An alternating method for the stationary Stokes system

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An alternating procedure for solving a Cauchy problem for the stationary Stokes system is presented. A convergence proof of this procedure and numerical results are included.

1 Introduction

Let a viscous incompressible fluid be contained in a bounded n-dimensional domain Ω, where $n \geq 2$. The problem of reconstructing the velocity and pressure of the fluid from boundary measurements, can mathematically be described as the solution to the following problem for the Stokes system:

$$\begin{cases} \Delta \boldsymbol{u} - \nabla p = 0 & \text{in} \quad \Omega\,, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega\,, \end{cases} \tag{1.1}$$

supplied with Cauchy data

$$\begin{cases} \boldsymbol{u} = \boldsymbol{\varphi} & \text{on} \quad \Gamma_0\,, \\ p\boldsymbol{\nu} - N\boldsymbol{u} = \boldsymbol{\psi} & \text{on} \quad \Gamma_0\,, \end{cases} \tag{1.2}$$

where Γ_0 is a part of the boundary $\Gamma = \partial\Omega$, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ is the outward unit normal to Γ, and

$$N\boldsymbol{u} = \left(\sum_{j=1}^{n} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \nu_j \right)_{1 \leq i \leq n} .$$

The vector $p\boldsymbol{\nu} - N\boldsymbol{u}$ is called the normal stress vector. It is clear that the above problem is ill-posed.

In 1989, Kozlov and Maz'ya [3] proposed an alternating iterative method for solving Cauchy problems for general strongly elliptic and formally self-adjoint systems. This method was further developed, notably by Baumeister and Leitao [1]. Its numerical implementation using the boundary element method has recently been developed in Mera et al. [9] for the Laplace-Belltrami equation in steady-state anisotropic heat conduction, Marin et al. [7] for the Lamé system in isotropic elasticity, and Marin et al. [8] for the Helmholtz equation in acoustics. The Stokes system does not satisfy the aforementioned requirements since it is not strongly elliptic. The aim of this paper is to extend the alternating method to the Stokes system.

The alternating method for system (1.1) with Cauchy data (1.2) runs as follows. We start with an arbitrary approximation of the normal stress vector $\boldsymbol{\xi}_0$ on $\Gamma_1 = \Gamma \setminus \overline{\Gamma}_0$. The first approximation $(\boldsymbol{u}_0, p_0)$ to the velocity and pressure is obtained by solving system (1.1) supplied with the boundary conditions $\boldsymbol{u} = \boldsymbol{\varphi}$ on Γ_0 and $(p\boldsymbol{\nu} - N\boldsymbol{u}) = \boldsymbol{\xi}_0$ on Γ_1. The next approximation $(\boldsymbol{u}_1, p_1)$ is obtained by solving system (1.1) with normal stress $(p\boldsymbol{\nu} - N\boldsymbol{u}) = \boldsymbol{\psi}$ on Γ_0 and $\boldsymbol{u} = \boldsymbol{u}_0$ on Γ_1. The next approximation solves the system (1.1) with velocity equal to $\boldsymbol{\varphi}$ on Γ_0 and normal stress on Γ_1 equal to the normal stress of $(\boldsymbol{u}_1, p_1)$ on Γ_1. Continuing this procedure, we get approximations $(\boldsymbol{u}_k, p_k)$ for $k = 0, 1, \ldots$, see Sect. 3.

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Each problem solved in this alternating procedure is well-posed, see Theorem 5.2. In Theorem 6.1, we prove that the sequence $(\boldsymbol{u}_k, p_k)$ converges to the exact solution if the measured data is given exactly. In the case of noisy data, we give a stopping rule, see Sect. 7, which brings stability to the reconstruction of the velocity and pressure from boundary measurements. Numerical results are also given, see Sect. 8.

2 Notations and definitions

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. We denote by Γ the boundary of the domain Ω. Let Γ_0 be an open part of Γ with Lipschitz boundary γ. This means that near each point $x \in \gamma$, there exists a Cartesian coordinate system $y = (y_1, \ldots, y_n)$ with the center at the point x, such that Ω is locally given by $y_n > \Phi(y_1, \ldots, y_{n-1})$ and the local equation of Γ_0 is $y_n = \Phi(y_1, \ldots, y_{n-1})$, where $y_{n-1} > \Psi(y_1, \ldots, y_{n-2})$. Here, Φ and Ψ are Lipschitz functions. Moreover, we assume that Γ_0 consists of a finite number of subdomains of Γ. We put $\Gamma_1 = \Gamma \setminus \overline{\Gamma}_0$.

As usual, $H^1(\Omega)$ denotes the Sobolev space of real valued functions in Ω with finite norm

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega u^2 \, dx + \int_\Omega |\nabla u|^2 \, dx \right)^{1/2} ,$$

where $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$. By $H_0^1(\Omega)$, we denote the subspace of functions of $H^1(\Omega)$ which vanish on Γ. Furthermore, the space of functions in $H^1(\Omega)$ vanishing on $\Gamma_0(\Gamma_1)$ is denoted by $H^1_{\Gamma_0}(\Omega)(H^1_{\Gamma_1}(\Omega))$.

The space of traces of functions from $H^1(\Omega)$ on Γ is denoted by $H^{1/2}(\Gamma)$. This space is equipped with the norm

$$\|u\|_{H^{1/2}(\Gamma)} = \left(\int_\Gamma u^2(x) \, dS(x) + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x-y|^n} \, dS(x) \, dS(y) \right)^{1/2} . \tag{2.1}$$

Restrictions of functions in $H^{1/2}(\Gamma)$ to the boundary part $\Gamma_0(\Gamma_1)$ form the space $H^{1/2}(\Gamma_0)(H^{1/2}(\Gamma_1))$ with norm defined by (2.1), where Γ is replaced by $\Gamma_0(\Gamma_1)$. The set of functions on Γ with compact support on Γ_0 (Γ_1) and bounded first derivatives are dense in $H^{1/2}(\Gamma_0)(H^{1/2}(\Gamma_1))$, see Chap. 1, Sect. 11 in Lions and Magenes [6]. We shall also use the space $H_{00}^{1/2}(\Gamma_0)$ which consists of functions from $H^{1/2}(\Gamma)$ vanishing on Γ_1. This is a subspace of $H^{1/2}(\Gamma)$ and one of the equivalent norms in this space is

$$\|u\|_{H_{00}^{1/2}(\Gamma_0)} = \left(\int_{\Gamma_0} \frac{u^2(x)}{\operatorname{dist}(x, \Gamma_1)} \, dS + \int_{\Gamma_0} \int_{\Gamma_0} \frac{|u(x) - u(y)|^2}{|x-y|^n} \, dS(x) \, dS(y) \right)^{1/2} .$$

Analogously, one can define $H_{00}^{1/2}(\Gamma_1)$. Let us mention that the space of restrictions of functions from $H_{00}^{1/2}(\Gamma_0)$ to Γ_0, is dense in $H^{1/2}(\Gamma_0)$. However, $H_{00}^{1/2}(\Gamma_0)$ does not coincide with $H^{1/2}(\Gamma_0)$, see Chap. 1, Sect. 11 in Lions and Magenes [6]. In what follows the product of n samples of a space X is denoted by X^n.

3 Formulation of the alternating procedure

In the iterative procedure for reconstruction of the solution to problem (1.1), (1.2), we use the following two problems

$$\begin{cases} \Delta \boldsymbol{u} - \nabla p = 0 & \text{in } \ \Omega \, , \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \ \Omega \, , \\ p\boldsymbol{\nu} - N\boldsymbol{u} = \boldsymbol{\xi} & \text{on } \ \Gamma_1 \, , \\ \boldsymbol{u} = \boldsymbol{\varphi} & \text{on } \ \Gamma_0 \, , \end{cases} \tag{3.1}$$

and

$$\begin{cases} \Delta \boldsymbol{u} - \nabla p = 0 & \text{in } \ \Omega \, , \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \ \Omega \, , \\ \boldsymbol{u} = \boldsymbol{\eta} & \text{on } \ \Gamma_1 \, , \\ p\boldsymbol{\nu} - N\boldsymbol{u} = \boldsymbol{\psi} & \text{on } \ \Gamma_0 \, . \end{cases} \tag{3.2}$$

Assume that $\boldsymbol{\varphi} \in H^{1/2}(\Gamma_0)^n$ and $\boldsymbol{\psi} \in (H_{00}^{1/2}(\Gamma_0)^n)^*$ are the same as in (1.2). The alternating iterative procedure runs as follows.

- The first approximation $(\boldsymbol{u}_0, p_0)$ to the solution $(\boldsymbol{u}, p)$ of (1.1), (1.2), is obtained by solving (3.1) with $\boldsymbol{\xi} = \boldsymbol{\xi}_0$ on Γ_1, where the function $\boldsymbol{\xi}_0 \in (H_{00}^{1/2}(\Gamma_1)^n)^*$ is arbitrary.
- Having constructed $(\boldsymbol{u}_{2k}, p_{2k})$, we find $(\boldsymbol{u}_{2k+1}, p_{2k+1})$ by solving problem (3.2) with $\boldsymbol{\eta} = \boldsymbol{u}_{2k}$ on Γ_1.
- Then we find the element $(\boldsymbol{u}_{2k+2}, p_{2k+2})$ by solving problem (3.1) with $\boldsymbol{\xi} = p_{2k+1}\boldsymbol{\nu} - N\boldsymbol{u}_{2k+1}$ on Γ_1.

In the next two sections, we prove that the problems (3.1) and (3.2) used in the alternating procedure are well-posed and that the restrictions of solutions to the boundary that appear in the procedure, are well-defined. We start by discussing the last condition in (3.2).

4 Properties of the trace $p\boldsymbol{\nu} - N\boldsymbol{u}$

We say that $\boldsymbol{u} \in H^1(\Omega)^n$ and $p \in L^2(\Omega)$ is a *Stokes pair* if $\operatorname{div} \boldsymbol{u} = 0$ and if

$$\int_\Omega [\boldsymbol{u}, \boldsymbol{v}]\, dx - \int_\Omega p \operatorname{div} \boldsymbol{v}\, dx = 0$$

holds for every $\boldsymbol{v} \in H_0^1(\Omega)^n$. Here,

$$\int_\Omega [\boldsymbol{u}, \boldsymbol{v}]\, dx = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) dx\,. \tag{4.1}$$

Observe that if $(\boldsymbol{u}, p) \in H^2(\Omega)^n \times H^1(\Omega)$ is a solution to (1.1), then it is also a Stokes pair. Indeed, let $(\boldsymbol{u}, p) \in H^2(\Omega)^n \times H^1(\Omega)$ be a solution to (1.1). Since $\operatorname{div} \boldsymbol{u} = 0$, one can write the first equation in (1.1) as

$$\sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{\partial p}{\partial x_i} = 0\,,$$

where $i = 1, \ldots, n$. Multiplying this equation by $v_i \in H^1(\Omega)$, integrating by parts, and summing over i, we obtain

$$\int_\Gamma (p\boldsymbol{\nu} - N\boldsymbol{u}) \cdot \boldsymbol{v}\, dS = -\int_\Omega [\boldsymbol{u}, \boldsymbol{v}]\, dx + \int_\Omega p \operatorname{div} \boldsymbol{v}\, dx\,. \tag{4.2}$$

Thus, if $\boldsymbol{v} \in H_0^1(\Omega)^n$, it follows that $(\boldsymbol{u}, p)$ is a Stokes pair.

Using relation (4.2), we shall now define $p\boldsymbol{\nu} - N\boldsymbol{u}$ for an arbitrary Stokes pair $(\boldsymbol{u}, p) \in H^1(\Omega)^n \times L^2(\Omega)$. First, we introduce a linear functional on $H^{1/2}(\Gamma)^n$ defined by

$$G(\boldsymbol{\zeta}) = -\int_\Omega [\boldsymbol{u}, \boldsymbol{v}]\, dx + \int_\Omega p \operatorname{div} \boldsymbol{v}\, dx\,,$$

where $\boldsymbol{v}|_\Gamma = \boldsymbol{\zeta}$ and $\boldsymbol{v} \in H^1(\Omega)^n$. This definition does not depend on $\boldsymbol{v}$, since $(\boldsymbol{u}, p)$ is a Stokes pair. For $\boldsymbol{\zeta} \in H^{1/2}(\Gamma)^n$, there exists a $\boldsymbol{v} \in H^1(\Omega)^n$ such that $\boldsymbol{v} = \boldsymbol{\zeta}$ on Γ and $\|\boldsymbol{v}\|_{H^1(\Omega)^n} \leq C\|\boldsymbol{\zeta}\|_{H^{1/2}(\Gamma)^n}$, with the constant C independent of $\boldsymbol{\zeta}$. This together with the definition of G gives

$$\begin{aligned} |G(\boldsymbol{\zeta})| &\leq \left(\|\boldsymbol{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} \right) \|\boldsymbol{v}\|_{H^1(\Omega)^n} \\ &\leq C \left(\|\boldsymbol{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} \right) \|\boldsymbol{\zeta}\|_{H^{1/2}(\Gamma)^n}\,. \end{aligned} \tag{4.3}$$

Hence, the functional G is bounded on the space $H^{1/2}(\Gamma)^n$. We shall use the notation $p\boldsymbol{\nu} - N\boldsymbol{u}$ for this functional belonging to $H^{-1/2}(\Gamma)^n$. Using the standard continuous extension of the scalar product in the space $L^2(\Gamma)^n$ to the space $H^{1/2}(\Gamma)^n \times H^{-1/2}(\Gamma)^n$, we write the functional as

$$G(\boldsymbol{\zeta}) = \int_\Gamma (p\boldsymbol{\nu} - N\boldsymbol{u}) \cdot \boldsymbol{\zeta}\, dS\,.$$

This extends relation (4.2) to all Stokes pairs $(\boldsymbol{u}, p) \in H^1(\Omega)^n \times L^2(\Omega)$. Estimate (4.3) implies that

$$\|p\boldsymbol{\nu} - N\boldsymbol{u}\|_{H^{-1/2}(\Gamma)^n} \leq C \left(\|\boldsymbol{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} \right)\,. \tag{4.4}$$

The restriction of $p\boldsymbol{\nu} - N\boldsymbol{u}$ to Γ_0, where $(\boldsymbol{u}, p) \in H^1(\Omega)^n \times L^2(\Omega)$ is a Stokes pair, is understood as a functional on $H_{00}^{1/2}(\Gamma_0)^n$, and hence it belongs to the space $(H_{00}^{1/2}(\Gamma_0)^n)^*$. In the following lemma, we give a condition which guarantees that $(p\boldsymbol{\nu} - N\boldsymbol{u})|_{\Gamma_0} \in (H^{1/2}(\Gamma_0)^n)^*$.

Lemma 4.1 *Suppose that* $(\boldsymbol{u}, p) \in H^1(\Omega)^n \times L^2(\Omega)$ *is a Stokes pair. If* $(p\boldsymbol{\nu} - N\boldsymbol{u})|_{\Gamma_1} \in (H^{1/2}(\Gamma_1)^n)^*$, *then* $(p\boldsymbol{\nu} - N\boldsymbol{u})|_{\Gamma_0} \in (H^{1/2}(\Gamma_0)^n)^*$.

Proof. Let $\boldsymbol{\zeta} \in H^{1/2}(\Gamma_0)^n$. Consider the functional F defined by

$$F(\boldsymbol{\zeta}) = -\int_\Omega [\boldsymbol{u}, \boldsymbol{v}]\, dx + \int_\Omega p \operatorname{div} \boldsymbol{v}\, dx - \int_{\Gamma_1} (p\boldsymbol{\nu} - N\boldsymbol{u}) \cdot \boldsymbol{v}\, dS\,, \tag{4.5}$$

where $\boldsymbol{v} \in H^1(\Omega)^n$ and $\boldsymbol{v} = \boldsymbol{\zeta}$ on Γ_0. Having (4.2) in mind, it is enough to prove that this functional is well-defined and bounded. We first prove that it is well-defined. So, we choose $\boldsymbol{v}_1, \boldsymbol{v}_2 \in H^1(\Omega)^n$ such that $\boldsymbol{v}_1 = \boldsymbol{v}_2$ on Γ_0, i.e., $(\boldsymbol{v}_1 - \boldsymbol{v}_2)|_\Gamma \in H^{1/2}_{00}(\Gamma_1)^n$. Since $(\boldsymbol{u}, p)$ is a Stokes pair, it follows from (4.2) that

$$-\int_\Omega [\boldsymbol{u}, \boldsymbol{v}_1 - \boldsymbol{v}_2]\, dx + \int_\Omega p \operatorname{div}(\boldsymbol{v}_1 - \boldsymbol{v}_2)\, dx = \int_{\Gamma_1} (p\boldsymbol{\nu} - N\boldsymbol{u}) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2)\, dS\,.$$

Thus, the functional F is well-defined. Let $\boldsymbol{v} \in H^1(\Omega)^n$ with $\boldsymbol{v}|_{\Gamma_0} = \boldsymbol{\zeta}$ and $\|\boldsymbol{v}\|_{H^1(\Omega)^n} \le C\|\boldsymbol{\zeta}\|_{H^{1/2}(\Gamma_0)^n}$. This estimate together with Cauchy's inequality applied to (4.5) gives

$$\begin{aligned} |F(\boldsymbol{\zeta})| &\le \left(\|\boldsymbol{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} + \|p\boldsymbol{\nu} - N\boldsymbol{u}\|_{(H^{1/2}(\Gamma_1)^n)^*}\right) \|\boldsymbol{v}\|_{H^1(\Omega)^n} \\ &\le C\left(\|\boldsymbol{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} + \|p\boldsymbol{\nu} - N\boldsymbol{u}\|_{(H^{1/2}(\Gamma_1)^n)^*}\right) \|\boldsymbol{\zeta}\|_{H^{1/2}(\Gamma_0)^n} \end{aligned}$$

which implies that the functional F is bounded on $H^{1/2}(\Gamma_0)^n$. Let $\boldsymbol{\zeta} \in H^{1/2}_{00}(\Gamma_0)^n$. Then, using (4.2) and (4.5) give $F = (p\boldsymbol{\nu} - N\boldsymbol{u})|_{\Gamma_0}$. From this, we deduce that $(p\boldsymbol{\nu} - N\boldsymbol{u})|_{\Gamma_0} \in (H^{1/2}(\Gamma_0)^n)^*$. □

Based on this lemma we are therefore allowed to define a weak solution to problem (1.1), (1.2) with $\boldsymbol{\varphi} \in H^{1/2}(\Gamma_0)^n$ and $\boldsymbol{\psi} \in (H^{1/2}_{00}(\Gamma_0)^n)^*$, as a Stokes pair satisfying (1.2).

5 Solutions to problems (3.1) and (3.2)

Let $\boldsymbol{\eta} \in H^{1/2}(\Gamma_1)^n$ and $\boldsymbol{\psi} \in (H^{1/2}_{00}(\Gamma_0)^n)^*$ be given. We say that the pair $(\boldsymbol{u}, p) \in H^1(\Omega)^n \times L^2(\Omega)$ is a *weak solution* to problem (3.2) if $\operatorname{div} \boldsymbol{u} = 0$, $\boldsymbol{u} = \boldsymbol{\eta}$ on Γ_1, and

$$\int_\Omega [\boldsymbol{u}, \boldsymbol{v}]\, dx - \int_\Omega p \operatorname{div} \boldsymbol{v}\, dx = -\int_{\Gamma_0} \boldsymbol{\psi} \cdot \boldsymbol{v}\, dS \tag{5.1}$$

holds for every $\boldsymbol{v} \in H^1_{\Gamma_1}(\Omega)^n$. In a similar way, one can define weak solutions to problem (3.1). Notice that a weak solution to problem (3.2) is a Stokes pair and $p\boldsymbol{\nu} - N\boldsymbol{u} = \boldsymbol{\psi}$ on Γ_0, where the operator $p\boldsymbol{\nu} - N\boldsymbol{u}$ is defined as in Sect. 4. On the other hand, if $(\boldsymbol{u}, p)$ is a Stokes pair with $\boldsymbol{u} = \boldsymbol{\eta}$ on Γ_1 and $p\boldsymbol{\nu} - N\boldsymbol{u} = \boldsymbol{\psi}$ on Γ_0, where $\boldsymbol{\eta} \in H^{1/2}(\Gamma_1)^n$ and $\boldsymbol{\psi} \in (H^{1/2}_{00}(\Gamma_0)^n)^*$, then $(\boldsymbol{u}, p)$ is a weak solution to problem (3.2).

In the solvability result that we shall prove, we need the following lemma.

Lemma 5.1 *Let* $g \in L^2(\Omega)$. *Then there exists a* $\boldsymbol{u} \in H^1_{\Gamma_1}(\Omega)^n$ *satisfying*

$$\operatorname{div} \boldsymbol{u} = g \quad in \quad \Omega\,, \tag{5.2}$$

and subject to

$$\|\boldsymbol{u}\|_{H^1(\Omega)^n} \le C\|g\|_{L^2(\Omega)} \tag{5.3}$$

with C *independent of* $\boldsymbol{u}$ *and* g.

Proof. Let $x \in \Gamma_0$. There exists a Cartesian coordinate system $(y_1, \ldots, y_n)$ in a neighborhood $\mathcal{U}$ of x, such that $\Omega \cap \mathcal{U}$ is given by the intersection of $\mathcal{U}$ and the graph domain $y_n > \alpha(y')$, where $y' = (y_1, \ldots, y_{n-1})$ and α is a Lipschitz function. Moreover, we choose $\mathcal{U}$ so small that it does not contain any point of $\overline{\Gamma}_1$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a non-negative function which is equal to zero outside $\mathcal{U}$. We choose ϕ such that

$$\int_{\mathbb{R}^{n-1}} \phi(y_1, \ldots, y_{n-1}, \alpha(y'))\, dy_1 \ldots dy_{n-1} = 1\,.$$

Define

$$\boldsymbol{\eta} = (0, \ldots, 0, \phi \int_\Omega g \, dx) .$$

Then $\boldsymbol{\eta} = 0$ on Γ_1. We seek a solution $\boldsymbol{u}$ to (5.2) in the form $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{\eta}$, where $\boldsymbol{v} \in H_0^1(\Omega)^n$. Then $\boldsymbol{v}$ must satisfy

$$\begin{cases} \operatorname{div} \boldsymbol{v} = g - \operatorname{div} \boldsymbol{\eta} & \text{in} \quad \Omega , \\ \quad \boldsymbol{v} = 0 & \text{on} \quad \Gamma . \end{cases} \tag{5.4}$$

By construction, $(g - \operatorname{div} \boldsymbol{\eta}) \in L^2(\Omega)$ and

$$\int_\Omega (g - \operatorname{div} \boldsymbol{\eta}) \, dx = \int_\Omega g \, dx - \int_\Gamma \boldsymbol{\eta} \cdot \boldsymbol{\nu} \, dS = 0 .$$

From Galdi [2, pp. 121–125] it follows that there exists a $\boldsymbol{v} \in H_0^1(\Omega)^n$ satisfying (5.4) and

$$\|\boldsymbol{v}\|_{H^1(\Omega)^n} \le C \left(\|g - \operatorname{div} \boldsymbol{\eta}\|_{L^2(\Omega)} \right) \le C \left(\|g\|_{L^2(\Omega)} + \| \operatorname{div} \boldsymbol{\eta}\|_{L^2(\Omega)} \right) .$$

Then $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{\eta}$ satisfies (5.2) and (5.3). □

Now, we are in a position to prove existence and uniqueness of a weak solution to problems (3.1) and (3.2).

Theorem 5.2

(i) *Let $\boldsymbol{\varphi} \in H^{1/2}(\Gamma_0)^n$ and $\boldsymbol{\xi} \in (H_{00}^{1/2}(\Gamma_1)^n)^*$. There exists a unique weak solution $(\boldsymbol{u}, p) \in H^1(\Omega)^n \times L^2(\Omega)$ to problem* (3.1)*, satisfying*

$$\|\boldsymbol{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} \le C \left(\|\boldsymbol{\varphi}\|_{H^{1/2}(\Gamma_0)^n} + \|\boldsymbol{\xi}\|_{(H_{00}^{1/2}(\Gamma_1)^n)^*} \right) . \tag{5.5}$$

(ii) *Let $\boldsymbol{\eta} \in H^{1/2}(\Gamma_1)^n$ and $\boldsymbol{\psi} \in (H_{00}^{1/2}(\Gamma_0)^n)^*$. There exists a unique weak solution $(\boldsymbol{u}, p) \in H^1(\Omega)^n \times L^2(\Omega)$ to problem* (3.2)*, satisfying*

$$\|\boldsymbol{u}\|_{H^1(\Omega)^n} + \|p\|_{L^2(\Omega)} \le C \left(\|\boldsymbol{\psi}\|_{(H_{00}^{1/2}(\Gamma_0)^n)^*} + \|\boldsymbol{\eta}\|_{H^{1/2}(\Gamma_1)^n} \right) . \tag{5.6}$$

Proof. We only prove (ii) since the proof of (i) is literally the same.

Uniqueness. Assume that $(\boldsymbol{u}, p) \in H_{\Gamma_1}^1(\Omega)^n \times L^2(\Omega)$ is a weak solution to (3.2), with $\boldsymbol{\eta} = 0$ and $\boldsymbol{\psi} = 0$. First, taking $\boldsymbol{v} = \boldsymbol{u}$ in (5.1), we arrive at

$$\int_\Omega [\boldsymbol{u}, \boldsymbol{u}] \, dx = 0 .$$

Using Korn's inequality for vector functions vanishing on a part of the boundary, see for e.g. Oleinik et al. [10, Theorem 2.5], it follows that $\boldsymbol{u} = 0$. Next, by Lemma 5.1, we can find a function $\boldsymbol{v} \in H_{\Gamma_1}^1(\Omega)^n$ with $\operatorname{div} \boldsymbol{v} = p$. Using this $\boldsymbol{v}$ in (5.1), we find that $\int_\Omega p^2 \, dx = 0$, i.e., $p = 0$. Hence, the solution is unique.

Existence. Assume first that $\boldsymbol{\eta} = 0$. It follows from (5.1) that a weak solution $(\boldsymbol{u}, p)$ must satisfy

$$\int_\Omega [\boldsymbol{u}, \boldsymbol{v}] \, dx = - \int_{\Gamma_0} \boldsymbol{\psi} \cdot \boldsymbol{v} \, dS \tag{5.7}$$

for every $\boldsymbol{v} \in V = \{\boldsymbol{v} \in H^1(\Omega)^n : \operatorname{div} \boldsymbol{v} = 0, \ \boldsymbol{v}|_{\Gamma_1} = 0\}$. Clearly, if $(\boldsymbol{u}, p)$ is a weak solution then $\boldsymbol{u} \in V$. Since the right-hand side in (5.7) is a linear bounded functional on the space V, there exists a $\boldsymbol{u} \in V$ satisfying (5.7) for every $\boldsymbol{v} \in V$, by Riesz representation theorem. This function $\boldsymbol{u}$ satisfies

$$\|\boldsymbol{u}\|_{H^1(\Omega)^n} \le C \|\boldsymbol{\psi}\|_{\left(H_{00}^{1/2}(\Gamma_0)^n\right)^*} . \tag{5.8}$$

We now construct an element $p \in L^2(\Omega)$ such that relation (5.1) holds. We define a linear functional P on $L^2(\Omega)$ by

$$P(q) = \int_\Omega [\boldsymbol{u}, \boldsymbol{v}] \, dx + \int_{\Gamma_0} \boldsymbol{\psi} \cdot \boldsymbol{v} \, dS ,$$

where $\boldsymbol{v} \in H^1_{\Gamma_1}(\Omega)^n$ with $\operatorname{div} \boldsymbol{v} = q$. In order to see that the functional P is well-defined, we take $\boldsymbol{v}_1,\ \boldsymbol{v}_2 \in H^1_{\Gamma_1}(\Omega)^n$ such that $\operatorname{div} \boldsymbol{v}_1 = \operatorname{div} \boldsymbol{v}_2$. Then

$$P(\operatorname{div} \boldsymbol{v}_1) - P(\operatorname{div} \boldsymbol{v}_2) = \int_\Omega [\boldsymbol{u}, (\boldsymbol{v}_1 - \boldsymbol{v}_2)]\ dx + \int_{\Gamma_0} \boldsymbol{\psi} \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2)\, dS\,.$$

Since $(\boldsymbol{v}_1 - \boldsymbol{v}_2) \in V$, the right-hand side above vanishes by (5.7). Hence, the functional P is well defined. For $q \in L^2(\Omega)$, by Lemma 5.1, there exists a $\boldsymbol{v} \in H^1_{\Gamma_1}(\Omega)^n$ such that $\operatorname{div} \boldsymbol{v} = q$ and $\|\boldsymbol{v}\|_{H^1_{\Gamma_1}(\Omega)^n} \le C\|q\|_{L^2(\Omega)}$. This gives the estimate

$$|P(q)| \le \left(\|\boldsymbol{u}\|_{H^1(\Omega)^n} + \|\boldsymbol{\psi}\|_{(H^{1/2}_{00}(\Gamma_0)^n)^*}\right) \|\boldsymbol{v}\|_{H^1(\Omega)^n} \le C\left(\|\boldsymbol{u}\|_{H^1(\Omega)^n} + \|\boldsymbol{\psi}\|_{(H^{1/2}_{00}(\Gamma_0)^n)^*}\right) \|q\|_{L^2(\Omega)}\,. \tag{5.9}$$

Thus, the functional P is bounded. By Riesz representation theorem, there exists a $p \in L^2(\Omega)$ such that

$$\int_\Omega p \operatorname{div} \boldsymbol{v}\, dx = \int_\Omega [\boldsymbol{u}, \boldsymbol{v}]\, dx + \int_{\Gamma_0} \boldsymbol{\psi} \cdot \boldsymbol{v}\, dS$$

for every $\boldsymbol{v} \in H^1_{\Gamma_1}(\Omega)^n$. So, $(\boldsymbol{u}, p) \in H^1_{\Gamma_1}(\Omega)^n \times L^2(\Omega)$ is a weak solution. Moreover, by (5.9), we have the estimate

$$\|p\|_{L^2(\Omega)} \le C\left(\|\boldsymbol{u}\|_{H^1(\Omega)^n} + \|\boldsymbol{\psi}\|_{(H^{1/2}_{00}(\Gamma_0)^n)^*}\right)\,.$$

Combining this with (5.8), the estimate (5.6) follows.

Consider then the case when $\boldsymbol{\eta}$ is an arbitrary element in $H^{1/2}(\Gamma_1)^n$. Denote by $\boldsymbol{\eta}_1 \in H^{1/2}(\Gamma)^n$ an extension of $\boldsymbol{\eta}$ to Γ such that

$$\int_\Gamma \boldsymbol{\eta}_1 \cdot \boldsymbol{\nu}\, dS = 0 \qquad \text{and} \qquad \|\boldsymbol{\eta}_1\|_{H^{1/2}(\Gamma)^n} \le C\|\boldsymbol{\eta}\|_{H^{1/2}(\Gamma_1)^n}\,.$$

From Galdi [2, p. 182], there exists a unique solution $(\boldsymbol{u}_1, p_1) \in H^1(\Omega)^n \times L^2(\Omega)$ to the problem

$$\begin{cases} \Delta \boldsymbol{u} - \nabla p = 0 & \text{in } \ \Omega\,, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \ \Omega\,, \\ \boldsymbol{u} = \boldsymbol{\eta}_1 & \text{on } \ \Gamma\,, \end{cases}$$

and

$$\|\boldsymbol{u}_1\|_{H^1(\Omega)^n} + \|p_1\|_{L^2(\Omega)} \le C\|\boldsymbol{\eta}_1\|_{H^{1/2}(\Gamma)^n} \le C\|\boldsymbol{\eta}\|_{H^{1/2}(\Gamma_1)^n}\,. \tag{5.10}$$

It follows that $(\boldsymbol{u}_1, p_1)$ is a Stokes pair. Therefore

$$\int_\Omega [\boldsymbol{u}_1, \boldsymbol{v}]\, dx - \int_\Omega p_1 \operatorname{div} \boldsymbol{v}\, dx = -\int_\Gamma \boldsymbol{\psi}_1 \cdot \boldsymbol{v}\, dS\,,$$

where $\boldsymbol{v} \in H^1_{\Gamma_1}(\Omega)^n$ and $\boldsymbol{\psi}_1 = (p_1\boldsymbol{\nu} - N\boldsymbol{u}_1)|_\Gamma \in H^{-1/2}(\Gamma)^n$. Let $(\boldsymbol{u}_2, p_2)$ be the weak solution to problem (3.2) with $\boldsymbol{\eta} = 0$ and $\boldsymbol{\psi}$ equal to $(\boldsymbol{\psi} - \boldsymbol{\psi}_1)$, which exists by the above arguments. Then $(\boldsymbol{u}, p) = (\boldsymbol{u}_1, p_1) + (\boldsymbol{u}_2, p_2)$ is the sought solution. Using (5.10) and the estimate for $(\boldsymbol{u}_2, p_2)$ proved above, the estimate (5.6) follows. □

We have then proved that the problems used in the alternating procedure are well-posed. Moreover, from Sect. 4, the restrictions to the boundary of solutions that appear in this procedure are well-defined. We now turn to the question of convergence of the procedure.

6 Convergence of the alternating procedure

We now state the main theorem of this paper.

Theorem 6.1 *Let $\boldsymbol{\varphi} \in H^{1/2}(\Gamma_0)^n$ and $\boldsymbol{\psi} \in (H^{1/2}_{00}(\Gamma_0)^n)^*$. Assume that problem* (1.1), (1.2) *has a solution $(\boldsymbol{u}, p) \in H^1(\Omega)^n \times L^2(\Omega)$. Let $(\boldsymbol{u}_k, p_k)$ be the k-th approximate solution in the alternating procedure described in Sect.* 3. *Then*

$$\lim_{k\to\infty} \|\boldsymbol{u} - \boldsymbol{u}_k\|_{H^1(\Omega)^n} = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \|p - p_k\|_{L^2(\Omega)} = 0$$

for any initial data element $\boldsymbol{\xi}_0 \in (H^{1/2}_{00}(\Gamma_1)^n)^*$.

Proof. If we start the alternating procedure with $\boldsymbol{\xi} = (p\boldsymbol{\nu} - N\boldsymbol{u})|_{\Gamma_1}$, then one can check that $(\boldsymbol{u}_k, p_k) = (\boldsymbol{u}, p)$ for every k. Therefore, it is enough to prove the theorem when $(\boldsymbol{u}, p) = 0$, i.e., when $\boldsymbol{\varphi} = 0$ and $\boldsymbol{\psi} = 0$.

First, we prove convergence of the iterates $\boldsymbol{u}_k$. We follow the idea of the proof of Theorem 1 in Kozlov, Maz'ya, and Fomin [4]. Since $\boldsymbol{u}_k|_{\Gamma_0} = 0$ or $(p_k\boldsymbol{\nu} - N\boldsymbol{u}_k)|_{\Gamma_0} = 0$, we have from (4.2) that

$$\int_\Omega [\boldsymbol{u}_k, \boldsymbol{u}_k]\, dx = -\int_{\Gamma_1} (p_k\boldsymbol{\nu} - N\boldsymbol{u}_k) \cdot \boldsymbol{u}_k\, dS\,. \tag{6.1}$$

In the same way, since $\boldsymbol{u}_{2k} = 0$ on Γ_0

$$\int_\Omega [\boldsymbol{u}_{2k-1}, \boldsymbol{u}_{2k}]\; dx = -\int_{\Gamma_1} (p_{2k-1}\boldsymbol{\nu} - N\boldsymbol{u}_{2k-1}) \cdot \boldsymbol{u}_{2k}\, dS\,. \tag{6.2}$$

Using that $(p_{2k}\boldsymbol{\nu} - N\boldsymbol{u}_{2k}) = (p_{2k-1}\boldsymbol{\nu} - N\boldsymbol{u}_{2k-1})$ on Γ_1 and (6.1), we obtain from (6.2) that

$$\int_\Omega [\boldsymbol{u}_{2k-1}, \boldsymbol{u}_{2k}]\; dx = \int_\Omega [\boldsymbol{u}_{2k}, \boldsymbol{u}_{2k}]\; dx\,.$$

This implies

$$\int_\Omega [\boldsymbol{u}_{2k} - \boldsymbol{u}_{2k-1}, \boldsymbol{u}_{2k} - \boldsymbol{u}_{2k-1}]\; dx = \int_\Omega [\boldsymbol{u}_{2k-1}, \boldsymbol{u}_{2k-1}]\; dx - \int_\Omega [\boldsymbol{u}_{2k}, \boldsymbol{u}_{2k}]\; dx\,. \tag{6.3}$$

In a similar way, one can show that

$$\int_\Omega [\boldsymbol{u}_{2k+1} - \boldsymbol{u}_{2k}, \boldsymbol{u}_{2k+1} - \boldsymbol{u}_{2k}]\; dx = \int_\Omega [\boldsymbol{u}_{2k}, \boldsymbol{u}_{2k}]\; dx - \int_\Omega [\boldsymbol{u}_{2k+1}, \boldsymbol{u}_{2k+1}]\; dx\,. \tag{6.4}$$

Introduce the bounded linear operator $B : (H_{00}^{1/2}(\Gamma_1)^n)^* \to (H_{00}^{1/2}(\Gamma_1)^n)^*$ by $B\boldsymbol{\xi} = (p_2(\boldsymbol{\xi})\boldsymbol{\nu} - N\boldsymbol{u}_2(\boldsymbol{\xi}))|_{\Gamma_1}$. One can check that

$$\boldsymbol{u}_{2k} = \boldsymbol{u}_0\left(B^k\boldsymbol{\xi}\right)\,.$$

As will be shown in Sect. 7.3, the operator B is self-adjoint, non-negative, non-expansive, and one is not an eigenvalue. This implies that $\boldsymbol{u}_{2k}$ tends to zero in $H^1(\Omega)^n$. Now, relation (6.4) gives that $\int_\Omega [\boldsymbol{u}_k, \boldsymbol{u}_k]\, dx$ tends to zero. Combining this with a variant of Korn's inequality used also in the proof of Theorem 5.2 implies that $\boldsymbol{u}_k$ converges to zero in $H^1(\Omega)^n$.

It remains to show the convergence of p_k to zero in $L^2(\Omega)$. We can, by Theorem 5.2 (ii), estimate the solution $(\boldsymbol{u}_{2k+1}, p_{2k+1})$ to (3.2) as

$$\|\boldsymbol{u}_{2k+1}\|_{H^1(\Omega)^n} + \|p_{2k+1}\|_{L^2(\Omega)} \le C\|\boldsymbol{u}_{2k}\|_{H^{1/2}(\Gamma_1)^n}\,,$$

which implies that p_{2k+1} converges to zero in $L^2(\Omega)$. In the same way, we can estimate the solution to problem (3.1) as

$$\|\boldsymbol{u}_{2k+2}\|_{H^1(\Omega)^n} + \|p_{2k+2}\|_{L^2(\Omega)} \le C\|p_{2k+1}\boldsymbol{\nu} - N\boldsymbol{u}_{2k+1}\|_{\left(H_{00}^{1/2}(\Gamma_1)^n\right)^*}\,.$$

Since $(\boldsymbol{u}_{2k+1}, p_{2k+1})$ converges to zero, this implies that p_{2k} tends to zero in $L^2(\Omega)$. The proof is complete. □

Corollary 6.2 *Let the assumptions of Theorem* 6.1 *be fulfilled and let Ω' be a domain such that $\overline{\Omega'} \subset \Omega$. Then, for $l = 1, 2, \ldots$,*

$$\lim_{k\to\infty} \|\boldsymbol{u} - \boldsymbol{u}_k\|_{H^{l+1}(\Omega')^n} = 0 \quad \text{and} \quad \lim_{k\to\infty} \|p - p_k\|_{H^l(\Omega')} = 0$$

for any initial data $\boldsymbol{\xi}_0 \in (H_{00}^{1/2}(\Gamma_1)^n)^*$.

Proof. Since $(\boldsymbol{u}_k - \boldsymbol{u}, p_k - p)$ satisfies system (1.1), we can use local estimates for Stokes system, which gives

$$\|\boldsymbol{u}_k - \boldsymbol{u}\|_{H^{l+1}(\Omega')^n} + \|p_k - p\|_{H^l(\Omega')} \le C\left(\|\boldsymbol{u}_k - \boldsymbol{u}\|_{H^1(\Omega)^n} + \|p_k - p\|_{L^2(\Omega)}\right)\,.$$

The reference to Theorem 6.1 completes the proof. □

Remarks:

1. If the boundary Γ_0 is sufficiently smooth, then Corollary 6.2 is valid for a domain $\Omega' \subset \Omega$ with $\overline{\Omega'} \cap \overline{\Gamma}_1 = \varnothing$. To prove it, it suffices to use local estimates for solutions to Stokes system near the boundary Γ_0.
2. Suppose that $\boldsymbol{\psi} \in (H^{1/2}(\Gamma_0)^n)^*$ and, as before, that $\boldsymbol{\varphi} \in H^{1/2}(\Gamma_0)^n$. Then it follows from Lemma 4.1 that $(p_k\boldsymbol{\nu} - N\boldsymbol{u}_k)|_{\Gamma_i} \in (H^{1/2}(\Gamma_i)^n)^*$ for $k = 1, 2, \ldots$, and $i = 0, 1$.

7 A stopping criterion for the alternating procedure

7.1 Reformulation of Cauchy problem (1.1), (1.2)

In order to rewrite the Cauchy problem (1.1) and (1.2), we introduce several auxiliary operators. Define an operator $D_{\Gamma_1} : H^{1/2}(\Gamma_1)^n \to H^1(\Omega)^n \times L^2(\Omega)$, by

$$D_{\Gamma_1}\boldsymbol{\eta} = (\boldsymbol{u}, p)\,, \tag{7.1}$$

where $(\boldsymbol{u}, p)$ solves problem (3.2) with $\boldsymbol{\psi} = 0$. Also, we introduce an operator $T_{\Gamma_1} : (H^{1/2}_{00}(\Gamma_1)^n)^* \to H^1(\Omega)^n \times L^2(\Omega)$, by

$$T_{\Gamma_1}\boldsymbol{\xi} = (\boldsymbol{u}, p)\,, \tag{7.2}$$

where $(\boldsymbol{u}, p)$ satisfies (3.1) with $\boldsymbol{\varphi} = 0$. From Theorem 5.2, it follows that the operators D_{Γ_1} and T_{Γ_1} are bounded. Similarly, one can introduce the operators D_{Γ_0} and T_{Γ_0}. We put $\Lambda_{\Gamma_i}(\boldsymbol{u}, p) = (p\boldsymbol{\nu} - N\boldsymbol{u})|_{\Gamma_i}$ and $\Pi_{\Gamma_i}(\boldsymbol{u}, p) = \boldsymbol{u}|_{\Gamma_i}$, for $i = 0, 1$. Moreover, $\Pi(\boldsymbol{u}, p) = \boldsymbol{u}$.

Now, we can rewrite the Cauchy problem (1.1), (1.2) in the following way. Let $\boldsymbol{\xi} \in (H^{1/2}_{00}(\Gamma_1)^n)^*$. Denote by $(\boldsymbol{u}_0, p_0)$ the solution to (3.1). Clearly, $(\boldsymbol{u}_0, p_0) = T_{\Gamma_1}\boldsymbol{\xi} + D_{\Gamma_0}\boldsymbol{\varphi}$. Let $(\boldsymbol{u}_1, p_1)$ be the solution to (3.2) with $\boldsymbol{\eta} = \boldsymbol{u}_0|_{\Gamma_1}$. This solution can be represented as $(\boldsymbol{u}_1, p_1) = T_{\Gamma_0}\boldsymbol{\psi} + D_{\Gamma_1}\Pi_{\Gamma_1}(\boldsymbol{u}_0, p_0)$. One can verify directly that $(\boldsymbol{u}_1, p_1) \in H^1(\Omega)^n \times L^2(\Omega)$ is a solution to (1.1), (1.2) if and only if $\boldsymbol{\xi}$ satisfies $\Lambda_{\Gamma_1}(\boldsymbol{u}_1, p_1) = \boldsymbol{\xi}$. We rewrite the last equation as

$$\boldsymbol{\xi} = B\boldsymbol{\xi} + G\,, \tag{7.3}$$

where

$$B\boldsymbol{\xi} = \Lambda_{\Gamma_1}\, D_{\Gamma_1}\Pi_{\Gamma_1}T_{\Gamma_1}\boldsymbol{\xi}\,, \tag{7.4}$$

$$G = \Lambda_{\Gamma_1}\left(D_{\Gamma_1}\Pi_{\Gamma_1}D_{\Gamma_0}\boldsymbol{\varphi} + T_{\Gamma_0}\boldsymbol{\psi}\right)\,. \tag{7.5}$$

The fact that the operators D_{Γ_i} and T_{Γ_i} are bounded, combined with the estimate (4.4) imply that the operator $B : (H^{1/2}_{00}(\Gamma_1)^n)^* \to (H^{1/2}_{00}(\Gamma_1)^n)^*$ is bounded and that $G \in (H^{1/2}_{00}(\Gamma_1)^n)^*$. We note that the operator B is exactly the same as that we have used in the proof of Theorem 6.1.

7.2 A factor space

Let $\boldsymbol{\xi} \in (H^{1/2}_{00}(\Gamma_1)^n)^*$ and define by $\|\cdot\|$, throughout the paper, the following functional

$$\|\boldsymbol{\xi}\| = \left(\int_\Omega [\boldsymbol{v}, \boldsymbol{v}]\,dx\right)^{1/2}\,, \tag{7.6}$$

where $(\boldsymbol{v}, q) = T_{\Gamma_1}\boldsymbol{\xi}$. From Theorem 5.2 (i), we find that

$$\|\boldsymbol{\xi}\| \le C\|\boldsymbol{\xi}\|_{\left(H^{1/2}_{00}(\Gamma_1)^n\right)^*}\,.$$

It is straightforward to check that $\|\boldsymbol{\xi}\| = 0$, if and only if $\boldsymbol{\xi} = C\boldsymbol{\nu}$, where, as before, $\boldsymbol{\nu}$ is the outward unit normal to Γ, or equivalently $(\boldsymbol{v}, q) = (0, C)$. We then introduce the factor (quotient) space W of $(H^{1/2}_{00}(\Gamma_1)^n)^*$ by the equivalence relation $\boldsymbol{\xi}_1 \sim \boldsymbol{\xi}_2$, which means that $\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 = C\boldsymbol{\nu}$ for some constant C. The corresponding factor space norm will be denoted by $\|\cdot\|_W$. Let us show that $\|\cdot\|_W$ and $\|\cdot\|$ are equivalent on the space W. It is enough to prove that

$$\|\boldsymbol{\xi}\|_W \le C\|\boldsymbol{\xi}\|\,. \tag{7.7}$$

Let $(\boldsymbol{u}, p) = T_{\Gamma_1}\boldsymbol{\xi}$. We can suppose, probably after replacing $\boldsymbol{\xi}$ with an equivalent element, that $\int_\Omega p\,dx = 0$. According to the definition of a weak solution

$$\int_\Omega [\boldsymbol{u}, \boldsymbol{v}]\,dx - \int_\Omega p\,\operatorname{div}\boldsymbol{v}\,dx = -\int_{\Gamma_1} \boldsymbol{\xi}\cdot\boldsymbol{v}\,dS \tag{7.8}$$

for all $\boldsymbol{v} \in H^1_{\Gamma_0}(\Omega)^n$. Then let $\boldsymbol{\zeta} \in H^{1/2}_{00}(\Gamma_1)^n$. By Chap. 3 in Galdi [2], there exists an element $\boldsymbol{v} \in H^1(\Omega)^n$ satisfying

$$\operatorname{div}\boldsymbol{v} = \frac{1}{|\Omega|}\int_\Gamma \boldsymbol{\zeta}\cdot\boldsymbol{\nu}\,dS$$

and $\boldsymbol{v} = \boldsymbol{\zeta}$ on Γ. Moreover, we can choose $\boldsymbol{v}$ such that

$$\|\boldsymbol{v}\|_{H^1(\Omega)^n} \leq C\|\boldsymbol{\zeta}\|_{H^{1/2}(\Gamma)^n}\,. \tag{7.9}$$

Now, (7.8) implies that

$$\int_\Omega [\boldsymbol{u}, \boldsymbol{v}]\, dx = -\int_{\Gamma_1} \boldsymbol{\xi} \cdot \boldsymbol{\zeta}\, dS\,. \tag{7.10}$$

The definition of an equivalent norm in $(H_{00}^{1/2}(\Gamma)^n)^*$ runs as $\sup|\int_{\Gamma_1} \boldsymbol{\xi} \cdot \boldsymbol{\zeta}\, dS|$, where the supremum is taken over all $\boldsymbol{\zeta} \in H_{00}^{1/2}(\Gamma_1)^n$ with norm equal to one. This combined with (7.9) and (7.10) implies that (7.7) holds.

In what follows, we consider the space W equipped with the norm $\|\cdot\|$ defined in (7.6) and the corresponding scalar product will be denoted by $(\cdot\,,\cdot)$.

7.3 Properties of the operator B

Since $B\boldsymbol{\nu} = 0$, the operator B in (7.4) is well-defined on the space W introduced in the previous section. Let us show that the operator B is self-adjoint and non-negative in W. Let $\boldsymbol{\xi} \in (H_{00}^{1/2}(\Gamma_1)^n)^*$. We then put $(\boldsymbol{v}_0, q_0) = T_{\Gamma_1}\boldsymbol{\xi}$, $(\boldsymbol{v}_1, q_1) = D_{\Gamma_1}(\boldsymbol{v}_0|_{\Gamma_1})$, and $(\boldsymbol{v}_2, q_2) = T_{\Gamma_1}\Lambda_{\Gamma_1}(\boldsymbol{v}_1, q_1)$. The elements $(\boldsymbol{w}_0, p_0)$, $(\boldsymbol{w}_1, p_1)$, and $(\boldsymbol{w}_2, p_2)$ are then constructed in the same way but with $\boldsymbol{\xi}$ replaced by $\boldsymbol{\zeta} \in (H_{00}^{1/2}(\Gamma_1)^n)^*$. Then using the boundary conditions for $\boldsymbol{v}_2$ and $\boldsymbol{w}_0$, we obtain that

$$(B\boldsymbol{\xi}, \boldsymbol{\zeta}) = -\int_\Gamma (q_2\boldsymbol{\nu} - N\boldsymbol{v}_2) \cdot \boldsymbol{w}_0\, dS = -\int_\Gamma (q_1\boldsymbol{\nu} - N\boldsymbol{v}_1) \cdot \boldsymbol{w}_1\, dS = \int_\Omega [\boldsymbol{v}_1, \boldsymbol{w}_1]\, dx\,.$$

In the same way, one can prove that

$$(\boldsymbol{\xi}, B\boldsymbol{\zeta}) = \int_\Omega [\boldsymbol{v}_1, \boldsymbol{w}_1]\ dx\,.$$

Thus, the operator B is self-adjoint. Let us then show that the operator B is non-expansive. We have

$$\|B\boldsymbol{\xi}\|^2 = \int_\Omega [\boldsymbol{v}_2, \boldsymbol{v}_2]\ dx\,.$$

Using (6.3) and (6.4) with $\boldsymbol{u}_k = \boldsymbol{v}_k$, $k = 0, 1, 2$, we obtain

$$\|B\boldsymbol{\xi}\|^2 \leq \int_\Omega [\boldsymbol{v}_0, \boldsymbol{v}_0]\ dx = \|\boldsymbol{\xi}\|^2\,.$$

Next, we show that the number one is not an eigenvalue of the operator B. Indeed, if one is an eigenvalue of B, then

$$B\boldsymbol{\xi} = \boldsymbol{\xi}$$

for a certain $\boldsymbol{\xi} \in W$. This means that $(q_1\boldsymbol{\nu} - N\boldsymbol{v}_1)|_{\Gamma_1} = \boldsymbol{\xi}$, which gives that $(\boldsymbol{v}_1, q_1) = (\boldsymbol{v}_0, q_0)$ because of the same Cauchy data on Γ_1. Furthermore, since $\boldsymbol{v}_0 = 0$ on Γ_0 and $q_1\boldsymbol{\nu} - N\boldsymbol{v}_1 = 0$ on Γ_0, we find that $\boldsymbol{v}_0 = 0$ and $q_0 = 0$, which implies that $\boldsymbol{\xi} = 0$.

7.4 A stopping criterion

Using the operators introduced above, we can rewrite the alternating procedure as

$$\begin{aligned}
(\boldsymbol{u}_0, p_0) &= T_{\Gamma_1}\boldsymbol{\xi} + D_{\Gamma_0}\boldsymbol{\varphi}\,,\\
(\boldsymbol{u}_{2k+1}, p_{2k+1}) &= D_{\Gamma_1}\boldsymbol{u}_{2k}|_{\Gamma_1} + T_{\Gamma_0}\boldsymbol{\psi}\,,\\
(\boldsymbol{u}_{2k+2}, p_{2k+2}) &= T_{\Gamma_1}(p_{2k+1}\boldsymbol{\nu} - N\boldsymbol{u}_{2k+1})|_{\Gamma_1} + D_{\Gamma_0}\boldsymbol{\varphi}\,.
\end{aligned}$$

We put $\Psi_k = \Lambda_{\Gamma_1}(\boldsymbol{u}_{2k}, p_{2k})$. Let B and G be defined by (7.4) and (7.5). Then noticing that $\Lambda_{\Gamma_1}T_{\Gamma_1} = I$ and $\Lambda_{\Gamma_1}D_{\Gamma_0} = 0$ we obtain

$$\Psi_{k+1} = B\Psi_k + G\,, \tag{7.11}$$

where $\Psi_0 = \boldsymbol{\xi}$. Thus, the iterative procedure for solving the Cauchy problem (1.1), (1.2) presented in Sect. 3, can be written as the operator equation (7.11), with a self-adjoint, non-negative, and non-expansive operator B. A stopping criterion for such an equation is well-known, see for example Chap. 3 Sect. 3 in Vainikko and Veretennikov [11]. For example, one can use the following discrepancy principle as a stopping rule. Let $b > 1$ and let the right-hand side G in eq. (7.11) be known with an error δ, i.e., $\|G - G^\delta\| \le \delta$. Then we consider the iterations

$$\Psi^\delta_{k+1} = B\Psi^\delta_k + G^\delta\,,$$

with initial approximation $\Psi^\delta_0 = \boldsymbol{\xi}$. Let $k = k(\delta) \ge 0$ be the first integer such that $\|\Psi^\delta_k - B\Psi^\delta_k - G^\delta\| \le b\delta$ or equivalently

$$\left\|\Psi^\delta_{k+1} - \Psi^\delta_k\right\| \le b\delta\,. \tag{7.12}$$

Then $\Psi^\delta_{k(\delta)}$ converges to the exact solution in the space W as $\delta \to 0$.

7.5 Reformulation of the stopping criterion

Let us reformulate the stopping criterion given in the previous section for the alternating procedure described in Sect. 3. We suppose that we know only approximations $\boldsymbol{\varphi}^{(\delta)} \in H^{1/2}(\Gamma_0)^n$ and $\boldsymbol{\psi}^{(\delta)} \in (H^{1/2}_{00}(\Gamma_0)^n)^*$ to the exact Cauchy data in (1.1), (1.2) with an error δ, i.e.,

$$\left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}^\delta\right\|_{H^{1/2}(\Gamma_0)^n} + \left\|\boldsymbol{\psi} - \boldsymbol{\psi}^\delta\right\|_{\left(H^{1/2}_{00}(\Gamma_0)^n\right)^*} \le C\delta\,, \tag{7.13}$$

where the constant C is chosen such that

$$\left\|\Lambda_{\Gamma_1}\left(D_{\Gamma_1}\Pi_{\Gamma_1}D_{\Gamma_0}\left(\boldsymbol{\varphi} - \boldsymbol{\varphi}^{(\delta)}\right) + T_{\Gamma_0}\left(\boldsymbol{\psi} - \boldsymbol{\psi}^{(\delta)}\right)\right)\right\| \le \delta\,. \tag{7.14}$$

Denote by $(\boldsymbol{u}_k, p_k)$ the approximations constructed in Sect. 3 from exact Cauchy data $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ with initial approximation $\boldsymbol{\xi}_0$. Then one can verify that

$$\Lambda_{\Gamma_1}(\boldsymbol{u}_1, p_1) = B\boldsymbol{\xi}_0 + G\,.$$

Analogously, let $\left(\boldsymbol{u}^\delta_k, p^\delta_k\right)$ be the approximations constructed from noisy Cauchy data $\boldsymbol{\varphi}^\delta$ and $\boldsymbol{\xi}^\delta$ with the same $\boldsymbol{\xi}_0$. Also,

$$\Lambda_{\Gamma_1}\left(\boldsymbol{u}^\delta_1, p^\delta_1\right) = B\boldsymbol{\xi}_0 + G^\delta\,.$$

Setting

$$(\boldsymbol{r}_\delta, q_\delta) = T_{\Gamma_1}\left(G^\delta - G\right) = T_{\Gamma_1}\Lambda_{\Gamma_1}\left(\boldsymbol{u}^\delta_1 - \boldsymbol{u}_1, p^\delta_1 - p_1\right)\,,$$

we can rewrite (7.14) as

$$\int_\Omega (\boldsymbol{r}_\delta, \boldsymbol{r}_\delta)\,, dx \le \delta^2\,.$$

Since

$$T_{\Gamma_1}\left(\Psi^\delta_{k+1} - \Psi^\delta_k\right) = \left(\boldsymbol{u}^\delta_{2k+2} - \boldsymbol{u}^\delta_{2k}, p^\delta_{2k+2} - p^\delta_{2k}\right)\,,$$

we shall stop the procedure, according to (7.12), for the first $k = k(\delta)$ such that

$$\|\Psi^\delta_{k+1} - \Psi^\delta_k\|^2 = \int_\Omega \left[\boldsymbol{u}^\delta_{2k+2} - \boldsymbol{u}^\delta_{2k}, \boldsymbol{u}^\delta_{2k+2} - \boldsymbol{u}^\delta_{2k}\right]\,dx \le b^2\delta^2\,. \tag{7.15}$$

Using a Green's formula, we can rewrite this as

$$\int_{\Gamma_1}\left(-\left(p^\delta_{2k+2} - p^\delta_{2k}\right)\boldsymbol{\nu} + N\left(\boldsymbol{u}^\delta_{2k+2} - \boldsymbol{u}^\delta_{2k}\right)\right)\cdot\left(\boldsymbol{u}^\delta_{2k+2} - \boldsymbol{u}^\delta_{2k}\right)\,dS \le b^2\delta^2\,. \tag{7.16}$$

Now, according to the previous section, we have convergence of $\Psi^\delta_{k(\delta)}$ to Ψ in the space W. Since

$$T_{\Gamma_1}\Psi^\delta_k = \left(\boldsymbol{u}^\delta_{2k}, p^\delta_{2k}\right) - D_{\Gamma_0}\boldsymbol{\varphi}^\delta$$

and

$$T_{\Gamma_1}\Psi_k = (\boldsymbol{u}_{2k}, p_{2k}) - D_{\Gamma_0}\boldsymbol{\varphi}\,,$$

we have

$$\|\Psi^\delta_k - \Psi_k\|^2 = \int_\Omega \left[\boldsymbol{u}^\delta_{2k} - \boldsymbol{u}_{2k}, \boldsymbol{u}^\delta_{2k} - \boldsymbol{u}_{2k}\right]\,dx - \int_\Omega \left[\Pi D_{\Gamma_0}\left(\boldsymbol{\varphi}^\delta - \boldsymbol{\varphi}\right), \Pi D_{\Gamma_0}\left(\boldsymbol{\varphi}^\delta - \boldsymbol{\varphi}\right)\right]\,dx\,. \tag{7.17}$$

Combining this with (7.13) implies the convergence of $\boldsymbol{u}^\delta_{2k(\delta)}$ to $\boldsymbol{u}$ in $H^1(\Omega)^n$. From inequality (5.5) applied to $(\boldsymbol{u}^\delta_{2k(\delta)} - \boldsymbol{u}, p^\delta_{2k(\delta)} - p)$, we obtain that $p^\delta_{2k(\delta)}$ converges to p in $L^2(\Omega)$. This in turn implies that the alternating procedure together with the stopping rule defined by the inequality in (7.15), is a regularization method for solving the Cauchy problem (1.1), (1.2).

8 Numerical results and discussion

In this section we discuss the numerical results obtained using the algorithm proposed for a Cauchy problem in a two-dimensional bounded domain Ω, although the same conclusions hold in higher dimensions. Let Γ_0 and Γ_1 be two infinitely long circular cylinders of radii R_0 and R_1 satisfying $0 < R_1 < R_0$. Let $\Omega = \{(x,y) \mid R_1^2 < x^2 + y^2 < R_0^2\}$ be the annular domain between these cylinders which is filled with viscous fluid flowing at low Reynolds number and further, suppose that both cylinders rotate. The inverse Cauchy problem is to determine the fluid velocity and the stress force at the inner cylinder $\Gamma_1 = \{(x,y) \mid x^2 + y^2 = R_1^2\}$ by taking measurements at the outer cylinder $\Gamma_0 = \{(x,y) \mid x^2 + y^2 = R_0^2\}$. Let us consider the following benchmark test example

$$\mathbf{u} = \left(4y^3 - x^2, 4x^3 + 2xy - 1\right)\,, \quad p = 24xy - 2x\,, \quad (x,y) \in \Omega \tag{8.1}$$

which satisfies the Stokes equations (1.1) and has been previously considered by Zeb et al. [13] in another type of inverse problem. Of course, in the practical fluid dynamics context the viscous flow assumption implies that the velocity at the boundary should equal the wall velocity, but in our benchmark test example we investigate a more general situation. The analytical example (8.1) generates the Cauchy data (1.2) as given by

$$\boldsymbol{\varphi} = \left(4y^3 - x^2, 4x^3 + 2xy - 1\right)\,, \quad (x,y) \in \Gamma_0\,, \tag{8.2}$$

$$\boldsymbol{\psi} = \left(-12y^3 - 2y^2 + 2x^2 + 12x^2y, -12x^3 + 12xy^2 - 8xy\right)/R_0\,, \quad (x,y) \in \Gamma_0\,. \tag{8.3}$$

For solving the mixed well-posed direct problems (3.1) and (3.2) we employ the boundary element method (BEM), see Zeb et al. [12]. Unlike domain discretisation methods such as the finite-difference method (FDM) or the finite element method (FEM), with the BEM we actually never need to calculate the interior solution ($\mathbf{u}^{(k)}$,$p^{(k)}$) in Ω, but solve only at the boundary the following boundary integral equation, see Ladyzhenskaya [5],

$$\tfrac{1}{2}\mathbf{u}(\mathbf{x}) = \int_{\partial\Omega} \left[\underline{\underline{K}}(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y}) - \underline{\underline{U}}(\mathbf{x},\mathbf{y})\mathbf{t}(\mathbf{y})\right] dS(\mathbf{y})\,, \quad \mathbf{x} \in \partial\Omega \tag{8.4}$$

where $\mathbf{t} = -p\boldsymbol{\nu} + N\mathbf{u}$ and, in two-dimensions, the tensors $\underline{\underline{K}}$ and $\underline{\underline{U}}$ are given by

$$K_{kl}(\mathbf{x},\mathbf{y}) = -\frac{(x_k - y_k)(x_l - y_l)}{\pi \mid \mathbf{x} - \mathbf{y} \mid^4} \sum_{m=1}^{2} (x_m - y_m)\nu_m\,, \quad k,l = \overline{1,2}, \tag{8.5}$$

$$U_{kl} = -\frac{1}{4\pi}\left[-\delta_{kl} ln \mid \mathbf{x} - \mathbf{y} \mid + \frac{(x_k - y_k)(x_l - y_l)}{\mid \mathbf{x} - \mathbf{y} \mid^2}\right]\,, \quad k,l = \overline{1,2}\,. \tag{8.6}$$

Here δ_{kl} is the Kronecker tensor, $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $\boldsymbol{\nu} = (\nu_1, \nu_2)$. Based on the BEM we discretise each of the boundaries Γ_0 and Γ_1 into a collection of uniformly distributed $M/2$ straight-line segments, and use piecewise constant approximations for $\mathbf{u}$ and $\mathbf{t}$ over $\partial\Omega$. This recasts the boundary integral eq. (8.4) as a system of $2M$ linear algebraic equations

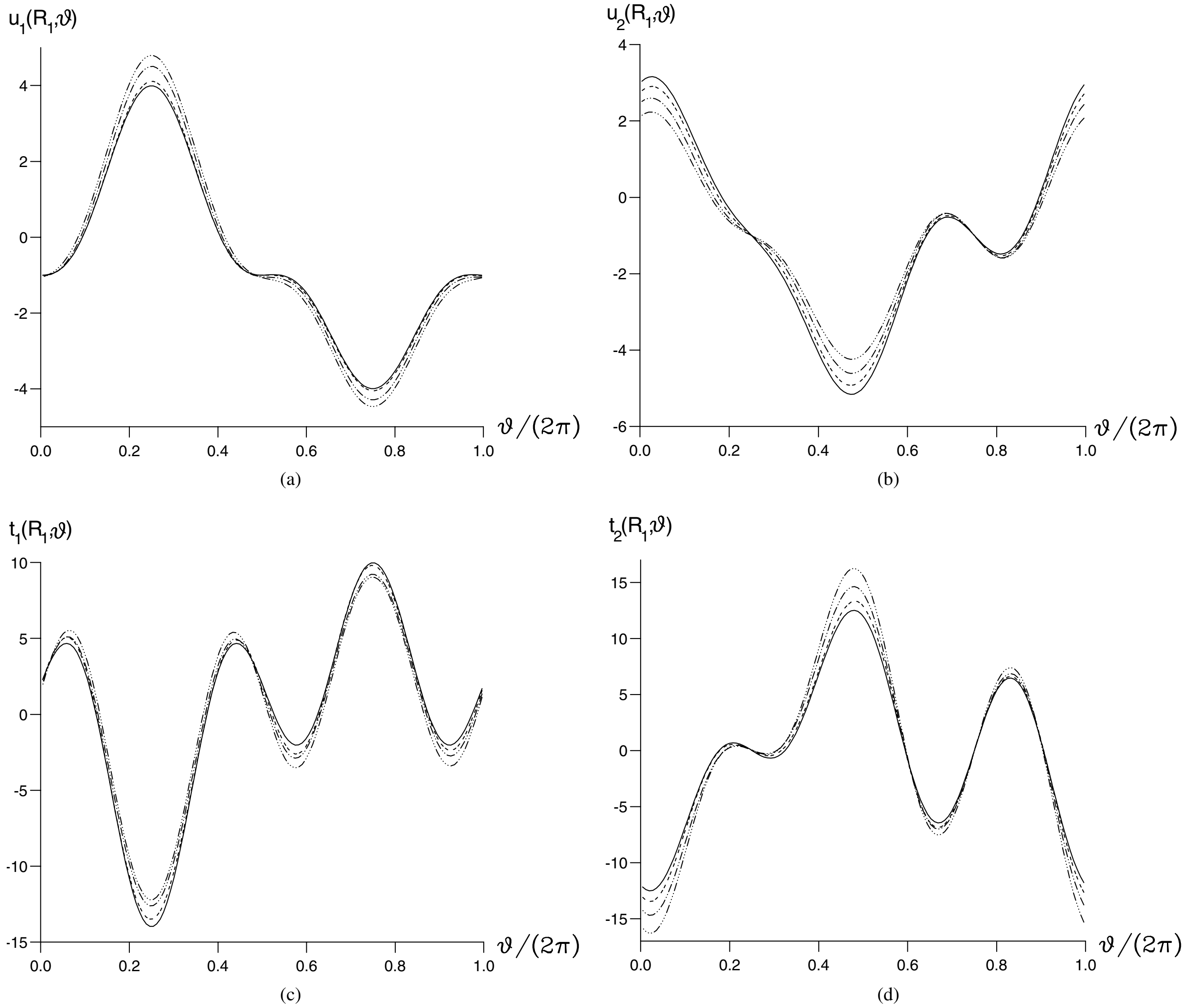

Fig. 1 The numerical values of (a) $u_1(R_1,\theta)$, (b) $u_2(R_1,\theta)$, (c) $t_1(R_1,\theta)$, and (d) $t_2(R_1,\theta)$ for various amounts of noise $\alpha = 1\%$ (— — —), $\alpha = 3\%$ (— · · · —), and $\alpha = 5\%$ (— · · · · · —), in comparison with the corresponding analytical solutions (——).

with $4M$ unknowns. However, $2M$ of these unknowns can be eliminated by imposing the boundary conditions of the mixed problems (3.1) or (3.2). The resulting system of equations at each iteration k can then be written in a generic form as

$$A\mathbf{X}^{(k)} = \mathbf{b}^{(k)} \tag{8.7}$$

where the coefficients of the matrix $A \in \mathbb{R}^{2M\times 2M}$ and the vector $\mathbf{b} \in \mathbb{R}^{2M}$ can be evaluated analytically, as described in Zeb et al. [13].

Remarks:

(i) The matrix A depends only on the type of mixed boundary conditions in (3.1) or (3.2) and thus it can be calculated only once and stored.

(ii) The system of eqs. (8.7) is well-conditioned although the original Cauchy problem is ill-posed.

For simplicity we take $R_1 = 1$ and $R_0 = 2$ and employ $M = 128$ boundary elements. In order to test the stability of the method we add, in the input boundary velocity data $\boldsymbol{\varphi}$, Gaussian random noise with mean zero and standard deviation $\sigma = max \mid \boldsymbol{\varphi} \mid \alpha/100$, where $\alpha\%$ is the amount of noise. An arbitrary initial guess such as $\boldsymbol{\xi}_0 = \mathbf{0}$ has been chosen to initiate the alternating algorithm described in Sect. 3.

Figs. 1a–d show the numerical values of $u_1(R_1,\theta)$, $u_2(R_1,\theta)$, $t_1(R_1,\theta)$, and $t_2(R_1,\theta)$, respectively, in comparison with the corresponding analytical solutions obtained from (8.1), for various amounts of noise $\alpha \in \{1,3,5\}\%$. According to the stopping criterion (7.16), the iterative process has been stopped after $k = 37$, $k = 16$, and $k = 11$ iterations, for $\alpha = 1$, 3, and 5%, respectively. From these figures it can be seen that the numerically obtained solutions are stable, and the accuracy of the numerical solution increases as α decreases.

At this stage, the boundary values for $\mathbf{u}$ and $\mathbf{t}$ are obtained over the whole boundary $\partial\Omega$. Then the fluid velocity $\mathbf{u}$ and the pressure p inside the domain Ω can easily be obtained from the integral equations

$$\mathbf{u}(\mathbf{x}) = \int_{\partial\Omega} \Big[\underline{\underline{K}}(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y}) - \underline{\underline{U}}(\mathbf{x},\mathbf{y})\mathbf{t}(\mathbf{y}) \Big] \, dS(\mathbf{y}) \,, \quad \mathbf{x} \in \Omega \,, \tag{8.8}$$

$$p(\mathbf{x}) = \int_{\partial\Omega} \left[\mathbf{L}(\mathbf{x},\mathbf{y}) \cdot \mathbf{v}(\mathbf{y}) - \mathbf{q}(\mathbf{x},\mathbf{y}) \cdot \mathbf{t}(\mathbf{y}) \right] dS(\mathbf{y}) \,, \quad \mathbf{x} \in \Omega \tag{8.9}$$

where, in two-dimensions, the vectors $\mathbf{q}$ and $\mathbf{L}$ are given by

$$q_k(\mathbf{x},\mathbf{y}) = -\frac{(x_k - y_k)}{2\pi \mid \mathbf{x} - \mathbf{y} \mid^2} \,, \quad k = \overline{1,2} \,, \tag{8.10}$$

$$L_k(\mathbf{x},\mathbf{y}) = \frac{n_k}{\pi \mid \mathbf{x} - \mathbf{y} \mid^2} - \frac{2(x_k - y_k)}{\pi \mid \mathbf{x} - \mathbf{y} \mid^2} \sum_{m=1}^{2} (x_m - y_m) n_m \,, \quad k = \overline{1,2} \,. \tag{8.11}$$

9 Conclusions

In this paper an alternating procedure for solving the Cauchy problem for the Stokes system, which is not strongly elliptic, was investigated. It was proved that the two mixed direct problems in the alternating procedure are well-posed and that the sequence of Stokes pairs obtained by the procedure converges. A stopping criterion for the iterative process was given. The computational implementation of the algorithm was based the boundary element method and the numerically obtained results support the theoretical results.

Future work will be concerned to extending the alternating method proposed in this study to the stationary Oseen system.

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