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THE STRUCTURE OF ROTATING STARS: THE J² METHOD AND RESULTS FOR UNIFORM ROTATION

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SUMMARY

The J^2 method for solving the structure of uniformly and non-uniformly rotating stars is briefly described and compared with other methods. The method is restricted to rotation laws of the form $\Omega = \Omega(\omega)$, but involves no approximation regimes and is capable of creating a self-consistent density and potential field. The structure equations are written in essentially the same form as they would take in the absence of rotation. The explicit effects of rotation are included through correction factors which depend only on the total potential.

Results are given for uniform rotation using a Roche approximation in order to calculate the correction factors. Corrections arising from a better approximation to the potential are shown to be small. The results show good agreement with other authors but some discrepancies are pointed out.

1. INTRODUCTION

The purpose of this paper is to present, briefly, a method for computing the internal structure of uniformly rotating and non-uniformly rotating stars; we shall refer to the method presented here as the J^2 technique. Several computational techniques already exist for solving the internal structure of rotating stars and are reviewed by Strittmatter (1969) and Fricke & Kippenhahn (1972). We present the J^2 technique since: (i) the method has been used already without explanation in several places (Whelan, Papaloizou & Smith 1971; Whelan 1972; Moss 1973), and (ii) we have received several enquiries about it.

The J^2 method treats the equations for rotating stellar models accurately and writes them in a form similar to those of a non-rotating stellar model, but containing three 'correction factors', which are functions of the total potential. A restriction is that the disturbing force due to rotation (or otherwise) must be derivable from a potential, which restricts the rotation law to the form: $\Omega = \Omega(\varpi)$, where ϖ is the cylindrical polar radius coordinate, and Ω is the angular velocity. The effects of a close companion star can be included also. The method makes no distinction, in principle, between uniform and non-uniform rotation. It also incorporates a procedure for determining the gravitational potential which is consistent with the density distribution of the model, thus creating a 'self-consistent field'.

The equations are presented in Section 2. For the sake of brevity, but to illustrate the technique, only one of the equations is derived. Section 3 contains an outline of the potential solution and Section 4 contains a comparison of the J^2 technique with three other techniques. Sections 5 and 6 contain results for uniform rotation and a comparison with results of other papers. Results for non-uniform rotation will be given separately.

2. THE J^2 FORMULATION OF THE EQUATIONS

The equations are:

$$f_1 \frac{1}{\rho}\frac{dP}{d\eta} = -\frac{GM}{\eta^2} + f_2 \tag{1}$$

$$\frac{dM}{d\eta} = 4\pi\eta^2 \rho f_3 \tag{2}$$

$$\frac{dL}{d\eta} = 4\pi\eta^2 \rho\epsilon f_3 \tag{3}$$

$$f_1 \frac{dT}{d\eta} = -\frac{3\kappa\rho L}{16\pi ac\eta^2 T^3} \quad \text{radiative} \tag{4}$$

$$\frac{d \log T}{d \log P} = \frac{\Gamma - 1}{\Gamma} \quad \text{convective} \tag{4a}$$

$$f_1 = \int \nabla\eta . d\mathbf{s}/4\pi\eta^2 \tag{5}$$

$$f_2 = \int \frac{1}{\varpi}\frac{d}{d\varpi}(\Omega^2(\varpi)\,\varpi)\, dV/4\pi\eta^2 \tag{6}$$

$$f_3 = \frac{dV}{d\eta}\Big/4\pi\eta^2 \tag{7}$$

$$\psi = F(\eta) = \phi_g - \phi_R + \phi_B \tag{8}$$

$$\nabla^2\phi_g = 4\pi G\rho \tag{9}$$

$$\phi_R = \int_0^\varpi \Omega^2(\varpi)\,\varpi\, d\varpi \tag{10}$$

There is also, of course, an equation of state. ψ, ϕ_g and ϕ_R are respectively total, gravitational and rotational potential; η is any arbitrary quantity constant on an equipotential surface; ϕ_B is any other potential, for example that due to a close companion star. The rest of the notation is standard; 'M' is, of course, '$M(\eta)$' and so on. Circulation and angular momentum diffusion effects are neglected, it is assumed that the angular velocity is constant on cylinders and it follows that P, ρ and T are constant on equipotential surfaces.

The equations may be derived from the basic equations for a rotating star (see e.g. Mark 1968, Section II(c)) by integrating over equipotential surfaces and using vector theorems. A basic transformation used is

$$\nabla \to \nabla\eta \frac{d}{d\eta}. \tag{11}$$

As an illustration equation (1) is derived. The equation for hydrostatic equilibrium neglecting circulation effects is

$$\frac{1}{\rho}\nabla P = -\nabla\psi \tag{12}$$

where ψ is given by equations (8), (9) and (10). Introduce η, which is constant on an equipotential surface, such that the total potential is some function of η. Integrating equation (12) over an equipotential surface labelled by η, using Gauss's theorem, taking quantities constant on equipotentials outside of integrals and using equation (11) gives

$$\frac{1}{\rho}\frac{dP}{d\eta}\int \nabla\eta . d\mathbf{s} = -\int 4\pi G\rho \, dV + \int \nabla^2 \int_0^{\varpi} \Omega^2(\varpi)\, d\varpi \, dV \tag{13}$$

where $\phi_B = 0$ for simplicity. After some manipulation, dividing both sides by $4\pi\eta^2$ gives

$$\frac{1}{\rho}\frac{dP}{d\eta}\left[\frac{\int \nabla\eta . d\mathbf{s}}{4\pi\eta^2}\right] = -\frac{GM(\eta)}{\eta^2} + \frac{\int \frac{1}{\varpi}\frac{d}{d\varpi}(\Omega^2(\varpi)\,\varpi^2)\, dV}{4\pi\eta^2} \tag{14}$$

which, using equations (5) and (6) is equation (1). The other equations are derived in a similar manner.

The influence of rotation on the effective gravity which enters the buoyancy force in the mixing-length theory of convection was included but any rotational correction to the Schwarzschild criterion was ignored. The surface boundary conditions used, where A is the surface area of the stellar model, M_* is the total mass and $T_{\mathrm{eff,\,average}}$ is an average effective temperature, were

$$L = A\sigma T^4_{\mathrm{eff,\,average}} \tag{15}$$

$$P_K = \frac{GM_*}{\eta^2}\frac{1}{f_1}\left(1 - \frac{f_2\eta^2}{GM_*}\right). \tag{16}$$

The J^2 form of the equations of rotational stellar structure looks very similar to the usual non-rotating set and the effects of rotation are entirely contained in the correction factors f_1, f_2 and f_3 which depend only on ψ. For the evaluation of f_1, f_2 and f_3, 16-point Gaussian integration was used.

3. THE POTENTIAL

The gravitational potential satisfies Poisson's equation. We can find a formal solution of Poisson's equation by expanding ϕ_g in the form

$$\phi_g = \sum_l \phi_l(r)\, P_l(\mu) \tag{17}$$

where $P_l(\mu)$ is the Legendre polynomial of order l and $\mu = \cos\theta$. We can then express ϕ_l as

$$\phi_l(r) = -2\pi G r^l \int_r^{\infty} dt \int_{-1}^{+1} \frac{\rho(t,\mu)\, P_l(\mu)\, d\mu}{t^{l-1}} - \frac{2\pi G}{r^{l+1}} \times \int_0^r t^{l+2}\, dt \int_{-1}^{+1} \rho(t,\mu)\, P_l(\mu)\, d\mu \tag{18}$$

The situation is now clear, at least in principle. The gravitational potential can be determined from equation (18) provided the density distribution is known. With the potential known the correction factors for rotation can be determined via equations (5), (6) and (7) and a new stellar model constructed. This model will have a density distribution which may or may not agree with the original. If it does not then the new density distribution will determine a new potential and an iteration procedure can be set up until the density and potential are self-consistent.

In order to proceed with the calculation of the stellar structure, we require a definition of η. We define the η associated with a particular potential surface to be the distance from the centre to the pole, i.e. $\psi(\eta, 0) = \psi(r, \theta)$. For a first guess at the potential surfaces, for the purpose of calculating f_1, f_2 and f_3, we pick the Roche potential. η is then given by

$$\frac{1}{\eta} = \frac{1}{r} + \frac{\phi_R}{GM_*} \tag{19}$$

The Roche potential surfaces correspond to a point mass in a rotating frame. They might, thus, be thought to be adequate for a centrally condensed star undergoing uniform rotation. It has been shown by several authors that the correction to the potential surfaces at the surface of the star is small. See for example Faulkner, Roxburgh & Strittmatter (1968), Sackmann & Anand (1970) and Jackson (1970a). We now give further justification of the validity of the Roche approximation both at the centre and surface of the star, for the purposes of the calculation of f_1, f_2 and f_3.

Equation (18) together with the surfaces defined by equation (19) enable us to estimate the correction to the Roche form of the potential at the surface. Let β be the ratio of the first two terms of the potential evaluated at the surface of the model, then

$$\beta = \frac{\phi_2(R_*)}{GM_*/R_*} \approx \frac{8}{15}\frac{\Omega^2 R_*^3}{GM_*}\int_0^1 \left(\frac{r}{R_*}\right)^5 d\left(\frac{M}{M_*}\right), \tag{20}$$

where R_* is the equatorial radius of the model. Equation (20) has been quoted here for the case of uniform rotation, but it holds also for non-uniform rotation provided Ω^2 is replaced by a suitably averaged Ω^2.

Now for uniform rotation

$$\frac{\Omega^2 R_*^3}{GM_*} \leqslant 1 \tag{21}$$

with equality holding at break-up. For typical stellar models the value of the integral is roughly between 0·01 and 0·03. A typical value then for uniform rotation near break-up is

$$\beta \approx 0{\cdot}01 \tag{22}$$

which is small. For non-uniform rotation in which the angular velocity increases inwards, the averaged angular velocity $\overline{\Omega^2}$ may be several times its surface value leading to a much larger value for β.

For the uniformly rotating models considered in this paper equation (19) was used. The potential was determined from equations (17) and (18) and shown to give sufficiently small corrections to f_1, f_2 and f_3 both at the centre and surface. Table I shows some functions of the gravitational potential for a typical case of $1\ M_\odot$; ϕ_r (r odd) $= 0$ by symmetry. $\lambda[= \Omega^2 R_*^3/GM_*]$ measures the importance of rotation with respect to gravity at the surface and $\lambda = 0{\cdot}93$ is effectively at break-up. $\Omega^2/2\pi G\rho_c$ measures the importance of rotation with respect to gravity very near the centre of the model. $(\delta\phi_2/\delta\phi_0)$ measures the ratio of the gradients of the second and first terms in the gravitational potential, evaluated at centre and surface. $(\phi_2/\phi_0)_s$ is simply the ratio of the two terms of the potential at the surface of the model. ϕ_4 and $\delta\phi_4$ were less than ϕ_2 and $\delta\phi_2$ by factors in the range 10–10^3.

TABLE I

Some functions of the gravitational potential

$\Omega^2/10^{-8}$ s^{-2}	λ	$\Omega^2/2\pi G\rho_c$	$(\delta\phi_2/\delta\phi_0)_c$	$(\delta\phi_2/\delta\phi_0)_s$	$(\phi_2/\phi_0)_s$
17·3	0·27	5	3	12	4
34·0	0·93	10	5·1	14·2	4·8

The right-hand four columns are in dimensionless units of 10^{-3}.

The numbers given here seem in fair agreement with those of Sackmann (1970). For the model near break-up the surface $(\delta\phi_2/\delta\phi_0)_s$ value is negligible. However, at the centre the value $(\delta\phi_2/\delta\phi_0)_c$ is small but it is of the same order as the rotation to gravity ratio. It may seem that ϕ_2 cannot be neglected there because it is producing effects of the same order as the rotation. This is not true however; ϕ_2 enters into f_1 and f_3 via quantities such as $\partial\eta/\partial r$ and $\partial\eta/\partial\theta$ and this gives higher order effects in f_1 and f_3 than does the $\Omega^2/2\pi G\rho_c$ term in f_2. Thus for calculating f_1, f_2 and f_3 deviation from the Roche form of the potential is not important in the centre either. For non-uniform rotation, this was not found to be the case, except for rather slow rotation speeds.

4. COMPARISON OF J^2 AND OTHER METHODS

FRS method

The FRS method has been described by Faulkner, Roxburgh & Strittmatter (1968) and by Sackmann & Anand (1970). They define a 'radius' of an equipotential surface, r_0, by

$$\psi(r, \theta) = \psi(r_0, \theta = \cos^{-1}(1/\sqrt{3})) \tag{23}$$

which gives approximately

$$V_0 \equiv \tfrac{4}{3}\pi r_0^3 \approx V_\psi \equiv V(\eta) \tag{24}$$

where V_ψ is the volume contained in the equipotential surface of potential ψ.

Equation (1) for the case of uniform rotation can be written as, where

$$\gamma = V(\eta)/\tfrac{4}{3}\pi\eta^3,$$

$$f_1 \frac{1}{\rho}\frac{dP}{d\eta} = -\frac{GM(\eta)}{\eta^2}\left[1 - \frac{2}{3}\frac{\Omega^2\eta^3}{GM(\eta)}\gamma\right]. \tag{25}$$

The FRS equation of hydrostatic equilibrium is equation (25) in which η is replaced by r_0, $\gamma = 1$ and $f_1 = 1$. In the inner regions of a uniformly rotating star deviations from non-sphericity are small and indeed $\gamma \rightarrow 1$ and $f_1 \rightarrow 1$ there. In the outer laters, γ and f_1 do not approach unity but there is little mass in this region. f_3 is taken as unity in the FRS method for the same reasons as discussed above for f_1. f_1 is included in the FRS method in the energy transport equation through the $\Lambda(r_0)$ term. The FRS method can thus be regarded as a good approximation to the J^2 method for the case of uniform rotation and numerical calculations confirm this. For the case of non-uniform rotation in which the angular velocity increases inwards both f_1 and f_3 become significantly different from unity in the inner regions of the model and the FRS method cannot be used except for very slow rotation speeds.

KT method

The KT method is due to Kippenhahn & Thomas (1970) and the J^2 method is quite similar. There are three correction factors in each method. KT use the variable r_ψ which is defined by

$$\tfrac{4}{3}\pi r_\psi{}^3 = V_\psi \equiv V(\eta). \tag{26}$$

They too use a Roche approximation as a first approximation in order to calculate their correction factors. Differences in results between the methods should be accountable as differences in input physics, composition, etc.

The self-consistent field method

The technique is given by Ostriker & Mark (1968) and Jackson (1970b). The equations are similar to those of the J^2 method but ψ itself is used as independent variable. The iterations to create the self-consistent field are carried out and the method has been successfully used to construct non-uniformly rotating models.

5. RESULTS FOR UNIFORM ROTATION

The J^2 method, as described in the previous sections has been used to compute the structure of uniformly rotating stars in a mass range 0·6 $M_\odot$–62·7 $M_\odot$. For $M \geqslant 10\,M_\odot$ the program of Papaloizou (1972) was used. In this case the opacity was essentially electron scattering, energy generation rates were from Reeves (1965) and the composition was $X = 0{\cdot}739$, $Z = 0{\cdot}021$. For $M \leqslant 2\,M_\odot$ the basic program of Moss (1968) as modified in Moss & Whelan (1970) was used. The opacity was based on Cox (1965), energy generation rates were also from Reeves (1965), the mixing-length, α, was always taken to be two pressure scale-heights and the composition was $X = 0{\cdot}70$, $Z = 0{\cdot}05$. Since we are concerned largely with differential effects the differences in composition of the models, though unfortunate, are not very significant. More severe changes occur when the mixing-length is changed.

The notation in Table II, which gives the results, is standard. The units are indicated though, for convenience, they are not always the same in each different part of Table II. R_P is the polar radius, R_vol is the volume radius defined by

$$\frac{4}{3}\pi R_\mathrm{vol}{}^3 = \text{volume of star} \tag{27}$$

and subscript '0' indicates the non-rotating values. Results in detail are given for the non-rotating and break-up models for each mass. Enough information is given for a grid of model atmospheres to be constructed to deduce observable properties. Typically five to six models of different angular velocity were calculated for each mass and the variation of their parameters between zero and break-up rotation was similar to that found by other authors. L/L_0 varied linearly with Ω^2, suggesting that indeed uniform rotation may be considered as a small perturbation to the structure of the star. For all the models considered, R_vol increases monotonically as Ω^2 increases, and with increasing gradient as $\Omega^2 \to \Omega^2$ (break-up).

The central temperature always decreases with increasing rotation speed but the central density increases for $M \gtrsim 1{\cdot}5\,M_\odot$ and decreases for $M \lesssim 1\,M_\odot$. This

TABLE II

Uniform rotation results—non-rotating and break-up models are given

$M/M_\odot$	0.6		0.8		1.0	
$\Omega^2/10^{-7}$ s^{-2}	0	7.75	0	4.03	0	3.42
$L/10^{33}$ erg s^{-1}	0.200	0.128	0.726	0.488	2.103	1.536
$(L_0-L)/L_0$	0	0.360	0	0.328	0	0.270
$R_P/R_\odot$	0.525	0.449	0.661	0.614	0.786	0.699
$(R_{P_0}-R_P)/R_{P_0}$	0	0.145	0	0.071	0	0.111
$R_{vol}/R_\odot$	0.525	0.546	0.661	0.748	0.786	0.850
$(R_{vol}-R_{vol_0})/R_{vol_0}$	0	0.040	0	0.132	0	0.081
$P_c/10^{17}$ dyne cm^{-2}	0.964	0.902	1.214	1.106	1.546	1.425
$T_c/10^7$ K	0.974	0.924	1.152	1.085	1.360	1.285
ρ_c/g cm^{-3}	73.8	72.9	78.6	76.0	84.8	82.7

$M/M_\odot$	1.5		2.0	
$\Omega^2/10^{-8}$ s^{-2}	0	6.77	0	4.65
$L/10^{34}$ erg s^{-1}	1.375	1.294	4.484	4.278
$(L_0-L)/L_0$	0	0.059	0	0.046
$R_P/R_\odot$	1.403	1.372	1.729	1.712
$(R_{P_0}-R_P)/R_{P_0}$	0	0.022	0	0.010
$R_{vol}/R_\odot$	1.403	1.671	1.729	2.085
$(R_{vol}-R_{vol_0})/R_{vol_0}$	0	0.191	0	0.206
$P_c/10^{17}$ dyne cm^{-2}	2.061	2.066	1.719	1.736
$T_c/10^7$ K	1.827	1.812	2.049	2.039
ρ_c/g cm^{-3}	84.2	85.1	62.6	63.5

$M/M_\odot$	10		20	
$\Omega^2/10^{-8}$ s^{-2}	0	2.23	0	1.407
$L/10^{38}$ erg s^{-1}	0.2140	0.1968	1.590	1.462
$(L_0-L)/L_0$	0	0.080	0	0.081
$R_P/10^{11}$ cm	2.672	2.595	3.920	3.821
$(R_{P_0}-R_P)/R_{P_0}$	0	0.029	0	0.025
$R_{vol}/10^{11}$ cm	2.672	3.159	3.920	4.631
$(R_{vol}-R_{vol_0})/R_{vol_0}$	0	0.182	0	0.181
$P_c/10^{16}$ dyne cm^{-2}	4.116	4.208	2.542	2.579
$T_c/10^7$ K	3.087	3.070	3.481	3.461
ρ_c/g cm^{-3}	9.159	9.440	4.54	4.66

$M/M_\odot$	28.25		40.0		62.7	
$\Omega^2/10^{-8}$ s^{-2}	0	1.14	0	0.92	0	0.68
$L/10^{38}$ erg s^{-1}	3.824	3.520	8.314	7.673	20.76	19.24
$(L_0-L)/L_0$	0	0.080	0	0.077	0	0.073
$R_P/10^{11}$ cm	4.725	4.617	5.668	5.558	7.217	7.140
$(R_{P_0}-R_P)/R_{P_0}$	0	0.023	0	0.019	0	0.011
$R_{vol}/10^{11}$ cm	4.725	5.620	5.668	6.767	7.217	8.692
$(R_{vol}-R_{vol_0})/R_{vol_0}$	0	0.189	0	0.194	0	0.204
$P_c/10^{16}$ dyne cm^{-2}	2.167	2.187	1.939	1.951	1.763	1.764
$T_c/10^7$ K	3.660	3.641	3.822	3.803	4.016	3.997
ρ_c/g cm^{-3}	3.408	3.488	2.668	2.722	2.007	2.042

effect is also well known and the change in density character is caused by the change in dominant mode of energy generation. T_c and ρ_c for a rotating model correspond very closely to the T_c and ρ_c of a non-rotating model of smaller mass. Sackmann (1970, Fig. 4) discussed this effect in detail. The polar radius decreases and the equatorial and volume radii increase as the rotation speed increases.

The luminosity decrease at break-up rotating varies from about 35 per cent to about 5 per cent. For low mass models the mixing-length significantly controls the stellar radius which in turn affects the break-up angular velocity and luminosity decrease.

6. COMPARISON OF J² AND OTHER RESULTS

We use the Table of Sackmann (1970, Table 3) and reproduce it here (Table III) including the J² results. We maintain her notation. We note that between any two groups of authors there are always two sources of differences: the *method* and the '*physics*'. By '*physics*' we mean opacities, energy generation, composition, equation of state, convection theory, etc.

We note that the general agreement between the J² and the other methods is quite good. Since the J² method is similar to that of KT, we expect differences between them to be accountable as 'physics' differences. Using the composition $X = 0{\cdot}70$, $Z = 0{\cdot}033$, $\alpha = 1{\cdot}3$, at $1\,M_\odot$ we find $(L_0-L)/L_0 \approx 0{\cdot}16$ in good agreement with the KT result. This illustrates the dependence on 'physics'; (KT, and Sackmann (1970), used the Munich Stellar Interior Program). In all our models R_{vol} increases monotonically with angular velocity, whereas KT found that, at $1\,M_\odot$, R_{vol} increases and then decreases just before break-up. We suggest that this effect is due to a subtle numerical error involving atmosphere grids in which one or two grid points are marginally beyond break-up. Such grids cause large decreases in the polar radius of the resultant stellar model thus causing the R_{vol} decrease. Further work is required to clear up this discrepancy.

Since the FRS method approximates the J² method we expect differences between our results and those of FRS, Sackmann & Anand (1970) and Sackmann (1970) to also be mostly 'physics' differences. This can be seen to be the case by noting the ease with which $(L_0-L)/L_0$ is changed by composition, etc. For the case of lower mass stars ($M \lesssim 2\,M_\odot$) we have computed models by the FRS method using the same low-mass stellar model program described above. With the same 'physics' the differences due to method were all less than 5 per cent in any quantity.

The rather large values for $-\Delta L/L$ at masses $1{\cdot}0\,M_\odot$–$0{\cdot}6\,M_\odot$ found by us are simply a consequence of the composition we chose there. Our models have a rather small initial radius which allows the insertion of a lot of angular velocity which leads to a large luminosity decrease.

At high masses ($M \geqslant 28{\cdot}25\,M_\odot$) the J² results differ significantly from those of Mark (1968). We believe that this has been caused by the use of polytropes to calculate thermal properties. The full discussion is given in Whelan *et al.* (1971).

7. CONCLUSIONS

The J² method seems to be a satisfactory method for computing the structure of rotating stars and the results given here show good agreement with those of other authors.

The proportionate luminosity decreases at break-up uniform rotation is between about 7 and 10 per cent for high mass models, somewhat smaller for models of low mass with a radiative envelope, and between about 15 and 35 per cent for low mass models with a convective envelope. The latter result is the most

TABLE III

A summary of the luminosity and polar radius changes at break-up rotation, the results of several authors. The quantities κ and β refer to the opacity and to the ratio of gas to total pressure. κ_e is electron scattering opacity, otherwise notation is as in Sackmann (1970)

Authors	M		FRS		SA		S		KT		J²		
Description of model	Polytropic P vs. ρ $\kappa = \kappa_e$ $0 < \beta \leq 1$		$X = 0{\cdot}709$ $Z = 0{\cdot}021$ Detailed κ surface $\beta = 1$		$X = 0{\cdot}67$ $Z = 0{\cdot}03$ Detailed κ $P = T = 0$ surface $0 < \beta \leq 1$		$X = 0{\cdot}739$ $Z = 0{\cdot}021$ Cox κ Detailed surface $0 < \beta \leq 1$		$X = 0{\cdot}739$ $Z = 0{\cdot}021$ Cox κ Detailed surface $0 < \beta \leq 1$		See text Cox κ Detailed surface $0 < \beta \leq 1$		
$M/M_\odot$	$-\Delta L/L$	$-\Delta R_P/R_P$	$-\Delta L/L$	$-\Delta R_P/R_P$	$-\Delta L/L$	$-\Delta R_P/R_P$	$-\Delta L/L$	$-\Delta R_P/R_P$	$-\Delta L/L$	$-\Delta R_P/R_P$	$-\Delta L/L$	$-\Delta R_P/R_P$	$M/M_\odot$
	in %		in %		in %		in %		in %		in %		
62·7	17·2	11·6									7·3	1·1	62·7
40·0											7·7	1·9	40·0
28·25	48·5										8·0	2·3	28·25
20·0							7·5	2·1			8·1	2·5	20·0
15·0					6·5	1·2	7·7	2·5					15·0
12·0					6·7	1·5							12·0
10·0					6·7	1·7	7·2	2·5	6·2	2·5	8·0	2·9	10·0
9·0					6·7	1·8	7·2	2·4					9·0
7·0					6·7	1·9	7·1	2·3					7·0
5·0			8·6	2·9	6·6	2·0	6·6	2·1	5·4	1·7			5·0
3·0							5·9	1·7					3·0
2·0			8·8	2·3			5·5	1·3			4·6	1·0	2·0
1·8							5·0	1·2					1·8
1·5							5·6	1·2			5·9	2·2	1·5
1·4							5·5	2·6					1·4
1·0			16·1	4·2			15	6·4	17·7	19·0	27·0	11·1	1·0
0·8							23	8·2			32·8	7·1	0·8
0·6											26·0	14·5	0·6

uncertain and is affected most seriously by composition, mixing-length and ' physics '.

Two discrepancies with previous results were noted: the values of $(L_0 - L)/L_0$ at high masses as given by Mark and the behaviour of $R_{\rm vol}$ near break-up at 1 $M_\odot$ in KT.

It was not important to improve upon the Roche approximation for calculating f_1, f_2 and f_3 for the case of uniform rotation but it is in the case of non-uniform rotation.

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