An effective relaxed alternating procedure for Cauchy problem connected with Helmholtz equation

F. Aboud[a], K. Berdawood[b,c], A. Nachaoui[c], M. Nachaoui[1d]

[a] *University of Diyala, Diyala,Iraq*
[b] *Department of Mathematics Salahaddin University-Erbil*
[c] *Laboratoire de Mathématiques Jean Leray UMR6629 CNRS / Université de Nantes 2 rue de la Houssinière, BP92208 44322 Nantes, France*
[d] *Laboratoire de Mathématiques et Applications Université Sultan Moulay slimane, Faculté des Sciences et Techniques, B.P.523, Béni-Mellal, Maroc.*

Abstract

This paper is concerned by the Cauchy problem for the Helmholtz equation. Recently, some new works [5] asked the convergence of the well-known alternating iterative method [28]. Our main result is to propose a new alternating algorithm based on relaxation technique. In contrast to the existing results, the proposed algorithm is simple to implement, converges for all choice of wave number, and it can be used as an acceleration of convergence in the case where the classical alternating algorithm converges. We present theoretical results of the convergence of our algorithm. The numerical results obtained using our relaxed algorithm and the finite element approximation show the numerical stability, consistency and convergence of this algorithm. This confirms the efficiency of the proposed method.

Keywords: Inverse Cauchy problem, Data completion, Helmholtz equation, Relaxed alternating iterative method, Numerical simulation.

1. Introduction

The Helmholtz equation is an elliptic Partial Differential Equation (PDE), which represents time-independent solutions of the wave equation. It is often encountered in many branches of science and engineering. This equation is used to model a wide variety of physical phenomena. These include among others, wave propagation, vibration phenomena, aeroacoustics, under-water acoustics, seismic inversion, electromagnetic, as well as heat conduction problems. Efficient and accurate numerical approximation to the Helmholtz equation is significant to scientific computation. This, explain the extensive works about the approximation of Helmholtz equation [10, 15, 17, 31, 33, 35, 36, 43]. In General, boundary value problems for the Helmholtz equation reads as follow

$$\Delta u + k^2 u = f,$$

[1]Principal & Corresponding author: nachaoui@gmail.com

where k is the wave number. In solving physical phenomenon governed by Helmholtz equation, some data, like the geometry of the domain of interest, the material properties, the external sources acting in the solution domain and the boundary and initial conditions must be completely known. These are referred to as direct problems and their well-posedness has been clearly established, see for example [23]. When one or more of the conditions for solving the direct problem are partially or entirely unknown then an inverse problem may be formulated to determine the unknowns from specified or measured system responses. A classical example of inverse problems is the Cauchy one. In this class of problems, boundary conditions for both the solution and its normal derivative are prescribed only on a part of the boundary of the solution domain, whilst no information is available on the other part of the boundary. The Cauchy problem for elliptic equation is known to be severely ill-posed in the sense of Hadamard [16] i.e., the solution does not depend continuously on the Cauchy data, and thus a small error in the given data can cause a large gap between the solution of the original problem and that of the problem with nosy data.

Over the last two decades, many methods have been proposed to deal with the Cauchy problem for the Helmholtz equation. For instance, the truncation method [42], the conjugate gradient method [35, 36], the fractional Tikhonov regularization method [41], the spherical wave expansion method [43], the mollification method [31], the Fourier regularization method [10, 15], the method of fundamental solution [38], and the quasi-reversibility regularization method [40].

As it is well known, the quality of discrete numerical solutions to the direct Helmholtz equation depends significantly on the physical parameter k. Indeed, many works has mentioned that the quality of the numerical solution deteriorates when increasing of the wave number k [18, 19]. As the resolution of the inverse problems generally depends on associated direct problems. This raises the question, to be answered in this work: how can obtained good approximation of Cauchy Helmholtz equation defined with high values of k?

Many authors concerned by Cauchy Helmholtz equation have remarked this effect of high values of wave number k [5, 20, 34, 41]. Some authors developed methods allowing to solve Cauchy Helmholtz problem, even if the value of k is high [3, 5, 6, 27, 41]. Indeed, in [27, 41] the approach concerns the regularisation technique depends on heuristics choice of regularisation term and parameter of regularisation.

Recently, a modified alternating algorithm for solving the Cauchy problem for the Helmholtz equation in this situation has been proposed [5]. This algorithm based on the choice of an artificial interior boundary inside the domain together with a jump condition that includes a parameter μ. They showed that by selecting an appropriate interior boundary and sufficiently large value for μ, one can get a convergent iterative regularization method. They have proved the convergence of this method. The analysis of the numerical results showed that this method is rather slow and requires a large number of iterations. This drawback is commonly known for iterative regularization methods such as the original alternating algorithm [25, 22]. Berntson et al. in [4] proposed to use conjugate gradient type methods to achieve much faster convergence. But the interior boundary is necessary in order to implement the conjugate gradient method for this Cauchy problem. In addition to the difficulty of solving problems with internal interfaces in the case of any domain, no practical proposal for the choice of these parameters (internal interface and mu parameter) ensuring convergence. Jourhmane and Nachaoui [24, 25, 26] proposed the relaxation of the given Dirichlet data in the case of the alternating iterative algorithm of KMF applied to the Cauchy problem for steady-state heat conduction in isotropic and anisotropic media, respectively. This procedure drastically reduced the number of iterations required to achieve convergence for the inverse problems considered. A relaxation of the alternating method in elasticity was both numerically and theoretically investigated in [11, 37], and for Cauchy problem governed by Stocks equation [7].

Encouraged by these results, we do further investigations and propose in this paper a relaxation of the data on the over-specified boundary, in the case of the Helmholtz equation. Moreover, we also prove the convergence of these schemes and introduce appropriate optimal stopping criterion. Our approach preserves not only the differential equation but introduces no modification of the domain. It consists of solving two well-posed auxiliary problems for the original differential equation on the same domain of study. The regularizing nature of our algorithm is ensured by an appropriate choice of the boundary conditions at every step.

The main advantage of our approach lies in the fact that there is no heuristic parameters in algorithms on the contrary, all used parameters are completely expressed as function of the given data. Moreover, through a theoretical result and numerical analysis we prove that for any value wave number k we find an interval of relaxation parameter in which the convergence is assured.

The remainder of this paper is organized as follow. In section 2, we presents the mathematical formulation, which takes into account both cases Helmholtz and modified Helmholtz equations. Then through an example, we prove that the Cauchy problem for Helmholtz equation is ill-posed. In section 3, we describe the classical alternating algorithms. The section 4 is devoted to description of the proposed algorithms. In the section 5, we present the convergence results and we prove the main theorem of convergence. Finally, the section 6 is devoted to the numerical results and discussions.

2. Mathematical formulation

Let Ω be a bounded domain in $\mathbb{R}^d$ with the Lipschitz boundary Γ, where d is the space dimension in which the problem is posed, usually $d \in \{1, 2, 3\}$. Let Γ be divided into two disjoint parts Γ_0 and Γ_1.We denote ν the outward unite normal to the boundary Ω and consider the following Cauchy problem for the Helmholtz equation:

$$\Delta u \pm k^2 u = 0, \quad \text{in} \quad \Omega, \tag{2.1}$$

subject to the the following boundary conditions

$$u = f_1, \quad \text{on} \quad \Gamma_1, \tag{2.2}$$

$$\partial_\nu u = f_2, \quad \text{on} \quad \Gamma_1, \tag{2.3}$$

where the wave number k is positive real constant, ∂_ν denotes the outward normal derivative of u, f_1 and f_2 are known Cauchy data on Γ_1. In the Eq. (2.1), the problem is known as the Helmholtz equation for the case

$$\Delta u + k^2 u = 0, \quad \text{in} \quad \Omega, \tag{2.4}$$

and the modified Helmholtz equation for

$$\Delta u - k^2 u = 0, \quad \text{in} \quad \Omega. \tag{2.5}$$

In this study, both cases will be considered. In the direct problems, the data on the boundary Γ_0 are available and enables us to determine u at any point inside the computational domain Ω. In contrast, in the inverse Cauchy problems, the boundary Γ_1 is over-specified because both the Dirichlet and Neumann conditions (see Eqs. (2.2) to (2.3)) are known whilst the remaining Γ_0 is under-specified since it has no condition and we have to determine it.

The Cauchy problem is an ill-posed problem in the sense of Hadamard [16]. Indeed, even if the solution exists and unique it does not depend continuously on the Cauchy data. Actually, consider the Cauchy problem for the Helmholtz equation Eq. (2.4) where $\Omega =]0, a[\times]0, b[$ with the following boundary conditions,

$$u(x, 0) = \phi(x), \quad \text{on} \quad 0 \le x \le a, \tag{2.6}$$

$$\partial_y u(x, 0) = 0, \qquad \text{on} \quad 0 \le x \le a, \tag{2.7}$$

$$u(0, y) = u(a, y) = 0, \qquad \text{on} \quad 0 \le y \le b. \tag{2.8}$$

It is easy to derive a solution of the problem Eq. (2.4) and Eqs. (2.6) to (2.8) by the method of separation of variables combined with the principle of superposition, which is widely used to solve initial boundary-value problems involving linear partial differential equations. Indeed, the basic idea is to seek a solution expressed in the separable form

$$u(x, y) = X(x)Y(y),$$

where X and Y are functions of x and y respectively. In many cases, the partial differential equation reduces to two ordinary differential equations for X and Y. Then, solving the Helmholtz equation Eq. (2.4) is reduced to

$$\frac{X''}{X} + \frac{Y''}{Y} + k^2 = 0.$$

The constraint that we must impose to solve the Helmholtz equation is

$$\alpha^2 - \beta^2 + k^2 = 0,$$

where α and β are two real constants verify

$$\frac{X''}{X} = -\beta^2 \text{ and } \frac{Y''}{Y} = \alpha^2.$$

The solutions here are familiar:

$$X(x) = A\exp(-\mathrm{i}\beta x) + B\exp(\mathrm{i}\beta x) \text{ and } Y(y) = C\exp(-\alpha y) + D\exp(\alpha y).$$

Then,

$$u(x,y) = [A\sin(\beta x) + B\cos(\beta x)]\,[C\sinh(\alpha y) + D\cosh(\alpha y)]\,.$$

Taking into account the boundary conditions Eqs. (2.6) to (2.8) we have

$$\begin{aligned} u(0,y) = 0 \quad &\Rightarrow \quad B = 0, \\ u(a,y) = 0 \quad &\Rightarrow \quad \sin(\beta a) = 0 \text{ and thus their exists } n \text{ such that } \beta_n = \beta = \frac{n\pi}{a}, \end{aligned}$$

then,

$$\forall n \geq 0, \quad u_n(x,y) = A_n \sin(\beta_n x)\,[C_n \sinh(\alpha_n y) + D_n \cosh(\alpha_n y)]$$

is solution of the Helmholtz equation. Then,

$$\partial_y u(x,0) = 0 \quad \Rightarrow \quad C_n = 0,$$

Since the problem Eq. (2.4) and Eqs. (2.6) to (2.8) is linear and homogeneous, then by superposition principal and using the orthogonality of functions $\{\sin(\frac{n\pi}{a}x)\}_{n=1}^{\infty}$, the solution is reduced to

$$u(x,y) = \sum_{n=0}^{\infty} A_n \sin(\beta_n x)\cosh(\alpha_n y),$$

where $A_n = A_n D_n$.

Then, from the condition $u(x,0) = \phi(x)$ we get

$$A_n = \frac{2}{a}\int_0^a \phi(x)\sin(\beta_n x)\,\mathrm{dx}\,.$$

Finally, the solution can be reads to,

$$u(x,y) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} A_n \sin(\beta_n x)\cosh\left(\sqrt{\lambda_n}y\right), & 0 < k < \dfrac{\pi}{a}, \\ \displaystyle\sum_{n=1}^{n_0} A_n \sin(\beta_n x)\cosh\left(\sqrt{-\lambda_n}y\right) + \sum_{n=n_0+1}^{\infty} A_n \sin(\beta_n x)\cosh\left(\sqrt{\lambda_n}y\right), & k \geq \dfrac{\pi}{a}. \end{cases} \tag{2.9}$$

where $n_0 > 0$ an integer such that $\frac{n_0\pi}{a} \geq k$ and $\lambda_n = (\beta_n)^2 - k^2$.

Then to prove that problem Eq. (2.4) and Eqs. (2.6) to (2.8) is unstable in Hadamard sense [16], may takes $\phi_n(x) = \frac{\sin(\beta_n x)}{n}$, which gives $u_n(x,y) = \frac{\sin(\beta_n x)}{n}\cosh\left(\sqrt{\lambda_n}y\right)$ as exact solution of the problem Eq. (2.4) and Eqs. (2.6) to (2.8), where $\frac{n\pi}{a} > k$ are positive integers. Since

$$\sup_{x\in[0,a]} |\phi_n(x)| \to 0, \text{ as } n \to \infty,$$

whilst

$$\sup_{x\in[0,a]} |u_n(x)| \to \infty \text{ as } n \to \infty,$$

then the Cauchy problem for Helmholtz equation is ill–posed. Another way of showing that the Helmholtz equation leads to ill–posed problem can be found in Lavrent'ev [29, 30]. Note that the same conclusion can be obtained for the Cauchy problem governed by the modified Helmholtz equation Eq. (2.5) with $\lambda_n = (\beta_n)^2 + k^2$.

This explains the need to develop stable methods that lead to obtain approximate solutions that are less sensitive to perturbations. In the sequel, we will be interested in some algorithms, called iterative alternating-algorithms that have shown their efficiencies for solving several Cauchy problems governed by different PDE.

3. Description of algorithms

The alternating procedure is an iterative algorithm for solving Cauchy problems. It consists in obtaining successive solutions of two well-posed problems for the original differential equation. The attractive advantages of such algorithms lie in the fact that they preserve the differential equations and there regularizing character ensured solely by an appropriate choice of boundary conditions in each iteration. These methods have been introduced by Kozlov-Maz'ya in [28]. Then it was implemented and improved by relaxation schemes in [24, 25, 26]. After that, different studies have been done using these algorithms for solving ill-posed problems governed by partial differential equations [1, 2, 7, 8, 9, 11, 12, 13, 39].

The classical alternating procedure for the problem Eqs. (2.1) to (2.3) consists in solving alternatively two auxiliary problems defined respectively by

$$\Delta u \pm k^2 u = 0, \quad \text{in } \Omega, \tag{3.1}$$

$$u = v, \quad \text{on } \Gamma_0, \tag{3.2}$$

$$\partial_\nu u = f_2, \quad \text{on } \Gamma_1, \tag{3.3}$$

and

$$\Delta u \pm k^2 u = 0, \quad \text{in } \Omega, \tag{3.4}$$

$$\partial_\nu u = \eta, \quad \text{on } \Gamma_0, \tag{3.5}$$

$$u = f_1, \quad \text{on } \Gamma_1, \tag{3.6}$$

where f_1 and f_2 are given in Eqs. (2.1) to (2.3), while v and η are two functions will be changing in each iteration. The operator $\pm$ used to take into consideration the two cases problem Eq. (2.4) and Eq. (2.5). Problems Eqs. (3.1) to (3.3) and Eqs. (3.4) to (3.6) should alternately be solved until a prescribed stopping criterion is satisfied. According to whether we start with the problem Eqs. (3.1) to (3.3) or Eqs. (3.4) to (3.6), this procedure gives rise to two classical iterative algorithms. The first algorithm can be summarized by

Algorithm 1: Classical approach as proposed in [28]

1: $n \leftarrow 0$, choose the initial guess $v = v^0$, then
2: Find $u^{(2n)}$ by solving the problem Eqs. (3.1) to (3.3) and compute $\eta = \partial_\nu u^{(2n)} |_{\Gamma_0}$.
3: Find $u^{(2n+1)}$ by solving the problem Eqs. (3.4) to (3.6).
4: If $\frac{2||u^{(2n)}-u^{(2n+1)}||_{L^\infty(\Gamma_0)}}{||u^{(2n+1)}||_{L^\infty(\Gamma_0)}+||u^{(2n)}||_{L^\infty(\Gamma_0)}} < \varepsilon$ then stop.
5: Else, $n \leftarrow n+1$, then
6: Compute $v = v^{(n)} = u^{(2n-1)} |_{\Gamma_0}$ and go to step 2.

While the second algorithm, it can be summarized by

Algorithm 2: Classical approach as proposed in [28]

1: $n \leftarrow 0$, choose the initial guess $\eta = \eta^0$, then,
2: Find $u^{(2n)}$ by solving the problem Eqs. (3.4) to (3.6).
3: Having computing $v = u^{(2n)}$, find $u^{(2n+1)}$ by solving the problem Eqs. (3.1) to (3.3).
4: If $\frac{2||u^{(2n)}-u^{(2n+1)}||_{L^\infty(\Gamma_0)}}{||u^{(2n+1)}||_{L^\infty(\Gamma_0)}+||u^{(2n)}||_{L^\infty(\Gamma_0)}} < \varepsilon$ then stop.
5: Else, $n \leftarrow n+1$, then,
6: compute $\eta = \eta^{(n)} = \partial_\nu u^{(2n+1)} |_{\Gamma_0}$ and go to step 2.

The algorithms 1 and 2, applied for solving Cauchy problem for Helmholtz equation, have been the subject of several studies [21, 32, 34]. In particular, it was used by Marin et el. [34] to solve Cauchy problem for modified Helmholtz equation Eq. (2.5). They also reported that the algorithms 1 and 2 does not converge for the differential operator $\Delta + k^2$ where k is real as is the case in the problem Eq. (2.4). This remark was confirmed by Johansson-Kozlov in [20] where they claimed that lack of coercivity leads to non- convergence of the alternating method. After that, Berntsson et al. in [5] gives an upper estimation of the value of the real k to ensure the convergence of algorithms 1 and 2 for the problem Eq. (2.4). They also proposed two modifications to ensure the convergence with any value of the real k (see [5, 3]).

In the sequel, we will propose two relaxation alternating algorithms for solving the problem Eqs. (2.1) to (2.3) and we show that these algorithms converge for all values of the real k.

4. Description of the modified algorithms

In this section, we consider the relaxation algorithm initially introduced by Jourhmane-Nachaoui [24, 25] to solve Cauchy problem in a bioelectric field. Their approach was aimed at showing an improvement of the mathematical algorithm 1 based on the use of a relaxation factor, and then proving the convergence of the FEM numerical implementation of the algorithm. Then, the convergence of the relaxed algorithm applied to solve the Cauchy problem for Poisson's equation is proved in [26]. This procedure drastically reduced the number of iterations required to achieve convergence for the inverse problems considered. Recently, this relaxation was accurately investigated to solve inverse Cauchy problem arising in many applications [11, 32, 37]. In particular this relaxation was used for solving Cauchy problem for modified Helmholtz [21, 32]. The same relaxation was used to solve Cauchy Helmholtz problem via a preconditioned Richardson algorithm [14].

In the following, we will describe two kind of relaxation algorithms to solve Cauchy Helmholtz problem. The aim of these relaxed algorithms is to enable the convergence in the case where the standard algorithms 1 and 2 diverge (large values of the wave number k). In the remainder cases, the goal is to improve the computational time of the standard algorithms 1 and 2, and at the same time maintaining the accuracy of the numerical results obtained with the latter.

First relaxation algorithm

The first relaxation algorithm proposed to solve the problem Eq. (2.4) and Eqs. (2.2) to (2.3) has the same computational schemes as the standard alternating algorithm 1 but the Dirichlet condition Eq. (3.2) is relaxed by some relaxation parameter $0 < \theta < 2$.

This algorithm is summarised as follows,

Algorithm 3:

Step 1: For $m \leftarrow 0$, specify an initial approximation $v = v^0$ of $u\mid_{\Gamma_0}$, and a relaxation parameter $0 < \theta < 2$.

Step 2: Find $u^{(2m)}$ by solving the well posed problem

$$\Delta u^{(2m)} \pm k^2 u^{(2m)} = 0, \quad \text{in } \Omega, \tag{4.1}$$

$$u^{(2m)} = v^{(m)}, \quad \text{on } \Gamma_0, \tag{4.2}$$

$$\partial_\nu u^{(2m)} = f_2, \quad \text{on } \Gamma_1, \tag{4.3}$$

with $v^{(0)} = v^0\mid_{\Gamma_0}$ and for $m > 0$,

$$v^{(m)} = \theta u^{(2m-1)}\mid_{\Gamma_0} + (1-\theta) v^{(m-1)}\mid_{\Gamma_0}. \tag{4.4}$$

Step 3: Having construct $u^{(2m)}$ for $x \in \Omega$ and $\eta^{(m)} = \partial_\nu u^{(2m)}\mid_{\Gamma_0}$ the flux on Γ_0, find $u^{(2m+1)}$ by solving the following well posed problem

$$\Delta u^{(2m+1)} \pm k^2 u^{(2m+1)} = 0, \quad \text{in } \Omega, \tag{4.5}$$

$$\partial_\nu u^{(2m+1)} = \eta^{(m)}, \quad \text{on } \Gamma_0, \tag{4.6}$$

$$u^{(2m+1)} = f_1, \quad \text{on } \Gamma_1. \tag{4.7}$$

Step 4: If $\frac{2\|u^{(2m)} - u^{(2m+1)}\|_{L^\infty(\Gamma_0)}}{\|u^{(2m+1)}\|_{L^\infty(\Gamma_0)} + \|u^{(2m)}\|_{L^\infty(\Gamma_0)}} < \varepsilon$ then stop. Else, $m \leftarrow m+1$, then, go to Step 2.

Remark 1. *The value $\theta = 1$ in Eq. (4.4) corresponds to the standard alternating iterative algorithm 1 with an initial guess for the Dirichlet data, whilst values $\theta \in]0,1[$ and $\theta \in]1,2[$ in Eq. (4.4) correspond to the standard alternating iterative algorithm 1 with an initial guess for the Dirichlet data and a constant under- and over-relaxation factor, respectively.*

Second relaxation algorithm

The second relaxation algorithm proposed to solve the problem Eq. (2.4) and Eqs. (2.2) to (2.3) has the same computational schemes as the standard alternating algorithm 2 but the Neumann condition Eq. (3.5) is relaxed by some relaxation parameter $0 < \theta < 2$.

This algorithm is summarized as follows,

Algorithm 4:

Step 1: For $m \leftarrow 0$, specify an initial approximation η^0 of the flux $\partial_\nu u\mid_{\Gamma_0}$, and a relaxation parameter $0 < \theta < 2$.

Step 2: Find $u^{(2m)}$ by solving the well posed problem

$$\Delta u^{(2m)} \pm k^2 u^{(2m)} = 0, \quad \text{in } \Omega, \tag{4.8}$$

$$\partial_\nu u^{(2n)} = \eta^{(m)}, \quad \text{on } \Gamma_0, \tag{4.9}$$

$$u^{(2m)} = f_1, \quad \text{on } \Gamma_1, \tag{4.10}$$

with $\eta^{(0)} = \eta^0\mid_{\Gamma_0}$ and for $m > 0$,

$$\eta^{(m)} = \theta \partial_\nu u^{(2m-1)}\mid_{\Gamma_0} \;+ (1-\theta)\eta^{(m-1)}\mid_{\Gamma_0}.$$

Step 3: Having construct $u^{(2m)}$ for $x \in \Omega$ and $v^{(m)} = u^{(2m)}\mid_{\Gamma_0}$ the trace on Γ_0, of $u^{(2m)}$ founded by solving the following well posed problem

$$\Delta u^{(2m+1)} \pm k^2 u^{(2m+1)} = 0, \quad \text{in } \Omega, \tag{4.11}$$

$$u^{(2m+1)} = v^{(m)}, \quad \text{on } \Gamma_0, \tag{4.12}$$

$$\partial_\nu u^{(2m+1)} = f_2, \quad \text{on } \Gamma_1. \tag{4.13}$$

Step 4: If $\frac{2\|u^{(2m)} - u^{(2m+1)}\|_{L^\infty(\Gamma_0)}}{\|u^{(2m+1)}\|_{L^\infty(\Gamma_0)} + \|u^{(2m)}\|_{L^\infty(\Gamma_0)}} < \varepsilon$ then stop. Else, $m \leftarrow m+1$, then, go to Step 2.

5. Convergence Results

In this section, we will discuss the convergence of the relaxation algorithms 3 and 4 for both cases Eq. (2.4) and Eq. (2.5) and for all values of k. In order to do this, we will start by showing the restriction of the standard algorithms 1 and 2 for solving the Cauchy problem for Helmholtz equation.

5.1. Non-convergence of standard algorithms

In this section we show through an example that the standard algorithms 1 and 2 do not converge for large values of the constant k^2 in the Helmholtz equation Eq. (2.4).

Consider the domain of the Cauchy problem the Helmholtz equation (2.1) to be a rectangle i.e. $\Omega = (0,a) \times (0,b)$. Without loss of generality, the given data f_1 and f_2 will be considered as the zeros functions, $\Gamma_1 = (0,a) \times \{0\}$ and $\Gamma_0 = (0,a) \times \{b\}$. Indeed, denotes by $w^{(2m)} = u - u^{(2m)}$ and

$w^{(2m+1)} = u - u^{(2m+1)}$, where u is solution of the problem Eqs. (2.1) to (2.3), $w^{(2m)}$ is solution the following problem

$$\Delta w^{(2m)} \pm k^2 w^{(2m)} = 0, \quad \text{in } \Omega, \tag{5.1}$$

$$w^{(2m)} = \theta w^{(2m-1)} + (1-\theta) w^{(2(m-1))}, \quad \text{on } \Gamma_0, \tag{5.2}$$

$$\partial_\nu w^{(2m)} = 0, \quad \text{on } \Gamma_1, \tag{5.3}$$

and $w^{(2m+1)}$ is solution the following problem

$$\Delta w^{(2m+1)} \pm k^2 w^{(2m+1)} = 0, \quad \text{in } \Omega, \tag{5.4}$$

$$\partial_\nu w^{(2m+1)} = \partial_\nu w^{(2m)}, \quad \text{on } \Gamma_0, \tag{5.5}$$

$$w^{(2m+1)} = 0, \quad \text{on } \Gamma_1. \tag{5.6}$$

Then, the original problem Eqs. (2.1) to (2.3) can be written as follows

$$\Delta u \pm k^2 u = 0, \quad \text{in} \quad]0,a[\times]0,b[, \tag{5.7}$$

$$u(x,0) = 0, \quad \text{on} \quad 0 \le x \le a, \tag{5.8}$$

$$\partial_y u(x,0) = 0, \quad \text{on} \quad 0 \le x \le a, \tag{5.9}$$

$$u(0,y) = u(a,y) = 0, \quad \text{on} \quad 0 \le y \le b. \tag{5.10}$$

First case : $\Delta u + k^2 u = 0$

In order to solve this Cauchy problem, we use the algorithm 1. Indeed, for $m = 0$ choose $v^{(0)}$ an initial approximation of the trace of $u^{(0)}(x,y)$ on Γ_0. Where $u^{(0)}(x,y)$ is the solution of the following well posed problem

$$\Delta u^{(0)} + k^2 u^{(0)} = 0, \quad \text{in} \quad]0,a[\times]0,b[, \tag{5.11}$$

$$\partial_y u^{(0)}(x,0) = 0, \quad \text{on} \quad 0 \le x \le a, \tag{5.12}$$

$$u^{(0)}(0,y) = u^{(0)}(a,y) = 0, \quad \text{on} \quad 0 \le y \le b. \tag{5.13}$$

$$u^{(0)}(x,b) = v^{(0)}, \quad \text{on} \quad 0 \le x \le a. \tag{5.14}$$

It is easy to see, by using the separation of variables combined with the principle of superposition method, that $u^0(x,y)$ reads as follows

$$u^{(0)}(x,y) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} B_n^{(0)} \sin(\beta_n x) \cosh\left(\sqrt{\lambda_n} y\right), & 0 < k < \dfrac{\pi}{a}, \\ \displaystyle\sum_{n=1}^{n_0} B_n^{(0)} \sin(\beta_n x) \cosh\left(\sqrt{-\lambda_n} y\right) + \sum_{n=n_0+1}^{\infty} B_n^{(0)} \sin(\beta_n x) \cosh\left(\sqrt{\lambda_n} y\right), & k \ge \dfrac{\pi}{a}. \end{cases} \tag{5.15}$$

where

$$B_n^{(0)} = \begin{cases} \frac{A_n}{\cosh(\sqrt{\lambda_n} b)}, & \text{if } k < \frac{\pi}{a} \text{ or } n > n_0, \\ \frac{A_n}{\cosh(\sqrt{-\lambda_n} b)}, & \text{otherwise}. \end{cases}$$

and

$$A_n = \frac{2}{a}\int_0^a v^{(0)} \sin\left(\frac{n\pi}{a}x\right).$$

Having construct the approximation $u^0(x,y)$, which allows to compute the approximation $\eta^{(0)} = \partial_y u^{(0)}$ of the flux on Γ_0, we find $u^{(1)}$ by solving the following well posed problem

$$\Delta u^{(1)} + k^2 u^{(1)} = 0, \quad \text{in} \quad]0,a[\times]0,b[, \tag{5.16}$$

$$u^{(1)}(x,0) = 0, \quad \text{on } 0 \leq x \leq a, \tag{5.17}$$

$$u^{(1)}(0,y) = u^{(1)}(a,y) = 0, \quad \text{on } 0 \leq y \leq b, \tag{5.18}$$

$$\partial_y u^{(1)}(x,b) = \eta^{(0)}, \quad \text{on } 0 \leq x \leq a. \tag{5.19}$$

Using the separation of variables method, we find that

$$u^{(1)}(x,y) = \begin{cases} \sum_{n=1}^{\infty} B_n^{(1)} \sin(\beta_n x) \sinh\left(\sqrt{\lambda_n} y\right), & 0 < k < \frac{\pi}{a}, \\ \sum_{n=1}^{n_0} B_n^{(1)} \sin(\beta_n x) \sinh\left(\sqrt{-\lambda_n} y\right) + \sum_{n=n_0+1}^{\infty} B_n^{(1)} \sin(\beta_n x) \sinh\left(\sqrt{\lambda_n} y\right), & k \geq \frac{\pi}{a}. \end{cases} \tag{5.20}$$

where

$$B_n^{(1)} = \begin{cases} \frac{A_n \tanh(\sqrt{\lambda_n} b)}{\cosh(\sqrt{\lambda_n} b)}, & \text{if } k < \frac{\pi}{a} \text{ or } n > n_0, \\ \frac{A_n \tanh(\sqrt{-\lambda_n} b)}{\cosh(\sqrt{-\lambda_n} b)}, & \text{otherwise.} \end{cases}$$

For $m \geq 1$, by using the mathematical induction we construct the sequences $u^{(2m)}$ and $u^{(2m+1)}$, solutions of the problems Eqs. (5.11) to (5.14) and Eqs. (5.16) to (5.19) respectively, as follow

$$u^{(2m)}(x,y) = \begin{cases} \sum_{n=1}^{\infty} B_n^{(2m)} \sin(\beta_n x) \cosh\left(\sqrt{\lambda_n} y\right), & 0 < k < \frac{\pi}{a}, \\ \sum_{n=1}^{n_0} B_n^{(2m)} \sin(\beta_n x) \cosh\left(\sqrt{-\lambda_n} y\right) + \sum_{n=n_0+1}^{\infty} B_n^{(2m)} \sin(\beta_n x) \cosh\left(\sqrt{\lambda_n} y\right), & k \geq \frac{\pi}{a}, \end{cases} \tag{5.21}$$

and

$$u^{(2m+1)} = \begin{cases} \sum_{n=1}^{\infty} B_n^{(2m+1)} \sin(\beta_n x) \sinh\left(\sqrt{\lambda_n} y\right), & 0 < k < \frac{\pi}{a}, \\ \sum_{n=1}^{n_0} B_n^{(2m+1)} \sin(\beta_n x) \sinh\left(\sqrt{-\lambda_n} y\right) + \sum_{n=n_0+1}^{\infty} B_n^{(2m+1)} \sin(\beta_n x) \sinh\left(\sqrt{\lambda_n} y\right), & k \geq \frac{\pi}{a}, \end{cases} \tag{5.22}$$

where

$$\begin{aligned} B_n^{(2m)} &= \begin{cases} B_n^{(2m-1)} \tanh(\sqrt{\lambda_n} b), & \text{if } k < \frac{\pi}{a} \text{ or } n > n_0, \\ B_n^{(2m-1)} \tanh(\sqrt{-\lambda_n} b), & \text{otherwise.} \end{cases} \\ &= \begin{cases} B_n^{(0)} \tanh(\sqrt{\lambda_n} b)^{2m}, & \text{if } k < \frac{\pi}{a} \text{ or } n > n_0, \\ B_n^{(0)} \tanh(\sqrt{-\lambda_n} b)^{2m}, & \text{otherwise.} \end{cases} \end{aligned}$$

Having construct the sequence $(u^m)_{m\geq 0}$, let's show that it diverges for large values of k. For this, notice that $u = 0$ is the solution of the Cauchy problem Eqs. (5.7) to (5.10). Let's inspect the behavior of the norm $\|u^{(2m)}\|^2_{L^2(\Omega)} = \int_0^b \int_0^a |u^{(2m)}(x, y)|^2 \,\mathrm{dxdy}$ when $m \to \infty$. First, we define

$$B_n(y) = B_n^{(2m)} \cosh(\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} y).$$

where sgn denotes the sign function of real numbers. Then by using the Parseval's identity, we have

$$\begin{aligned}
\|u^{(2m)}\|^2_{L^2(\Omega)} &= \int_0^b \int_0^a \left|u^{(2m)}\right|^2 \mathrm{dxdy}, \\
&= \int_0^b \frac{a}{2} \sum_{n=1}^{\infty} \left|B_n(y)\right|^2 \mathrm{dy}, \\
&= \int_0^b \frac{a}{2} \sum_{n=1}^{\infty} \left|B_n^{(2m)} \cosh(\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} y)\right|^2 \mathrm{dy}, \\
&= \frac{a}{2} \sum_{n=1}^{\infty} \left[B_n^{(2m)}\right]^2 \int_0^b \frac{e^{2\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} y} + e^{-2\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} y} + 2}{4} \mathrm{dy},
\end{aligned}$$

or

$$\begin{aligned}
\int_0^b \cosh(\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} y)^2 \mathrm{dy} &= \int_0^b \frac{e^{2\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} y} + e^{-2\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} y} + 2}{4} \mathrm{dy}, \\
&= \frac{1}{2\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n}} \left(\sinh(\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} b) \cosh(\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} b) + \sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} b\right).
\end{aligned}$$

Thereby,

$$\|u^{(2m)}\|^2_{L^2(\Omega)} = \frac{a}{2} \sum_{n=1}^{\infty} \left[B_n^{(0)}\right]^2 \left[\frac{\sinh(\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} b) \cosh(\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} b)}{2\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n}} + \frac{b}{2}\right] \left(\tanh(\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n} b)\right)^{4m}. \tag{5.23}$$

Note that in order to show that the sequence $\left(u^{(2m)}\right)_{m\geq 0}$ converges to 0 in $L^2(\Omega)$, one only has to verify that $|\tanh(\sqrt{\operatorname{sgn}(\lambda_1)\lambda_1} b)| < 1$. Then by using the fact that $\tanh(\mathrm{i}x) = \mathrm{i}\tan(x)$, we have

$$\tanh(\sqrt{\operatorname{sgn}(\lambda_1)\lambda_1} b) = \begin{cases} \tanh(\sqrt{\lambda_1} b), & \text{if } k < \frac{\pi}{a}, \\ \mathrm{i}\tan(\sqrt{-\lambda_1} b), & \text{otherwise}. \end{cases} \tag{5.24}$$

As it can be seen, the convergence of the sequence $\left(u^{(2m)}\right)_{m\geq 0}$ is conditioned by the values of k. Indeed, we have

$$\lim_{m\to\infty} \|u^{(2m)}\|_{L^2(\Omega)} = \begin{cases} 0, & \text{if } \quad 0 < k < \frac{\pi}{a}\sqrt{1 + \left(\frac{a}{4b}\right)^2}, \\ \infty, & \text{if } \quad k \geq \frac{\pi}{a}\sqrt{1 + \left(\frac{a}{4b}\right)^2}. \end{cases} \tag{5.25}$$

Therefore, we conclude that the algorithm 1 does not converge for the Helmholtz Cauchy problem with large values of k.

A similar conclusion can be founded for the algorithm 2. Indeed, starting by given η^m an initial approximation of the flux $\partial_\nu u$ on Γ_0. Then, we solve the following well posed problem

$$\Delta u^{(2m)} + k^2 u^{(2m)} = 0, \quad \text{in} \quad]0,a[\times]0,b[, \tag{5.26}$$

$$u^{(2m)}(x,0) = 0, \quad \text{on} \;\; 0 \leq x \leq a, \tag{5.27}$$

$$u^{(2m)}(0,y) = u^{(2m)}(a,y) = 0, \quad \text{on} \;\; 0 \leq y \leq b, \tag{5.28}$$

$$\partial_y u^{(2m)}(x,b) = w^{(m)}, \quad \text{on} \;\; 0 \leq x \leq a. \tag{5.29}$$

Having construct $u^{(2m)}$ which allow as to have the approximate trace $v^{(m)}$ on Γ_0, we solve the following well posed problem

$$\Delta u^{(2m+1)} + k^2 u^{(2m+1)} = 0 \quad \text{in} \quad]0,a[\times]0,b[, \tag{5.30}$$

$$\partial_y u^{(2m+1)}(x,0) = 0, \quad \text{on} \;\; 0 \leq x \leq a, \tag{5.31}$$

$$u^{(2m+1)}(0,y) = u^{(2m+1)}(a,y) = 0, \quad \text{on} \;\; 0 \leq y \leq b, \tag{5.32}$$

$$u^{(2m+1)}(x,b) = v^{(m)}, \quad \text{on} \;\; 0 \leq x \leq a. \tag{5.33}$$

Where $v^{(m)} = u^{(2m)} |_{\Gamma_0}$ and $\eta^{(m)} = \partial_y u^{(2m+1)} |_{\Gamma_0}$ for $m > 1$. By using the same tools as for the algorithm 1 we find that $u^{(2m)}$ and $u^{(2m+1)}$ solutions of problems Eqs. (5.26) to (5.29) and Eqs. (5.30) to (5.33) respectively reads as follow

$$u^{(2m)}(x,y) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} B_n^{(2m)} \sin(\beta_n x) \sinh\left(\sqrt{\lambda_n} y\right), & 0 < k < \dfrac{\pi}{a}, \\ \displaystyle\sum_{n=1}^{n_0} B_n^{(2m)} \sin(\beta_n x) \sinh\left(\sqrt{-\lambda_n} y\right) + \sum_{n=n_0+1}^{\infty} B_n^{(2m)} \sin(\beta_n x) \sinh\left(\sqrt{\lambda_n} y\right), & k \geq \dfrac{\pi}{a}, \end{cases} \tag{5.34}$$

and

$$u^{(2m+1)} = \begin{cases} \displaystyle\sum_{n=1}^{\infty} B_n^{(2m+1)} \sin(\beta_n x) \cosh\left(\sqrt{\lambda_n} y\right), & 0 < k < \dfrac{\pi}{a}, \\ \displaystyle\sum_{n=1}^{n_0} B_n^{(2m+1)} \sin(\beta_n x) \cosh\left(\sqrt{-\lambda_n} y\right) + \sum_{n=n_0+1}^{\infty} B_n^{(2m+1)} \sin(\beta_n x) \cosh\left(\sqrt{\lambda_n} y\right), & k \geq \dfrac{\pi}{a}, \end{cases} \tag{5.35}$$

where

$$\begin{aligned} B_n^{(2m)} &= \begin{cases} B_n^{(2m-1)} \tanh(\sqrt{\lambda_n} b), & \text{if } k < \frac{\pi}{a} \text{ or } n > n_0, \\ B_n^{(2m-1)} \tanh(\sqrt{-\lambda_n} b), & \text{otherwise.} \end{cases} \\ &= \begin{cases} B_n^{(0)} \tanh(\sqrt{\lambda_n} b)^{2m}, & \text{if } k < \frac{\pi}{a} \text{ or } n > n_0, \\ B_n^{(0)} \tanh(\sqrt{-\lambda_n} b)^{2m}, & \text{otherwise.} \end{cases} \end{aligned}$$

and

$$B_n^{(0)} = \begin{cases} \frac{A_n}{\lambda_n \cosh(\sqrt{\lambda_n} b)}, & \text{if } k < \frac{\pi}{a} \text{ or } n > n_0, \\ \frac{A_n}{\lambda_n \cosh(\sqrt{-\lambda_n} b)}, & \text{otherwise,} \end{cases}$$

$$A_n = \frac{2}{a}\int_0^a \eta^{(0)} \sin\left(\frac{n\pi}{a}x\right).$$

Thereby,

$$\|u^{(2m)}\|^2_{L^2(\Omega)} = \frac{a}{2}\sum_{n=1}^{\infty}\left[B_n^{(0)}\right]^2\left[\frac{\sinh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b)\cosh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b)}{2\sqrt{\text{sgn}(\lambda_n)\lambda_n}} - \frac{b}{2}\right]\left(\tanh(\sqrt{\text{sgn}(\lambda_n)\lambda_n}b)\right)^{4m}. \tag{5.36}$$

Consequently, we conclude that the convergence of algorithm 2, like algorithm 1, is conditioned by the values of k. Moreover, according to the value of k, the convergence or divergence of the algorithm 2 follows the same cases as defined at Eq. (5.25). Having showing the restriction of algorithm 1 and 2 for solving the Cauchy problem for the Helmholtz equation (case $\Delta u + k^2 u = 0$), it remains to inspect these algorithms for solving the Cauchy problem for the modified Helmholtz equation (case $\Delta u - k^2 u = 0$).

Second case: $\Delta u - k^2 u = 0$

Using the same tools as for the case $\Delta u - k^2 u = 0$, we see that the algorithm 1 leads the following sequence

$$u^{(2m)}(x,y) = \sum_{n=1}^{\infty} B_n^{(2m)} \sin(\beta_n x)\cosh\left(\sqrt{\lambda'_n}y\right)$$

and

$$u^{(2m+1)}(x,y) = \sum_{n=1}^{\infty} B_n^{(2m+1)} \sin(\beta_n x)\sinh\left(\sqrt{\lambda'_n}y\right),$$

where $\lambda'_n = (\beta_n)^2 + k^2$ in this case. Then the norm of the sequence $(u^{(2m)})_{m\geq 0}$ reads as

$$\|u^{(2m)}\|^2_{L^2(\Omega)} = \frac{a}{2}\sum_{n=1}^{\infty}\left[B_n^{(0)}\right]^2\left[\frac{\sinh(\sqrt{\lambda'_n}b)\cosh(\sqrt{\lambda'_n}b)}{2\sqrt{\lambda'_n}} + \frac{b}{2}\right]\left(\tanh(\sqrt{\lambda'_n}b)\right)^{4m}. \tag{5.37}$$

As we can see, since the λ'_n is positive for all values of k, this means that $|\tanh(\lambda'_1)| < 1,\ \forall k > 0$. Therefore, the algorithm 1 and similarly algorithm 2 converge without restriction for values of k in the case $\Delta u - k^2 u = 0$.

Having discuss the restriction of the convergence of the algorithms 1 and 2 for solving Cauchy problem for Helmholtz equation, in the following we well study the convergence of the relaxation algorithms 3 and 4.

5.2. Convergence of relaxation algorithms

In this section, we will discuss the convergence the algorithms 1 and 2 for solving Cauchy problem for Helmholtz equation. Starting by considering the same problem Eqs. (5.7) to (5.10) with the same data as is considered in the Section 5.1.

First case: $\Delta u + k^2 u = 0$

In order to solve the Cauchy problem for Helmholtz equation (case $\Delta u + k^2 u = 0$), we use the algorithm 3. Indeed, starting by given v^m an initial approximation of the trace of u on Γ_0. Then, we solve the following well posed problem

$$\Delta u^{(2m)} + k^2 u^{(2m)} = 0, \quad \text{in} \quad]0,a[\times]0,b[, \tag{5.38}$$
$$\partial_y u^{(2m)}(x,0) = 0, \quad \text{on} \quad 0 \le x \le a, \tag{5.39}$$
$$u^{(2m)}(0,y) = u^{(2m)}(a,y) = 0, \quad \text{on} \quad 0 \le y \le b, \tag{5.40}$$
$$u^{(2m)}(x,b) = v^{(m)}, \quad \text{on} \quad 0 \le x \le a. \tag{5.41}$$

Having construct $u^{(2m)}$ which allow as to have the approximate flux $\eta^{(m)}$ on Γ_0, we solve the following well posed problem

$$\Delta u^{(2m+1)} + k^2 u^{(2m+1)} = 0, \quad \text{in} \quad]0,a[\times]0,b[, \tag{5.42}$$
$$u^{(2m+1)}(x,0) = 0, \quad \text{on} \quad 0 \le x \le a, \tag{5.43}$$
$$u^{(2m+1)}(0,y) = u^{(2m+1)}(a,y) = 0, \quad \text{on} \quad 0 \le y \le b, \tag{5.44}$$
$$\partial_y u^{(2m+1)}(x,b) = \eta^{(m)}, \quad \text{on} \quad 0 \le x \le a. \tag{5.45}$$

Where $v^{(m)} = \theta u^{(2m-1)} \mid_{\Gamma_0} + (1-\theta) * v^{(m-1)}$ and $\eta^{(m)} = \partial_y u^{(2m)} \mid_{\Gamma_0}$ for $m > 1$ and $\theta > 0$. By using the same tools as in Section 5.1, we find that $u^{(2m)}$ and $u^{(2m+1)}$ solutions of problems Eqs. (5.38) to (5.41) and Eqs. (5.42) to (5.45) respectively reads as follow

$$u^{(2m)}(x,y) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} B_n^{(2m)} \sin(\beta_n x) \cosh\left(\sqrt{\lambda_n} y\right), & 0 < k < \dfrac{\pi}{a}, \\ \displaystyle\sum_{n=1}^{n_0} B_n^{(2m)} \sin(\beta_n x) \cosh\left(\sqrt{-\lambda_n} y\right) + \sum_{n=n_0+1}^{\infty} B_n^{(2m)} \sin(\beta_n x) \cosh\left(\sqrt{\lambda_n} y\right), & k \ge \dfrac{\pi}{a}, \end{cases} \tag{5.46}$$

and

$$u^{(2m+1)} = \begin{cases} \displaystyle\sum_{n=1}^{\infty} B_n^{(2m+1)} \sin(\beta_n x) \sinh\left(\sqrt{\lambda_n} y\right), & 0 < k < \dfrac{\pi}{a}, \\ \displaystyle\sum_{n=1}^{n_0} B_n^{(2m+1)} \sin(\beta_n x) \sinh\left(\sqrt{-\lambda_n} y\right) + \sum_{n=n_0+1}^{\infty} B_n^{(2m+1)} \sin(\beta_n x) \sinh\left(\sqrt{\lambda_n} y\right), & k \ge \dfrac{\pi}{a}, \end{cases} \tag{5.47}$$

where

$$B_n^{(2m)} = \begin{cases} B_n^{(0)} \left(\theta \left(\tanh(\sqrt{\lambda_n} b)\right)^2 + (1-\theta)\right)^m, & \text{if } k < \frac{\pi}{a} \text{ or } n > n_0, \\ B_n^{(0)} \left(\theta \left(\tanh(\sqrt{-\lambda_n} b)\right)^2 + (1-\theta)\right)^m, & \text{otherwise.} \end{cases}$$

and

$$B_n^{(0)} = \begin{cases} \frac{A_n}{\cosh(\sqrt{\lambda_n} b)}, & \text{if } k < \frac{\pi}{a} \text{ or } n > n_0, \\ \frac{A_n}{\cosh(\sqrt{-\lambda_n} b)}, & \text{otherwise,} \end{cases}$$

$$A_n = \frac{2}{a}\int_0^a v^{(0)} \sin\left(\frac{n\pi}{a}x\right).$$

Thereby, denotes by

$$\gamma_n = \left[\frac{\sinh(\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n}b)\cosh(\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n}b)}{2\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n}} - \frac{b}{2}\right],$$

then the $L^2(\Omega)$ norm of $u^{(2m)}$ reads as

$$\|u^{(2m)}\|^2_{L^2(\Omega)} = \frac{a}{2}\sum_{n=1}^{\infty}\left[B_n^{(0)}\right]^2\left[\theta\left(\tanh(\sqrt{\operatorname{sgn}(\lambda_n)\lambda_n}b\right)^2 + 1 - \theta\right]^{2m}\gamma_n. \tag{5.48}$$

Theorem 1. *Consider the Cauchy problem for Helmholtz equation Eqs. (5.7) to (5.10). The following statements hold*

(i) *for $k \leq \frac{\pi}{a}$, there exists $\theta_1^* = \min\left(\frac{2}{1-\left(\tanh(\sqrt{\lambda_1}b)\right)^2}, 2\right)$ such that, for all $\theta \in (0, \theta_1^*)$, the sequence $\left(u^{(m)}\right)_{m\geq 0}$ defined by Eqs. (5.46) to (5.47) converges to the solution of Eqs. (5.7) to (5.10) independently of the initial value $v^{(0)}$.*

(ii) *for $\frac{\pi}{a} < k < \frac{\pi}{a}\sqrt{1+\left(\frac{a}{4b}\right)}$, there exists $\theta_2^* = \frac{2}{1+\left(\tan(\sqrt{-\lambda_1}b)\right)^2}$ such that, for all $\theta \in (0, \theta_2^*)$, the sequence $\left(u^{(m)}\right)_{m\geq 0}$ defined by Eqs. (5.46) to (5.47) converges to the solution of Eqs. (5.7) to (5.10) independently of the initial value $v^{(0)}$.*

(iii) *for $k \geq \frac{\pi}{a}\sqrt{1+\left(\frac{a}{4b}\right)}$, there exists $\theta_3^* = \frac{2}{1+\left(\tan(\sqrt{-\lambda_1}b)\right)^2}$ such that, for all $\theta \in (0, \theta_3^*)$, the sequence $\left(u^{(m)}\right)_{m\geq 0}$ defined by Eqs. (5.46) to (5.47) converges to the solution of Eqs. (5.7) to (5.10) independently of the initial value $v^{(0)}$.*

proof: In order to show that the sequence $(u^{(m)})_{m\geq 0}$ converges to 0 in $L^2(\Omega)$ it suffices to show that

$$\mid \theta\left(\tanh(\sqrt{\operatorname{sgn}(\lambda_1)\lambda_1}b)\right)^2 + 1 - \theta \mid < 1. \tag{5.49}$$

But we have

$$\mid \theta\left(\tanh(\sqrt{\operatorname{sgn}(\lambda_1)\lambda_1}b)\right)^2 + 1 - \theta \mid = \begin{cases} \mid \theta\left(\tanh(\sqrt{\lambda_1}b)\right)^2 + 1 - \theta \mid, & \text{if } k \leq \frac{\pi}{a}, \\ \mid -\theta\left(\tan(\sqrt{-\lambda_1}b)\right)^2 + 1 - \theta \mid, & \text{if } k > \frac{\pi}{a}. \end{cases}$$

For different values of $k > 0$, we seek $0 < \theta < 2$ which ensures the convergence condition Eq. (5.49). We find three cases.

(i) : $k \leq \frac{\pi}{a}$.

To fulfill the condition Eq. (5.49), the relaxation variable θ must satisfy the following inequality,

$$-2 < \theta\left(\tanh(\sqrt{\lambda_1}b)^2 - 1\right) < 0$$

then, it suffices to take θ verify,

$$0 < \theta < \min\left(\frac{2}{1-\left(\tanh(\sqrt{\lambda_1}b)\right)^2}, 2\right).$$

Thus, by taking $\theta_1^* = \min\left(\frac{2}{1-\left(\tanh(\sqrt{\lambda_1}b)\right)^2}, 2\right)$ the insertion (i) is valid.

(ii) : $\frac{\pi}{a} < k < \frac{\pi}{a}\sqrt{1+\left(\frac{a}{4b}\right)^2}$.

In this case, to ensure the convergence condition Eq. (5.49), the relaxation variable θ must verify the following inequality,

$$-2 < -\theta\left(\tan(\sqrt{-\lambda_1}b)^2 + 1\right) < 0,$$

then, it suffice to take θ verify,

$$0 < \theta < \frac{2}{1+\left(\tan(\sqrt{-\lambda_1}b)\right)^2}.$$

Thus, by taking $\theta_2^* = \frac{2}{1+\left(\tan(\sqrt{-\lambda_1}b)\right)^2}$, the insertion (ii) is valid.

(iii) : $k > \frac{\pi}{a}\sqrt{1+\left(\frac{a}{4b}\right)^2}$.

The insertion (iii) is showed in the same way as (ii).

Then we take $\theta_3^* = \theta_2^*$.

□

Remark 2.

1. *Note that similar results, as in the algorithm 3, can be obtained by using the relaxation algorithm 4 for solving the Cauchy problem of Helmholtz equation Eqs. (5.7) to (5.10).*
2. *Note that in the cases (i) and (ii) of the Theorem 1, we have* $\theta_1^* > 1$ *and* $\theta_2^* > 1$*, which means that relaxation algorithms 3 and 4 coincide with standard algorithms 1 and 2, for theta* $= 1$*, respectively. Therefore, in these cases the role of the relaxation parameter is eventually the acceleration of convergence of the standards iterative schemes 1 and 2.*
3. *While in the case (iii) of the Theorem 1, we have* $\theta_3^* < 1$*, which corresponds to non-convergence case for the standard algorithms 1 and 2. Then, the role of the relaxation parameter* θ *is to ensure the convergence of the standards iterative schemes 1 and 2, and then possibly accelerate the convergence.*

Second case: $\Delta u - k^2 u = 0$

As is already showed in Section 5.1, standards algorithms 1 and 2 used for solving the Cauchy problem for the modified Helmholtz equation (case: $\Delta u - k^2 u = 0$) converge without any restriction on the value of k. Therefore, the role of the relaxation parameter is possible accelerate the convergence of the standards schemes 1 and 2. Indeed, if we consider the norm of $\|u^{(2m)}\|_{L^2(\Omega)}$ defined in Eq. (5.37) and reformulate it by taking into account the relaxation parameter, we find that

$$\|u^{(2m)}\|^2_{L^2(\Omega)} = \frac{a}{2}\sum_{n=1}^{\infty}\left[B_n^{(0)}\right]^2\left[\frac{\sinh(\sqrt{\lambda'_n}b)\cosh(\sqrt{\lambda'_n}b)}{2\sqrt{\lambda'_n}} + \frac{b}{2}\right]\left(\theta\tanh(\sqrt{\lambda'_n}b)^2 + 1 - \theta\right)^{2m}. \quad (5.50)$$

Then, its simple to see that there exists

$$\theta_4^* = \min\left(\frac{2}{1-\left(\tanh(\sqrt{\lambda'_1}b\right)^2}, 2\right),$$

such that, for all $\theta \in (0, \theta_4^*)$ relaxation algorithms 3 and 4 converge. Moreover, the standard algorithms 1 and 2 can obtained by taking $\theta = 1$ in the algorithms 3 and 4 respectively.

Having showing that, for solving Cauchy problem for Helmholtz equation, the standard algorithms 1 and 2 does not converge for large values of the wave number and that, in contrast the relaxed algorithms 3 and 4 converge for all values of the wave number, it remains to confirm these results through numerical experiments.

6. Numerical results and discussion

In this section, we discuss the numerical results obtained using the algorithms 1,2,3 and 4 proposed for solving a Cauchy problem for Helmholtz equation in a two dimensional bounded domain Ω, although the same conclusions hold in higher dimensions. The goal is to demonstrate that in the case of Helmholtz equation ($\Delta u + k^2 u = 0$) the algorithms 1 and 2 dose not converge for all $k \geq k^*$. Where k^* is the limit value determinates in the theorem 1. While the algorithms 3 and 4 converge without restriction of values of k. For the case of Modified Helmholtz equation ($\Delta u - k^2 u = 0$) we show that the algorithms 1 and 2 converge without restriction of k and that the algorithms 3 and 4 accelerate the convergence. Unlike to the anterior works, we will show, through a numerical analysis, that the algorithms 3 and 4 are stable, when the given data is perturbed, and have good convergence rate, independent of initial guess. Moreover, we will study the influence of the constant relaxation parameter θ on the convergence and the reduction of iterations number. The well–posed boundary value problems involved in each iteration are solved using the finite element method, which is more suitable especially for application involves complicated geometries. In order to show the goodness and efficiency of the proposed procedures compared to other existing schemes concerned by the same problem, we handle 3 examples [5, 34].

6.1. Example 1

In the first example we consider $\Omega =]0,1[\times]0,b[$ where $b > 0$. We shall solve the following Cauchy problem for Helmholtz equation

$$\Delta u \pm k^2 u = 0, \quad \text{in} \quad]0,1[\times]0,b[, \tag{6.1}$$

$$u(x,0) = f_1, \quad \text{on } 0 \leq x \leq 1, \tag{6.2}$$

$$\partial_y u(x,0) = f_2, \quad \text{on } 0 \leq x \leq 1, \tag{6.3}$$

$$u(0,y) = u(1,y) = 0, \quad \text{on } 0 \leq y \leq b. \tag{6.4}$$

We take $\Gamma_1 =]0,1[\times\{0\}$ and $\Gamma_0 =]0,1[\times\{b\}$. For the numerical computations, we particularly choose $b = 0.2, N = 400$ and $M = 80$, and select the boundary data $f_1(x) = u(x,0)$ on Γ_1 as

$$u(x,0) = \left(3\sin(\pi x) + \frac{\sin(3\pi x)}{19} + 9\exp(-30(x-b)^2\right) x^2(1-x)^2. \tag{6.5}$$

The exact boundary data on Γ_0, used to test the performance of the algorithm, is given by

$$ue(x,b) = 2\left(8\sin(\pi x) + \frac{\sin(3\pi x)}{17} + 20\exp(-50(x-b)^2\right)x^2(1-x)^2. \tag{6.6}$$

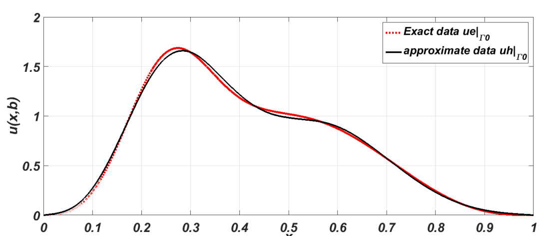
Note that the exact solution u is unknown, then first we construct it by solving the direct problem for the Helmholtz equation Eq. (6.1) with boundary data Eq. (6.4), Eq. (6.5) and Eq. (6.6).

First case: $\Delta u + k^2 u = 0$

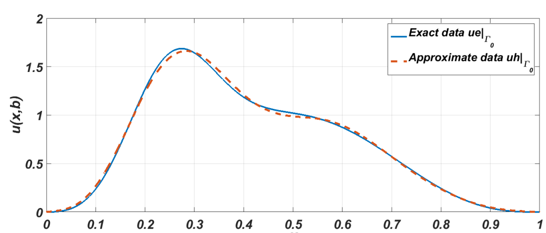
In this case, we illustrate through numerical examples, that the algorithms 1 and 2 dose not converge for all $k \geq k^*$. Where k^* is the limit value determinates in the theorem 1, while the algorithms 3 and 4 converge without restriction of values of k. Firstly, we see that according to the considered data the limit value is $k^* = \frac{\pi}{a}\sqrt{1+\left(\frac{a}{4b}\right)} \simeq 5,029$. In order to make the numerical discussion about convergence or not of the proposed algorithms according to the value of k and the influence of the parameter θ, we well solve the Cauchy problem for Helmholtz equation with four values of k. Namely, $k = \sqrt{15} < k^*$ and $k = \sqrt{25.5}$, $k = \sqrt{35}$ and $k = \sqrt{52}$ which are all upper to k^*. All algorithms will start from an arbitrary initial guess, $u(x,b) = 0$ on Γ_0 for algorithms 1 and 3 and $\partial_y u(x,b) = 0$ for algorithms 2 and 4 on Γ_0. In order to test the convergence of the proposed algorithms, we compute the following stopping criteria

$$\frac{2 \mid\mid u^{(2m)} - u^{(2m+1)} \mid\mid_{L^\infty(\Gamma_0)}}{\mid\mid u^{(2m+1)} \mid\mid_{L^\infty(\Gamma_0)} + \mid\mid u^{(2m)} \mid\mid_{L^\infty(\Gamma_0)}} < \varepsilon. \tag{6.7}$$

Firstly, we show that in the case of convergence ($k = \sqrt{15}$), all algorithms converge to a good



(a)



(b)

Figure 1: Exact solution at $y = b$ and the reconstructed one, obtained by the algorithms 1 and 2 in (a) and by algorithms 3 and 4 in (b), for $k = \sqrt{15}$.

approximation of the exact solution $ue(x,b)$ (see Fig. 1) for the same precision $\varepsilon = 1.14638e^{-05}$. However, the relaxation parameter θ allow an accelerate convergence. Indeed, as we can see in Fig. 2, by using the algorithm 3 with the same precision $\varepsilon = 1.14638e^{-05}$, the number of iterations varies according to $\theta \in]0,2]$. Moreover, we obtain an optimal $\theta^{op} = 1.6$, which allows the convergence with only 938 iterations (see Fig. 3 which presents the comparison of relative error at the convergence of algorithm 3, $\theta = 1$ and $\theta = 1.6$ for $k = \sqrt{15}$). Similar behavior is observed when we use the algorithm 4 and the optimal $\theta^{op} = 1.6$ allows a convergence with only 911 iterations. As we can clearly inspected the algorithms 3 and 4 accelerate significantly, by more the 1/3, the convergence. This confirms the benefits of these relaxed algorithms. Having illustrate the performance of the

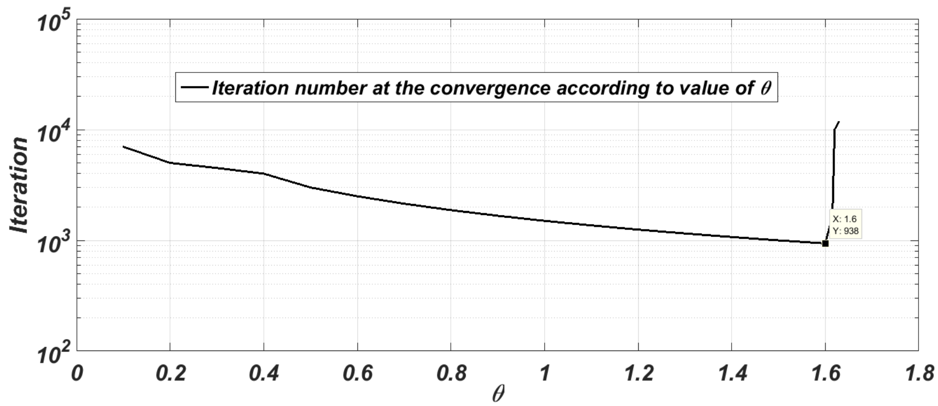


Figure 2: Variation of iterations number at the convergence of algorithm 3 with $\varepsilon = 1.14638e^{-05}$ for $k = \sqrt{15}$.

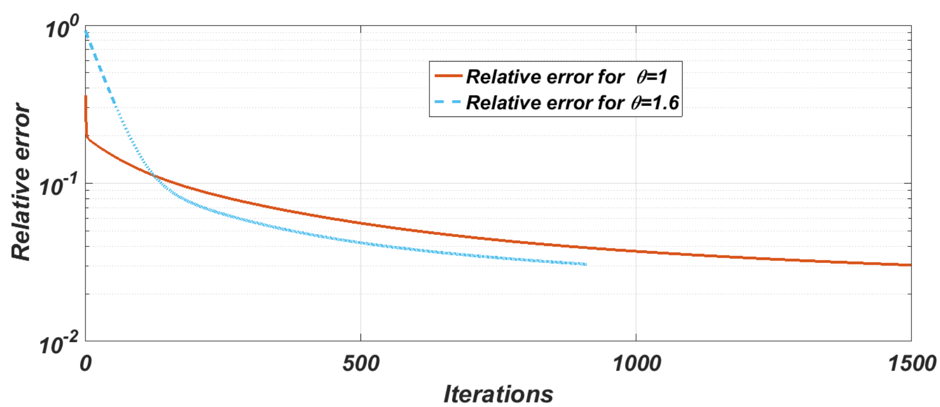


Figure 3: Comparison of relative error at the convergence of algorithm 3, $\theta = 1$ and $\theta = 1.6$ for $k = \sqrt{15}$.

relaxed algorithms for this example in the case of convergence ($k < k^*$), we will discuss in the sequel the case of non-convergence ($k \geq k^*$). Starting by seeing that for $k = \sqrt{25.5} > k^*$ the algorithms 1 and 2 do not converge as we can see in Fig. 4. In contrast, the algorithms 3 and 4 with $\theta \in]0, 1[$ converge to a good approximation, as it's can be expected in Fig. 5. A numerical study, similar to the one that allowed us to determine the optimal θ for $k = \sqrt{15}$, allowed us to determine the optimal $\theta^{op} = 0.98$ for $k = \sqrt{25.5}$. This optimal θ^{op}, allowed us to ensure the convergence of the algorithms 3 and 4. Moreover, this convergence is realized with an optimal number of iteration. Namely, 1474 iterations for the algorithm 3 and 1389 for the algorithm 4 (see Fig. 6 which represents of the comparison of relative error at the convergence of algorithm 3, $\theta = 0.1$ and $\theta = 0.98$ for $k = \sqrt{25.5}$).

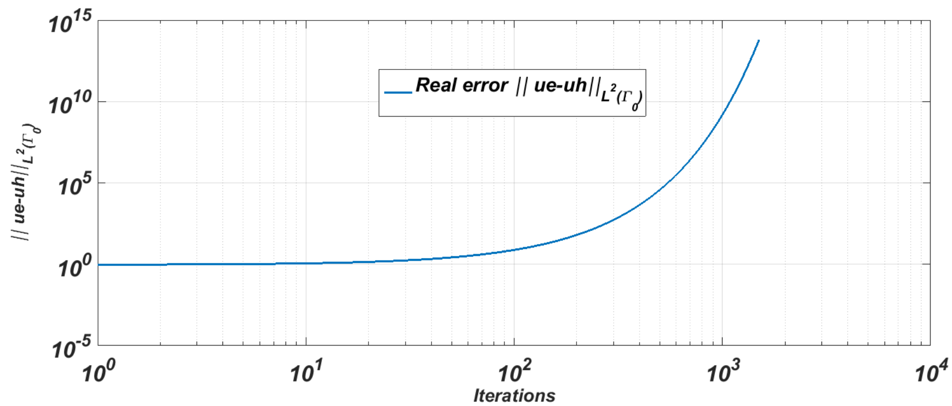
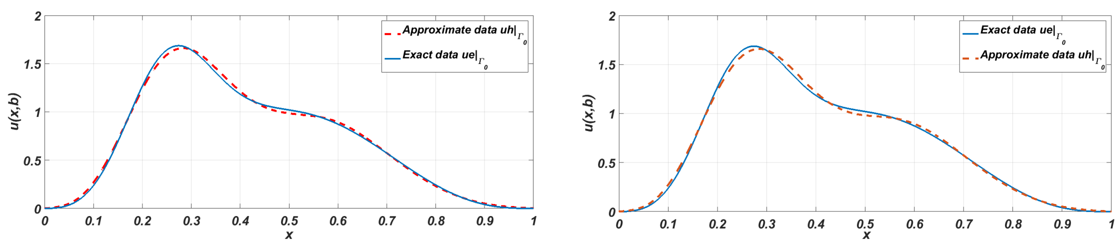


Figure 4: Real error for 1500 iterations, shown the divergence of the algorithms 1 and 2 for $k = \sqrt{25.5}$.



(a)

(b)

Figure 5: Exact solution at $y = b$ and the reconstructed one, obtained by the algorithm 3 in (a) and by algorithm 4 in (b), for $k = \sqrt{25.5}$.

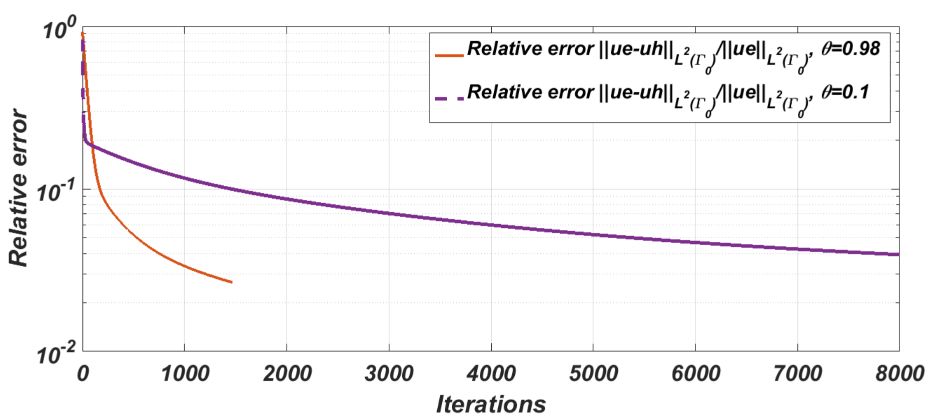


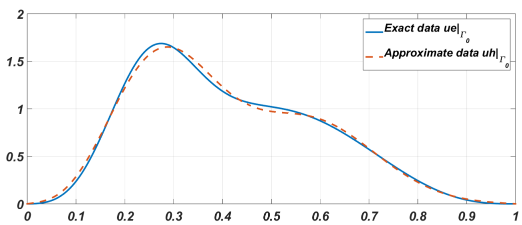
Figure 6: Comparison of relative error at the convergence of algorithm 3, $\theta = 0.1$ and $\theta = 0.98$ for $k = \sqrt{25.5}$.

Remark 3.

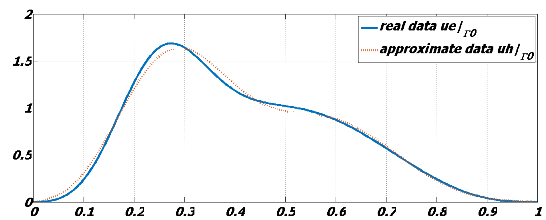
It was reported in [21] that, in the case of Cauchy problem for elasticity or for the modified Helmholtz equation, both these relaxation versions numerically produced similar results and this turns out to be valid also in the case of the Helmholtz equation $\Delta u + k^2 u = 0$ *(see Fig. 1 and*

Fig. 5). Thus, since it is easier and somewhat more natural to make a guess for function values on the boundary part Γ_0 than for normal derivatives on Γ_0, from now on, we shall mainly concentrate on numerical results for the algorithm 3.

Now we test the algorithms with relaxation in case $k = \sqrt{35}$ and $k = \sqrt{52}$. As it can be seen in figure 7, the algorithm 3 converge to a good approximation in both cases $k = \sqrt{35}$ and $k = \sqrt{52}$.



(a)



(b)

Figure 7: Exact solution at $y = b$ and the reconstructed one, obtained by the algorithm 3 in (a) for $k = \sqrt{35}$ and in (b) for $k = \sqrt{52}$.

Second case: $\Delta u - k^2 u = 0$

In this case, we illustrate through some numerical results that the algorithms 1 and 2 converge without any restriction. The goal then is to prove that the relaxation accelerates the convergence. To do this, we use the same data as in the first case and we look for $u|_{\Gamma_0}$ with three different values of wave number $k = \sqrt{15}, \sqrt{25.5}$ and $\sqrt{52}$. In order to show the efficiency of the procedure of relaxation we present in table 1 the comparison of relative error and the number of iterations with precision $\varepsilon = 1.14638e^{-05}$ for the three different value of k.

Wave number	Relaxation parameter	$\|u - ue\|_{L^2(\Omega)}/\|ue\|_{L^2(\Omega)}$	Number of iteration
$k = \sqrt{15}$	$\theta = 1.$	0.0130665	1691
	$\theta = 1.9$	0.0130666	**890**
$k = \sqrt{25.5}$	$\theta = 1$	0.014113	1766
	$\theta = 1.9$	0.0171063	**1032**
$k = \sqrt{52}$	$\theta = 1.$	0.0171054	1961
	$\theta = 1.9$	0.0171063	**1032**

Table 1: Comparison of relative error and number of iteration with $\theta = 1., 1.9$ and the precision $\varepsilon = 1.14638e^{-05}$ for $k = \sqrt{15}, \sqrt{25.5}$ and $\sqrt{52}$.

6.2. Example 2

In order to show the performance of the proposed approach, we solve, in this example, the Cauchy problem in different geometry, namely the unit disc $\Omega = \{(x, y) \in \mathbb{R}^2 \;\; / \;\; x^2 + y^2 < r^2\}$,

$r = 1$. We assume that the boundary $\partial\Omega := \Gamma_1 \cup \Gamma_2$, with

$$\Gamma_1 = \{(x, y) \in \partial\Omega \ \ / \ 0 < \vartheta(x, y) < \Theta\}, \quad \Gamma_2 = \{(x, y) \in \partial\Omega \ \ / \ \ \Theta < \vartheta(x, y) < 2\pi\},$$

where $\vartheta(x, y)$ is the polar angle of (x, y) and Θ is a given angle in $]0, 2\pi[$. Then, the Cauchy problem to be solved is defined as follow,

$$\Delta u \pm k^2 u = 0, \quad \text{in} \quad \Omega, \tag{6.8}$$

$$u(x, y) = f_1, \quad \text{on } \Theta \le \vartheta(x, y) \le 2\pi, \tag{6.9}$$

$$\partial_\nu u(x, y) = f_2, \quad \text{on } \Theta \le \vartheta(x, y) \le 2\pi, \tag{6.10}$$

We consider the following data

$$k = \sqrt{\alpha^2 - \beta^2}, \quad f_1 = \exp(a_1 x + a_2 y), \quad f_2 = (a_1 N.x + a_2 N.y) f_1, \quad \Theta = \frac{\pi}{2}, \tag{6.11}$$

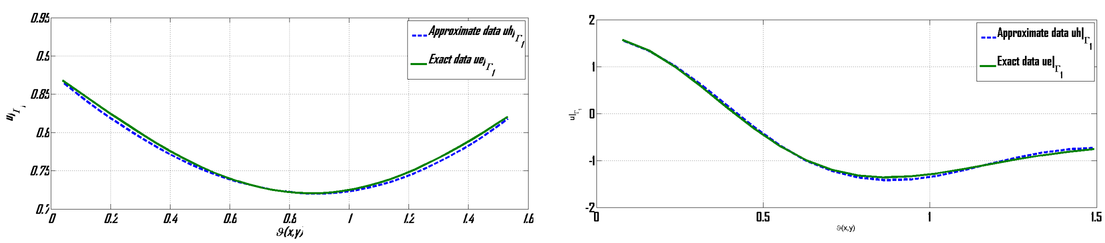
where

$$\alpha \in \{0, \sqrt{0.6}, \sqrt{15}, \sqrt{30}, \sqrt{100}\}, \quad \beta \in \{0, 1\}, \quad a_1 = 0.5, \quad a_2 = i\sqrt{a_1^2 + \alpha^2}, \tag{6.12}$$

and $(N.x, N.y)$ are the external normal components. With the given data (6.11)-(6.12) the exact solution of the problem (6.8-6.10) is $\exp((a_1 x + a_2 y))$.

First case: $\Delta u + k^2 u = 0$

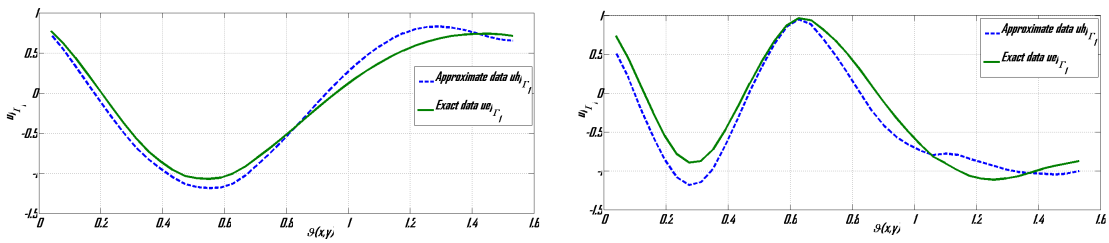
In this case $\beta = 0$ and we take $\alpha \in \{\sqrt{0.6}, \sqrt{15}, \sqrt{30}, \sqrt{100}\}$. In figures 8 and 9 we present the



(a) (b)

Figure 8: Exact solution at Γ_1 and the reconstructed one, obtained by the algorithm 3 in (a) for $k = \sqrt{0.6}$ and in (b) for $k = \sqrt{15}$.

exact and reconstructed solution at Γ_1 obtained by the algorithm 3 for different values of the wave number k. It should be noted that the same example was approximated in [34] but in the case of $\alpha = 0$ and $\beta \in \{1, 2\}$ which correspond to the case $\Delta u - k^2 u = 0$ namely Modified Helmholtz equation. They also reported that the classical alternating algorithms 1 and 2 does not converge in the case $\alpha \neq 0, \beta = 0$. In contrast, it is can be seen that the obtained results show the efficiency of the proposed method even with high value of wave number k, which confirm the theoretical results.

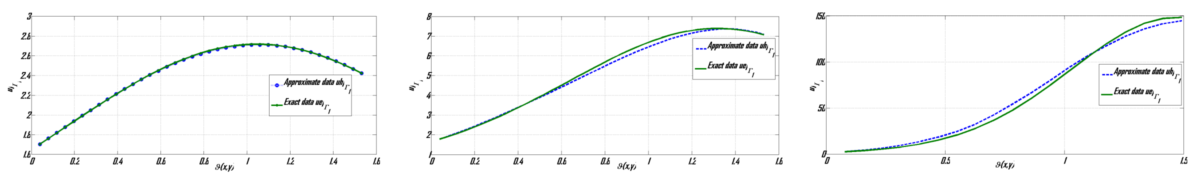


(a) (b)

Figure 9: Exact solution at Γ_1 and the reconstructed one, obtained by the algorithm 3 in (a) for $k = \sqrt{30}$ and in (b) for $k = \sqrt{100}$.

Second case $\Delta u - k^2 u = 0$

In this case, as the algorithms 1 and 2 converge without restriction for values of k, the goal is to show that the algorithms 3 and 4 are more speed. We take $\alpha = 0$ and $\beta \in \{1, 2, 5\}$. While $a_2 = \sqrt{\beta^2 - a_1^2}$ and $k = \beta$.



(a) (b) (c)

Figure 10: Exact solution at Γ_1 and the reconstructed one, obtained by the algorithm 3 in the case of modified Helmholtz for $k = 1$ in (a) and in (b) for $k = 2$ and in (c) for $k = 5$.

Wave number	Relaxation parameter	$\|u - ue\|_{L^2(\Omega)}/\|ue\|_{L^2(\Omega)}$	Number of iteration
$k = 1$	$\theta = 1.$	0.0018303	2000
	$\theta = 1.9$	0.00183012	**1052**
$k = 2$	$\theta = 1$	0.0226173	3000
	$\theta = 1.9$	0.0226168	**1579**
$k = 5$	$\theta = 1.$	0.0492136	4000
	$\theta = 1.9$	0.0492135991	**1679**

Table 2: Comparison of relative error and number of iteration with $\theta = 1.$, 1.9 and the precision $\varepsilon = 1.43538e^{-07}$ for $k \in \{1, 2, 5\}$.

In figure 10 we present the comparison between the exact solution on Γ_1 and the reconstructed one, obtained by the algorithm 3 for $\beta \in \{1, 2, 5\}$. While in the table 2 we present the comparison of relative error and number of iteration with $\theta = 1.$ and 1.9, for given precision $\varepsilon = 1.43538e^{-07}$ for $\beta \in \{1, 2, 5\}$. It's can be seen that, the obtained results are good in precision. Moreover, the proposed algorithms 3 and 4 converge with high-speed comparison to the classical algorithms 1 and 2. Actually, this allows us to reduce the cost of calculation by about half.

6.3. Example 3

In this example, we solve the Cauchy problem in different geometry, namely the annular domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 \ / \ r_0^2 < x^2 + y^2 < r_1^2\},$$

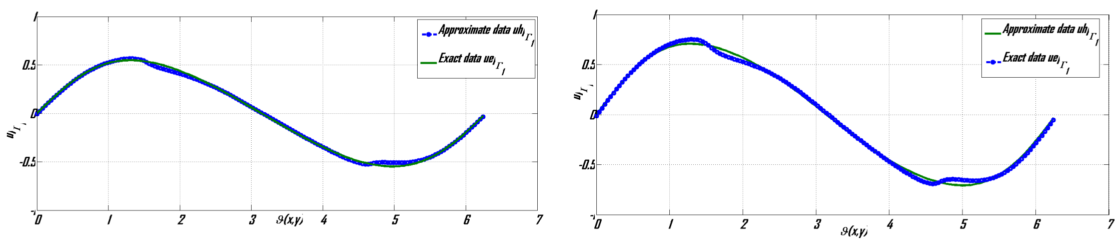
where $r_0 = 0.5$ and $r_1 = 1$. We assume that the boundary $\partial\Omega := \Gamma_1 \cup \Gamma_2$, with

$$\Gamma_1 = \{(x, y) \in \partial\Omega \ \ / \ \ x^2 + y^2 = r_0^2\}, \quad \Gamma_2 = \{(x, y) \in \partial\Omega \ \ / \ \ x^2 + y^2 = r_1^2\}.$$

We take the same data as in example 1 and we consider both case Helmholtz and Modified Helmholtz equation.

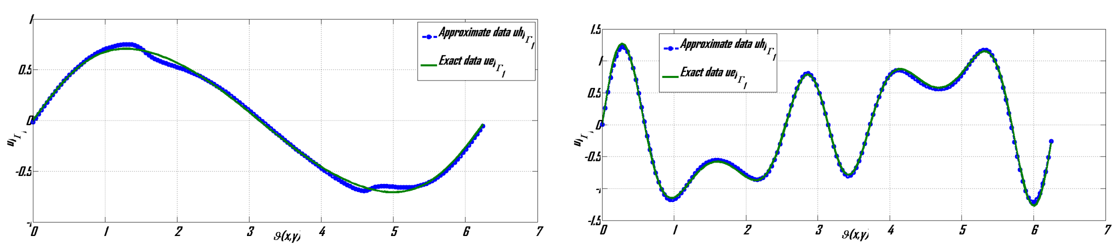
First case $\Delta u + k^2 u = 0$

In this case $\beta = 0$ and we take $\alpha \in \{1, \sqrt{2}, \sqrt{52}, \sqrt{128}\}$. In figures 11 and 12 we present the



(a) (b)

Figure 11: Exact solution at Γ_1 and the reconstructed one, obtained by the algorithm 3 in (a) for $k = 1$ and in (b) for $k = \sqrt{2}$.



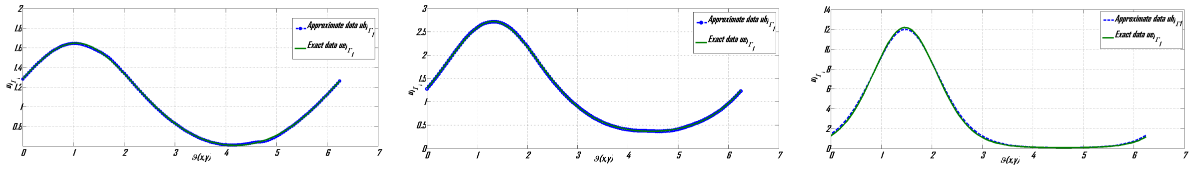
(a) (b)

Figure 12: Exact solution at Γ_1 and the reconstructed one, obtained by the algorithm 3 in (a) for $k = \sqrt{52}$ and in (b) for $k = \sqrt{128}$.

exact and reconstructed solution at Γ_1 obtained by the algorithm 3 for different values of the wave number k. It's can be seen that the obtained results show the efficiency of the proposed method even with high value of wave number k, which confirm the theoretical results.

Second case $\Delta u - k^2 u = 0$

As in the previews example the goal in this case is to discuss the convergence acceleration of the proposed algorithms. We take $\alpha = 0$ and $\beta \in \{1, 2, 5\}$. While $a_2 = \sqrt{\beta^2 - a_1^2}$ and $k = \beta$. In



(a) (b) (c)

Figure 13: Exact solution at Γ_1 and the reconstructed one, obtained by the algorithm 3 in the case of modified Helmholtz for $k = 1$ in (a) and in (b) for $k = 2$ and in (c) for $k = 5$.

figure 13 we present the comparison between the exact solution on Γ_1 and the reconstructed one, obtained by the algorithm 3 for $\beta \in \{1, 2, 5\}$. While in the table 2 we present the comparison of relative error and number of iteration with $\theta = 1.$ and 1.9, for given precision $\varepsilon = 2.63412e^{-05}$ for $\beta \in \{1, 2, 5\}$. It's can be seen that, the obtained results are good in precision. Moreover, the proposed algorithms 3 and 4 converge with high-speed comparison to the classical algorithms 1 and 2.

Wave number	Relaxation parameter	$\|u - ue\|_{L^2(\Omega)} / \|ue\|_{L^2(\Omega)}$	Number of iteration
$k = 1$	$\theta = 1.$	0.00393581	26
	$\theta = 1.9$	0.00398927	**14**
$k = 2$	$\theta = 1$	0.00397146	350
	$\theta = 1.9$	0.00395046	**185**
$k = 5$	$\theta = 1.$	0.025123398	1519
	$\theta = 1.9$	0.0251234	**798**

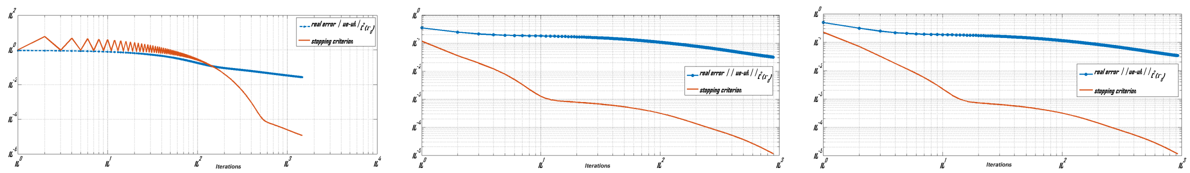
Table 3: Comparison of relative error and number of iteration with $\theta = 1.$, 1.9 and the precision $\varepsilon = 1.43538e^{-07}$ for $k \in \{1, 2, 5\}$.

6.4. Stopping criterion

Once the convergence of the algorithms 3 and 4 for different values of wave number k is proved, we show the goodness of the used stopping criterion. To do this, we show in figure 14 the comparison between the real error $\|uh - ue\|_{L^2(\Gamma_0)}$ and the stopping criterion (6.7) in example 1. As it's can be seen, the stopping criterion imitate the behavior of the real error.

6.5. Numerical stability

In the previous sections, we have developed some methods, for solving Cauchy problem associated to Helmholtz equation, based on relaxation. Thus, we have showing the effectiveness of these



(a) (b) (c)

Figure 14: Comparison between the real error and the used stopping criterion in the first case for $k = \sqrt{25.5}$ with $\theta = 0.98$ (a) and in second case for $k = \sqrt{15}$ (b) and $k = \sqrt{25.5}$ (c) with $\theta = 1.9$.

methods in terms of quality (accuracy of approximation) and quantitative (computation cost). Nevertheless, without a stability study, these methods can be irrelevant. Indeed, as already explained before, inverse problems are ill-posed problem in the sense of Hadamard [16]. The usual questions of existence and uniqueness can be ensured under certain conditions. In contrast, the not classic question is the stability, which is reflected by the fact that the solution depends continuously on the data. The lack of stability for inverse problems is in some way natural. Indeed, the experimental measurement noise is inevitable. Therefore, a naive resolution amplified noise of measurement in an uncontrolled manner, and leads to an unacceptable solution. For this reason, we must develop some sophisticated methods having some regularizing effect. In this section, we examine numerically the stability of our approach. For the sake of clarity, we will limit the presentation of results to the first example. We will examine the behavior of the algorithm 3 in the presence of small perturbations in the data. The Dirichlet and Neumann boundary conditions on Γ_1 are perturbed to simulate measurement errors such that

$$f_1^\delta = f_1 + \delta f_1, \qquad f_2^\delta = f_2 + \delta f_2, \tag{6.13}$$

where $\delta f_1 = f_1 * \delta * (2 * rand - 1)$ and $\delta f_2 = f_2 * \delta * (2 * rand - 1)$ are Gaussian noise with mean zero, generated by an appropriate function $rand$. While the δ is the noise level. The inverse Cauchy problem, which take into account these perturbations, is defined by

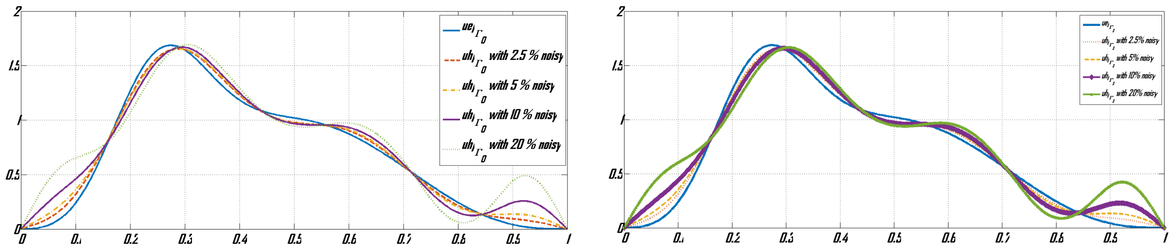
$$\Delta u^\delta \pm k^2 u^\delta = 0, \quad \text{in} \quad]0,1[\times]0,b[, \tag{6.14}$$

$$u^\delta(x,0) = f_1^\delta, \quad \text{on } 0 \le x \le 1, \tag{6.15}$$

$$\partial_y u^\delta(x,0) = f_2^\delta, \qquad 0 \le x \le 1, \tag{6.16}$$

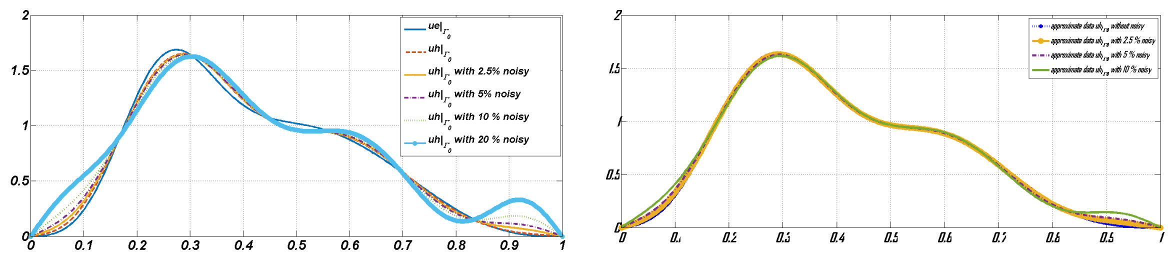
$$u^\delta(0,y) = u^\delta(1,y) = 0, \qquad \text{on } \; 0 \le y \le b. \tag{6.17}$$

In order to show that our methods are numerically stable we applied some different level of noise $\delta \in [2.5 * 10^{-2}, 2 * 10^{-1}]$ to given data and we compare the exact solution ue the approximate solution without noise and the approximate one u^δ with different noise level δ. Comparison between the exact solution, In figures 15 and 16 we present Comparison between the exact solution, the approximate solution without noise and the approximate one with different level of noise. For the



(a) (b)

Figure 15: Comparison between the exact solution, the approximate solution without noise and the approximate one with different level of noise for $k = \sqrt{15}$ with $\theta = 1.6$ (a) and for $k = \sqrt{25.5}$ (b) with $\theta = 0.98$.



(a) (b)

Figure 16: Comparison between the exact solution, the approximate solution without noise and the approximate one with different level of noise for $k = \sqrt{35}$ with $\theta = 0.5$ (a) and for $k = \sqrt{52}$ (b) with $\theta = 0.14$.

sake of clarity we present just the approximate solution u^δ four levels of noise $\delta \in \{2.5 * 10^{-2}, 5 * 10^{-2}, 10^{-1}, 2 * 10^{-1}\}$. As we can see in these figures, the obtained solution u^δ are note so far from the exact solution. Indeed, the error verify the following inequality

$$\|ue - u^\delta\|_{L^2(\Gamma_0)} \leq \delta$$

which shows that the algorithms are stable because the noise and obtained error are of the same order.

7. Conclusion

In this paper, we have investigated Cauchy problem associated with Helmholtz equation. We have developed some effective algorithms based on the relaxation. The novelty lies in the fact that these developed methods converge for all values of wave number, in the case of Helmholtz equation, and accelerate the convergence in the case of modified Helmholtz equation. Moreover, in these methods, all used parameters are completely expressed as function of the given data. Thus, the study is supported by theoretical results and numerical experiments. In particular, for the numerical experiments, many examples with different geometries and different values of wave number has been presented. An efficient stopping criterion has also presented. Finally, we have showed the numerical stability of the developed algorithms. Possible extensions of the present work is to generalize this approach for more kinds of inverse Cauchy problems.

References

[1] Avdonin, S., Kozlov, V., Maxwell, D., Truffer, M., 2009. Iterative methods for solving a nonlinear boundary inverse problem in glaciology. J. Inverse Ill-Posed Probl. 17 (3), 239–258.

[2] Bergam, A., Chakib, A., Nachaoui, A., Nachaoui, M., 2019. Adaptive mesh techniques based on a posteriori error estimates for an inverse cauchy problem. Applied Mathematics and Computation 346, 865–878.

[3] Berntsson, F., Kozlov, V., Mpinganzima, L., Turesson, B. O., 2018. Robin-Dirichlet algorithms for the Cauchy problem for the Helmholtz equation. Inverse Probl. Sci. Eng. 26 (7), 1062–1078.

[4] Berntsson, F., Kozlov, V. A., Mpinganzima, L., Turesson, B. O., 2014. An accelerated alternating procedure for the Cauchy problem for the Helmholtz equation. Comput. Math. Appl. 68 (1-2), 44–60.

[5] Berntsson, F., Kozlov, V. A., Mpinganzima, L., Turesson, B. O., 2014. An alternating iterative procedure for the Cauchy problem for the Helmholtz equation. Inverse Probl. Sci. Eng. 22 (1), 45–62.

[6] Berntsson, F., Kozlov, V. A., Mpinganzima, L., Turesson, B. O., 2017. Iterative Tikhonov regularization for the Cauchy problem for the Helmholtz equation. Comput. Math. Appl. 73 (1), 163–172.

[7] Chakib, A., Nachaoui, A., Nachaoui, M., Ouaissa, H., 2018. On a fixed point study of an inverse problem governed by stokes equation. Inverse Problems 35 (1), 015008.

[8] Chapko, R., Johansson, B. T., 2008. An alternating potential-based approach to the Cauchy problem for the Laplace equation in a planar domain with a cut. Comput. Methods Appl. Math. 8 (4), 315–335.

[9] Choulli, M., 2009. Une introduction aux problèmes inverses elliptiques et paraboliques. Vol. 65 of Mathématiques & Applications (Berlin) [Mathematics & Applications]. Springer-Verlag, Berlin.

[10] Eldén, L., Berntsson, F., Regińska, T., 2000. Wavelet and Fourier methods for solving the sideways heat equation. SIAM J. Sci. Comput. 21 (6), 2187–2205.

[11] Ellabib, A., Nachaoui, A., 2008. An iterative approach to the solution of an inverse problem in linear elasticity. Math. Comput. Simulation 77 (2-3), 189–201.

[12] Essaouini, M., Nachaoui, A., El Hajji, S., 2004. Numerical method for solving a class of nonlinear elliptic inverse problems. J. Comput. Appl. Math. 162 (1), 165–181.

[13] Essaouini, M., Nachaoui, A., El Hajji, S., 2004. Reconstruction of boundary data for a class of nonlinear inverse problems. J. Inverse Ill-Posed Probl. 12 (4), 369–385.

[14] Fatma, R. B., Azaïez, M., Abda, A. B., Gmati, N., 2007. Missing boundary data recovering for the helmholtz problem. Comptes Rendus Mécanique 335 (12), 787 – 792.

[15] Fu, C.-L., Feng, X.-L., Qian, Z., 2009. The Fourier regularization for solving the Cauchy problem for the Helmholtz equation. Appl. Numer. Math. 59 (10), 2625–2640.

[16] Hadamard, J., 1953. Lectures on Cauchy's problem in linear partial differential equations. Dover Publications, New York.

[17] Hua, Q., Gu, Y., Qu, W., Chen, W., Zhang, C., 2017. A meshless generalized finite difference method for inverse Cauchy problems associated with three-dimensional inhomogeneous Helmholtz-type equations. Eng. Anal. Bound. Elem. 82, 162–171.

[18] Ihlenburg, F., Babuška, I., 1995. Finite element solution of the helmholtz equation with high wave number part i: The h-version of the fem. Computers & Mathematics with Applications 30 (9), 9–37.

[19] Ihlenburg, F., Babuska, I., 1997. Finite element solution of the helmholtz equation with high wave number part ii: the hp version of the fem. SIAM Journal on Numerical Analysis 34 (1), 315–358.

[20] Johansson, B. T., Kozlov, V. A., 2009. An alternating method for Cauchy problems for Helmholtz-type operators in non-homogeneous medium. IMA J. Appl. Math. 74 (1), 62–73.

[21] Johansson, B. T., Marin, L., 2009. Relaxation of alternating iterative algorithms for the Cauchy problem associated with the modified Helmholtz equation. CMC Comput. Mater. Continua 13 (2), 153–189 (2010).

[22] Johansson, T., Lesnic, D., 2006. A variational conjugate gradient method for determining the fluid velocity of a slow viscous flow. Applicable Analysis 85 (11), 1327–1341.

[23] Joly, P., Lohrengel, S., Vacus, O., 1999. Un résultat d'existence et d'unicité pour l'équation de Helmholtz avec conditions aux limites absorbantes d'ordre 2. C. R. Acad. Sci. Paris Sér. I Math. 329 (3), 193–198.

[24] Jourhmane, M., Nachaoui, A., 1996. A relaxation algorithm for solving a cauchy problem. In: on Inverse Problems in Engineering, Engineering. Vol. 1 of Proc. of the 2nd Internat. Conf. Foundation Editor, pp. 151–158.

[25] Jourhmane, M., Nachaoui, A., 1999. An alternating method for an inverse Cauchy problem. Numer. Algorithms 21 (1-4), 247–260.

[26] Jourhmane, M., Nachaoui, A., 2002. Convergence of an alternating method to solve the Cauchy problem for Poisson's equation. Appl. Anal. 81 (5), 1065–1083.

[27] Karimi, M., Rezaee, A., 2017. Regularization of the cauchy problem for the helmholtz equation by using meyer wavelet. Journal of Computational and Applied Mathematics 320, 76–95.

[28] Kozlov, V. A., Maz'ya, V. G., Fomin, A. V., 1991. An iterative method for solving the Cauchy problem for elliptic equations. Zh. Vychisl. Mat. i Mat. Fiz. 31 (1), 64–74.

[29] Lavrent'ev, M. M., Romanov, V. G., Shishat·skiĭ, S. P., 1986. Ill-posed problems of mathematical physics and analysis. Vol. 64 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, translated from the Russian by J. R. Schulenberger, Translation edited by Lev J. Leifman.

[30] Lavrentiev, M. M., 2013. Some improperly posed problems of mathematical physics. Vol. 11. Springer Science & Business Media.

[31] Li, Z. P., Xu, C., Lan, M., Qian, Z., 2018. A mollification method for a Cauchy problem for the Helmholtz equation. Int. J. Comput. Math. 95 (11), 2256–2268.

[32] Marin, L., 2012. A relaxation method of an alternating iterative (mfs) algorithm for the Cauchy problem associated with the two-dimensional modified Helmholtz equation. Numer. Methods Partial Differential Equations 28 (3), 899–925.

[33] Marin, L., Elliott, L., Heggs, P., Ingham, D., Lesnic, D., Wen, X., 2004. Bem solution for the cauchy problem associated with helmholtz-type equations by the landweber method. Engineering Analysis with Boundary Elements 28 (9), 1025–1034.

[34] Marin, L., Elliott, L., Heggs, P. J., Ingham, D. B., Lesnic, D., Wen, X., 2003. An alternating iterative algorithm for the Cauchy problem associated to the Helmholtz equation. Comput. Methods Appl. Mech. Engrg. 192 (5-6), 709–722.

[35] Marin, L., Elliott, L., Heggs, P. J., Ingham, D. B., Lesnic, D., Wen, X., 2003. Conjugate gradient-boundary element solution to the Cauchy problem for Helmholtz-type equations. Comput. Mech. 31 (3-4), 367–377.

[36] Marin, L., Elliott, L., Heggs, P. J., Ingham, D. B., Lesnic, D., Wen, X., 2004. Comparison of regularization methods for solving the Cauchy problem associated with the Helmholtz equation. Internat. J. Numer. Methods Engrg. 60 (11), 1933–1947.

[37] Marin, L., Johansson, B. T., 2010. A relaxation method of an alternating iterative algorithm for the Cauchy problem in linear isotropic elasticity. Comput. Methods Appl. Mech. Engrg. 199 (49-52), 3179–3196.

[38] Marin, L., Lesnic, D., 2005. The method of fundamental solutions for the Cauchy problem associated with two-dimensional Helmholtz-type equations. Comput. & Structures 83 (4-5), 267–278.

[39] Nachaoui, A., 2003. An improved implementation of an iterative method in boundary identification problems. Numer. Algorithms 33 (1-4), 381–398.

[40] Qian, A.-L., Xiong, X.-T., Wu, Y.-J., 2010. On a quasi-reversibility regularization method for a Cauchy problem of the Helmholtz equation. J. Comput. Appl. Math. 233 (8), 1969–1979.

[41] Qian, Z., Feng, X., 2017. A fractional Tikhonov method for solving a Cauchy problem of Helmholtz equation. Appl. Anal. 96 (10), 1656–1668.

[42] Yang, F., Zhang, P., Li, X.-X., 2019. The truncation method for the Cauchy problem of the inhomogeneous Helmholtz equation. Appl. Anal. 98 (5), 991–1004.

[43] Yu, C., Zhou, Z., Zhuang, M., 2008. An acoustic intensity-based method for reconstruction of radiated fields. The Journal of the Acoustical Society of America 123 (4), 1892–1901.