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MECHANICS

Exact Solutions to the Navier–Stokes Equations with Generalized Separation of Variables

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In this paper, we find new multiparameter families of exact solutions (among them, periodic solutions) to the steady-state and unsteady Navier–Stokes equations. We also construct more general solutions depending on one or several arbitrary functions. Various modifications of the method of generalized separation of variables are employed for finding the exact solutions.

Self-similar and invariant solutions to the Navier–Stokes equations were considered in [1–6]. A number of exact solutions to nonlinear heat-conduction equations and other nonlinear equations of the second order with the generalized separation of variables were given in [5, 7–10].

1. EQUATION FOR THE STREAM FUNCTION

The two-dimensional nonstationary equations for a viscous incompressible fluid,

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u_1,$$

$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta u_2,$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

can be reduced to a nonlinear equation of the fourth order for the stream function w introduced by the formulas $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$ (with the subsequent elimination of the pressure from the first two equations by cross differentiation):

$$\frac{\partial}{\partial t}(\Delta w) + \frac{\partial w}{\partial y}\frac{\partial}{\partial x}(\Delta w) - \frac{\partial w}{\partial x}\frac{\partial}{\partial y}(\Delta w) = \nu\Delta\Delta w,$$
$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}. \tag{1}$$

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2. EXACT SOLUTIONS WITH THE GENERALIZED SEPARATION OF VARIABLES

New exact solutions to Eq. (1) with the generalized (incomplete) separation of its variables are described below. These solutions are sought in the form of finite sums,

$$w(x, y, t) = \sum_{k=1}^{n} f_k(x)g_k(y, t)$$

or

$$w(x, y, t) = \sum_{k=1}^{n} f_k(x, t)g_k(y),$$

where the functions $f_k(x)$ and $g_k(y, t)$ [or $f_k(x, t)$ and $g_k(y)$] should be chosen to satisfy the equation under consideration. For nonlinear equations, in contrast to linear ones, the functions $g_k(y, t)$ with different subscripts k are related to each other and to the functions $f_m(x)$.

We now consider the simplest case when a set of the functions depending on the coordinate [for example, $f_k(x)$] is described by linear differential equations with constant coefficients. In this paper, we use the most widespread solutions to such equations,

$$f_k(x) = x^k, \quad f_k(x) = e^{\lambda_k x},$$

$$f_k(x) = \sin(\alpha_k x), \quad f_k(x) = \cos(\beta_k x),$$

and their linear superpositions in order to find exact solutions to Eq. (1) (here, λ_k, α_k, and β_k are free parameters). Another set of the functions, $g_k(y, t)$, is determined by solving the corresponding nonlinear equations.

Remark. Solutions with another generalized separation of variables is given in Sections 3 (2°) and 4 (2° and 9°).

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3. STEADY-STATE SOLUTIONS IN THE CARTESIAN AND POLAR COORDINATE SYSTEMS

1°. There are the exact solutions with the generalized separation of variables:

$$w(x, y) = 6\nu x(y+\lambda)^{-1} + A(y+\lambda)^3 + B(y+\lambda)^{-1} + C(y+\lambda)^{-2} + D \quad (\nu \neq 0),$$

$$w(x, y) = (Ax + B)e^{-\lambda y} + \nu\lambda x + C,$$

$$w(x, y) = A\exp(-\lambda x) + B\exp(-\lambda y) + \nu\lambda(x - y) + C,$$

$$w(x, y) = A\exp(\lambda x) + B\exp(-\lambda y) + \nu\lambda(x + y) + C,$$

$$w(x, y) = [A\sinh(\beta x) + B\cosh(\beta x)]e^{-\lambda y} + \frac{\nu}{\lambda}(\beta^2 + \lambda^2)x + C,$$

$$w(x, y) = [A\sin(\beta x) + B\cos(\beta x)]e^{-\lambda y} + \frac{\nu}{\lambda}(\lambda^2 - \beta^2)x + C,$$

$$w(x, y) = Ae^{\lambda y + \beta x} + Be^{\gamma x} + \nu\gamma y + \frac{\nu}{\lambda}\gamma(\beta - \gamma)x + C,$$

$$\gamma = \pm\sqrt{\lambda^2 + \beta^2},$$

where A, B, C, D, β, and λ are arbitrary constants.

Taking in the second solution $A = -\nu\lambda$, $B = C = 0$, and $\lambda = \sqrt{\frac{k}{\nu}}$, we obtain

$$w = \sqrt{k\nu}\,x\left[1 - \exp\left(-\sqrt{\frac{k}{\nu}}y\right)\right].$$

This solution describes a steady-state fluid flow caused by the motion of the surface points $y = 0$ with the velocity $u_1|_{y=0} = kx$.

2°. There exists the more general exact solution with incomplete separation of its variables:

$$w(x, y) = F(z)x + G(z), \quad z = y + kx.$$

Here, the functions $F = F(z)$ and $G = G(z)$ are described by the system of ordinary differential equations of the fourth order,

$$F'_z F''_z - FF'''_{zzz} = \nu(k^2+1)F''''_{zzzz}, \tag{2}$$

$$G'_z F''_{zz} - FG'''_{zzz} = \nu(k^2+1)G''''_{zzzz} + 4k\nu F'''_{zzz} + \frac{2k}{(k^2+1)}FF''_{zz}. \tag{3}$$

Integrating these equations, we obtain the new system of equations of the third order,

$$(F'_z)^2 - FF''_{zz} = \nu(k^2+1)F'''_{zzz} + A, \tag{4}$$

$$G'_z F'_z - FG''_{zz} = \nu(k^2+1)G'''_{zzz} + \psi(z) + B, \tag{5}$$

where A and B are arbitrary constants and the function $\psi(z)$ is determined by the formula

$$\psi(z) = 4k\nu F''_{zz} + \frac{2k}{k^2+1}\int FF''_{zz}dz.$$

The order of autonomous equation (4) can be lowered by unity.

Equation (2) has the following particular solutions:

$$F(z) = az + b, \quad z = y + kx,$$

$$F(z) = 6\nu(k^2+1)(z+a)^{-1},$$

$$F(z) = ae^{-\lambda z} + \lambda\nu(k^2+1),$$

where a, b, and λ are arbitrary constants.

In general, the substitution $U = G'_z$ reduces Eq. (5) to a linear inhomogeneous equation of the second order, which has a nontrivial particular solution in the case of $\psi = B = 0$ (i.e., in the homogeneous case):

$$U = \begin{cases} F''_{zz}, & \text{if } F''_{zz} \neq 0 \\ F, & \text{if } F''_{zz} = 0. \end{cases}$$

Hence, its general solution can be expressed in terms of quadratures [11, 12].

3°. There is a solution with the generalized separation of variables in the polar coordinate system:

$$w(r, \theta) = f(r)\theta + g(r).$$

Here, $x = r\cos\theta$, $y = r\sin\theta$, and the functions $f = f(r)$ and $g = g(r)$ satisfy the system of ordinary differential equations

$$-f'_r\mathbf{L}(f) + f[\mathbf{L}(f)]'_r = \nu r\mathbf{L}^2(f), \tag{6}$$

$$-g'_r\mathbf{L}(f) + f[\mathbf{L}(g)]'_r = \nu r\mathbf{L}^2(g), \tag{7}$$

where $\mathbf{L}(f) = r^{-1}(rf'_r)'_r$.

The exact solution to Eqs. (6) and (7) takes the form

$$f(r) = C_1\ln r + C_2, \quad g(r) = C_3r^2 + C_4\ln r + C_5\int\left[\int rQ(r)dr\right]\frac{dr}{r} + C_6,$$

$$Q(r) = \int r^{(C_2/\nu)-1}\exp\left(\frac{C_1}{2\nu}\ln^2 r\right)dr,$$

where C_1, C_2, C_3, C_4, C_5, and C_6 are arbitrary constants.

4. UNSTEADY SOLUTIONS IN THE CARTESIAN AND POLAR COORDINATE SYSTEMS

1°. There is an exact solution with an incomplete separation of variables:

$$w(x, y, t) = F(y, t)x + G(y, t). \tag{8}$$

Here, the functions $F = F(y, t)$ and $G = G(y, t)$ are determined by the system of one-dimensional equations of the fourth order:

$$\frac{\partial^3 F}{\partial t \partial y^2} + \frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial y^2} - F\frac{\partial^3 F}{\partial y^3} = \nu\frac{\partial^4 F}{\partial y^4}, \tag{9}$$

$$\frac{\partial^3 G}{\partial t \partial y^2} + \frac{\partial G}{\partial y}\frac{\partial^2 F}{\partial y^2} - F\frac{\partial^3 G}{\partial y^3} = \nu\frac{\partial^4 G}{\partial y^4}. \tag{10}$$

Equation (9) is solved independently of Eq. (10). Integrating Eqs. (9) and (10) over y yields

$$\frac{\partial^2 F}{\partial t \partial y} + \left(\frac{\partial F}{\partial y}\right)^2 - F\frac{\partial^2 F}{\partial y^2} = \nu\frac{\partial^3 F}{\partial y^3} + f_1(t), \tag{11}$$

$$\frac{\partial^2 G}{\partial t \partial y} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} - F\frac{\partial^2 G}{\partial y^2} = \nu\frac{\partial^3 G}{\partial y^3} + f_2(t). \tag{12}$$

Here, $f_1(t)$ and $f_2(t)$ are arbitrary functions. Equation (12) is linear in the function G. After performing the substitution

$$G = \int U dy - hF + h'_t y, \qquad U = U(y, t), \quad F = F(y, t), \tag{13}$$

with the function $h = h(t)$ satisfying the linear ordinary differential equation

$$h''_{tt} - f_1(t)h = f_2(t), \tag{14}$$

equation (12) is reduced to the linear homogeneous equation of the second order,

$$\frac{\partial U}{\partial t} = \nu\frac{\partial^2 U}{\partial y^2} + F\frac{\partial U}{\partial y} - \frac{\partial F}{\partial y}U. \tag{15}$$

Thus if a particular solution to Eq. (9) or Eq. (11) is known, the determination of the function G is reduced to solving linear equations (14) and (15) with the subsequent integration in formula (13).

The exact solutions to Eq. (9) are listed in Table 1. The ordinary differential equations presented in the two last lines of Table 1 have the traveling-wave solution and the self-similar solution. These equations are autonomous; hence, their order can be lowered.

The general solution to the inhomogeneous equation (14) is found with the help of a fundamental set of solutions to the corresponding homogeneous equation (with $f_2 \equiv 0$). Necessary formulas and the fundamental solutions to homogeneous equation (14), which correspond to all exact solutions listed in Table 1, can be found in handbooks [11, 12].

For arbitrary function $F = F(y, t)$, Eq. (15) has a trivial solution. The expressions in Table 1 and formula (13) with $U = 0$ describe certain exact solutions of the form (8). A wider class of exact solutions can be obtained if nontrivial solutions to Eq. (15) are considered.

In Table 2, we present transformations that simplify Eq. (15) for some of the solutions to Eq. (9) [or (11)] listed in Table 1. It is seen that in the first two cases the solutions to Eq. (15) are expressed in terms of solutions to the conventional heat-conduction equation with constant coefficients. In the other three cases, Eq. (15) is reduced to an equation in separable variables.

2°. There is a more general exact solution with incomplete separation of variables:

$$w(x, y, t) = F(\xi, t)x + G(\xi, t), \quad \xi = y + kx.$$

Here, the functions $F(\xi, t)$ and $G = G(\xi, t)$ are determined from the system of one-dimensional equations of the fourth order:

$$\frac{\partial^3 F}{\partial t \partial \xi^2} + \frac{\partial F}{\partial \xi}\frac{\partial^2 F}{\partial \xi^2} - F\frac{\partial^3 F}{\partial \xi^3} = \nu(k^2+1)\frac{\partial^4 F}{\partial \xi^4}, \tag{16}$$

$$\frac{\partial^3 G}{\partial t \partial \xi^2} + \frac{\partial G}{\partial \xi}\frac{\partial^2 F}{\partial \xi^2} - F\frac{\partial^3 G}{\partial \xi^3} = \nu(k^2+1)\frac{\partial^4 G}{\partial \xi^4} + 4\nu k\frac{\partial^3 F}{\partial \xi^3} + \frac{2k}{k^2+1}\left(F\frac{\partial^2 F}{\partial \xi^2} - \frac{\partial^2 F}{\partial t \partial \xi}\right). \tag{17}$$

Integrating Eqs. (16) and (17) over ξ, we arrive at

$$\frac{\partial^2 F}{\partial t \partial \xi} + \left(\frac{\partial F}{\partial \xi}\right)^2 - F\frac{\partial^2 F}{\partial \xi^2} = \nu(k^2+1)\frac{\partial^3 F}{\partial \xi^3} + f_1(t), \tag{18}$$

$$\frac{\partial^2 G}{\partial t \partial \xi} + \frac{\partial F}{\partial \xi}\frac{\partial G}{\partial \xi} - F\frac{\partial^2 G}{\partial \xi^2} = \nu(k^2+1)\frac{\partial^3 G}{\partial \xi^3} + Q(\xi, t), \tag{19}$$

where $f_1(t)$ is an arbitrary function and $Q(\xi, t)$ is determined by the formula

$$Q(\xi, t) = 4\nu k\frac{\partial^2 F}{\partial \xi^2} - \frac{2k}{k^2+1}\frac{\partial F}{\partial t} + \frac{2k}{k^2+1}\int F\frac{\partial^2 F}{\partial \xi^2}d\xi + f_2(t),$$

with $f_2(t)$ being an arbitrary function.

Equation (19) is linear in the function G. The substitution $U = \frac{\partial G}{\partial \xi}$ reduces this equation to the linear equation of the second order

$$\frac{\partial U}{\partial t} = \nu(k^2+1)\frac{\partial^2 U}{\partial \xi^2} + F\frac{\partial U}{\partial \xi} - \frac{\partial F}{\partial \xi}U + Q(\xi, t). \tag{20}$$

Thus when a particular solution to Eq. (16) or (18) is known, the function G is determined by the linear equation (20) of the second order. With the help of the scaling of the independent variables, $\xi = (k^2 + 1)\zeta$ and

Table 1. Exact solutions to Eqs. (9) and (11). Here, $\varphi(t)$ and $\psi(t)$ are arbitrary functions, while A and λ are arbitrary constants

No.	Function $F = F(y, t)$ (the general form of solution)	Function $f_1(t)$ in Eq. (11)	Defining coefficients (or defining equation)
1	$F = \varphi(t)y + \psi(t)$	$f_1(t) = \varphi'_t + \varphi^2$	–
2	$F = \dfrac{6\nu}{y+\psi(t)} + \psi'_t(t)$	$f_1(t) = 0$	–
3	$F = A\exp[-\lambda y - \lambda\psi(t)] + \psi'_t(t) + \nu\lambda$	$f_1(t) = 0$	–
4	$F = Ae^{-\beta t}\sin[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = Be^{-2\beta t}$	$\beta = \nu\lambda^2$, $B = A^2\lambda^2 > 0$
5	$F = Ae^{-\beta t}\cos[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = Be^{-2\beta t}$	$\beta = \nu\lambda^2$, $B = A^2\lambda^2 > 0$
6	$F = Ae^{\beta t}\sinh[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = Be^{2\beta t}$	$\beta = \nu\lambda^2$, $B = A^2\lambda^2 > 0$
7	$F = Ae^{\beta t}\cosh[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = Be^{2\beta t}$	$\beta = \nu\lambda^2$, $B = -A^2\lambda^2 < 0$
8	$F = F(\xi)$, $\xi = y + \lambda t$	$f_1(t) = A$	$-A + \lambda F''_{\xi\xi} + (F'_\xi)^2 - FF''_{\xi\xi} = \nu F'''_{\xi\xi\xi}$
9	$F = t^{-1/2}[H(\xi) - \frac{1}{2}\xi]$, $\xi = yt^{-1/2}$	$f_1(t) = At^{-2}$	$\frac{3}{4} - A - 2H'_\xi + (H'_\xi)^2 - HH''_{\xi\xi} = \nu H'''_{\xi\xi\xi}$

Table 2. Transformations of Eq. (15) for corresponding exact solutions to Eq. (11). The numbers in the first column correspond to the numbers of the exact solutions $F = F(y, t)$ in Table 1

No.	Transformation of Eq. (15)	The equation obtained
1	$U = \dfrac{1}{\Phi(t)}u(z,\tau)$, $\tau = \int\Phi^2(t)dt$, $z = y\Phi(t) + \int\psi(t)\Phi(t)dt$, $\Phi(t) = \exp[\int\varphi(t)dt]$	$\dfrac{\partial u}{\partial\tau} = \nu\dfrac{\partial^2 u}{\partial z^2}$
2	$U = \zeta^{-3}u(\zeta, t)$, $\zeta = y + \psi(t)$	$\dfrac{\partial u}{\partial t} = \nu\dfrac{\partial^2 u}{\partial\zeta^2}$
3	$U = e^{\eta}Z(\eta, t)$, $\eta = -\lambda y - \lambda\psi(t)$	$\dfrac{\partial Z}{\partial t} = \nu\lambda^2\dfrac{\partial^2 Z}{\partial\eta^2} + (\nu\lambda^2 - A\lambda e^{\eta})\dfrac{\partial Z}{\partial\eta}$
8	$U = u(\xi, t)$, $\xi = y + \lambda t$	$\dfrac{\partial u}{\partial t} = \nu\dfrac{\partial^2 u}{\partial\xi^2} + [F(\xi) - \lambda]\dfrac{\partial u}{\partial\xi} - F'_\xi(\xi)u$
9	$U = t^{-1/2}u(\xi,\tau)$, $\xi = yt^{-1/2}$, $\tau = \ln t$	$\dfrac{\partial u}{\partial\tau} = \nu\dfrac{\partial^2 u}{\partial\xi^2} + H(\xi)\dfrac{\partial u}{\partial\xi} + [1 - H'_\xi(\xi)]u$

$t = (k^2 + 1)\tau$, Eq. (16) is reduced to Eq. (9), in which ζ and τ should be substituted for y and t (exact solutions to Eq. (9) are described in Table 1).

3°. There is an exact solution [a particular case of solutions taking form (8)]

$$w(x, y, t) = e^{-\lambda y}[f(t)x + g(t)] + \varphi(t)x + \psi(t)y + \chi(t),$$

$$f(t) = C_1E(t), \quad E(t) = \exp[\nu\lambda^2 t - \lambda\int\varphi(t)dt],$$

$$g(t) = C_2E(t) - C_1E(t)\int\psi(t)dt.$$

Here, $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are arbitrary functions, and C_1, C_2, and λ are arbitrary parameters.

4°. There is an exact solution

$$w(x, y, t) = e^{-\lambda y}[A(t)e^{\beta x} + B(t)e^{-\beta x}] + \varphi(t)x + \psi(t)y + \chi(t),$$

$$A(t) = C_1\exp[\nu(\lambda^2 + \beta^2)t - \beta\int\psi(t)dt - \lambda\int\varphi(t)dt],$$

$$B(t) = C_2\exp[\nu(\lambda^2 + \beta^2)t + \beta\int\psi(t)dt - \lambda\int\varphi(t)dt],$$

in which $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are arbitrary functions and C_1, C_2, λ, and β are arbitrary parameters.

5°. There is an exact solution

$$w(x, y, t) = e^{-\lambda y}[A(t)\sin(\beta x) + B(t)\cos(\beta x)] + \varphi(t)x + \psi(t)y + \chi(t).$$

Here, $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are arbitrary functions, λ and β are arbitrary parameters, and the functions $A(t)$ and $B(t)$ satisfy the linear nonautonomous system of ordinary differential equations

$$\begin{aligned} A'_t &= [\nu(\lambda^2 - \beta^2) - \lambda\varphi(t)]A + \beta\psi(t)B, \\ B'_t &= [\nu(\lambda^2 - \beta^2) - \lambda\varphi(t)]B - \beta\psi(t)A. \end{aligned} \tag{21}$$

The general solution to system (21) takes the form

$$A(t) = \exp[\nu(\lambda^2 - \beta^2)t - \lambda\int\varphi dt] \times [C_1\sin(\beta\int\psi dt) + C_2\cos(\beta\int\psi dt)],$$

$$B(t) = \exp[\nu(\lambda^2 - \beta^2)t - \lambda\int\varphi dt] \times [C_1\cos(\beta\int\psi dt) - C_2\sin(\beta\int\psi dt)],$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, and C_1 and C_2 are arbitrary constants. In particular, if $\varphi = \frac{\nu}{\lambda}(\lambda^2 - \beta^2)$ and $\psi = a$, we obtain the periodic solution

$$A(t) = C_1\sin(a\beta t) + C_2\cos(a\beta t),$$

$$B(t) = C_1\cos(a\beta t) - C_2\sin(a\beta t).$$

6°. There are exact solutions

$$w(x, y, t) = A(t)\exp(k_1x + \lambda_1y) + B(t)\exp(k_2x + \lambda_2y) + \varphi(t)x + \psi(t)y + \chi(t),$$

in which $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are arbitrary functions; k_1, λ_1, k_2, and λ_2 are arbitrary parameters related by one of the two equations

$k_1^2 + \lambda_1^2 = k_2^2 + \lambda_2^2$ (the first family of the solutions),

$k_1\lambda_2 = k_2\lambda_1$ (the second family of the solutions),

and the functions $A(t)$ and $B(t)$ satisfy the linear ordinary differential equations

$$A'_t = [\nu(k_1^2 + \lambda_1^2) + \lambda_1\varphi(t) - k_1\psi(t)]A,$$

$$B'_t = [\nu(k_2^2 + \lambda_2^2) + \lambda_2\varphi(t) - k_2\psi(t)]B.$$

These equations are easily solved:

$$A(t) = C_1\exp[\nu(k_1^2 + \lambda_1^2)t + \lambda_1\int\varphi(t)dt - k_1\int\psi(t)dt],$$

$$B(t) = C_2\exp[\nu(k_2^2 + \lambda_2^2)t + \lambda_2\int\varphi(t)dt - k_2\int\psi(t)dt].$$

7°. There is an exact solution

$$w(x, y, t) = [C_1\sin(\lambda x) + C_2\cos(\lambda x)] \times [A(t)\sin(\beta y) + B(t)\cos(\beta y)] + \varphi(t)x + \chi(t).$$

Here, $\varphi(t)$ and $\chi(t)$ are arbitrary functions, C_1, C_2, λ, and β are arbitrary parameters, and the functions $A(t)$ and $B(t)$ satisfy the linear non-autonomous system of ordinary differential equations

$$\begin{aligned} A'_t &= -\nu(\lambda^2 + \beta^2)A - \beta\varphi(t)B, \\ B'_t &= -\nu(\lambda^2 + \beta^2)B + \beta\varphi(t)A. \end{aligned} \tag{22}$$

The general solution to system (22) takes the form

$$A(t) = \exp[-\nu(\lambda^2 + \beta^2)t] \times [C_3\sin(\beta\int\varphi dt) + C_4\cos(\beta\int\varphi dt)], \quad \varphi = \varphi(t),$$

$$B(t) = \exp[-\nu(\lambda^2 + \beta^2)t] \times [-C_3\cos(\beta\int\varphi dt) + C_4\sin(\beta\int\varphi dt)],$$

where C_3 and C_4 are arbitrary constants.

8°. There is an exact solution

$$w(x, y, t) = [C_1\sinh(\lambda x) + C_2\cosh(\lambda x)] \times [A(t)\sin(\beta y) + B(t)\cos(\beta y)] + \varphi(t)x + \chi(t),$$

in which $\varphi(t)$ and $\chi(t)$ are arbitrary functions, C_1, C_2, λ, and β are arbitrary parameters, and the functions $A(t)$ and $B(t)$ satisfy the linear non-autonomous system of ordinary differential equations

$$\begin{aligned} A'_t &= \nu(\lambda^2 - \beta^2)A - \beta\varphi(t)B, \\ B'_t &= \nu(\lambda^2 - \beta^2)B + \beta\varphi(t)A. \end{aligned} \tag{23}$$

The general solution to system (23) takes the form

$$A(t) = \exp[\nu(\lambda^2 - \beta^2)t] \times [C_3\sin(\beta\int\varphi dt) + C_4\cos(\beta\int\varphi dt)], \quad \varphi = \varphi(t),$$

$$B(t) = \exp[\nu(\lambda^2 - \beta^2)t] \times [-C_3\cos(\beta\int\varphi dt) + C_4\sin(\beta\int\varphi dt)],$$

where C_3 and C_4 are arbitrary constants.

9°. There is an exact solution

$$w(x, y, t) = u(z, t) + \varphi(t)x + \psi(t)y, \quad z = kx + \lambda y.$$

Here $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary parameters, and the function $u(z, t)$ satisfies the linear differential equation of the fourth order:

$$\frac{\partial^3 u}{\partial t\partial z^2} + [k\psi(t) - \lambda\varphi(t)]\frac{\partial^3 u}{\partial z^3} = \nu(k^2 + \lambda^2)\frac{\partial^4 u}{\partial z^4}.$$

The transformation

$$U(\xi, t) = \frac{\partial^2 u}{\partial z^2}, \quad \xi = z - \int [k\psi(t) - \lambda\varphi(t)]dt$$

reduces this equation to the conventional heat-conduction equation

$$\frac{\partial U}{\partial t} = \nu(k^2 + \lambda^2)\frac{\partial^2 U}{\partial \xi^2}.$$

10°. There is a solution with the generalized separation of variables in the polar coordinate system:

$$w(r, \theta, t) = f(r, t)\theta + g(r, t).$$

Here, $x = r\cos\theta$ and $y = r\sin\theta$, the functions $f = f(r, t)$ and $g = g(r, t)$ satisfy the system of equations

$$\mathbf{L}(f_t) - r^{-1} f_r \mathbf{L}(f) + r^{-1} f[\mathbf{L}(f)]_r = \nu \mathbf{L}^2(f), \quad (24)$$

$$\mathbf{L}(g_t) - r^{-1} g_r \mathbf{L}(f) + r^{-1} f[\mathbf{L}(g)]_r = \nu \mathbf{L}^2(g), \quad (25)$$

the subscripts r and t imply the corresponding partial derivatives, and

$$\mathbf{L}(f) = r^{-1}(r f_r)_r, \quad \mathbf{L}^2(f) = \mathbf{L}\mathbf{L}(f).$$

For the particular solution $f = \varphi(t)\ln r + \psi(t)$ to Eq. (24) (φ and ψ are arbitrary functions), Eq. (25) is reduced to a linear equation of the second order by the substitution $U = \mathbf{L}(g)$.

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