The acceleration of convergence of the KMF algorithm to solve the Cauchy problem

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February 18, 2020

Abstract

In this article we propose a new demonstration of the convergence of the relaxed iterative JN algorithm for solving the inverse Cauchy problem. We give in particular the convergence interval as well as the interval of the convergence acceleration of KMF algorithm.

1 Introduction

In this work, we are interested in solving an inverse Cauchy problem for Poisson equation, which consists in the reconstruction of the data on a non accessible part of the boundary based on the measurements available on an other part. There are many approaches to solve this ill-posed problem[1–5, 9, 10]. One of them is the KMF alternating algorithm proposed in [9]. This approache is based on reducing the ill-posed Cauchy problem to a sequence of mixed well-posed boundary value problems, with Dirichlet and/or Neumann conditions on the non-accessible part. The authors theoretically show that the algorithm produces a convergent sequence whose limit is the solution of the Cauchy problem, but they give no numerical application. The first application of this algorithm is made by Jourhman and Nachaoui in [6] where they have shown that the algorithm was found to produce an accurate and stable numerical solution for the Cauchy problem. However, they pointed out a major drawback of the KMF algorithm, namely its slowness (a large number of iterations is necessary to reach convergence) in particular if the initial guess is far from the exact solution. They proposed to remedy this drawback by a relaxation of the given Dirichlet data. In [7] Jourhmane-Nachaoui showed numerically that, their JN relaxed algorithm not only does it keep the same advantages as the classic KMF algorithm but in addition this procedure drastically reduced the number of iterations required to achieve convergence for the considered Chauchy problems for some relaxation parameters θ chosen in an interval $(1, \theta_1)$ with $\theta_1 < 2$. In 2002 Jourhmane and Nachaoui gave a theoretical result of the convergence of their relaxed algorithm based on Krasnolskii's fixed point theory [8]. They proved that there is a real number θ_2 such that the algorithm converges to the exact solution, provided that relaxation parameter θ belongs to $(0, \theta_2)$ with $\theta_2 \leq 2$. However, the numerical tests performed revealed that these algorithms are convergent for a larger interval $(0, \theta_2)$. The numerical tests performed showed that within the interval of convergence $(0, \theta_2)$ there are large variations in the rate of convergence of the algorithms. Since these works, several authors have applied or have developed relaxed algorithms for the resolution of various Cauchy problems, we cite for example the works of Ellabib and Nachaoui (2008) [5] and Marin and Johansson (2010)[10]. However until now no estimates are available for the limits of the convergence and the accelration of convergence intervals $(0, \theta_2)$ and $(1, \theta_1)$. Therefore it is the purpose of this paper to revisit the convergence results of the relaxed JN algorithm [8] in order to find expressions for these two parameters.

In what follows, we use the theory of separation of variables to develop a simple and understandable demonstration of convergence. We also prove theoretically a result of convergence acceleration which, until now, was just investigated numerically. We give in particular an estimate of the parameters θ_1 and θ_2 according to the data of the Cauchy problem considered.

2 Cauchy problem for Poisson's equation and JN relaxed algorithm

Let Ω be a bounded domain in R^d with the Lipschitz boundary Γ. Let Γ be divided into four disjoint parts Γ_0, Γ_1, Γ_2 and Γ_3. Denote by ν the outward unit normal to the boundary Γ and consider the following Cauchy problem for the Poisson's equation:

$$\Delta u = f \text{ in } \Omega, \;\; u|_{\Gamma_1} = f_1, \;\; \partial_\nu u|_{\Gamma_1} = g_1, \;\; u|_{\Gamma_2} = f_2, \;\; u|_{\Gamma_3} = f_3. \tag{1}$$

2.1 The JN relaxed algorithm

The JN relaxation algorithm of Jourhmane-Nachaoui [6], [7], that we consider to solve the Cauchy problem ()1) is described as follows:

Step 1. Choose an initial approximation $\phi^{(0)}$ on the boundary (Γ_0), and solve the well-posed mixed boundary value problem

$$\Delta u^{(0)} = f \text{ in } \Omega, \;\; \partial_\nu u|_{\Gamma_1}^{(0)} = g_1, \;\; u|_{\Gamma_2}^{(0)} = f_2, \;\; u|_{\Gamma_3}^{(0)} = f_3, \;\; u|_{\Gamma_0}^{(0)} = \phi^{(0)} \tag{2}$$

Step 2. Solve the well-posed mixed boundary value problem for $n = 1, 2, 3, ...$

$$\Delta u^{(2n+1)} = f \text{ in } \Omega, \;\; u|_{\Gamma_1}^{(2n+1)} = f_1, \;\; u|_{\Gamma_2}^{(2n+1)} = f_2, \;\; u|_{\Gamma_3}^{(2n+1)} = f_3, \;\; \partial_\nu u|_{\Gamma_0}^{(2n+1)} = \partial_\nu u|_{\Gamma_0}^{(2n)} \tag{3}$$

Step 3. Solve the well-posed mixed boundary value problem for $n = 1, 2, 3, ...$

$$\Delta u^{(2n)} = f \text{ in } \Omega, \;\; \partial_\nu u|_{\Gamma_1}^{(2n)} = g_1, \;\; u|_{\Gamma_2}^{(2n)} = f_2, \;\; u|_{\Gamma_3}^{(2n)} = f_3, \;\; u|_{\Gamma_0}^{(2n)} = \phi|_{\Gamma_0}^{(n)} \tag{4}$$

$$\phi^{(n)} = \theta u^{(2n-1)} + (1-\theta)\phi^{(n-1)}, \quad for \;\; n \geq 1 \;\; \theta \in (0, 2]. \tag{5}$$

Step 4. Repeat step 2 and step 3 consequently untill a commended stopping criterion is achieved.

Note that for $\theta = 1$, we get the standard algorithm of Kozlov-Mas'ya [9].

The main result of this work is summarized in the following theorem:

Theorem 2.1 *Considering the Poisson problem defined in (2)-(4). We have: For all* $\theta \in \left(0, \frac{2}{1-\left(\tanh(\frac{\pi}{a}b)\right)^2}\right)$ *the sequence* $\{u^{(n)}\}$ *converges to the solution of the Cauchy problem in (1) independently of the initial value* $\phi^{(0)}$.

To prove the convergence of this last algorithm we recall the following lemma in which Jourhmane-Nachaoui [8] give the idea of proving the convergence of the algorithm to the solution in the domain Ω by proving its convergence on the boundary Γ_0 (for the proof see [8]):

Lemma 2.2 *[8] If the sequence $u|_{\Gamma_0}^{(2n)}$ converges in $\mathcal{L}^2(\Gamma_0)$, then the sequence $u|_{\Gamma_0}^{(n)}$ converges in $\mathcal{H}^1(\Omega)$ to the solution of (1), where $\mathcal{H}^1(\Omega)$ is the Sobolev space and $\mathcal{L}^2(\Gamma)$ is the space of square integrable functions on Γ.*

3 Convergence analysis

Note that, using (5) we can rewrite the non-homogeneous condition of (4) in the following form:

$$\begin{aligned} u^{(2n)} &= \theta u^{(2n-1)} + (1-\theta)\phi^{(n-1)} \\ &= \theta u^{(2n-1)} + (1-\theta)u^{(2(n-1))} \end{aligned} \tag{6}$$

Let $w^{(n)} = u - u^{(n)}$, then from the JN relaxed algorithm, $w^{(n)}$ is given as follows:

For $n = 0, w^{(0)}$ is the solution of the boundary value problem:

$$\Delta w^{(0)} = 0 \text{ in } \Omega, \;\; \partial_\nu w^{(0)}|_{\Gamma_1} = 0 \;\; w^{(0)}|_{\Gamma_2} = 0 \;\; w^{(0)}|_{\Gamma_3} = 0 \;\; w^{(0)}|_{\Gamma_0} = u|_{\Gamma_0} - \phi^{(0)} \tag{7}$$

For $n = 0, 2, 3, ...\, w^{(2n+1)}$ is the solution of the (odd) problem:

$$\Delta w^{(2n+1)} = 0 \text{ in } \Omega,\; w^{(2n+1)}|_{\Gamma_1} = 0,\; w^{(2n+1)}|_{\Gamma_2} = 0,\; w^{(2n+1)}|_{\Gamma_3} = 0,\; \partial_\nu w^{(2n+1)}|_{\Gamma_0} = \partial_\nu w^{(2n)}|_{\Gamma_0} \tag{8}$$

For $n = 1, 2, 3, ...\, w^{(2n)}$ is the solution of the (even) problem:

$$\Delta w^{(2n)} = 0 \;\text{ in } \Omega,\;\; \partial_\nu w^{(2n)}|_{\Gamma_1} = 0,\;\; w^{(2n)}|_{\Gamma_2} = 0,\;\; w^{(2n)}|_{\Gamma_3} = 0,\;\; w^{(2n)}|_{\Gamma_0} = v^{(n)} \tag{9}$$

where $v^{(n)} = u|_{\Gamma_0} - \phi^{(n)}$, with $\phi^{(n)}$ defined in (5). Using (6) and the fact that $u^{(2n)}|_{\Gamma_0} = \phi^{(n)}|_{\Gamma_0}$, we obtain

$$v^{(n)} = \theta w^{(2n-1)} + (1-\theta)w^{2(n-1)}, \quad for \;\; n \geq 1 \tag{10}$$

Remark 3.1 *Note that the study of the convergence of $u^{(n)}$ to u, the solution of 1, is equivalent to show that $w^{(n)}$ converges to 0.*

Based on this remark, our study of the convergence will be on $w^{(n)}$. To illustrate our technique and without lose of generality let Ω be the rectangular domain defined by $\Omega = (0,a) \times (0,b)$ with a and b are non negatives reals constantes. Let $\Gamma_1 = (0,a) \times \{0\}$, $\Gamma_2 = \{a\} \times (0,b)$, $\Gamma_3 = \{0\} \times (0,b)$ and $\Gamma_0 = (0,a) \times \{b\}$. The method of resolution of problems (7)-(9) by separation of the variables consists in looking for solutions of the form (we separate the variables x and y):

$$w^{(n)} = X^{(n)}(x)Y^{(n)}(y)$$

By replacing this expression in the Laplace equation of (7)-(9), we have:

$$\frac{d^2X^{(n)}}{dx^2}Y^{(n)} + \frac{d^2Y^{(n)}}{dy^2}X^{(n)} = 0 \tag{11}$$

Now, if we divide this expression by the product $X^{(n)}Y^{(n)}$, we get:

$$\frac{1}{X^{(n)}}\frac{d^2X^{(n)}}{dx^2} = -\frac{1}{Y^{(n)}}\frac{d^2Y^{(n)}}{dy^2} \tag{12}$$

The equality requires that both ratios to be constant, it is the only possibility for functions of different variables. We then introduce the separation constant $\lambda = \alpha^2$ which must be defined. So we have

$$\frac{1}{X^{(n)}}\frac{d^2X^{(n)}}{dx^2} = -\alpha^2 \ \ and \ \ \frac{1}{Y^{(n)}}\frac{d^2Y^{(n)}}{dy^2} = \alpha^2 \tag{13}$$

We are then brought to solve the following two problems for $X^{(n)}$ and $Y^{(n)}$,:

$$\begin{cases} d_x^2X^{(2n)} + \alpha^2X^{(2n)} = 0 & X^{(2n)}(0) = X^{(2n)}(a) = 0 \\ d_y^2Y^{(2n)} - \alpha^2Y^{(2n)} = 0 & d_yY^{(2n)}(0) = 0 \end{cases} \tag{14}$$

and

$$\begin{cases} d_x^2X^{(2n+1)} + \alpha^2X^{(2n+1)} = 0 & X^{(2n+1)}(0) = X^{(2n+1)}(a) = 0 \\ d_y^2Y^{(2n+1)} - \alpha^2Y^{(2n+1)} = 0 & Y^{(2n+1)}(0) = 0 \end{cases} \tag{15}$$

From the X-problem in (14) we get

$$X^{(2n)} = \big(A\cos(\alpha x) + B\sin(\alpha x). \tag{16}$$

Condition $X^{(2n)}(0) = 0$ imply that $A = 0$, so we obtain

$$X^{(2n)} = B\sin(\alpha x). \tag{17}$$

Condition $X^{(2n)}(a) = 0$ gives $B\sin(\alpha a) = 0$ which impose that (apart from $B = 0$, which is a trivial solution): $\alpha = \frac{m\pi}{a}$, with $n = 1, 2, 3, \cdots$.

Consequently, $\lambda_n = (\frac{m\pi}{a})^2$ are the eigenvalues of the X-problem. The eigenfunctions associated with the eigenvalues λ_n are written:

$$X_m^{(2n)} = B_m\sin(\frac{m\pi}{a}x). \tag{18}$$

Now the Y-problem in (14) imply that

$$Y_m^{(2n)} = \big(C_m\cosh(\frac{m\pi}{a}y) + D_m\sinh(\frac{m\pi}{a}y)\big). \tag{19}$$

Using the boundary condition $d_yY_m^{(2n)}(0) = 0$, we get that $D_m = 0$ and so

$$Y_m^{(2n)} = C_m\cosh(\frac{m\pi}{a}y). \tag{20}$$

The general solution $w^{(2n)}$ of the even problem is the superposition of the set of solutions (we must consider all the eigenfunctions) :

$$w^{(2n)}(x,y) = \sum_{m=1}^{\infty} A_m^{(2n)}\sin\big(\frac{m\pi}{a}x\big)\cosh\big(\frac{m\pi}{a}y\big). \tag{21}$$

We use the same technique for (15) to obtain that the solution for the odd problem is given by

$$w^{(2n+1)}(x,y) = \sum_{m=1}^{\infty} A_m^{(2n+1)} \sin\big(\frac{m\pi}{a}x\big) \sinh\big(\frac{m\pi}{a}y\big). \tag{22}$$

To obtain a solution of problems (7)-(9), the use of conditions on Γ_0 allows to find the expression of $A_m^{(2n)}$ and $A_m^{(2n+1)}$. This is illustrated in the following lemma:

Lemma 3.2 *Considering $w^{(2n)}$ and $w^{(2n+1)}$ in the problems (7)-(9), then the coefficients $A_m^{(2n)}$ and $A_m^{(2n+1)}$ in (21) and (22) are given by*

$$A_m^{(2n)} = A_m^{(0)} \Big[\theta\big(\tanh(\beta_m b)\big)^2 + (1-\theta)\Big]^n \tag{23}$$

$$A_m^{(2n+1)} = A_m^{(0)} \Big[\theta\big(\tanh(\beta_m b)\big)^2 + (1-\theta)\Big]^n \tanh(\beta_m b) \tag{24}$$

where $\beta_m = \frac{m\pi}{a}$ and

$$A_m^{(0)} = \frac{2}{a\cosh(\beta_m b)} \int_0^a \big(u(x,b) - \phi^{(0)}\big) \sin(\beta_m x) dx. \tag{25}$$

Proof:
We prove the lemma by induction on n. For $n = 0$, we apply the non-homogeneous condition of the problem (7) $w|_{\Gamma_0}^{(0)} = u|_{\Gamma_0} - \phi^{(0)}$ to get

$$\sum_{m=1}^{\infty} A_m^{(0)} \sin\big(\beta_m x\big) \cosh\big(\beta_m b\big) = u|_{\Gamma_0} - \phi^{(0)}, \quad \text{for } x \in (0,a). \tag{26}$$

Note that functions $\{\sin(\frac{n\pi}{a}x)\}_{n=1}^{\infty}$ are orthogonal and satisfies

$$\int_0^a \sin(\frac{l\pi}{a}x)\sin(\frac{m\pi}{a}x)dx = \frac{a}{2}\delta_{lm}, \tag{27}$$

where δ_{lm} is the Kronecker delta. Thus, multiply both side of (26) by $\sin(\beta_k x)$, integrate with respect to x and the orthogonality (27) leads to

$$A_k^{(0)} \frac{a}{2} \cosh\big(\beta_k b\big) = \int_0^a (u|_{\Gamma_0} - \phi^{(0)}) \sin(\beta_k x) dx,$$

which gives formula (25).

The coefficient $A_m^{(1)}$ is obtained using the non-homogeneous boundary conditions of problems (8)

$$\partial_\nu w^{(1)}|_{\Gamma_0} = \partial_\nu w^{(0)}|_{\Gamma_0}$$

written as

$$\sum_{m=1}^{\infty} A_m^{(1)} \beta_m \sin\big(\beta_m x\big) \cosh\big(\beta_m b\big) = \sum_{m=1}^{\infty} A_m^{(0)} \beta_m \sin\big(\beta_m x\big) \sinh\big(\beta_m b\big).$$

Multiplying by $\sin\big(\beta_k x\big)$ and integrating with respect to x, we get

$$A_k^{(1)} \frac{a}{2}(\beta_k) \cosh\big(\beta_k b\big) = A_k^{(0)} \frac{a}{2}(\beta_k) \sinh\big(\beta_k b\big), \quad \text{for } k = 1, 2, \cdots$$

so

$$A_m^{(1)} = A_m^{(0)} \tanh\big(\beta_m b\big), \quad \text{for } m = 1, 2, \cdots \tag{28}$$

which proves formula (24) for $n = 0$.

Now, for $n = 1$, we find $A_m^{(2)}$ and $A_m^{(3)}$ using the non-homogeneous boundary conditions of problems (9) and (8) respectively.

From $w|_{\Gamma_0}^{(2)} = \theta w^{(1)}(x, b) + (1 - \theta) w^{(0)}(x, b)$ we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} A_m^{(2)} \sin\big(\beta_m x\big) \cosh\big(\beta_m b\big) &= \theta \sum_{m=1}^{\infty} A_m^{(0)} \tanh\big(\beta_m b\big) \sin\big(\beta_m x\big) \sinh\big(\beta_m b\big) \\ &\quad + (1 - \theta) \sum_{m=1}^{\infty} A_m^{(0)} \sin\big(\beta_m x\big) \cosh\big(\beta_m b\big). \end{aligned}$$

Again, multiplying the above equality by $\sin(\beta_k x)$, integrating with respect to x and using the orthogonality (27), we get

$$A_k^{(2)} \frac{a}{2} \cosh\big(\beta_k b\big) = \theta A_k^{(0)} \tanh\big(\beta_k b\big) \frac{a}{2} \sinh\big(\beta_k b\big) + (1 - \theta) \frac{a}{2} \cosh\big(\beta_k b\big),$$

which gives

$$A_k^{(2)} = A_k^{(0)} \Big[\theta \Big(\tanh\big(\beta_k b\big)\Big)^2 + (1 - \theta)\Big], \quad \text{for } k = 1, 2, \cdots$$

This proves formula (23) for $n = 1$.

Now, $A_m^{(3)}$ is obtained in the same way as $A_m^{(1)}$ and this gives

$$A_m^{(3)} = A_m^{(2)} \tanh\big(\beta_m b\big) = A_m^{(0)} \Big[\theta \Big(\tanh\big(\beta_m b\big)\Big)^2 + (1 - \theta)\Big] \tanh\big(\beta_m b\big). \tag{29}$$

Finally, for $n > 1$ we suppose that these formulas are satisfied for $n = k$ and we show that they remain verified for $n = k + 1$.

From problem (9), we have for $n = k + 1$

$$w^{(2(k+1))}(x, y) = \sum_{m=1}^{\infty} A_m^{(2(k+1))} \sin\big(\beta_m x\big) \cosh\big(\beta_m y\big). \tag{30}$$

So, using the non-homogeneous boundary condition

$$\begin{aligned} w^{(2(k+1))}|_{\Gamma_0} &= \theta w^{(2(k+1)-1)}|_{\Gamma_0} + (1 - \theta) w^{(2((k+1)-1))}|_{\Gamma_0} \\ &= \theta w^{(2k+1))}|_{\Gamma_0} + (1 - \theta) w^{(2k)}|_{\Gamma_0}, \end{aligned}$$

thus

$$\begin{aligned} \sum_{m=1}^{\infty} A_m^{(2(k+1))} \sin\big(\beta_m x\big) \cosh\big(\beta_m b\big) &= \theta \sum_{m=1}^{\infty} A_m^{(0)} \Big[\theta \big(\tanh(\beta_m b)\big)^2 + (1 - \theta)\Big]^k \tanh(\beta_m b) \\ &\quad \sin\big(\beta_m x\big) \sinh\big(\beta_m b\big) \\ &\quad + (1 - \theta) \sum_{m=1}^{\infty} A_m^{(0)} \Big[\theta \big(\tanh(\beta_m b)\big)^2 + (1 - \theta)\Big]^k \\ &\quad \sin\big(\beta_m x\big) \cosh\big(\beta_m b\big). \end{aligned}$$

Again, multiplying the above equality by $\sin(\beta_l x)$, integrating and using the argument of orthogonality, we get

$$\begin{aligned} A_l^{(2(k+1))} &= \theta A_l^{(0)}\Big[\theta\big(\tanh(\beta_l b)\big)^2+(1-\theta)\Big]^k\Big[\tanh(\beta_l b)\Big]^2 \\ &\quad +(1-\theta)A_l^{(0)}\Big[\theta\big(\tanh(\beta_l b)\big)^2+(1-\theta)\Big]^k \\ &= A_l^{(0)}\Big[\theta\big(\tanh(\beta_l b)\big)^2+(1-\theta)\Big]^{k+1}, \end{aligned} \tag{31}$$

which proves (23) for $n=k+1$. From problem (8), we have for $n=k+1$

$$w^{(2(k+1)+1)}(x,y)=\sum_{m=1}^{\infty}A_m^{(2(k+1)+1)}\sin\big(\beta_m x\big)\sinh\big(\beta_m y\big) \tag{32}$$

Using the boundary condition $\partial_\nu w|_{\Gamma_0}^{(2(k+1)+1)}=\partial_\nu w|_{\Gamma_0}^{(2(k+1))}$ we obtain

$$\sum_{m=1}^{\infty}A_m^{(2(k+1)+1)}\sin\big(\beta_m x\big)\beta_m\cosh\big(\beta_m b\big)=\sum_{m=1}^{\infty}A_m^{(2(k+1))}\sin\big(\beta_m x\big)\beta_m\sinh\big(\beta_m b\big)$$

Again, Multiplication by $\sin(\beta_l x)$, integration over the interval $(0,a)$ and the orthogonality (27) lead to

$$A_l^{(2(k+1)+1)}=A_l^{(2(k+1))}\tanh\big(\beta_n b\big),\quad \text{for } l=1,2,\cdots$$

The introduction of the expression of $A_l^{(2(k+1))}$ defined in (31) in this last equality gives the formula (24) for $n=k+1$ and this completes the proof of the lemma. $\square$

Now, we return to the proof of the theorem.

Proof of (1) of the theorem 2.1:

Recall that from the lemma 2.2, we just need to show the convergence of $w^{(2n)}|_{\Gamma_0}$ to zero. So let us consider the norm of $w^{(2n)}$ in $\mathcal{L}^2(\Gamma_0)$:

$$\| w^{(2n)}(x,b)\|_{\mathcal{L}^2(\Gamma_0)}=\Big(\int_0^a|w^{(2n)}(x,b)|^2dx\Big)^{1/2},$$

By Parsaval's identity we have

$$\int_0^a|w^{(2n)}(x,b)|^2dx=\frac{a}{2}\sum_{m=1}^{\infty}\Big[A_m^{(0)}\Big[\theta\big(\tanh(\beta_m b)\big)^2+(1-\theta)\Big]^n\cosh(\beta_m b)\Big]^2,$$

thus

$$\| w^{(2n)}(x,b)\|^2_{\mathcal{L}^2(\Gamma_0)}=\frac{a}{2}\sum_{m=1}^{\infty}\big(A_m^{(0)}\big)^2\Big[\theta\big(\tanh(\beta_m b)\big)^2+(1-\theta)\Big]^{2n}\big(\cosh(\beta_m b)\big)^2,$$

and hence

$$\| w^{(2n)}(x,b)\|^2_{\mathcal{L}^2(\Gamma_0)}=\sum_{m=1}^{\infty}\hat{A}_m^{(0)}\Big[\theta\big(\tanh(\beta_m b)\big)^2+(1-\theta)\Big]^{2n}$$

where $\hat{A}_m^{(0)} = \left(\frac{a}{2}\right)\left(A_m^{(0)}\right)^2\left(\cosh(\beta_m b)\right)^2$. Therfore we obtain that $\lim_{n\to\infty} \parallel w^{(2n)} \parallel = 0$ when

$$\mid \theta\left(\tanh(\beta_m b)\right)^2 + (1-\theta) \mid < 1, \ \forall m \geq 1$$

i.e.

$$-1 < \theta\left(\tanh(\beta_m b)\right)^2 + (1-\theta) < 1, \ \forall m \geq 1$$

guiving

$$0 < \theta < \frac{2}{1 - \left(\tanh(\beta_m b)\right)^2}, \ \forall m \geq 1. \tag{33}$$

Frome $\beta_m = \frac{m\pi}{a}$ we get that $\tanh(\beta_1 b) \leq \tanh(\beta_m b), \ \forall m \geq 1$. Thus (33) is equivalent to

$$0 < \theta < \frac{2}{1 - \left(\tanh(\beta_1 b)\right)^2}$$

i.e. the convergence of $w^{(2n)}|_{\Gamma_0}$ to zero is ensured when $\theta \in (0, \theta^*)$ with $\theta^* = \dfrac{2}{1 - \left(\tanh(\beta_1 b)\right)^2}$ so the proof of the theorem is achieved. $\square$

4 Acceleration of convergence

Note that even if the JN relaxed algorithm is widely used in works dealing with applications modeled by Cauchy problems, there is no estimate of the interval in which the relaxation parameter must be chosen to accelerate convergence. We propose in the following theorem an identification of this convergence acceleration interval.

Theorem 4.1 *Considering the Poisson problem defined in (2)-(4). We have: The convergence of the JN relaxation algorithm is more faster than the convergence of the KMF algorithm* $\forall \theta \in], \theta_a[$ *with* $\theta_a = \frac{1+\left(\tanh(\frac{\pi}{a}b)\right)^2}{1-\left(\tanh(\frac{\pi}{a}b)\right)^2}$

Proof:
Recall that the standard KMF algorithm [9] can be obtained from the JN relaxated algorithm [7] by taking the relaxation parameter $\theta = 1$.

Denote by $\{w_\theta^{(n)}\}$ the sequence solutions obtained from the JN relaxation algorithm for the parameter θ.

Note that

$$w_\theta^{(2n)}(x,y) = \sum_{m=1}^{\infty} A_m^{(0)} \left[\theta\left(\tanh(\beta_m b)\right)^2 + (1-\theta)\right]^n \sin\left(\beta_m x\right)\cosh\left(\beta_m y\right) \tag{34}$$

Thus, for $\theta = 1$, we obtain

$$w_1^{(2n)}(x,y) = \sum_{m=1}^{\infty} A_m^{(0)} \left[\left(\tanh(\beta_m b)\right)^2\right]^n \sin\left(\beta_m x\right)\cosh\left(\beta_m y\right) \tag{35}$$

To find the condition for which the convergence of $\{w_\theta^{(n)}\}$ is faster than the convergence of the sequence $\{w_1^{(n)}\}$, we must verify the following condition

$$\mid \theta\left(\tanh(\beta_m b)\right)^2 + (1-\theta) \mid < \left(\tanh(\beta_m b)\right)^2, \ \forall m \geq 1$$

so

$$-\big(\tanh(\beta_m b)\big)^2 - 1 < \theta\Big(\big(\tanh(\beta_m b)\big)^2 - 1\Big) < \big(\tanh(\beta_m b)\big)^2 - 1,\ \forall m \geq 1.$$

Dividing by $\Big(1 - \big(\tanh(\beta_m b)\big)^2\Big)$, which is positive, we obtain

$$-\frac{\Big(1 + \big(\tanh(\beta_m b)\big)^2\Big)}{1 - \big(\tanh(\beta_m b)\big)^2} < -\theta < -1,\ \forall m \geq 1.$$

Which gives

$$\frac{1 + \big(\tanh(\beta_m b)\big)^2}{1 - \big(\tanh(\beta_m b)\big)^2} > \theta > 1,\ \forall m \geq 1.$$

Since $\tanh(\beta_1 b) \leq \tanh(\beta_m b),\ \forall m \geq 1$, these inequalities are equivalent to

$$1 < \theta < \frac{1 + \big(\tanh(\beta_1 b)\big)^2}{1 - \big(\tanh(\beta_1 b)\big)^2}.$$

This completes the proof of the theorem 4.1. □

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