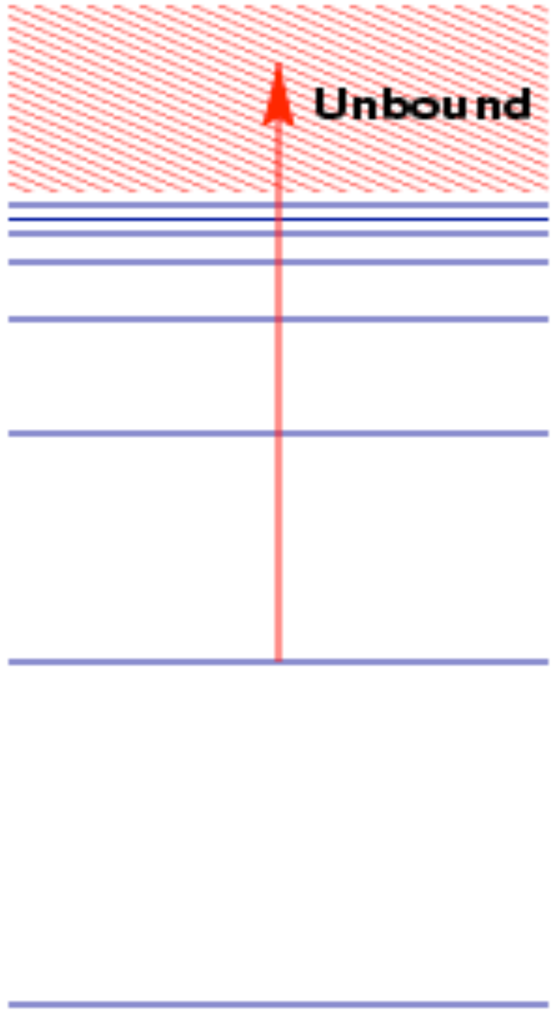
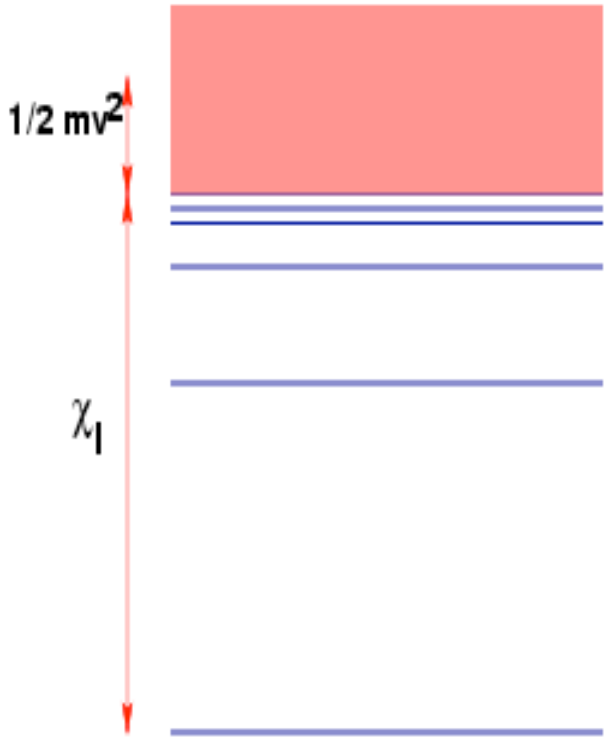
Bound-free transitions (Saha equation)

- **Bound-free transitions occur between atomic state and an unbound state.**
- **Free electron can have a range of kinetic energies => bound-free transitions produce continuous opacity (not just at lines).**
- **A minimum photon energy is needed to ionize an atom from a given level, eg need $\lambda \leq$ 91.2 nm to ionize hydrogen from the n=1 level.**



The Saha equation



- Gives the distribution of atoms in different stages of ionization. Simplest case: a neutral atom and its first stage of ionization.
- Energy difference between ground state of atom, and free electron having velocity υ, is:
- $$\Delta E = \chi_I + \frac{1}{2} m_e \upsilon^2$$
- where χ_I is the ionization potential.

Boltzmann

- **The Boltzmann law suggests:**

$$\frac{dN_0^+(\upsilon)}{N_0} = \frac{g}{g_0}\exp\left[-\frac{(\chi_\mathrm{I} + 0.5m_e\upsilon^2)}{kT}\right]d\upsilon$$

where:

  - **dN_0^+ (υ) is the number of ions in the ground state with the free electron having velocity between υ and υ +$d\upsilon$.**
  - **N_0 is number of atoms in ground level.**
  - **g_0 is the statistical weight of the atom in the ground state.**
  - **g is the product of the statistical weight of the ion in its ground state g_0^+, and the differential statistical weight of the electron g_e. ie $g = g_0^+ g_e$**

Statistical weight of free electron

- **For the electron, with two spin states,**

$$g_e = \frac{2dx_1dx_2dx_3dp_1dp_2dp_3}{h^3}$$

- **The volume $dx_1\ dx_2\ dx_3$ contains one electron, so $dx_1\ dx_2\ dx_3 = 1/N_e$, where N_e is the electron density.**
- **Since the electrons have an isotropic velocity distribution,**

$$dp_1dp_2dp_3 = 4\pi p^2 dp = 4\pi m_e^3 \upsilon^2 d\upsilon$$

- **which gives,**

$$\frac{dN_0^+(\upsilon)}{N_0} = \frac{g_0^+}{g_0}\frac{8\pi m_e^3}{N_e h^3}\exp\left[-\frac{(\chi_{\mathrm{I}} + 0.5 m_e \upsilon^2)}{kT}\right]\upsilon^2 d\upsilon$$

Eliminating velocity

- **We don't care about the electron velocity. Integrating over all possible υ gives,**

$$\frac{N_0^+ N_e}{N_0} = \frac{2g_0^+}{g_0}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\frac{\chi_I}{kT}}$$

- **where the integral**

$$\int_0^\infty e^{-x^2} x^2 dx = \frac{\pi^{1/2}}{4}, x \equiv \sqrt{\frac{m_e}{2kT}}\upsilon$$

was used.

Finally… the Saha equation

- **For the ground state, Boltzmann's law gives,**

$$\frac{N_0}{N} = \frac{g_0}{U(T)} \qquad \textbf{and} \qquad \frac{N_0^+}{N^+} = \frac{g_0^+}{U^+(T)}$$

- **Substituting these gives us Saha's equation,**
- $$\frac{N^+ N_e}{N} = \frac{2U^+(T)}{U(T)} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\frac{\chi_1}{kT}}$$
- **where *N* and N^+ are the number densities of neutral and once-ionized atoms, and *U* and U^+ are the corresponding partition functions.**
- **Saha's equation for any two neighbouring states of ionization is just the same, replace *N* by N^j, N^+ by N^{j+1} etc.**
-

Ionization of hydrogen - I

- Define the degree of ionization x by
- $$x = \frac{N^+}{N + N^+}$$
- ie for a neutral gas x=0, for a fully ionized gas x=1. Left hand side of Saha equation is then,
- $$\frac{N^+ N_e}{N} = \frac{x}{1-x} N_e$$
- Next, eliminate N_e by writing it in terms of the gas pressure.
- If $N_H = N + N^+$ is the total number of hydrogen nuclei, then can write the pressure of the electrons as:

$$P_e = N_e kT = (N_H + N_e)kT \frac{N_e}{N_H + N_e} = P_{gas} \frac{N_e}{N_H + N_e}$$

Ionization of hydrogen - II

- **Each ionized atom gives one electron, so for pure hydrogen $N_e = N^+$ and**

$$P_e = \frac{x}{1+x} P_{gas}$$

- **The Saha equation can then be written,**

$$\frac{x^2}{1-x^2} = \frac{1}{P_{gas}} \frac{2U^+(T)}{U(T)} \left(\frac{2\pi m_e}{h^2}\right)^{3/2} (kT)^{5/2} e^{-\frac{\chi_I}{kT}}$$

- **a quadratic equation for the degree of ionization. To apply, we need**
  - **P_{gas} and T. Ionization increases with the temperature (collisions become more violent) and decreases with increasing pressure at fixed T (more recombinations).**
  - **The partition functions. In practice, can take $U = 2$ (the ground state value) and $U^+ = 1$.**
- **Alas, even a small abundance of other elements can provide lots of electrons if the ionization potential is low. So the pure hydrogen case is of limited applicability.**

Bound-free absorption cross-section

- **Bound-free absorption provides an important source of continuum opacity. For a hydrogen-like atom in a level with principal quantum number *n*, with ionization potential χ_n, the bound-free absorption cross-section σ_{bf} is given by**

$$\sigma_{bf} = 0 \qquad \text{for } \nu < \frac{\chi_n}{h}$$

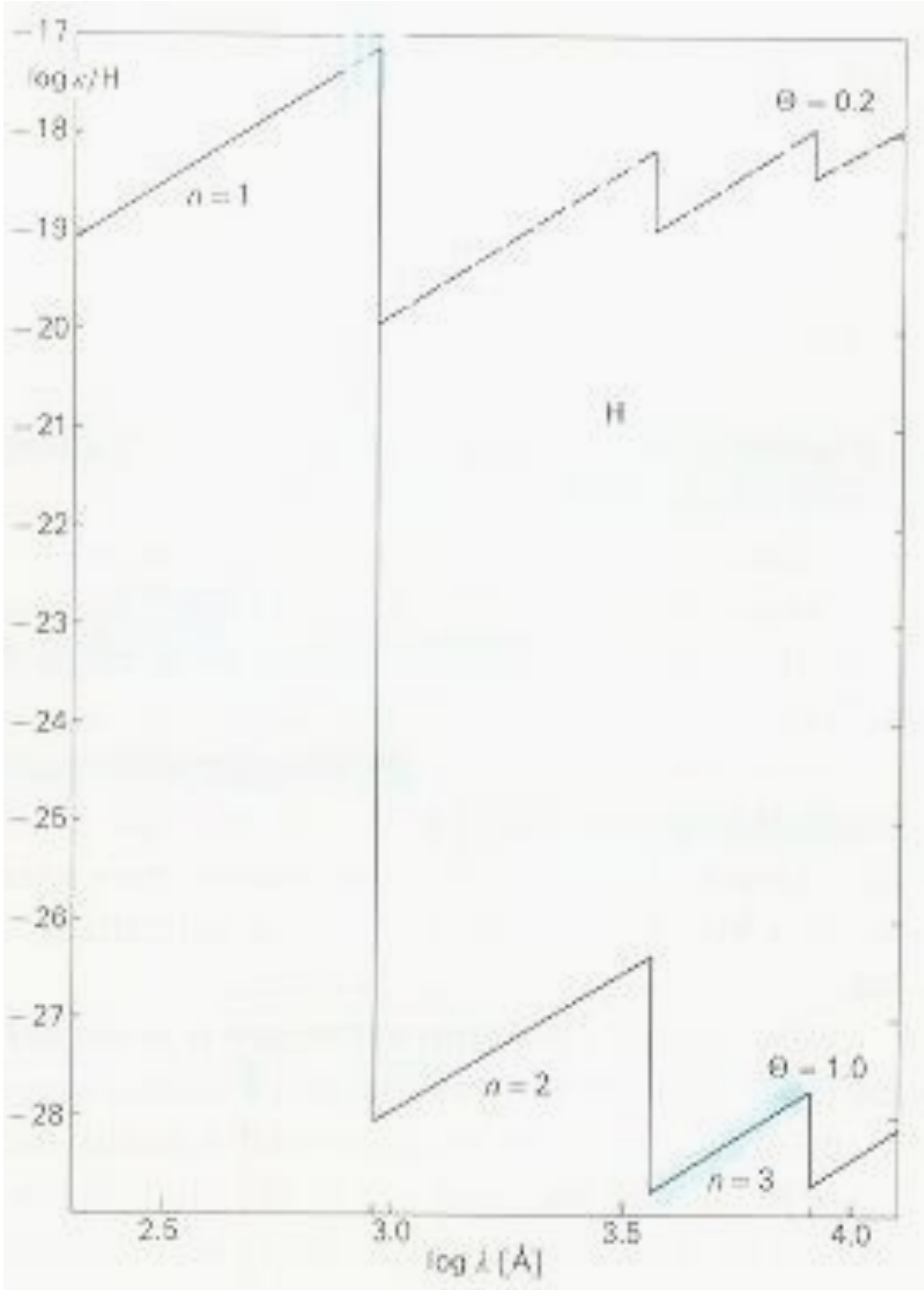
$$\sigma_{bf} \propto \frac{1}{n^5}\frac{1}{\nu^3} g(\nu,n,l) \qquad \text{otherwise}$$

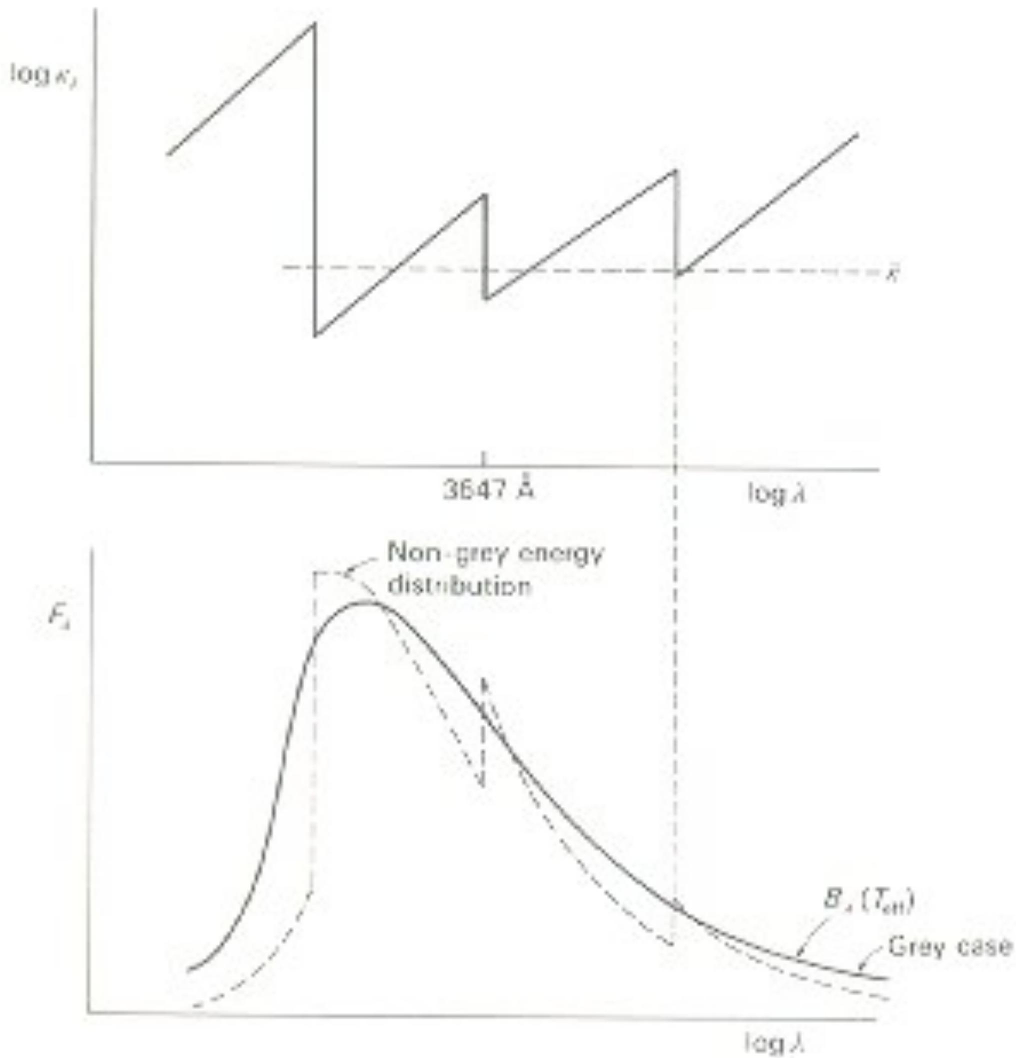
- **Here *g* is the bound-free Gaunt factor -- a slowly varying correction factor to the simple scaling.**
- **Properties:**
  - **Absorption cross-section has sharp rises, absorption edges, at the frequency where the atom in a given level can be ionized.**
  - **At frequencies higher than the edge:** $\sigma_{bf} \propto \nu^{-3}$
  - **The Gaunt factor is close to unity near the edge.**

The hydrogen absorption coefficient κ per hydrogen atom is shown as a function of wavelength for two temperatures 5040 and 25200 K.

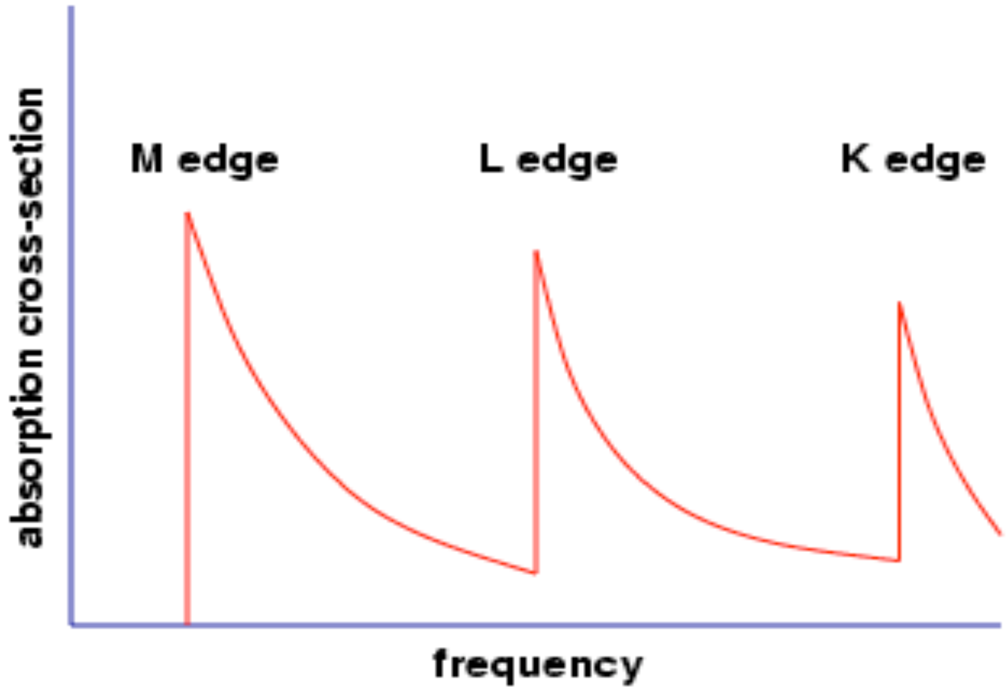
Higher temperatures lead to higher values of κ in the visual spectral region (Paschen continuum, absorption from the level $n = 3$).

The value of κ at the Lyman limit is ~ the cross section of the lowest orbital (0.5×10^{-8} cm) in the hydrogen atom.





Effect of wavelength-dependence of hydrogen absorption coefficient on the observed energy distribution of the star



An atom with many electrons will be characterized by a series of ionization edges as it loses electrons from successive shells.

- Heavy elements, either in the gas phase or in grains, have many inner-shell electrons. They provide large opacity to soft X-rays (below 1 keV).
- Hard X-rays (10 keV or more) see only the ν^{-3} tail (becoming closer to $\nu^{-3.5}$ at high ν). Very hard to absorb these.
- Seeing the absorption at low energies → measurement of the column density towards an X-ray source.

Example: absorption towards an Active Galactic nucleus

The *intrinsic* X-ray spectra of Active galaxies are often taken to be power laws. Superimposed on that we have,

- Absorption at low energy (here modelled as oxygen edges).
- Instrumental features than have not been calibrated quite right (a gold edge).
- Emission from fluorescent iron near the black hole.

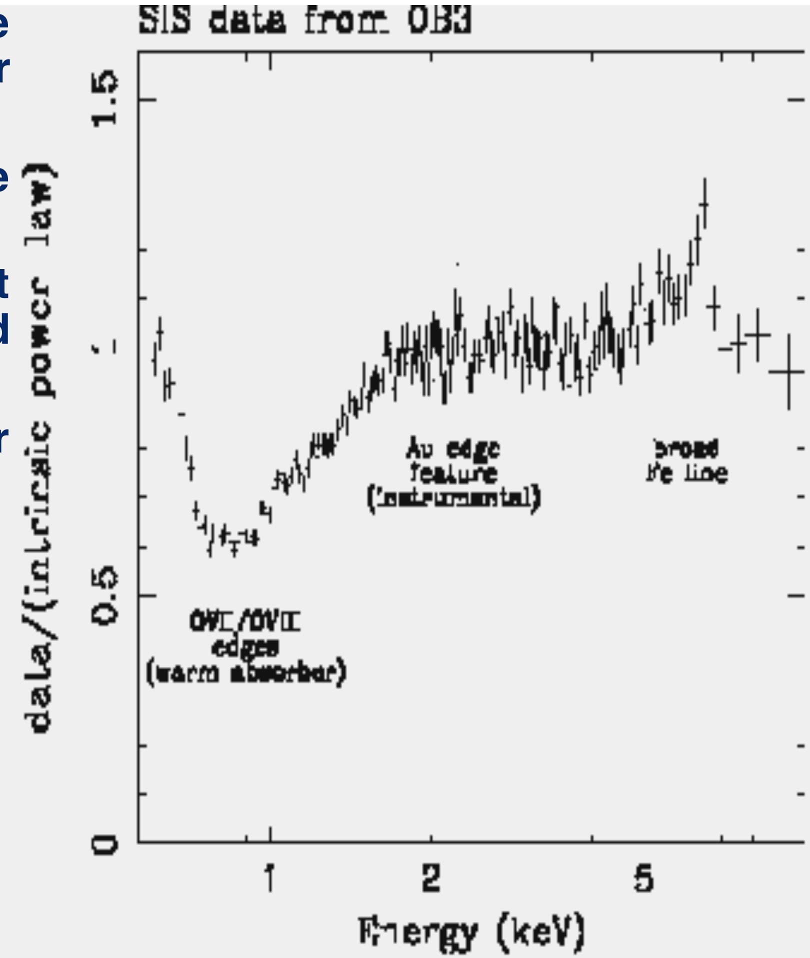


Figure 2.2: Ratio of the full band SIS0 spectrum of MCG−6-30-15 obtained during OB3 to the best fitting intrinsic power-law. The intrinsic continuum is defined by fitting a power-law to the 2–4 keV range (since there is negligible X-ray reprocessing over this range). Galactic absorption is included with a column density of $N_H = 4. \times 10^{20}\ \mathrm{cm}^{-2}$.

Lecture 9 revision quiz

- **Sanity-check integral with respect to v:**

$$\frac{dN_0^+(\upsilon)}{N_0} = \frac{g_0^+}{g_0}\frac{8\pi m_e^3}{N_e h^3}\exp\left[-\frac{(\chi_\mathrm{I} + 0.5 m_e \upsilon^2)}{kT}\right]\upsilon^2 d\upsilon$$

- **Plot the degree of ionization of hydrogen as a function of log(P_{gas}) at a fixed T=10^4 K.**
- **In the spectrum of an early-type star, why is there an abrupt change in flux with wavelength across hydrogen ionization boundaries?**
- **Do you expect the emergent intensity to be greater at higher or lower frequencies than the ionization threshold frequency? Why?**