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An Iterative Method for the Reconstruction of a Stationary Flow

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In this article, an iterative algorithm based on the Landweber-Fridman method in combination with the boundary element method is developed for solving a Cauchy problem in linear hydrostatics Stokes flow of a slow viscous fluid. This is an iteration scheme where mixed well-posed problems for the stationary generalized Stokes system and its adjoint are solved in an alternating way. A convergence proof of this procedure is included and an efficient stopping criterion is employed. The numerical results confirm that the iterative method produces a convergent and stable numerical solution. © 2007 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 23: 998–1017, 2007

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I. INTRODUCTION

In most boundary value problems arising in the slow viscous fluid flow the governing system of partial differential equations, i.e., the Stokes momentum and the continuity equations, have to be solved subject to appropriate boundary conditions, i.e., Dirichlet, Neumann, or mixed boundary conditions. These problems are called *direct problems* and their well-posedness have been well-established; see for example, Galdi [1]. However, there are other fluid flow engineering problems that do not belong to this category. It is well known that these are *inverse problems*. In particular, part of the boundary may be overspecified, giving rise to *Cauchy problems* generally ill-posed in the sense of Hadamard [2], i.e., the existence, uniqueness, and stability of a solution cannot be guaranteed. Prior to this study, Zeb et al. [3] and Curteanu [4] investigated some inverse problems for Stokes flow. In these problems the underspecification of one component of the velocity on a part of the boundary is compensated for by the measurement of the pressure on the remaining part of the boundary. Similar inverse problems formulated in terms of the streamfunction-vorticity have been investigated by Lesnic et al. [5], whereas Kozlov et al. [6] investigated the analogy in thermoelasticity.

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In this article, we consider a Cauchy problem for slow viscous fluids, where data are given on a part of the boundary. There exist various methods for solving Cauchy problems. One common approach is to use a Tikhonov regularization, which often leads to a change of the operator of problem; see Chapter 4 in Lattès and Lions [7]. Another way is to use iterative methods (see Kozlov and Maz'ya [8]), who proposed iterative methods for some ill-posed boundary value problems that preserve the corresponding operator. In Kozlov et al. [9], an alternating iterative method, obtained by changing the boundary conditions, is applied to Cauchy problems for the equations of anisotropic elasticity. One of the advantages of this method is that it preserves the original equation and that the regularizing character is achieved by changing appropriately the boundary conditions. For elliptic problems and systems of order $2m$, the regularizing properties are shown in the Sobolev space H^m. The main restriction is that the differential operator must be formally self-adjoint. An extension of the alternating method to a Cauchy problem for the Stokes system is given in Bastay et al. [10]. However, the method in [10] can only be applied to the case of the generalized Stokes system and cannot for example handle an Oseen type of flow.

In this article, we develop an iterative scheme, based on the Landweber-Fridman method, for the reconstruction of the velocity and pressure of an Oseen (generalized Stokes) type of fluid flow from the knowledge of the velocity and traction on a part of the boundary. The solution does not depend continuously on the data, so the problem is ill-posed. The algorithm we propose is a regularizing method where the regularizing character is obtained by solving a series of mixed well-posed boundary value problems for the generalized Stokes system and its adjoint (see Section III). A convergence proof of the method is given in Section V. An efficient stopping criterion is also employed; see Section VI. The numerical results obtained using the boundary element method confirm that the iterative scheme produces a convergent and stable numerical solution; see Section VIII.

Let us also mention that methods based on similar ideas have been proposed for Cauchy problems in heat transfer and in isotropic linear elasticity; see Bastay et al. [11] and Marin and Lesnic [12], respectively.

II. MATHEMATICAL FORMULATION

Consider a slow viscous flow of fluid that occupies a bounded domain Ω in R^n, where $n \geq 2$, with boundary $\Gamma = \partial\Omega$ of class C^2. We assume that the boundary Γ is the union of two closed and disjointed pieces Γ_0 and Γ_1. Although this assumption may look like an important restriction, in principle it can be overcome by expressing the analysis of this article in terms of some appropriate weighted Sobolev spaces, as described by Johansson [13] for the Cauchy problem of elliptic equations. In practice, the domain Ω can, for example, be the region between two infinitely long cylinders for $n = 2$ or between two spheres for $n = 3$, having outer boundary Γ_0 and inner boundary Γ_1. In nondimensional form, the generalized Stokes low Reynolds number incompressible flow equations can be written as follows:

$$\begin{cases} L\mathbf{u} - \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \end{cases}$$

where

$$L\mathbf{u} = \left(\Delta u_i + \sum_{j=1}^{n} b_j(x) \frac{\partial u_i}{\partial x_j} \right)_{1 \leq i \leq n}.$$

Here, $\mathbf{u} = (u_1, \ldots, u_n)$ is the fluid velocity, p is the pressure, and $\mathbf{b}$ is a coefficient function that is assumed of class $C^1(\overline{\Omega})^n$. Taking $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ to be the outward unit normal to the boundary of Ω, the traction $\mathbf{T}$ can be written as $\mathbf{T} = p\boldsymbol{\nu} - N\mathbf{u}$, where

$$N\mathbf{u} = \left(\sum_{j=1}^{n} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \nu_j \right)_{1 \le i \le n} = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)\boldsymbol{\nu}.$$

The so-called Cauchy problem which is investigated in this article corresponds to the case where the velocity and traction are both given on one of the boundary parts, say Γ_0. Altogether, we have the following problem to study:

$$\begin{cases} L\mathbf{u} - \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \boldsymbol{\varphi} & \text{on } \Gamma_0, \\ p\boldsymbol{\nu} - N\mathbf{u} = \boldsymbol{\psi} & \text{on } \Gamma_0. \end{cases} \tag{2.1}$$

Let us mention that the method developed in this article can handle the nonhomogeneous case, i.e., the case when $L\mathbf{u} - \nabla p = \mathbf{f}$ and $\operatorname{div}\mathbf{u} = g$, together with the boundary conditions in (2.1). Indeed, let $(\bar{\mathbf{u}}, \bar{p})$ be the solution to the nonhomogeneous equations in Ω, together with the homogeneous boundary condition that the velocity is zero on Γ. This is a well-posed problem. If we search for a solution to the nonhomogeneous problem of the form $(\mathbf{u}, p) = (\bar{\mathbf{u}}, \bar{p}) + (\tilde{\mathbf{u}}, \tilde{p})$, the pair $(\tilde{\mathbf{u}}, \tilde{p})$ solves a problem of the form (2.1).

A. Notations and Preliminaries

Let $L^2(\Omega)$ be the space of square integrable real-valued functions on Ω with the usual norm. The space $H^k(\Omega)$, where $k = 1, 2, \ldots$, denotes the standard Sobolev space on Ω, i.e., the space of functions with generalized derivatives of order $\le k$ in $L^2(\Omega)$. The dual space of $H^k(\Omega)$ with respect to the L^2-inner product is denoted by $(H^k(\Omega))^*$. By $H^{k-1/2}(\Gamma)$, we mean the space of traces of functions in $H^k(\Omega)$. We let the product of n samples of a space X be denoted by X^n.

We assume that the function $\mathbf{b}$ that appears in the operator L is chosen so that the problem, which consists of the first two equations of (2.1) supplied with the boundary conditions $\mathbf{u} = 0$ on Γ_1 and $p\boldsymbol{\nu} - N\mathbf{u} = 0$ on Γ_0, has only the trivial solution in $H^2(\Omega)^n \times H^1(\Omega)$.

III. A REGULARIZING PROCEDURE

Here, we present an iterative procedure for solving problem (2.1). Let us first introduce the problems

$$\begin{cases} L\mathbf{u} - \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \boldsymbol{\eta} & \text{on } \Gamma_1, \\ p\boldsymbol{\nu} - N\mathbf{u} = \boldsymbol{\psi} & \text{on } \Gamma_0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} L^*\mathbf{v} - \nabla q = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \Gamma_1, \\ q\boldsymbol{\nu} - N^*\mathbf{v} = \boldsymbol{\xi} & \text{on } \Gamma_0, \end{cases} \tag{3.2}$$

where

$$L^*\mathbf{v} = \left(\Delta v_i - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(b_j(x)v_i)\right)_{1\le i\le n},$$

$$N^*\mathbf{v} = \left(\sum_{j=1}^{n}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - b_j(x)v_i\right)\nu_j\right)_{1\le i\le n}.$$

The procedure is as follows:

- Choose an arbitrary function $\boldsymbol{\eta}_0 \in L^2(\Gamma_1)^n$. The first approximation $\mathbf{u}_0$ and p_0 is obtained by solving (3.1) with $\mathbf{u} = \boldsymbol{\eta}_0$ on Γ_1.
- Then we find $\mathbf{v}_0$ and q_0 by solving (3.2) with $\boldsymbol{\xi} = \mathbf{u}_0 - \boldsymbol{\varphi}$ on Γ_0.
- Having constructed $\mathbf{u}_{k-1}$, p_{k-1}, $\mathbf{v}_{k-1}$, and q_{k-1}, we obtain $\mathbf{u}_k$ and p_k by solving problem (3.1) with $\mathbf{u} = \boldsymbol{\eta}_k$ on Γ_1, where

$$\boldsymbol{\eta}_k = \boldsymbol{\eta}_{k-1} + \gamma(q_{k-1}\boldsymbol{\nu} - N^*\mathbf{v}_{k-1})$$

  and γ is a constant.
- Finally, $\mathbf{v}_k$ and q_k are obtained by solving (3.2) with $\boldsymbol{\xi} = \mathbf{u}_k - \boldsymbol{\varphi}$ on Γ_0.

This procedure converges if the constant γ is chosen in a certain interval (see Theorem 5.1).

IV. WEAK SOLUTIONS IN $L^2(\Omega)^N \times (H^1(\Omega))^*$

Most of the results for the direct mixed boundary value problem settings with the nonstandard choice of spaces for prescribed and unknown functions given in this section are described in detail in Johansson [14]. Herein, we only give the main definitions and proofs. We first specify what we mean by a solution to problem (3.1) and show that this problem is well posed. Note that (3.1) is elliptic in the sense of Douglis and Nirenberg [15]. To find a relation that a weak solution must satisfy, we need a suitable Green's formula. Using integration by parts, we find that the identity

$$\begin{aligned}\int_\Omega \mathbf{v}\cdot(L\mathbf{u} - \nabla p)\,dx - \int_\Omega \mathbf{u}\cdot(L^*\mathbf{v} - \nabla q)\,dx + \int_\Omega \mathbf{v}\cdot(\nabla\operatorname{div}\mathbf{u})\,dx - \int_\Omega \mathbf{u}\cdot(\nabla\operatorname{div}\mathbf{v})\,dx \\ = -\int_\Gamma \mathbf{v}\cdot(p\boldsymbol{\nu} - N\mathbf{u})\,dS + \int_\Gamma \mathbf{u}\cdot(q\boldsymbol{\nu} - N^*\mathbf{v})\,dS + \int_\Omega p\operatorname{div}\mathbf{v}\,dx - \int_\Omega q\operatorname{div}\mathbf{u}\,dx,\end{aligned} \tag{4.1}$$

holds for $\mathbf{u}, \mathbf{v} \in H^2(\Omega)^n$ and $p, q \in H^1(\Omega)$.

A. Definition of a Weak Solution

Definition. *Let $\boldsymbol{\eta} \in L^2(\Gamma_1)^n$ and $\boldsymbol{\psi} \in L^2(\Gamma_0)^n$. We call $\mathbf{u} \in L^2(\Omega)^n$ and $p \in (H^1(\Omega))^*$ a weak solution to* (3.1)*, if these functions satisfy*

$$\int_\Omega \mathbf{u} \cdot (L^*\mathbf{w} - \nabla q)\, dx + \int_\Omega p \operatorname{div}\mathbf{w}\, dx + \int_\Omega \mathbf{u} \cdot (\nabla \operatorname{div}\mathbf{w})\, dx \\ - \int_{\Gamma_0} \boldsymbol{\psi} \cdot \mathbf{w}\, dS + \int_{\Gamma_1} \boldsymbol{\eta} \cdot (q\boldsymbol{\nu} - N^*\mathbf{w})\, dS = 0, \quad (4.2)$$

for every $\mathbf{w} \in H^2(\Omega)^n$ and $q \in H^1(\Omega)$ that satisfy

$$\begin{cases} \mathbf{w} = 0 & \text{on } \Gamma_1, \\ q\boldsymbol{\nu} - N^*\mathbf{w} = 0 & \text{on } \Gamma_0. \end{cases}$$

Here, $\int_\Omega p \operatorname{div}\mathbf{w}\, dx$ is not to be interpreted as an integral in the usual sense. It denotes the value of the functional p on $\operatorname{div}\mathbf{w}$.

Similarly, we define a weak solution to (3.2).

Definition. *Let $\boldsymbol{\xi} \in L^2(\Gamma_0)^n$. We call $\mathbf{v} \in L^2(\Omega)^n$ and $q \in (H^1(\Omega))^*$ a weak solution to problem* (3.2)*, if these functions satisfy*

$$\int_\Omega \mathbf{v} \cdot (L\mathbf{w} - \nabla p)\, dx + \int_\Omega q \operatorname{div}\mathbf{w}\, dx + \int_\Omega \mathbf{v} \cdot (\nabla \operatorname{div}\mathbf{w})\, dx - \int_{\Gamma_0} \boldsymbol{\xi} \cdot \mathbf{w}\, dS = 0, \quad (4.3)$$

for every $\mathbf{w} \in H^2(\Omega)^n$ and $p \in H^1(\Omega)$ subject to

$$\begin{cases} \mathbf{w} = 0 & \text{on } \Gamma_1, \\ p\boldsymbol{\nu} - N\mathbf{w} = 0 & \text{on } \Gamma_0. \end{cases}$$

Here, $\int_\Omega q \operatorname{div}\mathbf{w}\, dx$ denotes the value of the functional q on $\operatorname{div}\mathbf{w}$.

If $\mathbf{u} \in H^2(\Omega)^n$ and $p \in H^1(\Omega)$ is a solution to (3.1), then it is clear from the above definition and the Green formula (4.1) that it is also a weak solution to (3.1).

B. Existence and Uniqueness of a Weak Solution to (3.1) and (3.2)

Lemma 4.1. *Let $\boldsymbol{\eta} \in L^2(\Gamma_1)^n$ and $\boldsymbol{\psi}, \boldsymbol{\xi} \in L^2(\Gamma_0)^n$. Then*

(i) *Problem* (3.1) *has a unique weak solution $\mathbf{u} \in L^2(\Omega)^n$ and $p \in (H^1(\Omega))^*$ and*

$$\|\mathbf{u}\|_{L^2(\Omega)^n} + \|p\|_{(H^1(\Omega))^*} \le C\big(\|\boldsymbol{\eta}\|_{L^2(\Gamma_1)^n} + \|\boldsymbol{\psi}\|_{L^2(\Gamma_0)^n}\big). \quad (4.4)$$

(ii) *Problem* (3.2) *has a unique weak solution $\mathbf{v} \in L^2(\Omega)^n$ and $q \in (H^1(\Omega))^*$ and*

$$\|\mathbf{v}\|_{L^2(\Omega)^n} + \|q\|_{(H^1(\Omega))^*} \le C\|\boldsymbol{\xi}\|_{L^2(\Gamma_0)^n}. \quad (4.5)$$

Proof. We only prove (i) since the proof of (ii) follows by similar arguments. Let $\mathbf{u}$ be a weak solution to (3.1). We first prove an inequality for $\mathbf{u}$. Consider the problem of finding $\mathbf{w} \in H^2(\Omega)^n$ and $q \in H^1(\Omega)$ that satisfy

$$\begin{cases} L^*\mathbf{w} - \nabla q = \mathbf{u} & \text{in } \Omega, \\ \operatorname{div}\mathbf{w} = 0 & \text{in } \Omega, \end{cases}$$

together with the boundary conditions of problem (3.2), where $\boldsymbol{\xi} = 0$. The operator for the corresponding problem for the Stokes system is Fredholm, which implies that the operator for this problem also is Fredholm. From the assumptions on the function $\mathbf{b}$ that appears in the operator L (see Section IIA), it then follows that there exists a unique solution $\mathbf{w} \in H^2(\Omega)^n$ and $q \in H^1(\Omega)$ to this problem. It also follows that a constant C exists such that

$$\|\mathbf{w}\|_{H^2(\Omega)^n} + \|q\|_{H^1(\Omega)} \le C\|\mathbf{u}\|_{L^2(\Omega)^n}. \tag{4.6}$$

Substituting this solution in (4.2) we obtain

$$\int_\Omega |\mathbf{u}|^2\,dx = \int_{\Gamma_0} \boldsymbol{\psi} \cdot \mathbf{w}\,dS - \int_{\Gamma_1} \boldsymbol{\eta} \cdot (q\boldsymbol{\nu} - N^*\mathbf{w})\,dS. \tag{4.7}$$

Applying Cauchy's inequality, the estimate (4.6), together with well-known trace estimates imply that

$$\|\mathbf{u}\|_{L^2(\Omega)^n} \le C\big(\|\boldsymbol{\eta}\|_{L^2(\Gamma_1)^n} + \|\boldsymbol{\psi}\|_{L^2(\Gamma_0)^n}\big). \tag{4.8}$$

We now derive an inequality for p. Let $\mathbf{w} \in H^2(\Omega)^n$ and $q \in H^1(\Omega)$ satisfy

$$\begin{cases} L^*\mathbf{w} - \nabla q = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{w} = g & \text{in } \Omega, \end{cases}$$

together with the boundary conditions of problem (3.2), where $\boldsymbol{\xi} = 0$. Here, $g \in H^1(\Omega)$ is arbitrary. As above, it follows that there exists a unique solution $\mathbf{w} \in H^2(\Omega)^n$ and $q \in H^1(\Omega)$ that satisfies these equations. Substituting this solution in (4.2) we obtain

$$\int_\Omega pg\,dx = -\int_\Omega \mathbf{u} \cdot \nabla g\,dx + \int_{\Gamma_0} \boldsymbol{\psi} \cdot \mathbf{w}\,dS - \int_{\Gamma_1} \boldsymbol{\eta} \cdot (q\boldsymbol{\nu} - N^*\mathbf{w})\,dS. \tag{4.9}$$

The functions $\mathbf{w}$ and q can be estimated as $\|\mathbf{w}\|_{H^2(\Omega)^n} + \|q\|_{H^1(\Omega)} \le C\|g\|_{H^1(\Omega)}$. This, estimate (4.8) for $\mathbf{u}$ and Cauchy's inequality, in (4.9) give that (4.4) holds, which in particular implies the uniqueness of a weak solution.

We now prove existence of a solution. Assume first that $\boldsymbol{\eta}$ and $\boldsymbol{\psi}$ are sufficiently smooth, i.e., $\boldsymbol{\eta} \in C^2(\Gamma_1)^n$ and that $\boldsymbol{\psi} \in C^2(\Gamma_0)^n$. Then there exists a solution $\mathbf{u} \in H^2(\Omega)^n$ and $p \in H^1(\Omega)$ to problem (3.1) with these $\boldsymbol{\eta}$ and $\boldsymbol{\psi}$ as data. Using (4.1), we see that $\mathbf{u}$ and p is also a weak solution to (3.1). The existence of a solution for general $\boldsymbol{\eta} \in L^2(\Gamma_1)^n$ and $\boldsymbol{\psi} \in L^2(\Gamma_0)^n$ follows if we approximate $\boldsymbol{\eta}$ and $\boldsymbol{\psi}$ with functions $\boldsymbol{\eta}_j \in C^2(\Gamma_1)^n$ and $\boldsymbol{\psi}_j \in C^2(\Gamma_0)^n$, in the respective L^2-norm. Using inequality (4.4), we obtain Cauchy sequences $\mathbf{u}_j$ and p_j, which converge to $\mathbf{u} \in L^2(\Omega)^n$ and $p \in (H^1(\Omega))^*$, respectively. That this solution is a weak solution follows from the construction of $\mathbf{u}_j$ and p_j. ■

We have thus proved that there exists a unique solution to problem (3.1) that depends continuously on the data. Hence, problem (3.1) is well posed and the same holds for the formal adjoint problem (3.2).

C. Traces on the Boundary of Weak Solutions

We start by proving the following lemma.

Lemma 4.2. *Let $\boldsymbol{\eta} \in L^2(\Gamma_1)^n$, $\boldsymbol{\psi} \in L^2(\Gamma_0)^n$, and let $\mathbf{u} \in L^2(\Omega)^n$, and $p \in (H^1(\Omega))^*$ be the weak solution to* (3.1).

(i) *Let $\Omega_0 \subset \Omega$ be a domain such that* $\operatorname{dist}(\Omega_0, \Gamma_0) > 0$. *If $\boldsymbol{\eta} = 0$, then $\mathbf{u} \in H^2(\Omega_0)^n$, $p \in H^1(\Omega_0)$ and*

$$\|\mathbf{u}\|_{H^2(\Omega_0)^n} + \|p\|_{H^1(\Omega_0)} \le C\|\boldsymbol{\psi}\|_{L^2(\Gamma_0)^n}.$$

(ii) *Let $\Omega_0 \subset \Omega$ be a domain such that* $\operatorname{dist}(\Omega_0, \Gamma_1) > 0$. *Then $\mathbf{u} \in H^1(\Omega_0)^n$, $p \in L^2(\Omega_0)$ and*

$$\|\mathbf{u}\|_{H^1(\Omega_0)^n} + \|p\|_{L^2(\Omega_0)} \le C\big(\|\boldsymbol{\eta}\|_{L^2(\Gamma_1)^n} + \|\boldsymbol{\psi}\|_{L^2(\Gamma_0)^n}\big).$$

Proof. We only prove (i) since (ii) follows by similar arguments. First, let $\boldsymbol{\psi} \in C^2(\Gamma_0)^n$. Then there exists a unique solution $\mathbf{u} \in H^2(\Omega)^n$ and $p \in H^1(\Omega)$ to problem (3.1) with $\boldsymbol{\eta} = 0$. This solution is also a weak solution according to Definition IVA. Using local estimates for Stokes system, it is straightforward to check that

$$\|\mathbf{u}\|_{H^2(\Omega_0)^n} + \|p\|_{H^1(\Omega_0)} \le C\big(\|\mathbf{u}\|_{L^2(\Omega)^n} + \|p\|_{(H^1(\Omega))^*}\big). \tag{4.10}$$

Combining this with estimate (4.4), we see that (i) holds for $\boldsymbol{\psi} \in C^2(\Gamma_0)^n$. Approximating a general $\boldsymbol{\psi} \in L^2(\Gamma_0)^n$ with $\boldsymbol{\psi}_j \in C^2(\Gamma_0)^n$ and using the above result, (i) follows.

We can now prove that weak solutions to (3.1) have traces on the boundary. ■

Lemma 4.3. *Let $\boldsymbol{\eta} \in L^2(\Gamma_1)^n$, $\boldsymbol{\psi} \in L^2(\Gamma_0)^n$ and let $\mathbf{u} \in L^2(\Omega)^n$, $p \in (H^1(\Omega))^*$ be the weak solution to problem* (3.1). *Then we have*

(i) $\mathbf{u}|_{\Gamma_0} \in H^{1/2}(\Gamma_0)^n$ *and the inequality*

$$\|\mathbf{u}\|_{H^{1/2}(\Gamma_0)^n} \le C\big(\|\boldsymbol{\eta}\|_{L^2(\Gamma_1)^n} + \|\boldsymbol{\psi}\|_{L^2(\Gamma_0)^n}\big)$$

holds.

(ii) *If $\boldsymbol{\eta} = 0$, then $N\mathbf{u}|_{\Gamma_1} \in L^2(\Gamma_1)^n$ and $p|_{\Gamma_1} \in H^{1/2}(\Gamma_1)$, and they satisfy*

$$\|N\mathbf{u}\|_{L^2(\Gamma_1)^n} + \|p\|_{H^{1/2}(\Gamma_1)} \le C\|\boldsymbol{\psi}\|_{L^2(\Gamma_0)^n}.$$

Proof. From Lemma 4.2, we know that $\mathbf{u} \in H^2(\Omega_0)^n$ and $p \in H^1(\Omega_0)$, where $\Omega_0 \subset \Omega$ is a nonempty domain of class C^2 with $\operatorname{dist}(\Omega_0, \Gamma_0) > 0$. The trace theorem and the inequality in Lemma 4.2 (i) then imply that (ii) holds. Using similar arguments, (i) follows. ■

Arguments similar to those used to prove Lemma 4.3 prove that weak solutions to the formally adjoint problem (3.2) have traces on the boundary.

Lemma 4.4. *Let $\boldsymbol{\xi} \in L^2(\Gamma_0)^n$ and let $\mathbf{v} \in L^2(\Omega)^n$, $q \in (H^1(\Omega))^*$ be the weak solution to problem* (3.2). *Then*

$$\text{(i)} \quad N^*\mathbf{v}|_{\Gamma_1} \in L^2(\Gamma_1)^n \quad \textit{and} \quad q|_{\Gamma_1} \in H^{1/2}(\Gamma_1).$$

Moreover,

$$\|N^*\mathbf{v}\|_{L^2(\Gamma_1)^n} + \|q\|_{H^{1/2}(\Gamma_1)} \leq C\|\boldsymbol{\xi}\|_{L^2(\Gamma_0)^n}.$$

$$\text{(ii)} \quad \mathbf{v}|_{\Gamma_0} \in H^{1/2}(\Gamma_0)^n \quad \textit{and} \quad \|\mathbf{v}\|_{H^{1/2}(\Gamma_0)^n} \leq C\|\boldsymbol{\xi}\|_{L^2(\Gamma_0)^n}.$$

V. CONVERGENCE OF THE PROCEDURE GIVEN IN SECTION III

In order to state the main result, let us introduce a linear operator $K : L^2(\Gamma_1)^n \to L^2(\Gamma_0)^n$ defined by

$$K\boldsymbol{\eta} = \mathbf{u}|_{\Gamma_0} \qquad \text{for} \quad \boldsymbol{\eta} \in L^2(\Gamma_1)^n, \tag{5.1}$$

where $\mathbf{u}$ solves (3.1) with $\boldsymbol{\psi} = 0$. The trace results in the preceding section imply that this operator is well defined. Moreover, combining the estimate in Lemma 4.3 (i) with the fact that $H^{1/2}(\Gamma_0)^n$ can be compactly embedded in $L^2(\Gamma_0)^n$, it follows that K is compact. Similarly, we introduce a linear operator $K_1 : L^2(\Gamma_0)^n \to L^2(\Gamma_0)^n$ by

$$K_1\boldsymbol{\psi} = \mathbf{u}|_{\Gamma_0} \qquad \text{for} \quad \boldsymbol{\psi} \in L^2(\Gamma_0)^n, \tag{5.2}$$

where $\mathbf{u}$ solves (3.1) with $\boldsymbol{\eta} = 0$. Using the definition of these operators, it is straightforward to check that the Cauchy problem (2.1) is equivalent with the equation

$$K\boldsymbol{\eta} = \boldsymbol{\varphi} - K_1\boldsymbol{\psi}. \tag{5.3}$$

Since K is compact, this recasts the ill-posedness of our model.

We can now state the main theorem.

Theorem 5.1. *Let $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ be given in $L^2(\Gamma_0)^n$. Assume that problem* (2.1) *has a solution $\mathbf{u} \in L^2(\Omega)^n$ and $p \in (H^1(\Omega))^*$. Let $\mathbf{u}_k$ and p_k be the kth approximate solution in the procedure presented in Section* III. *If $0 < \gamma < 1/\|K\|^2$, then*

$$\lim_{k\to\infty} \|\mathbf{u} - \mathbf{u}_k\|_{L^2(\Omega)^n} = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \|p - p_k\|_{(H^1(\Omega))^*} = 0 \tag{5.4}$$

for any initial data element $\boldsymbol{\eta}_0 \in L^2(\Gamma_1)^n$.

The rest of this section is devoted to a proof of this theorem.

Remark. In general, for $\boldsymbol{\varphi}, \boldsymbol{\psi} \in L^2(\Gamma_0)^n$, the inverse problem (2.1) has no solution. This is so because if $\boldsymbol{\psi} \in L^2(\Gamma_0)^n$ then the element $\boldsymbol{\varphi}$ is more smooth, for example $\boldsymbol{\varphi} \in H^{1/2}(\Gamma_0)^n$.

A. The Kernel of the Operator

Here, we show that the kernel of the operator defined in (5.1) consists of zero only.

In Regbaoui [16], the following lemma is proved.

Lemma 5.2. *Let Ω be a connected open subset of R^n containing 0. Assume that $|\mathbf{b}(x)| \leq C|x|^{-1+\varepsilon}$, where $\mathbf{b}$ is measurable and $\varepsilon > 0$. Let $\mathbf{u} \in H^1_{loc}(\Omega)^n$ and $p \in L^2_{loc}(\Omega)$ be a solution to*

$$\begin{cases} \Delta\mathbf{u} + (\mathbf{b}(x)\cdot\nabla)\mathbf{u} - \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega. \end{cases} \tag{5.5}$$

If such a solution to (5.5) *vanishes of infinite order at* 0*, i.e., if, for all $N > 0$,*

$$\int_{|x|<r} \left(|\mathbf{u}(x)|^2 + |p(x)|^2\right) dx = O(r^N) \qquad \text{as } r \to 0,$$

then $\mathbf{u}$ and p are equal to zero in $\overline{\Omega}$.

We can now prove the following lemma.

Lemma 5.3. *Assume that $\mathbf{u} \in H^2(\Omega)^n$ and $p \in H^1(\Omega)$ is a solution to* (2.1) *where $\boldsymbol{\varphi} = 0$ and $\boldsymbol{\psi} = 0$. Then $\mathbf{u} = 0$ and $p = 0$ in $\overline{\Omega}$.*

Proof. Let $\Omega_0 \subset \Omega$ be a domain with $\operatorname{dist}(\Omega_0, \Gamma_1) > 0$. Local regularity for elliptic equations implies that $\mathbf{u} \in C^1(\overline{\Omega}_0)^n$ and $p \in C(\overline{\Omega}_0)$. We then prove that $\partial^\alpha\mathbf{u} = 0$, for $|\alpha| \leq 1$, on Γ_0 and $p = 0$ on Γ_0. Let x_0 be any point on Γ_0. Without loss of generality, we can assume that x_n is the direction of the inward normal at x_0. Moreover, we can suppose that there exists a neighbourhood $\mathcal{U}$ of x_0 such that $\Omega \cap \mathcal{U}$ is given by the intersection of $\mathcal{U}$ and the graph domain $x_n > \alpha(x_1, \ldots, x_{n-1})$, where α is a function of class C^2. By combining the divergence condition and the boundary conditions, it follows that $\partial^\alpha\mathbf{u} = 0$ at x_0 for $|\alpha| \leq 1$. This, together with the condition $p\boldsymbol{\nu} - N\mathbf{u} = 0$ on Γ_0, imply that $p = 0$ at x_0. Since $x_0 \in \Gamma_0$ was arbitrary, it follows that $\partial^\alpha\mathbf{u} = 0$, for $|\alpha| \leq 1$, on Γ_0 and $p = 0$ on Γ_0.

These facts and the trace theorem imply that $\mathbf{u}$ and p can be extended by zero outside Γ_0. Also, the function $\mathbf{b}$ can be extended to a continuous function outside Γ_0. We denote this extended function by $\tilde{\mathbf{b}}$. We then have a solution to the problem

$$\begin{cases} \Delta\mathbf{u} + (\tilde{\mathbf{b}}\cdot\nabla)\mathbf{u} - \nabla p = 0 & \text{in } \Omega_1, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega_1, \end{cases}$$

where Ω_1 is a bounded domain in R^n with $\Omega \subset \Omega_1$, and $\mathbf{u}$ and p are zero in a part of Ω_1. Hence, there exist points where the solution vanishes of infinite order. Lemma 5.2 then implies that both $\mathbf{u} = 0$ and $p = 0$ in $\overline{\Omega}_1$. ∎

We can now prove an important property of the operator K defined in (5.1).

Lemma 5.4. *The kernel of K contains only* 0.

Proof. Assume that the kernel of K has a nonzero element $\boldsymbol{\eta} \in L^2(\Gamma_1)^n$. Let $\mathbf{u}$ and p be a weak solution to problem (3.1) with $\boldsymbol{\psi} = 0$. From the arguments used in the proof of Lemma 4.2, it can be seen that $\mathbf{u} \in H^2(\Omega_0)^n$ and $p \in H^1(\Omega_0)$, where $\Omega_0 \subset \Omega$ is a domain such that $\operatorname{dist}(\Omega_0, \Gamma_1) > 0$. Combining this with Lemma 5.3, it follows that $\mathbf{u} = 0$ and $p = 0$ in $\overline{\Omega}$. We therefore deduce that $\boldsymbol{\eta} = 0$. Thus, $\ker(K) = \{0\}$. ∎

B. The Adjoint Operator

The following lemma describes how problem (3.2) can be used to define the action of the adjoint operator K^*.

Lemma 5.5. *Let $\boldsymbol{\xi} \in L^2(\Gamma_0)^n$. The adjoint operator $K^* : L^2(\Gamma_0)^n \to L^2(\Gamma_1)^n$ to the operator K defined in* (5.1) *is given by $K^*\boldsymbol{\xi} = -(q\boldsymbol{\nu} - N^*\mathbf{v})|_{\Gamma_1}$, where $\mathbf{v} \in L^2(\Omega)^n$ and $q \in (H^1(\Omega))^*$ solve* (3.2).

Proof. Assume first that $\boldsymbol{\eta} \in C^2(\Gamma_1)^n$ and $\boldsymbol{\xi} \in C^2(\Gamma_0)^n$. Then there exists a unique solution $\mathbf{u} \in H^2(\Omega)^n$ and $p \in H^1(\Omega)$ to (3.1) with $\boldsymbol{\psi} = 0$ and a unique solution $\mathbf{v} \in H^2(\Omega)^n$ and $q \in H^1(\Omega)$ to problem (3.2). From (4.1), we find that

$$-\int_\Gamma \mathbf{v} \cdot (p\boldsymbol{\nu} - N\mathbf{u})\, dS + \int_\Gamma \mathbf{u} \cdot (q\boldsymbol{\nu} - N^*\mathbf{v})\, dS = 0. \tag{5.6}$$

Using this together with the boundary conditions, it follows that

$$\int_{\Gamma_0} \mathbf{u} \cdot \boldsymbol{\xi}\, dS + \int_{\Gamma_1} \boldsymbol{\eta} \cdot (q\boldsymbol{\nu} - N^*\mathbf{v})\, dS = 0. \tag{5.7}$$

Defining K^* by $K^*\boldsymbol{\xi} = -(q\boldsymbol{\nu} - N^*\mathbf{v})|_{\Gamma_1}$, we have

$$\int_{\Gamma_0} (K\boldsymbol{\eta}) \cdot \boldsymbol{\xi}\, dS = \int_{\Gamma_1} \boldsymbol{\eta} \cdot K^*\boldsymbol{\xi}\, dS. \tag{5.8}$$

For the general case, given $\boldsymbol{\eta} \in L^2(\Gamma_1)^n$ and $\boldsymbol{\xi} \in L^2(\Gamma_0)^n$, we approximate this data in the $L^2(\Gamma_i)^n$ sense by $C^2(\Gamma_i)^n$ functions, where $i = 0, 1$. It then follows that (5.8) still holds. This relation is the definition of the adjoint operator. Hence, the lemma follows. ■

C. End of the Proof of Theorem 5.1

From the procedure given in Section III and Lemma 5.5, it follows that

$$\begin{aligned}\boldsymbol{\eta}_k &= \boldsymbol{\eta}_{k-1} + \gamma(q_{k-1}\nu - N^*\mathbf{v}_{k-1})|_{\Gamma_1} = \boldsymbol{\eta}_{k-1} - \gamma K^*(\mathbf{u}_{k-1}|_{\Gamma_0} - \boldsymbol{\varphi})\\ &= \boldsymbol{\eta}_{k-1} - \gamma K^*(K\boldsymbol{\eta}_{k-1} - (\boldsymbol{\varphi} - K_1\boldsymbol{\psi})).\end{aligned}$$

This is the Landweber iteration for solving (5.3). By Theorem 6.1 in Engl et al. [17, p 155], the sequence $\boldsymbol{\eta}_k$ converges to $\boldsymbol{\eta}$ in $L^2(\Gamma_1)^n$ since $0 < \gamma < 1/\|K\|^2$. Notice that $\mathbf{u}$ and p satisfy problem (3.1). Applying the inequality (4.4), we see that the sequences $\mathbf{u}_k$ and p_k converge to $\mathbf{u}$ in $L^2(\Omega)^n$ and p in $(H^1(\Omega))^*$, respectively.

VI. A STOPPING RULE

From the proof of Theorem 5.1 it follows that the procedure in Section III is equivalent with the Landweber-Fridman method for solving equation (5.3). Since the Landweber-Fridman method is regularizing (see section 6 in Engl et al. [17]), the procedure proposed in this article is also a regularization method, and it therefore works with inexact Cauchy data. More precisely, consider the case when there is some error in $\boldsymbol{\varphi}$ in (2.1), namely

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}^\delta\|_{L^2(\Gamma_0)^n} \leq \delta, \tag{6.1}$$

with $\delta > 0$. The elements $\mathbf{u}_k^\delta$, p_k^δ and η_k^δ, are obtained by using the procedure of Section III with Cauchy data $\boldsymbol{\varphi}^\delta$ and $\boldsymbol{\psi}$.

Given the noise level, we can use the discrepancy principle of Morozov [18], to obtain a stopping criterion. This suggests determining the stopping index $k = k(\delta)$ as the smallest index for which

$$\|K\boldsymbol{\eta}_k^\delta - (\boldsymbol{\varphi}^\delta - K_1\boldsymbol{\psi})\|_{L^2(\Gamma_0)^n} = \|\mathbf{u}_k^\delta|_{\Gamma_0} - \boldsymbol{\varphi}^\delta\|_{L^2(\Gamma_0)^n} \approx \delta.$$

VII. A BOUNDARY ELEMENT FORMULATION

In practice, the direct, well-posed, mixed boundary value problems (3.1) and (3.2) of the iterative procedure described in Section III can seldom be solved analytically and therefore some form of numerical approximation is necessary. This can most advantageously be performed using the boundary element method (BEM). The advantages of this method over the domain discretisation methods, such as the finite-difference or finite element methods, consist of the following.

(i) Only the boundary $\partial\Omega$ needs to be discretized and therefore no domain discretisation is required. This in turn saves computational time and additional storage requirements.

(ii) It easily deals with irregular domains Ω since only the boundary $\partial\Omega$ needs to be discretized.

(iii) Once the boundary values have been found accurately, it evaluates the interior solution at any point in the domain Ω without the need of interpolation onto domain discretization cells.

(iv) For exterior problems in unbounded domains, e.g., the region outside Ω, it easily deals with the infinity condition without the need to include an arbitrary box or a circle at a large distance from Ω.

One possible disadvantage of implementing this method is that it depends on whether a fundamental solution for the governing partial differential equation can be found explicitly. Thus, in this section we illustrate the BEM only for the Stokes equations, i.e., $\mathbf{b} \equiv 0$ in the definition of the operator L, namely

$$\begin{cases} \Delta\mathbf{u} - \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega. \end{cases} \tag{7.1}$$

Using Green's formula, we can recast the Stokes equations (7.1) into a boundary integral form (see Chapter 3 in Ladyzhenskaya [19]):

$$c(\mathbf{x})\mathbf{u}(\mathbf{x}) = \int_{\partial\Omega} \left[K(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y}) - U(\mathbf{x},\mathbf{y})\mathbf{t}(\mathbf{y})\right] dS(\mathbf{y}), \qquad \text{for} \quad \mathbf{x} \in \overline{\Omega}, \tag{7.2}$$

where $c(\mathbf{x})$ is a coefficient function that is equal to 1 if $\mathbf{x} \in \Omega$, and 0.5 if $\mathbf{x} \in \partial\Omega$ and $\partial\Omega$ (smooth), $\mathbf{t} = (t_1, t_2) = -p\boldsymbol{\nu} + N\mathbf{u}$ ($\mathbf{t} = -\mathbf{T}$) is the fluid surface traction, and, in two dimensions, i.e.,

$n = 2$, the tensors K and U are given by

$$K_{kl}(\mathbf{x},\mathbf{y}) = -\frac{(x_k - y_k)(x_l - y_l)}{\pi \mid \mathbf{x}-\mathbf{y} \mid^4} \sum_{m=1}^{2}(x_m - y_m)\nu_m,$$

$$U_{kl}(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi}\left[-\delta_{kl} \ln \mid \mathbf{x}-\mathbf{y} \mid + \frac{(x_k - y_k)(x_l - y_l)}{\mid \mathbf{x}-\mathbf{y} \mid^2}\right],$$

for $k, l = 1, 2$. Here δ_{kl} is the Kronecker tensor, $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $\boldsymbol{\nu} = (\nu_1, \nu_2)$. Nevertheless, solving the boundary integral equation (7.2) analytically can be performed only in very simple cases and therefore numerical methods seem more appropriate. Based on the BEM, in two dimensions we discretize each of the boundaries Γ_0 and Γ_1 into a collection of uniformly distributed $M/2$ straight-line segments and use piecewise constant approximations for $\mathbf{u}$ and $\mathbf{t}$ over $\partial\Omega$. This recasts the boundary integral equation (7.2) as a system of $2M$ linear algebraic equations with $4M$ unknowns. However, $2M$ of these unknowns can be eliminated by imposing the boundary conditions of the mixed problem (3.1) or (3.2). The resulting system of linear algebraic equations at each iteration k can then be written in a generic form as follows:

$$A\mathbf{x}^{(k)} = \mathbf{b}^{(k)}, \tag{7.3}$$

where the coefficients of the matrix $A \in R^{2M\times 2M}$ and the vector $\mathbf{b} \in R^{2M}$ can be evaluated analytically, as described in Zeb et al. [20].

Remarks. (i) The matrix A depends only on the type of mixed boundary conditions in (3.1) or (3.2), i.e., Dirichlet on Γ_1 and Neumann on Γ_0, and thus it can be calculated only once and stored.

(ii) The system of equations (7.3) is well conditioned although the original Cauchy problem is ill posed.

(iii) Alternatively, instead of using the direct BEM described in this section, one can use an indirect second kind integral formulation for the solution of the mixed boundary problem (3.1) or (3.2) for the Stokes flow between two closed surfaces Γ_0 and Γ_1 (see Power and Gomez [21]).

VIII. NUMERICAL RESULTS AND DISCUSSION

In this section, we illustrate the numerical results obtained using the iterative method proposed in Section III combined with the BEM, described in the previous section. In addition, we investigate the stability of the method when the Cauchy data $\boldsymbol{\varphi}$ in (2.1) is perturbed by noise as in (6.1). In order to assess the convergence and stability of the algorithm, we define two different types of errors, namely,

$$e_a^{(k)} = \|\mathbf{u}|_{\Gamma_1} - \boldsymbol{\eta}_k^\delta\|_{L^2(\Gamma_1)^n}, \tag{8.1}$$

$$e_c^{(k)} = \|\mathbf{u}_k^\delta|_{\Gamma_0} - \boldsymbol{\varphi}^\delta\|_{L^2(\Gamma_0)^n}. \tag{8.2}$$

Here, we have used the notation of Section VI. Clearly, the accuracy error $e_a^{(k)}$ can be defined only when the analytical solution $\mathbf{u}|_{\Gamma_1} = \boldsymbol{\eta}$ is known. However, $e_c^{(k)}$ can always be calculated at each iteration k. According to the discrepancy principle (see Section VI), we stop the iteration at the first index k for which $e_c^{(k)} \approx \delta$.

We test three different benchmark examples in a two-dimensional bounded annular domain $\Omega = \{(x, y) : R_1^2 < x^2 + y^2 < R_0^2\}$, corresponding to a region between two infinitely long circular cylinders of radii $0 < R_1 < R_0$, which is filled with viscous fluid flowing at low Reynolds numbers. The Cauchy problem is to determine the fluid velocity $\mathbf{u}$ and the traction $\mathbf{t}$ at the inner cylinder

$$\Gamma_1 = \left\{(x, y) \mid x^2 + y^2 = R_1^2\right\}$$

by taking the measurements of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ at the outer cylinder

$$\Gamma_0 = \left\{(x, y) \mid x^2 + y^2 = R_0^2\right\}.$$

The model problems considered in the first two examples below have no physical meaning, but, because their solutions are explicitly known, the accuracy of the numerical results can easily be assessed. The third example is more physically realistic and uses the viscous flow assumption that the fluid velocity at the boundary should be equal to the wall velocity.

We now describe the different examples in more detail and then discuss the numerical results obtained for them.

Example 1. Consider the following example:

$$\mathbf{u} = (u_1, u_2) = (4y^3 - x^2, 4x^3 + 2xy - 1), \qquad p = 24xy - 2x, \quad (x, y) \in \Omega, \tag{8.3}$$

which satisfies the first two equations in (2.1) and has been previously considered by Zeb et al. [3] in another type of inverse problem. The analytical example (8.3) generates the Cauchy data in (2.1) as

$$\boldsymbol{\varphi} = (4y^3 - x^2, 4x^3 + 2xy - 1), \qquad (x, y) \in \Gamma_0,$$
$$\boldsymbol{\psi} = (-12y^3 - 2y^2 + 2x^2 + 12x^2y, -12x^3 + 12xy^2 - 8xy)/R_0, \qquad (x, y) \in \Gamma_0.$$

Example 2. Consider now a Leray-type solution (see Varnhorn [22]):

$$\mathbf{u} = (u_1, u_2) = (3 - x^2 - 2y^2, 2xy), \qquad p = -6x, \quad (x, y) \in \Omega. \tag{8.4}$$

This generates the Cauchy data

$$\boldsymbol{\varphi} = (3 - x^2 - 2y^2, 2xy), \qquad (x, y) \in \Gamma_0,$$
$$\boldsymbol{\psi} = (2y^2 - 2x^2, -8xy)/R_0, \qquad (x, y) \in \Gamma_0.$$

Example 3. Finally, we consider a more physical example given by the two-dimensional steady slow viscous flow between two co-axial cylinders of radii $0 < R_1 < R_0$ rotating about their axis with constant angular velocities ω_1 and ω_0. Then the analytical solution is given by (see Curteanu [4]) the following:

$$\mathbf{u} = (u_1, u_2) = \left(-y\left(C_0 + \frac{C_1}{x^2 + y^2}\right), \, x\left(C_0 + \frac{C_1}{x^2 + y^2}\right)\right), \qquad p = 0, \quad (x, y) \in \Omega. \tag{8.5}$$

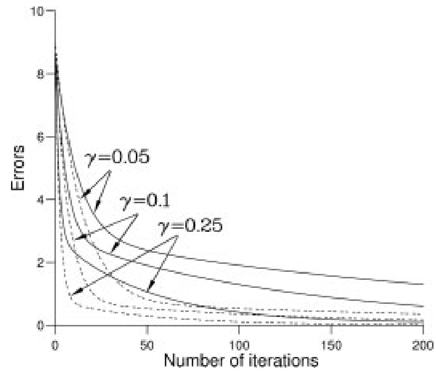

FIG. 1. The errors $e_a^{(k)}$ (—) and $e_c^{(k)}$ (– –), for various γ, for Example 1.

This generates the Cauchy data

$$\boldsymbol{\varphi} = \big(-y(C_0R_0^2 + C_1),\ x(C_0R_0^2 + C_1)\big)/R_0^2, \qquad (x, y) \in \Gamma_0,$$
$$\boldsymbol{\psi} = (-2C_1y, 2C_1x)/R_0^3, \qquad (x, y) \in \Gamma_0.$$

Choosing

$$C_0 = \frac{\omega_0R_0^2 - \omega_1R_1^2}{R_0^2 - R_1^2} \qquad \text{and} \qquad C_1 = \frac{R_0^2R_1^2(\omega_1 - \omega_0)}{R_0^2 - R_1^2},$$

we observe that $\mathbf{u} \cdot \boldsymbol{\nu} = 0$ on Γ, $\mathbf{u} \cdot \mathbf{s} = \omega_0R_0$ on Γ_0, and $\mathbf{u} \cdot \mathbf{s} = -\omega_1R_1$ on Γ_1, where $(\mathbf{s}, \boldsymbol{\nu})$ form a right-handed orthogonal set on the boundaries Γ_0 and Γ_1, with $\mathbf{s}$ along the tangent to the boundary.

For simplicity, we take $R_1 = 1$ and $R_0 = 2$, and in Example 3, $\omega_0 = 1$ and $\omega_1 = -2$, and employ $M = 128$ boundary elements. An arbitrary initial guess, such as $\boldsymbol{\eta}_0 = 0$, has been chosen to initiate the procedure.

Figures 1, 4, and 7 show the errors $e_a^{(k)}$ and $e_c^{(k)}$ given by Equations (8.1) and (8.2), as functions of the number of iterations k, for exact data, i.e., $\alpha = 0$, for various values $\gamma \in \{0.05, 0.1, 0.25\}$, for Examples 1, 2, and 3, respectively. In this case $\delta = 0$, and therefore no stopping criterion is

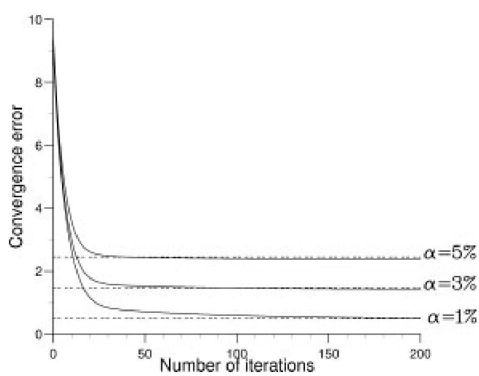

FIG. 2. The error $e_c^{(k)}$ for $\gamma = 0.1$ and various α, for Example 1. The dashed lines represent the corresponding values of δ.

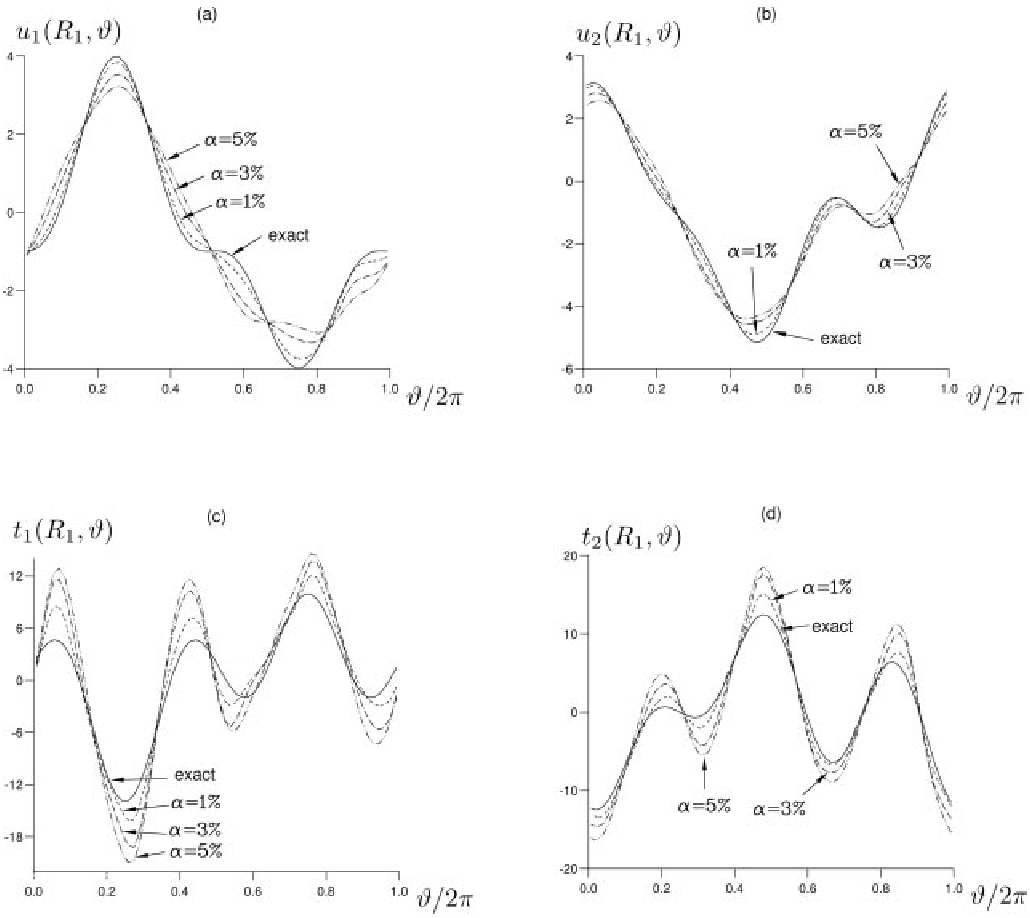

FIG. 3. (a–d) The numerical values of (a) $u_1(R_1, \vartheta)$, (b) $u_2(R_1, \vartheta)$, (c) $t_1(R_1, \vartheta)$, and (d) $t_2(R_1, \vartheta)$ for various amounts of noise $\alpha = 1\%$, $\alpha = 3\%$, and $\alpha = 5\%$, in comparison with the corresponding analytical solutions obtained from (8.3), for Example 1.

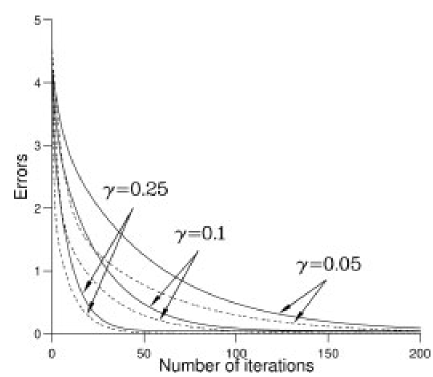

FIG. 4. The errors $e_a^{(k)}$ (—) and $e_c^{(k)}$ (– –), for various γ, for Example 2.

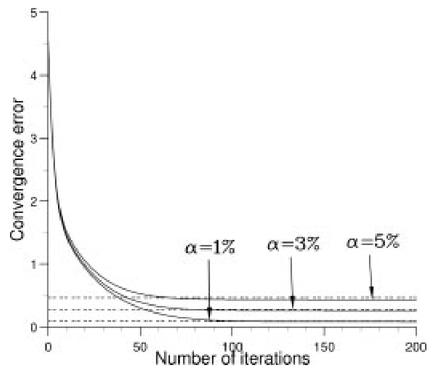

FIG. 5. The error $e_c^{(k)}$ for $\gamma = 0.1$ and various α, for Example 2. The dashed lines represent the corresponding values of δ.

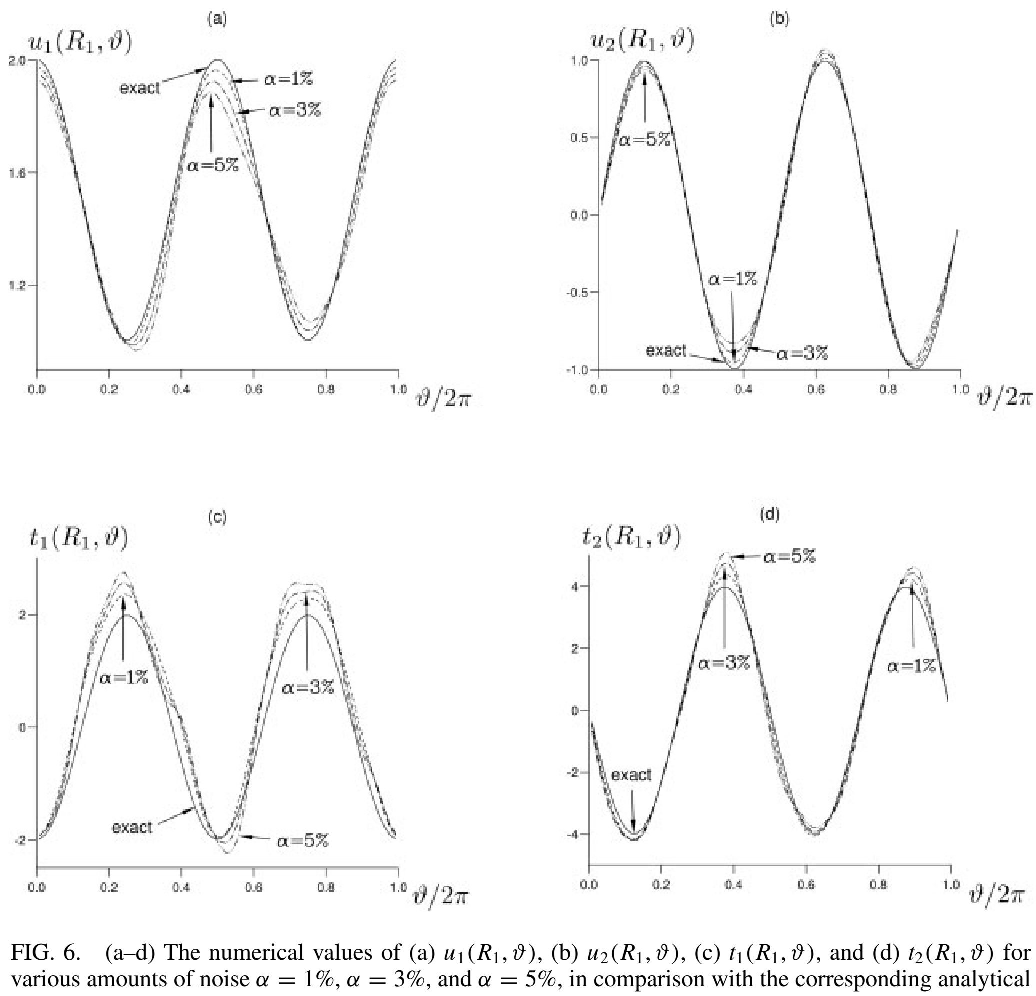

FIG. 6. (a–d) The numerical values of (a) $u_1(R_1, \vartheta)$, (b) $u_2(R_1, \vartheta)$, (c) $t_1(R_1, \vartheta)$, and (d) $t_2(R_1, \vartheta)$ for various amounts of noise $\alpha = 1\%$, $\alpha = 3\%$, and $\alpha = 5\%$, in comparison with the corresponding analytical solutions obtained from (8.4), for Example 2.

TABLE I. The values of δ and the stopping iteration indexes k for various percentages of noise α for Examples 1, 2, and 3.

	$\alpha = 1\%$	$\alpha = 3\%$	$\alpha = 5\%$
Example 1	$\delta = 0.490$ $k = 239$	$\delta = 1.470$ $k = 97$	$\delta = 2.450$ $k = 42$
Example 2	$\delta = 0.093$ $k = 108$	$\delta = 0.280$ $k = 80$	$\delta = 0.467$ $k = 67$
Example 3	$\delta = 0.038$ $k = 49$	$\delta = 0.115$ $k = 22$	$\delta = 0.192$ $k = 13$

required, up to the computer machine precision. Clearly, the rate of convergence depends on the relaxation parameter γ. From Figs. 1 and 4 it can be seen that, as expected, the smaller the values of γ (>0), the slower the iterative procedure converges. This is also shown in Fig. 7, with the note that a larger number of iterations is required to illustrate this conclusion. If γ is too large, i.e., if $\gamma \geq 1/\|K\|^2$, then the iterative process may diverge (see Theorem 5.1). In the examples considered, once γ is greater than about 0.25, it was found numerically that the iterative process diverges. According to Theorem 5.1 this suggests that $\|K\| \geq 2$ for the annular domain considered. Some analysis into this can be performed for Example 3, by remarking that $\mathbf{u}|_{\Gamma_0} = (-y, x)$ and $\mathbf{u}|_{\Gamma_1} = -2(-y, x)$. A direct calculation gives $\|\mathbf{u}\|_{L^2(\Gamma_1)^n} = \sqrt{2\pi}$ and $\|K\mathbf{u}\|_{L^2(\Gamma_0)^n} = 2\sqrt{2\pi}$. This implies that $\|K\| \geq 2$. From the above analysis, one can conjecture that the condition $0 < \gamma < 1/\|K\|^2$ in Theorem 5.1 is also sufficient for ensuring the convergence of the numerical solution.

Figures 2, 5, and 8 show for Examples 1, 2, and 3, respectively, $e_c^{(k)}$ given by equation (8.2), as functions of the number of iterations, for fixed $\gamma = 0.1$, and various percentages of noise level $\alpha \in \{1, 2, 3\}\%$ added into the Cauchy data $\boldsymbol{\varphi}$ in (2.1) of the form

$$\boldsymbol{\varphi}^\delta = \boldsymbol{\varphi}(1 + \alpha\epsilon),$$

where α represents the percentage of noise and ϵ is a pseudo-random real number taken from a uniform distribution over the interval $[-1, 1]$. This is generated using the NAG routine, G05DAF. In fact, in the discretized $\boldsymbol{\varphi}$, the routine will generate a vector of pseudo-random numbers.

The iterative process is stopped according to the discrepancy principle of Section VI, namely we stop the iteration at the smallest index k for which $e_c^{(k)} \approx \delta = \alpha\|\varepsilon\boldsymbol{\varphi}\|_{L^2(\Gamma_0)^n}$. These stopping values can be obtained graphically from Figs. 2, 5, and 8 and they are included in Table I.

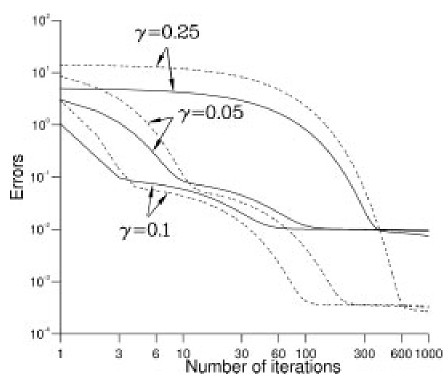

FIG. 7. The errors $e_a^{(k)}$ (—) and $e_c^{(k)}$ (– –), for various γ, for Example 3.

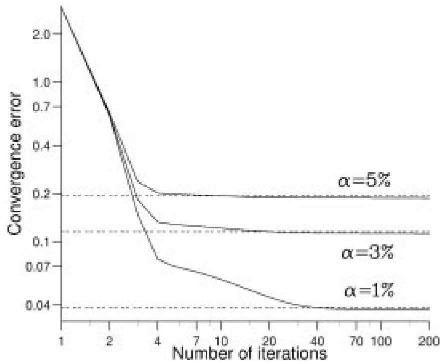

FIG. 8. The error $e_c^{(k)}$ for $\gamma = 0.1$ and various α, for Example 3. The dashed lines represent the corresponding values of δ.

With this stopping rule, Figs. 3 (a–d), 6 (a–d), and 9 (a–d) show the numerical values for the fluid velocity and the traction components on the inner boundary Γ_1 as functions of the angle ϑ, i.e., $u_1(R_1, \vartheta)$, $u_2(R_1, \vartheta)$, $t_1(R_1, \vartheta)$, and $t_2(R_1, \vartheta)$, respectively, in comparison with the

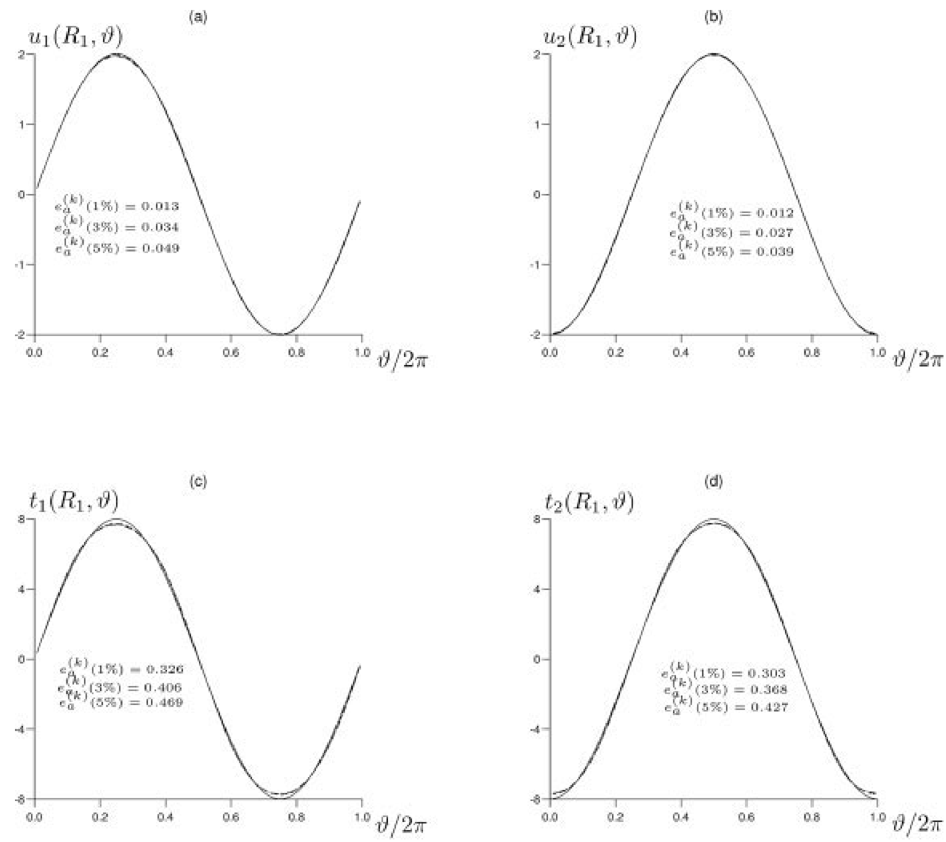

FIG. 9. (a–d) The numerical values of (a) $u_1(R_1, \vartheta)$, (b) $u_2(R_1, \vartheta)$, (c) $t_1(R_1, \vartheta)$, and (d) $t_2(R_1, \vartheta)$ for various amounts of noise $\alpha = 1\%$, $\alpha = 3\%$, and $\alpha = 5\%$, in comparison with the corresponding analytical solutions obtained from (8.5), for Example 3.

corresponding analytical solutions obtained from (8.3), (8.4), and (8.5), for various percentages of noise $\alpha \in \{1, 2, 3\}\%$, for Examples 1, 2, and 3. From these figures it can be seen that the numerically obtained solutions are stable, and the accuracy of the numerical solution increases as α decreases. Furthermore, in Figures 9 (a–d), the numerical results are difficult to distinguish for various α and therefore on these figures the values of $e_a^{(k)}$ have been included.

IX. CONCLUSIONS

In this article, we have investigated the Cauchy problem for the viscous stationary linear generalized Stokes system with L^2 boundary data. In order to handle the instability of the solution of this ill-posed problem, an iterative Landweber-Fridman type algorithm was developed. This approach reduces the Cauchy problem to solving a sequence of well-posed mixed boundary value problems. A discrepancy principle was used to give a stopping rule in the case of noise in the Cauchy data. For a two-dimensional Stokes viscous flow in an annular domain, the numerical results obtained for three different test examples using various amounts of noise added into the Cauchy data, showed that the iterative boundary element method produces a convergent and stable numerical solution. From a numerical point of view it might be difficult to choose the parameter γ, which occurs in the iterative procedure in the right interval to obtain convergence. However, it might be possible to propose a parameter free procedure, based on, for example, the conjugate gradient method or the minimal error method, but this is deferred to a future work.

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