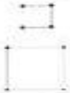
Geometrical Transformation

Alex Lin

Safety Sensing & Control Department
Intelligent Mobility Technology Division
Mechanical and Systems Research Laboratories
Industrial Technology Research Institute

Homography & other Geometric Transformations of 2D

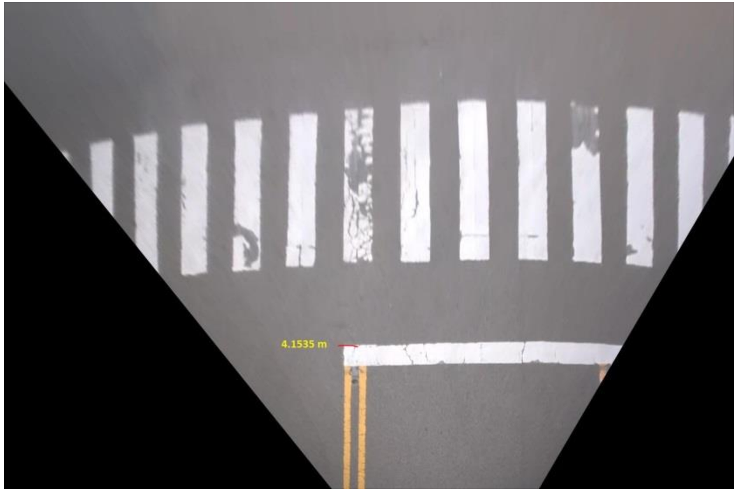
Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Goal

- To get yourself be familiar with
  - (1) Similarity Transformation
  - (2) Homography Transformation
- Application of Homography:
  - (1) Bird-view image transformation
  - (2) Panoramic image stitching
  - (3) Augmented Reality

Front-view to Bird-view

Homography



Panoramic image stitching

Augmented Reality

To be (self-contained) or not to be (self-contained),
that is the question.

Solving similarity transformation

$$\begin{bmatrix} wx'_1 \\ wy'_1 \\ w \end{bmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

$$\frac{wx'_1}{w} = \frac{ax_1 + by_1 + c}{1}, \quad \frac{wy'_1}{w} = \frac{dx_1 + ey_1 + f}{1}$$

If k correspondences are detected, the equation is

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_k & y_k & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_k & y_k & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} x'_1 \\ y'_1 \\ \vdots \\ x'_k \\ y'_k \end{bmatrix} => A_{2k\times 6} X_{6\times 1} = B_{2k\times 1}$$

If k=3,

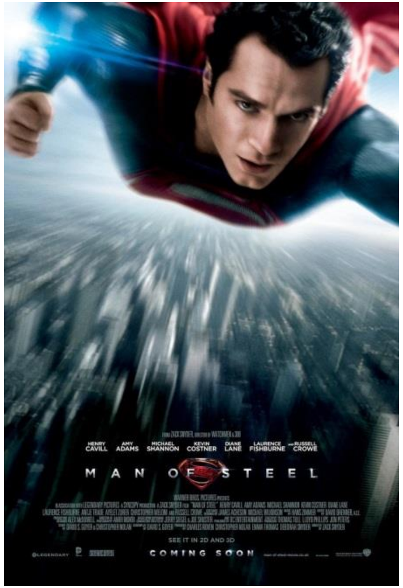
$$X = A^{-1}B$$

If k>>3,

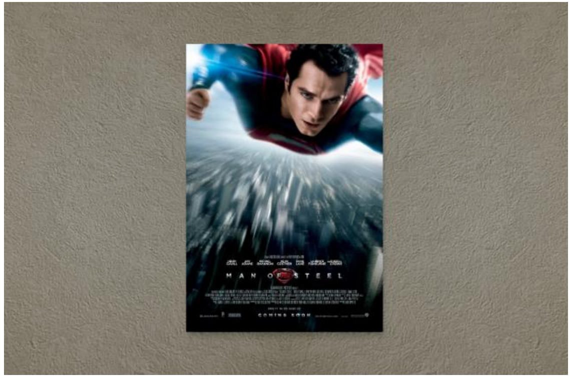
$$X = (A^t A)^{-1} A^t B$$

Example: similarity transform

- Q1: Can you revise the example to use only 3 sets of point pairs?
- Q2: Can we give “arbitrary” points to perform similarity transform?



+



Example: similarity transform

- Q3: If the input image is smaller, the result won’t meet our expectation! Do you know how to fix it?

+

Homography (1)

A popular algorithm, DLT (Direct Linear Transformation), to determine H is as follows.

$$X_2 = HX_1 \quad \Longleftrightarrow \quad \begin{bmatrix} wx'_1 \\ wy'_1 \\ w \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

In inhomogenous coordinates (x_2=wx_2/w and y_2=wy_2/w)

$$x'_1 = \frac{H_{11}x_1 + H_{12}y_1 + H_{13}z_1}{H_{31}x_1 + H_{32}y_1 + H_{33}z_1} \qquad x'_1 = \frac{H_{11}x_1 + H_{12}y_1 + H_{13}}{H_{31}x_1 + H_{32}y_1 + H_{33}}$$

Set z_1=1 $\Longrightarrow$

$$y'_1 = \frac{H_{21}x_1 + H_{22}y_1 + H_{23}z_1}{H_{31}x_1 + H_{32}y_1 + H_{33}z_1} \qquad y'_1 = \frac{H_{21}x_1 + H_{22}y_1 + H_{23}}{H_{31}x_1 + H_{32}y_1 + H_{33}}$$

After rearranging, we get

$$a_x^T h = 0 \qquad h = (H_{11}, H_{12}, H_{13}, H_{21}, H_{22}, H_{23}, H_{31}, H_{32}, H_{33})^T$$

$$a_x = (-x_1, -y_1, -1, 0, 0, 0, x'_1 x_1, x'_1 y_1, x'_1)^T$$

$$a_y^T h = 0 \qquad a_y = (0, 0, 0, -x_1, -y_1, -1, y'_1 x_1, y'_1 y_1, y'_1)^T$$

Homography (2)

By substituting 4 sets of correspondence,

$$A_{8*9}h_{9*1}=0$$

If I only know how to solve AX=B, how could we apply it to AX=0?

Ans: Substitute 5 sets of correspondence points and assume a very small **B (residual)**.

$$A_{10*9}h_{9*1}=B$$

$$AX=B, \overline{X}=(A^{t}A)^{-1}A^{t}B$$

However, correspondence points might be contaminated by noise and a assumed B will result in non-exact solution.

The best way is not to assume B but solve h with minimum sets of points while some “good” behavior of h could be preserved.

Homography (3):

Least Squares method on Homogeneous equations

The sum squared error can be written as,

$$f(h) = \frac{1}{2}(Ah - 0)^T (Ah - 0)$$

$$f(h) = \frac{1}{2}(Ah)^T (Ah)$$

$$f(h) = \frac{1}{2} h^T A^T Ah$$

Taking the derivative of f with respect to h and setting the result to zero, we get

$$\frac{d}{dh} f(h) = 0 = \frac{1}{2}(A^T A + (A^T A)^T)h$$

$$0 = A^T Ah$$

This is almost equivalent to the typical eigen vector problem of BX=λX with a constraint ||h||=1, while B is now A^TA

$$A^T Ah = 0h$$

Therefore, the solution of “h” corresponding to λ=0 of A^TA is the (rightmost) column vector corresponding to the smallest singular value so now that’s recall Linear Algebra .

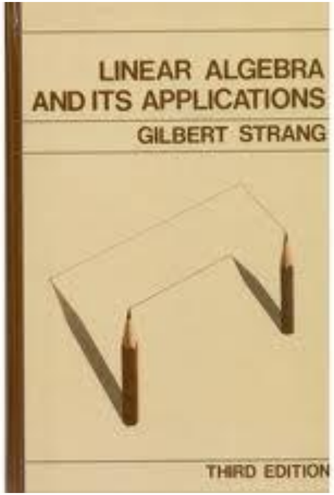
Example:Homography

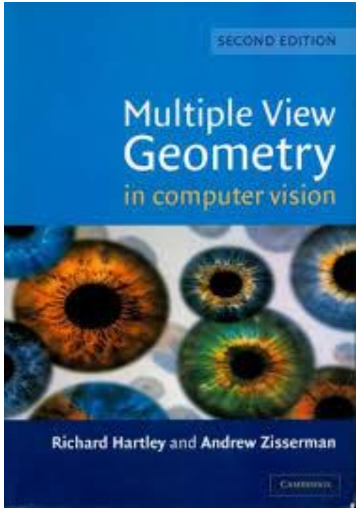
- Q1: The inverse transformation in our implementation would fail! Do you know why?

+

⇨

Reference


LINEAR ALGEBRA
AND ITS APPLICATIONS
GILBERT STRANG
THIRD EDITION


SECOND EDITION
Multiple View
Geometry
in computer vision
Richard Hartley and Andrew Zisserman

Supplementary Material 1:
From nullspace, Diagonalization to SVD

How to efficiently know if a set of equation in the form of AX=B has solution?

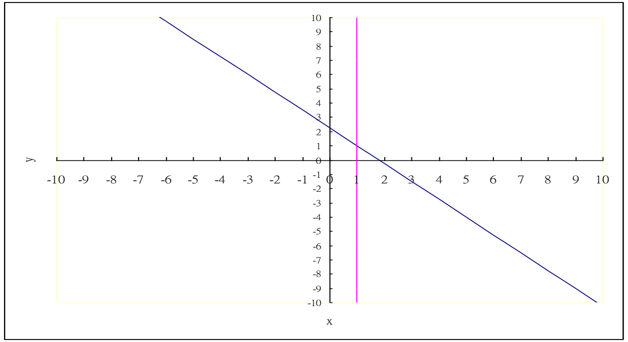
From traditional point of view: the intersection of two equation is the solution

Example 1:

$$x + 0y = 1$$

$$5x + 4y = 9$$

The solution is (1,1)



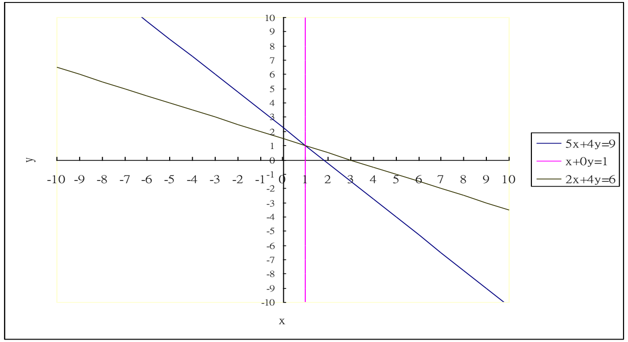
Example 2:

$$x + 0y = 1$$

$$5x + 4y = 9$$

$$2x + 4y = 6$$

The solution is also (1,1)



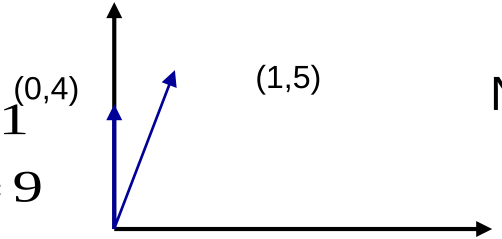
How to efficiently know if a set of equation in the form of AX=B has solution?

From vector point of view: Whether A has a solution depends on If B is the linear combination of the columns of A.

Example 1:

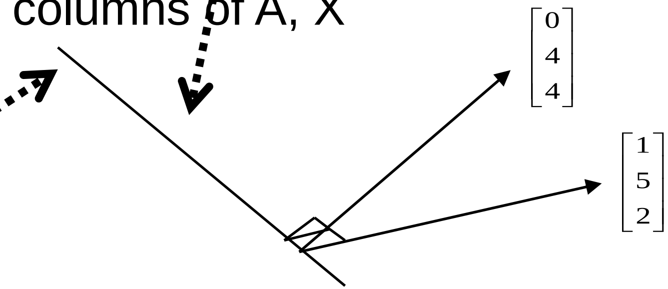
$$x + 0y = 1 \qquad 1\times1 + 0\times1 = 1$$

$$5x + 4y = 9 \qquad 5\times1 + 4\times1 = 9$$



Null space of $\begin{bmatrix}1 & 5 & 2\\ 0 & 4 & 4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

Yes, because B is the linear combination of the columns of A, X possesses a solution and the solution is (1,1)



Example 2:

$$\begin{aligned} x+0y&=1 \\ 5x+4y&=9 \\ 2x+4y&=6 \end{aligned} \Rightarrow \begin{aligned} 1\times1+0\times1&=1 \\ 5\times1+4\times1&=9 \\ 2\times1+4\times1&=6 \end{aligned}$$

If B is the linear combination of the columns of A, X possesses a solution and the solution is (1,1)

If B lies in a vector, such as c([0 4 4]^{t}x[1 5 2]^t) with arbitrary c, there is NO SOLUTION because B can't be the linear combination of the column vectors of A.

Does AX=0 posses solution?

- A is non-square matrix
- If dim(x)=Rank(A) The only solution is trivial solution and the null space contains only the zero vector
- If dim(x)≠Rank(A), there are infinite solutions lied in null space (since max rank is dim(x), dim(x) > Rank(A) is the possibility)

$$u\begin{bmatrix}1\\5\\2\end{bmatrix}+v\begin{bmatrix}0\\4\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

X is (0,0)

$$u\begin{bmatrix}1\\5\\2\end{bmatrix}+v\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

X is (0,c) and is a line

- A is square
  - (1) if determinant of A is not 0, there is only trivial solution.
  - (2) if determinant of A is 0, there are infinite solutions lied in null space.

$$u\begin{bmatrix}1\\5\\2\end{bmatrix}+v\begin{bmatrix}0\\4\\4\end{bmatrix}+w\begin{bmatrix}1\\9\\6\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Since $[1\ 9\ 6]^t$ is the linear combination of $[1\ 5\ 2]^t$ & $[0\ 4\ 4]^t$, det(A)=2 &
X is (c,c,-c) and is a line

Least Square method

- If AX=B ,rank(A)=dim(x) and B is the linear combination of the column vectors of A, there is an exact solution

$$\begin{aligned} x+0y&=1 \\ 5x+4y&=9 \end{aligned} \Rightarrow X=(1,1)$$

- If AX=B ,rank(A)=dim(x) and B is “NOT” the linear combination of the column vector of A, there is no exact solution (no X can fulfill AX=B) but a “APPROXIMATE” solution with minimum error is possible

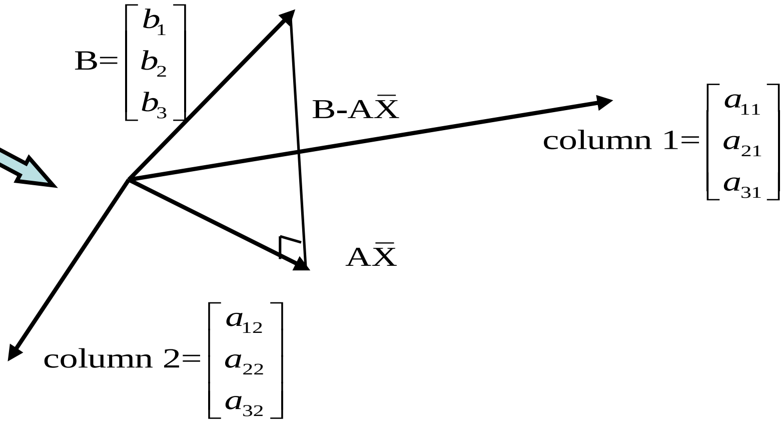
1. From traditional point of view

$E^2=(A\overline{X}-B)^t(A\overline{X}-B)=0$

By taking partial derivative, $2A^tA\overline{X}-2A^tB=0$

$\overline{X}=(A^tA)^{-1}A^tB$

$A_{3\times2}X_{2\times1}=B_{3\times1}$



2. From vector point of view

$A^t(B-A\overline{X})=0 \Rightarrow A^tA\overline{X}=A^tB$

$\overline{X}=(A^tA)^{-1}A^tB$

This is not scalar but vector

Least Square method

- If AX=B ,m>n, rank(A)<dim(x), for example A is 3x2, x is 2x1, rank(A)=1, then there are infinite solutions

$$u\begin{bmatrix}1\\2\\3\end{bmatrix}+v\begin{bmatrix}2\\4\\6\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$A^TAX=A^TB$$

$$\begin{bmatrix}14 & 28\\28 & 56\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}6\\12\end{bmatrix}$$

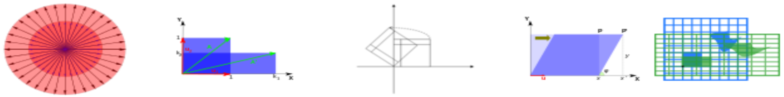
$$(u,v)=\left(\frac{6-28c}{14},c\right)$$

Eigenvalue and Eigenvector (1)

- An eigenvector of a **square matrix** A is a non-zero **vector** that, when the matrix is **multiplied** by *v*, yields a constant multiple of *v*, the multiplier being commonly denoted by λ . That is:

$$Av=\lambda v$$

- A could be any line ... rotation, horizontal shear &



	scaling	unequal scaling	rotation	horizontal shear	hyperbolic rotation
illustration					
matrix	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$	$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ $c = \cos\theta$ $s = \sin\theta$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} c & s \\ s & c \end{bmatrix}$ $c = \cosh\varphi$ $s = \sinh\varphi$
characteristic polynomial	$(\lambda - k)^2$	$(\lambda - k_1)(\lambda - k_2)$	$\lambda^2 - 2c\lambda + 1$	$(\lambda - 1)^2$	$\lambda^2 - 2c\lambda + 1$
eigenvalues λ_i	$\lambda_1 = \lambda_2 = k$	$\lambda_1 = k_1$ $\lambda_2 = k_2$	$\lambda_1 = e^{\mathbf{i}\theta} = c + s\mathbf{i}$ $\lambda_2 = e^{-\mathbf{i}\theta} = c - s\mathbf{i}$	$\lambda_1 = \lambda_2 = 1$	$\lambda_1 = e^{\varphi}$ $\lambda_2 = e^{-\varphi}$,
algebraic multipl. $\mu_i = \mu(\lambda_i)$	$\mu_1 = 2$	$\mu_1 = 1$ $\mu_2 = 1$	$\mu_1 = 1$ $\mu_2 = 1$	$\mu_1 = 2$	$\mu_1 = 1$ $\mu_2 = 1$
geometric multipl. $\gamma_i = \gamma(\lambda_i)$	$\gamma_1 = 2$	$\gamma_1 = 1$ $\gamma_2 = 1$	$\gamma_1 = 1$ $\gamma_2 = 1$	$\gamma_1 = 1$	$\gamma_1 = 1$ $\gamma_2 = 1$
eigenvectors	All non-zero vectors	$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$u_1 = \begin{bmatrix} 1 \\ -\mathbf{i} \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ +\mathbf{i} \end{bmatrix}$	$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

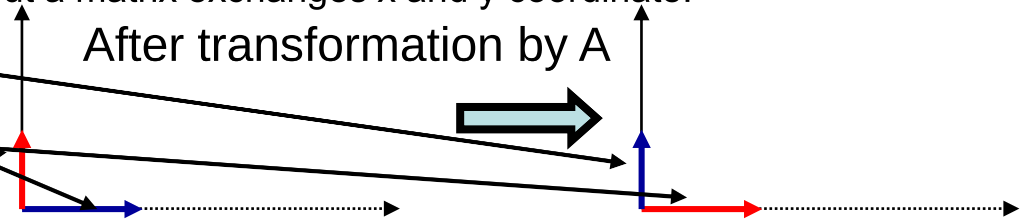
Eigenvalue and Eigenvector (2)

What is the eigenvalue & eigenvector of $A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

- What kind of linear transformation is A?
- Ans: it's not rotation matrix but a matrix exchanges x and y coordinate!

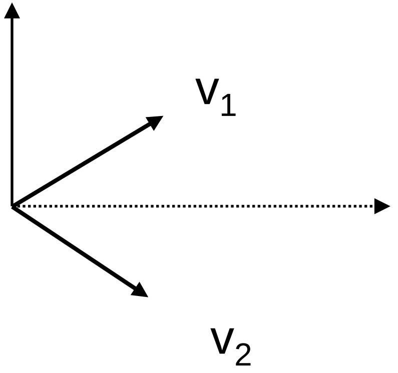
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$\lambda_1=1 \quad v_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2=-1 \quad v_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



two vectors, $v_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $v_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$,
only change a scalar multiplication
by $\lambda_1=1$ & $\lambda_2=-1$

Diagonalization

$P^{-1}AP=D$, where $P=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, P^{-1}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-1}{2}\end{bmatrix}$

The linear transformation effect expressed by two eigenvectors is D

The same linear transformation applied on an vector expressed in terms of P is simply a multiplication by a diagonal matrix D

Example:

What is the linear transformation result of $X=\begin{bmatrix}2 & 1\end{bmatrix}^t$ on the basis of P?

$$X|_P = P^{-1}X = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-1}{2}\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix} = \begin{bmatrix}1.5\\ 0.5\end{bmatrix}$$

$$X|_P' = P^{-1}AP\,X|_P = D\,X|_P = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1.5\\ 0.5\end{bmatrix} = \begin{bmatrix}1.5\\ -0.5\end{bmatrix}\Bigg|_P$$

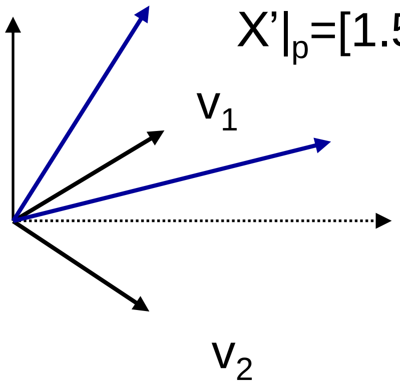
$$X' = P\,X'|_P = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1.5\\ -0.5\end{bmatrix} = \begin{bmatrix}1\\ 2\end{bmatrix}$$

$$X' = AX = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 2\end{bmatrix}$$

Matched!!

After transformation

$X'|_p=[1.5\ -0.5]^t$, $X'=[1\ 2]^t$



Before transformation

$X=[2\ 1]^t$, $X|_p=[1.5\ 0.5]^t$

Singular Value Decomposition

- Any mxn matrix A can be written as the product of three matrices: $A=UDV^T$
- The column vectors of U are mutually orthogonal and composed of the normalized eigenvectors of AA^T
- The column vectors of V are mutually orthogonal and composed of the normalized eigenvectors of A^TA
- The columns of the mxm matrix U are mutually orthogonal unit vectors, as are the columns of the nxn matrix V. The mxn matrix D is diagonal; its diagonal elements, σ_i, called singular values (the positive square root of the eigenvalues of U), are such that , $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n \geq 0$
- Three important application of SVD:
  - Solving system of nonhomogeneous linear equations;
  - Solving rank-deficient systems of homogeneous linear equations;
  - Guaranteeing that the entries of a matrix estimated numerically satisfy some given constraints (orthogonality)

The detail of implementation

- 1. What is the meaning of Diagonalization ($S^{-1}AS=D$) ?
  - To express a linear transformation matrix by the eigenvectors
- 2. What is the meaning of SVD $A=UDV^t$?
  - To express a linear transformation matrix by orthonormal vectors obtained by AA^t and A^tA.
- 3. What is the difference between Diagonalization and SVD?
  - The eigenvector matrix in diagonalization are linearly independent vectors but orthonormal ones
  - The U and V are “orthogonal” matrices whose column vectors are “orthonormal” ones
- 4. How do we correctly implement SVD?
  - The U in $A=UDV^t$ is composed of normalized eigenvectors obtained by $A*A^t$ and V the normalized ones from A^tA
  - Eigenvectors are not unique but mostly one-dimension null vector which means there are infinite solution.

Two examples to implement SVD

Example 1 (Not good)

$$A=\begin{bmatrix}5 & 5\\ -1 & 7\end{bmatrix},\ A^tA=\begin{bmatrix}26 & 18\\ 18 & 74\end{bmatrix},\ AA^t=\begin{bmatrix}50 & 30\\ 30 & 50\end{bmatrix}$$

$$\det(A^tA-\lambda I)=\det(\begin{bmatrix}26-\lambda & 18\\ 18 & 74-\lambda\end{bmatrix})=\lambda^2-100\lambda+1600=(\lambda-20)(\lambda-80)$$

$$\det(AA^t-\lambda I)=\det(\begin{bmatrix}50-\lambda & 30\\ 30 & 50-\lambda\end{bmatrix})=\lambda^2-100\lambda+1600=(\lambda-20)(\lambda-80)$$

$$A^tA-20I=\begin{bmatrix}6 & 18\\ 18 & 54\end{bmatrix},V_1=\begin{bmatrix}\frac{-3}{\sqrt{10}}\\ \frac{1}{\sqrt{10}}\end{bmatrix}\text{or}\begin{bmatrix}\frac{3}{\sqrt{10}}\\ \frac{-1}{\sqrt{10}}\end{bmatrix}$$

$$AA^t-20I=\begin{bmatrix}30 & 30\\ 30 & 30\end{bmatrix},\ U_1=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}\text{or}\begin{bmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$A^tA-80I=\begin{bmatrix}-54 & 18\\ 18 & -6\end{bmatrix},V_2=\begin{bmatrix}\frac{1}{\sqrt{10}}\\ \frac{3}{\sqrt{10}}\end{bmatrix}\text{or}\begin{bmatrix}\frac{-1}{\sqrt{10}}\\ \frac{-3}{\sqrt{10}}\end{bmatrix}$$

$$AA^t-80I=\begin{bmatrix}-30 & 30\\ 30 & -30\end{bmatrix},\ U_2=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\text{or}\begin{bmatrix}\frac{-1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}$$

$$V=\begin{bmatrix}\frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{bmatrix}\text{ or }\begin{bmatrix}\frac{-3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}}\end{bmatrix}\text{or }\begin{bmatrix}\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\ \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{bmatrix}\text{or }\begin{bmatrix}\frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\\ \frac{-1}{\sqrt{10}} & \frac{-3}{\sqrt{10}}\end{bmatrix}$$

$$U=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\text{ or }\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\text{or}\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\text{or}\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$$

There are 16 possible combination with only 4 solutions

If we randomly choose U and V

$$UDV^t=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}2\sqrt{5} & 0\\ 0 & 4\sqrt{5}\end{bmatrix}\begin{bmatrix}\frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{bmatrix}=\begin{bmatrix}-1 & 7\\ 5 & 5\end{bmatrix}$$

A is not reconstructed correctly!

But if we choose

$$UDV^t=\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}2\sqrt{5} & 0\\ 0 & 4\sqrt{5}\end{bmatrix}\begin{bmatrix}\frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{bmatrix}=\begin{bmatrix}5 & 5\\ -1 & 7\end{bmatrix}$$

A is successfully reconstructed!

Two examples to implement SVD

Example 2 (good and easy)

$$A=\begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix},\ A^tA=\begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix},\ AA^t=\begin{bmatrix} 50 & 30 \\ 30 & 50 \end{bmatrix}$$

randomly choose a V, for example $\begin{bmatrix} \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

We know $A=UDV^t$ but we can use A, V & D to make U unique

by $AVD^{-1}=U$ where D^{-1} could be obtained very easy

because $DD^{-1}=I$, $D=\begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{bmatrix}$,

$$D^{-1}=\begin{bmatrix} \frac{1}{2\sqrt{5}} & 0 \\ 0 & \frac{1}{4\sqrt{5}} \end{bmatrix}$$

$AVD^{-1}=\begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}\begin{bmatrix} \frac{1}{2\sqrt{5}} & 0 \\ 0 & \frac{1}{4\sqrt{5}} \end{bmatrix}=\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}=U$ which is absolutely correct!

However, the above case is just for square matrix.

For non-square matrix (as the following homography application), the method is different!

Will the non-uniqueness of singular vectors result in non-uniqueness solutions?

- Does the non-uniqueness singular vector interfere?
  - Ans:No, because the differentiation only relates to the minus sign

Since h_{9*1} is known,

it is actually the H matrix inside $X_2 = HX_1$

$$\text{which is } \begin{bmatrix} wx_2 \\ wy_2 \\ w \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}.$$

However, the transformed coordinates are

$$x_2 = \frac{H_{11}x_1 + H_{12}y_1 + H_{13}}{H_{31}x_1 + H_{32}y_1 + H_{33}}$$

$$y_2 = \frac{H_{21}x_1 + H_{22}y_1 + H_{23}}{H_{31}x_1 + H_{32}y_1 + H_{33}}.$$

In fact, either H or -H, that is $\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$ or $\begin{bmatrix} -H_{11} & -H_{12} & -H_{13} \\ -H_{21} & -H_{22} & -H_{23} \\ -H_{31} & -H_{32} & -H_{33} \end{bmatrix}$,is considered to be "correct" but the transformed coordinates

are the same because

$$x_2 = \frac{-H_{11}x_1 - H_{12}y_1 - H_{13}}{-H_{31}x_1 - H_{32}y_1 - H_{33}} = \frac{H_{11}x_1 + H_{12}y_1 + H_{13}}{H_{31}x_1 + H_{32}y_1 + H_{33}}$$

$$y_2 = \frac{-H_{21}x_1 - H_{22}y_1 - H_{23}}{-H_{31}x_1 - H_{32}y_1 - H_{33}} = \frac{H_{21}x_1 + H_{22}y_1 + H_{23}}{H_{31}x_1 + H_{32}y_1 + H_{33}}.$$

Supplementary Material 2: proof of SVD in finding minimum h with the constraint of $||h||=1$ in $AX=0$

The proof of SVD to find minimum h with the constraint of ||h||=1 (1)

Because Ah=0, our goal is to find $\min_{||h||=1} ||Ah||$

from the norm of Ah:

$||Ah||^2=(Ah)^t(Ah)=h^tA^tAh>=0$

we know that the range of Ah depends on A^tA which is a nxn semi-positive matrix

Its characteristic equation is

$A^tAv_i=\lambda_i v_i, i=1,....,n$ (1)

real symmetric matrix possesses non-negative eigenvalue, $\lambda_1>\lambda_2...>\lambda_n>0$ and

orthonormal eigenvectors, so $v_i^Tv_j=0, i\neq j, v_i^Tv_j=1, i=j$

the diagonal form of A^tA is

$A^tA=V\Lambda V^t$

where $\Lambda=diag(\lambda_1,....,\lambda_n), V=[v_1...v_n]$ is orthogonal matrix, $V^tV=I$

for simplification, let $z=V^th$ which means to express h in terms of V basis

$||Ah||^2=h^tA^tAh=h^tV\Lambda V^th=z^t\Lambda z$

and the norm with basis change doesn't change because

$||z||^2=z^tz=(V^th)^t(V^th)=h^tVV^th=h^th=1.$

so the original problem is equivalent to

$\min_{||z||=1} ||z^t\Lambda z||^{1/2}$ (2)

The proof of SVD to find minimum h with the constraint of ||h||=1 (2)

from the quadratic form of $z^t \Lambda z = \lambda_1 z_1^2 + .. + \lambda_n z_n^2 >= \lambda_n (z_1^2 + .. + z_n^2) = \lambda_n$

so

$$\min_{||h||=1} ||Ah|| = \min_{||z||=1} ||z^t \Lambda z||^{1/2} = \sqrt{\lambda_n}$$

Remind that what we want is a combination of the elements of z whose norm is 1 and

such element combination could result in minimum $||z^t \Lambda z||$.

If $z=(0,0,...,1)^t$,then $\min_{||h||=1} ||Ah|| = \sqrt{\lambda_n}$

Since $z=V^t h$, $h=Vz=V[0,0,...,1]^t$, $z=v_n$ (the singular vector corresponding to the minimum singular value)

another way to prove is to substitute h by v_i and see what happens

because $||Av_i||^2 = v_i^T (A^T A v_i) = v_i^T (\lambda_i v_i) = \lambda_i v_i^T v_i = \lambda_i$ (because λ_i is scalar,it is commutative)

we know that if $h=v_i$,its 2nd norm is λ_i

when λ_i is the minimum λ_n,the corrsponding h is v_i (orthonormal,$||v_i||=1$)

If A_{mxn} is not a square matrix and m<n, with r nonzero eigenvalue (rank=r),

$\lambda_1 \geq ,....,\lambda_r \geq 0, \lambda_{r+1} = ... = \lambda_n = 0$,

when i=1,...,r, $||Av_i|| = \sqrt{\lambda_i}$

when i>r, $||Av_i|| = 0$

The proof of SVD to find minimum h with the constraint of ||h||=1 (3)

What we know for now is that we need at least 4 points to find a solution for

$A_{8*9}h_{9*1}=0$

What if more correspondence points are fed?

1st we have to know the "rank"

if 5 ccorespondence points are given rank(A) is 8 for $A_{10*9}h_{9*1}=0$, the solution is the same as before.

if rank(A) is 9, there is no solution because the column vectors of A are all linearly independent.

Because Ah=0 could only be fulfilled when h is a null vector.

However, we could find an "approximate solution" which minimize ||Ah-0|| which is the error with the constraint ||h||=1

Fortunately, the proof is equivalent to the one when A is 8x9 and the solution is also the column vector of V which corresponds to minimum singular value

Please remind that

if A is 8x9 with rank=8, the corresponding singular vector of the last column of V is "0" because $\text{rank}(A^tA)_{9x9}$ is 8

if A is 10x9 with rank=9, the corresponding singular vector of the last column of V is mostly not "0" because $\text{rank}\ (A^tA)_{9x9}$ is 9.