Astron. Astrophys. 91, 175–180 (1980)

ASTRONOMY AND ASTROPHYSICS

The Time Scale of Thermohaline Mixing in Stars

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Received January 18, accepted March 31, 1980

Summary. The time scale of mixing due to thermohaline convection is estimated in the case of a gas in which the molecular weight decreases in the direction of gravity. The mass elements of higher molecular weight which sink into the regions below are hotter in the new surroundings. They therefore radiate into their neighborhood and create a circulation system which mixes the mass element with the surroundings before it has moved over a distance much bigger than its size. The "self destruction" reduces the mean free path of the sinking elements and therefore lengthens the time scale during which a (secularly) stable stratification is reached. The results are applied to a main sequence star which during mass exchange has received helium from a helium star companion. It is also applied to the case of the noncentral helium flash of a star of 1.3 $M_\odot$.

Key words: stellar structure – secular instability – thermohaline convection – circulation systems

1. Introduction

In a star, layers of higher molecular weight above a region of lower molecular weight are secularly unstable. The situation is similar to that of thermohaline convection, in which a layer of warm salt water is above a layer of fresh cold water of slightly higher density. There an instability occurs which forms "fingers" in which the hot salt water is cooling off and then – after having reached a higher density than the fresh water – sinks down (Stern, 1960; Veronis, 1965). The situation of a helium layer above a hydrogen region is similar. The onset of the instability has been discussed by Kato (1966) and others and the velocity of a helium blob sinking into the hydrogen region has been estimated by Kippenhahn (1974).

A situation of heavier material lying above lighter gas in a star can occur during the helium flash when helium burning does not start in the center but in a shell (Thomas, 1967). In close binary systems it may happen that helium-rich material is transferred to a main sequence star. Then a helium-rich outer layer is formed and the instability occurs at the interphase between that layer and the original stellar material.

In the following we try to estimate the effect of mixing due to this instability.

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2. Circulation in a Given μ-distribution

We now consider a region in a star where the molecular weight μ is variable. We assume that μ can be split into two parts, one being constant on equipotential surfaces $\phi=$const and a small perturbation which can vary on these surfaces and which therefore depends on the space vector $\boldsymbol{x}$ and on time t:

$$\mu(\boldsymbol{x},t)=\mu_0(\phi)+\mu_1(\boldsymbol{x},t). \tag{1}$$

In general in such a region the velocity field $\boldsymbol{v}$ will not vanish, and since the molecular weight does not vary in a co-moving frame, we have

$$d\mu/dt=0. \tag{2}$$

If we consider μ_1, and $\boldsymbol{v}$ as first order terms, and if we neglect second order terms – we have

$$\frac{\partial\mu_1}{\partial t}+\boldsymbol{v}\nabla\mu_0=0. \tag{3}$$

In the following we demand that the velocity be always very small compared to the speed of sound. We therefore can assume hydrostatic equilibrium to be a good approximation. We then have pressure and density constant on equipotential surfaces $P=P(\phi)$, $\varrho=\varrho(\phi)$. In the following we assume the perfect gas equation, neglecting radiation pressure and degeneracy. In appendix A we show how our results have to be modified for these effects. We therefore have for the temperature, T,

$$\frac{T_0+T_1}{\mu_0+\mu_1}=f(\phi), \tag{4}$$

and if we assume that for large distances on the equipotential surfaces the perturbations μ_1, T_1 vanish, we find that everywhere in space

$$T_1/T_0=\mu_1/\mu_0. \tag{5}$$

Due to the perturbations in μ and T we expect also the radiative flux $\boldsymbol{F}$ to be perturbed

$$\boldsymbol{F}=\boldsymbol{F}_0+\boldsymbol{F}_1. \tag{6}$$

For the quantities of the undisturbed case we have

$$\begin{aligned}\nabla P_0&=\varrho_0\boldsymbol{g},\\ \nabla T_0&=\boldsymbol{g}T_0\varrho_0\nabla/P_0\\ \boldsymbol{F}_0&=-\frac{4acT_0^4\nabla}{3\kappa_0P_0}\boldsymbol{g}=-4\boldsymbol{g}(1-\beta)c\nabla/\kappa_0,\end{aligned} \tag{7}$$

where $\boldsymbol{g}$ is the gravitational acceleration, κ is the opacity, a the radiation constant, c the velocity of light and $\nabla = (d \ln T_0 / d \ln P_0)$. In addition we have introduced β, the ratio of gas pressure to total pressure. According to our assumption this is a quantity very close to 1 and therefore $1-\beta$ is a small number. We now write down the equation for the conservation of energy in the first order

$$\nabla F_1 = -v(c_p \varrho_0 \nabla T_0 - \nabla P_0) - c_p \varrho_0 \frac{\partial T_1}{\partial t} \tag{8}$$

(c_p is the specific heat). With the thermodynamic relation

$$\nabla_{ad} = P_0 / T_0 \varrho_0 c_p, \quad (\nabla_{ad} = (d \ln T_0 / d \ln P_0)_{ad}) \tag{9}$$

and with Eq. (5) we can write

$$\nabla F_1 = -c_p \frac{\varrho_0 T_0}{\mu_0} \frac{\partial \mu_1}{\partial t} + \varrho_0 v g \frac{\nabla_{ad} - \nabla}{\nabla_{ad}}. \tag{10}$$

On the other hand we have

$$\frac{\partial \mu_1}{\partial t} = -v \nabla \mu_0 = -\varrho_0 \mu_0 \mu_p v g / P_0, \tag{11}$$

where we have introduced the quantity $\mu_p = (d \ln \mu_0 / d \ln P_0)$. We then can write

$$\nabla F_1 = \frac{\nabla_{\text{crit}}}{\nabla_{ad}} \varrho_0 v g, \quad \nabla_{\text{crit}} = \mu_p + \nabla_{ad} - \nabla. \tag{12}$$

We now write the equation for radiative transport in the form

$$F = -K \nabla T, \quad K = \frac{4ac}{3\kappa\varrho} T^3 \tag{13}$$

and have

$$\nabla F_1 = -K_0 \Delta T_1 - \nabla K_0 \nabla T_1 - \nabla K_1 \nabla T_0 - K_1 \Delta T_0 \tag{14}$$

and for small scale perturbations where the natural length of the perturbation is small compared to the scale height we have

$$\nabla F_1 = -K_0 \Delta T_1 = -K_0 \frac{T_0}{\mu_0} \Delta \mu_1. \tag{15}$$

3. Stability of a Stratified Layer

In the following we show that our formalism gives the instability of layers with molecular weight μ_0 decreasing in the direction of gravity. We show this in the plane parallel approximation where z increases in the direction opposite to gravity $\boldsymbol{g}$. With an exponential ansatz we have

$$\frac{\partial \mu_1}{\partial t} = \lambda \mu_1, \quad \Delta \mu_1 = -(k^2 + l^2 + m^2) \mu_1 = -\mu_1 / D^2 \tag{16}$$

where λ is the growth rate while, k, l, m are the wave numbers in the three directions. Equation (3) then gives

$$\lambda \mu_1 = -v_z \frac{d \mu_0}{dz} \tag{17}$$

and together with the Eqs. (12), (15) we have

$$v_z = -\frac{\nabla_{ad}}{\nabla_{\text{crit}}} \frac{K_0 T_0}{\varrho_0 g D^2 \mu_0} \mu_1. \quad (g = |\boldsymbol{g}|) \tag{18}$$

Combining Eq. (17) and Eq. (18) gives us the growth rate

$$\lambda^{-1} = \frac{\nabla_{\text{crit}}}{\nabla_{ad}} \frac{\varrho_0 g \mu_0 D^2}{K_0 T_0} \left(\frac{d \mu_0}{dz} \right)^{-1}. \tag{19}$$

The quantity ∇_{crit} is positive if the layer is dynamically stable according to the Ledoux-criterion. From this we see that if $d\mu_0/dz > 0$ the growth rate is positive, the layer is secularly unstable. The time of growth is determined by the time of thermal adjustment of the regions characterized by the wavelength of the perturbation. We can write

$$\lambda^{-1} = \nabla_{\text{crit}} \tau^* / \mu_p \tag{20}$$

where

$$\tau^* = \frac{c_p \varrho_0 D^2}{K_0} \tag{21}$$

is the time scale of thermal adjustment of regions of length scale D. The expression is similar to the expression

$$\tau^*_{KH} = \frac{c_p \kappa_0 \varrho_0^2 d^2}{16 ac T_0^3} = \frac{c_p \varrho_0 d^2}{12 K_0} \tag{22}$$

for spherical perturbations of diameter d (Kippenhahn, 1969). The subscript KH stands for Kelvin-Helmholtz. Indeed if the diameter of the blob approaches the diameter of the whole star, then τ^*_{KH} approaches the Kelvin-Helmholtz time scale of the whole star. One therefore can consider τ^*_{KH} as the Kelvin-Helmholtz time scale of the blob of size d. The instability grows with the Kelvin-Helmholtz time scale of regions whose size is of the order of the wavelength of the perturbation. The smaller the scale of the perturbation, the faster it grows.

4. Blob Theory

The linear theory does not give information on the fully developed instability. As an approach to learn more about this case we consider a blob of diameter d of material with a higher molecular weight, $\mu_0 + \mu_1$, in a region of (constant) lower molecular weight μ_0. We assume $\eta = \mu_1 / \mu_0$ to be small compared to 1. The velocity of the blob has been estimated by Kippenhahn (1974). We introduce the mean free path l of a photon and the pressure scale height H_p by

$$l = \frac{1}{\kappa\varrho}, \quad H_p = \frac{RT_0}{g\mu_0}, \tag{23}$$

where R is the gas constant. We can write for the velocity at which the blob sinks

$$\frac{v_\mu}{c} = \frac{H_p \eta}{(\nabla_{ad} - \nabla) \tau^*_{KH} c} = 48 \frac{(1-\beta)\nabla_{ad}}{(\nabla_{ad} - \nabla)} \frac{H_p l}{d^2} \eta \tag{24}$$

where τ^*_{KH} is the Kelvin-Helmholtz time scale of the blob defined in Eq. (22). Our expression for the velocity of the sinking blob does not take friction into account. We therefore define the Stokes velocity by

$$\frac{v_{St}}{c} = \frac{\eta d^2 g}{18 \nu c} = \frac{5}{36(1-\beta)\tilde{\nu}} \frac{d^2}{l H_p} \eta \tag{25}$$

where we have introduced a dimensionless viscosity $\tilde{\nu}$ which is just the normal viscosity ν in units of the radiative viscosity

$$\tilde{\nu} = \nu \frac{15 c \kappa_0 \varrho_0^2}{2 a T_0^4}. \tag{26}$$

We then obtain

$$\frac{v_\mu}{c} \frac{v_{St}}{c} = \frac{20 \nabla_{ad}}{3 \tilde{\nu} (\nabla_{ad} - \nabla)} \eta^2. \tag{27}$$

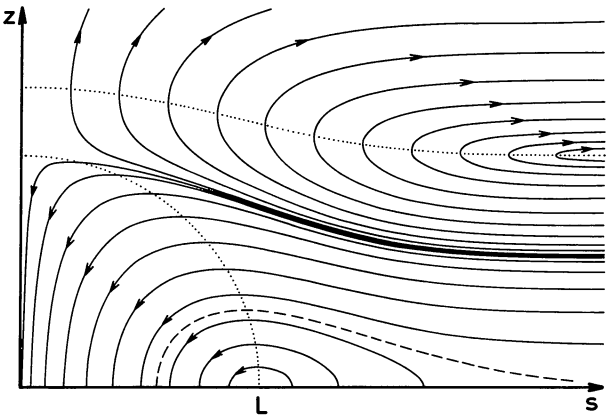

Fig. 1. The circulation pattern in the neighbourhood of a blob of heavier molecular weight. At the dotted lines the s- or the z-component of the velocity vanishes. Stream lines below the broken line are closed. The circulation corresponds to the undisturbed μ-distribution given by Eq. (32). It will be altered by the deformation of the distribution of molecular weight by the circulation itself

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The sinking blobs will have a maximum velocity $v_{\max}$ just before friction becomes important, that is if $v_\mu \approx v_{St}$. We then have

$$\frac{v_{\max}}{c}=\left(\frac{20\,\nabla_{ad}}{3\tilde{v}(\nabla_{ad}-\nabla)}\,\eta\right)^{1/2}. \tag{28}$$

The size of the blobs is then obtained by combining Eqs. (24) and (28) into

$$d^2=24\left(\frac{3}{5}\right)^{1/2}\left(\frac{\tilde{v}\nabla_{ad}}{\nabla_{ad}-\nabla}\right)^{1/2}(1-\beta)H_p l. \tag{29}$$

According to Eq. (28) the maximum velocity can be rather high, especially if η is not too small. Several mechanisms can become important before the maximum velocity is reached. It will take a certain time until the maximum velocity is reached from zero velocity after formation of the blob. The blob meanwhile might loose its identity. Further if v_μ approaches the velocity of sound, v_s, the Stokes formula is not valid since then sound waves carry away momentum.

In the picture of the sinking blob we have just estimated the mean velocity of the heavier material. We have assumed that all parts of the blob move with the same velocity. Actually our formula only holds if our blob is surrounded by a skin which allows pressure adjustment, but which does not allow mixing between the blob and the surrounding. But the blobs have no skin and since, due to hydrostatic equilibrium, our helium blob in the surrounding hydrogen material is always hotter than the neighborhood, energy will flow into the neighboring regions and heat up the material just outside the blob. This will cause motion. In the following we estimate this motion.

5. Self-destruction of the Blobs

Before we determine the full velocity field in the neighborhood of a blob, we give two other expressions for the velocity in a region of constant molecular weight μ_0. From Eqs. (12) and (15) we find

$$\frac{\nabla_{ad}-\nabla}{\nabla_{ad}}\,\varrho_0 v_z g=K_0 T_0 \Delta\eta \tag{30}$$

where v_z is the component of the velocity which points in the direction of $-\boldsymbol{g}$. By using Eqs. (12), (13), and (23) we find

$$\frac{v_z}{c}=\frac{4(1-\beta)\nabla_{ad}}{\nabla_{ad}-\nabla}\,lH_p\Delta\eta. \tag{31}$$

We now assume a continuous μ-distribution which describes a spherically symmetric perturbation of molecular weight

$$\eta=\eta^*\exp\left[-\left(\frac{r}{L}\right)^2\right] \tag{32}$$

where r is the distance from the center of that region. We then obtain from Eq. (31)

$$\frac{\nabla_{ad}-\nabla}{\nabla_{ad}}\,\varrho_0 g v_z=\frac{2K_0T_0}{L^2}\left(2\,\frac{r^2}{L^2}-3\right)\eta^*\exp\left[-\left(\frac{r}{L}\right)^2\right]. \tag{33}$$

This equation gives the spatial dependence of the velocity. It can be compared with the global velocity v_μ of Eq. (24) derived from the blob picture. We identify the η in that formula with a mean value $\bar{\eta}=\eta^*/e$ in the case of a continuous μ-distribution. Taking a blob of diameter $d=2L$ for comparison we find at $r=L$ that $|v_z|=2v_\mu/3$ and at $r=0$ we obtain $|v_z|=2v_\mu e$. We see that matter is sinking only in the central part of the perturbed region, namely for $r<\sqrt{3/2}\,L$, while it is rising in the outer part, where $r>\sqrt{3/2}\,L$. The velocity field is qualitatively given in Fig. 1.

The most striking result is that the velocity in the vertical direction vanishes on the sphere $r=\sqrt{3/2}\,L$. This means that the blob as a whole entity does not sink at all as in the case of the blob surrounded by a skin. Instead, a flow pattern occurs which mixes the blob with its surroundings. This is the *self destruction* of the blob. The velocities in the flow pattern are of the order of the old v_μ. We can estimate the life time of the blob by $\tau\sim L/v_\mu$, an estimate which is based on the assumption that the mass inside the sphere $v_z=0$ is mixed with an equal amount of mass in the surrounding in that time.

If we imagine the forming of a blob out of a linear perturbation we have to keep in mind that as long as the linear approximation holds, self destruction does not occur since it is a second order effect. But the more the "amplitudes" of the surfaces $\mu=$const. grow, the more it becomes a first order effect. But on the other hand, before a real blob is formed in a certain layer, self destruction begins to mix the blob with the surrounding matter of the layer. It seems to be a good estimate to assume that within the time τ out of a secularly unstable discontinuity in molecular weight (with $\mu=\mu_0+\mu_1$ above and with $\mu=\mu_0$ below) the mode of wavelength L begins to form fingers. But at the same time self destruction smears them out with neighboring material. Therefore they form a transition layer of thickness L in which the average molecular weight is $\mu=\mu_0+\mu_1/2$. Then instabilities occur between the transition layer and the regions above and below, but the time scales of the new instabilities are twice as big for the same wavelength since the difference in molecular weight with the region above and below are now only $\mu_1/2$. For this reason the value of v_μ is now half of that before. Therefore if we stick to a certain wavelength L of the perturbation, we have a picture in which below the initial discontinuity the transition layer is spreading inwards. The inward velocity of the inner boundary of the transition layer becomes smaller in proportion to the decreasing value of $|d\mu/dz|$. This is completely different from the old blob (with a skin) which is continually driven by its constant difference, μ_1, in molecular weight.

Before we discuss the consequences we should check the extent to which our picture depends on the Gaussian μ-distribu-

Table 1. Data for a 15 $M_\odot$ star with an outer helium envelope of one solar mass. For different regions radius, pressure scale height, photon mean free path, temperature gradient and the thermodynamic quantities $1-\beta$, ∇_{ad} and δ^2/ϕ are given

Location	$r/10^{10}$ cm	$H_p/10^{10}$ cm	l/cm	∇	$1-\beta$	∇_{ad}	δ^2/ϕ
Bottom of convective layer	22.8	0.012	$1.3\ 10^7$	0.269	0.220	0.290	4.53
Chemical discont.	17.6	2.0	6.0	0.282	0.048	0.355	1.44
Convective core	8.7	3.6	1.0	0.330	0.065	0.345	1.63

tion which we assumed in Eq. (32). We should consider blobs for which, as in the picture of the blob with skin, the molecular weight drops continuously from its value in the blob to the value in the surroundings.

Then the self destruction is even worse. If we think of a blob in which the molecular weight is constant inside the radius $r=L-q/2$, while in the shell $L-q/2 \leqq r \leqq L+q/2$ the molecular weight drops continuously (and with continuous gradient in order to avoid discussing singularities) from $\mu_0+\mu_1$ to μ_0, then outside the shell v_z vanishes. In the shell of thickness $q \ll L$ there is a circulation system with velocities which increase as $1/q^2$ if we make q smaller. Within a short time the matter in the shell will be mixed and circulation systems will appear around the two interfaces between blob interior and shell and between shell and surrounding region. They are now slower, because of smaller gradients in molecular weight. After a while, the circulation will have smeared out the originally sharp transition to a μ-distribution which is better described by our exponential μ-distribution of Eq. (33). Then the former considerations hold. We therefore in the following will work with that type of blob only.

In order to simplify the complicated situation, we replace the true process of mixing by the onset of an instability followed by the formation of blobs of size L which decay due to the circulation system which is created by the blob itself. The time of formation of such a blob and its lifetime are of the order $\tau \sim L/v_\mu$. We therefore can also say that a blob of size L is formed and it sinks with velocity v_μ and after its mean free path L the blob is destroyed and mixed with the surroundings.

6. Diffusion Theory

We therefore have the following picture: A blob is formed which moves downwards while other material rises and mixes with the neighboring gas out of which new rising and sinking elements are built. This is a diffusion process. We therefore define a diffusion coefficient

$$D=v_\mu L=\frac{H_p L}{(\nabla_{ad}-\nabla)\tau^*_{KH}}\,\eta \approx \frac{H_p L^2}{(\nabla_{ad}-\nabla)\tau^*_{KH}}\left|\frac{d\mu}{dr}\right|\frac{1}{\bar{\mu}}, \tag{34}$$

where we have replaced η by $L|d\mu/dr|/\bar{\mu}$, where $\bar{\mu}$ is a mean value of μ in the region of interest. Since $\tau^*_{KH} \sim L^2$ we see that in our approximation the diffusion coefficient does not depend on the size of the elements. We now consider a secularly unstable transition layer of matter whose molecular weight drops from its upper value μ_u to the value $\mu_u-\Delta\mu$ ($\Delta\mu>0$) at the bottom, over a distance W in depth. We then can define a diffusion time τ_{diff} during which diffusion doubles the geometrical thickness of this transition layer. We can estimate

$$\tau_{\text{diff}} \approx \frac{W^2}{D}=\left|\frac{dr}{d\mu}\right|\bar{\mu}\,\frac{(\nabla_{ad}-\nabla)}{H_p}\,\frac{W^2}{L^2}\,\tau^*_{KH} \approx \frac{\bar{\mu}}{\Delta\mu}\,\frac{W^3}{L^2 H_p}\,\tau^*_{KH}(\nabla_{ad}-\nabla). \tag{35}$$

We see that the diffusion time goes with the third power of the geometrical thickness of the region and is proportional to the inverse of the difference $\Delta\mu$ of the molecular weight between both boundaries of the transition layer.

Inserting τ^*_{KH} from Eq. (22) with $d=2L$ and replacing $1/\kappa\varrho$ by l [Eq. (23)] leads to

$$\tau_{\text{diff}}=\frac{\bar{\mu}}{\Delta\mu}\,\frac{c_p \varrho W^3}{4acT^3 H_p l}(\nabla_{ad}-\nabla). \tag{36}$$

Introducing β and $\nabla_{ad}=P/\varrho T c_p$, one finally has

$$\tau_{\text{diff}}=\frac{\bar{\mu}}{\Delta\mu}\,\frac{1}{12(1-\beta)\nabla_{ad}}\,\frac{W^3}{H_p lc}(\nabla_{ad}-\nabla). \tag{37}$$

Up to now our estimates are correct only for elements and transition layers with a thickness small compared to the scale height. But if we extrapolate our results to bigger elements and thicker transition layers we then can expect that we at least still get the right order of magnitude.

7. Application

a) A Massive Main Sequence Star with a Helium Envelope

As a specific example we take an originally 14 $M_\odot$ main sequence star with a composition of $X=0.739$, $Y=0.240$, which has accreted 1 $M_\odot$ of pure helium from a companion. It therefore now forms a 15 $M_\odot$ star with an envelope of heavier material above matter of lower molecular weight. Results of the stellar structure computations for such a model are presented in Table 1. Just below the photosphere the model is convective, however the convective layer is very thin (0.04% of the stellar radius) and the convective flux is negligible. At the discontinuity the layers are dynamically unstable since the Ledoux criterion

$$\nabla_{\text{crit}}=\nabla_{ad}-\nabla+\mu_p>0$$

is violated. We therefore expect first mixing on a dynamical time scale which smears out the discontinuity. As we estimate in appendix *B*, the layer involved contains about 2.5 $M_\odot$ and extends from the radius $14.8\ 10^{10}$ cm to $21.7\ 10^{10}$ cm. These numbers are estimated with the rough method described in appendix B. After this rapid mixing a smooth transition zone should have formed which is dynamically in a marginal state, but secularly it is still unstable. And then thermohaline mixing will occur. For the estimate of the diffusion time of the secular instability we will use the numerical values of the discontinuous model. Except for the product $H_p l$, which increases drastically close to the photosphere, the relevant parameters for calculating the diffusion time vary by less than an order of magnitude over the whole radiative envelope. We therefore will use the values at the chemical dis-

Table 2. Data for 1.3 $M_\odot$ star after the helium flash. For different regions the same quantities as in Table 1 are given

Location	$r/10^9$ cm	$H_p/10^8$ cm	$l/10^{-4}$ cm	∇	$1-\beta$	∇_{ad}	δ^2/ϕ
Bottom of helium burning shell	1.104	2.13	11.8	−1.099	$1.87\ 10^{-2}$	0.383	1.062
Temperature minimum	1.052	0.64	2.13	0.114	$2.11\ 10^{-6}$	0.399	0.048
Center	0	5.66	8.45	−0.005	$2.96\ 10^{-7}$	0.409	0.006

continuity to estimate diffusion times. In the following for our numerical estimates we take radiation pressure into account and use the formula (A7) which is derived in appendix A. Our principal conclusions could also be obtained by using the formulae derived in the main part of this paper. At what time will the diffusion front reach the surface of the star, bringing up hydrogen rich material? The matter is completely ionized in the region of interest and therefore $\bar{\mu}/\Delta\mu=1.31$. Since the diffusion front travels in both directions, we have to take $W=9.1\ 10^{10}$ cm, which is twice the distance to the surface and obtain $\tau_{\rm diff}=4.5\ 10^3$ yr. This value will be somewhat reduced by the fact that near the photosphere the diffusion front travels faster, but as an order of magnitude estimate it shows that mixing due to thermohaline convection is substantially reduced by taking into account the self destruction of the blobs (Ulrich, 1972 estimated a time scale of 400 yr for an appreciable reduction of the μ-gradient in a similar case). To reach the outer boundary of the convective core, diffusion will have to extend over the whole radiative envelope with $W=1.41\ 10^{11}$ cm, which gives $\tau_{\rm diff}=1.7\ 10^4$ yr. Here the actual time will be longer since after $4.5\ 10^3$ yr the value of $\Delta\mu$ starts to decrease.

b) Non Central Helium Flash

We now turn to the other example for which this process is of interest, the 1.3 $M_\odot$ star after the (noncentral) helium flash as computed by Thomas (1967). Here a layer of carbon on top of pure helium is produced in the interior of the star. The parameters are listed in Table 2. For the width W over which mixing should occur we take the radius of the helium burning shell. We assume that after the helium in this shell is exhausted that $\bar{\mu}/\Delta\mu=4$. If we take the values at the bottom of the carbon shell we find $\tau_{\rm diff}=4.1\ 10^5$ yr. This is already long compared to the time scale of exhaustion of helium in the shell which is of the order of 1000 yr. We conclude that thermohaline mixing is not important. The time scale of mixing becomes $5.6\ 10^8$ yr if we take the values at the temperature minimum between center and shell, which emphasizes the unimportance of mixing. Diffusion is fast near the chemical discontinuity but much slower after the temperature minimum because of the large drop in $1-\beta$. The time scales are so large that one can safely dismiss the possibility of carbon being mixed into the degenerate helium core, contrary to the assumption of Thomas (1967). Calculations with corrected neutrino rates (Thomas, 1970) have not been carried far enough into the post flash phase to be used for our estimates. But we think that our conclusions are not drastically modified by better models. For our numerical estimates we will use the formula (A7, appendix), which is also corrected for degeneracy.

Acknowledgements. Part of this work was carried out while one of the authors (R.K.) was Distinguished Visiting Professor at Ohio State University. He wants to thank his colleagues from the Astronomy Department in Columbus, Ohio, for their hospitality.

Appendix A

Effects of Radiation Pressure and Degeneracy

For those cases where the perfect gas equation of state is not a good approximation, we make use of the following logarithmic derivatives:

$$\alpha=\left(\frac{\partial\ln\varrho}{\partial\ln P}\right)_{T,\mu}\quad\delta=-\left(\frac{\partial\ln\varrho}{\partial\ln T}\right)_{P,\mu}\quad\phi=\left(\frac{\partial\ln\varrho}{\partial\ln\mu}\right)_{P,T}. \tag{A1}$$

Constancy of pressure and density on equipotential surfaces then requires that a change in molecular weight is compensated by a change in temperature according to

$$\frac{d\ln T}{d\ln\mu}=\frac{\phi}{\delta}. \tag{A2}$$

Treating the variations in molecular weight as a perturbation, we can neglect changes in ϕ and δ and obtain

$$\frac{T_1}{T_0}=\frac{\phi}{\delta}\frac{\mu_1}{\mu_0}, \tag{A3}$$

which replaces Eq. (5).

The energy equation contains a factor δ:

$$\nabla F_1=-v(c_p\varrho_0\nabla T_0-\delta\nabla P_0)-c_p\varrho_0\frac{\partial T_1}{\partial t} \tag{A4}$$

and so does the thermodynamic relation $\nabla_{ad}=\delta P_0/T_0\varrho_0 c_p$. The growth rate of the instability discussed in Sect. 3 can then be written as

$$\lambda^{-1}=\nabla_{\rm crit}\tau^*/\tilde{\mu}_p$$

where $\nabla_{\rm crit}=\tilde{\mu}_p+\nabla_{ad}-\nabla$ and $\tilde{\mu}_p=\phi\mu_p/\delta$ and the expression for τ^* remains unchanged.

For the estimates of diffusion times we have to use the correct velocity for sinking blobs as given by Kippenhahn (1974):

$$v_\mu=\frac{\phi H_p\eta}{\delta(\nabla_{ad}-\nabla)\tau^*_{KH}} \tag{A6}$$

which leads to a replacement of Eq. (37) by

$$\tau_{\rm diff}=\frac{\bar{\mu}}{\Delta\mu}\frac{\delta^2(\nabla_{ad}-\nabla)}{12\phi(1-\beta)\nabla_{ad}}\frac{W^3}{H_p l c}. \tag{A7}$$

We use this equation for our numerical estimates.

Appendix B

The Transition Layer Formed by the Dynamical Instability of the 15 $M_\odot$ Star

For marginal stability we obtain

$$\mu_p = \frac{\delta}{\phi}(\nabla - \nabla_{ad}). \tag{B1}$$

Therefore if throughout that zone $\Delta \log \mu$ is the difference in the logarithms of the molecular weight on top and at the bottom of the transition layer, the difference in pressure is given by

$$\Delta \log P = \frac{\phi}{\delta} \frac{\Delta \log \mu}{\nabla - \nabla_{ad}} \tag{B2}$$

and if we insert typical numerical values which can be found in Table 1 we find for the difference of the pressures on the top and at the bottom of the transition zone $\Delta \log P = 4$. The value of M_r at the discontinuity before dynamical mixing, M_{ro}, after dynamical mixing will correspond to a layer which is within the smooth transition zone which reaches from the inner boundary M_{ri} to the outer boundary M_{re}. Therefore

$$M_{ri} < M_{ro} < M_{re}. \tag{B3}$$

The amount of helium in that mass interval is the same before and after mixing since the other regions were not involved. If for the helium content $Y(M_r)$ we assume for our approximations a linear interpolation formula in M_r in the transition zone between the external value 1 and the internal value 0.24 the amount of helium in the transition zone is given by

$$\Delta M_{\mathrm{He}} = 0.38\,(\Delta M_{ri} + \Delta M_{re}). \tag{B4}$$

On the other hand the amount of helium in the same mass interval before mixing was $M_{re} - M_{ro}$. We therefore obtain for the boundaries of the transition layer

$$M_{ri} = 2.63\, M_{ro} - 1.63\, M_{re}. \tag{B5}$$

On the other hand our condition was that the difference in log P between these two boundaries is 4.00. These conditions should be fulfilled in the model after mixing. For our estimate of the thickness of the transition layer we use the numerical values of the discontinuous model. One then just has to look for two layers which fulfil the conditions (B5) and $\Delta \log P = 4$. We obtain the following values

$M_{ro} = 14 \quad M_\odot$, $(\log P)_o = 14.41$ $\quad r_o = 17.6\ 10^{10}$ cm

$M_{re} = 14.99\ M_\odot$, $(\log P)_e = 10.97$ $\quad r_e = 21.7\ 10^{10}$ cm

$M_{ri} = 12.37\ M_\odot$, $(\log P)_i = 14.97$ $\quad r_i = 14.8\ 10^{10}$ cm.

After such a layer has been formed the model is dynamically stable or at least in a marginal state, but ist is still secularly unstable since the molecular weight is decreasing inwards. Then the secular mixing discussed in the main part of the paper starts.

References

Kato, S.: 1966, *Publ. Astron. Soc.* Japan **18**, 374

Kippenhahn, R.: 1969, *Astron. Astrophys.* **2**, 309

Kippenhahn, R.: 1974, in *Late Stages of Stellar Evolution*, ed. R.J. Taylor, IAU *Symp.*No . **66**, 20

Stern, M.E.: 1960, *Tellus* **12**, 2

Thomas, H.-C.: 1967, *Z. Astrophys.* **67**, 420

Thomas, H.-C.: 1970, *Astrophys. Space Sci.* **6**, 400

Ulrich, R.K.: 1972, *Astrophys. J.* **172**, 165

Veronis, G.J.: 1965, *Marine Res.* **21**, 1