Quantification d'incertitudes

Part IV
Approximation of models

Forward and inverse problems in uncertainty quantification

Consider a (numerical or experimental) model depending on a set of random parameters $X = (X_1, \dots, X_d)$ that describe the uncertainties on the model, and let Y be some output variable of interest

$$Y = f(X)$$

- Propagation of uncertainties (forward problems): given the probability density p of X, evaluation of statistics, probability of events, sensitivity indices...

$$\mathbb{E}(h(Y)) = \mathbb{E}(h \circ f(X)) = \int h(f(x)) d\mu(x)$$

- Inverse problems (inference, data assimilation): from (partial) observations of Y, estimate the probability distribution μ of X

Approximation of models

Solving forward and inverse problems requires the evaluation of the model for many instances of X.

This is usually unaffordable when one evaluation requires a costly numerical simulation (or experiment).

In practice, we rely on approximations of the map

$$X \mapsto f(X)$$

which are used as predictive surrogate models (metamodels, reduced order models).

An approximation $\tilde{Y} = \tilde{f}(X)$ of $Y = f(X)$ can be directly used for obtaining an approximation of a quantity of interest, with a control of errors.
For example, for h a L-Lipschitz function

$$|\mathbb{E}(h(Y)) - \mathbb{E}(h(\tilde{Y}))| \le L \int |f(x) - \tilde{f}(x)| d\mu(x) = L\|f - \tilde{f}\|_{L^1_\mu}.$$

Also, it can be used to design variance reduction methods for Monte-Carlo methods, e.g. as a control variate

$$\mathbb{E}(Y) \approx \mathbb{E}(\tilde{Y}) + \frac{1}{n}\sum_{k=1}^{n}(f(X_k) - \tilde{f}(X_k)) := \hat{I}_n,$$

$$\mathbb{E}(|\hat{I}_n - \mathbb{E}(Y)|^2) = \mathbb{V}(\hat{I}_n) \le \frac{1}{n}\|f - \tilde{f}\|^2_{L^2_\mu}.$$

Approximation of models

The construction and use of surrogate models requires

- the choice of suitable approximation tools (polynomials, wavelets, kernels...),
- algorithms for constructing approximations from available information: model equations, pointwise evaluations (black box),
- a control of the error.

In this chapter, we will introduce some elements of approximation theory, present classical approximation tools and algorithms for the construction of approximations.