Some novel numerical techniques for an inverse Cauchy problem

A. Nachaoui[a], M. Nachaoui[1b], A. Chakib[b], M. A. Hilal[c]

[a]*Laboratoire de Mathématiques Jean Leray UMR6629 CNRS / Université de Nantes 2 rue de la Houssinière, BP92208 44322 Nantes, France*
[b]*Laboratoire de Mathématiques et Applications Université Sultan Moulay Slimane, Faculté des Sciences et Techniques, B.P.523, Béni-Mellal, Maroc.*
[c]*Middle Technical University, Baghdad, Iraq.*

Abstract

In this paper, we are interested in solving an elliptic inverse Cauchy problem. As it's well known this problem is one of highly ill posed problem in Hadamrd's sense [16]. We first establish formally a relationship between the Cauchy problem and an interface problem illustrated in a rectangular structure divided into two domains. This relationship allows us to use classical methods of non-overlapping domain decomposition to develop some regularizing and stable algorithms for solving elliptic inverse Cauchy problem. Taking advantage of this relationship we reformulate this inverse problem into a fixed point one, based on Steklov-Poincaré operator. Thus, using the topological degree of Leray-Schauder we show an existence result. Finally, the efficiency and the accuracy of the developed algorithms are discussed.

Keywords: Inverse Cauchy problem, Domain decomposition, Iterative algorithms, Steklov-Poincaré operator, Finite elements, Numerical simulation.

1. Introduction

Mathematical modelling of many natural phenomena and technological processes leads, in general, to boundary problems governed by partial differential equations. The resolution of these equations requires the knowledge of the geometry of the domain, the coefficients appear in the equations, as well as the boundary and initial conditions. However, in practice, these parameters are, in most cases, unknown, inaccessible or poorly measured. It is then necessary to identify some of these parameters. These kind of questions belong to a class of prblems called inverse problems. There are many inverse problems that appear in large number of engineering applications [1, 4, 15, 27, 28]. A classical example of inverse problems is the inverse Cauchy problem. This class of problems is encountered in many fields such as thermal inspection, electrical prospecting, geophysics and medical imaging and so on [8, 14, 24, 27]. In these situations, boundary conditions for both the solution and its normal derivative are prescribed only on a part of the boundary of the solution domain, whilst no information is available on a part of the boundary due to physical difficulties or the inaccessibility of geometry. Consequently, a special treatment of these problems is required. Indeed, unlike to the direct problems which are generally well posed, these inverse problems are ill-posed in Hadamard's sense [16]. This gives great importance to their formulation. One of the main difficulties is the lack of the stability, which is very important for the numerical resolution.Despite the complexity of these class of problems, it still attracting the interest of the scientific commonality. This explains the intense and immense contributions in the literature, as well as various approaches, to the theoretical and numerical solutions. For the inverse Cauchy problems, several numerical methods has been investigated in the last three decades. We mention the method of quasi-reversibility, first introduced by Lions and al. [6, 11, 12, 19, 21, 23, 22], to solve the ill-posed Cauchy problem with elliptic operator. The main idea of this method consists in transforming the ill-posed second-order initial problem into a family (depending on a small parameter ϵ) of fourth-order problems which are well posed. Some iterative methods are based on the use of a sequence of well-posed problems [9, 10, 17, 18, 20, 5, 25] and others on the minimization of a cost functional [2, 7, 8].

In this paper, we introduce new algorithms to solve the inverse Cauchy problem. These algorithms that we call domain decomposition-like methods are based on the reformulation of the Cauchy problem as a fixed point problem. For this, we start

[1]corresponding author: nachaoui@gmail.com

first by the following remark: consider an elliptic problem defined on a rectangular domain divided into two subdomains with interface conditions. By bending the domain at the interface and by aligning the two subdomains, this problem is formally reduced to solve an inverse Cauchy problem. This remark suggests that non-overlapping domain decomposition methods can be used for solving the Cauchy problem. An application of Dirichlet-Neumann method consolidates this idea, since the algorithm induced by this method is exactly the KMF algorithm (Kozlov-Mazya-Fomin [20]), widely used in the literature, especially its relaxed version, enabling the acceleration of the convergence, proposed by Jourhmane and Nachaoui [17, 18].

At this stage, taking advantage of this relationship we reformulate the inverse Cauchy problem into a problem of research of the inverse of an operator denoted by S obtained using Steklov-Poincaré operators [26]. However, the techniques used to show the existence of a solution of the problem reformulating domain decomposition problems cannot be applied to the problem reformulating the Cauchy one. The difficulty lies in the fact that the two domains are on the same side and this gives rise to the loss of coerciveness of the operator S. This, confirms the ill-posedness of the Cauchy problem and explains clearly the difference between this one and domain decomposition problem which is a direct problem and consequently is well posed. Therefore, we must proceed differently to overcome this difficulty. Our strategy consists in providing an equivalent fixed point problem for another operator denoted by F. We opt for Leray-Schauder degree tools [13], to show the existence of a fixed point. For this, we show first, an a priori estimate for any fixed point of the operator F, based on a result due to [3]. Then we show the continuity and a compactness result of F. Then based on an observation of the behavior of a norm of the discrepancy on a part of the boundary of the domain between two successive approximations, we propose a new stopping criterion which is more responsive and reflects the real behavior of the error on the inaccessible part of the boundary. Thanks to the proposed stopping criterion, we have been able to significantly reduce the number of iterations of KMF and all its variants. We extended the scope of the study to include other methods derived from domain decomposition for solving this inverse problem. In this context, we have conducted a numerical study, which shows that the proposed method accurately reflects the behavior of the error on inaccessible boundary. Finally, we confirm the efficiency of the proposed approach by a numerical stability analysis. These last method are presented for the method derived from the Agoshkov-Lebedev algorithm of domain decomposition, which generalizes in some way the other algorithms.

To the best of our knowledge, this is the first time that the domain decomposition, in the way investigated in this paper to solve inverse Cauchy problems, is proposed. More concretely, our contributions in this paper are:

1. A clear connection between interface problems and inverse Cauchy problems is established.
2. This connection allows us to understand the properties of this ill-posed problem and then to propose some regularizing and stable algorithms which are efficient and fast comparing to the existing methods.
3. A new stopping criteria is proposed and its numerical benefices are confirmed.
4. Theoretical study of the convergence related to this connection is established.

The paper is organized as follows, in section 2, we establish formally the relationship between the inverse Cauchy problem and the interface problem, and we show how domain decomposition method can be used to solve Cauchy problem. In section 3, we present the setting of the inverse problem and its formulation as fixed point one. Section 4, is devoted to existence of a fixed point problem. Finally, the section 5 is devoted to the numerical study of performance and stability for some algorithms derived from the proposed approach.

2. Inverse Cauchy problem & domain decomposition method.

In order to establish the relationship between the inverse Cauchy problem and the interface problem solved using non-overlapping domain decomposition technique, we start by defining the Cauchy problem in a rectangular domain (see Figure 1). Let $\Omega =]0,1[\times]0,1[$ with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_0$, then our problem is to find u solution of the following

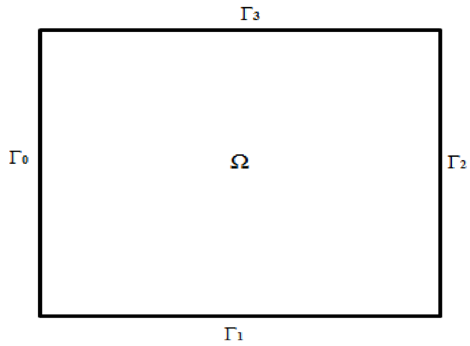

Figure 1: Geometry domain of inverse Cauchy problem

Cauchy problem on Ω :

$$\begin{cases} \mathcal{L}u = f & \text{in} \quad \Omega, \\ u = g_1 & \text{on} \quad \Gamma_1, \\ \frac{\partial u}{\partial n} = g_2 & \text{on} \quad \Gamma_2, \\ u = u_d & \text{on} \quad \Gamma_2, \\ \frac{\partial u}{\partial n} = g_3 & \text{on} \quad \Gamma_3, \end{cases} \tag{1}$$

where $\mathcal{L}$ is a generic second order elliptic operator, f, g_1, g_2, g_3, u_d are given functions, $\frac{\partial u}{\partial n} := \nabla u \cdot n$ denotes the directional derivative in the direction normal to the boundary $\partial\Omega$ conventionally pointing outwards, ∇ is the gradient operator, and u is the unknown field. As it can be inspected, in the above inverse Cauchy problem (1), the boundary conditions on the part Γ_0 are not known and must be determined from two boundary conditions on Γ_2. In order to solve the Cauchy problem (1), let us recall to the KMF algorithm [20]. For $k \geq 0$ and a given initial guess u^0, we construct two sequences $(u^{2k+1})_{k\geq 0}$ and $(u^{2k})_{k\geq 1}$ solutions of the problems (2) and (3) respectively.

$$(P_{2k+1}) \begin{cases} \mathcal{L}u^{2k+1} = f & \text{in} \quad \Omega, \\ u^{2k+1} = u_d & \text{on} \quad \Gamma_2, \\ u^{2k+1} = g_1 & \text{on} \quad \Gamma_1, \\ \frac{\partial u_1^{2k+1}}{\partial n} = g_3 & \text{on} \quad \Gamma_3, \\ u^{2k+1} = u^{2k} & \text{on} \quad \Gamma_0, \end{cases} \tag{2}$$

and

$$(P_{2k}) \begin{cases} \mathcal{L}u^{2k} = f & \text{in} \quad \Omega, \\ u^{2k} = g_1 & \text{on} \quad \Gamma_1, \\ \frac{\partial u^{2k}}{\partial n} = g_3 & \text{on} \quad \Gamma_3, \\ \frac{\partial u^{2k}}{\partial n} = g_2 & \text{on} \quad \Gamma_2, \\ \frac{\partial u^{2k}}{\partial n} = \frac{\partial u^{2k+1}}{\partial n} & \text{on} \quad \Gamma_0. \end{cases} \tag{3}$$

Now, Consider a rectangular $\Omega_d =]-1,1[\times]-1,1[$ divided into two sub-domains Ω_1 and Ω_2 (see Fig. 2), such that :

$$\partial\Omega_d = \Gamma_{11} \cup \Gamma_{12} \cup \Gamma_{22} \cup \Gamma_{32} \cup \Gamma_{31} \cup \Gamma_{21},$$

where

$$\Gamma_{11} = \{(x,y) \in \partial\Omega_1, y = 0\}, \quad \Gamma_{21} = \{(x,y) \in \partial\Omega_1, x = 1\}, \quad \Gamma_{31} = \{(x,y) \in \partial\Omega_1, y = 1\},$$
$$\Gamma_{32} = \{(x,y) \in \partial\Omega_2, y = 1\}, \quad \Gamma_{22} = \{(x,y) \in \partial\Omega_2, x = -1\}, \quad \Gamma_{12} = \{(x,y) \in \partial\Omega_1, y = 0\},$$

and the interface is defined by $\Gamma_0 = \{(x,y) \in \partial\Omega_1, /x = 0\}$. Then, the interface problem for an elliptic operator $\mathcal{L}$ defined in

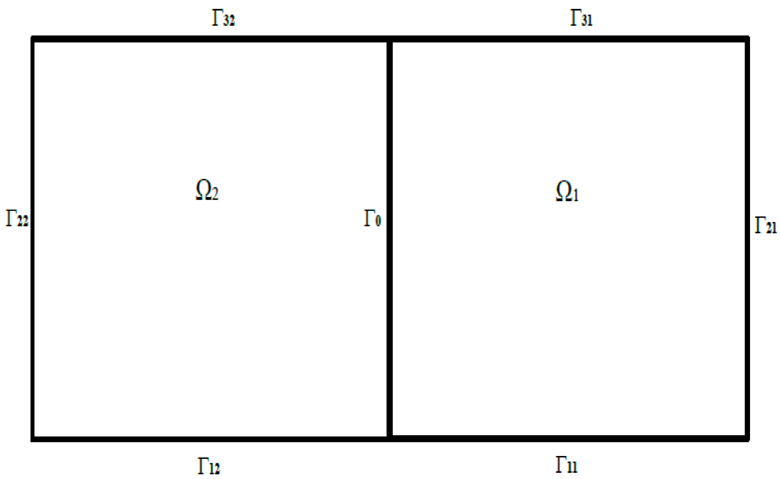

Figure 2: Non-overlapping domain decomposition

Ω_d is :

$$\begin{cases} \mathcal{L}u_i &= f \quad \text{in} \quad \Omega_i,\ i = 1,2, \\ u_1 &= g_1 \quad \text{on} \quad \Gamma_{11},\ u_2 = g_1 \quad \text{on} \quad \Gamma_{12}, \\ u_1 &= u_d \quad \text{on} \quad \Gamma_{21},\ \frac{\partial u_2}{\partial n_2} = g_2 \quad \text{on} \quad \Gamma_{22}, \\ \frac{\partial u_1}{\partial n_1} &= g_3 \quad \text{on} \quad \Gamma_{31},\ \frac{\partial u_2}{\partial n_2} = g_3 \quad \text{on} \quad \Gamma_{32}, \end{cases} \tag{4}$$

where n_i denotes the normal direction on $\partial\Omega_i \cap \Gamma_0,\ i = 1,2$, oriented outward.

In order to solve the interface problem (4), we present the classical non-overlapping domain decomposition method called Direchlet to Neumann algorithm [26]. For $k \geq 0$ and a given initial guess u_2^0, we construct two sequences $(u_1^{2k+1})_{k\geq 0}$ and $(u_2^{2k})_{k\geq 1}$ solutions of the problems (5) and (6) respectively.

$$\begin{cases} \mathcal{L}u_1^k &= f \quad \text{in} \quad \Omega_1, \\ u_1^k &= u_d \text{ on} \quad \Gamma_{21}, \\ u_1^k &= g_1 \quad \text{on} \quad \Gamma_{11}, \\ \frac{\partial u_1^{2k+1}}{\partial n_1} &= g_3 \quad \text{on} \quad \Gamma_{31}, \\ u_1^k &= u_2^k \quad \text{on} \quad \Gamma_0, \end{cases} \tag{5}$$

and

$$\begin{cases} \mathcal{L}u_2^k &= f \quad \text{in} \quad \Omega_2, \\ u_2^k &= g_1 \text{ on} \quad \Gamma_{12}, \\ \frac{\partial u_2^k}{\partial n} &= g_3 \quad \text{on} \quad \Gamma_{32}, \\ \frac{\partial u_2^k}{\partial n_2} &= g_2 \quad \text{on} \quad \Gamma_{22}, \\ \frac{\partial u_2^k}{\partial n_2} &= \frac{\partial u_1^k}{\partial n_1} \quad \text{on} \quad \Gamma_0. \end{cases} \tag{6}$$

Now let's bend the domain $\Omega_1 \cup \Omega_2$ at the interface Γ_0 and make the boundaries Γ_{ij} $i = 1\ldots,3$ $j = 1,2$ coincide. Denote the obtained domain by Ω and replace the Γ_{ij} by Γ_i, then the problem of interface becomes formally the Cauchy problem (1) and the Direchlet to Neumann algorithm can be seen as the KMF algorithm with $u_1^k = u^{2k+1}$ and $u_2^k = u^{2k},\ k \geq 0$. Note that this not means that the problems (5)-(6) and the problems (2)-(3) are the same. Indeed, the first one solve a direct a problem which is well posed and the second is an inverse problem which is ill-posed problem. This will be explained clearly in section 3 in which we will propose a new strategy to circumvent the complications that arise in both theoretical analysis and numerical approximations of Cauchy problem.

The above approach, which allowed us to find the KMF algorithm for solving the Cauchy problem from Dirichlet-Neumann non-overlapping domain decomposition method (DD), opens us ways in the study of the Cauchy problem. Indeed

1. The same approach can be applied to other DD methods to develop other algorithms for solving the Cauchy problem. We give examples in section 5.
2. This approach will allow us to develop appropriate algorithms for each type of PDE.
3. It will allow us to develop parallel algorithms and consequently faster algorithms.
4. It will also allow us to take advantage of the very rich literature on the DD methods for the study of the convergence of these algorithms.

Having introduce formally the analogy between the interface problem and the Cauchy one, in the following sections we will taking advantages of this analogy in order to understand the properties of this ill-posed problem and then, provides users some theoretical and numerical results which can efficiently help them in choosing the right numerical solver(s) to solve Cauchy problems.

In order to present the theoretical result for inverse Cauchy problem in general case and without loss of generality, in all that follows, we assume that Γ_3, the part of the boundary supporting only a Neumann condition is empty.

3. Statement and formulation of the inverse problem

3.1. Statement of the inverse problem

The considered Cauchy problem is stated as follows, let $\Omega \subset \mathbb{R}^2$ be an open bounded set with Lipschitz boundary $\partial\Omega$. Further, we assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where Γ_0, Γ_1 and Γ_2 are open disjoint parts of $\partial\Omega$. Find u solution of the following Cauchy problem:

$$(P)\left\{\begin{array}{rcccc} -\nabla.(\xi(x)\,\nabla u) & = & f & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \Gamma_1, \\ u & = & g & \text{on} & \Gamma_2, \\ \xi\frac{\partial u}{\partial n} & = & h & \text{on} & \Gamma_2, \end{array}\right. \tag{7}$$

where ξ, f, g, h and h are given functions. In the above inverse Cauchy problem (7), the boundary conditions on the part Γ_0 are not known and must be determined from two boundary conditions on Γ_2. In order to reconstruct these unknown boundary conditions based on the supplementary data provided on the other boundaries, we opt for a methodology proposed in section 2. Indeed, taking advantage of these methodology, the inverse Cauchy problem (7) can be considered as an interface problem. Then, we following step by step the procedure proposed in [26], we define the obtained interface problem in terms of the Steklov-Poincaré operator S that we are going to introduce. However, unlike to the case of domain decomposition, in the case of Cauchy problem we are face a lack of coerciveness of the operator S. This confirms the ill-posedness of such problem and makes the existence study of the fixed point more complicated. Thus, we must proceed differently. To overcome this difficulty, we propose an equivalent fixed point formulation for another operator F. At this stage, we note that since the image by F of a ball is not necessarily the same one, we cannot use the classical Schauder fixed point tools. We opt then for the degree of Leray Schauder to show the existence result [13].

3.2. Formulation of the inverse problem as a fixed point one

In this section, we will formulate the inverse Cauchy problem (7) as a fixed point one. To do this, firstly we give some definitions and assumptions on data. Then, we introduce the associated Steklov-Poincaré operators and we discuss their properties. Thus, we present the proposed fixed point formulation.

Firstly, let's suppose that $f \in L^2(\Omega)$ and $\xi \in C^1(\overline{\Omega})$, such that there exist $\alpha > 0$, satisfying

$$\xi(x) \geq \alpha \quad \text{a.e. } x \in \Omega.$$

Suppose also that

$$g \in H^{\frac{1}{2}}_{00}(\Gamma_2) \text{ and } h \in H^{-\frac{1}{2}}_{00}(\Gamma_2),$$

where the space $H^{\frac{1}{2}}_{00}(\Gamma_2)$ is defined by

$$H^{\frac{1}{2}}_{00}(\Gamma_2) = \{\varphi \in H^{\frac{1}{2}}(\Gamma_2) \ / \ \exists v \in H^1(\Omega) \text{ such that } v|_{\Gamma_2} = \varphi, \ \ v|_{\Gamma_c} = 0\},$$

with $\Gamma_c = \Gamma_1 \cup \Gamma_2$ and equipped with the norm

$$\|\varphi\|_{H^{1/2}_{00}(\Gamma_0)} = \inf_{\substack{\phi \in H^1_{\Gamma_c}(\Omega) \\ \phi|_{\Gamma_0} = \varphi}} \|\phi\|_{1,\Omega}, \tag{8}$$

where $H^{\frac{1}{2}}(\Gamma_2)$ is the space of trace function of the space $H^1(\Omega)$ and $H^1_{\Gamma_c}(\Omega)$ is the space defined by

$$H^1_{\Gamma_c}(\Omega) = \{v \in H^1(\Omega) / \quad v|_{\Gamma_c} = 0\}.$$

Denote by $H^{-\frac{1}{2}}_{00}(\Gamma_2)$ the dual space of $H^{\frac{1}{2}}_{00}(\Gamma_2)$. The norms in $L^2(\Omega)$ and $H^1(\Omega)$ are denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{1,\Omega}$ respectively.

Using the idea introduced in section 2, solving (7) can be reduced to solve an interface problem defined as follows :

$$\begin{cases} -\nabla.(\xi(x)\,\nabla u_1) & = & f & \text{in} & \Omega, \\ u_1 & = & 0 & \text{on} & \Gamma_1, \\ u_1 & = & g & \text{on} & \Gamma_2, \\ u_1 & = & u_2 & \text{on} & \Gamma_0, \end{cases} \tag{9}$$

$$\begin{cases} -\nabla.(\xi(x)\,\nabla u_2) & = & f & \text{in} & \Omega, \\ u_2 & = & 0 & \text{on} & \Gamma_1, \\ \xi\frac{\partial u}{\partial n} & = & h & \text{on} & \Gamma_2, \\ \xi\frac{\partial u_2}{\partial n} & = & \xi\frac{\partial u_1}{\partial n} & \text{on} & \Gamma_0. \end{cases} \tag{10}$$

Then, the equivalence between the problem (7) and the two problems (9) – (10) is obtained under the transmission condition (which guarantee the continuity of the solution on Γ_0) expressed by :

$$u_1 = u_2 \quad \text{and} \quad \xi\frac{\partial u_1}{\partial n} = \xi\frac{\partial u_2}{\partial n} \quad \text{on} \quad \Gamma_0. \tag{11}$$

Let φ denote the unknown value of u on Γ_0. Note that if φ is known, then the problem (7) is solved. Thus, in order to determine φ, we write u_1 as sum of two contributions

$$u_1 = H_1\varphi + G_1, \tag{12}$$

where $H_1\varphi = v_1$ is solution of :

$$\begin{cases} -\nabla.(\xi(x)\,\nabla v_1) & = & 0 & \text{in} & \Omega, \\ v_1 & = & \varphi & \text{on} & \Gamma_0, \\ v_1 & = & 0 & \text{on} & \Gamma_1, \\ v_1 & = & 0 & \text{on} & \Gamma_2, \end{cases} \tag{13}$$

and G_1 is solution of :

$$\begin{cases} -\nabla.(\xi(x)\,\nabla G_1) & = & f & \text{in} & \Omega, \\ G_1 & = & 0 & \text{on} & \Gamma_0, \\ G_1 & = & 0 & \text{on} & \Gamma_1, \\ G_1 & = & g & \text{on} & \Gamma_2. \end{cases} \tag{14}$$

In the same manner, we can write

$$u_2 = H_2\varphi + G_2, \tag{15}$$

where $H_2\varphi = v_2$ is solution of:

$$\begin{cases} -\nabla.(\xi(x)\,\nabla v_2) & = & 0 & \text{in} & \Omega, \\ v_2 & = & \varphi & \text{on} & \Gamma_0, \\ v_2 & = & 0 & \text{on} & \Gamma_1, \\ \xi\frac{\partial v_2}{\partial n} & = & 0 & \text{on} & \Gamma_2, \end{cases} \tag{16}$$

and G_2 is solution of:

$$\begin{cases} -\nabla.(\xi(x)\,\nabla G_2) & = & f & \text{in} & \Omega, \\ G_2 & = & 0 & \text{on} & \Gamma_0, \\ G_2 & = & 0 & \text{on} & \Gamma_1, \\ \xi\frac{\partial G_2}{\partial n} & = & h & \text{on} & \Gamma_2. \end{cases} \tag{17}$$

So the transmission condition (11) can be written as follows,

$$\xi\frac{\partial H_1\varphi}{\partial n} - \xi\frac{\partial H_2\varphi}{\partial n} = \xi\frac{\partial G_2}{\partial n} - \xi\frac{\partial G_1}{\partial n}.$$

Denote by S_1 an S_2 the operators defined by

$$S_1\varphi = \xi\frac{\partial H_1\varphi}{\partial n}, \qquad S_2\varphi = \xi\frac{\partial H_2\varphi}{\partial n}.$$

Then the inverse Cauchy problem can be reduced to solve the following Steklov-Poincaré kind's equation: find φ such that

$$S\varphi = \chi \quad \text{on} \quad \Gamma_0, \tag{18}$$

where

$$S = S_1 - S_2 \text{ and } \chi = \frac{\partial G_2}{\partial n} - \frac{\partial G_1}{\partial n} \quad \text{on} \quad \Gamma_0.$$

Remark 1. *Recall that in the case of domain decomposition the Stekolov-Poincaré operator is defined by $S = S_1 + S_2$. As it can be seen in the case of the Cauchy problem this operator is defined with a sign $(-)$. Note that this sign $(-)$ changes drastically the nature of the problem and induces dramatic complications in both theoretical analysis and numerical handling. These observations give us an explanation on the ill-posedness of the Cauchy problem.*

In order to prove the existence of the solution of the Steklov-Poincaré equation, let's give some proprieties of the operator S_i, for $i = 1, 2$.

Lemma 1. *(i) For all $i = 1, 2$, there exist two constants strictly positives C_1 and C_2, such that*

$$C_1\|\varphi\|_{H^{1/2}_{00}(\Gamma_0)} \le \|H_i\varphi\|_{1,\Omega} \le C_2\|\varphi\|_{H^{1/2}_{00}(\Gamma_0)}, \quad \forall\varphi \in H^{1/2}_{00}(\Gamma_0), \tag{19}$$

(ii) For all $i = 1, 2$, the operator S_i defined from $H^{1/2}_{00}(\Gamma_0)$ to $H^{-1/2}_{00}(\Gamma_0)$ is symmetric, continuous and coercive.

Proof. The left inequality of assertion (i) comes from the continuity of the trace operator from $H^1(\Omega)$ to $H^{1/2}_{00}(\Gamma_0)$ equipped with the norm (8). To show the right inequality in (19), for $i = 1$, let $\varphi \in H^{1/2}_{00}(\Gamma_0)$, then there exist $\phi \in H^1(\Omega)$ such that $\phi/_{\Gamma_0} = \varphi$ and $\phi/_{\Gamma_c} = 0$, we can write

$$H_1\varphi = \phi + u_0,$$

with $u_0 \in H^1_0(\Omega)$ the solution of the following problem

$$\int_\Omega \xi(x)\nabla u_0 \cdot \nabla v\,dx = -\int_\Omega \xi(x)\nabla\phi \cdot \nabla v, \qquad \forall v \in H^1_0(\Omega),$$

where the space $H_0^1(\Omega)$ is defined by

$$H_0^1(\Omega) = \left\{\varphi \in H^1(\Omega),\ \varphi|_{\partial\Omega} = 0\right\}.$$

Taking $v = u_0$, we have

$$\alpha \|\nabla u_0\|_{0,\Omega}^2 \leq \beta\|\nabla\phi\|_{0,\Omega}\|\nabla u_0\|_{0,\Omega},$$

where β is the upper bound of $\xi(x)$, then

$$\frac{\alpha}{\beta}\|\nabla u_0\|_{0,\Omega} \leq \|\nabla\phi\|_{0,\Omega}.$$

Using the Poincaré inequality, there exist $C_\Omega > 0$, such that

$$\frac{\alpha}{C_\Omega\beta}\|u_0\|_{1,\Omega} \leq \|\phi\|_{1,\Omega}.$$

Thus

$$\begin{aligned}
\|H_1\varphi\|_{1,\Omega} &= \|\phi + u_0\|_{1,\Omega} \\
&\leq \|\phi\|_{1,\Omega} + \|u_0\|_{1,\Omega} \\
&\leq \|\phi\|_{1,\Omega} + \frac{C_\Omega\beta}{\alpha}\|\phi\|_{1,\Omega} \\
&\leq (1 + \frac{C_\Omega\beta}{\alpha})\|\phi\|_{1,\Omega}.
\end{aligned}$$

This is satisfied for all $\phi \in H^1(\Omega)$ such that $\phi|_{\Gamma_0} = \varphi$ and $\phi|_{\Gamma_c} = 0$, then

$$\begin{aligned}
\|H_1\varphi\|_{1,\Omega} &\leq (1 + \frac{C_\Omega\beta}{\alpha}) \inf_{\substack{\phi\in H^1_{\Gamma_c}(\Omega)\\ \phi|_{\Gamma_0}=\varphi}} \|\phi\|_{1,\Omega} \\
&\leq (1 + \frac{C_\Omega\beta}{\alpha})\|\varphi\|_{H_{00}^{1/2}(\Gamma_0)}.
\end{aligned}$$

So the inequality is showed for $C_2 = 1 + \frac{C_\Omega\beta}{\alpha}$. We can use the same arguments to show the assertion for $i = 2$.
In order to show the assertion (ii), let $\eta, \varphi \in H_{00}^{1/2}(\Gamma_0)$, using the Green's formula and the fact that $H_i\varphi$ is solution of (13) and (16) according to $i = 1, 2$, we have

$$\langle S_i\varphi, \eta\rangle = \int_\Omega \xi(x)\nabla H_i\varphi \cdot \nabla R_i\eta\, dx,$$

where R_i denotes the extension operator from $H_{00}^{1/2}(\Gamma_0)$ to $H^1_{\Gamma_c}(\Omega)$, satisfying $R_i\eta|_{\Gamma_0} = \eta$, which can be taken such that $R_i\eta = H_i\eta$. Then

$$\langle S_i\varphi, \eta\rangle = \int_\Omega \xi(x)\nabla H_i\varphi \cdot \nabla H_i\eta\, dx.$$

So it's clear that S_i is symmetric and that

$$|\langle S_i\varphi, \eta\rangle| \leq \|\xi\|_\infty \|H_i\eta\|_{1,\Omega} \|H_i\varphi\|_{1,\Omega}.$$

Then according to the assertion (i), we have

$$|\langle S_i\varphi, \eta\rangle| \leq \|\xi\|_\infty C_2^2 \|\eta\|_{H_{00}^{1/2}(\Gamma_0)} \|\varphi\|_{H_{00}^{1/2}(\Gamma_0)}.$$

Thus

$$\|S_i\varphi\|_{H_{00}^{-1/2}(\Gamma_0)} \leq \|\xi\|_\infty C_2^2 \|\varphi\|_{H_{00}^{1/2}(\Gamma_0)},$$

which proves that S_i is continuous.
We have also

$$\langle S_i\eta, \eta\rangle \geq \frac{\alpha}{C_\Omega}^2 \|H_i\eta\|_{1,\Omega}^2.$$

Moreover by using the assertion (i), we obtain

$$\langle S_i\eta, \eta\rangle \geq \frac{\alpha}{C_\Omega}^2 C_1 \|\eta\|^2_{H_{00}^{1/2}(\Gamma_0)}.$$

This completes the proof. □

At this stage, we note that the study of the existence of the solution of (18) presents some difficulties. This is related to the non coercivity of the operator S. Indeed, even the operators S_1 and S_2 are coercive as is showed in Lemma 1, we can't conclude the same things for $S = S_1 - S_2$ because of the presence of the minus. To overcome this difficulties, we propose to formulate the problem (18) as fixed point one. Indeed, by using the fact that S_1 is coercive, symmetric and continuous, then S_1^{-1} exists and it is a continuous operator. Therefore, the equation (18) is equivalent to

$$S_1^{-1} S\varphi = S_1^{-1}\chi,$$

then

$$S_1^{-1}(S_1 - S_2)\varphi = S_1^{-1}\chi,$$

which means that

$$\varphi = S_1^{-1} S_2\varphi + S_1^{-1}\chi. \tag{20}$$

Let us denote by F the operator defined from $H_{00}^{1/2}(\Gamma_0)$ to $H_{00}^{1/2}(\Gamma_0)$ by

$$F(\varphi) = S_1^{-1} S_2\varphi + S_1^{-1}\chi,$$

we can then reformulate our inverse problem as a fixed point one: find $\varphi \in H_{00}^{1/2}(\Gamma_0)$, satisfying

$$F(\varphi) = \varphi. \tag{21}$$

4. Existence of fixed point

To show the existence of a fixed point of (21), we use Leray-Shauder degree. This requires to showing 1) that the operator F is well posed, continuous and compact, 2) the existence of an appropriate estimation of the solution of the problem (18).

In order to show that the operator F is well posed, it suffices to show that the operator S_1^{-1} is a well posed one. This is done in the following lemma.

Lemma 2. *For all $\psi \in H_{00}^{-1/2}(\Gamma_0)$, S_1^{-1} is defined by $S_1^{-1}\psi = v|_{\Gamma_0}$, where v is the solution of*

$$\begin{cases} -\nabla.(\xi(x)\,\nabla v) &= 0 \quad in \quad \Omega, \\ \xi\frac{\partial v}{\partial n} &= \psi \quad on \quad \Gamma_0, \\ v &= 0 \quad on \quad \Gamma_1, \\ v &= 0 \quad on \quad \Gamma_2, \end{cases} \tag{22}$$

Proof. First, we recall that for all $\varphi \in H_{00}^{1/2}(\Gamma_0)$,

$$S_1\varphi = \xi\frac{\partial u}{\partial n}|_{\Gamma_0},$$

where u is solution of

$$\begin{cases} -\nabla.(\xi(x)\,\nabla u) &= 0 \quad \text{in} \quad \Omega, \\ u &= \varphi \quad \text{on} \quad \Gamma_0, \\ u &= 0 \quad \text{on} \quad \Gamma_1, \\ u &= 0 \quad \text{on} \quad \Gamma_2. \end{cases} \tag{23}$$

We have to show that for all $\varphi \in H_{00}^{1/2}(\Gamma_0)$, $S_1^{-1} S_1\varphi = \varphi$. Indeed, we have

$$S_1^{-1} S_1\varphi = S_1^{-1}\xi\frac{\partial u}{\partial n}|_{\Gamma_0}.$$

Let v be the solution of

$$\begin{cases} -\nabla.(\xi(x)\,\nabla v) &= 0 \quad \text{in} \quad \Omega, \\ \xi\frac{\partial v}{\partial n} &= \xi\frac{\partial u}{\partial n} \quad \text{on} \quad \Gamma_0, \\ v &= 0 \quad \text{on} \quad \Gamma_1, \\ v &= 0 \quad \text{on} \quad \Gamma_2, \end{cases} \tag{24}$$

then $\omega = u - v$ is solution of

$$\begin{cases} -\nabla.(\xi(x)\,\nabla\omega) &= 0 \quad \text{in} \quad \Omega, \\ \xi\frac{\partial\omega}{\partial n} &= 0 \quad \text{on} \quad \Gamma_0, \\ \omega &= 0 \quad \text{on} \quad \Gamma_1, \\ \omega &= 0 \quad \text{on} \quad \Gamma_2, \end{cases} \tag{25}$$

thus $\omega = 0$ a.e. in Ω and consequently $u = v$ a.e. in Ω. This gives

$$v|_{\Gamma_0} = u|_{\Gamma_0} = \varphi.$$

□

The continuity of the operator F result clearly form the Lemma 1. The compactness of F is based on that of S_1^{-1} which is showed in the following lemma.

Lemma 3. *The operator S_1^{-1} is compact.*

Proof. In order to show that the operator S^{-1} is compact, it suffices to show that for each bounded sequence $(\varphi_k)_{k\geq 0}$ in $H_{00}^{-1/2}(\Gamma_0)$ we can extract a subsequence of $(S^{-1}(\varphi_k))_k$ which is convergent in $H_{00}^{-1/2}(\Gamma_0)$.Let $(\varphi_k)_k$ be a bounded sequence in $H_{00}^{-1/2}(\Gamma_0)$ and, for all k, v_k be the solution of the following problem

$$\begin{cases} -\nabla.(\xi(x)\,\nabla v_k) &= 0 \quad \text{in} \quad \Omega, \\ \xi\frac{\partial v_k}{\partial n} &= \varphi_k \quad \text{on} \quad \Gamma_0, \\ v_k &= 0 \quad \text{on} \quad \Gamma_1, \\ v_k &= 0 \quad \text{on} \quad \Gamma_2, \end{cases} \tag{26}$$

by the weak formulation of (26), we have

$$\int_\Omega \xi(x)\,\nabla v_k \cdot \nabla\omega\,dx = \langle\varphi_k, \omega\rangle_{H_{00}^{-1/2}(\Gamma_0),H_{00}^{1/2}(\Gamma_0)} \quad \forall\omega\in H^1_{\Gamma_c}(\Omega), \tag{27}$$

for $\omega = v_k$, using the coercivity of the bilinear form, the continuity of the trace operator and Poincaré inequality, there exists $C > 0$, such that

$$\|v_k\|^2_{1,\Omega} \leq C\,\|\varphi_k\|_{H_{00}^{-1/2}(\Gamma_0)}\,\|v_k\|_{1,\Omega},$$

thus $(v_k)_k$ is uniformly bounded in $H^1(\Omega)$, we can then extract a subsequence denoted again $(v_k)_k$, which converges to v^* weakly in $H^1(\Omega)$, and strongly in $L^2(\Omega)$. Moreover, it's easy to see that $-\nabla\cdot(\xi\nabla v_k) = -\nabla\xi\cdot\nabla v_k - \xi\Delta v_k$ converges to $-\nabla.(\xi\nabla v^*)$ in $\mathcal{D}'(\Omega)$ (in distributions sense). Since

$$-\nabla.(\xi\nabla v_k) = 0 \quad \text{in} \;\; \mathcal{D}'(\Omega),$$

then

$$-\nabla.(\xi\nabla v^*) = 0 \quad \text{in} \;\; \mathcal{D}'(\Omega),$$

and thus

$$-\nabla.(\xi\nabla v^*) = 0 \quad \text{a.e. in} \;\; \Omega.$$

Now since, for all $\omega\in H^1_{\Gamma_c}(\Omega)$, we have

$$\begin{aligned} \lim_{k\to\infty}\int_\Omega \xi(x)\,\nabla v_k\cdot\nabla\omega\,dx &= \int_\Omega \xi(x)\,\nabla v^*\cdot\nabla\omega\,dx \\ &= \int_\Omega -\nabla.(\xi\nabla v^*)\omega\,dx + \langle\xi\frac{\partial v^*}{\partial n}, \omega\rangle_{H_{00}^{-1/2}(\Gamma_0),H_{00}^{1/2}(\Gamma_0)}. \end{aligned}$$

Then

$$\lim_{k\to\infty}\langle\varphi_k, \omega\rangle_{H_{00}^{-1/2}(\Gamma_0),H_{00}^{1/2}(\Gamma_0)} = \langle\xi\frac{\partial v^*}{\partial n}, \omega\rangle_{H_{00}^{-1/2}(\Gamma_0),H_{00}^{1/2}(\Gamma_0)}.$$

Thus

$$\varphi_k \underset{k\to+\infty}{\longrightarrow} \xi \frac{\partial v^*}{\partial n} \quad \text{ in } H_{00}^{-1/2}(\Gamma_0).$$

This means that $(\varphi_k)_k$ is a convergent sequence in $H_{00}^{-1/2}(\Gamma_0)$, and then it is a Cauchy sequence in this space. To conclude, it remains to show that $(v_k|_{\Gamma_0})$ is Cauchy sequence in $H_{00}^{1/2}(\Gamma_0)$.
Let $p \neq q \in \mathbb{N}$ and v_p, v_q the associated solution to φ_p and φ_q, then there exists a constant $C > 0$ such that

$$\|v_p|_{\Gamma_0} - v_q|_{\Gamma_0}\|_{H_{00}^{1/2}(\Gamma_0)} \leq C\, \|\varphi_p - \varphi_q\|_{H_{00}^{-1/2}(\Gamma_0)},$$

therefore $(v_k|_{\Gamma_0})_k$ is a Cauchy sequence in $H_{00}^{1/2}(\Gamma_0)$ which is a complete space, so the sequence $(v_k|_{\Gamma_0})_k$ converges in $H_{00}^{1/2}(\Gamma_0)$. This achieves the proof. □

In order to show the existence of the fixed point of (21), we need an appropriate estimate on the solution i.e. we must show that if the solution of (18) (which is the solution of (21)) exists it is bounded in $H_{00}^{1/2}(\Gamma_0)$. For this, let us recall some definition and properties on the ill posed problem. So, for all $\eta \in H_{00}^{-1/2}(\Gamma_0)$, we define the size of incompatibility μ_s by

$$\mu_s(\eta) = \inf_{\varphi\in H_{00}^{1/2}(\Gamma_0)} \|S\varphi - \eta\|_{H_{00}^{-1/2}(\Gamma_0)}.$$

Definition 1. *We call that the problem* (18) *is η-compatible if*

$$\mu_s(\eta) = 0.$$

According to Badeva and Morozov [3], in particular the result concerned by minimum requirement to have the existence of solution of ill-posed problems, we show the following lemma.

Lemma 4. *If the problem* (18) *is η-compatible, then its solution φ is bounded in $H_{00}^{1/2}(\Gamma_0)$.*

Based on this result, we prove the following existence theorem.

Theorem 1. *If the problem* (18) *is η-compatible, then the fixed point problem* (21) *has a unique solution in $H_{00}^{1/2}(\Gamma_0)$.*

Proof. Note first that if φ belongs to a ball, we cannot succeed to show that $F(\varphi)$ belongs to the same ball. Hence, we cannot use the classical fixed-point theorems to show the existence of the fixed point. To overcome this difficulty, we opt for the topological degrees of Leray-Schauder. For all $t \in [0, 1]$ we define the operator F_t, as follows,

$$\begin{array}{rcl} F_t\,;\; H_{00}^{1/2}(\Gamma_0) & \longrightarrow & H_{00}^{1/2}(\Gamma_0) \\ \overline{\varphi} & \mapsto & F_t(\overline{\varphi}) = S_1^{-1}(t\,S_2\overline{\varphi} + \chi). \end{array}$$

It's clear that from Lemma 1, we have the operator F_t is continuous and according to Lemma 3, F_t is compact. On the other hand using the estimate of the solution φ (Lemma 4), there exists a constant $C > 0$, such that

$$\|\varphi\|_{H_{00}^{1/2}(\Gamma_0)} \leq C, .$$

Consider the open ball B defined by

$$B = \{\varphi \in H_{00}^{1/2}(\Gamma_0) \;\;/\;\; \|\varphi\|_{H_{00}^{1/2}(\Gamma_0)} \leq R\},$$

with $R = C + 1$. Clearly the operator has no fixed point on ∂B, the boundary of B. Thus, by virtue of the degree of Leray-Schauder [13], we have that $deg[I - F_t, B, 0]$ is defined and independent of t, where the function deg is well-known as degree of Leray-Schauder and I is the identity mapping on $H_{00}^{1/2}(\Gamma_0)$. Note that F_0 corresponds to the trivial solution

$$\varphi = S_1^{-1}\chi.$$

Therefore $deg[I - F_0, B, 0] = 1$. Consequently $deg[I - F_1, B, 0] = 1$ and there exists $\varphi \in H_{00}^{1/2}(\Gamma_0)$, such that $F_1(\varphi) = \varphi$, which means that the operator F_1 has a fixed point in the interior of B. Thus the problem (21) has a fixed point.
It's easy to see that the uniqueness of the fixed point follows from the equivalence between problem (21) and the Cauchy problem (7) and the fact that (7) admits a unique solution. □

Remark 2. *The study of the existence and uniqueness of the fixed point can also be seen as a convergence result of the proposed algorithm. This algorithm is nothing other than KMF algorithm and thus our fixed point proof is a new convergence of the KMF algorithm.*
The proposed approach offers opportunities to exploit other domain decomposition methods for solving the inverse problem. More precisely, based on classical domain decomposition algorithms, we can reformulate our inverse problem into a fixed point one, and using mainly the same above techniques we can show the existence of the fixed point.

Having proving the existence result, we present in the following section a numerical study of the classical alternating method (KMF) and some proposed methods based on domain decomposition method (DDM).

5. Numerical study of alternating method based on DDM

In this section, we will present some developed algorithms based on the relationship between Cauchy problem and the interface problem using techniques introduced in the previous sections. Considering the importance and difficulty of the inverse Cauchy problem, several methods have been developed. The development of these methods focus on improving two things: namely the stability and the computational. The stability is established through the study of the behavior of the computed solution, with small perturbations on the given data. As for the computing cost, the development aims to accelerate the convergence. We note that all the iterative methods previously used give arise to a common behavior of the approximation process. Indeed, the error on non-accessible boundary decreases until a threshold, then, it rises slightly before its becoming stationary. Classical stopping criterion does not detect this threshold. Therefore, these stopping criterions break the algorithms in the phase where the error stagnates. This observation prompted us to develop another stopping criterion, which is more responsive and reflects the real behavior of the error. Consequently, this reduces significantly the computational cost by decreasing the number of iterations.

We recall the Cauchy problem defined by

$$\begin{cases} -\nabla\cdot(\xi(x)\nabla u) & = f \quad \text{in } \Omega, \\ u & = g_1 \quad \text{on } \Gamma_1, \\ \xi(x)\frac{\partial u}{\partial n} & = g_2 \quad \text{on } \Gamma_2, \\ u & = u_d \quad \text{on } \Gamma_2, \\ \frac{\partial u}{\partial n} & = g_3 \quad \text{on } \Gamma_3. \end{cases} \tag{28}$$

The domain decomposition-like algorithm that we use to solve the Cauchy problem requires the following well posed problems.
Given an initial approximation $v^{(n)}$ of u on Γ_0 define

$$P^{(2n)}\begin{cases} -\nabla\cdot(\xi(x)\nabla u^{(2n)}) & = f \text{ in } \Omega, \\ u^{2n} & = v^{(n)} \text{ on } \Gamma_0, \\ u^{(2n)} & = g_1 \quad \text{on } \Gamma_1, \\ \xi(x)\frac{\partial u^{(2n)}}{\partial n} & = g_2 \text{ on } \Gamma_2, \\ \frac{\partial u^{(2n)}}{\partial n} & = g_3 \quad \text{on } \Gamma_3, \end{cases}$$

$$v^{(n+1)} = u^{(2n)},$$

$$P^{(2n+1)}\begin{cases} -\nabla\cdot(\xi(x)\nabla u^{(2n+1)}) & = f \text{ in } \Omega, \\ u^{(2n+1)} & = u_d \text{ on } \Gamma_2, \\ u^{(2n)} & = g_1 \quad \text{on } \Gamma_1, \\ \frac{\partial u^{(2n+1)}}{\partial n} & = \frac{\partial u^{(2n)}}{\partial n} \text{ on } \Gamma_0, \\ \frac{\partial u^{(2n+1)}}{\partial n} & = g_3 \quad \text{on } \Gamma_3. \end{cases}$$

Proposed stopping criterion

The classical stopping criterion commonly used is the successive difference between the solution of actual iteration and the preview one on Γ_0 defined by $\|u^{(2n)} - u^{(2n-2)}\|_{\Gamma_0}$. According to the established relationship between Cauchy problem and domain decomposition methods in rectangular domain, it's more significant that at the convergence of proposed algorithms the solutions $u^{(2n)}$ and $u^{(2n+1)}$ must be identical on all the boundary $\partial\Omega$. This allows us to develop a more efficient stopping criterion than those used in the previous works.

It makes more sense to use a stopping criterion on the norm of the gap $u^{(2n)} - u^{(2n+1)}$ in L^2 on all the boundary $\partial\Omega$. Where $u^{(2n)}$ is the solution of $P^{(2n)}$ and $u^{(2n+1)}$ is the solution of $P^{(2n+1)}$. We denote by $\Gamma_c := \Gamma_2 \cup \Gamma_3 \cup \Gamma_0$, the proposed stopping criterion is

$$\|u^{(2n)} - u^{(2n+1)}\|_{\Gamma_c}. \tag{29}$$

In the sequel we seek to approach the solution of the inverse Cauchy problem (28) defined by the following data

$$\begin{cases} f = 0, \ g1 = \cos(x), \ g_2 = \cos(x)\sinh(y) + \sin(x)\cosh(y), \\ u_d = \cos(1)\cosh(y) + \sin(1)\sinh(y), \ g_3 = -\sin(x)\cosh(y) + \cos(x)\sinh(y). \end{cases} \tag{30}$$

With the above data the exact solution is $u_e = \cos(x)\cosh(y) + \sin(x)\sinh(y)$. The approximation is made by the following algorithms and P_1 finite elements method.

In order to illustrate the efficient of the stopping criterion, denote by n_e the number of iterations made by an algorithm when the criterion is satisfied. We start by the classical KMF.

Algorithm 5.1. Classical KMF

1. *Given $\epsilon > 0$ and an initial approximation $v^{(0)}$ of u on Γ_0 satisfying the compatibility conditions.*
2. *Solve the problem $P^{(2n)}$.*
3. *Solve the problem $P^{(2n+1)}$ with $\frac{\partial u^{(2n+1)}}{\partial n} = \frac{\partial u^{(2n)}}{\partial n}$, on Γ_0.*
4. *Actualize $v^{(n+1)} = u^{(2n-1)}, \quad n \geq 1$.*
5. *If $\|u^{(2n)} - u^{(2n+1)}\|_{L^2(\Gamma_c)} \leq \epsilon$ stop. Else go to step 2.*

In the figure 3 we present the comparison of the gap between u_e and the approximate solution u_h.

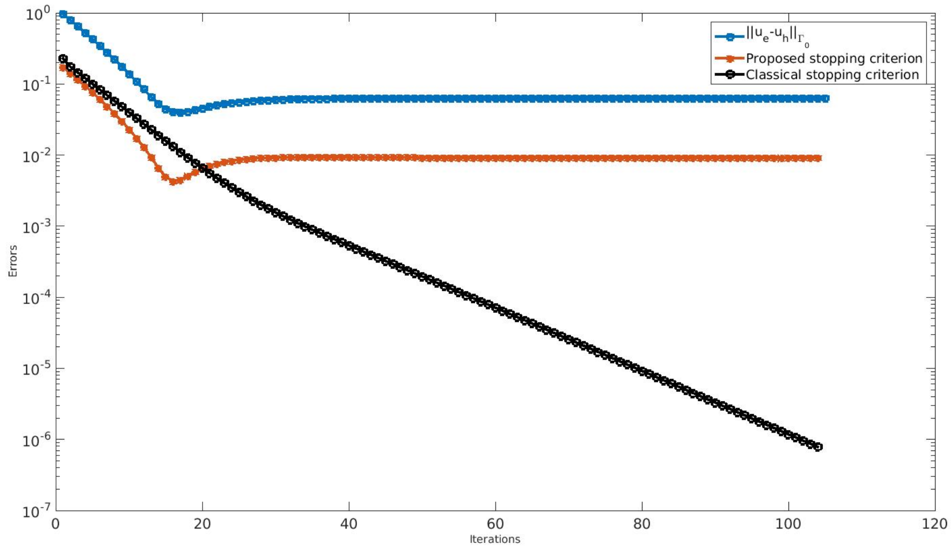

Figure 3: The behavior of proposed stopping criterion, the classical stopping criterion and the error on Γ_c obtained by the algorithm 5.1

As clearly shown in the figure 3, the classical stopping criterion does not follow the behavior of the error $(u_e - u_h)|_{\Gamma_0}$, which explains the large number of iterations in order to achieve the convergence. While the proposed stopping criterion follows faithfully the behavior of error. This allows to stop the algorithm with the best precision and with a minimum number of iterations equal to $n_e = 17$ iterations in this case.
In the following we relax KMF with dynamic θ proposed by Jourhmane and Nachaoui [17].

Algorithm 5.2. Relaxed KMF with dynamic θ.

1. *For $n = 0$ give $\epsilon > 0$, $\theta^{(n)} > 0$ and an initial approximation $v^{(n)}$ of u on Γ_0 satisfying the compatibility conditions.*
2. *Solve the problem $P^{(2n)}$.*
3. *Solve the problem $P^{(2n+1)}$ with $\frac{\partial u^{(2n+1)}}{\partial n} = \frac{\partial u^{(2n)}}{\partial n}$, on Γ_0.*
4. *Compute $e^{(2n)} = u^{(2n)}|_{\Gamma_0} - u^{(2(n-1))}|_{\Gamma_0}$, $e^{(2n+1)} = u^{(2n+1)}|_{\Gamma_0} - u^{(2(n-1))}|_{\Gamma_0}$.*
5. $\theta^{(n)} = \frac{\langle e^{(2n)}, e^{(2n)} - e^{(2n+1)} \rangle}{\|e^{(2n)} - e^{(2n+1)}\|_{L^2(\Gamma_0)}}, n \geq 1.$
6. *Compute $v^{(n)} = \theta^{(n)} u^{(2n-1)} + (1 - \theta^{(n)}) v^{n-1}, \quad n \geq 1.$*
7. *If $\|u^{(2n)} - u^{(2n+1)}\|_{L^2(\Gamma_c)} \leq \epsilon$ stop. Else go to step 2.*

In order to examine our proposed stopping criterion for the relaxed KMF with dynamic θ we present in the figure 4 the comparison of the gap between u_e and the approximate solution u_h obtained by the algorithm 5.2.

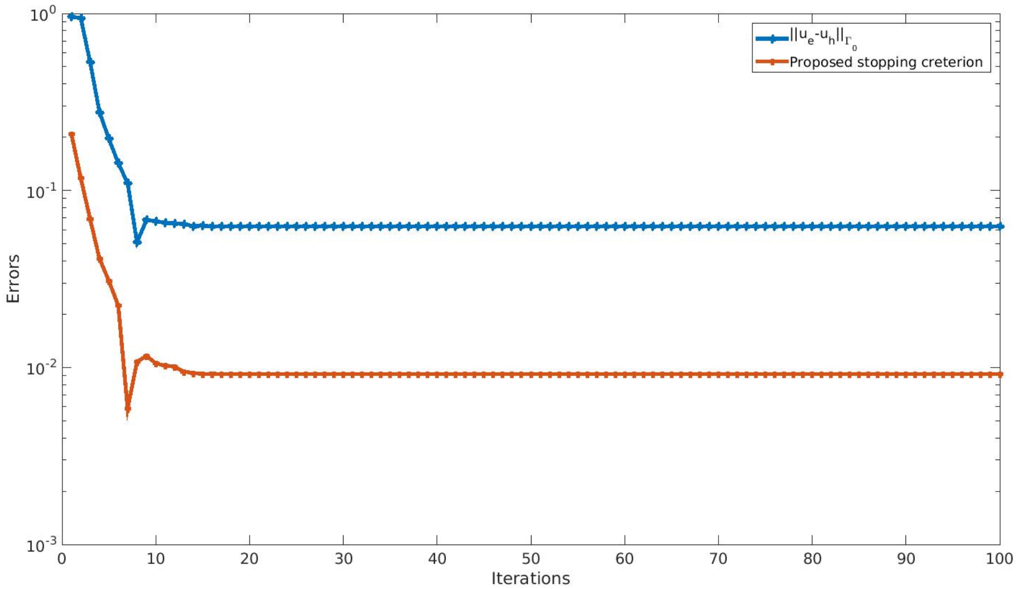

Figure 4: Comparison between the proposed stopping criterion and the error on Γ_c by the algorithm 5.2

Here also, we observe from Figure 4 that the new stopping criterion has accurately reflects the behavior of error on Γ_0. Furthermore, the number of iterations at the convergence is reduced by half, with $n_e = 8$ iterations.

In the following, we present an algorithm that we developed based on the decoupled Dirichlet-Neumann domain decomposition method. The advantage of this algorithm is that it can be coded in parallel computer. The idea is to decouple the two problems ($P^{(2n)}$) and ($P^{(2n+1)}$) in such way that at the iteration n, the normal derivative $\frac{\partial u^{2n}}{\partial n}$ on Γ_0 appear in the problem

$(P^{(2n+1)})$, depends on the solution of $(P^{(2(n-1))})$ at the previous iteration $n-1$. This leads that at the iteration n, the two problems $(P^{(2n)})$ and $(P^{(2n+1)})$ have become decoupled and therefore can be solved in parallel. This development allows reducing the computation time of the algorithm. Two configurations are presented: relaxed with fixed θ and with dynamic θ. These new methods are summarized in the following algorithms.

Algorithm 5.3. (Decoupled Dirichlet-Neumann (parallelized) with fixed θ)

1. *For* $n = 0$ *give* $\epsilon > 0,\ \theta > 0$ *and an initial approximation* $v^{(n)}$ *of* u *on* Γ_0 *satisfying the compatibility conditions.*
2. *Solve the problem* $P^{(2n)}$.
3. *Solve the problem* $P^{(2n+1)}$ *with* $\frac{\partial u^{(2n+1)}}{\partial n} = \frac{\partial u^{(2n-2)}}{\partial n},$ *on* Γ_0.
4. *Compute* $v^{(n)} = \theta u^{(2n-1)} + (1-\theta)v^{(n-1)}, \quad n \geq 1.$
5. *If* $\|u^{(2n)} - u^{(2n+1)}\|_{L^2(\Gamma_c)} \leq \epsilon$ *stop. Else go to step 2.*

Algorithm 5.4. (Decoupled version of Dirichlet-Neumann (parallelized) with dynamic θ)

1. *For* $n = 0$ *give* $\epsilon > 0,\ \theta^{(n)} > 0$ *and an initial approximation* $v^{(n)}$ *of* u *on* Γ_0 *satisfying the compatibility conditions.*
2. *Solve the problem* $P^{(2n)}$.
3. *Solve the problem* $P^{(2n+1)}$ *with* $\frac{\partial u^{(2n+1)}}{\partial n} = \frac{\partial u^{(2n-2)}}{\partial n},$ *on* Γ_0.
4. *compute* $e^{(2n)} = u^{(2n)}|_{\Gamma_0} - u^{(2(n-1))}|_{\Gamma_0},\ e^{(2n+1)} = u^{(2n+1)}|_{\Gamma_0} - u^{(2(n-1))}|_{\Gamma_0}$.
5. $\theta^{(n)} = \frac{\langle e^{(2n)}, e^{(2n)} - e^{(2n+1)} \rangle}{\|e^{(2n)} - e^{(2n+1)}\|_{L^2(\Gamma_0)}}, n \geq 1.$
6. *Compute* $v^{(n)} = \theta^{(n)} u^{(2n-1)} + (1-\theta^{(n)})v^{(n-1)}, \quad n \geq 1.$
7. *If* $\|u^{(2n)} - u^{(2n+1)}\|_{L^2(\Gamma_c)} \leq \epsilon$ *stop. Else go to step 2.*

In order to show the effectiveness of our developed algorithm compared to the classical versions of KMF, we will study three things: the number of iterations and the quality of the solution, at the convergence, as well as the behavior of the proposed stopping criterion compared to the error on Γ_0. In both cases: relaxation with fixed θ (algorithm 5.3) and dynamic θ (algorithm 5.4).

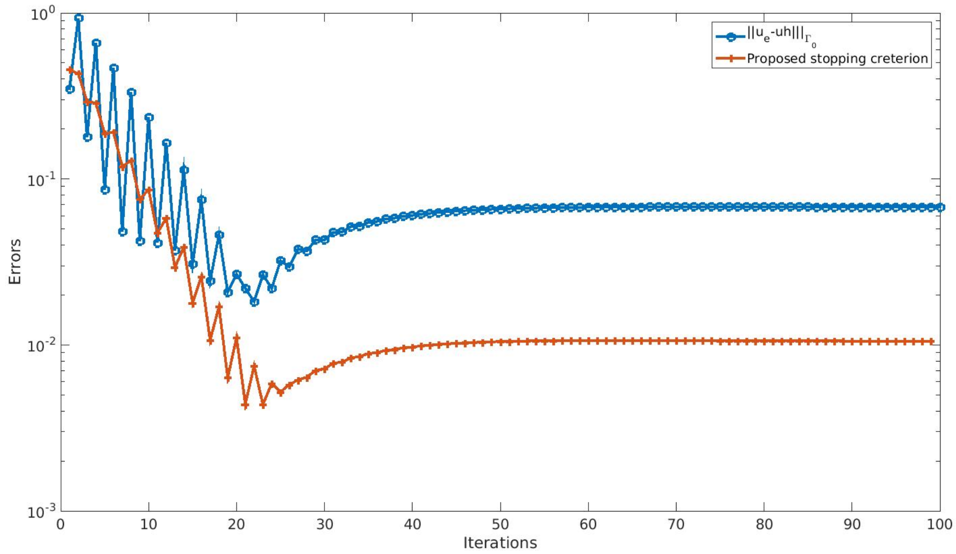

Figure 5: Comparison between the proposed stopping criterion and the error on Γ_c by the algorithm 5.3

In the figure 5 we present the comparison between the new stopping criterion and the error on Γ_0 obtained by the algorithm 5.3.

As we see in the figure 5 the error increase with some oscillations. This due to fact that the algorithm is decoupled. However, the proposed stopping criterion still a very good indication because it mimics the behavior of the error on Γ_0 even if it oscillate. Moreover, the number of iteration at the convergence is 19. Taking into account that at each iteration the two problems are decoupled which means that they can be solved in parallel. Consequently, the gain is about 40% of execution time compared to the classical one. In fact, if T is the time for solving a linear system. Then for the classical KMF algorithm the required time to reach convergence is $T_t = 17 * 2 * T$ while for decoupled algorithm 5.3, this time is $T_t' = 19 * T + T_c$, where T_c is time of communication between two processor. As it's known that T_c is very small compared to T, thus T_c can be neglected and the gain of time is (1-20/34)*100=42%. Moreover, with our criterion we can obtain a more accurate approximation with a more good agreement with the exact solution.

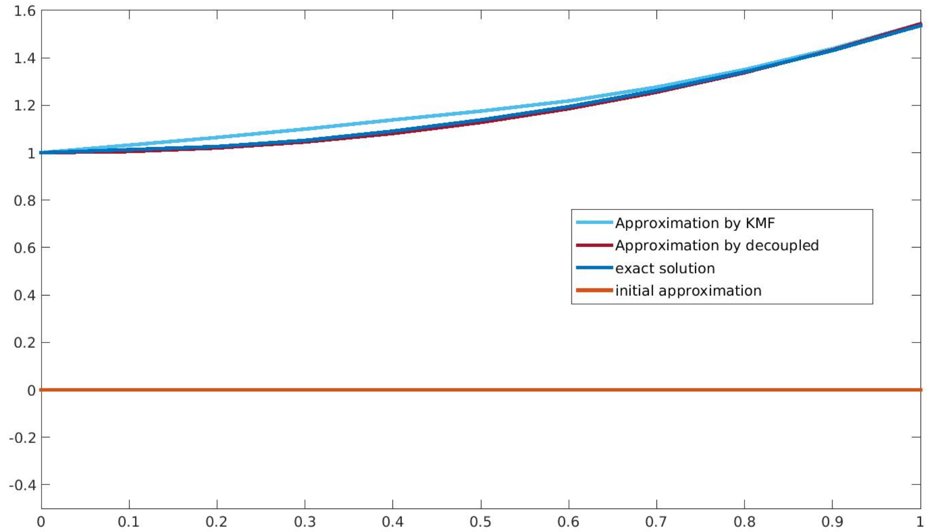

Figure 6: Exact solution, the approximate solution obtained by both KMF algorithm and decoupled 5.3 with fixed θ

It remains to shows the quality of the approximation for the two algorithms. In figure 6 we present the comparison of the exact solution and the approximate one obtained by both KMF Algorithms and decoupled 5.3. We observe that the decoupled algorithm gives a good approximation with the same order of accuracy.

Similar results are obtained using algorithm 5.4 as we can see in figure 7. 13 iteration was necessary to reach convergence for this algorithm. A similar reasoning as for the preview algorithm shows that this algorithm is more faster than algorithm 5.3.

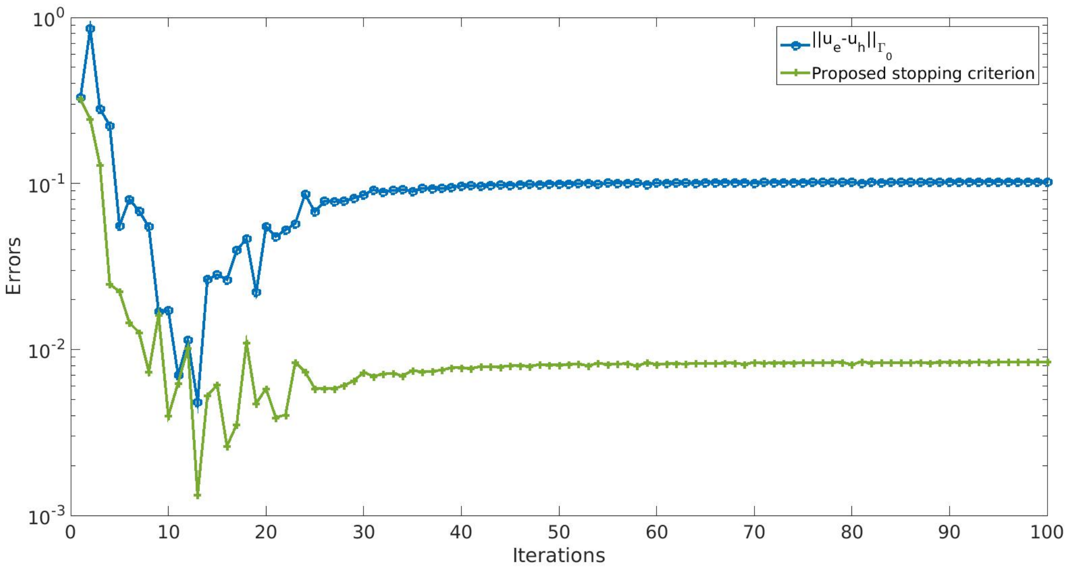

Figure 7: The proposed stopping criterion (29) and the error on Γ_c produced by the algorithm 5.4

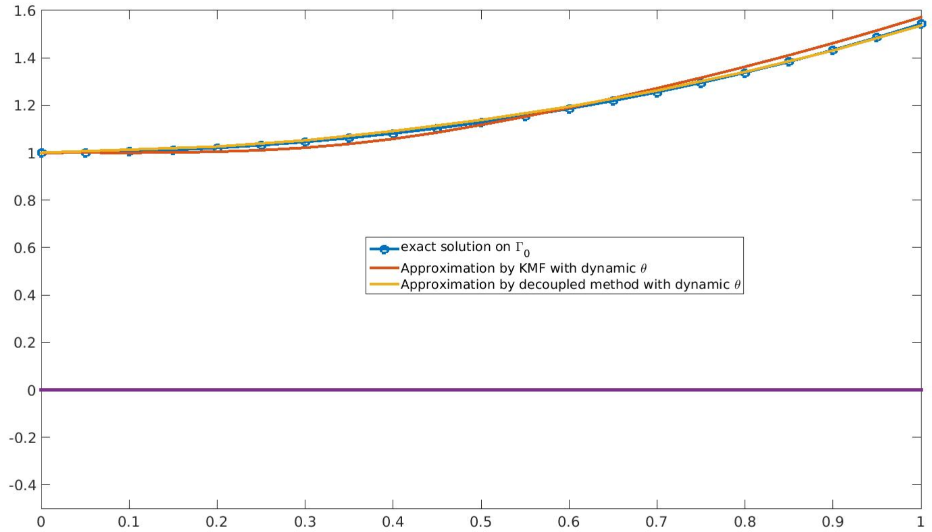

Figure 8: Exact solution and its approximate obtained by both KMF Algorithm and decoupled 5.4 with θ dynamic

In the following, we will make a comparative study to some domain decomposition methods to solve the inverse problem 28. Our study has two goals: first, we investigate the new stopping criterion on these methods then, carried out qualitative and quantitative comparisons between these methods and the methods presented above.

We start by the Schwartz algorithm with θ fixed and dynamic.

Algorithm 5.5. (Domain decomposition non-overlapping: generalized Dirichlet-Neumann algorithm) with fixed θ

1. *For $n = 0$ give $\epsilon > 0,\ \lambda^{(n)} > 0,\ \theta > 0$ and an initial approximation $v^{(n)}$ of u on Γ_0 satisfying the compatibility conditions*
2. *Solve the problem $P^{(2n)}$.*
3. *Solve the problem $P^{(2n+1)}$ with the following condition on Γ_0 : $\frac{\partial u^{(2n+1)}}{\partial n} = \lambda^{(n)}$.*
4. *Compute $\lambda^{(n)} = \lambda^{(n-1)} + \frac{u^{(2n)} - u^{(2n+1)}}{2},\ n \geq 1$.*
5. *Compute $v^{(n)} = \theta u^{(2n-1)} + (1-\theta)v^{n-1}, \quad n \geq 1$.*
6. *If $\|u^{(2n)} - u^{(2n+1)}\|_{L^2(\Gamma_c)} \leq \epsilon$ stop. Else $n = n+1$ go to step 2.*

Algorithm 5.6. (Domain decomposition non-overlapping: generalized Dirichlet-Neumann algorithm) with dynamic θ)

1. *For $n = 0$ give $\epsilon > 0,\ \lambda^{(n)} > 0,\ \theta > 0$ and an initial approximation $v^{(n)}$ of u on Γ_0 satisfying the compatibility conditions.*
2. *Solve the problem $P^{(2n)}$.*
3. *Solve the problem $P^{(2n+1)}$ with the following condition on Γ_0 $\frac{\partial u^{(2n+1)}}{\partial n} = \lambda^{(n)}$.*
4. *Compute $e^{(2n)} = u^{(2n)}|_{\Gamma_0} - u^{(2(n-1))}|_{\Gamma_0}$, $e^{(2n+1)} = u^{(2n+1)}|_{\Gamma_0} - u^{(2(n-1))}|_{\Gamma_0}$.*
5. $\theta^{(n)} = \frac{\langle e^{(2n)}, e^{(2n)} - e^{(2n+1)} \rangle}{\|e^{(2n)} - e^{(2n+1)}\|_{L^2(\Gamma_0)}}, n \geq 1.$
6. *Compute $\lambda^{(n)} = \lambda^{(n-1)} + \frac{u^{(2n)} - u^{(2n+1)}}{2},\ n \geq 1$.*

7. *Compute* $v^{(n)} = \theta^{(n)} u^{(2n-1)} + (1 - \theta^{(n)}) v^{n-1}, \quad n \geq .$

8. *If* $\|u^{(2n)} - u^{(2n+1)}\|_{L^2(\Gamma_c)} \leq \epsilon$ *stop. Else go to step 2.*

In figure 9 and figure 10 we present the comparison between the proposed stopping criterion and the error on Γ_c produced by algorithm 5.5 and algorithm 5.6. As we can see, the new stopping criterion allows us to stop the algorithms before it becomes stationary. Thus, the number of iterations at the convergence is 5 for the algorithm 5.5 and 4 iterations for the algorithm 5.6. Thus, we observe that the algorithm we developed from that of Schwartz allows to reduce considerably the computing cost. Indeed, compared to KMF with fixed θ the number of iteration is divided by 3, while compared to KMF with theta dynamic, the number of iteration is divided by 2.

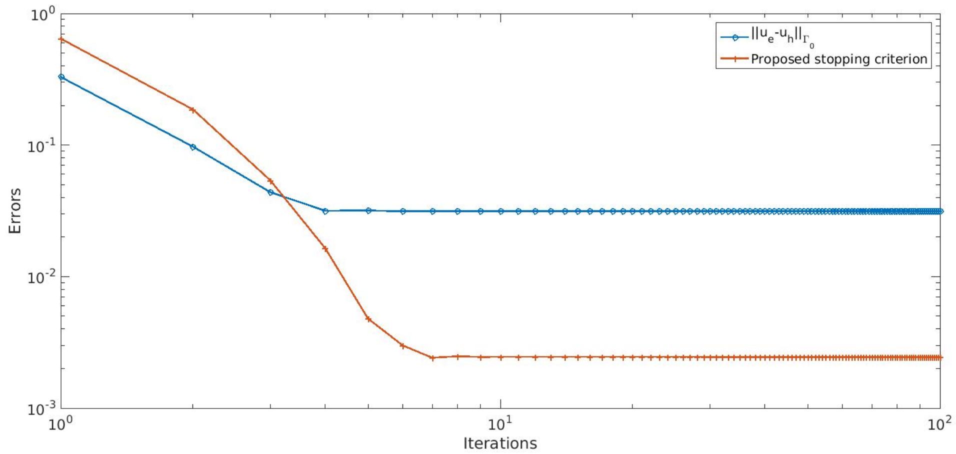

Figure 9: The proposed stopping criterion and the error produced on Γ_c by algorithm 5.5

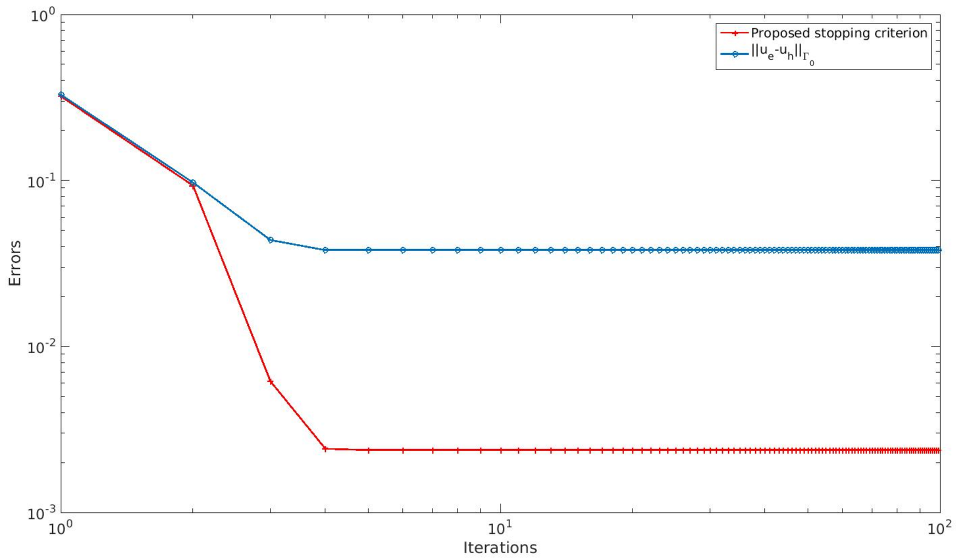

Figure 10: The proposed stopping criterion and the error on Γ_c by the algorithm 5.6

In the following, we state and examine some algorithms for solving the Cauchy problem derived from Neumann-Neumann DD method and its variants, namely Robin method and Agoshkov-Lebedev.

Algorithm 5.7. (Algorithm derived from Neumann-Neumann DD method)

1. *For n = 0 give* $\epsilon > 0,\ \lambda^{(n)} > 0,\ \theta > 0$ *and an initial approximation* $v^{(n)}$ *of u on* Γ_0 *satisfying compatibility conditions.*
2. *Solve the problem* $P^{(2n)}$.
3. *Solve the problem* $P^{(2n+1)}$ *with the following condition on* Γ_0 $u^{(2n+1)} = \lambda^{(n)}$.
4. *Solve the problem* $P^{(2n)}$ *for* $\psi^{(2n)}$ *with the following condition on* Γ_0
$$\psi^{(2n)} = \frac{\partial u^{(2n)}}{\partial n} - \frac{\partial u^{(2n+1)}}{\partial n}.$$
5. *Solve the problem* $P^{(2n+1)}$ *for* $\psi^{(2n+1)}$ *with the following condition on* Γ_0
$$\psi^{(2n+1)} = \frac{\partial u^{(2n)}}{\partial n} - \frac{\partial u^{(2n+1)}}{\partial n}.$$
6. *Compute* $\lambda^{(n)} = \lambda^{(n-1)} - \theta(\psi^{(2n)}|_{\Gamma_0} - \psi^{(2n+1)}|_{\Gamma_0}),\ n \geq 1.$
7. *Compute* $v^{(n)} = v^{(n)} - \theta(\psi^{(2n)}|_{\Gamma_0} - \psi^{(2n+1)}|_{\Gamma_0}), \quad n \geq 1.$
8. *If* $\|u^{(2n)} - u^{(2n+1)}\|_{L^2(\Gamma_c)} \leq \epsilon$ *stop. Else* $n = n + 1$ *go to step 2.*

Algorithm 5.8. (Algorithm derived from Robin-Robin DD method)

1. *For n = 0 give* $\epsilon > 0,\ \lambda^{(n)} > 0,\ \theta > 0$ *and an initial approximation* $v^{(n)}$ *of u on* Γ_0 *satisfying the compatibility conditions.*
2. *Solve the problem* $P^{(2n)}$ *with the following condition on* Γ_0
$$\frac{\partial u^{(2n)}}{\partial n} + \gamma_1 u^{(2n)} = \frac{\partial u^{(2n+1)}}{\partial n} + \gamma_1 u^{(2n+1)}.$$
3. *Solve the problem* $P^{(2n+1)}$ *with the following condition on* Γ_0
$$\frac{\partial u^{(2n+1)}}{\partial n} - \gamma_2 u^{(2n+1)} = \frac{\partial u^{(2n)}}{\partial n} + \gamma_2 u^{(2n)}.$$
4. *If* $\|u^{(2n)} - u^{(2n+1)}\|_{L^2(\Gamma_c)} \leq \epsilon$ *stop. Else* $n = n + 1$ *go to step 2.*

Algorithm 5.9. (Algorithm derived from Agoshkov-Lebedev DD method)

1. *For n = 0 give* $\epsilon > 0,\ \alpha, \beta, p, q > 0,$ *and* $u^{(0)}$ *and* $u^{(1)}$.
2. *Solve the problem* $P^{(2n+\frac{1}{2})}$ *with the following condition on* Γ_0
$$\frac{\partial u^{(2n+\frac{1}{2})}}{\partial n} + p u^{(2n+\frac{1}{2})} = \frac{\partial u^{(2n-1)}}{\partial n} + p u^{(2n-1)}.$$
3. *Solve the problem* $P^{(2n+\frac{3}{2})}$ *with the following condition on* Γ_0
$$-q\frac{\partial u^{(2n+\frac{3}{2})}}{\partial n} + u^{(2n+\frac{3}{2})} = -q\frac{\partial u^{(2n)}}{\partial n} + u^{(2n)}.$$
4. $u^{(2n)} = u^{(2n-2)} + \alpha(u^{(2n+\frac{1}{2})} - u^{(2n-2)})$ *and* $u^{(2n+1)} = u^{(2n-1)} + \beta(u^{(2n+\frac{3}{2})} - u^{(2n-1)})$.
5. *If* $\|u^{(2n)} - u^{(2n+1)}\|_{L^2(\Gamma_c)} \leq \epsilon$ *stop. Else* $n = n + 1$ *go to step 2.*

In figures 11,12 and 13, we present the Comparison between the proposed stopping criterion and the error on Γ_c produced by the algorithms 5.7,5.8 and 5.9. We observe from these figures that our stopping criterion can be used to stop these algorithms with reduced cost and best accuracy since the two norms have the same profile. The convergence is reached after 2 iterations for Neumann-Neumann like method. However, this algorithm requires to solve four problems in each iteration. Thus the time needed for an iteration of this algorithm is equivalent to twice the time required for iteration of the other algorithms.

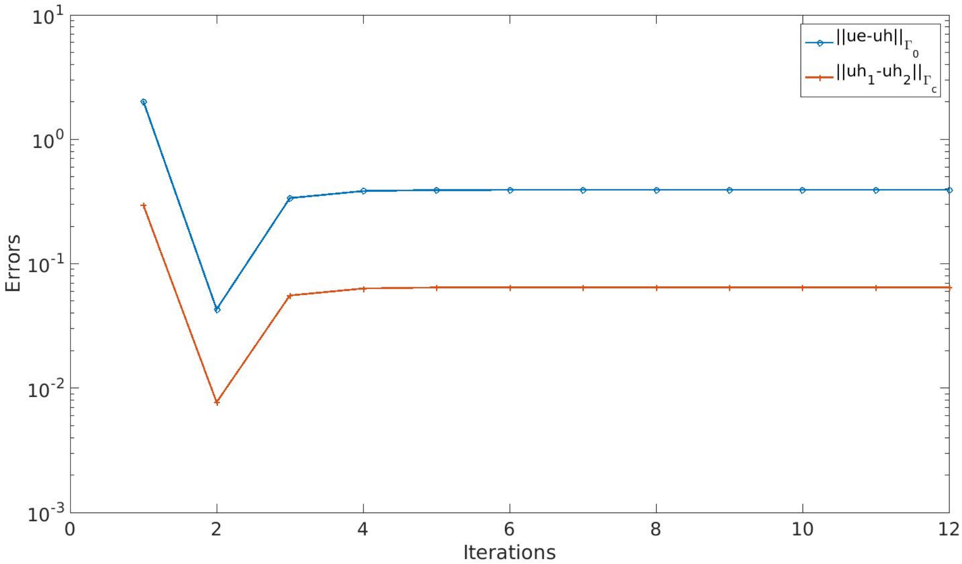

Figure 11: The proposed stopping criterion and the error on Γ_c produced by the algorithm 5.7

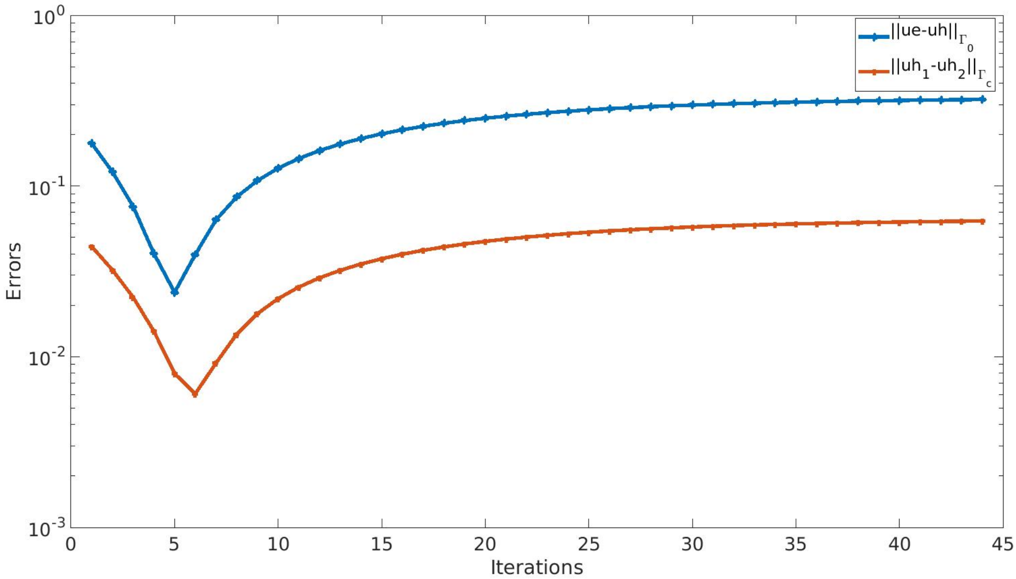

Figure 12: The proposed stopping criterion and the error on Γ_c by produced the algorithm 5.8

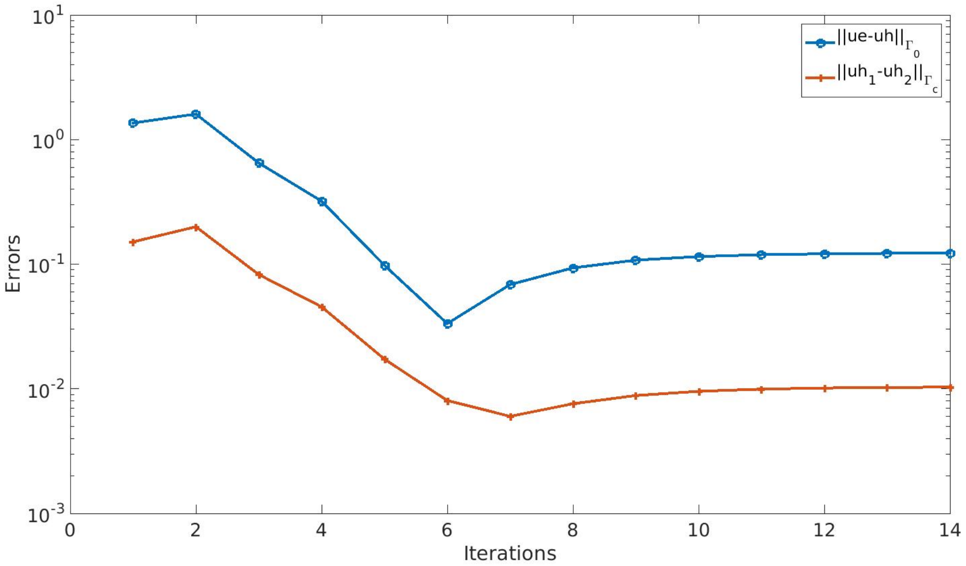

Figure 13: The proposed stopping criterion and the error on Γ_c by produced the algorithm 5.9

In the following, we present comparison results concerning the quality of the solution produced by KMF algorithm and each of the eight proposed algorithms. For this we present in Figure 14 the exact solution on the inaccessible part of the boundary, the common initial solution to all algorithms and approximations produced by each algorithm. As we see the quality of the approximations obtained by all algorithms is good. All approximation are in good agreement with the exact solution. The error $||u_e - u_h||_{\Gamma_0}$ vary from an algorithm to other but in all case belong $[10^{-2}, 5*10^{-2}]$ which is a good accuracy for such complex problem and with such size of mesh $h = 1/10$.

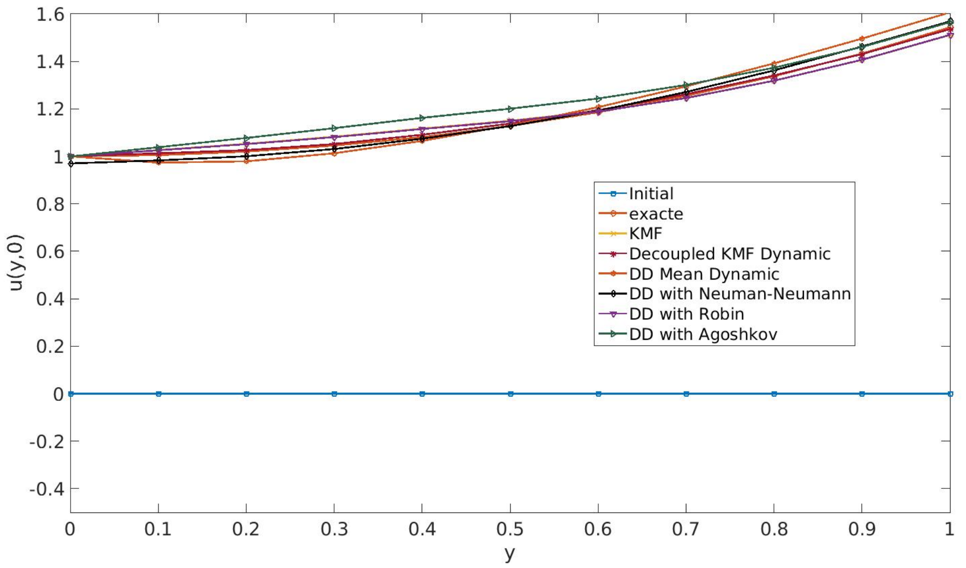

Figure 14: The exact solution and its approximation obtained by each DD-like method

In the table 1 we present the number of iterations at the convergence for each domain decomposition-like algorithms. As we see, algorithms derived from domain decomposition methods are all faster than the classical KMF algorithm 5.1.

Algorithm	Iteration number
5.1	17
5.2	8
5.3	19
5.4	13
5.5	4
5.6	4
5.7	2
5.7	5
5.9	6

Table 1: Number of iteration for each algorithm form the preview presented methods

In the next section, we will study the numerical stability of our new methods.

5.1. *Numerical stability*

In the previous sections, we have showed the effectiveness of the proposed methods in terms of quality (accuracy of approximation) and quantitative (computation cost). Since the solution of the inverse Cauchy problem does not depends continuously on the data, which justifies that this problem is ill-posed in Hdamard's sense [16], the methods we developed would not be relevant without a regularizing character. We will therefore present a study of the numerical stability of these algorithms. To simplify the presentation, we limit ourselves to examining the method derived from the Agoshov-Lebedev algorithm, which is some way a generalization of the others.

We examine the behavior of the Algorithm 5.9 in the presence of small perturbations in the data. The Dirichlet and Neumann boundary conditions on Γ_2 are perturbed to simulate measurement errors such that

$$g_2^\delta = g_2 + \delta g_2, \quad u_d^\delta = u_d + \delta u_d \quad \text{on } \Gamma_2,$$

where $\delta g_2 = g_2 * \delta * (2 * rand - 1)$ and $\delta u_d = u_d * \delta * (2 * rand - 1)$ are Gaussian noise with mean zero, generated by an appropriate function *rand*. While the δ is the noise level. The inverse Cauchy problem, which take into account these perturbations is defined by

$$\begin{cases} -\nabla \cdot (\xi(x)\nabla u^\delta) & = f \quad \text{in } \Omega, \\ u^\delta & = g_1 \quad \text{on } \Gamma_1, \\ \xi(x)\frac{\partial u^\delta}{\partial n} & = g_2^\delta \quad \text{on } \Gamma_2, \\ u^\delta & = u_d^\delta \quad \text{on } \Gamma_2, \\ \frac{\partial u^\delta}{\partial n} & = g_3 \quad \text{on } \Gamma_3. \end{cases} \tag{31}$$

In order to show that our methods are numerically stable we considered some different level of noise $\delta \in [10^{-2}, 2 * 10^{-1}]$ and we compared the exact solution u_e the approximate solution without noise and the approximate one u_h^δ with different level δ.

In figure 15 we present the exact Dirichlet data and the perturbed one with 20% of noise level.

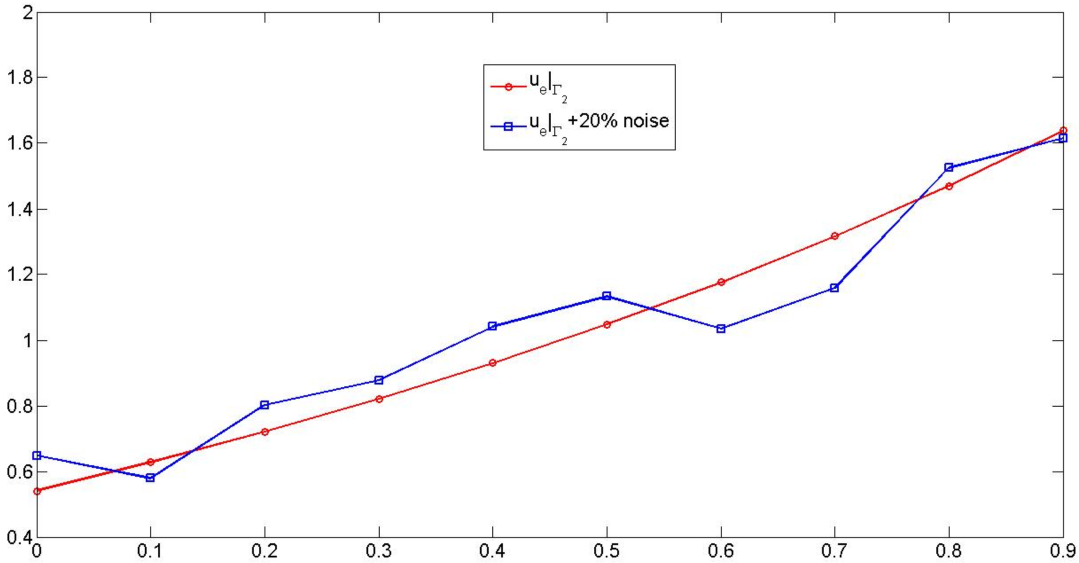

Figure 15: The exact Dirichlet data and the perturbed one with 20% of noise level

In figure 16 we present the comparison between the exact solution, the approximate solution without noise and the approximate one with three levels of noise $\delta = 10^{-2}, 10^{-1}, 2 * 10^{-1}$. As we can be seen in this figure, the obtained solution u_h^δ with 1% of noise coincides with the obtained solution without noise. One can also observe that the difference between the exact solution and that obtained from noisy data is of the same order as the noise added to the data. Indeed, the error satisfies the following inequality

$$\|u_e - u_h^\delta\|_{\Gamma_0} \leq \delta,$$

which shows that the algorithm is stable. That's why we can conclude that the developed algorithms have regularizing properties.

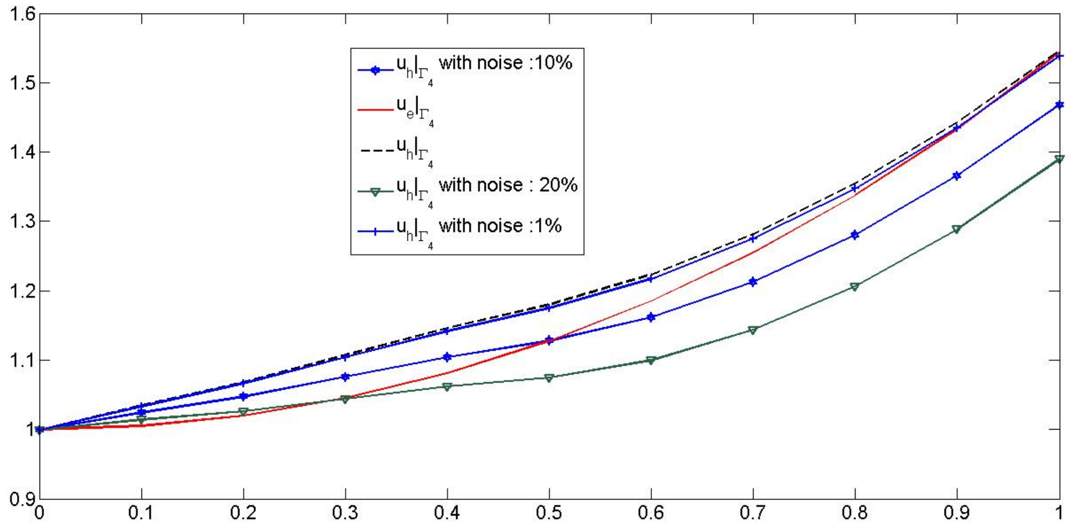

Figure 16: Comparison between the exact solution, the approximate solution without noise and the approximate one with different level of noise

Conclusion and remarks

In conclusion, we have considered an inverse Cauchy problem. In this case, we have established formally a relationship between this problem and the interface problem illustrated in a rectangular geometry. We have shown how the domain decomposition methods developed to solve the interface problem can be modified to build new methods to solve the Cauchy problem. Inspired by this relationship we have showing in general framework a theoretical study given a constructive proof of the convergence of what's we have called decomposition domain-like methods, for the inverse Cauchy problem. Then we have investigate the developed methods based on this relationship and proposed an efficient stopping criterion. A numerical study focus on the quality of approximate solutions and the cost of computation has been presented. Finally, we have showed the numerical stability of the developed algorithms. We mention that the results investigated in this work can easily be extended to other Cauchy problems governed by other elliptic equations.

Acknowledgments

The paper was developed when the second author was visiting professor at Laboratory of Mathematics Jean Leray UMR6629 CNRS, Nantes University, in June-July 2019. The authors are grateful to anonymous referees for their careful review of our manuscript and for their constructive comments.

References

[1] Aceto, L., Ghelardoni, P., Marletta, M., 2007. Numerical solution of forward and inverse sturm–liouville problems with an angular momentum singularity. Inverse Problems 24 (1), 015001.

[2] Andrieux, S., Baranger, T. N., Ben Abda, A., 2006. Solving Cauchy problems by minimizing an energy-like functional. Inverse Problems 22 (1), 115–133.

[3] Badeva, V. Morozov, V., 1997. Problèmes incorrectement posées, théorie et Applications. Masson, Paris.

[4] Bai, Z.-Z., Buccini, A., Hayami, K., Reichel, L., Yin, J.-F., Zheng, N., 2017. Modulus-based iterative methods for constrained tikhonov regularization. Journal of Computational and Applied Mathematics 319, 1–13.

[5] Bergam, A., Chakib, A., Nachaoui, A., Nachaoui, M., 2019. Adaptive mesh techniques based on a posteriori error estimates for an inverse Cauchy problem. Appl. Math. Comput. 346, 865–878.

[6] Bourgeois, L., 2005. A mixed formulation of quasi-reversibility to solve the Cauchy problem for Laplace's equation. Inverse Problems 21 (3), 1087–1104.

[7] Chakib, A., Nachaoui, A., 2006. Convergence analysis for finite element approximation to an inverse Cauchy problem. Inverse Problems 22 (4), 1191–1206.

[8] Chang, L., Gong, W., Sun, G., Yan, N., 2015. PDE-constrained optimal control approach for the approximation of an inverse Cauchy problem. Inverse Probl. Imaging 9 (3), 791–814.

[9] Cimetière, A., Delvare, F., Jaoua, M., Pons, F., 2000. Une mthode inverse rgularisation vanescente. Comptes Rendus de l'Acadmie des Sciences - Series {IIB} - Mechanics 328 (9), 639 – 644.

[10] Cimetière, A., Delvare, F., Jaoua, M., Pons, F., 2001. Solution of the Cauchy problem using iterated Tikhonov regularization. Inverse Problems 17 (3), 553–570.

[11] Clason, C., Klibanov, M. V., 2007/08. The quasi-reversibility method for thermoacoustic tomography in a heterogeneous medium. SIAM J. Sci. Comput. 30 (1), 1–23.

[12] Dardé, J., Hannukainen, A., Hyvönen, N., 2013. An H_{div}-based mixed quasi-reversibility method for solving elliptic Cauchy problems. SIAM J. Numer. Anal. 51 (4), 2123–2148.

[13] Deimling, K., 1985. Nonlinear functional analysis. Springer-Verlag.
URL `https://books.google.fr/books?id=RYo_AQAAIAAJ`

[14] Fan, C.-M., Li, P.-W., Yeih, W., 2015. Generalized finite difference method for solving two-dimensional inverse Cauchy problems. Inverse Probl. Sci. Eng. 23 (5), 737–759.

[15] Gong, R., Cheng, X., Han, W., 2014. A fast solver for an inverse problem arising in bioluminescence tomography. Journal of Computational and Applied Mathematics 267, 228–243.

[16] Hadamard, J., 1953. Lectures on Cauchy's problem in linear partial differential equations. Dover Publications, New York.

[17] Jourhmane, M., Nachaoui, A., 1999. An alternating method for an inverse Cauchy problem. Numer. Algorithms 21 (1-4), 247–260, numerical methods for partial differential equations (Marrakech, 1998).

[18] Jourhmane, M., Nachaoui, A., 2002. Convergence of an alternating method to solve the cauchy problem for poisson's equation. Applicable Analysis 81 (5), 1065–1083.

[19] Klibanov, M. V., Santosa, F., 1991. A computational quasi-reversibility method for Cauchy problems for Laplace's equation. SIAM J. Appl. Math. 51 (6), 1653–1675.

[20] Kozlov, V. A., Mazya, V. G., Fomin, A. V., 1991. An iterative method for solving the cauchy problem for elliptic equations. Zh. Vychisl. Mat. i Mat. Fiz. 31 (1), 64–74.

[21] Lattès, R., Lions, J.-L., 1967. Méthode de quasi-réversibilité et applications. Travaux et Recherches Mathématiques, No. 15. Dunod, Paris.

[22] Lattès, R., Lions, J.-L., 1969. The method of quasi-reversibility. Applications to partial differential equations. Translated from the French edition and edited by Richard Bellman. Modern Analytic and Computational Methods in Science and Mathematics, No. 18. American Elsevier Publishing Co., Inc., New York.

[23] Lattès, R., Lions, J. l., 1969. The method of quasi-reversibility. Applications to partial differential equations. Translated from the French edition and edited by Richard Bellman. New York, NY: Elsevier.

[24] Li, P., Liu, K., Zhong, W., 2018. Marching schemes for inverse scattering problems in waveguides with curved boundaries. Journal of Computational and Applied Mathematics 328, 287–301.

[25] Liu, J.-C., Wei, T., 2013. A quasi-reversibility regularization method for an inverse heat conduction problem without initial data. Appl. Math. Comput. 219 (23), 10866–10881.

[26] Quarteroni, A., Valli, A., 1999. Domain decomposition methods for partial differential equations. Numerical Mathematics and Scientific Computation. The Clarendon Press, Oxford University Press, New York, oxford Science Publications.

[27] Ramm, A. G. A. G., 2004. Inverse problems : mathematical and analytical techniques with applications to engineering. New York : Springer, includes bibliographical references and index.
URL http://site.ebrary.com/id/10139637

[28] Wen, Y.-W., Ching, W.-K., Ng, M., 2017. A semi-smooth newton method for inverse problem with uniform noise. Journal of Scientific Computing, 1–20.