12}\text{C}$ across the ${}^4\text{He}$ core which ranges from pure ${}^4\text{He}$ to pure ${}^{12}\text{C}$. This gives $l_s d \ln \mu_0/dr \approx 0.1$. The factor $(\partial \ln \rho / \partial \ln \mu)_{P,T}$ is roughly the same magnitude as Q since P depends primarily on ρ/μ_e and μ_e does not change when ${}^4\text{He}$ is converted to ${}^{12}\text{C}$. A more accurate value of $(\partial \ln \rho / \partial \ln \mu)_{P,T}$ is not required since the μ gradient is arbitrary. A sharp gradient in chemical composition can increase β_S locally but cannot alter the conclusions concerning general mixing of the helium core and the carbon shell.

b) The Aquatic Case

Since this case is included as a test of the theory, I have taken parameters from an experiment performed by Stommel and Faller, the results of which were published as part of the work by Stern (1960). The salt gradient was across a region about 5 cm in depth and involved a change of salinity of $3 \times 10^{-4} \text{ g cm}^{-3}$. The temperature in the upper and lower layers was 38°C and 18°C , respectively. Physical constants for water were taken from Weast and Selby (1967). The values used are $\alpha = 2.8 \times 10^{-4} \text{ deg}^{-1}$, $\nu = 1.0 \times 10^{-2} \text{ cm}^2 \text{ s}^{-1}$, $K = 1.5 \times 10^{-3} \text{ cm}^2 \text{ s}^{-1}$, $D = 1.5 \times 10^{-5} \text{ cm}^2 \text{ s}^{-1}$.

c) Dimensionless Constants

The dimensionless constants have been obtained from the physical constants given above by applying the equations of Table 1. The results are given in Tables 3 and 4. It is immediately apparent that for comparable gradients the diffusive properties of the stellar matter are much smaller. Furthermore, molecular viscosity is much less important in the stellar cases than in the aquatic case. The consequences of these differences are explored below in some detail.

IV. MOST RAPIDLY GROWING MODES

These modes arise in the laboratory first and dominate the motion. Stern (1960) has shown that the rigorous theory for the growth of the thermohaline instability yields a length scale for the salt fingers in good agreement with experiment. I derive in this

TABLE 3
SCALING FACTORS

Case	l_s (s)	l_s (cm)	v_s (cm s ⁻¹)	T_s (° K)
1.....	3.2×10^2	9.1×10^9	3.2×10^7	7.2×10^6
2.....	5.5×10^2	3.1×10^{10}	5.3×10^7	1.6×10^7
3.....	4.4	3.8×10^8	8.5×10^7	3.5×10^8
4.....	0.1	5	71	3.5×10^3

TABLE 4
DIMENSIONLESS PARAMETERS

Case	β_T	β_S	ϕ	K	D
1.....	0.15	10^{-4}	5.2×10^{-17}	2.8×10^{-11}	1.3×10^{-17}
2.....	0.05	0.01	4.6×10^{-16}	3.1×10^{-9}	4.6×10^{-17}
3.....	0.06	0.01	3.0×10^{-18}	1.1×10^{-14}	...
4.....	5.6×10^{-3}	3.0×10^{-4}	2.8×10^{-5}	4.2×10^{-6}	4.2×10^{-8}

section a formula identical to Stern's result in order to demonstrate that the present less rigorous theory gives correct results for the case in which empirical evidence is available. I then extend Stern's result to show that the velocity of the salt fingers may also be explained by the theory. I designate these modes the viscosity-limited modes.

a) Length Scale

The most rapidly growing modes are those which satisfy the condition that W' be a maximum with respect to variations in k^2 :

$$(CK + CD + \phi)W_m'^2 + 2k_m^2(K\phi + KD + D\phi)W_m' - \beta_S K + \beta_T D + 3k_m^4 \phi KD = 0. \quad (16)$$

The subscript m denotes the value at maximum. Subtraction of equation (16) from equation (10) gives

$$CW_m'^3 + [\beta_T - \beta_S - k_m^4(K\phi + KD + D\phi)]W_m' - 2k_m^6 \phi KD = 0. \quad (17)$$

In all cases the terms involving D may be dropped in equation (17). Equations (7)–(9) show that $C(W')^2 \leq \beta_S$. To the desired accuracy we may neglect β_S relative to β_T and obtain

$$k_m^2 = (\beta_T / K\phi)^{1/2}. \quad (18)$$

Recall that k^2 is the total horizontal wavenumber. For a roughly cylindrical disturbance the wavenumber in any direction k_x^2 say, is roughly $0.5 k^2$. Hence the dimensional wavelength λ in an arbitrary direction in centimeters is

$$\lambda = 2^{3/2} \pi \left\{ \frac{GM_r}{r^2 \nu K} \left(\frac{\partial \ln \rho}{\partial T} \right)_{P, \mu} \left[\frac{dT_0}{dr} - \left(\frac{dT}{dr} \right)_{ad} \right] \right\}^{-1/4}. \quad (19)$$

This formula is identical to that given by Stern and in good agreement with experiment. Values of k_m and λ are given in Table 5.

TABLE 5
DYNAMICAL PROPERTIES OF THE MODES

Case	k_m	λ_m (cm)	δ	$(W')_m$	$(\Delta r)_{\text{laminar}}$ (cm)	η (cm ² s ⁻¹)	r^2/η (10 ⁶ yrs)
1.....	3.2×10^6	2.5×10^4	7.1×10^2	7.1×10^{-3}	2.9×10^3	3.6×10^6	11
2.....	2.4×10^6	1.1×10^6	2.7×10^5	7.1×10^{-2}	6.5×10^3	8.4×10^{11}	4.1×10^{-4}
3.....	3.7×10^7	91	1.2×10^3	7.1×10^{-2}	8.0	4.8×10^4	1.2
4.....	82	0.54	5.6×10^{-2}	6.3×10^{-4}	2.4×10^2

b) *Velocity*

The maximum rate of growth of the velocity may be found by inserting k_m from equation (18) into equation (16). The positive root is

$$W_m' = \frac{(\beta_T K \phi)^{1/2}}{CK + \phi} \{ [1 + (1 + CK/\phi)(\beta_s/\beta_T - D/K)]^{1/2} - 1 \}. \quad (20)$$

When

$$\delta \equiv (1 + CK/\phi)(\beta_s/\beta_T - D/K) \gg 1$$

and D is small, the value of W_m' is $(\beta_s/C)^{1/2}$. Conversely when δ is small, the square root may be expanded with the result

$$W_m' = \frac{\beta_s - \beta_T D/K}{2k_m^2 \phi}. \quad (21)$$

Values of W' and δ are given in Table 5 for $C = 2$. The aquatic and stellar cases clearly represent different limiting classes of motion.

Since this study aims at determining the rate of mixing induced by thermohaline convection, a check on the theoretical velocity is crucial. Such a check was not made in detail by Stern. When the salt finger is just leaving the region of the medium where the gradients exist, the value of ξ is unity. At this point the velocity is $v = v_s W_m'$. For case 4 above, $v = 4.5 \times 10^{-2} \text{ cm s}^{-1}$. Note that $v \propto l_s^{1/2}$ for constant temperature and salinity differences. Stommel and Faller observe $v \approx 1 \times 10^{-2} \text{ cm s}^{-1}$, so there seems to be a discrepancy. In order to understand this discrepancy we must examine the behavior of the matter in the region where both gradients are zero. Upon setting β_T and β_s equal to zero in equation (7)–(9), we see that S and θ go to zero with e -folding times of $k^2 K$ and $k^2 D$, respectively, while W approaches $(\theta + S)/k^2 \phi$ with an e -folding time of $k^2 \phi$. According to Tables 3–5 the e -folding times for velocity, temperature, and salinity are 0.53, 3.5, and 350 s, respectively. In 3.5 s the salt finger moves only 0.15 cm, so the motion during the transition is not observable. After the transition to the terminal velocity, we may set $W = S/k^2 \phi$. Hence the salinity obeys the equation $S' = -k^4 D \phi$. The salt finger should come to a stop after it has traveled a distance $\Delta \xi = S'_m/k^4 D \phi$. From equation (9) and Tables 3–5, I find $S'_m = 2.1 \times 10^{-4}$ and $\Delta \xi = 4.0$ for the present case. The total possible distance of travel is then roughly 25 cm. It is likely that the observed velocity was the result of the components of sea water which diffuse more slowly than NaCl. Additional experiments would be desirable to check the present theory more completely.

It is worth noting briefly that the distribution of velocity with wavenumber is much flatter about the maximum when δ is large than it is when δ is small. As a consequence we would not expect a single length scale to be dominant in the stellar cases near k_m as in the aquatic case.

c) *Stability of the Flow*

The stability of any laminar flow against the onset of internal turbulence is governed by the Reynolds number $\mathcal{R} = v\lambda/4\nu$. In a free flow such as thermohaline convection the critical Reynolds number is about 30. Since the velocity grows linearly with Δr , we may use stability considerations to set an upper limit to Δr . This statement of course implies that $\Delta r \gg \lambda$ since any unstable cell should be able to move a distance comparable to its own diameter regardless of its internal Reynolds number. The limiting distance is then

$$(\Delta r)_{\text{laminar}} = \mathcal{R}_{\text{crit}} l_s \frac{2^{1/2} k_m \phi}{\pi W_m'}. \quad (22)$$

Values of $(\Delta r)_{\text{laminar}}$ computed from equation (22) and Tables 3–5 are given in Table 5 for $\mathcal{R}_{\text{crit}} = 30$. In fact, it is immediately clear that none of these numbers is meaningful. We saw above how diffusion limits Δr to much less than 240 cm in the aquatic case, and in the stellar cases the Reynolds number predicts that the flow breaks up almost before it starts. Once we admit the possibility of a flow which is not laminar, then there is no particular reason to select the modes that grow most rapidly as being the ones responsible for mixing.

V. THERMALLY LIMITED MODES

The thermally limited modes are those whose horizontal scale is so large that the growth of the velocity is limited by the rate at which energy may diffuse out of the matter. Under these conditions we may put $\phi = D = 0$. According to equation (7) the minimal condition for the continued motion of such a mode is $S = -\theta +$ (a small quantity). Equation (8) then gives

$$W' = \frac{k^2 K \beta_s}{\beta_T - \beta_s}. \quad (23)$$

This equation may be obtained from equation (10) by equating the terms in $(W')^0$ and $(W')^1$. As k^2 increases, the term in $(W')^2$ becomes more important and the ratio W'/k^2 decreases.

Denote the average of a quantity by angular brackets. The diffusion coefficient which results from thermohaline convection is $\langle v \Delta r \rangle$. In order to determine this average I assume that the unstable region is filled with matter which has moved various distances Δr and that all values of Δr are equally probable between $\pm (\Delta r)_{\text{max}}$. I further assume that $(\Delta r)_{\text{max}}$ is proportional to λ : $(\Delta r)_{\text{max}} = \alpha \lambda = 2^{3/2} \pi \alpha l_s / k$. The parameter α plays a role similar to the ratio of mixing length to scale height in the theory of thermal convection. The combination of this assumption and the fact that W'/k^2 decreases as k increases leads to the conclusion that the largest modes are most effective in the diffusion process. Since $v = v_s W' \Delta r / l_s$, the diffusion coefficient is

$$\eta = v_s l_s 8 \pi^2 \alpha^2 K \beta_s / 3 (\beta_T - \beta_s). \quad (24)$$

The values of η for cases 1–3 computed from equation (24) and Tables 3 and 4 are given in Table 4 for $\alpha = 5$. An estimate of the length of time required to mix the star is given in the last column of Table 4.

The assumption in the previous paragraph of a typical ratio of maximum displacement to wavelength introduces some uncertainty into the present theory. This assumption may be justified by noting that the shear flow between the center and the edge of the salt finger is unstable in all the astrophysical cases. The shear instability is of a different nature from the thermohaline instability and must lead to some turbulence within an individual salt finger. Lateral velocities associated with this internal turbulence may disrupt the finger when it becomes excessively long and narrow. Kippenhahn (1969) has used a model of this type in his analysis of the Goldreich and Schubert (1967) angular-momentum instability. In the case of thermohaline convection, the turbulent motions associated with the shear flow grow at a rate the same order as the horizontal velocity gradient. Thus if this internal turbulence disrupts the flow, it will do so when the displacement of the salt finger is a relatively small multiple of the width. Alternatively, the internal turbulence might not actually disrupt the salt finger but might merely enhance the molecular diffusion coefficient D . In accordance with this model the diffusion coefficient can be written as $D = \epsilon W \lambda / 4 l_s$, where $\lambda/4$ is the radius of the finger. The salt finger can then maintain its identity even for very large ratios of displacement to width. However, the enhanced value of D requires that the salt finger eventually reach

a steady terminal state in which diffusion exactly balances the buildup of S by the chemical gradient: $S = 2^{1/2} \beta_s (\pi \epsilon k)$. The minimal condition, $\theta = -S$, for the continued motion of thermally limited modes then gives $W = (2^{1/2} k K \beta_s) / (\pi \epsilon \beta_T)$. If the unstable region is filled with such salt fingers, then the flux in a radial direction of the species giving rise to S is just WS . A diffusion coefficient η may then be defined as the coefficient which relates WS to β_s . This coefficient is identical to that given in equation (24) provided $\alpha = 3^{1/2} / (2\pi^2 \epsilon)$. If the typical length and velocity of the internal turbulence is one-tenth the salt-finger radius and velocity, then $\epsilon \approx 0.01$ and $\alpha \approx 9$. In this model, diffusion sets an effective maximum displacement for the chemical perturbation even though the salt finger itself may be arbitrarily long.

A cautionary note concerning the applicability of equation (24) is warranted. Before using this equation, the Reynolds number of the fastest growing mode at $\Delta r = \alpha \lambda$ should be checked. If this number is substantially less than 30, then the viscosity-limited modes may be more important to the mixing process than the thermally limited modes. In such a case the diffusion coefficient is given by

$$\eta = \frac{2\Re_{\text{crit}}^2}{3\pi^2} l_s v_s \frac{\beta_T^{1/2} \phi^{3/2}}{W_m' K^{1/2}}, \quad (25)$$

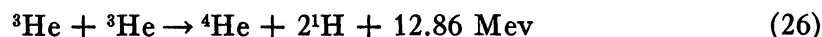
where W_m' is given by equation (20). Note that when δ is large, $\eta \propto \nu^{3/2}$, whereas when δ is small, $\eta \propto \nu^2$. In either case the fact that an increase in viscosity enhances diffusion is contrary to what one might expect. The explanation for this result is simply that viscosity does not limit the velocity significantly but is important in stabilizing the flow. In all cases below, equation (24) rather than equation (25) is used.

VI. APPLICATIONS

a) ${}^3\text{He}$ Burning (Case 1)

The entry for case 1 in the last column of Table 4 indicates that the mixing time is comparable to the time scale for depletion of ${}^3\text{He}$. Consequently, a detailed discussion of this case is necessary. I assume that the initial ${}^3\text{He}$ abundance is large and ask the question: What is the difference in ${}^3\text{He}$ abundance between the center and the surface of the star required to allow a uniform rate of depletion throughout?

Just prior to the arrival of a pre-main-sequence star on the ${}^1\text{H}$ burning main sequence, the reaction



can temporarily halt gravitational contraction. Furthermore, this reaction converts two particles into three and thus decreases the mean molecular weight. After the maximum gradient of X_3 , the mass fraction of ${}^3\text{He}$, has been achieved, the thermohaline-convection mechanism discussed above permits X_3 to decrease at a uniform rate throughout the star. Thus,

$$\left(\frac{\partial X_3}{\partial t}\right)_{\text{actual}} = \left(\frac{\partial X_3}{\partial t}\right)_{\text{nuc}} + \left(\frac{\partial X_3}{\partial t}\right)_{\text{diff}} = \frac{L_*}{QM_*} \quad (27)$$

where Q is the energy released per unit mass of ${}^3\text{He}$ which undergoes reaction (26); its value is 2.07×10^{18} ergs per g ${}^3\text{He}$ consumed. Also, L_* and M_* are the total luminosity and mass.

In order to discuss the diffusion process quantitatively, I define a flux ψ_3 which is the mass of ${}^3\text{He}$ flowing inward across a sphere of radius r per unit time. This flux obeys the transport and conservation relations:

$$\text{Transport:} \quad \psi_3 = 4\pi r^2 \rho \eta \frac{\partial X_3}{\partial r}; \quad (28)$$

Conservation:
$$\left(\frac{\partial \psi_3}{\partial r}\right) = 4\pi r^2 \rho \left(\frac{\partial X_3}{\partial t}\right)_{\text{diff}}. \quad (29)$$

(Note that no minus signs appear because ψ_3 and r are measured in opposite directions.) Upon combining equations (27) and (29) and integrating from 0 to \bar{M}_r , I find

$$\psi_3 = \frac{L_r}{Q} - \frac{L_* \bar{M}_r}{Q \bar{M}_*}. \quad (30)$$

In order to determine the value of $\partial X_3 / \partial r$ required to maintain ψ_3 in accordance with equation (30), write η as

$$\eta = \frac{8\pi^2 \alpha^2}{3} \frac{4acT^3}{3\rho\kappa} \frac{1}{\rho C_P (\nabla_{\text{ad}} - \nabla)} \left(-\frac{\partial \ln \mu}{\partial \ln P}\right). \quad (31)$$

The gradient of μ is related to the gradient in X_3 by

$$\frac{\partial \ln \mu}{\partial r} = \frac{1}{6} \mu \frac{\partial X_3}{\partial r}. \quad (32)$$

Also, $4acT^3/3\rho\kappa$ is given by

$$\frac{4acT^3}{3\rho\kappa} = \frac{L_r \rho}{(\partial \bar{M}_r / \partial r)(-\partial T / \partial r)}. \quad (33)$$

Combining equations (28) and (31)–(33), I find

$$\psi_3 = \frac{4\alpha^2 \pi^2}{9} \frac{L_r \mu}{C_P T (\nabla_{\text{ad}} / \nabla - 1)} \left(\frac{\partial X_3}{\partial \ln T}\right)^2. \quad (34)$$

The gradient of X_3 could of course be expressed in terms of r instead of $\ln T$, but the resulting formula would be more cumbersome. Equation (34) is somewhat unusual in that the flux depends on the square of the gradient. The cause is simply that the diffusive motions are caused by the gradient in the quantity which is diffusing.

Upon combining equations (30) and (34), I find that the change in X_3 across the model is

$$\Delta X_3 = \frac{3}{2\pi\alpha Q^{1/2}} \int_{T_{\text{surf}}}^{T_{\text{cent}}} \left[\frac{C_P}{\mu T} \left(\frac{\nabla_{\text{ad}}}{\nabla} - 1\right) \left(1 - \frac{L_* \bar{M}_r}{L_r \bar{M}_*}\right) \right]^{1/2} dT. \quad (35)$$

During ${}^3\text{He}$ burning, the gas is perfect so C_P is $\frac{5}{2} k/\mu m_{\text{H}}$. The numerical value of ΔX_3 is then, for $\alpha = 5$,

$$\Delta X_3 = 9.0 \times 10^{-4} \int_{T_{\text{surf}}}^{T_{\text{cent}}} \left[\frac{T_6}{\mu^2} \left(\frac{\nabla_{\text{ad}}}{\nabla} - 1\right) \left(1 - \frac{L_* \bar{M}_r}{L_r \bar{M}_*}\right) \right]^{1/2} d \ln T. \quad (36)$$

The value of the integral is roughly constant with the total mass of the model and varies between 1.5 for stars of mass $\sim 1 M_{\odot}$ to 2.0 for stars of mass $2 M_{\odot}$. Thus stars with an initial mass fraction of ${}^3\text{He}$ greater than about 0.002 will be completely mixed. This required initial abundance is inversely proportional to α , the ratio of maximum distance of travel to horizontal wavelength. Since inertial terms play no role in governing the development of the instability, the flow may be relatively orderly even though it is not stabilized by viscosity. Hence, I believe my value of $\alpha = 5$ is reasonable and may in fact be a considerable underestimate of the actual ratio. The value quoted here of 0.002 for ΔX_3 is greater than that of 0.00075 quoted in my earlier work partly because

of this choice here of $M = 2.0 M_{\odot}$ instead of $1.0 M_{\odot}$ and partly because I had used $\alpha = 10$ in the earlier estimate.

b) Accretion of ${}^4\text{He}$ -rich Matter (Case 2)

The mechanism proposed by Stothers and Simon (1969) to account for the β Cephei variables involves a ${}^4\text{He}$ -rich layer on the surface of a high-mass star. The associated increase in μ was needed to explain the observed periods of pulsation. In order to obtain the ${}^4\text{He}$ -rich layer these authors invoked a binary hypothesis. They also showed that pulsation is possible in these stars by the β mechanism, but felt that the μ mechanism accounted for the observations more satisfactorily. The theoretical discussion in the previous section showed that an inverted μ gradient can be expected to survive only 400 years after mass transfer is ended. Since the total main-sequence lifetime for a $15 M_{\odot}$ star is 10^7 years (Iben 1966; Simpson 1971b), only 4×10^{-3} percent of all B stars which have accreted a substantial layer of ${}^4\text{He}$ -rich matter as a result of mass transfer could be influenced by the μ mechanism. In fact, van Hoof (1965) has pointed out that between 5 and 10 percent of *all* B stars in the correct spectral range are β Cephei variables. Clearly the hypothesis of rapid mass transfer must be rejected as being responsible for a ${}^4\text{He}$ -rich layer on the surface of all the β Cephei variables. Also note that the mixing time decreases as the mass increases since $K \propto L_*$. Additional difficulties with the binary hypothesis in this context have been discussed by Plavec (1971).

A related problem of perhaps more general interest involves the determination of the distribution of ${}^4\text{He}$ within the mass-accreting component of a close binary during mass transfer. Very often the mass transferred has undergone some hydrogen burning and consequently is rich in ${}^4\text{He}$. There are two limiting cases of this process: (1) mass transfer occurs very rapidly for a short time interval, and (2) mass transfer occurs slowly over an extended period of time. In the first case the diffusion of the ${}^4\text{He}$ -rich matter into the star must be followed in detail numerically by using equations analogous to equations (28), (29) and (31), since the initial distribution of chemical composition is important. In the second case a unique distribution of ${}^4\text{He}$ can be found which depends only on the rate of mass transfer, since a steady-state flow may be achieved in the mass-gaining star. The choice between these two treatments can be made by comparing the rate of mass transfer to the mixing rate obtained from the theory of § V.

The quantitative theory of slow mass accretion of ${}^4\text{He}$ -rich matter proceeds in a manner analogous to the derivation of equation (35) above. The flux of ${}^4\text{He}$ is defined to be ψ_4 . For a steady state to exist in the distribution of ${}^4\text{He}$, the local rate of change of X_4 exclusive of any nuclear transformations must obey

$$\partial X_4 / \partial t = \psi_4 / M_r \quad (37)$$

and

$$\partial X_4 / \partial t = [(X_4)_{ac} - (X_4)_*](\partial M / \partial t)_{ac} / M_*, \quad (38)$$

where $(X_4)_{ac}$ and $(\partial M / \partial t)_{ac}$ are respectively the ${}^4\text{He}$ mass fraction in the accreted matter and the rate of accretion. The sensitivity of μ to the conversion of ${}^1\text{H}$ to ${}^4\text{He}$ gives $d \ln \mu / dX_4 = 5\mu/4$. The distribution of X_4 is then given by

$$\left| \frac{\partial X_4}{\partial T} \right| = \left\{ \frac{0.3[(X_4)_{ac} - (X_4)_*](\partial M / \partial t)_{ac} C_P M_r (\nabla_{ad} / \nabla - 1)}{\pi^2 \alpha^2 M_* \mu L_r T} \right\}^{1/2}. \quad (39)$$

For the parameters of case 2, the total change of X_4 across the star is roughly $0.0019 [(X_4)_{ac} - (X_4)_*]^{1/2} (\partial M / \partial t)_{ac}^{1/2} / M_*^{1/2}$, where time is measured in 10^6 years.

It is possible that the present theory of thermohaline convection is not applicable to these stars if rapid rotation is induced by the mass accretion as suggested by Huang (1966), Plavec (1970), and van den Heuvel (1970). The importance of the rotation problem is not clear, however, since the thermally limited modes are not excessively thin.

c) Degenerate Cores (Case 3)

The time scale for mixing by thermohaline convection given in Table 5 is longer than the evolutionary time scale for the appropriate models. Consequently, it is not possible to draw any conclusions about the influence of the inverted μ gradient on the course of evolution without reference to a specific evolutionary sequence. It is important to note that the diffusion coefficient η is proportional to the conductivity K . As the density increases, K and η increase while the size of the region decreases. Hence it is possible that mixing could occur at a later stage of evolution.

VII. CONCLUSIONS

The important conclusions of this study are:

1. Thermally limited modes dominate mixing by thermohaline convection in stellar interiors.
2. An inverted gradient of mean molecular weight is too short-lived to play an important role in the β Cephei pulsations.
3. Mixing between the central ${}^4\text{He}$ region and the ${}^{12}\text{C}$ shell after a degenerate ${}^4\text{He}$ shell flash takes longer than 10^6 years.

The last conclusion may have interesting consequences since it suggests that a central helium core may persist until near the end of the life of a low-mass star. If this ${}^4\text{He}$ -rich region is present during a subsequent thermal runaway in the central regions, the course of evolution could be significantly different from what one would expect with a pure ${}^{12}\text{C}$ core.

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