CONTINUUM EMISSION FROM A RAPIDLY ROTATING STELLAR ATMOSPHERE

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ABSTRACT

A method for describing the continuum emission of a rapidly rotating star is derived. Explicit results for a model star of assumed initial spectral type B3V are included. Assumptions are made which, the author feels, do not remove the results too far from the realm of physical reality. Detailed calculations imply that a variation in color with angle of inclination exists for hot stars rotating near the breakup velocity. The effect becomes more significant as one approaches the far ultraviolet. The results of rotation upon the observed total flux from a star are also discussed. In addition some calculations are included which give insight into the behavior of the Balmer and Lyman jumps for a rapidly rotating star.

I. INTRODUCTION

The problem of deriving the continuous spectrum of a star undergoing rapid rotation may be divided neatly in two parts. In the first part we must develop a theory which will enable us to formulate the fundamental parameters determining a stellar atmosphere appropriate to the physical conditions prevailing at a given point on the surface of a rotating star. The second part of our task is then to determine the atmosphere and the specific intensity at a given point on the surface and then to integrate this intensity over the visible hemisphere of the star. As the visible hemisphere depends on the angle of inclination, we must repeat the procedure for several different angles of inclination. To complete the picture we must also do this for several rotational velocities. We will then have a description of the monochromatic flux, and total flux of a star of given mass under the influence of rotation and variations of axis of rotation to the line of sight.

Before proceeding with a description of how the above was done and a discussion of the results, it is appropriate to say a few words about the interpretation of the results. In constructing a set of models of rotating stars one has a certain amount of freedom concerning the way in which he wishes to compare two models. The author has chosen to use as a basis for comparison the effects of rotation alone. This is clearly not the only manner in which models may be compared. For instance, we might compare models having equal mean surface gravity or equal mean effective temperature. These criteria impose definite restrictions on the method of construction. Slettebak (1949) found that imposing the condition that the mean density remained constant with respect to varying rotational velocity led to results that the mean effective temperature remained approximately constant from model to model. This restriction will not be used in this paper; rather it will be assumed that the polar radius of the star will remain constant with varying rotation. It will be found that this is completely consistent with other assumptions made in this study.

II. DETERMINATION OF LOCAL EFFECTIVE TEMPERATURE AND SURFACE GRAVITY FOR A ROTATING ATMOSPHERE

In order to simplify the theory it will be necessary to make the following assumptions about the models we are about to construct:

1. The distribution of mass within the model is that of a Roche model
2. The model will be assumed to rotate as a rigid body.
3 The energy-producing mechanism of the star will not be seriously affected by the rotation. This implies that the total luminosity of the star will be unaffected by rotation.

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Under these assumptions we may define a rotational potential which shall include only gravitational and rotational terms.

$$\phi \equiv \frac{GM}{R} + \frac{\omega^2 (x^2 + y^2)}{2} . \tag{1}$$

Eddington (1926) and others have shown that at the surface of a freely rotating star

$$\phi = \text{const.} \tag{2}$$

If we rewrite equation (2) in terms of polar coordinates where θ is the polar angle or colatitude, it is clear that the rotational term will vanish at the pole. Under the assumption that the polar radius will not change, the following equation will be valid for all values of angular velocity ω and values of θ.

$$\frac{GM}{R} + \tfrac{1}{2}\omega^2 R^2 \sin^2 \theta = \frac{GM}{R_p} . \tag{3}$$

This gives us a means of determining R for a given rotational velocity ω and colatitude θ.

The surface gravity may be obtained as a function of R and θ immediately by taking the negative gradient of equation (1) yielding

$$| \boldsymbol{g} | = \left[\left(\frac{GM}{R^2} - \omega^2 R \sin^2 \theta \right)^2 + \left(\omega^4 R^2 \sin^2 \theta \cos^2 \theta \right) \right]^{1/2} . \tag{4}$$

Setting $\boldsymbol{g}$ equal to zero at the equator in equation (4) we arrive at the well-known expression for the critical rotational velocity of breakup ω_c.

$$\omega_c{}^2 = \frac{GM}{R_e} . \tag{5}$$

Substitution of this result into equation (3) and evaluating the expression at the equator yields

$$R_e = \tfrac{3}{2} R_p . \tag{6}$$

Thus if we define $x = R/R_p$ we may normalize equation (3) and compute the rotational shapes for a star in units of the breakup velocity once and for all. The results of this computation are shown in Figure 1. To facilitate the speed of further computation, the results were thus approximated by a product polynominal in θ and ω/ω_c of the form

$$x = \sum_{i=0}^{2} C_i \theta^{2i} , \tag{7}$$

where

$$C_i = \sum_{j=0}^{10} b_{ij} (\omega / \omega_c)^j . \tag{8}$$

The values for b_{ij} were determined from the principle of least squares. Some experimentation was required to find the form of the product polynomial which would give the best approximation. The resulting polynomial yields a maximum error in x of 0.7 per cent while the mean error was more nearly 0.2 per cent. Thus we can rewrite equation (7) as follows

$$R = R_p \sum_{i=0}^{2} \left[\sum_{j=0}^{10} b_{ij} (\omega / \omega_i)^j \right] \theta^{2i} , \tag{9}$$

where the values of b_{ij} are given by Table 1.

All values not listed in Table 1 are to be taken to be equal to zero. We need now only determine the effective temperature as a function of ω and θ and we may proceed to the actual construction of the rotating atmosphere. To do this we need to make use of assumption 2 and a result of the theorem of Von Zeipel (1924) that the total flux is proportional to the local surface gravity. We shall assume that the atmosphere of a rotating star may be approximated at any point on its surface by a plane parallel atmosphere with the local temperature and gravity. Thus it follows that the effective temperature will be given by Stefan's Law. Therefore

$$T_e \propto g^{1/4} . \tag{10}$$

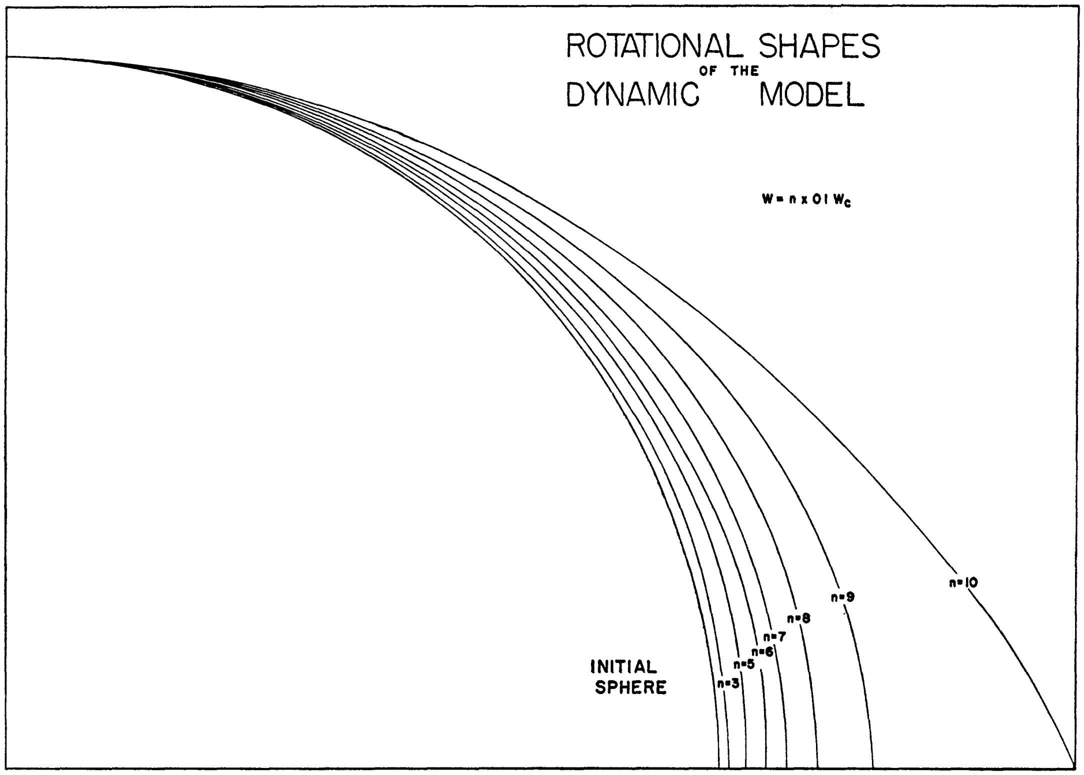

FIG 1 —Cross-sections of rotating stars under the assumptions of constant angular velocity and Roche model.

TABLE 1

PARAMETERS DETERMINING $R(\theta, \omega)$

	b_{ij}				
	$j=0$	$j=1$	$j=2$	$j=9$	$j=10$
$i=0$	1 0001956	0 0	0 0	-8 0843068$\times10^{-2}$	$+8$ 0509197$\times10^{-2}$
$i=1$	0 0	0 0	4 2098469$\times10^{-5}$	$+2$ 5709925$\times10^{-4}$	-2 545361$\times10^{-4}$
$i=2$	0 0	-1 183952$\times10^{-9}$	0 0	-5 598946$\times10^{-8}$	$+5$ 9308193$\times10^{-8}$

Thus we need only determine the constant of proportionality in equation (10), and our task of describing R, g, and T_e as functions of θ and ω is complete. Here the approach used is the same as that used by Slettebak (1949).

The total luminosity of the star is

$$L_\omega = \sigma \int_A T^4 dA = \sigma \int_A C_\omega g_\omega dA \,. \tag{11}$$

Thus the Von Zeipel constant is

$$C_\omega = \frac{L_\omega}{\sigma}\left[\int_A g_\omega dA\right]^{-1}. \tag{12}$$

But since by assumption 3 the total luminosity is not affected by rotation we have

$$C_\omega = \frac{L_0}{\sigma}\left[\int_A g_\omega dA\right]^{-1}, \tag{13}$$

where L_0 is the given non-rotating value of the total luminosity of the star.

TABLE 2*a*

DYNAMIC MODEL

a_0	$= 9.127209\times10^{12}$
a_3	$= 1.307018\times10^{13}$
a_4	$= -2.697141\times10^{13}$
a_6	$= 4.398956\times10^{13}$
a_9	$= -1.043987\times10^{14}$
a_{10}	$= 8.113702\times10^{13}$

TABLE 2*b*

INITIAL PARAMETERS FOR THE NON-ROTATING MODEL

$\log T_e$....	4.27
$M/M_\odot$.	8
$R_p/R_\odot$..	4
$L/L_\odot$..	1835
μ.....	1.127
ω_c...	1.203×10^{-4} rad/sec

If one rewrites equation (4) in terms of the dimensionless variables ω/ω_c and x by making use of equations (5) and (6), it follows that, for a particular ω, the integral in equation (13) may be written as a constant times the mass M. Thus we may write the following:

$$C_\omega' = \frac{ML_0'C_\omega}{M'L_0}. \tag{14}$$

Thus it is possible to compute the constants in Von Zeipel's theorem for a given set of models and by a simple transformation arrive at the appropriate constants for another set of models. To facilitate the ease of computing the C_ω's, the values were computed for a given model by means of equation (13) and Gaussian quadrature and thus fit by means of an approximating polynomial in ω/ω_c of the form

$$C_\omega = \sum_{j=0}^{10} a_j(\omega/\omega_i)^j. \tag{15}$$

The values of the a_j's for the specific set of models to be discussed are given in Table 2*a* and the defining parameters of the set of models in Table 2*b*.

As with Table 1, all values not listed in Table 2*a* are to be taken to be zero. As described earlier the set of models to be discussed will be based on the values for a non-rotating star. The mass, radius, and effective temperature for the model is that of Slettebak (1949). The spectral type and bolometric magnitude agree with those of Schwarzschild (1958) and the mean molecular weight is based on an 85 per cent hydrogen to 15 per cent helium mixture.

We have now constructed all the material required to determine the local radius, surface gravity, and effective temperature on the surface of a star given the mass, total luminosity, polar radius, rotational velocity, and colatitude. Figures 2 and 3 display

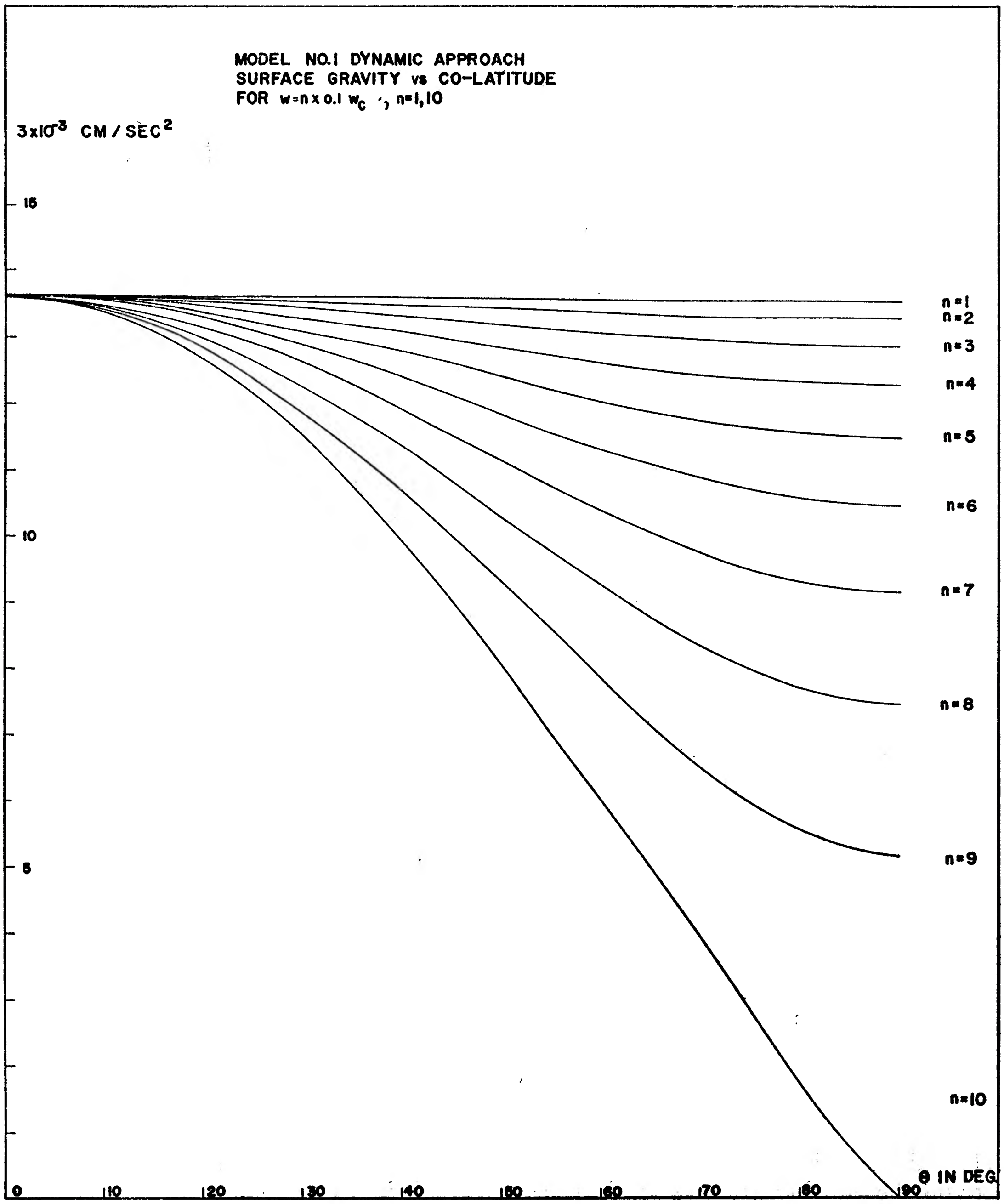

FIG. 2.—Variation of surface gravity with colatitude for a model with $R_p = 4R_\odot$ and $M = 8M_\odot$

the results of these calculations. Thus we may proceed to Section III and determine the specific intensity, monochromatic flux, and integrated flux from such a model star.

III. DETERMINATION OF THE MONOCHROMATIC FLUXES AND COLORS

The determination of the colors and monochromatic fluxes for the set of models described in Section II introduce another free parameter. Since a star undergoing rotation is no longer a spherical body, we would expect a variation in color with a change in the angle of inclination of the axis of rotation of the star. We must also make some assumption as to the type atmosphere we wish to use at a local point on the surface of the star.

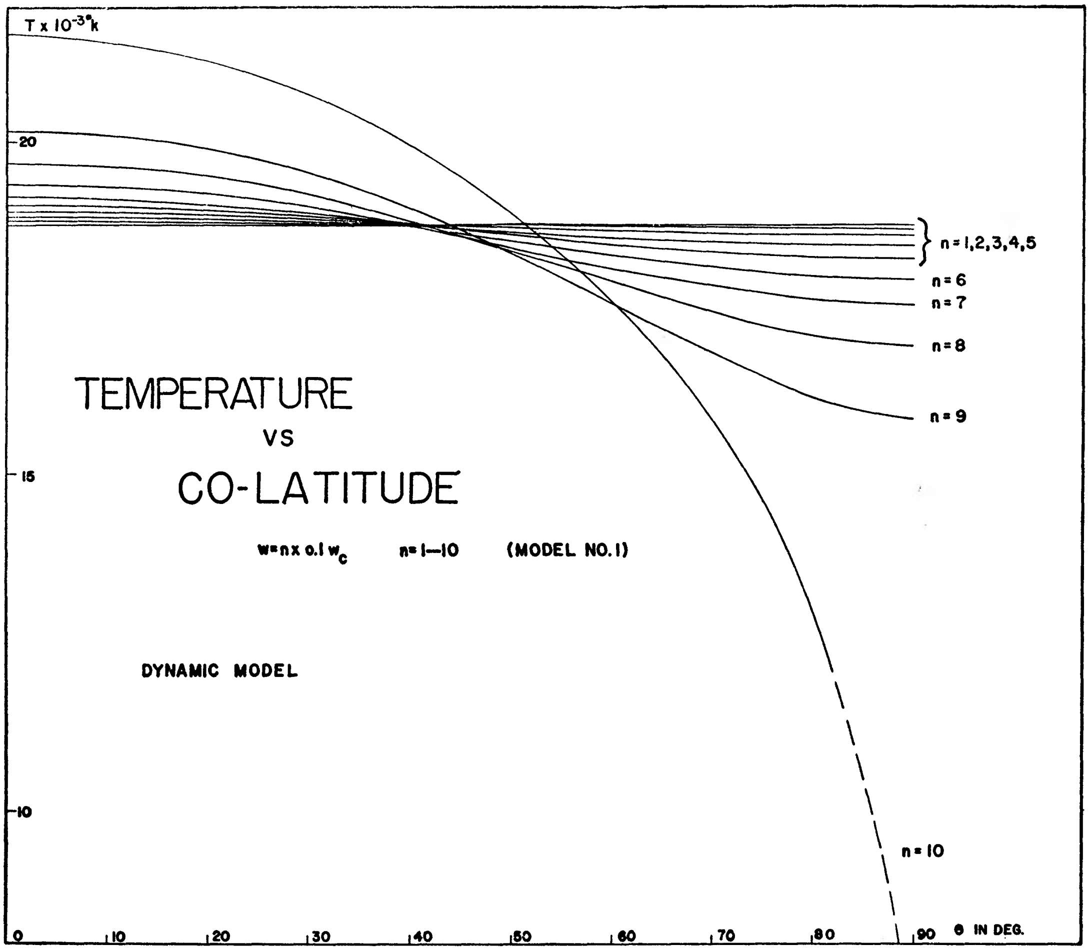

FIG. 3.—Variation of effective temperature with colatitude for a model with $R_p = 4R_{\odot}$, $M = 8M_{\odot}$, and $\log T_e(\omega = 0) = 4.27$.

For the purpose of this paper it will be sufficient to assume that the atmosphere of the star is gray. The author hopes in the near future to replace the gray atmosphere used in this study with a set of atmosphere described by Underhill (1961).

The classical solution to the equation of transfer for a gray atmosphere yields the following

$$I_{\nu}(\mu, 0) = \int_0^{\infty} B_{\nu}(T)\, e^{-\tau/\mu} \frac{d\tau}{\mu}, \tag{16}$$

where the relationship between T and τ is given by

$$T^4 = \tfrac{3}{4} T_e^4 [\tau + q(\tau)] . \tag{17}$$

Chandrasekhar (1950) has shown that $q(\tau)$ may be approximated by

$$q(\tau)=Q+\sum_{a=1}^{n-1} L_a e^{-\tau k_a}, \tag{18}$$

where n is the order of the approximation and the values of k_a are given by the solution to the characteristic equation

$$1=\sum_{i=1}^{n} \frac{a_i}{1-u_i^2 k^2}, \tag{19}$$

TABLE 3

PARAMETERS DETERMINING $q(\tau)$

$n=5$			
a	k_a	L_a	$Q=0\ 708268$
1..	1 05942613	$-4\ 58184\times10^{-3}$	.
2	1 29781394	$-1\ 72589\times10^{-2}$	
3	1 98733010	$-3\ 81283\times10^{-2}$	
4	5 72117528	$-7\ 09314\times10^{-2}$	

where a_i and u_i are the weights and points of Gaussian division in the interval -1 to $+1$. Chandrasekhar has also shown that values for Q and L_a may be determined from the following set of equations:

$$\sum_{a=1}^{n-1} \frac{L_a}{1-u_i k_a}-u_i+Q=0 \qquad (i=1, 2, \ldots, n). \tag{20}$$

For purposes of this paper it was felt that taking $n = 5$ would yield a sufficient approximation for $q(\tau)$. The values for L_a, k_a, and Q determined in this manner are given below in Table 3.

We need yet to determine a value for μ in equation (16) as a function of position on the surface of the star. Since the star is no longer spherical the expression arrived at will be more complicated than might be expected. The method, however, is very simple. Since μ is defined as the cosine or the angle between the observer and the normal to the emitting surface, we need only compute the inner product of $-g$ with a unit vector in the direction of the observer and divide by $|g|$. When this is done in the coordinate system of the star (Fig. 4) we have the following result:

$$\mu=\frac{a(\sin\theta\sin i\cos\phi+\cos\theta\cos i)-b(\sin i\cos\phi\cos\theta-\sin\theta\cos i)}{|g|} \tag{21}$$

where $a = [(GM/R^2) - \omega^2 R \sin^2\theta]$ and $b = (\omega^2 R \sin\theta\cos\theta)$. We are now in a position to evaluate the monochromatic flux for a particular model as a function of ω and i. The appropriate expression is

$$L_\lambda(\omega, i)=R_p^2\int_{-\pi/2}^{\pi/2}\int_0^{\pi} I_\lambda(\theta', \phi')\, x^2(\theta')\sin\theta'\mu d\theta' d\phi'. \tag{22}$$

It should be noted that equation (22) is written in terms of a coordinate system attached to the star but oriented toward the observer. However, all the quantities in the integral have been determined with respect to a coordinate system oriented with respect to the axis of rotation of the star. Since it will be necessary to perform the integration numerically, we shall have to pick certain values of θ' and ϕ' and determine the value of the integrand at those points. Thus we shall require a transformation from the primed coordinate system to the unprimed coordinate system. From symmetry of the integrals about the meridional and equatorial planes in the unprimed coordinate systems, the following transformations will be sufficient and unambiguous.

$$\cos\theta = \cos\theta' \sin i + \sin\theta' \cos\phi' \cos i$$

and (23)

$$\tan\phi = \sin\theta' \sin\phi' / (\sin\theta' \cos\phi' \sin i - \cos\theta' \cos i) .$$

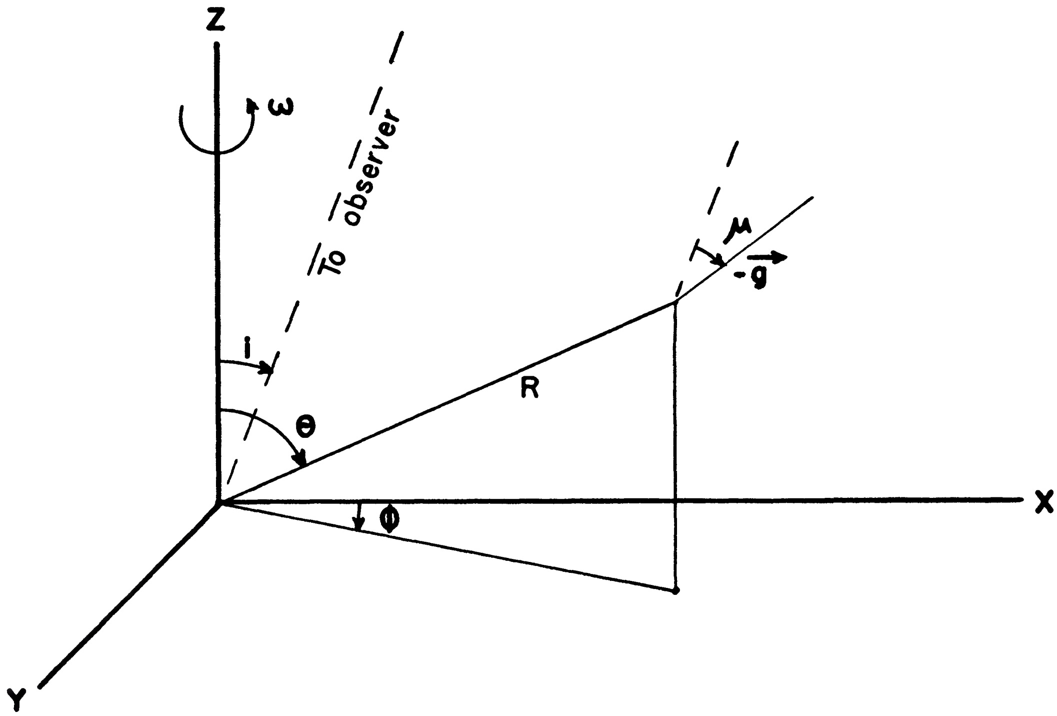

FIG. 4.—Coordinate system defining angles used in this study. The Z-axis is taken to be the rotational axis of the star.

In evaluating (22) numerically it would be desirable to use a quadrature formula with the same degree of precision as the Gaussian quadrature formulae for one dimension. Secrest and Stroud (1961) have shown from the work of Peirce (1956) that a spherical product formula which is exact for polynomials of degree $(2n - 1)(2n' - 1)$ defined on the surface of a sphere may be found in the following manner. Let $V_i = \sin\phi_i$ and $U_j = \cos\theta_j$, then if one chooses V_i to be the zeros of the Chebyshev polynomials in the interval -1 to $+1$ and U_j to be the zeros of the Legendre polynomials in the interval -1 to $+1$ the following approximation formula for equation (22) results with the required degree of precision:

$$L_\lambda(\omega, i) = \frac{\pi^2}{n} R_p \sum_{i=1}^{n} \sum_{j=1}^{n'} I_\lambda(\theta' \,|\, U_j, \phi' \,|\, V_i)\, x^2(\theta' \,|\, U_j)\, \mu B_j \,, \tag{24}$$

where B_j represents the appropriate weights for Gauss-Legendre quadrature.

The results of these calculations are displayed in Figures 5, 6, and 7.

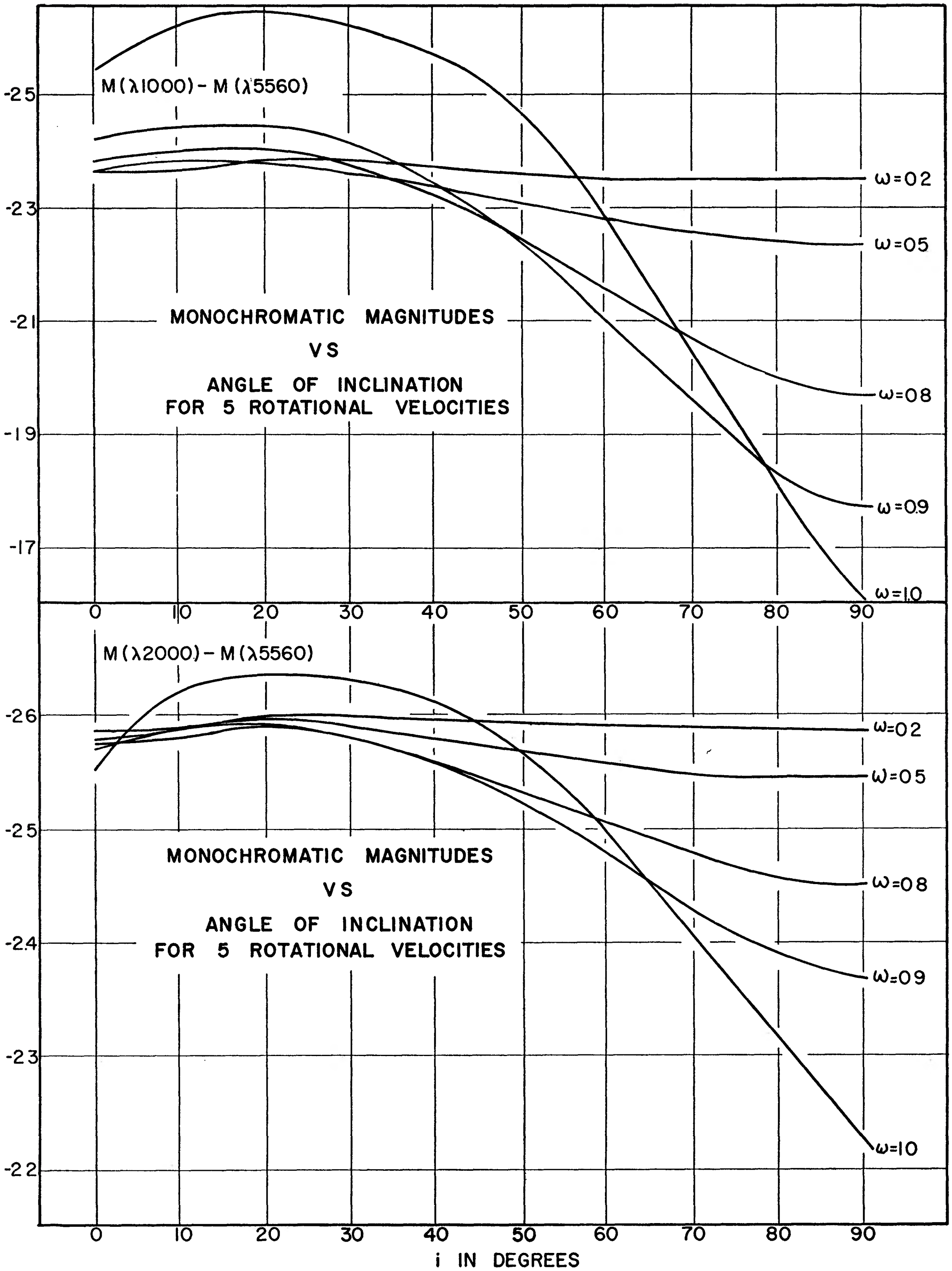

FIG. 5.—Relative monochromatic magnitudes in energy per unit wavelength for model described in Table 2b.

Having completed the continuum study it was decided to investigate variations of the Balmer and Lyman jumps that would be observed as a function of given angle of inclination for a rotational velocity equal to the velocity of breakup. To facilitate the calculations it was assumed that $\kappa_\lambda/\bar{\kappa}$ would be constant with optical depth. The equation analogous to equation (16) is

$$I_\lambda(\mu, 0) = \int_0^\infty \frac{\kappa_\lambda}{\bar{\kappa}} B_\lambda(T)\, e^{-\kappa_\lambda \tau/\mu\bar{\kappa}} \frac{d\tau}{\mu}. \quad (25)$$

This, as in the case of equation (16) was evaluated numerically by means of Gauss-Laguerre quadrature. Since for any given rotation T_e is a function of g alone, it is possible

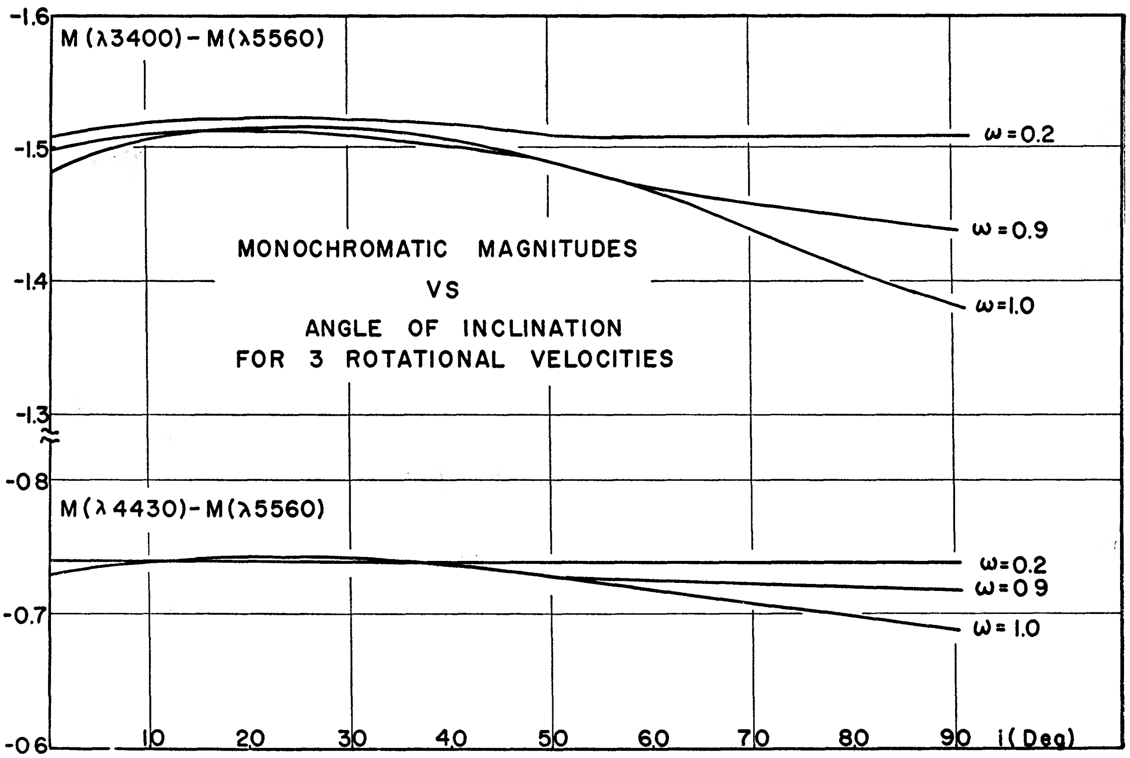

FIG. 6.—Relative monochromatic magnitudes in energy per unit wavelength for model described in Table 2*b*.

to find for each model $\kappa_\lambda/\bar{\kappa}$ as a function of g alone. This was done for breakup velocity with the aid of the graphs of Unsöld (1938). The results of this computation for the Balmer and Lyman jumps are given in Figure 7.

IV. DISCUSSION OF RESULTS

The immediate results of this study may be divided into three main areas:

1. The variation of the observed flux with ω and i.
2. The variation of the theoretical monochromatic colors with ω and i.
3. The variation of the theoretical Balmer and Lyman jumps for a star rotating with breakup velocity as a function of i.

The first of these effects is clearly the most striking. As one would expect, the maximum brightness is observed when the star is seen pole-on while rotating at the breakup veloc-

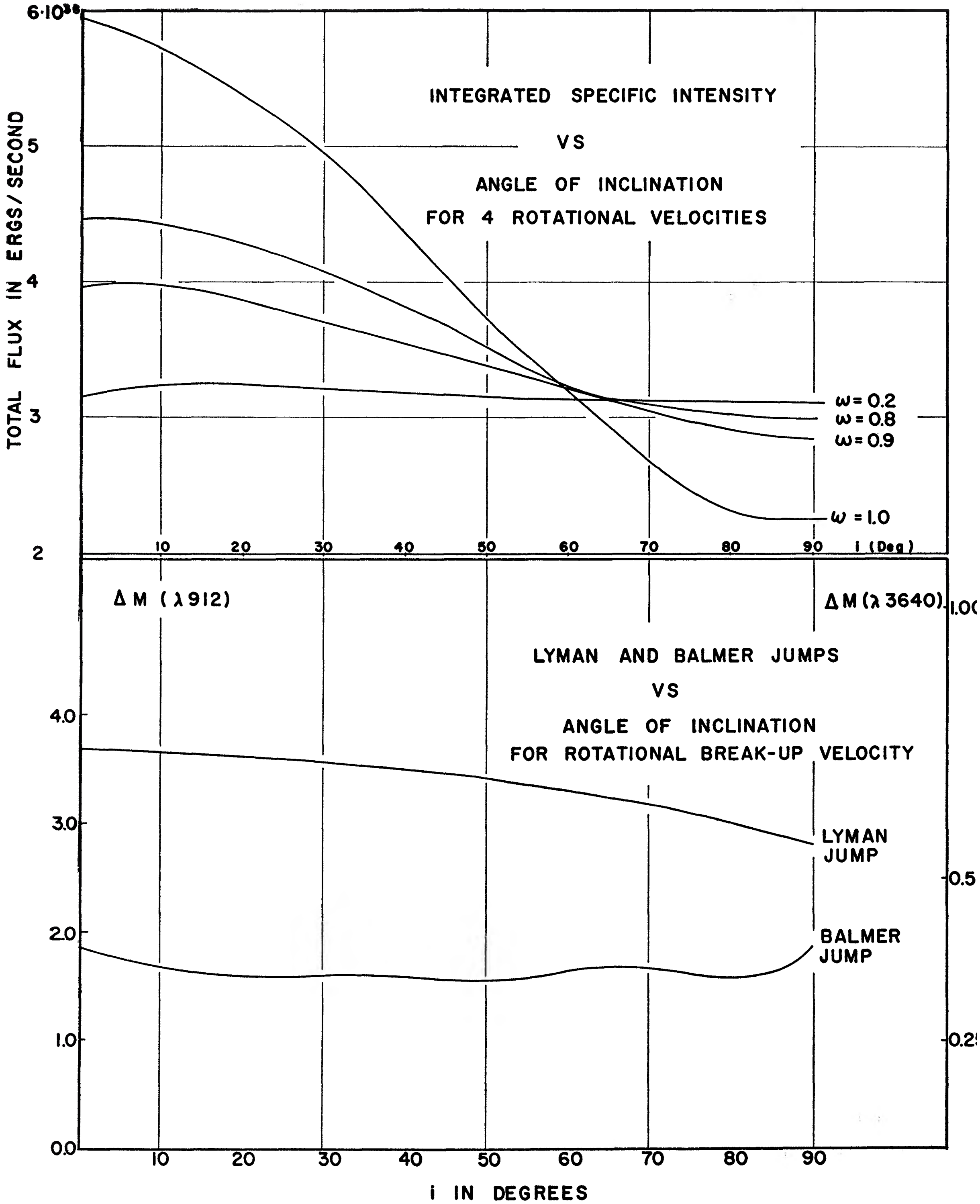

FIG. 7.—Variation of observed integrated specific intensity, Balmer and Lyman jumps, with angle of inclination for the model described in Table 2*b*.

ity. When the star is observed equator-on while rotating with the velocity of breakup, the absolute luminosity is a minimum. Perhaps the only surprising fact is the range of the luminosity. This range corresponds to a little over a magnitude, but is not centered about the non-rotating value. Thus if the distribution of the angles of inclination were uniform in angle, and rotation did not introduce a significant change in spectral type, one would expect to find the mean absolute magnitude of stars rotating at the breakup velocity to be above the main sequence by about half a magnitude at spectral class B3V. However, if the distribution of rotational axis is random in space, then the lower angles of inclination are more likely. This would tend to diminish the prominence of the magnitude shift above the main sequence.

The second quantity determined in this study yields effects which are not nearly as striking as the change in the absolute luminosity. It should be noted that the monochromatic magnitudes displayed in Figures 4 and 5 are computed in wave length normalized to λ 5560. Thus in order to arrive at values in units of frequency it is necessary to add a factor of $5 \log_{10}(5560/\lambda_c)$.

If these values are applied to the monochromatic colors for the non-rotating star, excellent agreement is achieved with the observed color for B3V stars given by Code (1960). This leads one to believe that the assumptions of the gray atmosphere do not seriously affect the results for the monochromatic colors in the visual region of the spectrum. Due to the large differences in the flux near the Lyman limit between the gray and non-gray atmospheres, one should not take the far-ultraviolet colors too seriously. However, the trends indicated by these colors at least provide one with a feeling of what to expect from the use of non-gray atmospheres. Therefore, any serious attempt to measure far-ultraviolet colors will have to deal with the effects of rotation. Although the effects of rotation are small, it is clear that they may well contribute to the "cosmic scatter" in the upper main sequence. Indeed there are a number of effects for which a suitable explanation may be found in the interpretation of the effects of rotation. A few such effects which the author is presently studying are the so-called blue A stars, the apparent subluminous B0ne star, and the systematic displacement above the normal main sequence of the Be stars.

The results of the investigation of the Balmer and Lyman jump lead one to wonder if the same result holds true for spectral lines. Of particular interest to the author are lines strongly affected by pressure broadening as the marked variation in the local surface gravity for large rotational velocities should affect these lines. This will have to wait, however, until the completion of the non-gray atmospheres referred to earlier. The most striking feature of the result of the calculation of the Balmer and Lyman jumps is the lack of variation with respect to angle of inclination of the Balmer jump.

This may be explained by observing that the magnitude of the Balmer jump is affected conversely by increasing surface gravity and effective temperature. Since the effect of changing the angle of inclination for a star rotating at breakup velocity is equivalent to changing the mean gravity and effective temperature in the same direction, one should not be surprised that the net effect on the variation of the Balmer jump is small. It is interesting to note that the value for the Balmer jump agrees well with $-2.5D$ where D is the $\log_{10}$ of the ratio of the observed intensity each side of the jump as given Chalonge and Divan (1952) for a star of this spectral type.

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REFERENCES

Chalonge, D., and Divan, L. 1952, *Ann. d'ap.*, **15**, 201.
Chandrasekhar, S. 1950, *Radiative Transfer* (New York: Dover Publications).
Code, A. 1960, *Stellar Atmospheres* (Chicago: University of Chicago Press).
Eddington, A. 1926, *The Internal Constitutions of the Stars* (New York: Dover Publications).
Peirce, W. 1956, "Numerical Integrations over Planar Regions," unpublished thesis, University of Wisconsin.
Schwarzschild, M. 1958, *Structure and Evolution of the Stars* (Princeton, N J.: Princeton University Press).
Secrest, D., and Stroud, A. 1961, *Wis. A.F. Series*, **9**, 18.
Slettebak, A. 1949, *Ap. J.*, **110**, 498–514.
Underhill, A. 1961, *Pub. Dom. Ap. Obs.*, **11**, no. 19, 363–383.
Unsöld, A. 1938, *Physik d. Sternatmosphären*, **2**, 186.
Von Zeipel, H. 1924, *M.N.*, **84**, 665.