CONTINUUM EMISSION FROM A ROTATING NON-GRAY STELLAR ATMOSPHERE

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ABSTRACT

This paper is an attempt to extend and revise the results of an earlier paper In the earlier work, the continuum emission of an early, rapidly rotating star was predicted under the assumptions of a gray atmosphere, Roche model, and Von Zeipel's theorem In this study the assumption of a gray atmosphere has been replaced by a series of non-gray atmospheres calculated by means of the Strömgren-Underhill Atmosphere-Computer Program. A more accurate formulation of the problem, along with a more sophisticated numerical integration scheme has been employed.

The results of the earlier paper have, by and large, been substantiated. That is, the gray atmosphere proved to be a moderately good approximation in the visible range and a rather poor approximation in the ultraviolet. However, the effects indicated by the earlier study have been shown to be of the correct form but of too small a magnitude. A variation of up to 1.4 mag in the few ultraviolet colors due to effects of rotation may be expected.

I. INTRODUCTION

This paper is essentially an extension of an earlier work by the author (1963) in which the continuum emission from a rapidly rotating early-type star was assumed to be well approximated by a gray atmosphere. Some of the results of that study indicated that the effects of rotation upon the continuum of an early-type star could be significant, particularly in the far ultraviolet. The increased interest in the ultraviolet continuum of early-type stars has prompted the author to continue this investigation and attempt to remove some of the more objectionable assumptions which existed in the earlier study.

We may divide the assumptions made in this study into two areas: those concerned with the structure of the model, and those concerned with the intercomparison of two or more models. Perhaps the most objectionable assumption in either area is the approximation of the emission by means of a gray atmosphere. This assumption has been completely removed by using a large number of non-gray model atmospheres computed by means of the Strömgren-Underhill Atmosphere-Computer Program. In addition to removal of this assumption, considerable progress has been made in simplifying the formulation of the problem and in extending the results to stars throughout the B spectral class.

We shall first consider construction of the model in terms of normalized parameters and then discuss the method used to obtain the flux. Some confusion may be generated by the difference between the total luminosity and the integrated specific intensity. The reader should keep in mind that, because of the asymmetry in the radiation field produced by rotation, these quantities are not the same. The total luminosity refers to the total energy radiated by the star in all directions.

The integrated specific intensity is just the flux per unit solid angle seen by the observer. In the second half of the paper an attempt will be made to compare different models and group them into "families" in order to demonstrate the effects of rotation. One of these families is presented in detail, but a discussion of the variation of these effects with spectral type shall be handled as a separate topic in a later paper.

II. CONSTRUCTION OF THE MODELS

We may divide the construction of the models into two parts. The first is concerned with the formulation of the shape and physical parameters which specify the atmosphere of the star at any point. The second part will deal with calculation of the specific intensity appropriate to that point on the surface and direction of the observer, and with the integration of the specific intensity over the surface.

It is perhaps useful to first state the assumptions on which investigation into the first region will be based: (1) The star's effective gravity field will be assumed to be that of a Roche model modified by rigid rotation. (2) The flux at any point on the surface of the rotating star will be proportional to the gravity.

Since these assumptions are of primary importance to the construction of the models, it is appropriate that we consider their validity. First, the Roche model assumption enters only in the determination of the gravity field due to the mass distribution of the star. The Schwarzschild (1958) models for early main-sequence stars show that approximately 90 per cent of the mass lies within a sphere having 50 per cent of the radius. One may estimate the distortion of such a sphere due to a rotational angular velocity equivalent to the breakup velocity of the entire star. Under the assumption of rigid rotation this distortion is less than 1 per cent. Thus, the perturbation to the gravity field that will be at least of second order in terms of the distortion will be less than 0.01 per cent. Any rotation law that does not increase toward the center will yield an even smaller perturbation. Indeed, Epps (1964) has used the actual mass distribution to show that the perturbation to the gravitational potential will be about 1 part per million.

Some concern may be generated by the use of rigid rotation for construction of the models, as Von Zeipel's theorem forbids it being realized in the case of rapidly rotating stars. Since the rotation only enters the problems in determining the effective gravity, one must only worry about departures from rigid rotation which are an appreciable percentage of the equatorial velocity. Schwarzschild (1958) has estimated that the meridional circulation currents will be many orders of magnitude less than the rotational velocity and hence we may ignore them. The variation of rotational velocity with latitude presents somewhat more of a problem. If we assume that the angular velocity is a monotone decreasing function with decreasing latitude, then the departure of the shape from that obtained from a rigidly rotating model having the same equatorial velocity will be confined to the region near the poles. However, since the rotational distortion is small near the poles, the percentage error in the radius introduced by departures from rigid rotation should also be small.

Finally, the assumption that H(i.e., the luminous flux) is proportional to g may be objected to as it is a result of Von Zeipel's theorem, which, as Eddington (1926) has pointed out, cannot strictly hold for rapidly rotating stars. However, it appears that Von Zeipel's theorem may be a sufficient, but not necessary condition for the flux to be proportional to the surface gravity. Von Zeipel (1924*b*) has also shown that the law $H \propto g$ also applies to stars distorted by tidal forces, and Jeans (1919) reached a similar conclusion for very massive rotationally distorted stars. Thus, it appears that the $H \propto g$ law may perhaps be the result of the distortion of the star and not the rotation, and since rapid rotation, rigid or otherwise, will distort a star, it does not seem unreasonable that the law will hold at least as a good approximation. Indeed, Eddington (1926) concludes his discussion of Von Zeipel's theorem by stating that the approximation will justify itself.

Partly for computational reasons it was found desirable to normalize the surface gravity in terms of the surface gravity at the pole. The resulting gravity then proves to be invariant under changes in the basic parameters of a family of models (i.e., M, R_p, L). This normalization may be accomplished with the aid of the normalized rotational velocity $w = (\omega/\omega_c)$ where the critical angular velocity ω_c is given by

$$\omega_c^2 = \frac{GM}{R_e^3} = \frac{8GM}{27R_p^3}, \tag{1}$$

and where R_e and R_p are the equatorial and polar radii, respectively. In addition to the normalized rotational velocity, we shall need to normalize the radius so that

$$x = R/R_p . \tag{2}$$

The expression for the surface gravity derived in the previous study (Collins 1963) is

$$g = \left[\left(\frac{GM}{R^2} - \omega^2 R \sin^2\theta\right)^2 + (\omega^4 R^2 \sin^2\theta \cos^2\theta)\right]^{1/2}, \tag{3}$$

where θ is the co-latitude measured from the pole of rotation. Normalizing equation (3) in terms of the polar gravity we have

$$g = \frac{GM}{R_p^2} g_n , \tag{4}$$

where

$$g_n = \frac{8}{27}\left[\left(\frac{27}{8x^2} - w^2 x \sin^2\theta\right)^2 + (w^4 x^2 \sin^2\theta \cos^2\theta)\right]^{1/2}, \tag{5}$$

which may be obtained by combining equations (1), (2), and (3). This function has essentially been calculated in the previous study (Collins 1963) and may be obtained by

TABLE 1

NORMALIZED VON ZEIPEL CONSTANT APPROXIMATION PARAMETERS

i	a_i	i	a_i
0	+1 0000000	5	+0 0000000
1	+0 0000000	6	− 1162862
2	+0 0000000	7	+ 0000000
3	+0 7770854	8	+ 0000000
4	−0 5863339	9	+0 4589665

dividing the ordinate of Figure 2 in that work by 13612 cm/sec^2. As derived in the earlier work (Collins 1963), the effective temperature of various points on the surface of the star may be expressed as

$$T_e^4(w, \theta) = C_\omega g , \tag{6}$$

and C_ω shall be known as the Von Zeipel constant and is given by

$$C_\omega = \frac{L_\omega}{\sigma}\left[\int_A g\, dA\right]^{-1} = \frac{L_\omega}{4\pi GM\sigma}\left[\int_0^1 \frac{g_n x^2(\mu)\, d\mu}{\hat{g}_n \cdot \hat{r}}\right]^{-1} \tag{7}$$

and μ is just $\cos\theta$. We shall call the reciprocal of the term in the brackets the normalized Von Zeipel constant $C_n(w)$ and determine it explicitly from equation (7) or from

$$C_n(w) = (A\, \bar{g}_n)^{-1} . \tag{8}$$

In this expression A is a normalized area (i.e., normalized by $4\pi R_p^2$) and $\bar{g}_n$ is the mean value of the normalized surface gravity. For purposes of computation $C_n(w)$ was approximated by

$$C_n(w) = \sum_{i=0}^{5} a_i w^i , \tag{9}$$

where the values of a_i are given in Table 1. Figure 1 contains a plot of A, $\bar{g}_n$, and $C_n(w)$ as functions of w, while Figure 2 defines the coordinate system for the model. It is interesting to note that C_n (w) is very nearly numerically equal to A (w). Since both A and $\bar{g}_n$ are invariant under changes in R_p, M, and L_ω, C_n (w) is also invariant. Thus, variations of R_p and L_ω with ω which might result from a detailed study of rotating stellar interiors will in no way affect $C_n(w)$. We shall return to this result in Section IV with the discussion of families of models.

III. DETERMINATION OF THE NON-GRAY FLUXES AND COLORS

The method used to determine the monochromatic magnitudes and fluxes is quite similar to the method used in the earlier study (Collins 1963). The main differences lie in the type of atmospheres used and the frame of reference in which the integrations were carried out. The model atmospheres used in this study are similar to those described by Underhill (1962, 1963). Indeed, the Strömgren-Underhill Atmosphere-Computer Program was slightly modified for use on the CDC 1604 computer and was used exclusively

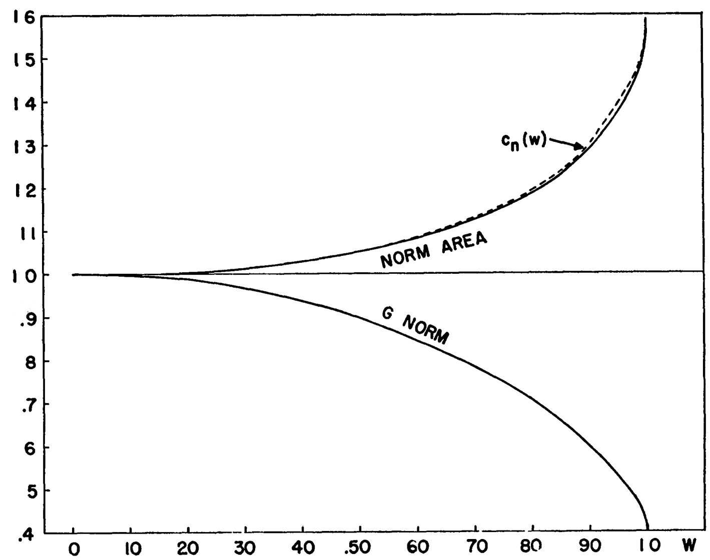

FIG. 1.—Variation of the normalized gravity, area, and Von Zeipel constants with normalized rotational velocity. These dimensionless quantities are invariant to changes in L, M, and R_p.

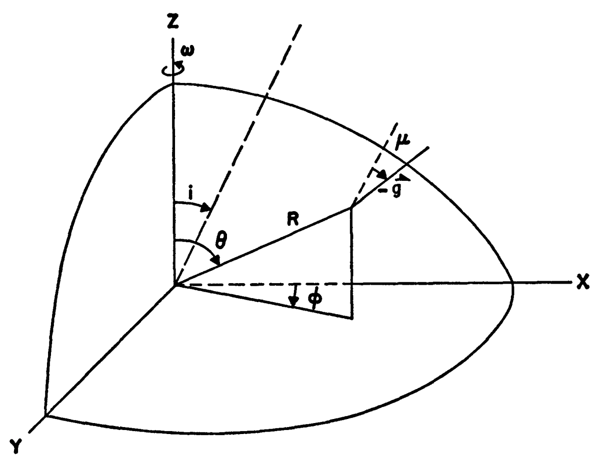

FIG. 2.—Coordinate systems defining angles used in this study. The observer is contained in the X-Z plane.

for the computations of the model atmospheres. The iteration for constant flux was carried out until the following inequality was satisfied:

$$|1 - [F(\tau)/F(0.6)]|e^{-\tau} < 0.01 . \tag{10}$$

Satisfying this inequality at all optical depths used in this study insured that the surface fluxes were constant within 1 per cent while the mean deviation of the flux from a constant value was generally less than 0.5 per cent. Table 2 contains a list of the effective temperatures and surface gravities for the atmospheres used in this study, and Figures 3–8 show the emergent monochromatic flux for a selected number of the atmospheres.

TABLE 2

ATMOSPHERIC PARAMETERS FOR ROTATION APPROXIMATION $X = 0\ 68$, $Y = 0\ 32$

$\log_{10} g$	T_e (° K)					
2 0	14501	17847	21617	26482		
2 5	14575	18004	21177	27581	. . . .	
3 0	14616	18053	22391	27449	31753	. .
3 5	14595	18055	22433	27583	32127	39734
4 0	14596	18039	22415	27678	32165	40086
4 5	14595	17893	22385	27678	31906	39760

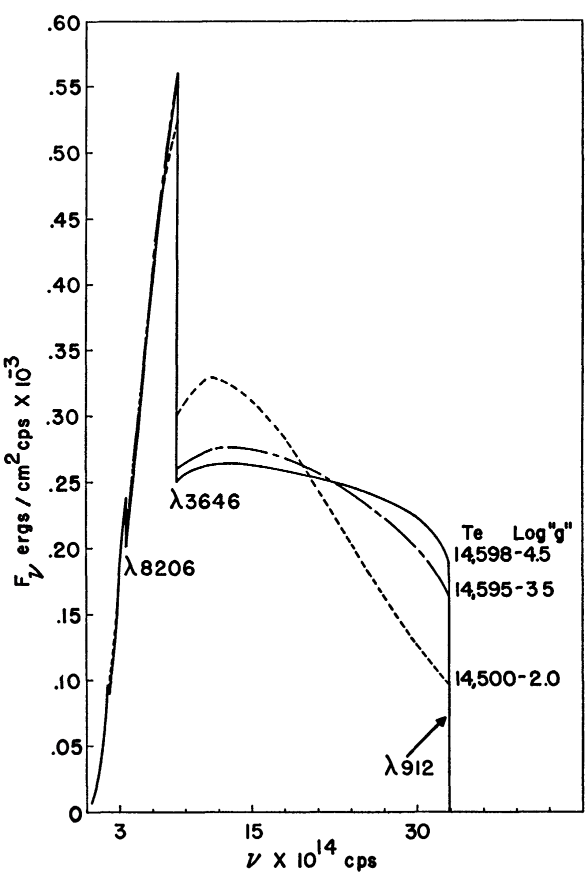

FIG. 3.—Energy distribution for model atmospheres having various log g's and an effective temperature (T_e) of approximately 14600° K.

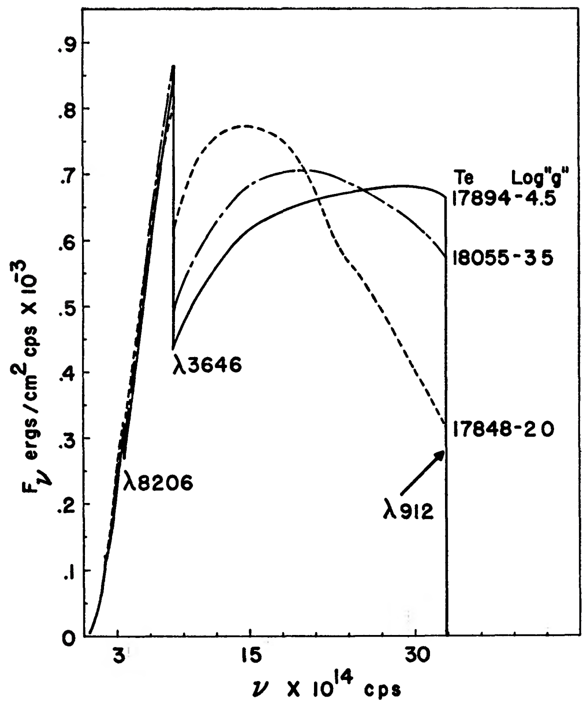

Fig. 4.—Same as Fig. 3 but with T_e approximately 17900° K

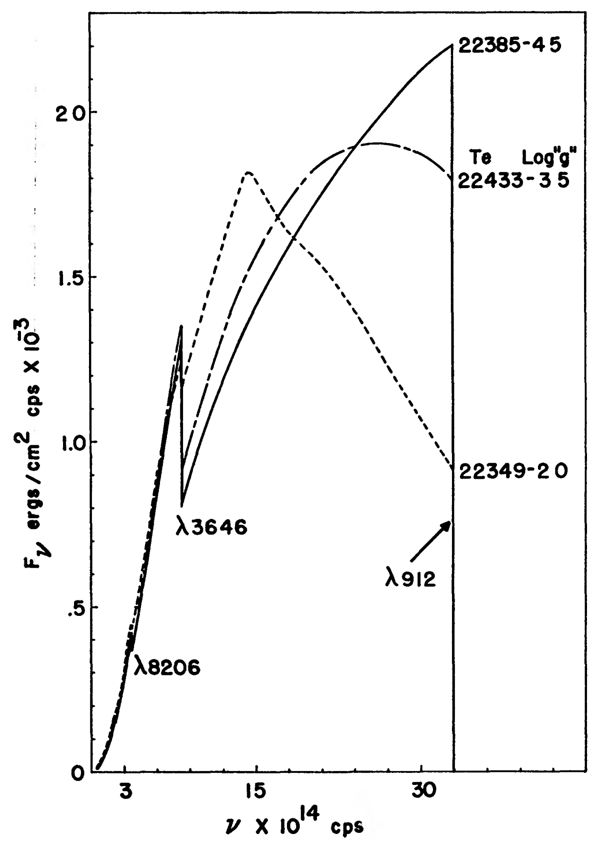

Fig. 5.—Same as Fig. 3 but with T_e approximately 22400° K

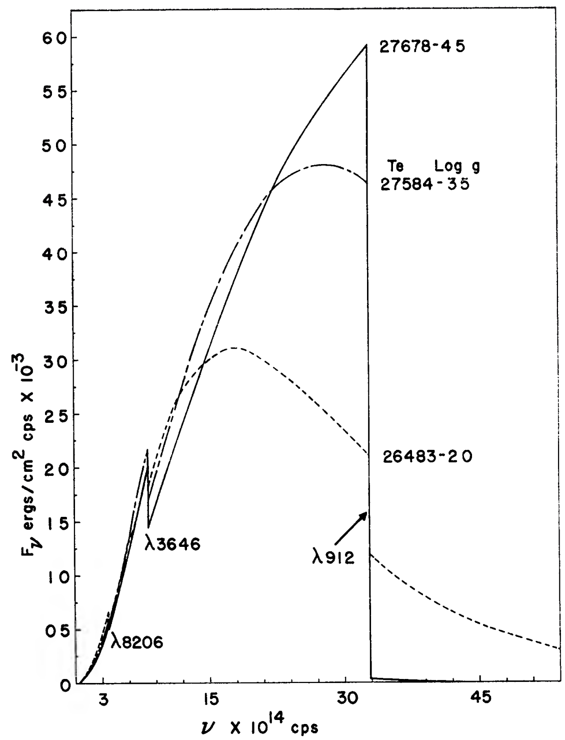

FIG 6 —Same as Fig 3 but with T_e approximately 27000° K

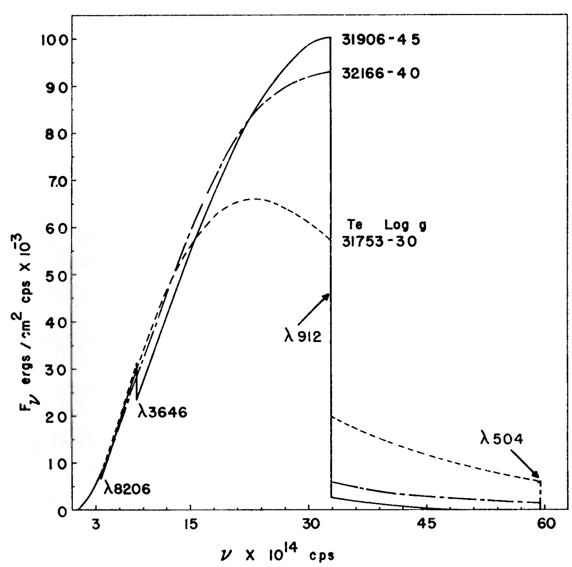

FIG 7 —Same as Fig 3 but with T_e approximately 32000° K

The spacing of the temperatures and gravities is sufficiently small to allow a reliable atmosphere to be obtained anywhere within the array by means of linear interpolation. Thus, by obtaining the local effective temperatures and surface gravity by means of equations (4), (5), and (6), it is possible to find non-gray model atmospheres which adequately represent the atmosphere at a given point on the surface of a rotating star.

The flux emitted by a rotating star inclined at an angle i in the direction of an observer (see Fig. 2), will be

$$F_\nu(i, w) = \pi R_p^2 \int_A I_\nu(\theta, \phi)\, x^2(\theta) \sin\theta \,|\, \mu/\hat{g}\cdot\hat{r}) \,|\, dA\,, \tag{11}$$

where A is the area of the star which can be seen by the observer and μ is the cosine of the angle between the normal to the surface (g) and the observer. It can be shown that, if the

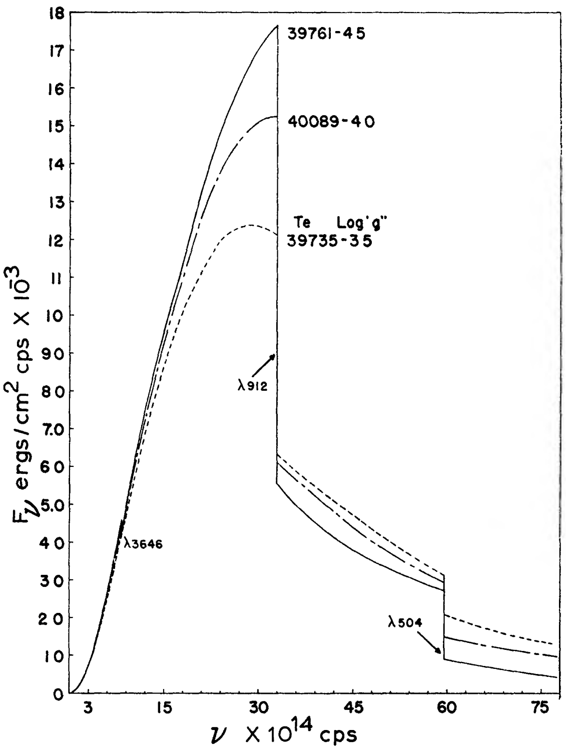

FIG 8—Same as Fig 3 but with T_e approximately 40000° K

integrand of equation (11) is symmetric about the axis of rotation and the equatorial plane, the integral may be performed over any half of the star defined by a plane passing through the center of the star and containing the Y-axis of the star's coordinate system (see Fig. 2).

Under these conditions, equation (11) becomes

$$F_\nu(i, w) = R_p^2 \int_{-\pi/2}^{\pi/2} \int_0^{\pi} [\,(I_\nu(\theta, \phi)\, x^2(\theta) \sin\theta \,|\, \mu/(\hat{g}\cdot\hat{r}) \,|\,]\, d\theta\, d\phi\,, \tag{12}$$

where the quantity $\mu/(\hat{g}\cdot\hat{r})$ is just

$$\frac{\mu}{\hat{g}\cdot\hat{r}} = (\sin\theta \sin i \cos\phi + \cos\theta \cos i) - a(\sin i \cos\phi \cos\theta - \sin\theta \sin i) \tag{13}$$

and

$$a = \frac{8w^2 x^3(\theta) \sin\theta \cos\theta}{[\,27 - 8w^2 x^3(\theta) \sin^2\theta\,]}\,. \tag{14}$$

Thus, once the intensity is determined at any part on the star's surface we may determine the monochromatic flux by straightforward integration of equation (12).

The classical solution to the transfer equation yields

$$I_\nu(\mu, 0) = \int_0^\infty S_\nu(T)\, e^{-\tau/\mu} \frac{d\tau}{\mu}. \tag{15}$$

Both the $T(\tau)$ distribution and the $S_\nu(T)$ functions are obtained directly by interpolation from the model atmospheres previously mentioned. The integral in equation (15) was obtained by Gauss-Laguerre quadrature so that

$$I_\nu(\mu, 0) = \sum_{i=0}^{M} S_\nu(T_i)\gamma_i, \tag{16}$$

where

$$T_i = T(\tau_i) = T(\mu x_i), \tag{17}$$

the x_i's are the zeros of the Laguerre polynomials of order M in the interval 0 to ∞, while the γ_i's are the appropriate weights.

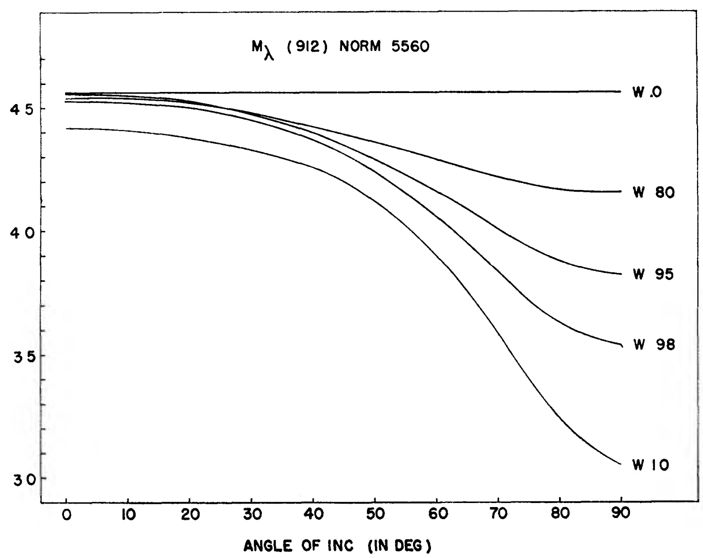

FIG 9.—Relative monochromatic magnitudes in energy per unit wavelength for the family of models described in this study.

The numerical integration of the specific intensity was performed in the same way as it was in the earlier study, except that the projection factor omitted earlier was included. The appropriate approximation formula used to evaluate equation (12) is

$$F_\nu(w, i) = \frac{\pi^2 R_p^{\,2}}{N'} \sum_{i=1}^{N} \sum_{j=1}^{N'} I_\nu(\theta_i, \phi_j)\, x^2(\theta_i) \sin(\theta_i)\, |\, \mu_i/(\hat{g}_i \cdot \hat{r}_i)\, |\, B_i\,, \tag{18}$$

where $\cos\theta_i$'s are the zeros of the Legendre polynomials of order N in the interval -1 to $+1$, B_i's are the appropriate weights and $\sin\phi_j$'s are the zeros of the Chebyschev polynomials of order N' in the interval of -1 to $+1$. A total of 132 points were used to cover the surface of the model which yielded an error consistent with the interpolation error incurred within the atmosphere grid. The integrations were carried out on an IBM 7094 computer at the Numerical Computation Laboratory of the Ohio State University. Figures 9–13 show the results of these integrations.

IV. DISCUSSION OF RESULTS AND COMPARISON OF MODELS

It is difficult indeed to separate the discussion of the results from the mode of comparison of various models. However, we may make some general statements concerning the results, most of which are not too surprising.

1. Stars of equal mass, luminosity, and angle of inclination will appear fainter and redder as the rotational velocity increases.
2. Stars of equal mass, luminosity, and rotational velocity will appear fainter and redder as the angle of inclination increases from 0° to 90°.
3. These effects are more pronounced in the blue (down to 912 Å) than in the red.
4. The earlier the spectral type, the more pronounced are the effects.

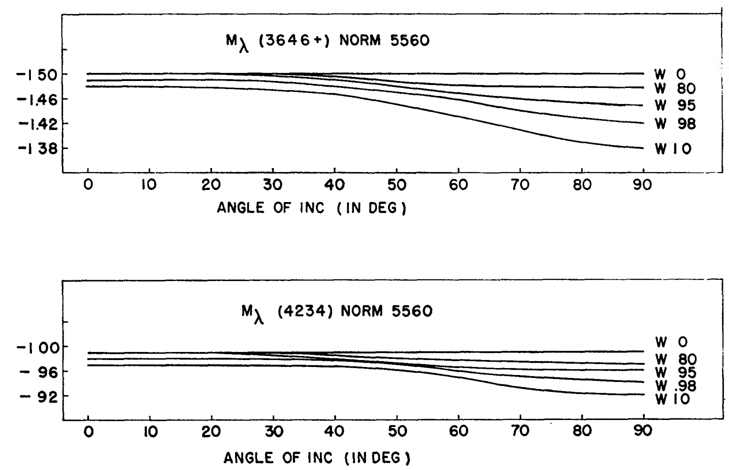

FIG. 10 —Same as Fig 9

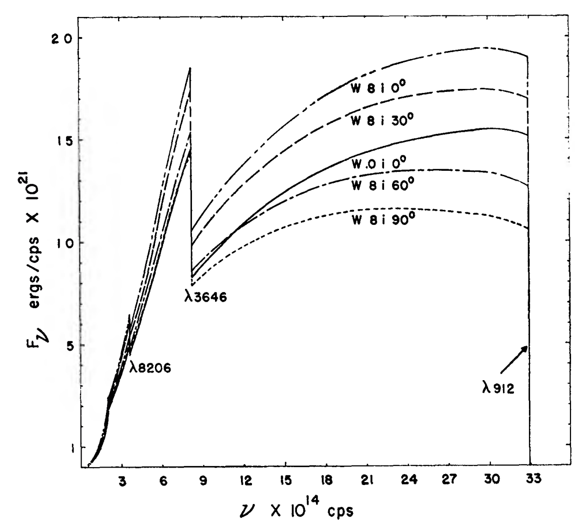

FIG 11.—Energy distribution for a rotating model atmosphere with $w = 0.8$ and various angles of inclination The solid line in the center of the figure represents the energy distribution for the non-rotating atmosphere of the same family as the rotating atmospheres The calculations include limb darkening, "gravity darkening," and deviations from spherical shape of the rotating star.

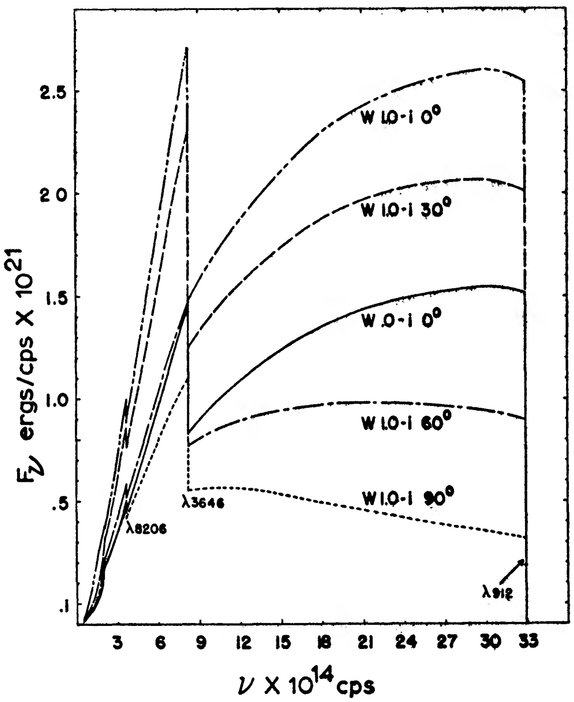

FIG. 12.—Same as Fig. 11 but with $w = 1.0$

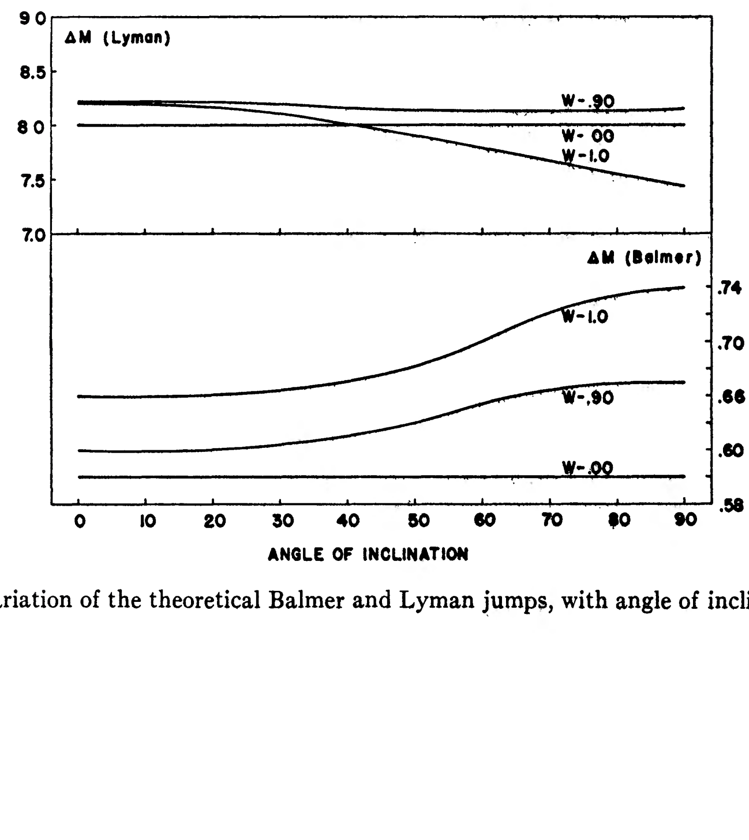

FIG. 13.—Variation of the theoretical Balmer and Lyman jumps, with angle of inclination and rotational velocity.

5. The mean effective temperature and mean surface gravity determined for rapidly rotating stars by means of spectral analysis will not be an accurate reflection of the true mean value of these quantities.

6. With the exception of color effects in the far ultraviolet, no rotational effects on the continuum radiation are significant unless $w > 0.8$.

Only the first of these results depends in any way on the mode of comparison of various models. A detailed study of the effects of rotation will require the construction of model rotating stellar interiors in order to determine how L_w and R_p vary with angular velocity and chemical composition for a star of a given mass. However, until such models are available it appears instructive to apply the present considerations to the comparison of rotating stars having the same mass luminosity and chemical composition. This leads to the last two assumptions which are as follows:

1. $L_w = L_0 =$ const., where L_0 is the value for the total luminosity of a non-rotating model.

2. $R_p(w) = R_p(0) =$ const., where $R_p(0)$ is the polar radius of the non-rotating model.

The first of these assumptions seems quite reasonable and consistent with the earlier assumptions. Since the central structure is not seriously distorted by rotation, it seems reasonable to suggest that the sources of energy generation which are highly concentrated in the center should be relatively unaffected by rotation. The second assumption is somewhat harder to justify and is made here mainly out of necessity. However, since the rotational forces vanish along the axis of rotation, one might hope that the structure would not be seriously disturbed from that of the non-rotating case.

Adopting the above assumptions we may now define a basis for comparing models. That is, we may define a family of models to be those models having the same mass, total luminosity, and polar radius. The results may now be divided into effects of rotation within a family and effects between families. Most of our discussion will be limited to the first area since the numerical effort required to generate more than two families has so far been prohibitive. The first family investigated is based on the same parameters as those used in earlier work by the author (1963) (i.e., $M = 8M_\odot$, $L = 1835\ L_\odot$, $R_p = 4R_\odot$). Figures 9 and 10 show the monochromatic magnitudes for several wavelengths and rotational velocities as they vary with angle of inclination. These graphs bear out the general conclusion concerning the effects of rotation on the continuum, but give somewhat more quantitative results. Figures 11 and 12 show the variation of the entire flux distribution under various angles of inclination for two rotational velocities. These figures clearly demonstrate that the rotational effects should be detectable in the near ultraviolet and will be extremely strong in the far ultraviolet. Indeed, in rapidly rotating stars, the rotational effects will be one of the most prominent features of the continuum emission below 3000Å . This effect will have to be considered by anyone making observations of rapidly rotating early-type stars in this region of the spectrum. This fact appears to be borne out by the observations of Stecher (1964), who apparently finds a depressed ultraviolet continuum in the rapidly rotating stars η UMa and α Aql.

If one corrects the value given here for the non-rotating star by $5 \log_{10}(5560/\lambda_c)$, he will arrive at values for the colors $M(1/\lambda)$ which agree very well with the observed colors for a B3 V star as given by Code (1960). Unfortunately, no accurate monochromatic colors for the far ultraviolet and stars of this spectral type were available to the author, but the theoretical models agree well with other models in this range of T_e and g. The final figure presented here demonstrates the variation of the Balmer and Lyman jumps with angle of inclination. Although some variation was found to exist for the Balmer jump, it is small and well within the limits set by the errors in the earlier paper (Collins 1963).

It would not be appropriate to close a discussion of the results without a few comments concerning the internal errors of the computation. Although it is impossible to give a rigorous analysis of the propagation of errors throughout the problem, an effort has been made to keep approximation errors to a minimum. Errors in the computation of the radius and normalized Von Zeipel constants have been kept to less than 1 per cent and are generally about 0.1 per cent. Varying the order of approximation to the integrals used to determine F_ν results in a change of less than 1 per cent in F_ν and hence much less in the monochromatic color. One check on the over-all accuracy of the computation is provided by the recomputation of the total luminosity for the non-rotating star. Since this input parameter may be computed from F_ν, we may use any error in it as an upper limit for the error in F_ν. In every case tried, this error was found to be less than 10 per cent, implying an over-all error in the effective temperature of about 2 per cent. This implies that the maximum error in the colors should be of the order of 1 per cent or less. In light of the assumptions and approximations made in this study, the author does not feel higher accuracy is warranted.

The author would like to acknowledge the co-operation of the Numerical Computation Laboratories of the Ohio State University, the University of Wisconsin, and the Institute for Space Studies. Special thanks are due Professor B. Strömgren for the use of the Strömgren-Underhill Atmosphere-Computer Program. Appreciation is also extended to J. P. Harrington for much of the programming of this problem and for many valuable discussions in connection with its present formulation.

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