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A Geometric Nonlinear Observer for Simultaneous Localisation and Mapping

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***Abstract*— The Simultaneous Localisation and Mapping (SLAM) problem involves estimating the pose of a robot relative to landmarks observed in the environment while at the same time estimating the location of those landmarks in the environment. This paper introduces a framework in which the landmarks and robot pose can be modelled in a single geometric structure, that of a homogeneous space obtained as the quotient of a novel Lie-group that we term the $\mathrm{SLAM}_n(3)$ group. Using this formulation we apply techniques from observer design for symmetric systems to derive a novel observer for the SLAM problem posed in continuous-time.**

I. Introduction

Simultaneous Localisation and Mapping (SLAM) has been a core problem in robotics for the last thirty years [1], [2] and remains a key technology for mobile robotics. The problem involves estimation of an environment map while simultaneously estimating the pose of a robot with respect to that map. If the robot's pose is known *a-priori* then the environment estimation problem is purely a mapping problem and classical solutions are well established and highly effective [3]. If a map is available *a-priori* then the problem is one of pose-estimation, a problem that is well known in the robotics community [4], and has attracted attention in the non-linear observer community [5], [6], [7]. The dual estimation problem, especially for full 3D-robot pose and dynamic environments, is highly non-linear and continues to attract significant research effort [8]. Known issues are consistency of filtering based approaches [9], computational complexity and scalability of optimization based approaches [8], especially in the case the environment is not static [2], [10], and non-linearity of the underlying optimization [8], [2], [1]. The inherent non-linearity of robot pose has lead a number of authors to exploit the Lie-group structure of the special Euclidean group $\mathrm{SE}(3)$ in formulating classical SLAM problems [11], [12].

In the last ten years, the non-linear observer community has made significant contribution in attitude estimation [13], [14], [15] and full pose estimation [16], [17], [5]. There have been several works that have considered applying non-linear observer concepts to the SLAM problem. Bonnabel exploited the natural invariance of the SLAM problem to changes in the frame-of-reference to derive an invariant EKF [18]. In recent work Lourenço *et al.* [19] propose a "robo-centric" nonlinear observer that estimates environment points expressed in the body-fixed-frame of the robot. An alternative approach taken by Johanssen *et al.* [20] is to use an Attitude Heading Reference System (AHRS) to provide a rotation estimate, which is used to render the remaining mapping and robot position estimation linear and then apply Kalman Filtering. A similar approach is taken in Le Bras *et al.* [21] where bearing measurements are used rather than landmarks.

In this paper we present a highly robust, simple, and computationally cheap nonlinear observer for the general landmark SLAM problem posed in continuous time. The approach is based on a novel formulation of the SLAM state-space as a quotient manifold of equivalent landmark and pose configuration (under change of reference frame) and the identification of a transitive group action on this space for which the SLAM kinematics are equivariant. We term the underlying Lie-group, associated with a SLAM problem for a rigid-body robotic system moving in Euclidean 3-space with n landmarks, the $\mathrm{SLAM}_n(3)$ group. This group appears to be novel and to the authors knowledge has not been studied before in the literature. The structure introduced allows direct application of the authors previous work [22] in development of non-linear observers to yield a novel observer for continuous-time SLAM. The approach taken has several advantages: It accounts for the known invariance of the SLAM problem to change of reference frame by considering a quotient structure on the raw coordinates of the SLAM problem. The formulation naturally allows for the consideration of independent motion of points in the environment in the structure of the $\mathfrak{slam}_n(3)$ Lie-algebra, although that extension is not included in the present paper. The inherent symmetry of the approach should offer advantages in global robustness and convergence properties of algorithms based on this formulation.

The paper is organised in six sections along with introduction and conclusion. In Section II we introduce notation and definitions before formulating the SLAM problem in a geometric setting in §III. The SLAM group is introduced in §IV and equivariance of the SLAM kinematics is proved. Section V uses the geometric structure introduced to derive a non-linear observer for the SLAM problem and Section VI provides a simple simulation to demonstrate the observer evolution. Due to space constraints, we have not been able to include proofs. The present simulation is intended only to demonstrate the underlying behaviour of the proposed observer, and the authors will undertake more detailed experimental studies on real world data in future work.

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II. NOTATION

We denote 3-dimensional Euclidean space by $\mathbb{E}^3$, a copy of $\mathbb{R}^3$ without the normal vector space structure; that is, without the addition operation, scalar multiplication or identified zero element. One should think of elements in $\mathbb{E}^3$ as physical points in space. In order to provide coordinates for points $p \in \mathbb{E}^3$ it is necessary to choose a reference frame; that is a reference point and a set of orthonormal axes by which a point $p = (p_1, p_2, p_3)$ is written as a triple of coordinates associated with its offset distance from the reference in each of the axes directions. For a given frame of reference, we will use homogeneous coordinates

$$\overline{p} \in \mathbb{E}^3, \qquad \overline{p} = \begin{pmatrix} p \\ 1 \end{pmatrix}$$

to distinguish points in $\mathbb{E}^3$.

We use the notation $\mathbf{R}^3$ to denote the real 3-dimensional additive group and we distinguish between this and the full vector space structure of $\mathbb{R}^3$. Elements of $\mathbf{R}^3$ do not represent points in space, rather, they represent translations of Euclidean space. We write $a = (a_1, a_2, a_3) \in \mathbf{R}^3$ as a three vector with respect to the canonical basis, noting that since $\mathbf{R}^3$ is not physical, there is no concept of a frame of reference. Elements of the additive group $\mathbf{R}^3$ can be added and subtracted in the natural manner corresponding to the Abelian vector addition.

There is a natural binary operation $\boxplus : \mathbf{R}^3 \times \mathbb{E}^3 \to \mathbb{E}^3$ by

$$a \boxplus \overline{p} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \boxplus \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 + p_1 \\ a_2 + p_2 \\ a_3 + p_3 \\ 1 \end{pmatrix} \tag{1}$$

that represents translations of $\overline{p} \in \mathbb{E}^3$ by $a \in \mathbf{R}^3$. We abuse notation and write $\overline{p} \boxplus a = a \boxplus \overline{p}$ relying on the over-bar notation to identify the element of $\mathbb{E}^3$. One has

$$(\overline{p} \boxplus a) \boxplus b = \overline{p} \boxplus (a + b). \tag{2a}$$

$$\overline{p} \boxplus 0 = \overline{p}. \tag{2b}$$

The special Euclidean group $\mathrm{SE}(3)$ is the set of rigid-body transformations of $\mathbb{E}^3$. For $A \in \mathrm{SE}(3)$ we use the notation $R_A \in \mathrm{SO}(3)$ and $x_A \in \mathbf{R}^3$ to denote the rotation and translation components of the rigid-body transformation. The classical (homogeneous) matrix representation of A is

$$A = \begin{pmatrix} R_A & x_A \\ 0 & 1 \end{pmatrix}, \tag{3}$$

and the rigid-body transformation on homogeneous coordinates is matrix multiplication $\overline{q} = A\overline{p} = \overline{R_A p + x_A}$. One has $R_{AB} = R_A R_B$, and $R_{A^{-1}} = R_A^{-1} = R_A^\top$ since the normal $\mathrm{SE}(3)$ structure is direct on the $\mathrm{SO}(3)$ subgroup. A key formulae that we exploit is

$$A(\overline{p} \boxplus a) = A\overline{p} \boxplus R_A a. \tag{4}$$

Observe that there can only ever be a single element of $\mathbb{E}^3$ in a $\boxplus$ summation and it is this element that attracts the full action of A. Any translations associated with elements of the group $\mathbf{R}^3$ attract only the rotation R_A.

Consider a point $\overline{p}(t) \in \mathbb{E}^3$ that is moving. Exploiting its affine structure, we identify the tangent vector space $T_{\overline{p}}\mathbb{E}^3 \equiv \mathring{\overline{\mathbb{R}}}^3$ to the $\mathbb{R}^3$ vector subspace of $\mathbb{R}^4$ obtained by fixing the fourth element to be zero. Thus, the velocity $\mathring{\overline{u}} \in \mathring{\overline{\mathbb{R}}}^3$ of a point $\overline{p} \in \mathbb{E}^3$ is written

$$\mathring{\overline{u}} = \begin{pmatrix} u \\ 0 \end{pmatrix} \in \mathring{\overline{\mathbb{R}}}^3,$$

where the "bar-circle" notation ($\mathring{\overline{u}}$) is similar to the homogeneous coordinates "bar" notation ($\overline{p}$) except that the fourth entry of the vector is zero rather than one. One has the convenient algebra

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{p} = \mathring{\overline{u}}, \tag{5}$$

although this a function of the coordinate realisation chosen and is not intrinsic geometric structure. Note that whereas Euclidean space $\mathbb{E}^3$ has no vector space structure, the tangent space $\mathring{\overline{\mathbb{R}}}^3$ is a 3-dimensional vector space in the normal manner associated with tangent constructions.

Remark 2.1: Equation (5) concerns the differential structure of landmark kinematics. We note that to specify a physical velocity, one must also take account of the frame of reference in which the velocity is measured or expressed. In particular, in this paper we will consider both landmark velocities $\mathring{\overline{u}}_i$ expressed in coordinates of the stationary reference frame and the same physical velocity $\mathring{\overline{v}}_i$ expressed in coordinates of the body-fixed frame. One has

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{p}_i = \mathring{\overline{u}}_i = \mathring{\overline{R_P v_i}} = P\mathring{\overline{v}}_i \tag{6}$$

where we write the coordinate change as matrix multiplication $P\mathring{\overline{v}}_i$, exploiting the zero in the fourth element of $\mathring{\overline{v}}_i$ to ensure that only the rotation component of P is applied to transform body-fixed frame coordinates for velocity $\mathring{\overline{v}}_i$ into reference frame coordinates $\mathring{\overline{u}}_i$.

We write $\mathcal{SE}(3)$ to denote the SE(3)-torsor, that is elements of $\mathrm{SE}(3)$ considered without the group structure. The pose of a vehicle moving in Euclidean 3D space, that is the position $\overline{x}_P \in \mathbb{E}^3$ and orientation R_P of a body-fixed frame $\{P\}$ attached to the vehicle relative to some fixed reference frame $\{0\}$, is an element $P \in \mathcal{SE}(3)$.

In this paper, we choose to express the velocities of the robot in the body-fixed frame. This formulation is the most common encountered in the SLAM literature and is most interesting for the present geometric study. The linear velocity[1] of x_P is denoted $v \in \mathbb{R}^3$ and the angular velocity of R_P is denoted $\Omega \in \mathbb{R}^3$ as elements of the full vector space $\mathbb{R}^3$. The kinematics of the motion of the rigid-body are given by

$$\dot{x}_P = R_P v, \qquad \dot{R}_P = R_P \Omega_\times \tag{7}$$

[1]The position of the robot frame $\overline{x}_P \in \mathbb{E}^3$ is an element of the Euclidian space, however, we do not use the homogeneous representation here since it is subsumed by the matrix $\mathcal{SE}(3)$ representation of pose (3). Similarly, the velocity representation $\dot{\overline{x}}_P = P\mathring{\overline{v}}$ is not used since this algebra is subsumed in the $(v, \Omega)_\wedge$ notation introduced in (9).

where

$$\Omega_\times = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

One has that $\Omega_\times w = \Omega \times w$ for any $w \in \mathbb{R}^3$ and $\times$ the vector product. Equation (7) can be written compactly

$$\dot{P} = PV, \tag{8}$$

where

$$V = (\Omega, v)_\wedge := \begin{pmatrix} \Omega_\times & v \\ 0 & 0 \end{pmatrix}. \tag{9}$$

The velocity V is an element of the tangent space of $\mathrm{SE}(3)$ at the identity

$$T_I\mathrm{SE}(3) = \{(\Omega, v)_\wedge \mid \Omega, v \in \mathbf{R}^3\}$$

where I is the $\mathbb{R}^{4\times4}$ identity matrix. Note that $T_I\mathrm{SE}(3) \equiv \mathfrak{se}(3)$ is identified with the Lie algebra of $\mathrm{SE}(3)$ and we will make this identification throughout the paper, although only the vector space structure of $\mathfrak{se}(3)$ is relevant to the velocity of a pose. The tangent space of $\mathcal{SE}(3)$ at a point P is given by

$$T_P\mathcal{SE}(3) = \{PV \mid V = (\Omega, v)_\wedge \in \mathfrak{se}(3)\}.$$

We will make extensive use of smooth Lie-group actions α, ϕ, Φ and ρ. A 'right' action $\beta : \mathbf{G} \times \mathcal{M} \to \mathcal{M}$ of a Lie group $\mathbf{G}$ on a smooth manifold $\mathcal{M}$ is a smooth mapping with properties

$$\beta(A, \beta(B, \xi)) = \beta(B \cdot A, \xi)$$
$$\beta(\mathrm{id}, \xi) = \xi$$

where $\mathrm{id} \in \mathbf{G}$ is the identity element. We use the notation $\beta_A : \mathcal{M} \to \mathcal{M}$ and $\beta_\xi : \mathbf{G} \to \mathcal{M}$ where

$$\beta_A(\xi) := \beta(A, \xi) =: \beta_\xi(A)$$

for $A \in \mathbf{G}$ and $\xi \in \mathcal{M}$. Observe that $\boxplus$ is a group action of $\mathbf{R}^3$ on $\mathbb{E}^3$.

For a group $\mathbf{G}$ and its associated Lie algebra $\mathfrak{g}$ the adjoint map is $\mathrm{Ad}_A V = AVA^{-1}$.

Let $\shortmid : \mathbb{R}^{4\times4} \to \mathfrak{se}(3)$ denote the unique orthogonal projection of $\mathbb{R}^{4\times4}$ onto $\mathfrak{se}(3)$ with respect to the Frobenius inner product $\langle\langle A, B\rangle\rangle = \mathrm{tr}\,(A^\top B)$. That is for all $V \in \mathfrak{se}(3)$, $M \in \mathbb{R}^{4\times4}$, one has

$$\langle\langle V, M\rangle\rangle = \langle\langle V, \shortmid(M)\rangle\rangle = \langle\langle \shortmid(M), V\rangle\rangle\,.$$

One has that for all $M_1 \in \mathbb{R}^{3\times3}, m_{2,3} \in \mathbf{R}^3, m_4 \in \mathbb{R}$,

$$\shortmid\left(\begin{bmatrix} M_1 & m_2 \\ m_3^\top & m_4 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}(M_1 - M_1^\top) & m_2 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

III. Problem formulation

A. SLAM Manifold

We begin by defining *raw* coordinates for the SLAM problem using a fixed but arbitrary reference $\{0\}$. Let $P \in \mathcal{SE}(3)$ represent the body-fixed frame coordinates of this reference frame and let

$$\overline{p}_i \in \mathbb{E}^3, \qquad i = 1, \dots, n,$$

be sparse points in the environment expressed with respect to the reference frame $\{0\}$. The *total space* of the SLAM problem is the product space

$$\mathcal{T}_n(3) = \mathcal{SE}(3) \times \mathbb{E}^3 \times \cdots \times \mathbb{E}^3$$

made up of these raw coordinates $(P, \overline{p}_1, \dots, \overline{p}_n)$. The subscript denotes the number of landmarks while the bracket denotes that the dimension of Euclidean space with which the group is associated.

The raw coordinates of the SLAM problem are not intrinsic since they depend on the arbitrary choice of inertial frame $\{0\}$. Indeed, one may consider any rigid transformation $S \in \mathrm{SE}(3)$ of frame $\{0\}$ to a new reference $\{1\}$ (as shown in Figure 1) and generate new 'raw' coordinates $(S^{-1}P, S^{-1}\overline{p}_1, \dots, S^{-1}\overline{p}_n)$ with respect to a new reference $\{1\}$ that represents the same SLAM configuration.

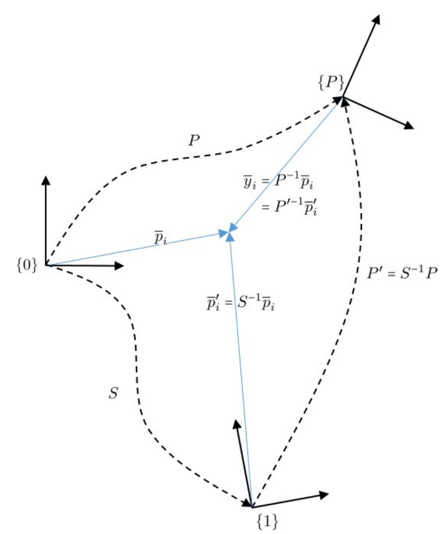

Fig. 1. The raw coordinates $P \in \mathcal{SE}(3)$ and $\overline{p}_i \in \mathbb{E}$ are measured relative to a reference frame $\{0\}$. An arbitrary $\mathrm{SE}(3)$ transformation S transforms $\{0\}$ to a new reference $\{1\}$ and new coordinates $P' \in \mathcal{SE}(3)$ and $\overline{p}'_i \in \mathbb{E}$ describe the robot and environment. Note that the output $\overline{y}_i$ does not change and refers to the same physical location as $\overline{p}_i$ and $\overline{p}'_i$.

Lemma 3.1: Consider the map $\alpha : \mathrm{SE}(3) \times \mathcal{T}_n(3) \to \mathcal{T}_n(3)$ defined by

$$\alpha(S, (P, \overline{p}_1, \dots, \overline{p}_n)) := (S^{-1}P, S^{-1}\overline{p}_1, \dots, S^{-1}\overline{p}_n). \tag{11}$$

This map defines a (non-transitive) proper right group action of $\mathrm{SE}(3)$ on $\mathcal{T}_n(3)$. The quotient (see Definition 3.2)

$$\mathcal{M}_n(3) = \mathcal{T}_n(3)/\alpha \tag{12}$$

where n denotes the number of points in the environment that are considered, is a smooth manifold of dimension $3n$.

Definition 3.2: The group action (11) is termed the *SLAM invariance* action, or just the invariance action where the context is clear. An equivalence class of an element $(P, \overline{p}_1, \dots, \overline{p}_n) \in \mathcal{T}_n(3)$ is the set

$$[P, \overline{p}_1, \dots, \overline{p}_n] = \left\{ (S^{-1}P, S^{-1}\overline{p}_1, \dots, S^{-1}\overline{p}_n) \,\middle|\, S \in \mathrm{SE}(3) \right\} \tag{13}$$

obtained by applying $\alpha(S, \cdot)$ to $(P, \overline{p}_1, \dots, \overline{p}_n)$ for all elements $S \in \mathrm{SE}(3)$. An equivalence class $[P, \overline{p}_1, \dots, \overline{p}_n]$ is termed a (SLAM) configuration. The manifold (12) $\mathcal{M}_n(3) = \mathcal{T}_n(3)/\alpha$ consists of the set of all configurations

$$\mathcal{M}_n(3) = \{[P, \overline{p}_1, \dots, \overline{p}_n] \,|\, (P, \overline{p}_1, \dots, \overline{p}_n) \in \mathcal{T}_n(3)\}$$

This manifold is termed the *SLAM manifold*. We will denote elements of $\mathcal{M}_n(3)$ by configurations

$$\xi = [P, \overline{p}_1, \dots, \overline{p}_n] \in \mathcal{M}_n(3)$$

where compressed notation is appropriate. We will also use the notation $\pi : \mathcal{T}_n(3) \to \mathcal{M}_n(3)$,

$$\pi(P, \overline{p}_1, \dots, \overline{p}_n) := [P, \overline{p}_1, \dots, \overline{p}_n] \tag{14}$$

for the intrinsic projection associated with the quotient projection. □

The key advantage of this formulation is that the ambiguity associated with specification of the reference frame is not present for configurations in the SLAM manifold. In particular, since a configuration is an equivalence class $\xi = [P, \overline{p}_1, \dots, \overline{p}_n]$, then for any choice of raw coordinates $(P', \overline{p}'_1, \dots, \overline{p}'_n) = (S^{-1}P, S^{-1}\overline{p}_1, \dots, S^{-1}\overline{p}_n)$ related by rigid-body translation of the reference frame, the associated configuration $\xi = [S^{-1}P, S^{-1}\overline{p}_1, \dots, S^{-1}\overline{p}_n]$ is the same element of $\mathcal{M}_n(3)$.

The outputs we will consider in this paper are body-fixed frame observations of points in the environment. The output space is denoted $\mathcal{N}_n(3) = \mathbb{E}^3 \times \dots \times \mathbb{E}^3$ and an output $y \in \mathcal{N}_n(3)$ is a collection of n body-fixed frame environment points in $\mathbb{E}^3$

$$y = (\overline{y}_1, \dots, \overline{y}_n) = (P^{-1}\overline{p}_1, \dots, P^{-1}\overline{p}_n).$$

In order that the output can be thought of as a function $h : \mathcal{M}_n(3) \to \mathcal{N}_n(3)$ then it is necessary to verify that the map

$$h(\xi) = h([P, \overline{p}_1, \dots, \overline{p}_n]) = (P^{-1}\overline{p}_1, \dots, P^{-1}\overline{p}_n) \tag{15}$$

is well defined. That is

$$\begin{aligned} h(\xi) &= h([\alpha(S, (P, \overline{p}_1, \dots, \overline{p}_n))]) \\ &= h([S^{-1}P, S^{-1}\overline{p}_1, \dots, S^{-1}\overline{p}_n]) \\ &= (P^{-1}SS^{-1}\overline{p}_1, \dots, P^{-1}SS^{-1}\overline{p}_n) \\ &= (P^{-1}\overline{p}_1, \dots, P^{-1}\overline{p}_n) = h(\xi) \end{aligned}$$

as required. In particular, the output does not depend on the particular element of the equivalence class used to represent the configuration $\xi \in \mathcal{M}_n(3)$.

B. Tangent Space of $\mathcal{M}_n(3)$

The tangent space of $\mathcal{M}_n(3)$ at a point $\xi = \pi(\Xi)$ is formally a quotient of the tangent space of the total space [23]

$$T_{\pi(\Xi)}\mathcal{M}_n(3) := T_\Xi \mathcal{T}_n(3) / \ker \mathrm{d}\pi_\Xi$$

where $\mathrm{d}\pi_\Xi : T_\Xi \mathcal{T}_n(3) \to T_\xi \mathcal{M}_n(3)$ is the differential of π at the point Ξ and $\ker \mathrm{d}\pi_\Xi$ is the kernel of the linear map.

To make this more explicit, consider a smooth curve $\Xi(t) = (P(t), \overline{p}_1(t), \dots, \overline{p}_n(t)) \in \mathcal{T}_n(3)$. The tangent vector at a point $\Xi = \Xi(0)$ can be identified with the derivative

$$\dot{\Xi}(0) = (\dot{P}(0), \dot{\overline{p}}_1(0), \dots, \dot{\overline{p}}_n(0)) \in \mathbb{R}^{4\times4} \times \mathbb{R}^4 \times \dots \times \mathbb{R}^4.$$

That is we use the embedded nature of the total coordinates to define $T_\Xi \mathcal{T}_n(3) \subset \mathbb{R}^{4\times4} \times \mathbb{R}^4 \times \dots \times \mathbb{R}^4$ as a matrix vector subspace. Recalling (8) and (5), and noting that these kinematics fully characterise all motion in the total space, then it is possible to write an algebraic form for a tangent vector at $\Xi \in \mathcal{T}_n(3)$ as

$$\dot{\Xi} = (PV, \vec{\overline{u}}_1, \dots, \vec{\overline{u}}_n)$$

for $V \in \mathfrak{se}(3)$ and $\vec{\overline{u}}_i \in \vec{\overline{\mathbb{R}}}^3$.

Fixing $\Xi_0 = (P, \overline{p}_1, \dots, \overline{p}_n)$ constant, consider a family of curves

$$\Xi(t) = \alpha(S(t), \Xi_0) = (S^{-1}(t)P, S^{-1}(t)\overline{p}_1, \dots, S^{-1}(t)\overline{p}_n)$$

where $S(t) \in \mathrm{SE}(3)$ is a curve with $\dot{S}(0) = W \in \mathfrak{se}(3)$ and $S(0) = I_4$. Then $\pi(\Xi(t)) = \Xi_0$ by construction. Taking the differential of $\Xi(t)$ at $t = 0$ we get

$$(-WP, -W\overline{p}_1, \dots, -W\overline{p}_n) \in \ker \mathrm{d}\pi_{\Xi_0} \subset T_{\Xi_0}\mathcal{T}_n(3) \tag{16}$$

that, for $W \in \mathfrak{se}(3)$, fully characterises the kernel of $\mathrm{d}\pi$ at Ξ_0 since $\frac{\mathrm{d}}{\mathrm{d}t}\pi(\Xi(t)) = \mathrm{d}\pi(\dot{\Xi}(0)) = 0$.

A representation of the tangent space $T_\xi \mathcal{M}_n(3)$ is now given by

$$T_\xi \mathcal{M}_n(3) = \{\lfloor \dot{\Xi} \rfloor_\Xi \mid \Xi \in \mathcal{T}_n(3) \text{ such that } \pi(\Xi) = \xi\}$$

where

$$\lfloor \dot{\Xi} \rfloor_\Xi = \{(PV - WP, \vec{\overline{u}}_1 - W\overline{p}_1, \dots, \vec{\overline{u}}_n - W\overline{p}_n) \mid W \in \mathfrak{se}(3)\}$$

is an algebraic construction for the equivalence class in $T_\Xi \mathcal{T}_n(3)/\ker \mathrm{d}\pi$ associated with a tangent vector $\dot{\Xi} = (PV, \overline{u}_1, \dots, \overline{u}_n)$.

This representation of $T_\xi \mathcal{M}_n(3)$ depends on the particular choice of $\Xi \in \mathcal{T}_n(3)$ (where $\pi(\Xi) = \xi$) that is used in the construction. It is, however, straightforward to map between different representations of the same tangent space. In particular, for a given Ξ, consider the point $\alpha(S, \Xi) \in \mathcal{T}_n(3)$ for $S \in \mathrm{SE}(3)$. Then for a time varying trajectory $\Xi(t)$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\alpha(S, \Xi) &= \mathrm{d}\alpha_S(\dot{\Xi}) \\ &= (S^{-1}PV, S^{-1}\vec{\overline{u}}_1, \dots, S^{-1}\vec{\overline{u}}_n) \in T_{\alpha(S,\Xi)}\mathcal{T}_n(3) \end{aligned}$$

By construction $\pi(\alpha(S, \Xi(t))) = \pi(\Xi(t))$ and thus $\lfloor \frac{\mathrm{d}}{\mathrm{d}t}\alpha(S, \Xi) \rfloor_{\alpha(S,\Xi)}$ and $\lfloor \frac{\mathrm{d}}{\mathrm{d}t}\Xi \rfloor_\Xi$ are the same element of $T_{\pi(\Xi)}\mathcal{M}_n(3)$ expressed in different representations of the tangent space. Thus, given a tangent vector $\lfloor\dot{\Xi}\rfloor_\Xi$ then the equivalent algebraic representation of the same tangent vector at a different point in the fibre $\alpha(S,\Xi)$ is $\lfloor \mathrm{d}\alpha_S \dot{\Xi}\rfloor_{\alpha(S,\Xi)}$. In explicit coordinates

$$\lfloor (PV, \vec{u}_1, \dots, \vec{u}_n)\rfloor_\Xi = \lfloor (S^{-1}PV, S^{-1}\vec{u}_1, \dots, S^{-1}\vec{u}_n)\rfloor_{\alpha(S,\Xi)} \tag{17}$$

Note that $\mathrm{d}\pi : T_\Xi \mathcal{T}_n(3) \to T_\xi \mathcal{M}_n(3)$. This map is given by $\mathrm{d}\pi(\dot{\Xi}) = \lfloor \dot{\Xi}\rfloor_\Xi$, which can be written as

$$\mathrm{d}\pi(PV, \vec{u}_1, \dots, \vec{u}_n) = \lfloor (PV, \vec{u}_1, \dots, \vec{u}_n)\rfloor_\Xi \tag{18}$$

with respect to the natural representation.

C. SLAM Kinematics

Recalling (8) and (5) the velocity measurements of the SLAM problem consist of a rigid-body velocity $V \in \mathfrak{se}(3)$ and n landmark velocities $\vec{v}_i \in \vec{\mathbb{R}}^3$. These velocities are physical quantities associated with measuring the velocity of the quantity (robot pose in the case of V and landmark position in the case of $\vec{v}_i$) and expressing this velocity in the body-fixed frame $\{P\}$ coordinates. Define

$$\mathbb{V} = \{(V, \vec{v}_1, \dots, \vec{v}_n) \mid V \in \mathfrak{se}(3), \vec{v}_i \in \vec{\mathbb{R}}^3\} \tag{19}$$

and note that $\mathbb{V}$ inherits a natural linear vector space structure from the product of the underlying structures on $\mathfrak{se}(3)$ and $\vec{\mathbb{R}}^3$. As discussed in (6) we will use the matrix multiplication notation $P\vec{v}_i$ to transform $\vec{v}_i$ from body-fixed frame coordinates to reference frame coordinates.

Lemma 3.3: The map

$$\begin{aligned} &f : \mathcal{M}_n(3) \times \mathbb{V} \to T\mathcal{M}_n(3) \\ &f(\xi, (V, \vec{v}_1, \dots, \vec{v}_n)) \coloneqq \lfloor (PV, P\vec{v}_1, \dots, P\vec{v}_n)\rfloor_\Xi \end{aligned} \tag{20}$$

for any $\Xi = (P, \bar{p}_1, \dots, \bar{p}_n) \in \mathcal{T}_n(3)$ such that $\pi(\Xi) = \xi$ is well defined.

The kinematics

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi = f(\xi, (V, \vec{v}_1, \dots, \vec{v}_n)), \qquad \xi(0) = \xi_0, \tag{21}$$

are the kinematics of the SLAM problem for physical velocities $(V, \vec{v}_1, \dots, \vec{v}_n)$. Note that for a static scene assumption then $\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}_i = \vec{u}_i = 0$ and hence $P^{-1}\vec{u}_i = \vec{v}_i = 0$. We model the more general case where $\vec{v}_i \neq 0$ since the observer innovation will contain landmark velocities.

IV. Symmetry of the SLAM problem

In addition to the invariance properties discussed in Section III the SLAM problem has a separate, and independent, equivarient symmetry structure. The symmetry is associated with a group structure that acts transitively on the SLAM manifold, mapping configurations to configurations.

A. Symmetry of the SLAM Manifold

Consider the set of elements

$$\begin{aligned} \mathrm{SLAM}_n(3) = \{&(A, a_1, \dots, a_n) \mid \\ &A \in \mathrm{SE}(3), a_i \in \mathbf{R}^3, i = 1, \dots, n\}. \end{aligned}$$

We will use compact notation $X = (A, a_1, \dots, a_n) \in \mathrm{SLAM}_n(3)$ to denote elements of this set.

Lemma 4.1: The set $\mathrm{SLAM}_n(3)$ is a Lie group, with the product differential structure on $\mathrm{SE}(3) \times \mathbf{R}^3 \times \dots \times \mathbf{R}^3$, under the binary operation

$$\begin{aligned} &(A, a_1, \dots, a_n) \cdot (B, b_1, \dots, b_n) \\ &\qquad = (AB, a_1 + R_A b_1, \dots, a_n + R_A b_n), \end{aligned} \tag{22}$$

with identity $\mathrm{id} = (I_4, 0, \dots, 0)$ and inverse

$$(A, a_1, \dots, a_n)^{-1} = (A^{-1}, -R_{A^{-1}}a_1, \dots, -R_{A^{-1}}a_n).$$

We distinguish strongly between $\mathrm{SLAM}_n(3)$ and the total space $\mathcal{T}_n(3)$ although elements of both sets look similar as sets. We will term $\mathrm{SLAM}_n(3)$ the *SLAM group*.

The SLAM group acts as a symmetry group on the SLAM manifold.

Lemma 4.2: The mapping $\phi : \mathrm{SLAM}_n(3) \times \mathcal{M}_n(3) \to \mathcal{M}_n(3)$

$$\begin{aligned} &\phi((A, a_1, \dots, a_n), [P, \bar{p}_1, \dots, \bar{p}_n]) \\ &\qquad \coloneqq ([PA, \bar{p}_1 \boxplus R_P a_1, \dots, \bar{p}_n \boxplus R_P a_n]) \end{aligned} \tag{23}$$

is a transitive right group action of $\mathrm{SLAM}_n(3)$ on $\mathcal{M}_n(3)$.

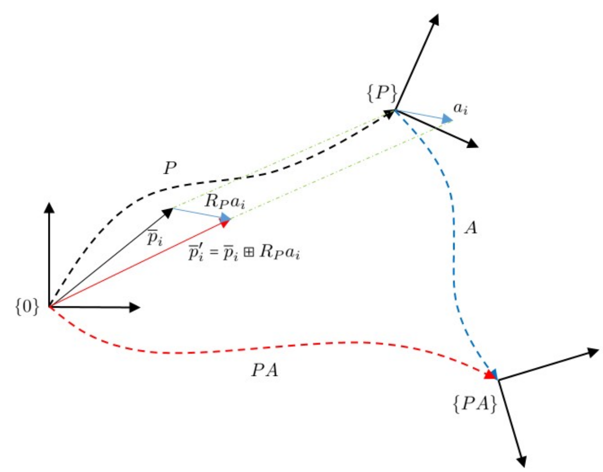

Fig. 2. Group action $\phi((A, a_1, \dots, a_n), [P, \bar{p}_1, \dots, \bar{p}_n])$. The pose $P \mapsto PA$, that is the tip point of the pose is updated by the correction $A \in \mathrm{SE}(3)$. The environment points $\bar{p}_i \mapsto \bar{p}_i \boxplus R_P a_i$, are updated by the corrections a_i rotated back into the inertial frame.

Associated with the group action ϕ there is a related group action $\Phi : \mathrm{SLAM}_n(3) \times \mathcal{T}_n(3) \to \mathcal{T}_n(3)$ given by

$$\begin{aligned} &\Phi((A, a_1, \dots, a_n), (P, \bar{p}_1, \dots, \bar{p}_n)) = \\ &\qquad (PA, \bar{p}_1 \boxplus R_P a_1, \dots, \bar{p}_n \boxplus R_P a_n) \end{aligned}$$

An analogous argument to that for proving Lemma 4.2 yields

$$\phi_X(\pi(\Xi)) = \pi(\Phi_X(\Xi)) \tag{24}$$

where Ξ is any element of the total space $\mathcal{T}_n(3)$. If we now consider a trajectory $\Xi(t)$ and take the time differential of (24) one has

$$\mathrm{d}\phi_X \circ \mathrm{d}\pi(\dot{\Xi}) = \mathrm{d}\pi \circ \mathrm{d}\Phi_X(\dot{\Xi})$$

Using the above and recalling (18) one has

$$\mathrm{d}\phi_X(\lfloor \dot{\Xi} \rfloor_{\Xi}) = \mathrm{d}\phi_X \circ \mathrm{d}\pi(\dot{\Xi}) = \mathrm{d}\pi \circ \mathrm{d}\Phi_X(\dot{\Xi}) = \lfloor \mathrm{d}\Phi_X(\dot{\Xi}) \rfloor_{\Phi_X(\Xi)} \tag{25}$$

for any tangent vector $\dot{\Xi} \in T_{\Xi}\mathcal{T}_n(3)$. This will be important in the following subsection.

B. Equivariance of the SLAM kinematics

We consider the question of equivariance of the SLAM kinematics in the sense discussed in Mahony *et al.* [22]. In particular, we will require a group action on the input space $\mathbb{V}$ in order to model equivariance.

We will continue to use the notation introduced in (6), $A\mathring{\overline{v}} = \overset{\circ}{\overline{R_A v}}$, to denote coordinate change of a landmark velocity by a rotation R_A associated with an element $A \in \mathrm{SE}(3)$. In addition, we introduce a similar matrix multiplication notation

$$W\mathring{\overline{a}} = \overset{\circ}{\overline{\Omega_W a}} \in \mathring{\overline{\mathbb{R}}}^3, \qquad a \in \mathbf{R}^3 \text{ and } W \in \mathfrak{se}(3).$$

where Ω_W is the skew-symmetric upper-left 3×3 block of $W \in \mathfrak{se}(3)$ that encodes the rotational velocity of a rigid-body motion.

Lemma 4.3: The mapping $\psi : \mathrm{SLAM}_n(3) \times \mathbb{V} \to \mathbb{V}$, defined by

$$\begin{aligned}\psi((A, a_1, \ldots, a_n), (V, \mathring{\overline{v}}_1, \ldots, \mathring{\overline{v}}_n)) &:= \\ (\mathrm{Ad}_{A^{-1}} V, A^{-1}(\mathring{\overline{v}}_1 + V\mathring{\overline{a}}_1), &\ldots, A^{-1}(\mathring{\overline{v}}_n + V\mathring{\overline{a}}_n))\end{aligned} \tag{26}$$

is a right group action of $\mathrm{SLAM}_n(3)$ on the space of inputs $\mathbb{V}$.

The following Lemma shows that the SLAM kinematics are equivariant with respect to action by the SLAM group. This is the key symmetry result that will enable us to go on and apply the observer design methodology developed in prior work by the authors [22].

Lemma 4.4: The SLAM kinematics (20) are equivariant under the group actions (23) and (26) in the sense that

$$\mathrm{d}\phi_X f(\xi, v) = f(\phi_X(\xi), \psi_X(u)).$$

C. Compatibility of the output

A key property of the geometric SLAM formulation is that there is also a compatible group operation on the output y of the system.

Lemma 4.5: The action $\rho : \mathrm{SLAM}_n(3) \times \mathcal{N}_n(3) \to \mathcal{N}_n(3)$ defined by

$$\begin{aligned}\rho((A, a_1, \ldots, a_n), &(\overline{y}_1, \ldots, \overline{y}_n)) \\ &:= (A^{-1}(\overline{y}_1 \boxplus a_1), \ldots, A^{-1}(\overline{y}_n \boxplus a_n))\end{aligned}$$

is a transitive right group action on $\mathcal{N}$. Furthermore, one has

$$\begin{aligned}\rho((A, a_1, \ldots, a_n), &h([P, \overline{p}_1, \ldots, \overline{p}_n]) \\ &= h(\phi((A, a_1, \ldots, a_n), [P, \overline{p}_1, \ldots, \overline{p}_n]))\end{aligned}$$

V. Observer design

We will design an observer by lifting the system kinematics onto the symmetry group and designing the observer on the $\mathrm{SLAM}_n(3)$. Let $\xi(t) = [P(t), \overline{p}_1(t), \ldots, \overline{p}_n(t)]$ be the true configuration of the SLAM problem as a trajectory $\xi(t) \in \mathcal{M}_n(3)$. Let $X(t) = (A(t), a_1(t), \ldots, a_n(t)) \in \mathrm{SLAM}_n(3)$ and define the *lifted kinematics* [22]

$$\frac{\mathrm{d}}{\mathrm{d}t}(A(t), a_1(t), \ldots, a_n(t)) = (AV, 0, \ldots, 0), \tag{27}$$

on the SLAM group where V is the velocity of the robot. Choose an arbitrary reference configuration

$$\xi^\circ = [P^\circ, \overline{p}_1^\circ, \ldots, \overline{p}_n^\circ] \in \mathcal{M}_n(3).$$

In the ideal case, where $X(0) \in \mathrm{SLAM}_n(3)$ satisfies $\phi(X(0), \xi^\circ) = \xi(0)$, and there is no noise on the velocity $V \in \mathfrak{se}(3)$, then the lifted kinematics (27) induce a trajectory that satisfies [22]

$$\xi(t) = \phi(X(t), \xi^\circ) \in \mathcal{M}_n(3)$$

for all time. Of course, the velocity measurements will always be corrupted by noise, and since the true initial configuration $\xi(0) \in \mathcal{M}_n(3)$ is not known, it is impossible to choose $X(0)$ *a-priori* to satisfy the initial condition.

We choose the state space of the observer to lie on the SLAM group

$$\hat{X} = (\hat{A}, \hat{a}_1, \ldots, \hat{a}_n) \in \mathrm{SLAM}_n(3).$$

The configuration estimate is then given by

$$\hat{\xi} = [\hat{P}, \hat{\overline{p}}_1, \ldots, \hat{\overline{p}}_n] = \phi(\hat{X}, \xi^\circ) \in \mathcal{M}_n(3).$$

The goal of the observer design is to estimate both the relative symmetry that takes $\xi^\circ = [P^\circ, \overline{p}_1^\circ, \ldots, \overline{p}_n^\circ]$ to $\xi(0) = [P(0), \overline{p}_1(0), \ldots, \overline{p}_n(0)]$ as well as encode the ongoing evolution of $X(t)$ given by (27) while correcting for any errors introduced by the noisy measurement of $V \in \mathfrak{se}(3)$.

The proposed observer kinematics are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{X} &= \frac{\mathrm{d}}{\mathrm{d}t}(\hat{A}, \hat{a}_1, \ldots, \hat{a}_n) \\ &= (\hat{A}V + \Delta\hat{A}, \delta_1, \ldots, \delta_n), \quad \hat{X}(0) = \mathrm{id},\end{aligned} \tag{28}$$

where $\Delta \in \mathfrak{se}(3)$ and $\delta_i \in \mathbb{R}^3$ are the innovation terms.

Theorem 5.1: Fix an arbitrary reference $\xi^\circ = [P^\circ, \overline{p}_1^\circ, \ldots, \overline{p}_n^\circ] \in \mathcal{M}_n(3)$. Consider the SLAM kinematics (21) with output $y = h(\xi(t)) \in \mathcal{N}_n(3)$ (15). For $\hat{X} \in \mathrm{SLAM}_n(3)$ define an error $\overline{e} = (\overline{e}_1, \ldots, \overline{e}_n)$ by

$$\overline{e} = \rho(\hat{X}^{-1}, (\overline{y}_1, \ldots, \overline{y}_n)) \in \mathcal{N}_n(3). \tag{29}$$

Consider the observer kinematics (28) with innovation terms

$$\Delta := \shortmid \left(\sum_{i=1}^{n} k_i (\overline{e}_i - \overline{p}_i^\circ) \overline{e}_i^\top \right) \in \mathfrak{se}(3) \tag{30}$$

$$\mathring{\overline{\delta}}_i := -\frac{l_i}{k_i}(\overline{e}_i - \overline{p}_i^\circ) + \Delta \mathring{\overline{\hat{a}}}_i \in \mathring{\overline{\mathbb{R}}}^3. \tag{31}$$

where k_i and $l_i \in \mathbb{R}$ are positive gains. Then the configuration estimate $\hat{\xi}(t) = \phi(\hat{X}(t), \xi^\circ) \in \mathcal{M}_n(3)$ converges exponentially to the true state $\xi(t) \in \mathcal{M}_n(3)$.

Remark 5.2: Note that $\Delta \in \mathfrak{se}(3)$ and hence $\Delta\bar{\hat{a}}_i$ applies the skew-symmetric upper-left 3×3 block of Δ to $\hat{a}_i$ and expresses the resulting vector in $\bar{\mathbb{R}}^3$ with a zero in the 4th element. Similarly the error $(\bar{e}_i - \bar{p}_i^\circ) \in \bar{\mathbb{R}}^3$.

Proof: Let $X(t) \in \mathrm{SLAM}_n(3)$ satisfy the lifted kinematics (27) such that $\phi(X(t), \xi^\circ) = \xi(t)$, noting that $X(t)$ is not available to the observer. Define $E = \hat{X}X^{-1}$ and note that

$$\bar{e} = \rho(\hat{X}^{-1}, (\bar{y}_1, \dots, \bar{y}_n)) = \rho(E^{-1}, (\bar{p}_1^\circ, \dots, \bar{p}_n^\circ))$$

From (22), one can verify that:

$$E = (\tilde{A}, \tilde{a}_1, \dots, \tilde{a}_n), \text{ with } \tilde{A} := \hat{A}A^{-1}, \text{ and } \tilde{a}_i := \hat{a}_i - R_{\tilde{A}} a_i$$

It is straightforward to verify that

$$\dot{E} = (\Delta\tilde{A}, \bar{\delta}_1 + \Delta(\bar{\hat{a}}_1 - \bar{\tilde{a}}_1), \dots, \bar{\delta}_n + \Delta(\bar{\hat{a}}_n - \bar{\tilde{a}}_n)). \quad (32)$$

Using the fact that $E^{-1} = (\tilde{A}^{-1}, -R_{\tilde{A}^{-1}}\tilde{a}_1, \dots, -R_{\tilde{A}^{-1}}\tilde{a}_n)$, it follows that

$$\begin{aligned} \bar{e} &= (\tilde{A}(\bar{p}_1^\circ \boxplus (-R_{\tilde{A}^{-1}}\tilde{a}_1)), \dots, \tilde{A}(\bar{p}_n^\circ \boxplus (-R_{\tilde{A}^{-1}}\tilde{a}_n)) \\ &= (\tilde{A}(\bar{p}_1^\circ \boxminus R_{\tilde{A}^{-1}}\tilde{a}_1), \dots, \tilde{A}(\bar{p}_n^\circ \boxminus R_{\tilde{A}^{-1}}\tilde{a}_n)) \\ &= (\tilde{A}\bar{p}_1^\circ \boxminus \tilde{a}_1, \dots, \tilde{A}\bar{p}_n^\circ \boxminus \tilde{a}_n) \end{aligned} \quad (33)$$

where $\boxminus$ has the obvious meaning.

Note that the set of output errors $\{\bar{e}_i(t)\}$ are estimates of the set of reference points $\{\bar{p}_i^\circ\}$. Formally, if $E = \mathrm{id}$ (and hence $\tilde{A} = I_4$ and $\hat{a}_i = a_i$), one can easily verify that $(\bar{e}_1, \dots, \bar{e}_n) \equiv (\bar{p}_1^\circ, \dots, \bar{p}_n^\circ)$.

Based on (32), it is straightforward to show that the derivative of each element of (33) fulfills

$$\dot{\bar{e}}_i = \Delta\bar{e}_i - \Delta\bar{\hat{a}} + \bar{\delta}_i.$$

Define the following candidate (positive definite) Lyapunov function

$$\mathcal{L} = \sum_{i=1}^n \frac{k_i}{2} |\bar{e}_i - \bar{p}_i^\circ|^2 \quad (34)$$

Differentiating $\mathcal{L}$, it yields:

$$\begin{aligned} \dot{\mathcal{L}} &= \sum_{i=1}^n k_i (\bar{e}_i - \bar{p}_i^\circ)^\top \dot{\bar{e}}_i \\ &= \sum_{i=1}^n k_i (\bar{e}_i - \bar{p}_i^\circ)^\top \left(\Delta\bar{e}_i - \Delta\bar{\hat{a}} + \bar{\delta}_i\right) \\ &= \mathrm{tr}\left(\Delta \sum_{i=1}^n k_i \bar{e}_i (\bar{e}_i - \bar{p}_i^\circ)^\top\right) + \sum_{i=1}^n k_i (\bar{e}_i - \bar{p}_i^\circ)^\top \left(\bar{\delta}_i - \Delta\bar{\hat{a}}\right). \end{aligned}$$

Introducing the expression of the innovation terms (30) and (31) for Δ and δ_i respectively, one obtains

$$\dot{\mathcal{L}} = -\left|\mathbb{P}\left(\sum_{i=1}^n k_i(\bar{e}_i - \bar{p}_i^\circ)\bar{e}_i^\top\right)\right|^2 - \sum_{i=1}^n l_i |\bar{e}_i - \bar{p}_i^\circ|^2. \quad (35)$$

The derivative of the Lyapunov function is negative definite and equal to zero when $\bar{e}_i = \bar{p}_i^\circ$, and therefore one can ensure that $E = (\tilde{A}, -\tilde{a}_1, \dots, -\tilde{a}_n)$ converges exponentially to a constant and $\bar{e}_i - \bar{p}_i^\circ$ converges exponentially to zero. Using (33), one has at the limit:

$$\tilde{A}\bar{p}_i^\circ \boxminus \tilde{a} = \bar{p}_i^\circ, \quad \forall i = 1, \dots, n$$

By exploiting the expression of $\tilde{a}_i := -R_{\tilde{A}} a_i + \hat{a}_i$, one gets:

$$\tilde{A}(\bar{p}_i^\circ \boxplus a_i) = \bar{p}_i^\circ \boxplus \hat{a}_i$$

This in turn implies that:

$$A^{-1}(\bar{p}_i^\circ \boxplus a_i) = \hat{A}^{-1}(\bar{p}_i^\circ \boxplus \hat{a}_i)$$

and this concludes the proof of the theorem. ■

VI. Simulation results

We consider the case of a vehicle equipped with a 3D-sensor, such as a stereo camera, observing four landmarks in the environment with unknown locations. The vehicle moves along a circular trajectory at a fixed altitude ($z = 2m$) above the ground. The vehicle frame is chosen to coincide with the sensor system frame of reference to keep the formulation simple. We choose a reference configuration $\xi^\circ = [I, \bar{y}_1(0), \dots, \bar{y}_4(0)]$ that corresponds to fixing $A(0) = I_4$ and $\bar{y}_i(0) = \bar{p}_i^\circ \boxplus a_i$, for $i = 1 \dots 4$ where a_i is a constant bias that corrupts the initial measurements. The measurement of the velocity $V = (\Omega, v)_\wedge$ is such that $v = (1,\ 0,\ 0)^\top$ m/s and $\Omega = (0,\ 0,\ 0.2)^\top$ rad/s, both measured in the body-fixed-frame and corresponding to the circular trajectory discussed above. The observer estimate is $\hat{\xi}(t) = \phi(\hat{X}, \xi^\circ)$ where $\hat{X}$ is computed according to to (28) along with (30) and (31). The observer gains $k_i = 4$ and $l_i = 1$ are chosen for $i = 1 \dots 4$. The coordinates of the landmark points are $\mathrm{p}_1 = (4,\ 0,\ 0)^\top$, $\mathrm{p}_2 = (-4,\ 0,\ 0)^\top$, $\mathrm{p}_3 = (0,\ 4,\ 0)^\top$, $\mathrm{p}_4 = (0,\ -4,\ 0)^\top$.

Figure 3 shows the evolution of the $\log_{10}(\mathcal{L})$ of the Lyapunov function. Note the initial steep descent during transient followed by local exponential convergence. The evolution of the system eventually becomes corrupted by numerical error as the error approaches 10^{-12}.

It is not expected that $\tilde{A} = \hat{A}A^{-1}$ will converge to the identity as there will always be a residual rigid-body transformation associated with the invariance inherent in the SLAM problem. However, this residual error itself provides the required invariance transformation that can be used to rewrite the observer estimate in the reference frame associated with the total coordinates. Applying this correction transformation we can visualize the output of the observer as shown in Figure 4.

In Figure 4 the initial transient of the observer is clearly shown converging asymptotically to the true trajectory while the landmark estimates converge to the true landmark points. The exponential convergence is not visible in the scale of Figure 4 and it appears simply that the estimate tracks the true robot position exactly and the landmarks are correctly identified.

VII. Conclusions

To the authors' knowledge, this paper presents the first full symmetry structure for a classical landmark point SLAM problem. The underlying Lie-group appears to be novel and has not been studied before in the literature. The development presented also accounts for the known invariance of the SLAM problem to change of reference frame by introducing a quotient structure in the state representation of the SLAM

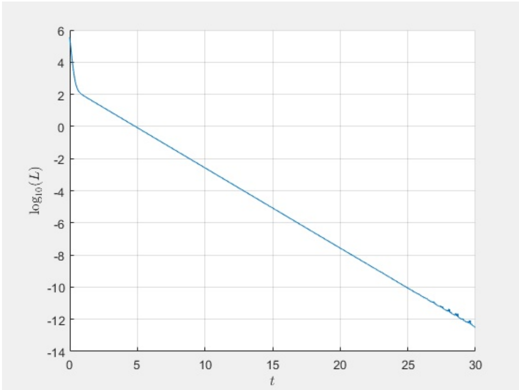

Fig. 3. Evolution of $\log_{10}$ of the Lyapunov function (34) with respect to time

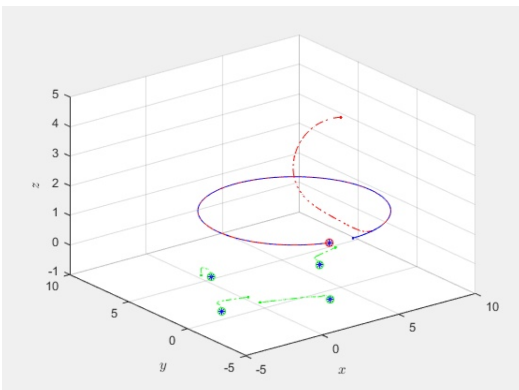

Fig. 4. Evolution of trajectories of the observer and true system for the example considered. The true robot trajectory (origin of the body-fixed-frame) is shown in blue (solid-line) with its final point shown as a $\star$. The trajectory of the estimation of the robot frame origin is shown in red (dash-dot) with its final condition shown as a circle. The true landmark positions are blue $\star$ symbols. The landmark estimates estimated positions are four trajectories shown in green (dash-dot) finishing in a circle.

problem. The structure introduced allows direct application of the authors previous work in development of non-linear observers to yield a novel new algorithm for continuous-time SLAM. Although this work is still in its infancy, the authors believe that the approach offers significant advantages in global robustness as compared to current state-of-the-art formulations of the SLAM problem.

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