Complex Networks

Project 2

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1 Introduction

The Kuramoto model is a mathematical model used to describe the collective behavior of N coupled oscillators. Each oscillator is represented by a phase angle θ_i, which evolves over time according to the following equation[1] :

$$\frac{d}{dt}\theta_i(t) = \omega_i + \frac{K}{N}\sum_j \sin(\theta_j(t) - \theta_i(t)) \tag{1}$$

with $i = 1, ..., N$, the frequencies ω_i are i.i.d random variables and K the coupling constant.

If the coupling K is small each oscillator rotates with its natural frequency ω_i, whereas for large coupling K almost all angles θ_i will be entrained by the mean field and the oscillators synchronize. This model emphasises a phase transition, such that it will exist a critical value of the coupling K_c where for $K > K_c$ the system is synchronized ($\dot{\theta}_i = \dot{\theta}$), while for $K < K_c$ no such state exists.

In this project we study first a simple case where the natural frequencies of the rotators are known and in the second part, when they are sampled from a probability distribution $g(\omega)$.

2 Theory: Kuramoto model

2.1 Lyapunov Function

In this first half we concentrate on phase locking state and exhibit a Lyapunov function. One can define the arithmetical[2] mean frequency to be:

$$\bar{\omega} := \frac{1}{N}\sum_{i=1}^{N}\omega_i \tag{2}$$

and intuitively think that it is the common frequency of the phase locked component. In fact,

$$\frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^{N}\theta_i(t)\right) = \left(\frac{1}{N}\sum_{i=1}^{N}\frac{d}{dt}\theta_i(t)\right)$$

from equation **??**:

$$= \left(\frac{1}{N}\sum_i \omega_i\right) + \frac{K}{N^2}\sum_{i,j}\sin(\theta_j(t) - \theta_i(t))$$

Since $i, j = 1, \dots, N$, in the calculations one encounters the terms $\theta_i - \theta_j$ as well $\theta_j - \theta_i$ and since the sum is and odd function:

$$\sin(\theta_i - \theta_j) = -\sin(\theta_j - \theta_i)$$

hence the term i, j and j, i cancels out. More precisely, we can separate the initial sum:

$$\sum_{i>j}\sin(\theta_j - \theta_i) + \sum_{i<j}\sin(\theta_j - \theta_i)$$

exchanging i with j in the second one we get $\sum_{j<i}\sin(\theta_i - \theta_j) = -\sum_{j<i}\sin(\theta_j - \theta_i)$ which is exacly equal to the first term in the sum. we then are left with:

$$\frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^{N}\theta_i(t)\right) = \frac{1}{N}\sum_{i=1}^{N}\omega_i = \bar{\omega} \tag{3}$$

Now one can consider the new variables ϕ_i in the reference frame rotating with $\omega = \bar{\omega}$:

$$\phi_i(t) = \theta_i(t) - om\bar{e}gat$$

so that the Kuramoto model becomes:

$$\frac{d\phi_i}{dt}(t) = (\omega_i - \bar{\omega}) + \frac{K}{N}\sum_{j=1}^{N}\sin\left(\phi_j(t) - \phi_i(t)\right) \tag{4}$$

At this point we can introduce the Lyapunov function to study the nature of stable points (if they exists):

$$\mathcal{H} := -\frac{K}{2N}\sum_{i,j}\cos\left(\phi_i - \phi_j\right) - \sum_{i=1}^{N}\left(\omega_i - \bar{\omega}\right)\phi_i. \tag{5}$$

To be a good guess for a Lyapunov Function must satisfy:

$$\dot{\mathcal{H}} = \sum_{l=1}^{N}\frac{\partial\mathcal{H}}{\partial\phi_l}\frac{d\phi_l}{dt} \le 0, \tag{6}$$

for any solution ϕ_i.

The term $\frac{d\phi_l}{dt}$ is given by eq. (**??**), while the term $\frac{\partial\mathcal{H}}{\partial\phi_l}$ is:

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial\phi_l} &= +\frac{K}{2N}\sum_{i,j}\sin\left(\phi_i - \phi_j\right)\delta_{i,l} \\ &\quad -\sum_{i=1}^{N}\left(\omega_i - \bar{\omega}\right)\delta_{i,l} \\ &= -\frac{K}{2N}\sum_j \sin\left(\phi_j - \phi_l\right) - \left(\omega_l - \bar{\omega}\right)\end{aligned}$$

one can notice that this term is equal to eq. **??** except for a global minus sign[3]. Therefore:

$$\dot{\mathcal{H}} = -\sum_{l=1}^{N}\left(\frac{\partial\mathcal{H}}{\partial\phi_l}\right)^2$$

which is non positive.

Moreover we can write down the equation for the extreme points, and check their nature:

$$\nabla\mathcal{H} = 0 = \frac{K}{2N}\sum_j \sin\left(\phi_j - \phi_l\right) + \left(\omega_l - \bar{\omega}\right)$$

[1] The original Kuramoto model was developed with the sin function. Nonetheless Kuramoto showed that for any system of weakly coupled, nearly limit-cycle oscillators, long term dynamics are given by phase equations where the sin term is replaced by a generic function $\Gamma_{ij}(\theta_j - \theta_i)$, generalysing the model

[2] This is an aritmetical mean, however as $N \to \infty$ the quantity $\bar{\omega}$) is a nonrandom number and equals the mean $\int d\mu(\omega)\omega$ by the strong law of large numbers.

[3] This means that $\phi_i = -\nabla\mathcal{H}$ we can represent the dynamics of the system thought the Lyapunov function which now can be considered as a potential