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Circulation and turbulence in rotating stars

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Abstract. We examine the interaction between meridian circulation and turbulence in rotating, non-magnetic stars. That turbulence is assumed to be anisotropic, with stronger transport in the horizontal directions than in the vertical, thereby enforcing a rotation rate which depends only on depth, to first approximation. This conjecture is supported by the interior rotation of the Sun, which is now being revealed through acoustic sounding. We calculate the meridian flow and derive the partial differential equation which governs the transport of angular momentum. This equation allows for asymptotic regimes that are briefly described. The main result is that both the meridian circulation and the turbulence are determined by the loss of angular momentum, which we ascribe here solely to a stellar wind. When there is no wind, the meridian flow is very weak, and it can even vanish in slow rotators. When the wind is active, it drives the circulation in order to transport angular momentum to the surface, but the advection of chemical species by this flow is partly inhibited because of the horizontal turbulence. Salient properties of the low-mass main-sequence stars are explained in a coherent way: *i*) the helium settling and subsequent abundance anomalies in A and early F stars, below a certain rotation speed; *ii*) the observed depletion of lithium, and its strong correlation with the angular momentum loss. We can therefore conclude that, since the old population II stars are slow rotators, they do not display at their surface the original abundance of lithium. It is also anticipated that the Sun has kept a rapidly spinning core, although this needs to be confirmed by actually integrating the evolution equations. All these results are rather insensitive to the assumptions made about the turbulent transport.

Key words: turbulence; Sun: interior, rotation – stars: abundances, interior, rotation

1. Introduction

It was Eddington (1926) who pointed out that some mixing must occur in stellar radiation zones, in order to prevent the gravitational settling and the thermal diffusion of heavier elements, which is not observed as a rule. He suggested that this mixing was mediated through the meridian circulation caused by the rotation of the star, which had just been discovered (Eddington 1925; Vogt 1925), and this still remains the broadly accepted explanation.

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In its standard formulation, which was shaped by Sweet (1950) and Mestel (1953), the meridian flow is calculated assuming that the star is rotating uniformly. This was done mostly for simplicity, although Mestel emphasized from the outset that a weak magnetic field suffices to keep the rotation nearly uniform. But in the absence of such a field, and this is the case we shall consider here, the flow advects angular momentum, and thereby induces differential rotation. For this reason, other rotation laws have also been examined, notably by Baker and Kippenhahn (1959), who pointed out that uniform rotation is not a typical case, when it comes to estimate the flow speed in the upper part of the star.

It also became clear that the rotation state must ultimately result from the balance between meridian advection and turbulent stresses, since viscosity is far too weak to act on the large scale flows. This was recognized, more or less explicitely, by Randers (1941), Osaki (1972), Zahn (1974, 1975), Sakurai (1975), Kippenhahn and Thomas (1981), Tassoul and Tassoul (1982), but the lack of knowledge about the turbulent transport prevented them from going much beyond.

Turbulent motions are unavoidable in the nearly inviscid stellar material, due to various instabilities that are produced by differential rotation. A very natural candidate is the familiar shear instability (Spiegel & Zahn 1970; Zahn 1974, 1975). In particular, nothing can prevent a horizontal shear from becoming turbulent at such low viscosity, although the instability may not be a linear one, but rather of finite amplitude (Drazin & Reid 1981); for recent contributions to this problem, see Lerner and Knobloch (1988) and Dubrulle and Zahn (1991). Other instabilities certainly occur as well, such as the GSF instability (Goldreich & Schubert 1967; Fricke 1968), or similar baroclinic instabilities (Knobloch & Spruit 1982, 1983; Zahn 1983).

To our frustration, little is known about the properties of the turbulent motions which are then sustained, except that they are presumably more vigorous in the horizontal directions than in the vertical, as illustrated by the so-called geostrophic turbulence observed on Earth, in the oceans and in the atmosphere (see Rhines 1979). Such highly anisotropic turbulence appears to be present beneath the solar convection zone (Spiegel & Zahn 1992); we shall examine this piece of evidence below in §2.5. Other constraints, both on the meridian flow and on the supected turbulent mixing, are provided by the existence, or the absence, of chemical abundance anomalies.

Various prescriptions have been proposed for the turbulent transport, often with adjustable parameters (G. Vauclair 1976; Schatzman 1977; Zahn 1983; S. Vauclair 1988; Pinsonneault et al.

1989), and some fared rather well, considering the approximations made. But they do not make up for a coherent theory, based on a minimum of plausible assumptions. Moreover, one can only feel uncomfortable with the way meridian circulation has been treated in recent work: it was either neglected as such (Schatzman & Maeder 1981; Baglin et al. 1985; Lebreton & Maeder 1987; S. Vauclair 1988; Tassoul & Tassoul 1989), or modeled as a diffusive process (Pinsonneault et al. 1989, 1990), or calculated with a stipulated, uniform rotation law (Charbonneau & Michaud 1988). Contradictions abound, and some who were among the first to advocate rotation-induced turbulence as the main mixing mechanism in radiative interiors, confess that they are losing faith (Schatzman 1991a).

We are in a different mood. We feel that the moment has come to gather all the information which is available, and to draw a self-consistent picture of the transport of chemicals and angular momentum in stellar interiors. By self-consistency we mean that neither the rotation rate nor the turbulent transport will be postulated, as was done previously; instead, they will be let to adjust interactively. Neither will a magnetic field be invoked to achieve the balance in the azimuthal direction. *Our only conjecture, throughout this paper, will be that turbulence is anisotropic, with stronger horizontal transport than vertically.*

In §2, we examine the properties of such turbulence. The most striking is the erosion of horizontal inhomogeneities, which partly inhibits the advection of chemical species by the meridian flow. It tends also to establish a rotation rate which depends little on latitude. Assuming that such a shellular rotation state has been achieved throughout the radiative zone, we derive in §3 an explicit expression for the meridian velocity. It differs from the classical treatments so far by the inclusion of baroclinic terms, which involve up to third-order derivatives of the angular velocity, with respect to depth. The reader who is only mildly interested in technical details may skip these two sections, and head directly to §4, where we examine the properties of the partial differential equation which governs the transport of angular momentum. We describe the asymptotic states that are reached when the circulation timescale is short enough. These regimes are shown in §5 to explain the salient properties of low-mass main-sequence stars, namely the observed depletion of lithium, and the abundance anomalies in A and early F stars, below a certain rotation speed. In §6 we verify how sensitive our results are to the assumptions made about the turbulence. We conclude by summarizing the main results and by stressing the determining role of the angular momentum loss.

2. Properties of the rotation-induced turbulence

In this section, we shall assume that rotation-induced turbulence actually occurs in stellar radiation zones, with motions more vigorous in the horizontal than in the vertical directions, and we set forth to describe the consequences of such turbulence. The likelihood of this conjecture will be discussed later in §6. We describe the properties of this anisotropic turbulence, discuss the proof of its existence in the Sun, and make up for our scanty knowledge by some crude, but we hope reasonable parametrization of the turbulent transport.

Within this section, we are allowed to neglect the oblateness of the star, and the horizontal surfaces, which we also call level surfaces, will be represented by concentric spheres. Throughout this article, we use the spherical coordinate system centered on the rotation axis.

2.1. Transport of matter

A crucial point has been recently clarified, namely the interference of turbulence with the large-scale meridian flow. Horizontal turbulent motions of sufficient strength work against the advection of chemicals by the meridian flow, because they tend to homogenize horizontal layers. The vertical transport of matter then behaves as a diffusion process, as it has been established by Chaboyer & Zahn (1992) (hereafter referred to as CZ). This turbulent erosion is very similar to the shear dispersion observed in pipe and channel flows, which was explained by Taylor (1953).

The effective diffusion coefficient may be expressed in terms of the meridional velocity and of the horizontal diffusivity D_h, which is assumed to be constant on a level surface. When the rotation rate depends solely on depth, the vertical component of the circulation velocity involves only the Legendre function of order 2:

$$u(r,\theta) = U(r)\, P_2(\cos\theta)\,, \tag{2.1}$$

r being the radial coordinate and θ the colatitude. We shall assume that this constitutes still a reasonable approximation in the more general case. Then the horizontal variation δc of the concentration of such a chemical is given by

$$\delta c(r,\theta) = -\frac{r^2}{6\,D_h}\frac{\mathrm{d}\bar{c}}{\mathrm{d}r}\, U(r)\, P_2(\cos\theta)\,, \tag{2.2}$$

and the mean concentration $\bar{c}(r)$ diffuses vertically with a total diffusivity

$$D_t = D_v + D_{\text{eff}} = D_v + \frac{|rU(r)|^2}{30\,D_h}\,. \tag{2.3}$$

These expressions are valid when the horizontal diffusivity exceeds the vertical diffusivity: $D_h \gg D_v$. Moreover, that horizontal diffusivity must dominate the advective transport, thus $D_h \gg |rU|$; this ensures that the horizontal variations $\delta c/\bar{c}$ remain small, as is assumed in the derivation of (2.3). Charbonneau (1992) has since performed extensive computer simulations which confirm these results, and which show that the requirement put on the horizontal diffusivity is rather mild: the one-dimensionalization of the chemical composition is achieved to a good degree as soon as $D_h/|rU| \gtrsim 1$. The rotation rate displays a similar behavior, with a tendency to shellular rotation when this condition is satisfied for the horizontal viscosity. This is the problem we examine next.

2.2. Transport of angular momentum

As a first approximation, we shall assume that the effect of the turbulent stresses on the large scale flow are adequately described by an anisotropic eddy viscosity, whose components are ν_v and ν_h respectively in the vertical and horizontal directions.

The transport of angular momentum is then governed by an advection/diffusion equation

$$\frac{\partial}{\partial t}[\rho r^2 \sin^2\theta\, \Omega] + \nabla \cdot [\rho r^2 \sin^2\theta\, \Omega\, \boldsymbol{u}]$$
$$= \frac{\sin^2\theta}{r^2}\frac{\partial}{\partial r}\left[\rho \nu_v r^4 \frac{\partial\Omega}{\partial r}\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\rho\nu_h \sin^3\theta \frac{\partial\Omega}{\partial\theta}\right]\,, \tag{2.4}$$

where $\Omega(r,\theta)$ is the angular velocity.

On each level surface, which can be approximated here by a sphere of radius r, we split the angular velocity in its mean and latitudinal (zonal) parts

$$\Omega(r,\theta) = \overline{\Omega}(r) + \widehat{\Omega}(r,\theta) \quad \text{with} \quad \overline{\Omega}(r) = \frac{\int \Omega \sin^3\theta \, d\theta}{\int \sin^3\theta \, d\theta} . \tag{2.5}$$

We assume that the horizontal component of the turbulent viscosity is large enough so that the departures from spherical rotation may be neglected to lowest order:

$$|\widehat{\Omega}| \ll \overline{\Omega} ; \tag{2.6}$$

we shall use this approximation in all derivations which follow, leaving for later on to verify under what conditions it is fulfilled.

With such shellular rotation, the meridian circulation is given by (2.1), as we shall see in §3. After horizontal averaging of (2.4), one finds that the vertical transport of angular momentum is governed by the following equation (Chaboyer & Zahn 1992)

$$\frac{\partial}{\partial t}[\rho r^2 \overline{\Omega}] = \frac{1}{5r^2}\frac{\partial}{\partial r}\left[\rho r^4 \overline{\Omega} U\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left[\rho \nu_v r^4 \frac{\partial \overline{\Omega}}{\partial r}\right] . \tag{2.7}$$

The two terms on the right are the divergence of respectively the advected flux of angular momentum and of the viscous flux. As was pointed out by CZ, the transport of angular momentum is little affected by the horizontal diffusion, since it subsists when Ω is constant on a level surface. It will therefore prove much more efficient than the transport of matter, in the presence of this anisotropic turbulence.

2.3. Zonal flow

To establish an expression for the horizontal variation $\widehat{\Omega}$ of the rotation, we follow the same procedure as CZ when they estimated the horizontal fluctuation of the concentration of chemical species. We multiply (2.7) by $\sin^2\theta$ and subtract it from the original equation (2.4) to get

$$\begin{aligned} &\frac{\partial}{\partial t}[\rho r^2 \sin^2\theta \widehat{\Omega}] + \nabla \cdot [\rho r^2 \sin^2\theta \overline{\Omega} \boldsymbol{u}] + \frac{\sin^2\theta}{5r^2}\frac{\partial}{\partial r}\left[\rho r^4 \overline{\Omega} U\right] \\ &= \frac{\sin^2\theta}{r^2}\frac{\partial}{\partial r}\left[\rho \nu_v r^4 \frac{\partial \widehat{\Omega}}{\partial r}\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\rho \nu_h \sin^3\theta \frac{\partial \widehat{\Omega}}{\partial \theta}\right] . \end{aligned} \tag{2.8}$$

When horizontal diffusion operates faster than the meridian advection, i.e. when $\nu_h \gtrsim |rU|$, a stationary state is achieved after a time of order r^2/ν_h, which for $\nu_h \gg \nu_v$ obeys

$$\begin{aligned} &\frac{1}{r^2}\frac{d}{dr}\left(\rho r^4 \overline{\Omega} U\right)\left[\sin^2\theta\left(P_2(\cos\theta) + \frac{1}{5}\right)\right] \\ &+ \frac{1}{6}\overline{\Omega}\frac{d}{dr}(\rho r^2 U)\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin^3\theta \frac{dP_2}{d\theta}\right] \\ &= \rho\nu_h \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin^3\theta \frac{\partial\widehat{\Omega}}{\partial\theta}\right] . \end{aligned} \tag{2.9}$$

The assumption $|\widehat{\Omega}| \ll \overline{\Omega}$ has simplified the advective terms in (2.8), and the equation of continuity has been used to make explicit the horizontal component $u_\theta = V(r)\, dP_2(\theta)/d\theta$ of the meridian circulation:

$$\frac{1}{r}\frac{d}{dr}[\rho r^2 U] - 6\rho V = 0 . \tag{2.10}$$

This equation (2.9) is readily solved to yield the following expression for the zonal rotation rate

$$\widehat{\Omega}(r,\theta) = \Omega_2(r)\, P_2(\cos\theta) , \tag{2.11a}$$

with

$$\begin{aligned} \nu_h \Omega_2(r) &= -\frac{1}{10\rho r^2}\frac{d}{dr}\left[\rho r^4 \overline{\Omega}(r)\, U(r)\right] + \frac{\overline{\Omega}(r)}{6\rho}\frac{d}{dr}\left[\rho r^2 U(r)\right] \\ &= \frac{1}{5}\overline{\Omega}(r)\, rU(r)\left[\frac{1}{3}\frac{d\ln \rho r^2 U}{d\ln r} - \frac{1}{2}\frac{d\ln r^2\overline{\Omega}}{d\ln r}\right] \\ &= \frac{1}{5}\overline{\Omega}(r)\, r\, [2V - \alpha U] , \end{aligned} \tag{2.11b}$$

where we have introduced

$$\alpha = \frac{1}{2}\frac{d\ln r^2\overline{\Omega}}{d\ln r} \tag{2.12}$$

for brevity ($\alpha = 1$ for uniform rotation). This expression for $\widehat{\Omega}$ is similar to (2.2), but it involves also the horizontal component of the meridian flow. From now on, since there is no longer ambiguity, we shall omit the overbar from the mean rotation rate, and designate it just by $\Omega(r)$.

2.4. The vertical component of the turbulent viscosity

We now come to grips with what still remains the most delicate point of the theory, namely to derive a plausible estimate for the turbulent viscosity, which we assumed from the outset to be highly anisotropic. We take the view that, among all competitors, shear instabilities are the most likely to dominate, because they proceed on a dynamical time-scale. In that case the vertical component of the turbulent viscosity is due either to the horizontal shear we have just examined, or to the vertical shear. We shall consider separately the two contributions.

2.4.1. Turbulence produced by the vertical shear

Instabilities from the vertical variation of the horizontal velocity are strongly hindered by the stable stratification, unless there is sufficient radiative smoothing. This effect has been discussed by Townsend (1958), who modified the original Richardson criterion to account for radiative leakage. In an optically thick medium, it is the Péclet number $v\ell/K$ which decides whether an eddy of velocity v and size ℓ behaves adiabatically or not (K is the thermal diffusivity). There, the stratification is unable to prevent the instability when (Dudis 1974; Zahn 1974)

$$\frac{N^2}{(dV/dz)^2}\frac{v\ell}{K} < Ri_c, \quad \text{with} \quad N^2 = \frac{g}{H_P}(\nabla_{ad} - \nabla) , \tag{2.13}$$

where V is the amplitude of the horizontal flow, z the vertical coordinate, g the gravity, H_P the pressure scale-height and N the buoyancy frequency; $Ri_c \lesssim 1/4$ is the critical Richardson number. In the present case, the shear rate (dV/dz) is $(r \sin\theta\, d\Omega/dr)$.

The eddy viscosity in the vertical direction is dominated by the largest eddies that satisfy this condition; therefore

$$\nu^{\dagger}_{v,v}(r,\theta) = \frac{1}{3}v\ell \approx \frac{Ri_c}{3}K\left(\frac{r\sin\theta}{N}\frac{d\Omega}{dr}\right)^2 . \tag{2.14}$$

We see that the turbulent viscosity is not constant in latitude, as was implicitely assumed when averaging (2.4) to obtain (2.7).

But eq. (2.7) is still valid, if we define the viscosity $\nu_{v,v}$ as the weighted average

$$\nu_{v,v} = \frac{\int \nu^{\dagger}_{v,v} \sin^3\theta \, d\theta}{\int \sin^3\theta \, d\theta} \, ; \tag{2.15}$$

the result is

$$\nu_{v,v} = \frac{8 Ri_c}{45} K \left(\frac{r}{N} \frac{d\Omega}{dr} \right)^2 . \tag{2.16}$$

Note that this instability only occurs if $3\nu_{v,v} > \nu Re_c$, with Re_c being the critical Reynolds number of that shearing flow. Moreover, it is suppressed by a gradient of molecular weight, when $g\,|d\ln\mu/dr| > Ri_c\,|d\ln\Omega/d\ln r|^2$.

2.4.2. Turbulence produced by the horizontal shear

With our neglect of the vertical component of the turbulent viscosity in (2.9), the viscous dissipation in the zonal shear reduces to

$$\varepsilon_t(r,\theta) = \nu_h \left[\sin\theta \frac{\partial \widehat{\Omega}}{\partial \theta} \right]^2 , \tag{2.17}$$

per unit mass and unit time. Since ν_h represents here the effect of the turbulent stresses on the mean zonal flow, this ε_t can also be interpreted as the local energy injection rate into the turbulent motions.

If we assume that all this energy is dissipated by viscous friction within the zonal strip in which it is injected, we may infer from that dissipation rate an estimate of the vertical component of the turbulent viscosity, in the same way as was done earlier (Zahn 1983, 1987). In our anisotropic turbulence, the motions are much more vigorous in the horizontal than in the vertical direction. Among all eddies of velocities and scales $[v, \ell]$, only the smallest, for which the turn-over rate v/ℓ exceeds the vertical component $\Omega\cos\theta$ of the rotation vector, will behave as in isotropic turbulence, because they are not dominated by the Coriolis force. They will therefore obey Kolmogorov's law, which in physical space takes the form $v^3/\ell = \mathrm{cst}\cdot\varepsilon_t$, and cascade their kinetic energy to the small scales where it can be dissipated through viscous friction.

These nearly isotropic eddies will be responsible for the transport of momentum in the vertical direction, which can be expressed by a turbulent viscosity $\nu^{\dagger}_{v,h} = v\ell/3$, with v and ℓ characterizing the largest of them. Thus from

$$\frac{v}{\ell} = c_1 \, 2\Omega\cos\theta \quad \text{and} \quad \frac{v^3}{\ell} = c_2\,\varepsilon_t \tag{2.18}$$

we derive an estimate for $\nu^{\dagger}_{v,h}(r,\theta)$:

$$\nu^{\dagger}_{v,h} = \frac{1}{3} v\ell = c_4 \frac{\varepsilon_t}{(\Omega\cos\theta)^2} = c_4\,\nu_h \left(\frac{\tan\theta}{\Omega} \frac{\partial\widehat{\Omega}}{\partial\theta} \right)^2 \tag{2.19}$$

where $c_4 = c_2/3(2c_1)^2$. The coefficient $c_2 = 0.55$ is related to the Kolmogorov constant (Pao 1965), and $c_1 = 0.2$ has been determined in the laboratory (Hopfinger et al. 1982); hence $c_4 = 1.15$. Performing the same horizontal average as before in (2.15), we finally get, with use of (2.11b),

$$\nu_{v,h} = \frac{216}{875} c_4 \frac{r^2}{\nu_h} |2V - \alpha U|^2 . \tag{2.20}$$

In general, buoyancy cannot prevent the motions from becoming three-dimensional, because it is weakened through radiative damping. But there are circumstances (exceptionally large ε_t, as in the tachocline considered below) where the buoyancy is stronger than the Coriolis force; this situation was discussed by Schatzman and Baglin (1991).

In the spirit of Townsend (1958), they replace the criterion (2.18*a*) by

$$\left(\frac{v}{\ell} \right)^2 \approx N^2 \left(\frac{v\ell}{K} \right) , \tag{2.21}$$

which leads to the following turbulent viscosity, in the vertical direction:

$$\nu^{\dagger}_{v,h} = \frac{1}{3} v\ell = \frac{c_5}{3} \frac{(\varepsilon_t K)^{1/2}}{N} = \frac{c_5}{3} \frac{(\nu_h K)^{1/2}}{N} \left| \sin\theta \frac{\partial\widehat{\Omega}}{\partial\theta} \right| , \tag{2.22}$$

from (2.17), where c_5 is a constant of order unity. Performing again the horizontal average, and referring back to (2.11*b*), one obtains

$$\nu_{v,h} = \frac{1}{10} c_5 \left(\frac{\Omega}{N} \right) \left(\frac{K}{\nu_h} \right)^{1/2} r|2V - \alpha U| . \tag{2.23}$$

This expression applies whenever it yields a *smaller* value for the viscosity than (2.20). Furthermore, a stable gradient of molecular weight is very efficient in suppressing the vertical turbulent diffusion, in which case yet another formula must be used for $\nu_{v,h}$ (Zahn 1983, 1987).

These prescriptions rest on the assumption that all the kinetic energy will be dissipated through viscous friction. That is not necessarily the case, however. Presumably, internal-inertial waves are emitted too in this anisotropic turbulence, and since they involve small scales in the vertical direction, they may dissipate energy very efficiently through radiative damping. Unfortunately, we do not know what fraction of the energy will be diverted by such waves, and how much will remain to be channelled through the Kolmogorov cascade. Only the observations can tell if the expressions (2.20) and (2.23) above give the right order of magnitude for the vertical viscosity, or whether they grossly overestimate it.

2.5. The observational evidence: the solar tachocline

Below the solar convection zone, there is a transition layer in which the rotation changes from the differential regime observed at the surface to an apparently uniform rotation deeper down. With the present helioseismological data, this layer is not resolved yet, which means that it is thinner than about one tenth of the solar radius (Brown et al. 1989; Goode et al. 1991).

This transition layer, which we now call the *tachocline*, was first identified by Spiegel (1972); its structure and evolution have been studied recently in more detail (Spiegel & Zahn 1992). If it were controlled solely by radiative diffusion, the layer would spread in time as $t^{1/4}$ and by now its thickness would reach 300,000 km, by the most conservative estimate. This is far deeper than is allowed by the observations. But the expansion of the layer can be prevented by sufficiently anisotropic turbulence, which then balances the advection of angular momentum, and thereby achieves a stationary state. For this to occur, the horizontal component of the turbulent viscosity must be $O(r/h)^2$ larger than the vertical one, h being the thickness of the tachocline.

This is the strongest evidence so far that differential rotation is able to generate horizontal turbulence, of the type which can enforce shellular rotation. The tachocline thickness is given approximately by

$$\frac{h}{r} \approx \left(\frac{\Omega}{N}\right)^{1/2} \left(\frac{K}{\nu_h}\right)^{1/4}, \tag{2.24}$$

where K is again the thermal diffusivity. Therefore we already have a lower bound for the horizontal component of the turbulent viscosity, which is $\nu_h > 1.5\,10^8$ cm^{2}s^{-1}, in the present Sun.

Even more can be learned from the solar tachocline. It is easy to check that the turbulent viscosity due to the vertical shear, $\nu_{v,v}$, plays a negligible role in the spread of that layer, which leaves us with the contribution due to the horizontal shear, $\nu_{v,h}$. This is a case where it is the buoyancy which prevents the turbulent motions from becoming three-dimensional, and we have to use criterion (2.21).

If the turbulent motions dissipate all their energy through viscous friction, the ratio between the vertical and horizontal components of the turbulent viscosity is related to the differential rotation by (cf. 2.22 and 2.15):

$$\frac{\nu_{v,h}}{\nu_h} = \frac{c_5}{3} \left(\frac{K}{\nu_h}\right)^{1/2} \int \left|\sin\theta \frac{\mathrm{d}\Omega}{\mathrm{d}\theta}\right| \sin^3\theta\, \mathrm{d}\theta \Big/ N \int \sin^3\theta\, \mathrm{d}\theta\,. \tag{2.25}$$

With the differential rotation deduced from helioseismology (Goode et al. 1991), that ratio would be

$$\frac{\nu_{v,h}}{\nu_h} = 0.024\, c_5 \left(\frac{K}{\nu_h}\right)^{1/2} \left(\frac{\Omega}{N}\right) = 0.024\, c_5 \left(\frac{h}{r}\right)^2, \tag{2.26}$$

at the base of the convection zone. It thus complies with the requirement stated above, that $\nu_{v,h}/\nu_h \ll (h/r)^2$.

However, unless the tachocline turns out to be much thinner than the present resolution limit ($h/r \lesssim 1/10$), the turbulent diffusivity which accompanies this viscosity would be too large to tolerate the observed surface abundance of lithium (cf. Schatzman 1977). We thus suspect that the turbulent viscosity is probably overestimated by expression (2.23).

There are two possible explanations. Either gravitational settling of helium, together with thermal diffusion, has established there a gradient of molecular weight which is sufficient to suppress turbulent diffusion in the vertical direction. But for this to occur, the vertical component of the turbulent viscosity had to be sufficiently weak at the onset. This leaves us with the other possibilty, which was suggested above in §2.4.2: presumably a large fraction of the kinetic energy is channelled into waves, to be dissipated by radiative damping, and not through viscous friction.

Not much else can be gleaned from the observed depletion of lithium. It is true that the tachocline is the seat of a local meridian circulation, whose overturn time is shorter than the Eddington-Sweet time (3.41) by a factor $(h/r)^4$. However, the advective transport of chemicals is strongly inhibited by the horizontal turbulence (§2.1), and one verifies that it is insignificant in the present Sun, unless the tachocline is very thin (less than 2,000 km !), in which case the mixing would be accomplished on the required timescale of about 1 Gyr.

The most plausible explanation is that the Li depletion has occurred much earlier, when the Sun was spun down by a strong wind. We shall discuss this later in §5.2.

2.6. Parametrizing the horizontal turbulent transport

As we have seen, two transport coefficients remain, which cannot be derived from first principles, namely the horizontal component of the turbulent viscosity ν_h, and its companion, the horizontal diffusivity D_h. If we wish to proceed, we must content with some parametrization, whose arbitrariness can fortunately be limited by the few constraints that we have encountered.

Referring back to (2.11*b*), we note that the amplitude of the differential rotation will remain small only as long as ν_h is of the order of $|2V - \alpha U|$, or larger. The simplest way to implement this is to take

$$\nu_h = \frac{1}{c_h} r\, |2V - \alpha U|\,, \tag{2.27}$$

with $c_h \lesssim 1$; it will also ensure that $\nu_{v,h}/\nu_h \ll 1$, since according to (2.20)

$$\nu_{v,h} = \frac{216}{875} c_4 c_h\, r\, |2V - \alpha U| = C_\nu\, r\, |2V - \alpha U|\,. \tag{2.28}$$

The last parameter we have introduced, C_ν, contains all the uncertainties. It cannot exceed $\approx 1/4$, but it is presumably much less, as we just learned from the lithium depletion in the Sun (§2.5).

Likewise, we set

$$D_h = \frac{1}{C_h} r\, |2V - \alpha U|\,, \tag{2.29}$$

again with $C_h \lesssim 1$. When this horizontal turbulence interfers with the meridian circulation, the resulting effective diffusivity will be (2.3)

$$D_{\rm eff} = \frac{C_h}{30} \frac{r U^2}{|2V - \alpha U|}\,. \tag{2.30}$$

In the event that the denominator in $D_{\rm eff}$ vanishes at some depth, as it happens when the circulation consists of superposed cells, we may avoid the singularity by replacing $|2V - \alpha U|$ with $(|2V - \alpha U|^2 + U^2)^{1/2}$, or any similar expression.

We are fully aware of the crudeness of such prescriptions. For instance, they predict a differential rotation whose amplitude is the constant $c_h/5$, whereas its level ought to be linked with the strength of the turbulence, as experienced in the solar tachocline. Furthermore, there are good reasons to believe that the horizontal transport will depend on whether the differential rotation increases towards the equator or towards the poles. In the first case turbulence is submitted to cyclonic forcing, and one knows that it is then rather gentle. The opposite is true for anticyclonic forcing, where the turbulent transport is probably enhanced (Salmon, private communication; see also Griffa & Salmon 1989). The prescriptions above do not distinguish between these two situations.

Thus our parametrization of the horizontal transport leaves much to be desired. But we keep it as is, and wait for observational tests to guide us in the necessary improvements.

3. The meridian circulation for shellular rotation

The scope of this section is to establish an expression for the meridian velocity in a star which has a shellular rotation law, with the angular velocity depending only on depth (and time).

As is well known, the circulation velocity $\boldsymbol{u}$ is deduced from the heat equation:

$$\rho T \frac{\partial S}{\partial t} + \rho T\, \boldsymbol{u} \cdot \nabla S = \nabla \cdot (\chi \nabla T) + \rho \varepsilon, \tag{3.1}$$

with the usual notation for the temperature T, the specific entropy S and the production rate ε of nuclear energy, per unit mass. The radiative conductivity χ is related to the thermal diffusivity by $\chi = \rho C_P K$, C_P being the specific heat at constant pressure.

We shall omit the time derivative, since we are interested only in the behavior of the meridian flow on timescales which exceed the thermal relaxation time. That time is of the order of the Kelvin-Helmholtz time, or less, when steeper temperature gradients are involved. The terms which then remain express the balance between the advective and the diffusive transport of the thermal energy.

When the temperature field departs from spherical symmetry, as it does in a rotating star, the radiative flux is non solenoidal in general, and it is that thermal imbalance which drives the meridian flow. Such departures from sphericity may be forced from the boundary of a radiation zone, by the adjacent convection zone, as we have just seen in §2.5. But in rotating stars, they occur also within their radiative regions, due to the centrifugal force. This is the case which has been considered originally by Eddington (1925) and Vogt (1925), and by most authors since.

In the standard treatment, which was described by Sweet (1950) as a special case of his more general formulation, the rotation is assumed to be uniform. Sweet also neglected higher order terms, although these become the most important in the upper part of the star (cf. §3.5), as was recognized already by Gratton (1945), and later again by Öpik (1951). This shows that some care must be taken when calculating the meridian velocity.

More general expressions have been derived by Baker and Kippenhahn (1959), followed by several others which will be quoted below; they apply to cylindrical rotation laws, for which the centrifugal force derives from a potential. Here we shall treat instead the case of shellular rotation, where the angular velocity varies with depth, and is constant over level surfaces. We thus assume that the horizontal differential rotation is smoothed out by the anisotropic turbulence examined above – at least to a degree which allows this shellular idealization.

3.1. Linearization

As we just stated, we take the rotation rate as constant on horizontal surfaces. But we retain the horizontal variation of all other physical quantities, since they take part in the radiative imbalance which causes the meridian circulation. We assume that they are small enough to be treated as first order perturbations around their mean value.

Here we can no longer ignore the departures from sphericity of the horizontal surfaces, or level surfaces, as we call them also. Depending on the problem we deal with, we use the most convenient of two possible expansions. The surface of reference is either a spherical layer, as illustrated here for the density:

$$\rho(r, \theta) = \rho_0(r) + \widehat{\rho}(r)\, f(\theta)\,, \tag{3.2}$$

or it is a level surface, or isobar:

$$\rho(P, \theta) = \overline{\rho}(P) + \widetilde{\rho}(P)\, f(\theta)\,. \tag{3.3}$$

We assume, and shall verify later, that the horizontal variation of all physical quantities can be represented by the same function $f(\theta)$ of zero horizontal mean, and we anticipate of course that this function will be the spherical harmonic $P_2(\cos\theta)$.

Let us briefly examine how these two representations are related to each other. The departure of pressure from spherical symmetry can be written likewise as

$$P(r, \theta) = P_0(r) + \widehat{P}(r)\, f(\theta)\,, \tag{3.4}$$

from which we deduce the variation $\zeta(r) f(\theta)$ of the radial coordinate on an isobar. Since, by first order Taylor expansion,

$$P(r_0 + \zeta f(\theta), \theta) = P_0(r_0) + \zeta \frac{\mathrm{d}P_0}{\mathrm{d}r} f(\theta) + \widehat{P}(r_0)\, f(\theta)\,, \tag{3.5}$$

which is constant on a isobar, we get

$$\zeta = -\frac{\widehat{P}}{(\mathrm{d}P_0/\mathrm{d}r)}\,. \tag{3.6}$$

Applying the same procedure to (3.2) and comparing with (3.3), we find that to lowest order

$$\widetilde{\rho}(P) = \widehat{\rho} - \left(\frac{\mathrm{d}\rho_0}{\mathrm{d}P_0}\right) \widehat{P}\,, \tag{3.7}$$

and $\rho_0(r_0) = \overline{\rho}(P)$, which defines the mean radius r_0 of that isobar. At that level of approximation, we do not distinguish between $\mathrm{d}\rho_0/\mathrm{d}P_0$ and $\mathrm{d}\overline{\rho}/\mathrm{d}P$.

3.2. Baroclinicity

Due to the centrifugal force, which is conservative only in the case of uniform or cylindrical rotation, all physical quantities vary in latitude on a level surface, except of course the pressure. To link that variation to the rotation rate, we start from the equation of hydrostatic equilibrium

$$\frac{1}{\rho} \nabla P = \boldsymbol{g} = \nabla \Phi + \frac{1}{2} \Omega^2\, \nabla (r \sin\theta)^2\,, \tag{3.8}$$

where Φ is the gravitational potential and $\boldsymbol{g}$ the local effective gravity. We then take the curl of this equation:

$$-\frac{1}{\rho^2} \nabla\rho \times \nabla P = -\frac{1}{\rho} \nabla\rho \times \boldsymbol{g} = \frac{1}{2} \nabla\Omega^2 \times \nabla (r \sin\theta)^2\,; \tag{3.9}$$

it relates the angle between gravity and the density gradient to the gradient of the rotation rate (fig. 1).

To lowest order, this baroclinic equation reduces to

$$\overline{g}\, \frac{\widetilde{\rho}}{\overline{\rho}}\, \frac{1}{r} \frac{\mathrm{d}f}{\mathrm{d}\theta} = -\frac{\mathrm{d}\Omega^2}{\mathrm{d}r}\, r \sin\theta \cos\theta\,, \tag{3.10}$$

where $\overline{g}$ is the mean value, on the isobar, of $g = |\boldsymbol{g}|$. At that level of approximation, we can neglect the departure of the polar coordinate r from the mean radius r_0 of the isobar; hence we use the same notation for them. Integrating in θ, we finally get

$$\frac{\widetilde{\rho}}{\overline{\rho}} = \frac{1}{3} \frac{r^2}{\overline{g}} \frac{\mathrm{d}\Omega^2}{\mathrm{d}r} \quad \text{and} \quad f(\theta) = P_2(\cos\theta)\,. \tag{3.11}$$

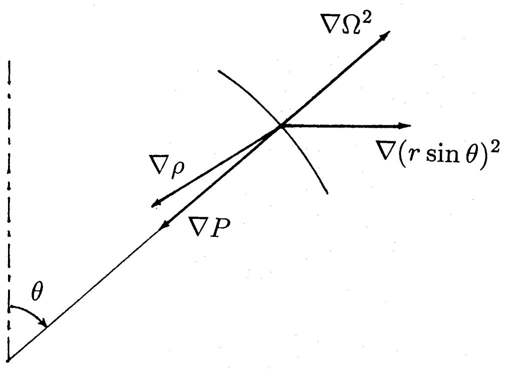

Fig. 1. When the angular velocity of a star varies with depth (only), the surfaces of equal density no longer coincide with the isobars: in this baroclinic state, the density gradient and that of the temperature (not shown) both have a horizontal component.

This provides us with an estimate of the relative density perturbation, which has to be small to allow for the linearization we have performed.

Since the horizontal fluctuation of the concentration of chemical species is described by the same spherical harmonic of order 2 (cf. 2.2), we conclude that this will be the case for all physical variables, such as the molecular weight μ, the temperature T, and therefore also the radiative conductivity χ, the nuclear generation rate ε, etc.

From the equation of state we deduce that

$$\frac{\widetilde{\mu}}{\overline{\mu}} - \frac{\widetilde{T}}{\overline{T}} = \frac{\widetilde{\rho}}{\overline{\rho}} = \frac{1}{3}\frac{r^2}{\overline{g}}\frac{\mathrm{d}\Omega^2}{\mathrm{d}r} \,. \tag{3.12}$$

For simplicity, we have used the perfect gas law; corrections apply when the radiation pressure becomes important.

3.3. Gravity

For later purpose, we need to know also how the effective gravity g varies in latitude on a level surface. We anticipate that

$$g(P,\theta) = \overline{g}(P) + \widetilde{g}(P)\,P_2(\cos\theta) \,, \tag{3.13}$$

and we proceed to determine the amplitude $\widetilde{g}$ of the fluctuation.

The treatment is easier in the spherical coordinate system, and therefore we shall expand the density, the pressure and the gravitational potential as in (3.2) and (3.4). We first seek an equation for the perturbation of the gravitational potential, and we follow for this the classical procedure outlined by Gratton (1945), and later by Sweet (1950) and Öpik (1951).

We rewrite (3.11) in terms of the spherical variations, as in (3.7), and obtain thereby a first relation between the pressure and density fluctuations:

$$g_0\frac{\widehat{\rho}}{\rho_0} + \frac{\widehat{P}}{\rho_0^2}\frac{\mathrm{d}\rho_0}{\mathrm{d}r} = \frac{r^2}{3}\frac{\mathrm{d}\Omega^2}{\mathrm{d}r} \,. \tag{3.14}$$

A second relation can be drawn from the hydrostatic balance in the θ-direction:

$$\frac{\widehat{P}}{\rho_0} = \widehat{\Phi} - \frac{1}{3}r^2\Omega^2 \,. \tag{3.15}$$

Eliminating $\widehat{P}$ between the two, we get

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\widehat{\Phi} + g_0\widehat{\rho} = \frac{r^2}{3}\frac{\mathrm{d}}{\mathrm{d}r}(\rho_0\Omega^2) \,. \tag{3.16}$$

Finally, applying Poisson's equation to eliminate $\widehat{\rho}$,

$$\nabla^2[\widehat{\Phi}(r)\,P_2(\cos\theta)] = -4\pi G\widehat{\rho}(r)\,P_2(\cos\theta) \,, \tag{3.17}$$

we reach the following second order o.d.e. for $\widehat{\Phi}$ (Sweet 1950)

$$\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\widehat{\Phi}) - 6\frac{\widehat{\Phi}}{r^2} - \frac{4\pi G}{g_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\widehat{\Phi} = -\frac{4\pi G r^2}{3g_0}\frac{\mathrm{d}}{\mathrm{d}r}(\rho_0\Omega^2) \,; \tag{3.18}$$

it determines the quadrupolar moment of a star whose rotation rate is a function of depth only.

We are now ready to evaluate the horizontal variation of the gravity. Starting from the hydrostatic equation, we expand the effective gravity as

$$\begin{aligned} \boldsymbol{g} &= \nabla\Phi + \frac{1}{2}\Omega^2\nabla(r\sin\theta)^2 \\ &= \nabla\Phi_0 + f\,\nabla\widehat{\Phi} + \widehat{\Phi}\nabla f + \frac{1}{2}\Omega^2\nabla(r\sin\theta)^2 \,. \end{aligned} \tag{3.19}$$

We then square the members of that equation, and keep only the first order terms, thereby establishing that g^2 varies on a spherical layer as

$$\begin{aligned} g^2(r,\theta) &= \left[g_0 + \widehat{g}f(\theta)\right]^2 = (g_0)^2 + 2g_0\widehat{g}f(\theta) \\ &= \left[\left(\frac{\mathrm{d}\Phi_0}{\mathrm{d}r}\right)^2 + \frac{4}{3}r\Omega^2\frac{\mathrm{d}\Phi_0}{\mathrm{d}r}\right] - 2g_0\left(\frac{\mathrm{d}\widehat{\Phi}}{\mathrm{d}r} - \frac{2}{3}r\Omega^2\right)f(\theta) \,. \end{aligned} \tag{3.20}$$

We next ask how g varies on a level surface, whose radial coordinate is $r = r_0 + \zeta\,f(\theta)$; the answer is

$$g(P,\theta) = g(r_0 + \zeta f(\theta), \theta) = g_0 + \frac{\mathrm{d}g_0}{\mathrm{d}r}\,\zeta\,f(\theta) + \widehat{g}f(\theta) \,. \tag{3.21}$$

Referring back to (3.6) and (3.13), we find that

$$\widetilde{g}(P) = \frac{\mathrm{d}g_0}{\mathrm{d}r}\left(\frac{\widehat{P}}{g_0\rho_0}\right) - \frac{\mathrm{d}\widehat{\Phi}}{\mathrm{d}r} + \frac{2}{3}r\,\Omega^2 \,. \tag{3.22}$$

Using (3.15) to eliminate the pressure variation, we finally get

$$\frac{\widetilde{g}}{\overline{g}} = \frac{1}{3}\Omega^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{g_0}\right) - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\widehat{\Phi}}{g_0}\right) \,. \tag{3.23}$$

Besides g_0, which to lowest order is the gravity of the star in spherical hydrostatic equilibrium, it involves the non-spherical perturbation of the gravitational potential. The latter vanishes in a point mass star, where $g_0 = GM/r^2$, and one has there the simple result

$$\frac{\widetilde{g}}{\overline{g}} \approx \frac{4}{3}\left(\frac{\Omega^2 r^3}{GM}\right) \,. \tag{3.24}$$

For many purposes, this relation represents an adequate approximation in the upper part of any star.

3.4. Thermal imbalance

Many of the preceding results have been derived by Sweet (1950) and by McDonald (1972), who considered the most general (hence baroclinic) rotation law. But we now depart from their treatment in order to capture the higher order Gratton-Öpik terms, which are overlooked in the standard procedure.

The task is to calculate the divergence of the radiative flux. As in (3.3), we expand all physical quantities around their horizontal average on a level surface, or isobar; thus

$$T(P,\theta) = \overline{T}(P) + \widetilde{T}(P) f(\theta) \quad \text{with} \quad f(\theta) = P_2(\cos\theta)\,. \tag{3.25}$$

Likewise, the temperature gradient will be

$$\begin{aligned}\nabla T &= \nabla\overline{T} + \nabla\widetilde{T} f(\theta) + \widetilde{T}\,\nabla f(\theta)\\ &= \rho\left[\frac{\mathrm{d}\overline{T}}{\mathrm{d}P} + \frac{\mathrm{d}\widetilde{T}}{\mathrm{d}P} f(\theta)\right]\frac{\nabla P}{\rho} + \widetilde{T}\,\nabla f(\theta)\,.\end{aligned} \tag{3.26}$$

Multiplying by the radiative conductivity χ and taking the divergence, we get

$$\begin{aligned}&\nabla\cdot(\chi\nabla T)\\ &= \rho\chi\left[\frac{\mathrm{d}\overline{T}}{\mathrm{d}P} + \frac{\mathrm{d}\widetilde{T}}{\mathrm{d}P} f(\theta)\right]\left[-4\pi G\rho + \nabla\cdot\left(\frac{1}{2}\Omega^2\,\nabla(r\sin\theta)^2\right)\right]\\ &+ \nabla\left\{\rho\chi\left[\frac{\mathrm{d}\overline{T}}{\mathrm{d}P} + \frac{\mathrm{d}\widetilde{T}}{\mathrm{d}P} f(\theta)\right]\right\}\cdot\frac{\nabla P}{\rho}\\ &+ \chi\widetilde{T}\nabla^2 f(\theta) + \nabla(\chi\widetilde{T})\cdot\nabla f(\theta)\,.\end{aligned} \tag{3.27}$$

In the process, we have also taken the divergence of the hydrostatic equation

$$\frac{\nabla P}{\rho} = \nabla\Phi + \frac{1}{2}\Omega^2\,\nabla(r\sin\theta)^2 \tag{3.28}$$

and have used Poisson's equation to substitute for the divergence of the gravitational potential. We next split all quantities in their mean and horizontal variation, including

$$\nabla\cdot\left(\frac{1}{2}\Omega^2\,\nabla(r\sin\theta)^2\right) = 2\overline{\Omega^2}(P) + 2\widetilde{\Omega^2}(P)\,f(\theta)\,, \tag{3.29}$$

and keep only the first order terms. Adding the nuclear energy production, we get, after some arrangement:

$$\begin{aligned}&\nabla\cdot(\chi\nabla T) + \rho\varepsilon\\ &= \left\langle \overline{\rho\chi}\frac{\mathrm{d}\overline{T}}{\mathrm{d}P}\left(-4\pi G\overline{\rho} + 2\overline{\Omega^2}\right) + \overline{\rho\varepsilon} + \overline{\rho}\frac{\mathrm{d}}{\mathrm{d}P}\left(\overline{\rho\chi}\frac{\mathrm{d}\overline{T}}{\mathrm{d}P}\right)\overline{g}^2\right\rangle\\ &+ \left[\overline{\rho\chi}\frac{\mathrm{d}\overline{T}}{\mathrm{d}P}\left(-4\pi G\widetilde{\rho} + 2\widetilde{\Omega^2}\right)\right.\\ &+ \left(\widetilde{\rho\chi}\frac{\mathrm{d}\overline{T}}{\mathrm{d}P} + \overline{\rho\chi}\frac{\mathrm{d}\widetilde{T}}{\mathrm{d}P}\right)\left(-4\pi G\overline{\rho} + 2\overline{\Omega^2}\right)\\ &+ \overline{\rho}\frac{\mathrm{d}}{\mathrm{d}P}\left(\overline{\rho\chi}\frac{\mathrm{d}\overline{T}}{\mathrm{d}P}\right)2\overline{g}\,\widetilde{g} + \overline{\rho}\frac{\mathrm{d}}{\mathrm{d}P}\left(\widetilde{\rho\chi}\frac{\mathrm{d}\overline{T}}{\mathrm{d}P} + \overline{\rho\chi}\frac{\mathrm{d}\widetilde{T}}{\mathrm{d}P}\right)\overline{g}^2\\ &\left. + \chi\widetilde{T}\nabla^2 + \widetilde{\rho\varepsilon}\right] f(\theta)\,.\end{aligned} \tag{3.30}$$

But the star adjusts its structure so as to cancel the mean part of this expression, which is in $\langle ... \rangle$, as it does when it is non-rotating, to achieve radiative equilibrium. This yields the value of the second derivative term which multiplies the fluctuating gravity $\widetilde{g}$. The first derivative is easily evaluated too, through integration over a level surface:

$$\begin{aligned}\overline{\rho\chi}\frac{\mathrm{d}\overline{T}}{\mathrm{d}P} = \overline{\rho}\,\overline{\chi}\frac{\mathrm{d}\overline{T}}{\mathrm{d}P} &= \frac{\iint \overline{\chi}\nabla T\cdot\mathrm{d}\Sigma}{\iint \mathbf{g}\cdot\mathrm{d}\Sigma}\\ &= \frac{\iint \overline{\chi}\nabla T\cdot\mathrm{d}\Sigma}{\iiint(\nabla^2\Phi + 2\Omega^2)\,\mathrm{d}\tau} = \frac{L(P)}{4\pi G M_*(P)}\,,\end{aligned} \tag{3.31}$$

where we have introduced the "reduced" mass $M(1-\Omega^2/2\pi G\rho_m) = M_*$, with ρ_m being the mean density inside the considered surface.

The result of these manipulations is

$$\begin{aligned}&\frac{1}{\overline{\rho}}\left(\nabla\cdot(\chi\nabla T) + \rho\varepsilon\right)\\ &= \left\{2\left[\frac{L}{M_*}\left(1 - \frac{\overline{\Omega^2}}{2\pi G\overline{\rho}}\right) - \overline{\varepsilon}_n\right]\frac{\widetilde{g}}{\overline{g}} + \frac{L}{M_*}\,\frac{\widetilde{\Omega^2}}{2\pi G\overline{\rho}}\right\} f(\theta)\\ &+ \left[\frac{\overline{g}^2}{4\pi G}\,\frac{L}{M_*}\,\frac{\mathrm{d}}{\mathrm{d}P}\left(\frac{\mathrm{d}\widetilde{T}}{\mathrm{d}\overline{T}} + \frac{\widetilde{\rho\chi}}{\overline{\rho\chi}}\right) - \overline{\varepsilon}\left(\frac{\mathrm{d}\widetilde{T}}{\mathrm{d}\overline{T}} + \frac{\widetilde{\rho\chi}}{\overline{\rho\chi}}\right) + \frac{\widetilde{\rho\varepsilon}}{\overline{\rho}}\right.\\ &\left. - \frac{L}{M_*}\,\frac{\widetilde{\rho}}{\overline{\rho}} - \frac{6\overline{\chi}\widetilde{T}}{r^2\overline{\rho}}\right] f(\theta)\,.\end{aligned} \tag{3.32}$$

Only the first expression in braces is present in a barotropic star; it has been derived successively by Baker and Kippenhahn (1959), Mestel (1965), Smith (1966), Maheswaran (1968), Kippenhahn and Möllenhoff (1974), Pavlov and Yakovlev (1978), Kippenhahn and Thomas (1981), with various degrees of approximation.

We should mention here that we have ignored the turbulent transport of heat, which is justified as long as the Péclet number D_t/K is smaller than one. This is certainly the case for the vertical component of the turbulent diffusivity, but not necessarily for its horizontal component. If it turns out that $D_h > K$, a turbulent conductivity $\rho C_P D_h$ must be added to $\overline{\chi}$ in the term describing the horizontal transport of heat, i.e. the last term in (3.32).

We are ready to cast the departure from radiative equilibrium in its final form. We transform the derivatives with respect to pressure into r-derivatives: $\overline{\rho}\,\overline{g}\,\mathrm{d}/\mathrm{d}P \to -\mathrm{d}/\mathrm{d}r$, and express all baroclinic terms with the functions $\Theta = \widetilde{\rho}/\overline{\rho}$ and $\Lambda = \widetilde{\mu}/\overline{\mu}$, again neglecting the higher order terms. Thus the temperature variation will be given by (3.12)

$$\Lambda(r) - \frac{\widetilde{T}}{\overline{T}} = \Theta(r) = \frac{1}{3}\,\frac{r^2}{\overline{g}}\,\frac{\mathrm{d}\Omega^2}{\mathrm{d}r}\,. \tag{3.33}$$

Likewise, recalling (3.29)

$$\begin{aligned}\nabla\cdot\left(\frac{1}{2}\Omega^2\,\nabla(r\sin\theta)^2\right) &= 2\Omega^2 + \frac{2}{3}\,r\frac{\mathrm{d}\Omega^2}{\mathrm{d}r}\,(1 - P_2(\cos\theta))\\ &= 2\overline{\Omega^2}(P) + 2\widetilde{\Omega^2}(P)\,f(\theta)\,,\end{aligned}$$

$$\text{we have}\quad \overline{\Omega^2} = \Omega^2 + \frac{\overline{g}}{r}\,\Theta \quad\text{and}\quad \widetilde{\Omega^2} = -\frac{\overline{g}}{r}\,\Theta\,. \tag{3.34}$$

Furthermore, we expand the radiative conductivity and the nuclear energy generation rate horizontally as

$$\begin{aligned}&\frac{\widetilde{\chi}}{\overline{\chi}} = \left[-\chi_T\,\Theta + (\chi_\mu + \chi_T)\,\Lambda\right] \quad\text{and}\\ &\frac{M}{L}\,\frac{\widetilde{\rho\varepsilon}}{\overline{\rho}} = \frac{\overline{\varepsilon}}{\varepsilon_m}\left[(1 - \varepsilon_T)\,\Theta + (\varepsilon_\mu + \varepsilon_T)\,\Lambda\right]\,,\end{aligned} \tag{3.35}$$

with $\varepsilon_m(r) = L(r)/M(r)$ and the logarithmic derivatives

$$\chi_\mathrm{T} = \left(\frac{\partial \ln \chi}{\partial \ln T}\right)_{P,\mu}, \quad \chi_\mu = \left(\frac{\partial \ln \chi}{\partial \ln \mu}\right)_{P,T},$$
$$\varepsilon_\mathrm{T} = \left(\frac{\partial \ln \varepsilon}{\partial \ln T}\right)_{P,\mu}, \quad \varepsilon_\mu = \left(\frac{\partial \ln \varepsilon}{\partial \ln \mu}\right)_{P,T}.$$

For simplicity, the perfect gas law has again been assumed when performing these expansions.

Following Mestel (1953), we separate the result in two components, one which exists in the homogeneous star, and the other which arises from the horizontal variations of the molecular weight:

$$\nabla \cdot (\chi \nabla T) + \rho\varepsilon = \overline{\rho}\, \frac{L}{M} (E_\Omega + E_\mu)\, P_2(\cos\theta)\,. \tag{3.36}$$

The component due to the centrifugal force is

$$\begin{aligned} E_\Omega = {} & 2\left[1 - \frac{\Omega^2}{2\pi G\rho} - \frac{\varepsilon}{\varepsilon_m}\right] \frac{\widetilde{g}}{g} \\ & - \frac{\rho_m}{\rho}\left[\frac{r}{3}\frac{\mathrm{d}}{\mathrm{d}r}\left(H_\mathrm{T}\frac{\mathrm{d}\Theta}{\mathrm{d}r} - \chi_\mathrm{T}\,\Theta\right) - 2\frac{H_\mathrm{T}}{r}\Theta + \frac{2}{3}\Theta\right] \\ & - \frac{\varepsilon}{\varepsilon_m}\left[H_\mathrm{T}\frac{\mathrm{d}\Theta}{\mathrm{d}r} + (\varepsilon_\mathrm{T} - \chi_\mathrm{T} - 1)\,\Theta\right] - \Theta\,, \end{aligned} \tag{3.37}$$

with $\widetilde{g}/g$ being given by (3.23) and

$$\Theta = \frac{1}{3}\frac{r^2}{g}\frac{\mathrm{d}\Omega^2}{\mathrm{d}r} = \frac{2}{3}\left(\frac{\Omega^2 r}{g}\right)\frac{\mathrm{d}\ln\Omega}{\mathrm{d}\ln r}\,.$$

Likewise, the component induced by the inhomogeneities is

$$\begin{aligned} E_\mu = {} & \frac{\rho_m}{\rho}\left[\frac{r}{3}\frac{\mathrm{d}}{\mathrm{d}r}\left(H_\mathrm{T}\frac{\mathrm{d}\Lambda}{\mathrm{d}r} - (\chi_\mu + \chi_\mathrm{T} + 1)\,\Lambda\right) - 2\frac{H_\mathrm{T}}{r}\Lambda\right] \\ & + \frac{\varepsilon}{\varepsilon_m}\left[H_\mathrm{T}\frac{\mathrm{d}\Lambda}{\mathrm{d}r} + (\varepsilon_\mu + \varepsilon_\mathrm{T} - \chi_\mu - \chi_\mathrm{T} - 1)\,\Lambda\right], \end{aligned} \tag{3.38}$$

with $\Lambda = \widetilde{\mu}/\mu$; it was first derived by Mestel (1953, 1957).

These expressions have been somewhat simplified by noting that $\Omega^2/2\pi G\rho_m \ll 1$, and by keeping of course only the linear terms in Θ and Λ. Also, we have introduced the temperature scale height $H_\mathrm{T} = |\mathrm{d}r/\mathrm{d}\ln\overline{T}|$, and have dropped the overbar from the averages of ρ and ε, since there is no longer ambiguity.

We see that when the rotation rate depends on depth, the baroclinic terms intervene as soon as the logarithmic gradient $\mathrm{d}\ln\Omega/\mathrm{d}\ln r$ becomes of order unity. More importantly, owing to their behavior in $1/\rho$, these terms will tend to dominate in the outer part of the star, where they are felt already when the rotation gradient is comparable with the oblateness, i.e. when $|\mathrm{d}\ln\Omega/\mathrm{d}\ln r| \gtrsim \Omega^2R^3/GM$. A similar conclusion was reached by Baker and Kippenhahn (1959) for cylindrical rotation, in which the stratification remains barotropic. It underscores the fact that uniform rotation is a very special case, which cannot be regarded as a typical one.

3.5. *The meridian velocity*

Let us now write down the amplitude of the meridian flow, for such a shellular rotation law. As anticipated in (2.1), the vertical velocity is $u(r,\theta) = U(r)\,P_2(\cos\theta)$, and we express its amplitude in terms of the functions E_Ω and E_μ introduced above:

$$U(r) = \frac{L}{Mg}\left(\frac{P}{C_P\rho T}\right)\frac{1}{\nabla_{ad} - \nabla}\,(E_\Omega + E_\mu)\,. \tag{3.39}$$

(We have replaced the mean entropy gradient by the more familiar logarithmic gradients $\nabla = \partial\ln T/\partial\ln P$.) In a *uniformly rotating star*, this reduces to

$$U(r) = 2\,\frac{L}{Mg}\left(\frac{P}{C_P\rho T}\right)\frac{1}{\nabla_{ad} - \nabla}\left[1 - \frac{\varepsilon}{\varepsilon_m} - \frac{\Omega^2}{2\pi G\rho}\right]\frac{\widetilde{g}}{g}\,, \tag{3.40}$$

as was established by Gratton (1945), Öpik (1951) and Mestel (1966). Since according to (3.24) $\widetilde{g}/g$ is of order (Ω^2R^3/GM), the timescale for the circulation is the familiar Eddington-Sweet time t_ES:

$$\frac{1}{t_\mathrm{ES}} = \frac{LR}{GM^2}\left(\frac{\Omega^2R^3}{GM}\right). \tag{3.41}$$

The term $\Omega^2/2\pi G\rho$ in (3.40) plays a crucial role in Von Zeipel's (1924) celebrated paradox. However it has often been neglected in the expression of the meridian flow, on the ground that it is second order in (Ω^2R^3/GM). In this truncated form, the circulation velocity in a uniformly rotating star is given by the *standard formula*

$$U_\mathrm{st}(r) = 2\,\frac{L}{Mg}\left(\frac{P}{C_P\rho T}\right)\frac{1}{\nabla_{ad} - \nabla}\left[1 - \frac{\varepsilon}{\varepsilon_m}\right]\frac{\widetilde{g}}{g}\,, \tag{3.42}$$

which was derived by Sweet (1950). But there is no physical justification whatsoever for the neglect of the Gratton-Öpik term, which dominates near the surface. Moreover, as we have seen above, a slight departure from uniform rotation will introduce terms which are even larger than the neglected Gratton-Öpik correction.

The Tassoul and Tassoul (1982) solution is essentially the same as (3.42), with some viscous corrections at the boundary with a convective region, where this expression seems to predict a singularity. But such singularities are avoided because the convective motions penetrate into the stable domain, and therefore the subadiabatic gradient $(\nabla_{ad} - \nabla)$ has still a finite positive value at the edge of a radiation zone (Zahn 1991a).

4. Asymptotic regimes

In the two preceding sections, we have gathered all the ingredients which are necesssary to calculate the rotation state in radiative stellar interiors. The evolution of the angular velocity is governed by (2.7), which expresses the conservation of angular momentum, in the absence of magnetic or tidal torques. We assume shellular rotation, and therefore omit the overbar on $\Omega(r)$, and we split the turbulent viscosity in the two components we have identified in §2.4:

$$\frac{\partial}{\partial t}\left[\rho r^2\,\Omega\right] = \frac{1}{5r^2}\frac{\partial}{\partial r}\left[\rho r^4\,\Omega\,U\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left[\rho(\nu_{v,v} + \nu_{v,h})\,r^4\frac{\partial\Omega}{\partial r}\right]. \tag{4.1}$$

The strength $U(r)$ of the meridian flow is given by (3.39), with E_Ω and E_μ being expressed by (3.37) and (3.38), in terms of the baroclinic functions $\Theta(r) = (r^2/3\overline{g})\,\mathrm{d}\Omega^2/\mathrm{d}r$ and $\Lambda(r) = \widetilde{\mu}/\overline{\mu}$.

Sakurai (1975) was the first to point out that the advective term involves the fourth-order derivative of the angular velocity. Indeed, it is easy to check that the p.d.e. behaves to leading order as

$$t_r \frac{\partial \Omega}{\partial t} = -\frac{\partial^4 \Omega}{\partial x^4}, \tag{4.2}$$

with $x = r/R$. The relaxation time t_r is of the order of the Eddington-Sweet time (3.41); in the upper part of the star it varies as

$$\begin{aligned} t_r(r,t) &= \frac{45}{2}\, t_{\mathrm{ES}}(t)\, \frac{\nabla_{ad} - \nabla}{\nabla_{ad}}\, \frac{R}{H_{\mathrm{T}}}\, \frac{\rho}{\rho_m} \\ &= \frac{45}{2}\, t_{\mathrm{KH}} \left(\frac{\Omega_s^2(t) R^3}{GM}\right)^{-1} \frac{\nabla_{ad} - \nabla}{\nabla_{ad}}\, \frac{R}{H_{\mathrm{T}}}\, \frac{\rho}{\rho_m}, \end{aligned} \tag{4.3}$$

with $\Omega_s(t)$ being the surface velocity and $t_{\mathrm{KH}} = GM^2/LR$ the Kelvin-Helmholtz time.

In Sakurai's approach, which was fully developed later on (Sakurai 1986, 1991), the turbulent stresses are neglected, and the angular velocity remains a function of latitude. The problem is thus two-dimensional in space, and hence it is less tractable than our one-dimensional simplification. The other differences bear on the treatment of the tachocline.

Our one-dimensional equation (4.1) has a very stable behavior: both spatial operators, the advective and the diffusive, act to smooth out any narrow feature in the rotation rate. Therefore, the system will settle in a steady state after a time of order t_{ES}, provided such a solution exists that satisfies the boundary conditions. A similar differential equation controls the spread of the tachocline (Spiegel & Zahn 1992).

This partial differential equation must be solved with suitable initial and boundary conditions, together with a stellar evolution code and an advection/diffusion equation for the molecular weight. This task was undertaken by Endal and Sofia (1976, 1978, 1981), and their effort was pursued later by Pinsonneault et al. (1989). They solve a similar equation, but they do not include as such the advection of angular momentum: they rely only on turbulent diffusion for all the transports.

Three of the boundary conditions are of homogeneous type, and they need not to be discussed here. The fourth is obtained by integrating (4.1)

$$-\frac{1}{5}\,\rho r^4 \Omega\, U - \rho(\nu_{v,v} + \nu_{v,h})\, r^4 \frac{\partial \Omega}{\partial r} = -\frac{3}{8\pi} \frac{\partial J(r,t)}{\partial t}, \tag{4.4}$$

and by applying this relation at the surface, where the loss of angular momentum, $-dJ/dt$, matches that carried away by the stellar wind. If this boundary condition varies with a sufficiently long timescale t_s, the upper part of the star, where $t_r(r) \ll t_s$, will still reach an asymptotic regime in which the rotation profile evolves too with that slow time t_s.

Here we shall only consider those asymptotic regimes, whose properties can be sketched without having to integrate the full p.d.e. above. Our scope is necessarily restricted to main-sequence stars, whose evolution is sufficiently slow to permit such asymptotic states. Furthermore, in this first survey we shall ignore the effect of chemical inhomogeneities, which remain rather slight in the stellar envelope. Work is in progress to deal with the full problem (Matias & Zahn, in preparation).

At this point already, it is clear that when the wind is present, it requires a meridian velocity varying as $1/\rho$ in the envelope to carry the net angular momentum flux, since there is no viscous stress applied at the surface. We thus suspect that the solution will depend very much on whether there is such a wind, or not. We shall examine in turn the three cases: no wind, moderate wind, strong wind.

4.1. No wind

We assume fast rotation, to ensure that the local thermal adjustment time t_r is short enough compared to the evolution timescale, at least in the upper part of the star. Then a steady regime will be reached there, with zero net angular momentum flux.

We are primarily interested in the behavior near the top of a *radiative envelope*, where $\rho \ll \rho_m$. There, according to (3.37) and (3.39), the circulation velocity is given by

$$\begin{aligned} \frac{\rho}{\rho_m} U = -\frac{L}{Mg} \frac{\nabla_{ad}}{\nabla_{ad} - \nabla} \Bigg[2\, \frac{\Omega^2}{2\pi G \rho_m} \frac{\widetilde{g}}{g} \\ + \frac{r}{3} \frac{\mathrm{d}}{\mathrm{d}r} \left(H_{\mathrm{T}} \frac{\mathrm{d}\Theta}{\mathrm{d}r} - \chi_{\mathrm{T}}\,\Theta \right) - 2 \frac{H_{\mathrm{T}}}{r} \Theta + \frac{2}{3} \Theta \Bigg], \end{aligned} \tag{4.5}$$

where we have applied for simplicity the perfect gas law. We need to balance the advective flux with the diffusive flux; hence, from (4.4),

$$-\frac{1}{5} U = \left[\nu_{v,v} + \nu_{v,h}\right] \frac{\mathrm{d} \ln \Omega}{\mathrm{d}r}. \tag{4.6}$$

Apparently, it was Randers (1941) who first wrote down such an equation, and he concluded that the circulation velocity must be of order ν/R. He was close to the correct answer: R ought to be replaced by the scale height $|\mathrm{d}r/\mathrm{d}\ln\Omega|$, which becomes much larger near the surface, as we shall see. The same approach was chosen by Kippenhahn and Thomas (1981), who anticipated that the Gratton-Öpik term (the first on the r.h.s. of eq. 4.5) would be compensated by a suitable differential rotation, in order to avoid the $1/\rho$ singularity at the surface. We shall proceed much as they did, but for the turbulent viscosity we shall use our prescriptions established in §2, instead of theirs, which they inferred from the GSF instability.

Since we do not allow for a viscous stress at the surface, nor for mass flows, we are seeking a solution for which ρU vanishes there. We expand all quantities near the surface in the normalized depth $z = (R - r)/R$:

$$\ln \Omega(z) = (1 + \omega_1 z + \omega_2 z^2 + \ldots), \quad \rho r^2\, U = W_1 z + W_2 z^2 + \ldots, \tag{4.7}$$

which enables us to rewrite (4.6), using (2.28), as

$$\frac{1}{5} \rho r U = \left[C_v\, \rho r\, |2V - \alpha U| + \rho \nu_{v,v}\right] (\omega_1 + 2\omega_2 z + \ldots). \tag{4.8}$$

(We recall that $\alpha = 1/2\, \mathrm{d}\ln r^2\Omega/\mathrm{d}\ln r = 1 + \omega_1/2 + \omega_2 z + \ldots$)

Now if $\rho r^2 U = W_j z^j$ to lowest order, it follows from continuity (2.10) that $\rho r |2V - \alpha U|$ will be of order z^{j-1}, and therefore this equation (4.8) has no solution when $\omega_1 \neq 0$. We thus impose $\omega_1 = 0$, which entrains that $\Theta = 0$ too at the surface.

Moreover, because the coefficient C_v is rather small, as we have mentioned in §2.6, it means that we cannot balance the advective flux, at any order, with the diffusive flux due to this component $\nu_{v,h}$ of the turbulent viscosity, which is caused by the horizontal (zonal) shear. But the other component $\nu_{v,v}$ is able to achieve this, albeit only at order z^3, as can be checked by rewriting (2.16) as

$$\rho v_{v,v} = \frac{8Ri_c}{45}\rho K \left(\frac{\Omega}{N}\right)^2 \left|\frac{\mathrm{d}\ln\Omega}{\mathrm{d}\ln r}\right|^2$$
$$= \frac{8Ri_c}{45}\frac{L}{Mg}\frac{\Omega^2}{4\pi G}\frac{H_\mathrm{T}\nabla_{ad}}{\nabla_{ad}-\nabla}\left|\frac{\mathrm{d}\ln\Omega}{\mathrm{d}\ln r}\right|^2 . \tag{4.9}$$

Since $H_\mathrm{T} \propto z$ and $\mathrm{d}\ln\Omega/\mathrm{d}\ln r \propto z$, the result follows, all other quantities being of order zero near the surface. The consequence is that ρU will behave there as z^4, in a radiative envelope.

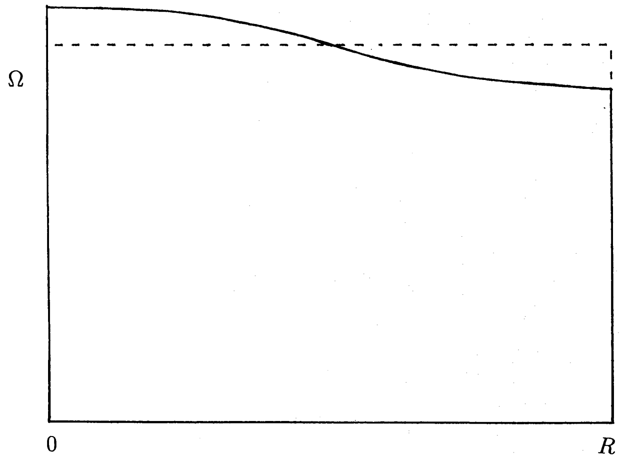

Fig. 2. The asymptotic rotation profile in a star (with a radiative envelope) that does not lose angular momentum. The dotted line indicates the initial rotation, which we assumed to be uniform.

The last step is to calculate the actual value of ω_2. Replacing Θ, H_T and χ_T in (4.5) by similar expansions $\Theta(z) = \Theta_1 z + \ldots$, $H_\mathrm{T}(z) = H_{\mathrm{T},1} z + \ldots$, $\chi_\mathrm{T} = \chi_{\mathrm{T},0} + \ldots$, we find that

$$\left[H_{\mathrm{T},1} + \chi_{\mathrm{T},0}\right]\Theta_1 = -4\frac{\Omega_s^2 R^3}{GM}\left(\frac{\tilde{g}}{g}\right)_s \approx -\frac{16}{3}\left(\frac{\Omega_s^2 R^3}{GM}\right)^2 .$$

We thus get the value of the second derivative of Ω:

$$\frac{1}{2\Omega_s}\frac{d^2\Omega}{dz^2} = \omega_2 \approx C_\mathrm{T}\frac{\Omega_s^2 R^3}{GM} \quad \text{with} \quad C_\mathrm{T} = \frac{4}{H_{\mathrm{T},1}+\chi_{\mathrm{T},0}} . \tag{4.10}$$

In a polytropic atmosphere, $H_{\mathrm{T},1} = |\mathrm{d}H_\mathrm{T}/\mathrm{d}r| = 1$, and $\chi_{\mathrm{T},0}$ is a positive constant depending on the opacity law: for constant opacity $C_\mathrm{T} = 0.8$, whereas for Kramers' law $C_\mathrm{T} = 0.42$. The departure from uniform rotation is thus of the order of the oblateness, and it depends little on the turbulent viscosity. A typical rotation profile is sketched in fig. 2.

Replacing all terms in (4.8) by the appropriate expressions, we finally get

$$U(z) = \frac{128Ri_c}{27}C_\mathrm{T}^3\frac{LR^2}{GM}\frac{\nabla_{ad}}{\nabla_{ad}-\nabla}\left(\frac{\Omega_s^2 R^3}{GM}\right)^4\left(\frac{\rho_m}{\rho}z^4\right) . \tag{4.11}$$

In terms of the standard value U_st of the meridian flow, this is

$$U(z) = \frac{16Ri_c}{9}U_\mathrm{st}(z)\,C_\mathrm{T}^3\left(\frac{\Omega_s^2 R^3}{GM}\right)^3\left(\frac{\rho_m}{\rho}z^4\right) . \tag{4.12}$$

Since the polytropic index is generally less than 4, the $1/\rho$ dependence will be over-compensated by the factor z^4, and thus the meridian velocity will actually be much smaller that predicted by the standard formula (3.42). The flow rises at the poles and sinks in the equatorial plane, as in the standard Sweet-Eddington circulation. The angular momentum is carried downwards by the circulation, and it diffuses upwards through turbulent viscosity.

Within the validity range of our Taylor expansions, the turbulent viscosity is given by

$$\nu_{v,v} = \frac{1}{10\,\omega_2\,z}R\,U(z)$$
$$= \frac{64Ri_c}{135}\frac{LR^3}{GM^2}\frac{\nabla_{ad}}{\nabla_{ad}-\nabla}C_\mathrm{T}^2\left(\frac{\Omega_s^2 R^3}{GM}\right)^3\left(\frac{\rho_m}{\rho}z^3\right) \tag{4.13}$$
$$= \frac{8Ri_c}{45}R\,U_\mathrm{st}\,C_\mathrm{T}^2\left(\frac{\Omega_s^2 R^3}{GM}\right)^2\left(\frac{\rho_m}{\rho}z^3\right) .$$

But we must keep in mind that the turbulent motions will be sustained only if their Reynolds number is larger than some critical value: $3\nu_{v,v}/\nu > Re_c$. We see that this condition depends very sensitively on the rotation speed Ω_s. If it is not fulfilled, there will be no turbulence (except, perhaps, close to the surface where $\nu_{v,v}$ formally diverges when the polytropic index is larger than 3). The star then reaches a state in which the meridian flow vanishes altogether. This zero-circulation case has been considered by Schwarzschild (1942), Kippenhahn (1963) and Roxburgh (1964), but it was Busse (1981, 1982) who proved that it would indeed be achieved in an inviscid star (with a stress-free surface).

Let us assume on the contrary that the rotation speed is high enough to induce such shear turbulence. It is then the main cause for the vertical transport of chemicals, as can be easily verified. The turbulent diffusivity is comparable with the eddy-viscosity, but we leave open the possibility of a different value by introducing the ratio $\sigma_t = \nu_t/D_t$. Advection through the meridian circulation is inhibited by the horizontal diffusion, as we have explained in §2.1, and the resulting effective diffusivity is indeed much smaller than D_t:

$$\frac{D_\mathrm{eff}}{D_t} = \frac{1}{4}Ri_c\,\sigma_t\,C_h\,C_\mathrm{T}\left(\frac{\Omega_s^2 R^3}{GM}\right)z^2 . \tag{4.14}$$

We have used our parametrization (2.30); let us recall that all the parameters above are of order unity.

4.2. Moderate wind

In the presence of a wind whose strength varies with time, no stationary state can be achieved. However, an asymptotic regime may still be reached in the outer part of the star, provided the local relaxation time t_r (4.3) is shorter than the time t_s characterizing the boundary condition (4.4). We shall consider here specifically the case where the wind is caused by magnetic activity, which enhances the loss of angular momentum through the rigid rotation of the star's magnetosphere, as was suggested by Schatzman (1962). The mass flux inside the star can then be neglected, as we have done implicitly so far. Such stars possess a convective envelope, and we shall describe the circulation and the turbulent transport in the radiative zone below.

The region which settles into the asymptotic regime has a low enough density to allow the neglect of its moment of inertia; therefore the flux of angular momentum is nearly constant there with depth. Let us further assume that the flux carried by the turbulent viscosity is negligible compared to the advected flux; we shall verify later under which conditions this approximation is justified. In the asymptotic region, (4.4) then reduces to

$$-\frac{1}{5}\rho r^4 \Omega U = -\frac{3}{8\pi}\frac{\partial J(r)}{\partial t} \approx -\frac{3}{8\pi}\frac{\mathrm{d}J(R)}{\mathrm{d}t}, \tag{4.15}$$

which permits to express the meridian velocity in terms of the angular momentum loss rate:

$$U(r) = -\frac{5}{2}\frac{\Omega_s}{\Omega(r)}\frac{k^2R^2}{r\,t_J}\frac{\rho_m}{\rho}. \tag{4.16}$$

The spindown time is defined as $t_J = k^2 M R^2 \Omega_s/(-\mathrm{d}J/\mathrm{d}t)$, with

$$k^2 = \frac{2}{3}\frac{\int r^2 \,\mathrm{d}M}{MR^2}; \tag{4.17}$$

if $\Omega(r)$ were constant throughout the star, its angular momentum would be $k^2MR^2\Omega_s$.

We note that here the circulation rises in the equatorial plane and sinks at the poles, which is required by the upward transport of angular momentum. The apparent singularity of $U(r)$ at the surface causes no problem, since the flow reconnects horizontally in the convection zone, where it deposits the angular momentum it has carried so far.

Let us examine the variation with depth of the rotation profile which is required to drive the circulation given above in (4.16). The Taylor expansions used before would be too stretched here, since we are dealing with stars which possess a substantial convective envelope. Instead, we write down the differential equation obeyed by the scaled angular velocity $\omega = \Omega/\Omega_s$ in the upper part of the star; combining (4.5), (4.16) and (3.24), we get

$$\begin{aligned}\frac{16}{9}x^6\left(\frac{\Omega_s^2R^3}{GM}\right) + \frac{x}{3}\frac{\mathrm{d}}{\mathrm{d}x}\left[h_\mathrm{T}\frac{\mathrm{d}y}{\mathrm{d}x} - \chi_\mathrm{T}\, y\right] - 2\frac{h_\mathrm{T}}{x}\,y + \frac{2}{3}\,y \\ = \frac{5}{2}k^2\frac{t_\mathrm{ES}}{t_J}\frac{\nabla_{ad}-\nabla}{\nabla_{ad}}\frac{1}{x^3\omega}.\end{aligned} \tag{4.18}$$

We have scaled r and H_T by the radius: $r = xR$, $H_\mathrm{T} = h_\mathrm{T}R$; also, consistently with the neglect of terms in ρ/ρ_m, we have ignored the mass of the envelope. The function y is the scaled Θ:

$$\Theta = \left(\frac{\Omega_s^2R^3}{GM}\right) y = \left(\frac{\Omega_s^2R^3}{GM}\right)\frac{x^4}{3}\frac{\mathrm{d}\omega^2}{\mathrm{d}x}. \tag{4.19}$$

Equation (4.18) tells us that, as long as $t_J \gtrsim k^2 t_\mathrm{ES}$, it does not require steep gradients of the rotation rate to drive the meridian circulation: the scale height of Ω will remain of the order of the radius (or larger). We take this as our criterion for deciding whether the wind is moderate, or strong.

Likewise, we shall define as weak a wind for which the Gratton-Öpik term, i.e. the first on the l.h.s. of (4.18), comes into play: this occurs when the rate of loss of angular momentum becomes so small that $k^2 t_\mathrm{ES}/t_J \lesssim (\Omega_s^2R^3/GM)$.

A remarkable property of the moderate wind regime is that the rotation profile $\omega = \Omega(r)/\Omega_s$ depends only on the ratio t_ES/t_J between the two characteristic times, as we can see in (4.18), since the Gratton-Öpik term is negligible then. Presumably, the gradient of Ω will be weak in the convection zone, as it is in the Sun. Below, according to (4.18), the rotation rate increases with depth.

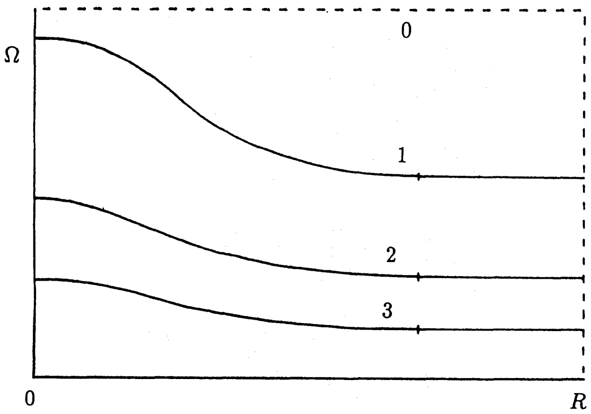

Fig. 3. Evolution in time of the rotation profile, in a solar-like star submitted to a moderate wind which obeys Skumanich's spindown law. The star then reaches an asymptotic regime, in which $\ln\Omega(r)$ no longer depends on the initial condition. In the convection zone, as below in the radiative interior, it is the horizontal average of the angular velocity which has been depicted.

In general t_ES/t_J itself is a function of time, and it varies with a timescale that we have defined earlier as t_s. Thus the rotation profile evolves also with this t_s, until it "freezes", when the angular velocity has become so slow that the local adjustment time t_r exceeds t_s.

An interesting situation arises when the rate of angular momentum loss varies as Ω_s^3, which asymptotically yields Skumanich's $t^{-1/2}$ law for Ω_s (Skumanich 1972): both characteristic times then scale as Ω_s^{-2}, and t_ES/t_J is constant. In that case, the whole star reaches the asymptotic regime, and according to (4.18) its rotation profile remains unchanged during the subsequent spindown (fig. 3). This case was considered also by Sakurai (1991), for which he gave a laminar, two-dimensional solution.

With the Skumanich law, the determining ratio can be written conveniently in terms of the initial and present surface velocities Ω_i and Ω_s (again ignoring the variation of the moment of inertia as the star evolves):

$$\frac{t_J}{t_\mathrm{ES}} = 2\frac{t}{t_\mathrm{KH}}\left(\frac{\Omega_s^2R^3}{GM}\right)\left[1-\left(\frac{\Omega_s}{\Omega_i}\right)^2\right]^{-1}; \tag{4.20}$$

t is the age on the main-sequence and $t_\mathrm{KH} = GM^2/LR$ the Kelvin-Helmholtz time. Note that t_J/t_ES is insensitive to the initial Ω_i, once $\Omega_s \ll \Omega_i$. With the solar parameters, this ratio is 6.5 10^{-3}, whereas $k^2 = 5.9\ 10^{-2}$. One is thus tempted to conclude that the Sun does not qualify as a moderate wind star; however more detailed calculations show that it still reaches the asymptotic regime (Matias & Zahn, in preparation).

Let us examine further the properties of the moderate wind case. If there were no horizontal turbulence, the chemicals would be transported by the circulation at the same rate as the angular momentum. As pointed out by Law et al. (1984), this is not observed, at least not in solar-type stars, for the depletion of lithium would be much more pronounced.

Our explanation for the lesser efficiency of the chemical transport is the erosion due to horizontal turbulence, which has been discussed by Chaboyer and Zahn (1992) (see §2.1). In the bulk of the radiation zone, where the differential rotation is induced only through the meridional flow forced by the wind, we use our parametrization of §2.6 to quantify this effect; it yields an effective diffusivity (2.30)

$$D_{\text{eff}} = \frac{|rU(r)|^2}{30\,D_h} = \frac{C_h}{30}\,r|U|\,\frac{|U|}{|2V - \alpha U|}\,.$$

We differentiate (4.15) to eliminate V: this yields $6V + 2\alpha U = 0$, again with $\alpha = 1/2\ \mathrm{d}\ln r^2\Omega/\mathrm{d}\ln r$. It remains to express $U(r)$ through (4.16) to reach the result

$$D_{\text{eff}} = \frac{C_h}{50}\,\frac{r|U|}{\alpha} = \frac{C_h}{20}\,\frac{\Omega_s}{\Omega(r)}\,\frac{k^2}{\alpha}\,\frac{R^2}{t_J}\,\frac{\rho_m}{\rho}\,. \tag{4.21}$$

We can now compare the flux of a chemical element of concentration c with that of angular momentum. The diffusion speed is $D_{\text{eff}}\,(\mathrm{d}\ln c/\mathrm{d}r)$, when neglecting microscopic diffusion, and the advection velocity for angular momentum is the horizontal average of the vertical velocity $U(r)P_2(\cos\theta)$ weighted by $\sin^2\theta$. The ratio between the two is

$$\mathscr{R} = \frac{15}{2}\,\frac{D_{\text{eff}}}{|rU|}\left|\frac{\mathrm{d}\ln c}{\mathrm{d}\ln r}\right| = \frac{3C_h}{20}\,\frac{1}{\alpha}\left|\frac{\mathrm{d}\ln c}{\mathrm{d}\ln r}\right| = \frac{3C_h}{10}\left|\frac{\mathrm{d}\ln c}{\mathrm{d}\ln r^2\Omega}\right|\,; \tag{4.22}$$

since $C_h < 1$, $\mathscr{R}$ is smaller than unity when the gradients are of comparable strength. We shall discuss in §5.2 how D_{eff} may be further reduced in the vicinity of a convection zone, within the tachocline.

It remains to check under what conditions we can neglect the flux which is transported by turbulence, as we did in (4.15). Since the logarithmic gradient of Ω is of order unity (or less), the turbulent flux due to the horizontal shear is just a small fraction of the advective flux, as it is in the absence of wind. To evaluate the importance of the other component, which is due to the vertical shear, we calculate the ratio

$$\frac{5\nu_{v,v}}{|rU|}\left|\frac{\mathrm{d}\ln\Omega}{\mathrm{d}\ln r}\right| = \frac{2Ri_c}{15}\,\frac{t_J}{k^2 t_{\text{ES}}}\,\frac{\nabla_{ad}}{\nabla_{ad}-\nabla}\left(\frac{rM(r)}{RM_s}\right)^2\frac{\Omega(r)}{\Omega_s}\left|\frac{\mathrm{d}\ln\Omega}{\mathrm{d}\ln r}\right|^3\frac{H_{\text{T}}(r)}{R}\,; \tag{4.23}$$

it tells us the level above which turbulent viscosity may be omitted in deriving the meridian velocity. Below that depth, turbulent diffusion takes its share of the transport, and the meridian flow is somewhat slower than predicted by (4.16).

4.3. Strong wind

If t_J happens to be substantially shorter than $k^2\,t_{\text{ES}}$, we are challenged by a strong wind, and the treatment above does not apply. The spindown is then too fast to allow an asymptotic regime, and we have to solve the full partial differential equation (4.1).

The internal rotation of such stars does never settle into an asymptotic profile. It keeps evolving, with a rather steep front progressing downwards, until the local relaxation time t_r becomes longer than the main-sequence life span, as the star slows down (fig. 4). These stars are even more prone to conserve a fast spinning core than those with a moderate wind. But we cannot say much more before we actually solve the time-dependent problem, and take into account the production of helium through hydrogen burning.

We also note that the angular momentum decreases outwards in the sharp Ω-gradients; these are liable therefore to the powerful Rayleigh-Taylor instability, and they must be treated accordingly.

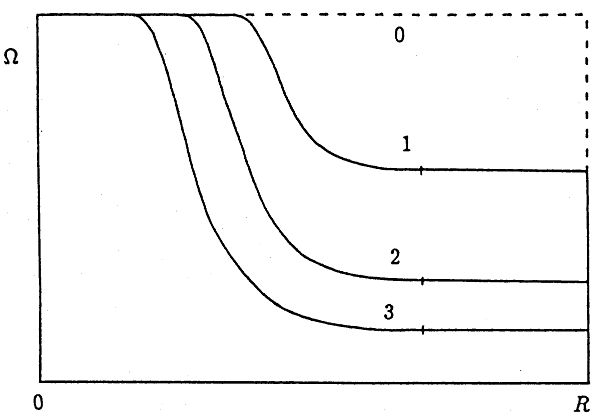

Fig. 4. Evolution of the rotation profile in a solar-like star which emits a strong wind. Such a star may keep a very rapidly rotating core.

5. Interpreting the observations

We shall limit our ambition here to confront the properties of the asymptotic regimes which have just been described with those observations that provide the most crucial tests for the mixing processes which operate in the envelope of rotating stars. One is the settling of helium in A and early F type stars, which is believed to control the abundance anomalies in such stars, and another is the depletion of lithium in low-mass stars.

5.1. Helium settling in A and early F stars

It has been recognized by Vauclair et al. (1974) that helium settling below the shallow convection zones of A and F stars plays a crucial role in fostering the abundance anomalies which are observed in many of these stars. Once the helium abundance has decreased by about a factor of 3, the convection zone due to the second ionization of helium disappears. Then the diffusion of chemical species takes place closer to the surface, and therefore on a much faster rate, which makes it more easy to produce the observed anomalies.

The interference of this process with turbulent mixing had been anticipated by Schatzman (1969), and it was thoroughly discussed by Vauclair et al. (1978). They showed that helium settling is inhibited when the turbulent diffusivity D_t becomes larger than 20 to 100 times the particle diffusivity D_{12} of helium, depending on the variation with depth of D_t.

The adverse effect of meridional circulation was examined by Michaud et al. (1983), first in a two-column approximation. Their results were confirmed by Charbonneau & Michaud (1988, 1991), with detailed two-dimensional simulations. They found a limiting equatorial velocity of about 100 km s^{-1} for A and early F stars, above which the meridional circulation would prevent helium settling. This value is in excellent agreement with the observations, and it was concluded that turbulent diffusion must be negligible in such stars.

However, in all these comparisons the meridian circulation was calculated in the standard way, assuming uniform rotation and ignoring the Gratton-Öpik term (Sweet 1950; Tassoul & Tassoul 1982). We therefore conclude that the agreement found by Michaud et al. (1983) and Charbonneau & Michaud (1988, 1991) must be due to a coincidence, unless a magnetic field is present, which induces the right amount of differential rotation needed to suppress the $1/\rho$ singularity (see Mestel & Moss 1977).

We favor another interpretation: these stars have reached the steady state described above in §4.1, with no loss of angular momentum, and below some critical rotation speed the vertical shear is insufficient, inside, to produce the turbulent mixing which would forbid the settling of helium.

If we follow Vauclair et al. (1978), we require the turbulent diffusivity D_t to exceed about 20 times the molecular diffusivity D_{12} in order to prevent helium settling. This modest ratio corresponds to our case, where D_t varies little with depth, as we have seen in §4.1. Therefore, we expect the critical rotation rate to be given by

$$3\nu_t(\Omega_c) = \max\,[60\,\sigma_t D_{12},\, Re_c\,\nu]\ . \tag{5.1}$$

The coefficient σ_t is the ratio ν_t/D_t between the turbulent viscosity and diffusivity, a number which is of order unity. The presence of the threshold $Re_c\,\nu$ in (5.1) reminds us that the Reynolds number characterizing the turbulent motions must exceed a critical value Re_c, which is of order 1000, for this shear turbulence to be sustained. Since the microscopic viscosity ν is comparable with D_{12}, it is presumably that latter condition which is the most stringent. *In other words, when turbulence does occur, then its strength is sufficient to suppress helium settling.*

If the star had a radiative envelope up to the surface, we would be able to use (4.13) to express the turbulent viscosity ν_t in terms of the rotation speed. This would yield a limiting equatorial velocity for turbulent mixing of $V_{\rm eq} \approx 200$ km s^{-1}, in this range of the main sequence (with the model data kindly provided by G. Michaud). This value is too high to comply with the observations, which is hardly a surprise, since our Taylor expansions in §4.1 do not take into account the shallow superficial convection zones. Presumably, the differential rotation rate at the base of the helium convection zone is higher than implied by these expansions[1], and therefore $\nu_{v,v}$ is underestimated in (4.13), leading to an overestimate of the critical rotation speed. Below that threshold velocity, there will be neither turbulence nor meridian circulation. Hence, there seems no need to invoke Mestel's (1953) mechanism, the so-called μ-barrier, to explain the inhomogeneous evolution of such stars towards the giant branch. However these composition gradients will presumably play their role in early-type stars, which are known to also have a wind.

5.2. *Lithium depletion in low mass stars*

5.2.1. The blue side of the lithium dip

As we have just seen, little mixing occurs in stars when they are not losing angular momentum, and this probably explains why lithium keeps its original value on the hot side of the dip discovered by Boesgaard and Tripicco (1986), at least in stars which do not rotate above the threshold we have just mentioned.

Another possibility has been proposed earlier by S. Vauclair (1988): if the meridian flow consists of two superposed cells, their interface has a shielding effect on the transport. Such a split of the circulation in two cells exists for uniform rotation, as was shown by Gratton (1945) and Öpik (1951), and this was the case originally considered by Vauclair. Two similar cells may also occur in our weak wind limit, according to (4.18); however, the location of their interface would be a very sensitive function of the wind strength, and this would be reflected in the lithium depletion. On closer examination, Charbonneau and Michaud (1990) showed that this putative interface is rather permeable, and Charbonnel and Vauclair (1992) confirmed that it would play a limited role. More importantly, these latest simulations indicate that the transport of lithium is much weaker than expected from a uniform rotation law, particularly in the upper part of the envelope. To be precise, Charbonnel and Vauclair used an effective diffusivity of the form $D_{\rm eff} = c\,|rU|$, with $U(r)$ being given by (3.40), and they had to lower that parameter c to $\approx 10^{-3}$ on the blue side of the Li dip, whereas a value of order 1 accounted rather well for the red side of that dip (see below). As we have seen, our solution without angular momentum loss explains well this apparent drop in the transport efficiency, without having to adjust a free parameter.

5.2.2. The red side of the lithium dip

Stars on the cool side of the Li dip have slower rotation rates, which indicates that they have lost a fraction of their initial angular momentum. Consequently, they have also undergone some mixing, as we have seen in §4.2, and this is revealed by their lower lithium abundance.

That strong correlation between Li depletion and angular momentum loss has been modeled in a very convincing way by Pinsonneault et al. (1989, 1990). They describe both transports by a diffusion equation, and they implement a turbulent diffusivity which they infer from the various instabilities that may occur in the star. Some flexibility is provided by adjustable parameters, which are calibrated with the Sun.

We have already mentioned that the Li depletion observed on the Sun proves that the transport of chemicals is less efficient than the transport of angular momentum. This was first pointed out by Law et al. (1984). To allow for this disparity, Pinsonneault et al. (1989) assume that the ratio $f_c = D_t/\nu_t$ is smaller than unity, and they determine it with the Sun. Their best fit is obtained for $f_c = 0.046$. This value is used in all their simulations, and it leads to an excellent agreement with the observed lithium abundance of a large variety of stars, within reasonable assumptions about their initial angular momentum.

Our interpretation is that in these stars all transports are accomplished by the meridian circulation, but that the advection of chemical elements is partly inhibited by the horizontal turbulence, as described by Chaboyer and Zahn (1992). This was explained in §4.2, where we gave the ratio between the two transport velocities (4.22):

$$\mathscr{R} = \frac{15}{2}\frac{D_{\rm eff}}{|rU|}\left|\frac{{\rm d}\ln c}{{\rm d}\ln r}\right| = \frac{3C_h}{10}\left|\frac{{\rm d}\ln c}{{\rm d}\ln r^2\Omega}\right|\ . \tag{5.2}$$

The parameter C_h is related to the level of differential rotation in latitude (see §2.6), and it may be calibrated with the Sun, to be applied to other stars, much as was done by Pinsonneault et al. (1989) for their coefficient f_c. In their simulations, both transports are modeled by diffusion equations, and the ratio of their efficiencies is given by

$$\mathscr{R} = f_c\left|\frac{{\rm d}\ln c}{{\rm d}\ln\Omega}\right|\ . \tag{5.3}$$

[1] In the solar convection zone, the angular velocity increases by 5 % in the first 20,000 km, but this may be partly due to magnetic stresses. However, convection itself induces large-scale horizontal flows (Krishnamurti & Howard 1981), and work is in progress to fully understand this mechanism (Howard & Krishnamurti 1986; Massaguer et al. 1992).

In spite of the differences in the underlying physics, the two expressions are sufficiently close to warrant a great resemblance when it comes to predict the lithium abundances. Therefore we may borrow from Pinsonneault et al. the calibrated value of their f_c to estimate crudely the coefficient used in our parametrization (2.29): $C_h \approx 0.15$.

In the Appendix, we show that in the asymptotic regime, the surface abundance of lithium evolves as

$$-\mathrm{d}\ln c = -K_{\mathrm{Li}}\,\mathrm{d}\ln J\,, \tag{5.4}$$

with J being the angular momentum. The constant K_{Li} is proportional to our C_h; it depends on the location of the base of the convection zone and of the level at which lithium is destroyed through nuclear burning, *but it does not depend on the spindown law itself.* Therefore, when one observes a dispersion of the Li abundances, in a cluster, it is likely to reflect the dispersion of the initial values of the angular momentum. Another conclusion bears on the lithium abundance of old population II stars: since these are slow rotators, they must have depleted some of their lithium, and the presently observed value (Spite & Spite 1982) is not the original abundance.

We wish to mention at this point that the effective diffusivity D_{eff} may be even lower than stated above, in the vicinity of a convection zone. There, the differential rotation is coupled through radiative diffusion with that of the convection zone, within a boundary layer that has been named the tachocline (Spiegel & Zahn 1992), as we explained in §2.5. This shearing flow produces a horizontal turbulence whose diffusivity could well exceed that predicted by the parametrization (2.29) introduced in §2.6, which led to (5.2). In that case, D_{eff} would drop below the value given in (2.30), albeit in a relatively thin layer. Helioseismic observations of higher resolution will tell us soon whether the turbulent motions in the tachocline are indeed strong enough to interfer significantly with the vertical transport of lithium.

Note that the choking effect of the turbulent tachocline would be effective only below deep convection zones. Since the circulation speed increases dramatically near the surface, so does also the effective diffusivity (4.21). Therefore, its reduction in the tachocline below a shallow convection zone would not affect much the transport, which is then regulated mainly by the deeper layers, close to the level where lithium is destroyed, because D_{eff} takes there its smallest values. The fading of this choking effect in stars with thinner convection zones could contribute to the decrease of the lithium abundances as one aproaches the Li dip on its cool side.

Finally, let us ask why the red shoulder of the dip is rather well rendered when one calculates the circulation speed in the classical way, assuming uniform rotation. This was done recently by Charbonnel et al. (1992): invoking horizontal turbulence, they used an effective diffusivity of the form $D_{\mathrm{eff}} = c\,|rU|$, with the parameter $c = 3/8$. This prescription is identical with that established in §4.2, although the meridian velocities have not the same profile with depth (the coefficients c and $C_h/50$ differ too). But in both cases the flow speed $U(r,t)$ varies in time as $\Omega_s^2(t)$, when the spindown obeys Skumanich's law, and that presumably explains the lucky coincidence.

6. Discussion

The results presented in this paper have been obtained by ascribing some properties to the turbulent motions that occur, we believe, in rotating stars. Let us recall the assumptions made, and let us examine their impact on the solutions we have outlined.

First, we have assumed that the turbulence is sufficiently anisotropic to keep the differential rotation in latitude at a rather modest level. This enabled us to treat the rotation rate as if it depended only on depth, which greatly simplified the derivation of the meridian velocity, because it involves then only the spherical function $P_2(\cos\theta)$. For that approximation to be valid, the differential rotation in latitude $\delta\Omega/\Omega$ must satisfy

$$\frac{\delta\Omega}{\Omega} \ll \max\left[\left|\frac{\partial\ln\Omega}{\partial\ln r}\right|, \frac{\Omega^2 R^3}{GM}\right], \tag{6.1}$$

as can be seen by returning to relation (3.9), and checking the impact of the baroclinic terms on the meridian flow (§3.4). This condition does not seem too severe in the bulk of a radiative region. It is of course violated in the vicinity of a convection zone, in which the differential rotation is maintained by the turbulent convective stresses. But we know how to handle that tachocline, with a specific treatment mentioned in §2.5 (Spiegel & Zahn 1992). We thus feel that the assumption of shellular rotation is a reasonable one, at least one which is worth to confront with the observational material. But let us emphasize that our main result, namely that the meridian flow is controlled by the wind, does not rely on this assumption: for any rotation law, except cylindrical rotation, does the meridian flow carry a net flux of angular momentum.

In fact, the assumption of shellular rotation is used mainly to predict the internal rotation rate in the no-wind case (§4.1), where it led to our prediction (5.1) for the critical rotation velocity above which a radiative envelope would remain homogeneous through turbulent mixing. One ingredient which enters in this criterion is the turbulent viscosity that is allowed by the stable stratification. It was derived from the classical Richardson criterion, modified to include radiative damping (Townsend 1958; Dudis 1974; Zahn 1974), and its validity is well established in atmospheric sciences. One may quibble that this criterion involves a critical Richardson number Ri_c which depends somewhat on the actual profile of the shearing flow, but that uncertainty is rather minor.

Let us turn to the horizontal turbulence, which we have already invoked to establish a shellular rotation state. It assumes another important role, as we saw in §2.1: it erodes the horizontal inhomogeneities that are built up by the circulation, and thereby it inhibits the advective transport of chemical species. This effect is the straightforward explanation of the observed disparity between the transport of chemicals and that of angular momentum.

Such anisotropic turbulence occurs in the solar tachocline, where it prevents that layer from spreading into the radiative core (Spiegel & Zahn 1992). But we have no direct evidence, as yet, bearing on other stars. Throughout this paper, we adopted the view, at least as a working hypothesis, that this anisotropic turbulence pervades the whole radiation zone. The physical reason we may invoke in favor of such widespread occurrence is that the horizontal shear produces turbulent motions which have a net vertical vorticity; by inverse cascade, these tend to generate larger eddies which have the same vorticity, and thus will remain nearly horizontal. However, we are ready to accept the fact, if it should be established by the observations, that the existence of such turbulence is restricted to tachoclines. Future will tell!

Having conjectured that this predominantly horizontal turbulence exists in the bulk of a radiation zone, it remains to assess its strength. This is clearly the weakest point of our theory. In §2.6, we have proposed a parametrization which complies with the constraints identified so far, but which admittedly leaves much to be desired. It reduces our ignorance to a single coefficient C_h, for the horizontal diffusivity, which is common practice in astrophysical fluid dynamics – remember the mixing-length treatment for stellar convection! Here again we have to interrogate the observations, and to check if a single value of C_h is able to cover a broad variety of stars. Obviously, this inquiry will be related with the question raised above: whether such anisotropic turbulence is widespread or confined in the tachoclines.

An even more uncertain point is the strength of the vertical component of that turbulence, and the turbulent viscosity $\nu_{v,h}$ which is associated with it; we discussed this in §2.4.2, and suggested a parametrization in §2.6 (2.28). The main uncertainty here is the branching of the kinetic energy between the two possible dissipation processes: viscous friction or radiative damping. The latter seems to dominate in the solar tachocline, as we saw in §2.5, but we cannot say much about other stars. Fortunately, this viscosity coefficient plays a very minor role: at most, its effect is to increase slightly the meridian flow velocity.

Turbulence arises also from the vertical shear, but then its viscosity $\nu_{v,v}$ is well defined, in terms of the vertical gradient of angular velocity (§2.4.1), as we have mentioned already. This coefficient intervenes in the no-wind case, to determine the level of residual circulation and the small amount of mixing which is associated with it (§4.2).

Finally, we have to explain why we have deliberately ignored the possibility of a magnetic field in our radiative interiors, although it could modify drastically the dynamics, as we are well aware. Only a weak field is needed to balance the advection of angular momentum, as was shown by Mestel (1953, 1961, 1965), Roxburgh (1963), and discussed again in more recent papers. The reason for our neglect is that we did not need to invoke such a field to obtain a self-consistent picture of the internal rotation of stars, except in the outer envelope, where it amplifies the loss of angular momentum. To take magnetic effects into account at this point would entail the introduction of additional free parameters. We adopted the same philosophy towards the internal waves (gravity modes), which might also contribute to the transport (Press 1981; Zahn 1991b; Gracía Lopéz & Spruit 1991; Schatzman 1991b). We prefer to wait until crucial observational tests force us to increase the complexity of the problem. Obviously, one of them will be the internal rotation rate of the Sun, which we should know soon with much higher accuracy, thanks to helioseismology and to the global networks that are being implemented.

7. The relevant concept: the wind-driven circulation

Since the pioneering works of Eddington and Vogt, we have been accustomed to consider the meridian circulation as being determined, if not forced, by the rotation of the star. We knew of course that the rotation law would be modified through the advection of angular momentum, and that it would feed back on the meridian flow. It was even anticipated that this could lead to a stationary regime, with no circulation at all (Busse 1981, 1982). And that is precisely what we find here, provided the star behaves dynamically as a closed system and conserves its angular momentum.

When there is no loss of angular momentum, a stationary regime is reached within an Eddington-Sweet time, in which a feeble, inefficient circulation balances the turbulent stresses that may be caused by the residual differential rotation. The internal rotation is nearly uniform throughout the radiative envelope, until it meets the composition gradient built in the core by the production of helium. Only for fast rotators is the shear sufficient to produce a weak turbulence which is then responsible for mixing the stellar material – the meridian flow contributes little to it. At lower rotation speed, both circulation and turbulence vanish, and there is no mixing at all, safe perhaps just below the surface (cf. 4.13). In a star with a radiative envelope, the equatorial velocity for this transition to turbulence is about 200 $\mathrm{km\,s^{-1}}$, but it is likely to be lower when there are superficial convection zones.

However the picture changes drastically when the star loses angular momentum through a stellar wind, as it does when it is destined to become a slow rotator. *Then the circulation arises from the need of transporting that angular momentum from the interior to the surface, and the rotation profile adjusts slavishly to yield the required meridian flow.*

When the wind is moderate, an asymptotic regime is reached after an Eddington-Sweet time: then the rotation profile does not depend on the initial conditions, but is determined entirely by the ratio between the spindown time t_J and the Eddington-Sweet time t_ES (§4.2). The core of the star is spinning more rapidly than its convective envelope, and thus a fraction of its angular momentum is concealed in the inner region. In particular, we anticipate that the Sun has kept a rapidly rotating core, as seems to be indicated by the most recent observations carried out from a spacecraft (Toutain & Frölich 1992).

In the case of a strong wind, i.e. when $t_\mathrm{J}/t_\mathrm{ES} \ll k^2$ (k^2 is the moment of inertia scaled by MR^2), the star cannot reach such an asymptotic regime, and its rotation profile bears the marks of the initial conditions. But otherwise the properties of the moderate wind case apply also here.

Most of the mixing inside the stars that generate a wind is accomplished during their initial spindown phase, whose duration is rather short (only a few millions years for solar-type stars, if we dare to extrapolate Skumanich's law). It is mediated through the meridian circulation which, as we have seen, is enforced by the wind, but it is inhibited to some extent by the anisotropic, predominantly horizontal turbulence, which tends to smooth out the horizontal inhomogeneities. Thus the transport of chemical species proceeds at the same pace as that of angular momentum, although with lesser efficiency. Consequently, the depletion of fragile elements, such as lithium, is strongly correlated with the loss of angular momentum (cf. 5.4). All slow rotators have undergone some lithium depletion, and this applies in particular to the old population II stars observed by Spite and Spite (1982). After that initial phase, mixing becomes negligible, and nothing should further prevent microscopic diffusion and gravitational settling.

This notion, that it is the torque applied to the rotating star which drives the circulation, and not the rotation as such, is well known for viscous fluids, as the so-called Ekman pumping (Bondi & Lyttleton 1948; Greenspan & Howard 1963). Its relevance for stars was fully recognized during the heated debate on the solar oblateness, which took place in the late sixties: Howard et al. (1967) showed that a similar mechanism operates when the coupling is achieved through radiative diffusion, and Bretherton and Spiegel (1968) saw that the role of the Ekman layer is then assumed by the entire convection zone. Surprisingly, however, that concept of wind-driven circulation did not catch on in the astronomical community, as demonstrated by the literature which later appeared. But we feel that the observational evidence has become so substantial by now that the correct treatment can no longer be ignored.

The case of a strong wind, which requires the integration of the evolution equations, is currently being investigated, as are the effects of a composition gradient. The results will be reported in a forthcoming paper.

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Appendix : Lithium depletion in the asymptotic regime

In the asymptotic regime described in §4.2, the depletion of a fragile element, such as lithium, can be related directly to the loss of angular momentum.

The mean concentration $c(r,t)$ of the considered element, averaged horizontally, obeys the diffusion equation

$$\frac{\partial}{\partial t}\rho c = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho\, D_{\rm eff}\,\frac{\partial c}{\partial r}\right), \tag{A.1}$$

with the notations used before. Only turbulent mixing is considered here, with an effective diffusivity that depends on time as described in (4.21), which we rewrite here as

$$D_{\rm eff}(r,t) = -D_0(r)\, t_{\rm J}\,\frac{{\rm d}\ln J}{{\rm d}t}, \tag{A.2}$$

where D_0 and $t_{\rm J}$ are the present values of $D_{\rm eff}$ and of the spin-down time $-{\rm d}t/{\rm d}\ln J$. In this expression, we have neglected the variation with depth of the rotation rate.

The boundary conditions applied to (A.1) are

$$c(r_n,t) \equiv 0 \quad \text{at the nuclear destruction level} \qquad r = r_n, \tag{A.3}$$

and the following flux condition at the base of the convective envelope ($r = r_b$), whose mass is M_{zc}:

$$4\pi r^2 \rho_b\, D_{\rm eff}\,\frac{\partial c}{\partial r} = -M_{zc}\,\frac{\partial c}{\partial t}. \tag{A.4}$$

Since the p.d.e. (A.1) is linear, the solutions are of the type

$$c(r,t) = \sum c_j(r)\, h_j(t)\,; \tag{A.5}$$

the functions $h_j(r)$ and $c_j(t)$ verify separately

$$\frac{\rm d}{{\rm d}t}\ln h_j = K_j\,\frac{\rm d}{{\rm d}t}\ln J \quad \text{and} \tag{A.6}$$

$$\frac{1}{r^2}\frac{\rm d}{{\rm d}r}\left(r^2\rho\, D_0 t_{\rm J}\,\frac{{\rm d}c_j}{{\rm d}r}\right) + K_j\,\rho\, c_j = 0. \tag{A.7}$$

The eigenvalues K_j are obtained by solving (A.7) with the boundary conditions (A.3) and (A.4), the latter rewritten here as

$$\frac{{\rm d}c_j}{{\rm d}r} = K_j\left[\frac{M_{zc}}{4\pi r^2 \rho_b\, D_0 t_{\rm J}}\right] c_j \quad \text{at} \quad r = r_b.$$

When the distance $\Delta r = r_b - r_n$ is small enough to be treated as a narrow gap, and to neglect the variation of ρ and D_0, the eigenvalues are

$$K_j = \lambda_j^2\,\frac{D_0 t_{\rm J}}{(\Delta r)^2},$$

where the λ_j are the solutions of

$$\lambda_j \tan\lambda_j = \frac{4\pi r^2 \rho_b\,\Delta r}{M_{zc}}. \tag{A.8}$$

Unless Δr is very small compared with the size of the convective envelope, the first eigenvalue $(\lambda_1)^2$ is of order unity; in any case, the next is at least 9 times larger. Therefore the surface abundance of lithium varies as

$${\rm d}\ln c = K_{\rm Li}\,{\rm d}\ln J, \tag{A.9}$$

after the quick damping of the transients described by the higher eigensolutions. Neglecting again the variation with depth of the rotation rate, the coefficient $K_{\rm Li}$ is given by

$$K_{\rm Li} = K_1 = (\lambda_1)^2\frac{D_0 t_{\rm J}}{(\Delta r)^2} = (\lambda_1)^2\frac{C_h}{20}\left(\frac{kR}{\Delta r}\right)^2\frac{\rho_m}{\rho_*}, \tag{A.10}$$

with ρ_* being a density in that interval Δr, closer to ρ_n at the lithium destruction level, than to ρ_b at the base of the convective envelope. Note that the spindown law, as such, does not intervene in this result.

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