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AN ANALYTICAL METHOD FOR THE INVERSE CAUCHY PROBLEM OF LAPLACE EQUATION IN A RECTANGULAR PLATE

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ABSTRACT

The present paper reveals an analytically computational method for the inverse Cauchy problem of Laplace equation. For the sake of analyticity, and also for the frequent use of rectangular plate in engineering structure, we only consider the analytical solution in a two-dimensional rectangular domain, wherein a missing boundary condition is recovered from a partial measurement of the Neumann data on an accessible boundary. The Fourier series is used to formulate a first-kind Fredholm integral equation for the unknown function of data. Then, we consider a Lavrentiev regularization amended to a second-kind Fredholm integral equation. The termwise separable property of kernel function allows us to obtain a closed-form solution of the regularization type. The uniform convergence and error estimation of the regularization solution are proven. The numerical examples show the effectiveness and robustness of the new method.

Keywords: Laplace equation, Inverse Cauchy problem, Fourier series, Analytical regularization solution, Error estimation, Separable kernel, Lavrentiev regularization.

1. INTRODUCTION

The inverse Cauchy problem is to solve the boundary value problem of elliptic type partial differential equations given by some overspecified Cauchy data on a partial portion of the boundary, which is proved to have a unique solution if the solution exists. However, the problem of numerical instability lent the researchers a headache, because the inverse of the original operator is not obtainable. It means that the continuous dependence of the solution on the given data is not satisfied in the Hardmard sense. To treat this kind of problem, many techniques were proposed. The most famous one is the Tikhonov's regularization technique, which transforms the original problem into a constrained minimization problem. The Lagrangian multiplier (the regularization parameter) is not known in advance, which can be determined by the L-curve concept [1] or through an engineering judgement. Except this method, the truncated singular value decomposition method had also been used. All those methods try to improve the ill-conditioning of the resulting leading matrix of original system. There are two major drawbacks of the previous approaches: Some kind of matrix inversion is required for the resulting ill-conditioned matrix and they cannot be directly extended to nonlinear elliptic systems.

It is well-known that the Cauchy problem of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 ,$$

endowed with "initial conditions" (one-side boundary conditions)

$$u(x, 0) = 0 ,$$
$$\frac{\partial u}{\partial y}(x, 0) = n^{-1} \sin(nx)$$

is highly ill-posed, since the work of Hadamard.

It is easily verified that the exact solution

$$u(x, y) = n^{-2} \sin (nx) \sinh (ny)$$

of the above problem does not become small for any nonzero y, even the initial condition $n^{-1} \sin(nx)$ can be arbitrarily small by increasing n. It is obvious that the solution does not depend continuously on the initial data. Thus, it is not a well-posed problem in the sense of Hadamard.

The use of electrostatic image in the non-destructive testing of metallic plates leads to an inverse boundary value problem for the Laplace equation in two-dimension. In order to detect the unknown shape of the inclusion within a conducting metal, the over-determined Cauchy data, for example the voltage and current, are imposed on the accessible exterior bound-

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ary [2-4]. This amounts to solving an inverse Cauchy problem from available data on partial boundary. There had been many studies on this type problems in the open literature [5-14].

The Cauchy problem is difficult to solve both numerically and analytically, since its solution, if exists, does not depend continuously on the given data. Therefore, we must treat this type problem with a different numerical algorithm from that used in the direct problem, which compromises accuracy and stability.

In this work we focus on a numerical computation based on an analytically and regularizedly solving methodology. In the past several years there already were many numerical methods proposed to solve the Cauchy problems [15-22], to name a few. Among the many numerical methods, the schemes based on iteration have also been developed previously by Jourhmane and Nachaoui [23,24], Essaouini *et al*. [25], Nachaoui [26], and Jourhmane *et al*. [27]. In contrast to those methods the present method aims to provide an analytically regularization solution without resorting-on iteration. Liu [28] has applied a modified collocation Trefftz method in the inverse Cauchy problem in a circular domain. In order to achieve a stable numerical solution for recoverning the discontinuous data, a regularization by truncating the higher modes of the Fourier series of the input data is necessary. The present paper will consider a similar truncation of the Fourier series but under a different approach. In [29,30], a similar method has been named the Fourier regularization method.

In this paper, we cast the Cauchy problem in a rectangular domain into a first-kind Fredholm integral equation, and then we propose a Lavrentiev type regularization to transform it into a second-kind Fredholm integral equation. By utilizing the separating characteristic of kernel function and eigenfunctions expansion technique we can derive a closed-form regularization solution of the second-kind Fredholm integral equation. Liu *et al.* [31] have used this technique to solve the inverse geometric problem, and Liu [32] has solved the backward heat conduction problem. This method was first used by Liu [33] to solve a direct problem of elastic torsion of a bar with arbitrary cross-section, where it was called a meshless regularized integral equation method. Then, Liu [34,35] extended it to solve the Laplace direct problem in arbitrary plane domains. Liu [36] has developed a modified Trefftz method by a simple collocation technique to treat the inverse Cauchy problem of Laplace equation in arbitrary plane domain.

This paper is organized as follows. In Section 2 we derive a first-kind Fredholm integral equation for the Cauchy problem. By a direct regularization of the first-kind Fredholm integral equation, in Section 3 we derive a two-point boundary value problem, which helps us to derive a closed-form regularization solution of the integral equation in Section 4. In Section 5 we prove the uniform convergence of the regularization solution, as well as give an error estimation. In Section 6 we use some numerical examples to test the new method. Then, we give conclusions in Section 7.

2. THE FIRST-KIND FREDHOLM INTEGRAL EQUATION

We consider an inverse Cauchy problem given as follows:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\,,\quad 0<x<\ell\,,\quad 0<y<b\,, \tag{1}$$

$$u(x,b)=0\,,\quad 0\le x\le \ell\,, \tag{2}$$

$$-\frac{\partial u}{\partial y}(x,b)=h(x)\,,\quad 0\le x\le \ell\,, \tag{3}$$

$$u(0,y)=0\,,\quad u(\ell,y)=0,\quad 0\le y\le b, \tag{4}$$

where $h(x)$ is a given function.

Instead of Eq. (3) we consider the following boundary condition:

$$u(x,0)=f(x)\,,\quad 0\le x\le \ell\,, \tag{5}$$

where $f(x)$ is an unknown function to be determined.

Inverse Cauchy Problem. Suppose that the datum $f(x)$ in Eq. (5) is unknown, but the Neumann datum $h(x)$ in Eq. (3) is overspecified. Determine the unknown function $f(x)$.

We can write a series expansion of $u(x, y)$ satisfying Eqs. (1), (2), (4) and (5):

$$u(x,y)=\sum_{k=1}^{\infty}\frac{a_k\sinh\left[(b-y)k\pi/\ell\right]}{\sinh(bk\pi/\ell)}\sin\frac{k\pi x}{\ell}\,, \tag{6}$$

where

$$a_k=\frac{2}{\ell}\int_0^{\ell}f(\xi)\sin\frac{k\pi\xi}{\ell}\,d\xi\,. \tag{7}$$

Taking the differential of Eq. (6) with respect to y, we obtain

$$\frac{\partial u(x,y)}{\partial y}=\sum_{k=1}^{\infty}\frac{\pi k a_k\cosh\left[(b-y)k\pi/\ell\right]}{\ell\sinh(bk\pi/\ell)}\sin\frac{k\pi x}{\ell}\,, \tag{8}$$

and further imposing condition (3) on the above equation we have

$$\frac{\pi}{\ell}+\sum_{k=1}^{\infty}\frac{k a_k}{\sinh(bk\pi/\ell)}\sin\frac{k\pi x}{\ell}=h(x)\,, \tag{9}$$

Substituting Eq. (7) for a_k into Eq. (9) and assuming that the order of summation and integral can be interchanged, it follows that

$$\int_0^{\ell}K(x,\xi)\,f(\xi)\,d\xi=h(x)\,, \tag{10}$$

where

$$K(x,\xi)=\frac{2\pi}{\ell^2}\sum_{k=1}^{\infty}\frac{k}{\sinh(bk\pi/\ell)}\sin\frac{k\pi x}{\ell}\sin\frac{k\pi\xi}{\ell} \tag{11}$$

is a kernel function.

Up to here we can see that the ill-posedness of this inverse Cauchy problem is fully reflected in a solution of the first-kind Fredholm integral equation. In order to obtain $f(x)$ we have to solve the first-kind Fredholm integral Eq. (10). This is however a quite difficult task, since this integral equation is highly ill-posed [37].

If the boundary condition $u(x, b) = 0$ in Eq. (2) is replaced by a nonhomogeneous one:

$$u(x, b) = g(x), \tag{12}$$

then we have

$$u(x,y) = \sum_{k=1}^{\infty}\left[a_k \frac{\sinh[(b-y)k\pi/\ell]}{\sinh(bk\pi/\ell)} + b_k \frac{\sinh(k\pi y/\ell)}{\sinh(bk\pi/\ell)}\right]\sin\frac{k\pi x}{\ell}, \tag{13}$$

where

$$b_k = \frac{2}{\ell}\int_0^\ell g(\xi)\sin\frac{k\pi\xi}{\ell}\,d\xi. \tag{14}$$

We can derive the same integral Eq. (10) but replace $h(x)$ by

$$h(x) + \frac{\pi}{\ell}\sum_{k=1}^{\infty}\frac{kb_k \sinh(k\pi y/\ell)}{\sinh(bk\pi/\ell)}\sin\frac{k\pi x}{\ell}.$$

The method we apply to solve Eq. (10) is given below.

3. TWO-POINT BOUNDARY VALUE PROBLEM

Instead of Eq. (10), we solve $f(x)$ by

$$\alpha f(x) + \int_0^\ell K(x,\xi)f(\xi)\,d\xi = h(x), \tag{15}$$

where $\alpha > 0$ is a regularization parameter. The above equation is a second-kind Fredholm integral equation. This type of regularization is known as the Lavrentiev regularization.

We also assume that the kernel function can be approximated by m terms with

$$K(x,\xi) = \frac{2\pi}{\ell^2}\sum_{k=1}^{m}\frac{k}{\sinh(bk\pi/\ell)}\sin\frac{k\pi x}{\ell}\sin\frac{k\pi\xi}{\ell}. \tag{16}$$

Usually, the series solution is dominated by the first few terms. By an inspection of Eq. (16) we can write

$$K(x,\xi) = \mathbf{P}(x)\cdot\mathbf{Q}(\xi), \tag{17}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are m-vectors given by

$$\mathbf{P} := \frac{2\pi}{\ell^2}\begin{bmatrix} \frac{1}{\sinh(b\pi/\ell)}\sin\frac{\pi x}{\ell} \\ \frac{2}{\sinh(2b\pi/\ell)}\sin\frac{2\pi x}{\ell} \\ \vdots \\ \frac{m}{\sinh(mb\pi/\ell)}\sin\frac{m\pi x}{\ell} \end{bmatrix}, \quad \mathbf{Q} := \begin{bmatrix} \sin\frac{\pi\xi}{\ell} \\ \sin\frac{2\pi\xi}{\ell} \\ \vdots \\ \sin\frac{m\pi\xi}{\ell} \end{bmatrix}, \tag{18}$$

and the dot between $\mathbf{P}$ and $\mathbf{Q}$ denotes the inner product, which is sometimes written as $\mathbf{P}^T\mathbf{Q}$, where the superscript T signifies the transpose.

Equation (15), with the aid of Eq. (17), can be decomposed as

$$\alpha f(x) + \int_0^x \mathbf{P}^T(x)\,\mathbf{Q}(\xi)f(\xi)\,d\xi + \int_x^\ell \mathbf{P}^T(x)\,\mathbf{Q}(\xi)f(\xi)\,d\xi = h(x). \tag{19}$$

Upon defining

$$\mathbf{u}_1(x) := \int_0^x \mathbf{Q}(\xi)f(\xi)\,d\xi, \tag{20}$$

$$\mathbf{u}_2(x) := \int_\ell^x \mathbf{Q}(\xi)f(\xi)\,d\xi, \tag{21}$$

Equation (19) can be expressed as

$$\alpha f(x) + \mathbf{P}^T(x)[\mathbf{u}_1(x) - \mathbf{u}_2(x)] = h(x). \tag{22}$$

Taking the differentials of Eqs. (20) and (21) with respect to x we can obtain

$$\mathbf{u}_1'(x) = \mathbf{Q}(x)f(x), \tag{23}$$

$$\mathbf{u}_2'(x) = \mathbf{Q}(x)f(x). \tag{24}$$

Inserting Eq. (22) for $f(x)$ into the above two equations we have

$$\alpha\mathbf{u}_1'(x) = \mathbf{Q}(x)\mathbf{P}^T(x)[\mathbf{u}_2(x) - \mathbf{u}_1(x)] + h(x)\mathbf{Q}(x), \quad \mathbf{u}_1(0) = 0, \tag{25}$$

$$\alpha\mathbf{u}_2'(x) = \mathbf{Q}(x)\mathbf{P}^T(x)[\mathbf{u}_2(x) - \mathbf{u}_1(x)] + h(x)\mathbf{Q}(x), \quad \mathbf{u}_2(\ell) = 0, \tag{26}$$

where the last two boundary conditions follow from Eqs. (20) and (21) readily. The above two equations constitute a two-point boundary value problem, which can be used to solve $\mathbf{u}_1(x)$ and $\mathbf{u}_2(x)$, and then $f(x)$ can be calculated from Eq. (22). However, a more simplified approach is available in the next section.

4. A CLOSED-FORM REGULARIZED SOLUTION

In this section we will find a closed-form solution of $f(x)$. From Eqs. (23) and (24) it can be observed that

$$\mathbf{u}_1 = \mathbf{u}_2 + \mathbf{c} \tag{27}$$

where c is a constant vector to be determined. By using the final condition $\mathbf{u}_2(\ell) = 0$ in Eq. (26) we find that

$$\mathbf{u}_1(\ell) = \mathbf{u}_2(\ell) + \mathbf{c} = \mathbf{c}\,. \tag{28}$$

Substituting Eq. (27) into (25) we have

$$\alpha \mathbf{u}_1'(x) = -\mathbf{Q}(x)\,\mathbf{P}^T(x)\mathbf{c} + h(x)\,\mathbf{Q}(x)\,, \quad \mathbf{u}_1(0) = 0\,. \tag{29}$$

Integrating the above equation and using the initial condition it follows that

$$\mathbf{u}_1(x) := \frac{-1}{\alpha}\int_0^x \mathbf{Q}(\xi)\,\mathbf{P}^T(\xi)\,d\xi\,\mathbf{c} + \frac{1}{\alpha}\int_0^x h(\xi)\,\mathbf{Q}(\xi)\,d\xi\,. \tag{30}$$

Taking $x = \ell$ in the above equation and imposing the condition (28), one obtains a governing equation for c:

$$\left(\alpha \mathbf{I}_m + \int_0^\ell \mathbf{Q}(\xi)\,\mathbf{P}^T(\xi)\,d\xi\right)\mathbf{c} = \int_0^\ell h(\xi)\,\mathbf{Q}(\xi)\,d\xi\,. \tag{31}$$

It is straightforward to write

$$\mathbf{c} = \left(\alpha \mathbf{I}_m + \int_0^\ell \mathbf{Q}(\xi)\,\mathbf{P}^T(\xi)\,d\xi\right)^{-1}\int_0^\ell h(\xi)\,\mathbf{Q}(\xi)\,d\xi\,. \tag{32}$$

On the other hand, from Eqs. (22) and (27) we have

$$\alpha f(x) = h(x) - \mathbf{P}(x)\cdot\mathbf{c}\,. \tag{33}$$

Inserting Eq. (32) into the above equation we can obtain

$$\alpha f(x) = h(x) - \mathbf{P}(x)\cdot\left(\alpha \mathbf{I}_m + \int_0^\ell \mathbf{Q}(\xi)\,\mathbf{P}^T(\xi)\,d\xi\right)^{-1}\int_0^\ell h(\xi)\,\mathbf{Q}(\xi)\,d\xi\,. \tag{34}$$

Due to the orthogonality of

$$\int_0^\ell \sin\frac{j\pi\xi}{\ell}\sin\frac{k\pi\xi}{\ell}\,d\xi = \frac{\ell}{2}\,\delta_{jk}\,, \tag{35}$$

where δ_{jk} is the Kronecker delta, the $m \times m$ matrix can be written as

$$\int_0^\ell \mathbf{Q}(\xi)\,\mathbf{P}^T(\xi)\,d\xi = \frac{\pi}{\ell}\operatorname{diag}\left[\frac{1}{\sinh(b\pi/\ell)}, \frac{2}{\sinh(2b\pi/\ell)}, \cdots, \frac{m}{\sinh(mb\pi/\ell)}\right], \tag{36}$$

where diag means a diagonal matrix.

Inserting Eq. (36) into Eq. (34) we thus obtain

$$f(x) = \frac{1}{\alpha}h(x) - \frac{1}{\alpha}\mathbf{P}^T(x)\operatorname{diag}\left[\frac{1}{\alpha + \frac{\pi}{\ell\sinh(b\pi/\ell)}}, \frac{1}{\alpha + \frac{2\pi}{\ell\sinh(2b\pi/\ell)}}, \cdots, \frac{1}{\alpha + \frac{m\pi}{\ell\sinh(mb\pi/\ell)}}\right]\int_0^\ell h(\xi)\,\mathbf{Q}(\xi)\,d\xi\,. \tag{37}$$

By using Eq. (18) for **P** and **Q**, we can obtain

$$f(x) = \frac{1}{\alpha}h(x) - \frac{2}{\alpha\ell}\sum_{k=1}^m \frac{k\pi}{\alpha\ell\sinh(bk\pi/\ell) + k\pi}\int_0^\ell \sin\frac{k\pi x}{\ell}\sin\frac{k\pi\xi}{\ell}h(\xi)\,d\xi\,. \tag{38}$$

The kernel function appeared here is slightly different from that in Eq. (16). For a given $h(x)$, through some integrals one may employ the above equation to calculate $f(x)$ very efficiently.

Moreover, if $f(x)$ in Eq. (38) is available we can insert it into Eq. (7) and utilize the orthogonality equation (35) to obtain

$$a_k^\alpha = \frac{2\sinh(bk\pi/\ell)}{\alpha\ell\sinh(bk\pi/\ell) + k\pi}\int_0^\ell \sin\frac{k\pi\xi}{\ell}h(\xi)\,d\xi\,. \tag{39}$$

Then, from Eq. (6) with the above a_k^α we can calculate $u^\alpha(x, y)$ by

$$u^\alpha(x, y) = \sum_{k=1}^\infty \frac{a_k^\alpha \sinh\left[(b-y)k\pi/\ell\right]}{\sinh(bk\pi/\ell)}\sin\frac{k\pi x}{\ell}\,, \tag{40}$$

where we use $u^\alpha(x, y)$ to denote an analytically regularization solution.

In principle, we can let $m = \infty$ in Eq. (38). However, when k approaches ∞ the coefficient a_k^α also approaches zero, by using the following result:

$$\int_0^\ell \sin\frac{k\pi\xi}{\ell}h(\xi)\,d\xi \to 0\,,\ k \to \infty\,. \tag{41}$$

In practice we cannot let $k \to \infty$ for the calculation really performed in a computer machine. Therefore, we really adopt a truncation of the higher modes in the regularization solution, and hence, the present method can be viewed as a Fourier regularization.

5. ERROR ESTIMATION

In the previous section we have derived a regularizedly analytical solution $u^\alpha(x, t)$ of Eqs. (1) ~ (4) under the regularization in Eq. (15) with a regularization parameter $\alpha > 0$. We can prove the following main results.

Theorem 1. If the Neumann datum $h(x)$ is bounded in the interval $x \in [0, \ell]$, then for any $\alpha > 0$ and $y_0 > 0$, the regularization solution $u^\alpha(x, y)$ converges uniformly to an exact solution for all $x \in [0, \ell]$ and $y \in [y_0, b]$.

Proof. Because $\alpha > 0$, and $h(x)$ is bounded, say $|h(x)| \le C^*$, $x \in [0, \ell]$, for some $C^* > 0$, from Eq. (39) we have

$$|a_k^\alpha| = \frac{2\sinh(bk\pi/\ell)}{\alpha\ell\sinh(bk\pi/\ell) + k\pi}\left|\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)\, d\xi\right|$$

$$\le \frac{2}{\alpha\ell}\int_0^\ell |h(\xi)|\, d\xi \le \frac{2C^*}{\alpha} =: C_1 \tag{41}$$

By noting that

$$\frac{\sinh[(b-y)k\pi/\ell]}{\sinh(bk\pi/\ell)} = e^{-k\pi y/\ell}\frac{1-e^{-(2k\pi/\ell)(b-y)}}{1-e^{-2k\pi b/\ell}} \le C_2 e^{-k\pi y/\ell}, \tag{42}$$

we have

$$\left|a_k^\alpha \frac{\sinh[(b-y)k\pi/\ell]}{\sinh(bk\pi/\ell)}\sin\frac{k\pi x}{\ell}\right| \le Ce^{-k\pi y/\ell}, \tag{43}$$

where $C = C_1C_2$ is a constant.

Hence, the series for $u^\alpha(x, y)$ is dominated by the series

$$\sum_{k=1}^\infty Ce^{-k\pi y_0/\ell}, \quad y \ge y_0 > 0. \tag{44}$$

Through the ratio test it is obvious that the series $e^{-k\pi y_0/\ell}$ converges. Hence, by the Weierstrass M-test, the series in Eq. (40) with a_k^α given by Eq. (39) converges uniformly with respect to x and y whenever $y \in [y_0, b]$ and $x \in [0, \ell]$. This ends the proof.

Taking $\alpha = 0$ in Eq. (39) and inserting it into Eq. (6) we have an exact solution of Eqs. (1) ~ (4):

$$u(x, y) = \sum_{k=1}^\infty \frac{a_k^* \sinh[(b-y)k\pi/\ell]}{\sinh(bk\pi/\ell)}\sin\frac{k\pi x}{\ell}, \tag{45}$$

where

$$a_k^* = \frac{2\sinh(bk\pi/\ell)}{k\pi}\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)\, d\xi, \tag{46}$$

Theorem 2. Assume that the Neumann datum $h(x) \in L^2(0, \ell)$. Then the sufficient and necessary condition that the Cauchy problem (1) ~ (4) has a solution is that

$$\sum_{k=1}^\infty \frac{\sinh^2(bk\pi/\ell)}{k^2\pi^2}\left(\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)\, d\xi\right)^2 < \infty. \tag{47}$$

Proof. Inserting $y = 0$ into Eq. (45) and noting Eqs. (5) and (46), we have

$$f(x) = u(x, 0)$$

$$= \sum_{k=1}^\infty \frac{2\sinh(bk\pi/\ell)}{k\pi}\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)\, d\xi \sin\frac{k\pi x}{\ell}, \tag{48}$$

where $f(x) \in L^2(0, \ell)$. The above is a Fourier sine expansion of $f(x)$, and by the Parseval equality we have

$$\sum_{k=1}^\infty \frac{2\ell}{\left(\dfrac{k\pi}{\sinh(bk\pi/\ell)}\right)^2}\left(\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)\, d\xi\right)^2$$

$$= \| f(x) \|^2_{L^2(0,\ell)} < \infty, \tag{49}$$

where $\| f(x) \|^2_{L^2(0,\ell)}$ is the L^2-norm of $f(x)$ in the interval $(0, \ell)$. This proves the sufficient and necessary condition.

Theorem 3. If the Neumann datum $h(x)$ satisfies condition (47) and there exists an $\varepsilon \in (0, 1)$, such that

$$4\sum_{k=1}^\infty \left[\frac{\ell^2\sinh^2(bk\pi/\ell)}{k^2\pi^2}\right]^{1+\varepsilon}\left(\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)\, d\xi\right)^2 := M^2(\varepsilon) < \infty, \tag{50}$$

then for any $\alpha > 0$ the regularization solution $u^\alpha(x, y)$ satisfies the following error estimation:

$$\| u^\alpha(x, y) - u(x, y) \|_{L^2(0,\ell)} \le \alpha^\varepsilon M(\varepsilon). \tag{51}$$

Proof. From Eqs. (40), (39), (45) and (46) it follows that

$$u^\alpha(x, y) - u(x, y) = \sum_{k=1}^\infty b_k \frac{\sinh[(b-y)k\pi/\ell]}{\sinh(bk\pi/\ell)}\sin\frac{k\pi x}{\ell}, \tag{52}$$

where

$$b_k = \frac{2\alpha}{\dfrac{k\pi}{\sinh(bk\pi/\ell)}\left[\alpha + \dfrac{k\pi}{\ell\sinh(bk\pi/\ell)}\right]}\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)\, d\xi. \tag{53}$$

Therefore, for any $\varepsilon \in (0, 1)$ we have the following estimation:

$$\begin{aligned}
\| u(x, y) - u^\alpha(x, y) \|^2_{L^2(0,\ell)} &= 4\alpha^2\sum_{k=1}^\infty \left(\frac{\sinh[(b-y)k\pi/\ell]}{\sinh(bk\pi/\ell)}\right)^2 \\
&\frac{\ell^2\sinh^2(bk\pi/\ell)}{k^2\pi^2}\left[\left(\alpha + \frac{k\pi}{\ell\sinh(bk\pi/\ell)}\right)^\varepsilon\left(\alpha + \frac{k\pi}{\ell\sinh(bk\pi/\ell)}\right)^{1-\varepsilon}\right]^{-2}\left(\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)\, d\xi\right)^2 \\
&\le 4\alpha^2\sum_{k=1}^\infty \frac{\ell^2\sinh^2(bk\pi/\ell)}{k^2\pi^2}\left[\frac{\ell^2\sinh^2(bk\pi/\ell)}{k^2\pi^2}\right]^\varepsilon [\alpha^{1-\varepsilon}]^{-2}\left(\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)\, d\xi\right)^2 \\
&= 4\alpha^{2\varepsilon}\sum_{k=1}^\infty \left[\frac{\ell^2\sinh^2(bk\pi/\ell)}{k^2\pi^2}\right]^{1+\varepsilon}\left(\int_0^\ell \sin\frac{k\pi\xi}{\ell} h(\xi)\, d\xi\right)^2.
\end{aligned} \tag{54}$$

This completes the proof.

Theorem 3 is vital, which indicates that the regularization solution approaches to the exact solution in an L^2-norm sense when the regularization parameter α tends to zero.

6. NUMERICAL TESTS

In this section we compare the analytical and numerical solutions through some given examples. The results show that the accuracy of the new method is reasonable.

Example 1. For the test of the new method we first consider a simple problem as follows:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\,,\quad 0<x<\pi\,,\quad 0<y<b\,, \tag{55}$$

$$u(x,b)=0\,,\quad -\frac{\partial u}{\partial y}(x,b)=h(x)=\frac{1}{n}\sin(nx)\,,\quad 0\le x\le \pi\,, \tag{56}$$

$$u(0,y)=0\,,\ u(\pi,y)=0,\quad 0\le y\le b\,, \tag{57}$$

where n is a positive integer. As mentioned in Section 1, it is an ill-posed problem, and when n is larger, the problem is worser of its ill-posedness.

It is easily verified that the exact solution of this problem is

$$u_e(x,y)=n^{-2}\sin(nx)\sinh\left[n(b-y)\right]. \tag{58}$$

Correspondingly, the exact $f_e(x)$ is

$$f_e(x)=u_e(x,0)=\frac{\sinh(nb)}{n^2}\sin(nx)\,. \tag{59}$$

Substituting Eq. (56) for $h(x)$ into Eq. (38), noting $\ell=\pi$ and through some manipulations, we can obtain

$$f(x)=\frac{\sinh(nb)}{\alpha n\sinh(nb)+n^2}\sin(nx)\,. \tag{60}$$

Remarkably, $f(x)=f_e(x)$ when $\alpha=0$, and our method leads to the same result as the exact solution.

Substituting the above $f(x)$ into Eq. (7) and utilizing Eq. (35) one has

$$a_n^\alpha=\frac{\sinh(nb)}{\alpha n\sinh(nb)+n^2}\,. \tag{61}$$

The other $a_k^\alpha=0$; $k\ne n$. Then we obtain the numerical solution

$$u^\alpha(x,y)=\frac{\sin(nx)\sinh\left[n(b-y)\right]}{\alpha n\sinh(nb)+n^2}\,. \tag{62}$$

Fixing $b=1$ and $n=1$, we plot $f_e(x)$ and $f(x)$ in Fig. 1 for $\alpha=10^{-3}$ and $\alpha=10^{-5}$. The errors as can be seen

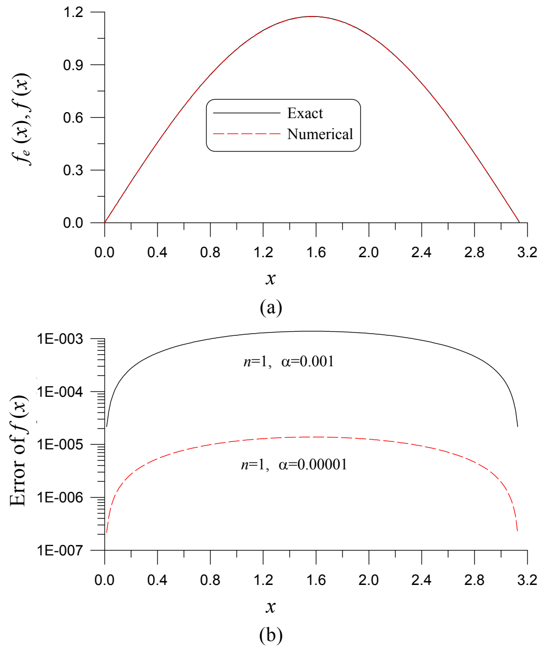


Fig. 1 For the Cauchy inverse problem comparing regularized and exact Cauchy data in (a), and the numerical errors in (b) for Example 1 with $n=1$

are in the order of α. Correspondingly, the exact solution and the numerical solution are compared in Fig. 2(a) for a fixed $y=0.5$. Because they are almost coincident, we cannot see the difference between these two solutions. The absolute errors are plotted in Fig. 2(b). It can be seen that very accurate results are obtained.

Next, we consider a slightly ill-posed case with $n=5$. The exact solution and the numerical solutions with $\alpha=10^{-4}$ and $\alpha=10^{-6}$ are compared in Fig. 3(a) for a fixed $y=0.5$. Similarly, we cannot see the difference between these two solutions, because they are almost coincident. The absolute errors are plotted in Fig. 3(b). It can be seen that very accurate results are obtained.

Example 2. In this example we consider a non-smooth case with

$$f(x)=\begin{cases}2x\,, & \text{for}\quad 0\le x\le 0.5\,,\\ 2(1-x)\,, & \text{for}\quad 0.5\le x\le 1\,.\end{cases} \tag{63}$$

The coefficient a_k is obtained from Eq. (7) by

$$a_k=\frac{8\sin(k\pi/2)}{k^2\pi^2}\,, \tag{64}$$

and the exact solution is given by Eq. (6) after inserting the above a_k.

The function $h(x)$ required in our method to solve the Cauchy problem is given by

$$h(x)=\pi\sum_{k=1}^{\infty}\frac{ka_k}{\sinh(bk\pi)}\sin(k\pi x)\,. \tag{65}$$

Inserting it into Eq. (39) we have

$$a_k^\alpha=\frac{2\sinh(bk\pi)}{\alpha\sinh(bk\pi)+k\pi}\int_0^1\sin(k\pi\xi)\,h(\xi)\,d\xi=\frac{\pi k a_k}{\alpha\sinh(bk\pi)+k\pi}\,. \tag{66}$$

If $\alpha = 0$, we can deduce $a_k^\alpha = a_k$, and the solution in Eq. (6) with the above a_k^α is coincident with the exact solution.

The numerical solution of $f(x)$ can be recovered from

$$f(x) = \sum_{k=1}^{\infty} a_k^\alpha \sin(k\pi x) . \tag{67}$$

In Fig. 4(a) we plot the function $f(x)$ for some α, which are compared with the exact $f(x)$. In practice, we have calculated the above series up to 30 terms. Fixing $b = 1$, it can be seen that the numerical solutions of $f(x)$ with $\alpha = 10^{-10}$ and 10^{-15} are very close to the exact one, except that there is a little difference happened at the non-smooth point $x = 0.5$.

To simulate the measurement errors of real boundary data we add the random noises with a zero mean and different levels of amplitude in the exact data a_k defined by Eq. (64), such that the function $h(x)$ in Eq. (65) becomes

$$h(x) = \pi \sum_{k=1}^{\infty} \frac{k\left[a_k + sR(k)\right]}{\sinh(bk\pi)} \sin(k\pi x) , \tag{68}$$

where $R(k)$ are random numbers in $[-1, 1]$. The noise is obtained by multiplying $R(k)$ by a factor s. The numerical results with $s = 0.01$ and 0.02 were compared

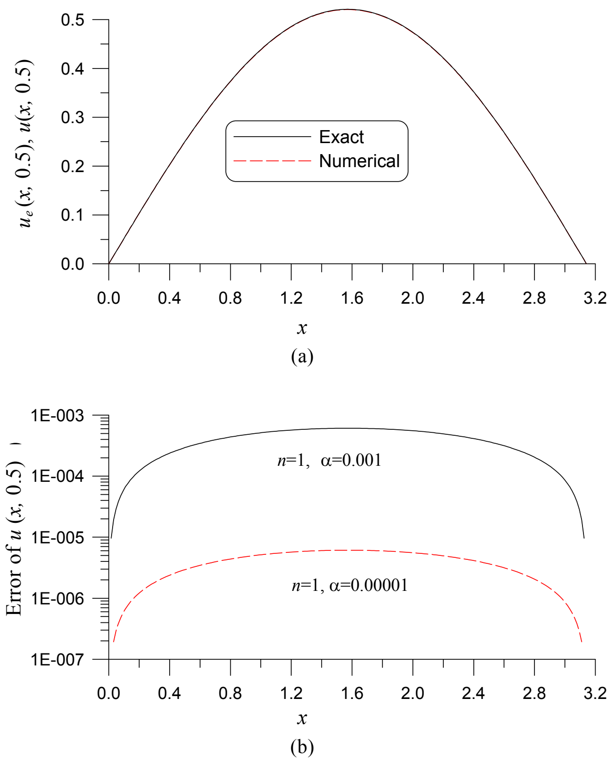


Fig. 2 For Example 1 with $n = 1$: (a) comparing regularized and exact solutions, and (b) the numerical errors

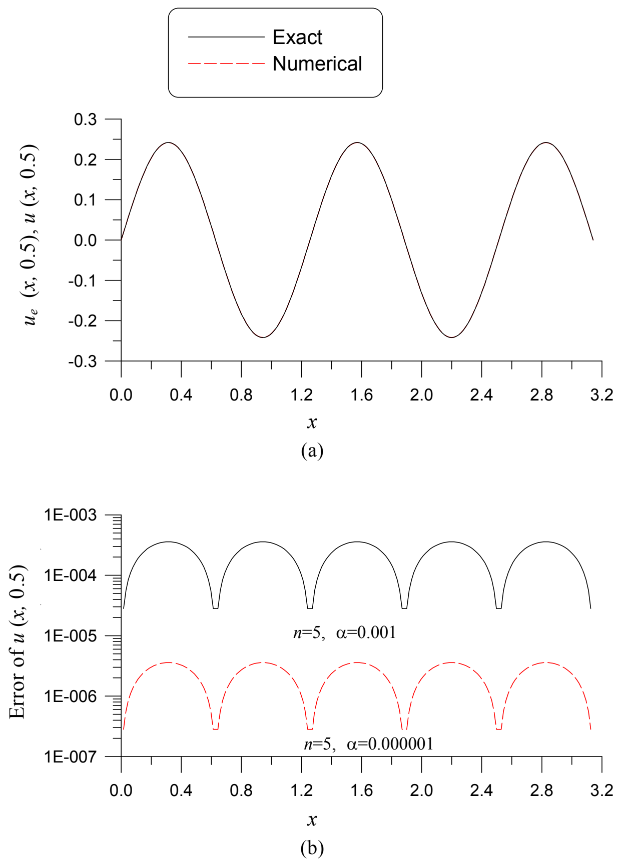


Fig. 3 For Example 1 with $n = 5$: (a) comparing regularized and exact solutions, and (b) the numerical errors

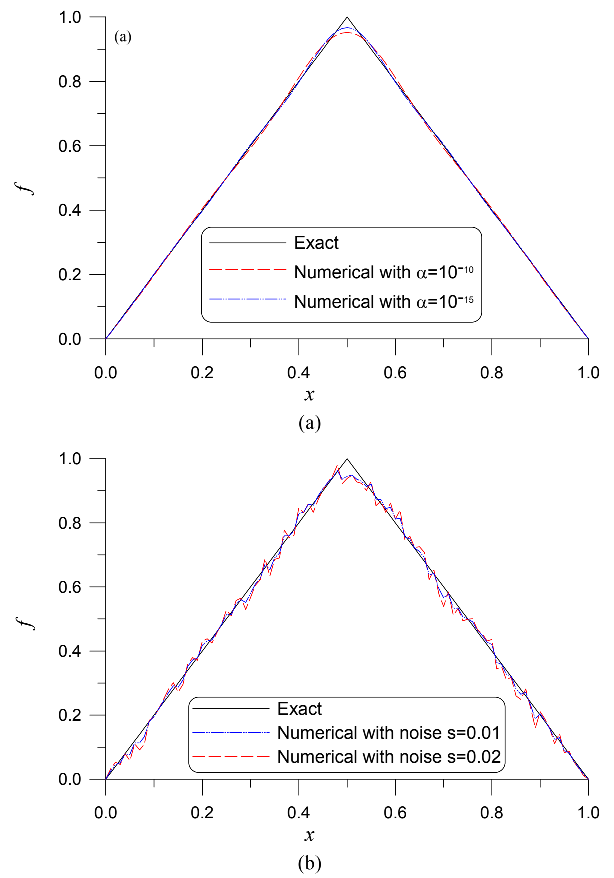


Fig. 4 For Example 2 with non-smooth data we compare regularized and exact solutions without considering noise in (a), and with noises in (b)

with the exact data in Fig. 4(b). It can be seen that the noises disturb the numerical solutions deviating from the exact solution quite small. Also it can be seen that the present approach is robust against the noise, even the noise is large up to 0.02.

Example 3. In a practical use the data may be offered through measurement and thus some numerical integrations to obtain the regularization coefficients are required. In this case we test the last problem:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\,,\ \ 0<x<\pi\,,\ \ 0<y<b\,, \tag{69}$$

$$u(x,b)=0\,,\ \ -\frac{\partial u}{\partial y}(x,b)=h(x)=\sin(x)+\frac{1}{10}\sin(10x)\,,\ \ 0\le x\le\pi\,, \tag{70}$$

$$u(0,y)=0\,,\ u(\pi,y)=0,\ \ 0\le y\le b\,, \tag{71}$$

where b is fixed to be b = 1. First the data of $h(x)$ are sampled at the nodal points, which are subjected to random noise:

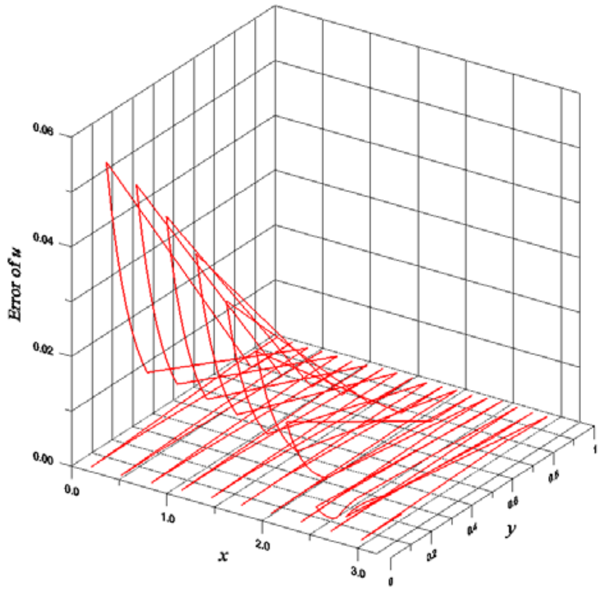
$$\hat{h}(x_i)=h(x_i)+sR(i)\,, \tag{72}$$

$$x_i=(i-1)\pi/100\,, \tag{73}$$

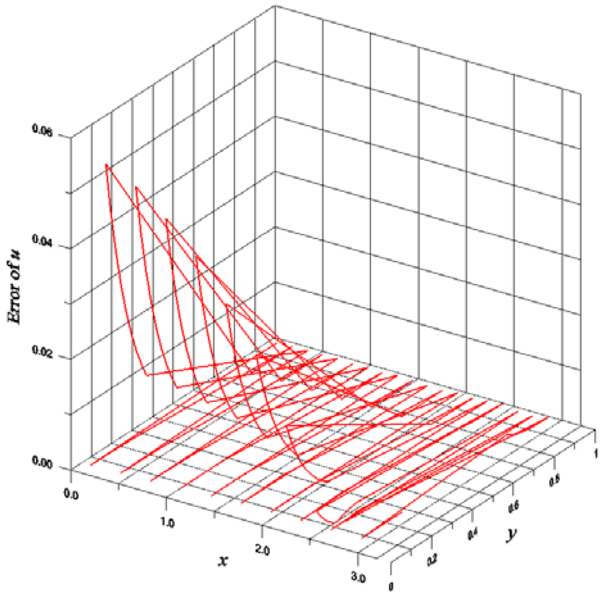
where s = 0.001. Then we insert $\hat{h}(x_i)$ into Eq. (39) to obtain the coefficients a_k^α by using the trapezoidal quadrature. When a_k^α are available, from Eq. (40) we can compute the regularization solution. Upon comparing with the exact solution:

$$u_e(x,y)=\sin(x)\sinh[b-y]+\sin(10x)\sinh\left[10(b-y)\right], \tag{74}$$

we show the relative numerical errors in Figs. 5(a) and 5(b), respectively, for α = 0 and $\alpha = 10^{-6}$. We found that no matter which α are used the numerical solution is rather accurate with the maximum relative error smaller than 5.64×10^{-2}. In a real practice, if the datum h includes higher modes in its Fourier series we suggest to use very small α.



(a)



(b)

Fig. 5 For Example 3 with different regularization parameters the relative numerical errors are compared

7. CONCLUSIONS

The present paper has displayed a new analytical method to solve the inverse Cauchy problem of Laplace equation by a first-kind Fredholm integral equation. For the sake of analyticity, we were restricted ourselves to a simple rectangular domain for the frequent use of rectangular domain in engineering structure. Because of the use of separable kernel it can avoid the singularity, which might occur if one would employ the fundamental solution in the integral equation. The ill-posedness of the inverse Cauchy problem is fully reflected in the first-kind Fredholm integral equation. Fortunately, by employing the Fourier series expansion technique and a termwise separable property of kernel function, an analytical regularization solution for approximating the real solution can be derived exactly. The influence of regularization parameter on the perturbed solution is clear. The regularization solution was shown to be uniformly convergent to the exact solution and the error estimation was provided. In a practical use the data may be offered through measurement and thus some numerical integrations to obtain the regularization coefficients are required. When the input data have higher modes the best choice of regularization parameter must be a small number. The numerical examples have shown that the new method could retrieve very well the missing boundary data, and the numerical results against the disturbance of noise are rather better.

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