FYS3500 Midterm exam

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Main contribution with candidate nr. 15522
Others minor contributions with candidates nr: 15506, 15517 , 15521.

Exercise 1: Short question

a) According to the shell model the spin of the ground state of even even nuclei is 0 and the parity is +1.This result conventionally is written: 0^+. Since in the ground state of an even-even all the neutrons are coupled, and all the protons are coupled as well, they don't contribute to the spin. On the other hand, the parity is positive because protons and neutrons tend to pair up and each particle in the nucleus gives a contribute equal to $(-1)^l$, where l is the eigenvalue of the orbital angular momentum giving a parity +1.

b) The decay is governed by the stong interacrion,in which the quark flavors numbers and the baryon number must be conserved. In particular, we find that the hadron must have all quarks flavours null. Since the baryon number $B = 0$, calculated looking at the products of the decay, has to be conserved we find that the hadron was a meson. Looking at the table of $M\&S$ for a meson with null strangeness there are only 2 possible value for the isospin: or $I = 0$ or $I = 1$. The isospin projections I_3 is related to the up and down quark content of particles, since there are same number of quark and antiquark of the same flavor I_3 has to be equal to 0.

Hence, the possible values for the couple (I, I_3) are $(1, 0)$ and $(0, 0)$.

c) The nuclear potential is the same for the neutron and the proton, because the nuclear force is charge independent. The proton will in addition feels a Coulomb barrier due to the elecrtic repulsion of the nucleus, while that's not present for the neutron because it is not a charge particle.

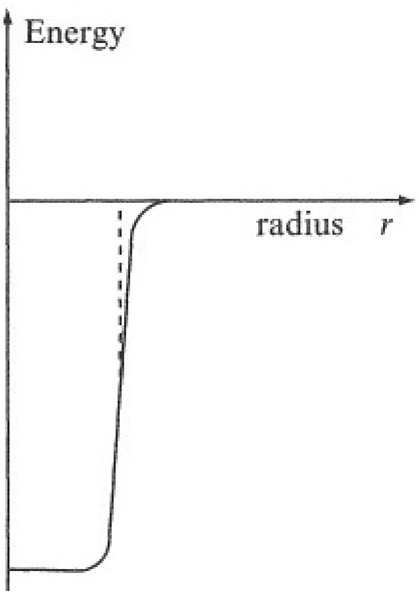

Figure 1: Potential felt by the neutron

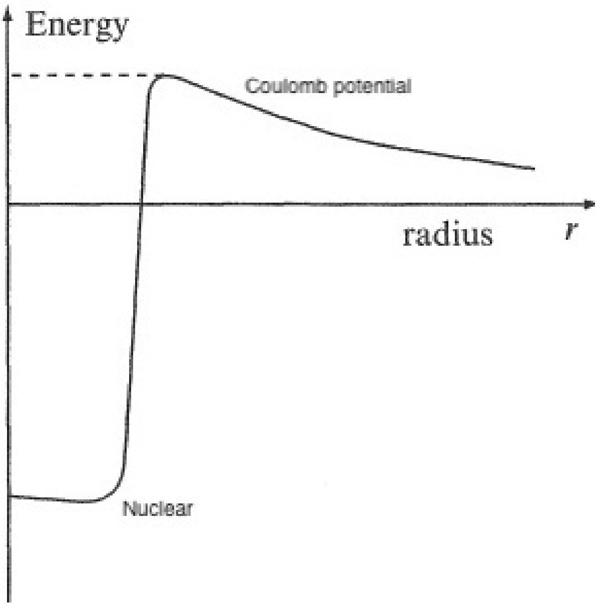

Figure 2: Potential felt by the proton

d) The only other two possible baryons with strangeness $S = 0$ are uuu and ddd. Assuming the rule that the the total wavefunction of our identical fermions must be totally symmetric, if the spacial part is symmetric, also the spin part must be so. In order to respect this condition is to have all the spins aligned, i.e. $J = 3/2 \neq 1/2$.

e) By comparing the difference among the binding energy of the different nuclei before and after stealing a neutron. Using the formulas from M&S,which is the same equation reported in Exercise nr.2 eq:2. We find that the biggest difference in binding energy is from the ^{16}O.

f) Using the relation ($N = \sigma\mathcal{L}$) that connect the number of evenst, the luminosity and the cross section, and paying attention to the efficiency ϵ and the to the number of predicted background events N_{back}, we can find the solution to be:

$$\sigma = \frac{(N_{observed} - N_{back})}{\epsilon\mathcal{L}} = \frac{984}{0.25 \cdot 5} pb \approx 0.787pb$$

g) Considering the 3 decay:

$$1)\quad \gamma e^- \to \gamma e^- \qquad 2)\quad \gamma e^- \to \gamma\gamma e^- \qquad 3)\quad \gamma e^- \to \gamma\gamma\gamma e^-$$

We have to find the approximate ratio of the cross sections. As defined, the cross section is a measure of the probability that a specific process will take place due to interact with something. This measure of probability depends on the coupling constant as is possible to see in M&S, in this case since we are dealing with the electromagnetic interaction $\alpha_{em} \approx \frac{1}{137}$, and each vertex of the feynman diagram introduce a multiplicative factor α. So, drawing properly the feynman diagram for each scattering process, we notice that the probability of second scattering is $\frac{1}{137}$ compared to the first one, and the probability that the third process take place is $(\frac{1}{137})^2)$ compared to the first. Hence, the cross section ratio among these 3 process is:

$$1 : \frac{1}{137} : \frac{1}{137^2}$$

Exercise 2: Nuclear Binding energy

a)The semi-empirical mass formula is:

$$m({}^A_Z X_N) = Z(m_p + m_e) + Nm_n - \frac{B}{c^2} \tag{1}$$

where the last term indicates the Binding energy. Using the liquid drop model this is:

$$\frac{B}{c^2} = \underbrace{a_v A}_{\text{Volume Term}} - \underbrace{a_s A^{\frac{2}{3}}}_{\text{Surface Term}} - \underbrace{a_c \frac{Z(Z-1)}{A^{\frac{1}{3}}}}_{\text{Coulomb term}} - \underbrace{a_{sym} \frac{(Z-N)^2}{A}}_{\text{Symmetry term}} + \underbrace{\delta}_{\text{pairing term}} \tag{2}$$

where δ is the pairing therm:

$$\delta \begin{cases} a_p\, A^{-1/2} & \text{for even-even nuclei} \\ 0 \quad \text{odd A} \\ -a_p\, A^{-1/2} & \text{for odd-odd nuclei} \end{cases}$$

All the a_k coefficients are determined experimentally ans in this exercise I use, according to Wikipedia ("citation") :

$$a_v = 15.8, \quad a_s = 18.3, \quad a_c = 0.714, \quad a_{sym} = 23.4, \quad a_p = 12$$

All the coefficients are expressed in Mev/c^2. The aim of the first point is to evaluate approximately the nuclear mass of ^{48}Ca, which has $Z = 20$ and $N = 28$. In the light of the above, we can evaluate B calculating each term:

$$a_v A = 758.4 \quad a_s A^{\frac{2}{3}} = 241.7 \quad a_c \frac{Z(Z-1)}{A^{\frac{1}{3}}} = 74.7 \quad a_{sym} \frac{(Z-N)^2}{A} = 31.2 \quad \delta = 1.7$$

As a result, substituting these therms in 2 we get the following value for the Binding energy of ^{48}Ca: $B = 412.5Mev/c^2$. At this point the other therms of eq(1) are left to evaluate. For the sake of units involved, to calculate the left terms we used as a proton mass $m_p = 938,8$ and for the neutron we used $m_n = 939,6$, and since $m_p \gg m_e$ we can neglect the electron contribution in mass. Now we can use these values to evaluate the mass with eq(1):

$$m({}^{48}_{20}Ca_{28}) = 20(938,8) + 28(939,6) - 412.5 = 44.472,3Mev/c^2 = 47,7u$$

b) We are dealing with:

$$^{44}_{20}Ca_{24} \rightarrow ^{43}_{20}Ca_{23} + n$$

from energy conservation we have:

$$\begin{aligned} E(^{44}_{20}Ca_{24}) &= E(^{43}_{20}Ca_{23}) + E(n) + \frac{\Delta E}{c^2} \\ \cancel{20(m_p)} + \cancel{24m_n} - \frac{B(44)}{c^2} &= \cancel{20(m_p)} + \cancel{23m_n} - \frac{B(43)}{c^2} + \cancel{m_n} + \frac{\Delta E}{c^2} \\ \frac{\Delta E}{c^2} &= \frac{B(43)}{c^2} - \frac{B(44)}{c^2} \end{aligned} \tag{3}$$

using the equation for the binding energy eq(2), hence we obtain for $\frac{B(43)}{c^2}$:

$$a_v A = 679.4 \quad a_s A^{\frac{2}{3}} = 224.7 \quad a_c \frac{Z(Z-1)}{A^{\frac{1}{3}}} = 77.4 \quad a_{sym} \frac{(Z-N)^2}{A} = 4.9 \quad \delta = 0$$

and for $\frac{B(44)}{c^2}$:

$$a_v A = 695.2 \quad a_s A^{\frac{2}{3}} = 228.1 \quad a_c \frac{Z(Z-1)}{A^{\frac{1}{3}}} = 76.9 \quad a_{sym} \frac{(Z-N)^2}{A} = 8.5 \quad \delta = 1.8$$

substituting the results in eq(3) we obtain the energy that needs to remove a neutron:

$$|\Delta E| = |\frac{B(43)}{c^2} - \frac{B(44)}{c^2}| \approx 11 Mev/c^2$$

c) There are 2 types of β decay: β^+ and β^-. The former consist in a proton decay in a neutron and a positron ($p \rightarrow n + e^+$), this process occur only for protons bound in nuclei; they are energetically forbidden for free protons or for protons in hydrogen atoms. On the other hand, the β^- decay consists in a neutron that decay into a proton and an electron and an anti-neutrino. The most basic β decay process in a nucleus, β decay changes both Z and N by one unit: $Z \pm 1$ and $N \mp 1$ so that A remains constant.Thus β decay provides a convenient way for an unstable nucleus to get to the minimum of the mass parabola of constant A (figure 3) and to approach the stable isobar. Given A the most stable nucleus is when the binding energy is maximised, or equivalently when the mass is minimized. Hence we start calculating the derivative of the mass as a function of Z:

$$\begin{aligned} \frac{\partial}{\partial Z} m(^A_Z X_{N=A-Z}) &= \frac{\partial}{\partial Z}(Z(m_p + m_e) + (A-Z)m_n - \frac{B}{c^2}) \\ &= m_p + m_e - m_n - \frac{\partial}{\partial Z}(\frac{B}{c^2}) \\ &= m_p + m_e - m_n - (-a_c \frac{(2Z-1)}{A^{\frac{1}{3}}} - a_{sym} \frac{4(2Z-A)}{A})) \end{aligned}$$

and check for which Z is 0:

$$\begin{aligned} Z(\frac{2a_c}{A^{\frac{1}{3}}} + \frac{8a_{sym}}{A})) &= m_n - m_p - m_e + a_c A^{-\frac{1}{3}} + 4a_{sym} \\ Z &= \frac{m_n - m_p - m_e + a_c A^{-\frac{1}{3}} + 4a_{sym}}{(2a_c A^{-\frac{1}{3}} + 8a_{sym} A^{-1})} \end{aligned}$$

By putting the correct values for each term in the last equation we obtain that the minimum is when $Z = 56$. Actually in the Chart of nuclides, there is an element (Bario) that corresponds to the description with $Z = 56$ and $A = 136$: $^{136}_{56}Ba$.

Making the calculations also for the element with $A = 135$, as is possible to see in the graph, there is a slight difference in mass for odd A, as $A = 135$, or for even $A = 136$. This is because the binding energy of even nuclei has a non null pairing term (δ) contribution, on the contrary for odd A this term is 0.

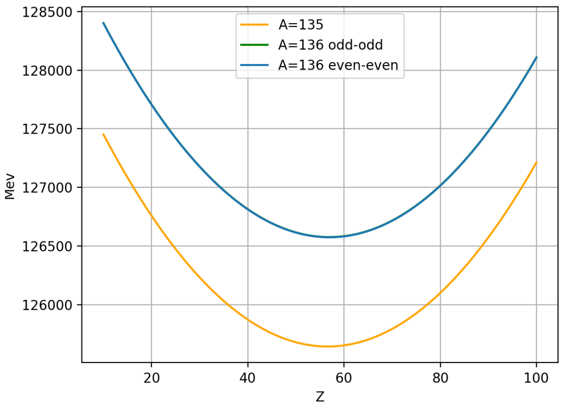

Figure 3: mass parabola for A=136 even-even, A=136 odd-odd and A=135. The difference between the A=136 even-even and the A=135 odd-odd consist only in the pairing factor, which contributes with only $1Mev$. In fact, it is very difficult to appreciate the difference between A=136 odd-odd and A=136 even-even. The pairing term gives to A=136 odd-odd a slightly higher mass parabola compared to the mass parabola of A=136 even-even.

Exercise 3: Higgs boson decay

a)

The invariant mass of the Higgs boson is a Lorentz invariant. As a consequence, it is a characteristic of the system's total energy and momentum that is the same in all frames of reference related by Lorentz transformations. The expression of this quantity is:

$$W = \sqrt{(\sum_i E_i)^2 - |\sum \bar{p}_i|^2} \tag{4}$$

Considering the case of Higgs boson, which is at rest in his frame of reference, it yields:

$$W = \sqrt{(\sum_i E_i)^2 - \cancel{|\sum \bar{p}_i|^2}} = E_H = m_H \quad \text{atomic units} \tag{5}$$

Since this quantity is Lorentz invariant it must be the same if it is calculate after the Higgs decay using the electrons data, measured in the lab frame, hence:

$$W = \sqrt{(\sum_i E_i)^2 - |\sum \bar{p}_i|^2} \tag{6}$$

Equating the last two expressions (eq:6 and eq:5) and by inserting the given values we yield:

$$m_h \simeq 125.2GeV$$

Which is a quite good approximation of the experimental value tabulated here.

b) There are 4 different ways to combine the couples of electron-positron generated by the bosons. In fact, Z boson can decay in an electron and a positron, and each pair of the given electron-positron, in principle, can be generated, without violating any rules.

$$\begin{array}{llll} 1) & Z \to e_1^+ + e_1^-, \quad Z^* \to e_2^+ + e_2^- & 2) & Z \to e_1^+ + e_2^-, \quad Z^* \to e_2^+ + e_1^- \\ 3) & Z \to e_2^+ + e_1^-, \quad Z^* \to e_1^+ + e_2^- & 4) & Z \to e_2^+ + e_2^-, \quad Z^* \to e_1^+ + e_1^- \end{array}$$

Imposing the tetra-momentum conservation in the bosons' decay and in particular, for the Z boson, this means

$$E_Z = \sqrt{m_Z^2 + p_Z^2} = E_{e^+} + E_{e^-} = \sqrt{m_{e^+}^2 + p_{e^+}^2} + \sqrt{m_{e^-}^2 + p_{e^-}^2}$$

this equation yields to the following expression:

$$p_Z^2 = (E_{e^+} + E_{e^-})^2 - m_Z^2$$

this gives the value of the momentum that the Z boson must have in order to satisfy the conservation of the energy. The value of m_Z is given from the tabuled values of Z boson. By explicit calculation for the 4 cases one obtains

$$1) \quad p_Z^2 \simeq -2414\ GeV/c \quad 2) \quad p_Z^2 \simeq -7350\ GeV/c$$
$$3) \quad p_Z^2 \simeq 744\ GeV/c \quad 4) \quad p_Z^2 \simeq -5872\ GeV/c$$

Only in the third case the particle can be regarded as physical accepted quantity.

c)

From the conservation of quadrimomentum, given that the momentum $\vec{p}$ of the Higgs boson in its rest frame is null, we get that $\vec{p}_{Z^*} = -\vec{p}_Z$. We can then write a system of equation, using the energy-momentum relations and the conservation of energy ($c = 1$):

$$\begin{cases} m_H = E_Z + E_{Z^*} \\ E_Z = \sqrt{m_Z^2 + p_Z^2} \\ E_{Z^*} = \sqrt{m_{Z^*}^2 + p_{Z^*}^2} \end{cases} \tag{7}$$

Just going a bit through the math and imposing $p_Z^2 = p_{Z^*}^2$ we get:

$$p_Z^2 = p_{Z^*}^2$$
$$E_Z^2 - m_Z^2 = E_{Z*}^2 - m_{Z^*}^2$$
$$E_{Z*}^2 - E_Z^2 = m_{Z^*}^2 - m_Z^2$$

then from the first equation of the system 7:

$$(E_{Z^*} - E_Z) = \frac{m_{Z^*}^2 - m_Z^2}{m_H}$$

then note that $E_{Z^*} - E_Z = m_H - 2E_Z$, it follows:

$$E_Z = \frac{m_H}{2} - \frac{m_{Z^*}^2 - m_Z^2}{2m_H}$$

In conclusion, we can obtain the expression for E_Z in the same way:

$$\begin{cases} E_Z = \frac{m_H}{2} - \frac{m_{Z*}^2 - m_Z^2}{2m_H} \\ E_{Z^*} = \frac{m_H}{2} + \frac{m_{Z*}^2 - m_Z^2}{2m_H} \end{cases}$$

We notice that in the case $m_Z = m_{Z^*}$ the energy is equally split.

d) More generally, we can always use the conservation of 4-momentum, which has to be conserved at every step of the process and in every reference frame. Hence, for convenience is possible to study the process from the rest frame of the Higgs boson:

$$(m_H, \vec{0}) = (E_Z + E_{Z^*}, \vec{p}_Z + \vec{p}_{Z^*}) = \left(\sum_i E_i, \sum_i \vec{p}_i \right)$$

where the last summation index runs over the electrons. Moreover, for each particle, one can use the energy triangle relation $E = \sqrt{p^2 + m^2}$. Since all the leptons are non-virtual, so the relation holds, hence using the tabulated mass of a positron/electron $m = 0.512\text{MeV}/c^2$. With these information, and putting $E = \sqrt{p^2 + m_e^2}$ in the previous equations is possible to estimate the energy of the 4 leptons.

Exercise 4: Lepton universality

The Feynman diagram of the decays considerate in the problem are shown in figure (4) and figure (5).

Is shown that the decay ratio Γ, for each decay, is proportional to the modulus square of the amplitude $\mathcal{M}$, which is directly related to the probability of the process occurring. $\mathcal{M}$ in Born approximation, is given by:

$$\mathcal{M}(\bar{q}) = \int d^3 r V(r) e^{i \frac{\bar{q} \cdot \bar{r}}{\hbar}}$$

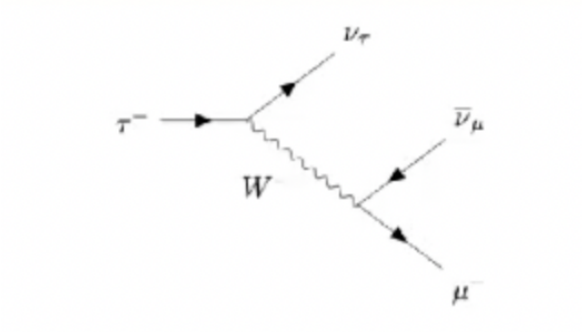

Figure 4

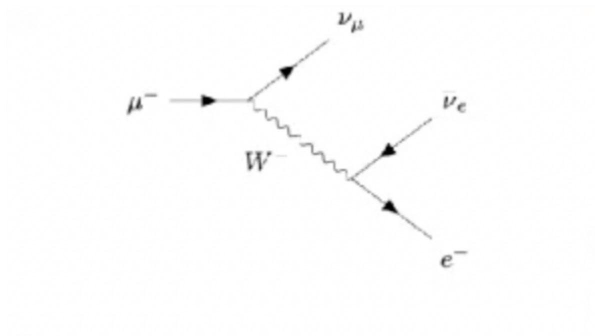

Figure 5

Where $\bar{q} = q_f - q_i$ (difference in momentum). For these interaction, we substitute V with the static Yukawa potential,and this yields to:

$$\mathcal{M}(Q^2) = \frac{-g_X^2 \hbar^2}{Q^2 + m_X^2 c^2}$$

Where $Q^2 = (q_f - q_i)^2 - (E_f - E_i)^2/c^2$, and m_X is the mass of the force mediator. This decay is governed by the weak interaction, which is mediated by the W boson. This is times heavier than the mass of all others particles in the two decay that we consider. Hence $m_W \gg m_{\tau,\mu,e}$, from this follows that that the weak interaction is the same for all leptons, they interact in the same way with the W boson. This is called leptons universality. In this reasonable approximation the $\mathcal{M}$ value is:

$$\mathcal{M} \approx \frac{-g^2 \hbar^2}{m_W^2 c^2} \sqrt{2} = G_F \approx 1.166 \times 10^{-5} \left[\frac{(\hbar c)^3}{GeV^2} \right]$$

where we have taken into account a $\sqrt{2}$ factor due to the spin. The equation shows the Fermi coupling constant G_F, which has dimension of $[\frac{1}{GeV^2}]$ in natural units and is the same for all the leptons (lepton universality).

The decay rate, which follows from Fermi's golden rule has dimension of energy, and must be proportional to the square of the amplitude, and thus the square of Fermi's coupling constant G_F that has the dimension of $[\frac{(\hbar c)^3}{GeV^2}]$, with over-all dimension of inverse fourth power of energy. By dimensional analysis, this leads to:

$$\Gamma = K G_F^2 \frac{(\Delta E)^5}{(\hbar c)^6} \tag{8}$$

where in natural units:

$$\Gamma = K G_F^2 (\Delta E)^5$$

where the dimensionless constant K must have the same value for muon and tauon decay if universality holds. This ΔE represents the released energy (in literature usually is indicates as Q value which represents the kinetic energy released in the decay at rest). When the products of the decay have masses negligible respect the initial element, ΔE is only the rest energy ($m_l c^2$) of the initial element. For the μ decay (fig.(5)): ($\mu \to e^- \nu_\mu \nu_e$), the factor is only $\Delta E = (m_\mu - m_e)c^2$,since $m_\mu \gg m_e$, as a result the decay ratio for this is:

$$\Gamma(\mu \to e^- \nu_\mu \nu_{\nu_e}) = K G_F^2 m_\mu^5$$

On the other hand, if the mass of the particles after the decay are heavy in principle we cannot neglect them. In principle the contribution in energy of μ is not negligible, but experimentally we see that $BR(\tau \to \mu^- \nu_\mu \nu_\tau) \approx BR(\tau \to e^- \nu_e \nu_\tau)$. Hence, using this approximation ΔE becomes: $m_\tau c^2 - m_\mu c^2 \approx m_\tau c^2$, or m_τ in natural units.

In the light of above this yields to:

$$\Gamma(\tau \to \mu^- \nu_\mu \nu_\tau) = K G_F^2 (m_\tau)^5$$

In conclusion:

$$\frac{\Gamma(\tau \to \mu^- \nu_\mu \nu_\tau)}{\Gamma(\mu \to e^- \nu_\mu \nu_{\nu_e})} = \left(\frac{m_\tau}{m_\mu} \right)^5 \approx 1.37 \times 10^6 \tag{9}$$

b) Experimentally we don't measure decay rates, so the observable quantities that we need to measure to test this ratio for lepton universality are the masses of τ and μ leptons, or manipulating the terms in eq(9) we obtain:

$$\frac{\Gamma_\tau}{\Gamma_\mu}\frac{\Gamma_\mu}{\Gamma(\mu\to e^-\nu_\mu\nu_{\nu_e})}\frac{\Gamma(\tau\to\mu^-\nu_{mu}\nu_\tau)}{\Gamma_\tau}=\frac{\Gamma_\tau}{\Gamma_\mu}\frac{BR(\tau\to\mu^-\nu_\mu\nu_\tau)}{BR(\mu\to e^-\nu_\mu\nu_{\nu_e})}=\frac{\tau_\mu}{\tau_\tau}\frac{BR(\tau\to\mu^-\nu_\mu\nu_\tau)}{BR(\mu\to e^-\nu_\mu\nu_{\nu_e})}=\frac{\tau_\mu}{\tau_\tau}BR(\tau\to\mu^-\nu_\mu\nu_\tau)$$

this suggests that the needed quantities are the lifetime $\tau_l=\frac{1}{\Gamma_l}$ (in opportune units) and the branching ratios.

Exercise 5: Quantum numbers

1. **Parity**

   Let us consider a wavefunction $\psi(\vec{r})$. The parity operator $\hat{P}$, as defined, acts on the wavefunction in this way:

   $$\hat{P}\psi(\vec{r})=\psi(-\vec{r})$$

   and if we apply it 2 times, it must return the initial wavefunction:

   $$\hat{P^2}\psi(\vec{r})=\hat{P}\psi(-\vec{r})=\psi(\vec{r}) \tag{10}$$

   the eigenvalues of the parity operator is $+1$ for even eigenfunctions and -1 for odd eigenfunctions. If the wavefunction describes the state of particle, and the state is an eigenstate of the parity operator, the corresponding eigenvalue π_ψ is also called (intrinsic) parity of the particle.
   In order to understand better how parity acts on a wavefunction describing a system, let's take a look on this example with two particles p_1+p_2 described by a wavefunction $\psi_{p_1p_2}(\vec{r}_{p_1},\vec{r}_{p_2})$. It can be proven that the parity of the system is given by

   $$\hat{P}\psi_{p_1p_2}=\pi_{p_1}\pi_{p_2}(-1)^l\psi_{p_1p_2}$$

   where l denotes quantum number associated to the orbital angular momentum. If the parity commute with the hamiltonian H, then it is a conserved quantity along the reaction. For example, if in a reaction of the type $p_1+p_2\to p_3+p_4$ the parity is conserved

   $$\pi_{p_1}\pi_{p_2}(-1)^{l_{p_1p_2}}=\pi_C\pi_D(-1)^{l_{p_3p_4}}$$

   This, for example, is not the case of the weak interaction where, in general, the hamiltonian operator does not commute with the parity operator.

   Charge conjugation

   The charge conjugation C operator transforms a particle into its antiparticle changing , e.g. $C|p\rangle=|p^-\rangle$, $C|\pi^-\rangle=|\pi^+\rangle$, $C|e^-\rangle=|e^+\rangle$, etc...
   more in general:

   $$C|\psi_a\rangle=|\bar{\psi_a}\rangle=C_{\psi_a}|\psi_a\rangle$$

   where C_{ψ_a} is a multiplicative quantum number (C-parity) that is conserved in strong and electromagnetic interactions, but not in the weak interaction. The C operator changes the sign of all of the additive quantum numbers describing the particle as Charge, Baryon number, Lepton number, Isospin, Strangeness, Charm etc... However, variables such as momentum and spin do not change sign under C. Acting twice C operator must give $C^2|\psi\rangle=|\psi\rangle$. Hence, we have a condition on the possible eigenvalues of C operator. The eigenvalue is the quantum number of the charge conjugation C and the only possible eigenvalues are: ±1. Neutral particles (like photons γ) and particle-antiparticle systems ($p\bar{p}$) can be eigenstates of C, i.e., be their own antiparticle ($|\psi\rangle=|\bar{\psi}\rangle$). In addition, under the action of this operator, the orbital angular momentum (L) and the spin (S) can contribute with a phase factor.

2. $J/\psi(3097)$ is a meson made of a charm quark (c) and an anti-charm quark ($\bar{c}$). This is a bound state of 2 fermions. The problem gives this meson at the state $^3S_1$, this means that the orbital $L=0$, $S=1$ and $J=1$. Applying the charge conjugation operator (C) on a two-fermion system, when C is applied there are factors consider: one comes from the spin part of the wave function $(-1)^{s+1}$, one from the orbital angular momentum $(-1)^L$ and the last one from the exchange of a fermion by its antifermion (-1), we obtain:

   $$C|\psi_{c\bar{c}}\rangle=(-1)^L(-1)^{S+1}(-1)|\psi_{c\bar{c}}\rangle=-1|\psi_{c\bar{c}}\rangle$$

   So the quark model predicts the number -1 for C-parity of $J/\psi(3097)$ meson.

3. ^{15}N atom has 7 protons and 8 neutrons. All the neutrons (in the ground state) are paired up (each sub-shell admits two nucleons) so they do not contribute to the spin and the parity contribution is $+1$. On the other hand,the ground state of ^{15}N is $(1s)^2\,(1p_{3/2})^2\,(1p_{1/2})^1$, this shows that there is only one proton unpaired, which contributes to spin and parity of the nitrogen, as it is shown in figure 6,. The orbital p is associated to the quantum number $l = 1$ the parity of the the nucleus is the product of the neutrons an protons contribution, that is $(+1) \cdot (-1)^1 = -1$. The spin is then 1/2, entirely due to the unpaired proton.
There are 3 alternatives for the configuration of the first excited state, all of them consist in moving a proton or a neutron to another level starting from the ground state configuration.

The first possibility consist in moving the $1p_{1/2}$ proton to the $1d_{5/2}$ level, so we end up with a new proton configuration is $(1s)^2\,(1p_{3/2})^4\,(1p_{1/2})^{-2}\,(1d_{5/2})^1$ and the spin-parity is $5/2^+$.

The second possibility consist in moving the $1p_{1/2}$ neutron to the $1d_{5/2}$ level, so we end up with the new neutron configuration corresponding to $(1s)^2\,(1p_{3/2})^4\,(1p_{1/2})^{-1}\,(1d_{5/2})^1$. The parity here is non-trivial because of the angular momenta addition rules.

And the last possible configuration for the first excited state consist in moving the $1p_{3/2}$ proton to the $1p_{1/2}$ level, so we end up with the new proton configuration is $(1s)^2\,(1p_{3/2})^3\,(1p_{1/2})^2$ and the spin-parity is $3/2^-$.

These 3 scenarios of the first exited states can be viewed in figure 7

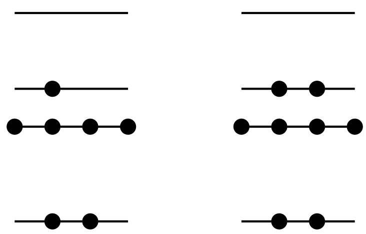

Figure 6: Ground states of ^{15}N

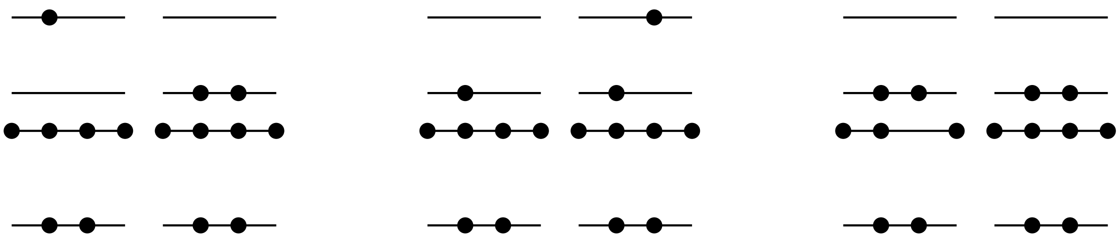

Figure 7: From left to right: the first drawing shows one of the possibilities of exited states where a proton has been promoted to higher energy, the second shows the first exited states when a neutron has been promoted to higher energy, and the last scheme shows when a proton has been promoted from $1p_{3/2}$ to the $1p_{1/2}$

4. The third excited state is the one that has $J^p = \frac{3}{2}^-$. From this state the nucleus can decay into the second excited state, the first excited state and the ground state. Let us start with considering the decay from the third state to to the ground state. When the excited proton returns to the $1p_{3/2}$ level there is no change in parity and one has that $\Delta J = J_{exc} - J_{ground} = 1$. In other words this yields that one must have odd-L magnetic fields and even-L electric fields.
Because of the general selection rules in the gamma decays one has that, said L the photon's angular momentum

$$|J_{exc} - J_{ground}| \leq L \leq |J_{exc} + J_{ground}| \qquad \longrightarrow \qquad 1 \leq L \leq 2$$

Hence, the allowed states are $M_{L=1}, E_{L=2}$ For $A = 15$ one has that

$$\frac{\lambda(E_2)}{\lambda(M_1)} \approx 10^{-3} \tag{11}$$

As written in Krane E and M indicate the multipole terms($L = 1$ for dipole, $L = 2$ for quadrupole, and so on), with E for electric and M for magnetic. Hence, we expect that the M_1 decay to be the most probable one, but with a non-neglectable contribution of E_2.
In similar way we can find that for the decays to second and first excited states. The relations

Transition	ΔE	$\Delta\pi$	L	Fields	Dominants
$3 \to 0$	$\approx 6.3\ MeV$	no	$1 \le L \le 2$	M_1, E_2	M_1, E_2
$3 \to 1$	$\approx 1\ MeV$	yes	$1 \le L \le 4$	E_1, M_2, E_3, M_4	E_1
$3 \to 2$	$\approx 1\ MeV$	yes	$1 \le L \le 2$	E_1, M_2	E_1

Table 1: This table rapresent the possible gamma transitions from the third excited state

are reported in Table: 1. For the transition $2 \to 1$ one has that the dominant field is E_2, due to the condition to the angular momentum L the parity is unchanged, while for the decay $2 \to 0$ the dominant field is E_1, and the parity change. In conclusion, from the second excited state the nucleus can decay to the ground state via E_1 radiation, or it can decay to the first excited state via E_2 radiation. By comparing probabilities (the data has been taken from Krane):

$$\frac{P_{2\to 0}}{P_{2\to 1}} \approx \frac{\lambda(E_1)}{\lambda(E_2)} = \frac{1}{7.3} \cdot 10^7 \cdot A^{-2/3} > 1$$

Exercise 6: Allowed, suppressed and forbidden processes

1) The first interaction consists in the decay of the charmonium $\psi(3686)$ in a photon and a less energetic charmonium $\chi_0(3145)$:

$$\psi(3686) \to \chi_0(3145) + \gamma$$

The process is allowed, and the Feynman diagram is:

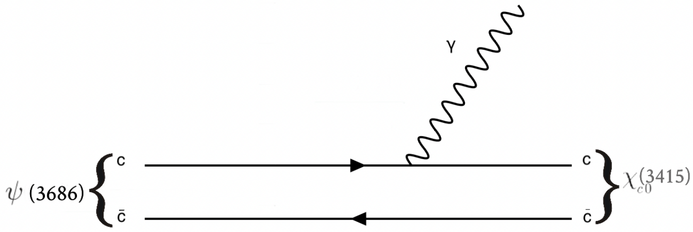

Figure 8: Charmonium radioactive decay

The process involves the electromagnetic interaction. The characteristic time of electromagnetic force is approximately the same time of the decay occurs according to PDG.

2)

$$pp \to \Sigma^+ p \bar{K}^0 \pi^-$$

This process is **forbidden**, the charge is not conserved.

3)

$$b \to s\gamma$$

This process is allowed, but **suppressed** because is extremely rare and is not the most common decay for the bottom quark. Nonetheless the Feynman diagram is:

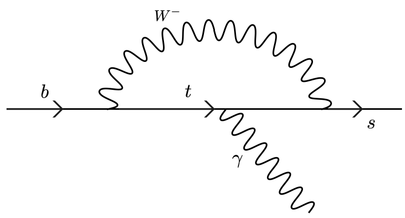

Figure 9: Feynman diagram of $b \to s\gamma$

This process happens thanks to the electroweak interaction, where a virtual W^- boson and a top quark are generated in order to release the photon, which interact only with charged particle, although the top quark and the W^- boson in this case are virtual. The lifetime of the bottom quark is approximately

$10^{-12}s$. This is in accordance with the characteristics time of the weak interaction, which can vary in a range from $10^{-8}s$ to $10^{-13}s$. Theoretically we can estimate the time decay from the decay rate Γ

4)

$$\Lambda^0 \to p + e^- + \bar{\nu_e}$$

Allowed, the decay involves neutrinos and is therefore a weak interaction. Thus this is the feynman diagram:

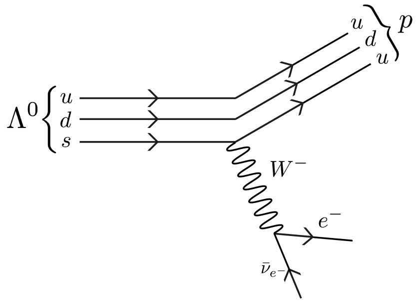

Figure 10: Feynman diagram of λ_0 baryon decay

Also in this case the W boson is the W^- in order to conserve the charge in every vertex. The lifetime of Λ^0 is approximately $10^{-10}s$, this value is compatible with the characteristics time of the weak interaction that spaces from $10^{-8}s$ to $10^{-13}s$.

5)

$$\Delta^+ \to p\pi^0$$

This process is **allowed**, because all the quantity are conserved.

The interaction is the strong force and the mediator is the gluon. The Feynman diagram is:

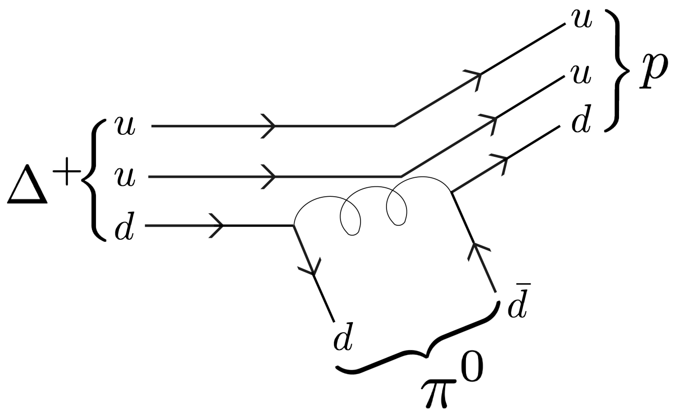

Figure 11: Feynman diagram of Δ^+ baryon

The characteristic time for the strong interaction is approximately 10^{24} s. This can be seen by imagining the following "thought experiment". Suppose two particles interacting with strong force approach one another, moving at speeds comparable to the speed of light. If the particles approach within $10^{16}m$ of each other, they will interact via the strong interaction. The amount of time that either particle spends near the other is thus approximately the time it takes for one particle to traverse a distance of $10^{-15}m$, i.e. $10^{-23}s$. From that little thought experiment we are led to conclude that whenever hadrons spend around $10^{24}s$ within the range of their mutual strong interactions they will interact. This is in accordance with the scheduled value of the Δ^+ lifetime, which is approximately $10^{-24}s$, compatible with what just said.

6)

$$e^+e^- \to q\bar{q}g$$

Allowed, this process happens thanks to the electroweak interaction and strong interaction. The photon is virtual.

Since this process is not a decay so the point (c) is not required.

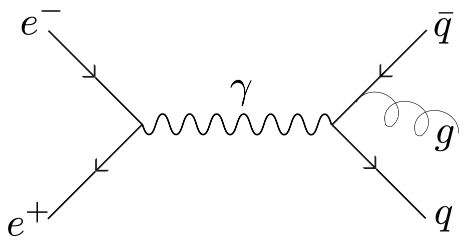

Figure 12: Feynman diagram of process 6

7) and 8)

We are considering the following double beta decay:

$$\textbf{7:}\quad 2n \to 2p + 2e^- + 2\bar{\nu}_e \qquad \textbf{8:}\quad 2n \to 2p + 2e^-$$

This is the decay of 2 neutrons via β decay in 2 different ways. The first process ($\beta\beta 2\bar{\nu}$ (7)) is **allowed**, it conserves the lepton number and the feynman diagram is shown in fig.(13). The second process (8), however, is **forbidden**, because it violates lepton number. However, the double beta decay without the anti-neutrino can occur for Majorana neutrinos of nonzero mass, which are their own antiparticles, by the mechanism shown in the feynman diagram in figure 14. Both process involve the weak interaction.

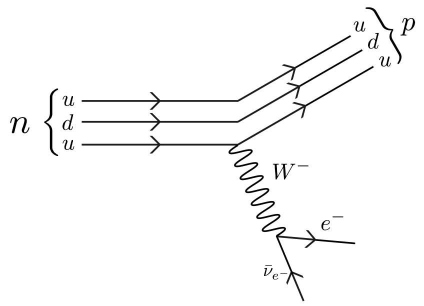

Figure 13

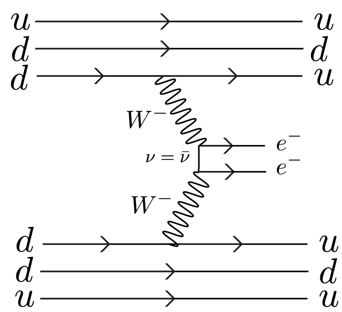

Figure 14

As said before, the observation of $\beta\beta 0\bar{\nu}$ decays would demonstrate the Majorana nature of neutrinos and the violation of total lepton number. It can in principle be distinguished from $\beta\beta 2\bar{\nu}$ decays by measuring the energies of the emitted electrons. In $\beta\beta 2\bar{\nu}$ decays, a part of the total energy is carried away by the neutrinos, resulting in a continuous spectrum for the combined energy of the electrons, which can be estimated experimentally. Whereas in $\beta\beta 0\bar{\nu}$ decays the electrons carry off all the available energy, resulting in a sharp line, almost a delta, in their combined energy. Observation of the reaction would therefore be strong evidence for the existence of Majorana neutrinos.

Exercise 7:Radioactive decay chain

a)

$N(t)$ is the number of atoms present at time t, the radioactive decay law states that

$$\mathcal{A} = -\frac{dN}{dt} = \lambda N \tag{12}$$

where λ is called the decay constant and $\mathcal{A}$ is the activity.
Integrating in time one obtains

$$N(t) = N_0\, e^{-\lambda t} \tag{13}$$

Now if we consider a chain decay of the type $1 \to 2 \to 3$ with initial conditions: $N_1(t=0) = N_0$ and $N_2(t=0) = N_3(t=0) = 0$. It follows straightforwardly the expression for the number of atoms N_1 as a function of time by applying the decay law (12)

$$N_1(t) = N_0 e^{-\lambda_1 t}$$

To get the expression for N_2 we have to consider: the number of atoms produced by decay 1, and the number of atoms decayed into 3. This means that the radioactive law is:

$$N_2(t) = -\frac{1}{\lambda_2}\frac{dN_2(t)}{dt} + \lambda_1 N_{1\to 2}(t)$$

the derivative term rapresents the atoms that decay in 3 and the las term are the atoms from the decay number 1.
Equaling the last expression with the general expression showed in eq:13 we get:

$$\frac{dN_2(t)}{dt} + \lambda_2 N_2(t) = N_0\, e^{-\lambda_1 t}$$

and with a bit of sleight of hand we can get the general solution for this differential equation of the first order:

$$N_2(t) = e^{-\alpha(t)}\, \left(\cancel{N_2(0)}^0 + \lambda_1 \int_0^t N_0 e^{-\lambda_1 x} e^{\alpha(t')}\, dt'\right)$$

where $\alpha(t) = \lambda_2 \int_0^t dt' = \lambda_2 t$. Hence

$$N_2(t) = e^{-\lambda_2 t}\, \lambda_1 \int_0^t N_0 e^{-(\lambda_1-\lambda_2)t'}\, dt' = \frac{\lambda_1 N_0\, e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\, \left(1 - e^{-(\lambda_1-\lambda_2)t}\right) =$$
$$= N_0\, \frac{\lambda_1 (e^{-\lambda_1 t} - e^{-\lambda_2 t})}{\lambda_2 - \lambda_1}$$

That is the result we are looking for.

b)

From eq.13 we can see that λ has units inverse of time. The time $t_{1/2}$ is defined as the time necessary to halve the number of atoms from N_0. This can be found by imposing $N(t) = N_0/2$ and solving for t we obtain

$$t_{1/2} = \frac{1}{\lambda}\, \log 2 \equiv \tau \log 2$$

where $\tau \equiv 1/\lambda$ is the mean lifetime. This last parameter τ gives information on the expected life of a particle. To visualise it better we can calculate the probability to find a certain number of decayed particles at time t:

$$p(t) = \frac{N(t)}{\int N(t)\, dt}$$

Hence, since it is a poissonian process it follows the poisson distribution, the expected lifetime is given by $\frac{1}{\lambda}$. Furthermore, it is also possible to get that from some math calculations:

$$\langle t \rangle = \frac{\int tN(t)\, dt}{\int N(t)\, dt} = \frac{\int_0^\infty te^{-\lambda t}\, dt}{\int_0^\infty e^{-\lambda t}} = \frac{1}{\lambda}$$

which it is exactly τ.

Another important quantity in decay phenomena is the decay width Γ. It is usually measured in eV. This is because is strictly related to the Breit Wigner distribution. In fact, if we draw the amplitude probability of a decay as a function of energy we end up with the Breit Wigner distribution, and the FWHM is the the decay width Γ. This quantity is related to the average lifetime of the particle trough the energy-time uncertainty relation getting: $\Gamma = \frac{\hbar}{\tau}$

c) A sample with activity at $t = 0$ equal to 1 milli Curie corresponds to $1mCi = 37MBq$. To calculate the total number of ^{139}Cs at the beginning we can substitute this in the equation (13) and imposing $\mathcal{A}(t = 0) = -\frac{dN_1(t)}{dt}|_{t=0} = 37MBq$ we obtain:

$$N_0 = \frac{37MBq}{\lambda_{Cs}} \simeq 29689798324 \simeq 3 \times 10^{10}$$

d) The plots are shown in figure 15 and in figure 16

e) The ^{139}Ba has a maximum activity of $3.14MBq$ and it reaches its maximum after 0.55 hours, i.e. 33 minutes.

f) The activities of ^{139}Cs and ^{139}Ba becomes equal after 0.55 hours. Moreover, this value is in correspondence to the maximum of the Activity of ^{139}Ba, as it possible to appreciate also from a first look at the figure 16. To get more precise value the code that I implemented is shown in the next section.

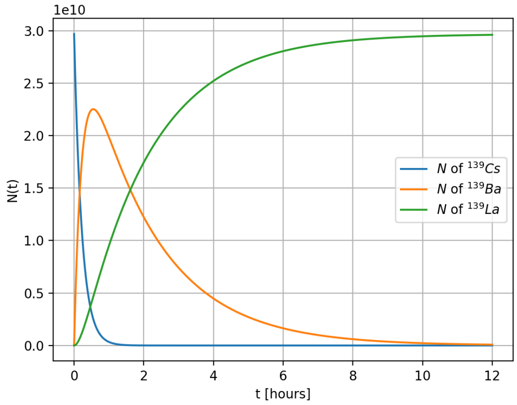

Figure 15: Radioactive decay chain. This plot shows the number of particle for each element as a function of time (measured in hours).

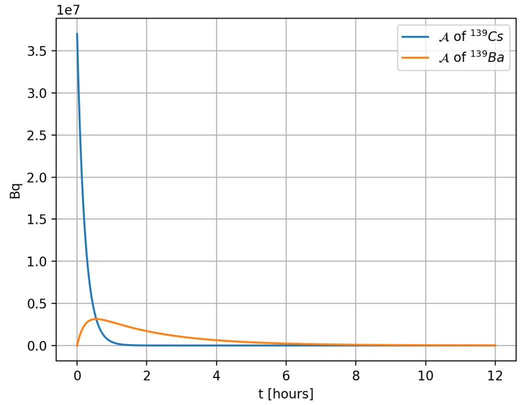

Figure 16: Activities of decay chain. The activity is defined as $\mathcal{A}_\rangle = \lambda_i N_i$
. This figure shows the trend of the activity over the time of the first and second decay. Since, the third element is stable $\lambda_3 = 0$, hence $\mathcal{A}$ of ^{139}La is equal to 0.

Script python exercise 2

```python
import matplotlib.pyplot as plt
#from typing_extensions import Required
import numpy as np
#######################
#Variables initialisation
#######################
z=np.linspace(10,100,1000)
a=136
a_v=15.8
a_s=18.3
a_c=0.714
a_sym=23.4
a_p=12
m_p=938.28
m_n=939.28
m_e=0.51
def mass(A,Z):
    if A%2==0:
        m=Z*(m_p+m_e) +(A-Z)*m_n
        b=( a_v*A - a_s*pow(A,2/3) - a_c*(Z*(Z-1))/pow(A,1/3) - a_sym*(pow((2*Z-A),2))/A + a_p*(pow(A,-1/2)))
        return m-b
    else:
        return Z*(m_p+m_e) +(A-Z)*m_n -(a_v*A - a_s*pow(A,2/3) - a_c*(Z*(Z-1))/pow(A,1/3) - a_sym*(pow((2*Z-A),2))/A)
def massodd(A,Z):
        m=Z*(m_p+m_e) +(A-Z)*m_n
        b=(a_v*A - a_s*pow(A,2/3) - a_c*(Z*(Z-1))/pow(A,1/3) - a_sym*(pow((2*Z-A),2))/A - a_p*(pow(A,-1/2)))
        return  m-b

Z_min= (m_n-m_p-m_e + a_c*pow(a,-1/3)+4*a_sym)/(2*a_c*pow(a,-1/3) + 8*a_sym*pow(a,-1))
print("The minimum Z is: " ,Z_min)

plt.plot(z,mass(135,z)*0.001,label='A=135', color="orange" )
plt.plot(z,massodd(136,z)*0.001,label='A=136 odd-odd', color= "green"  )
plt.plot(z,mass(136,z)*0.001,label='A=136 even-even' )
plt.legend()
plt.xlabel("Z")
plt.xticks()
plt.ylabel("Mev")
plt.grid()
plt.show()
```

Script python exercise 7

```
import matplotlib.pyplot as plt
#from typing_extensions import Required
import numpy as np


########################################
#Radioactive decay chain
########################################
# const declaration
halflife1=9.27/60 #time converted in hour
halflife2=82.73/60 #time converted in hour
lambda1 = np.log(2)/(halflife1)
lambda2 = np.log(2)/(halflife2)
n0=(37e6)/(lambda1/3600)
print(n0,"\n")
t = np.linspace(0,12,12000)
#N decay
n1=n0*np.exp(-lambda1*t)
n2= n0*(lambda1*(np.exp(-lambda1*t)-np.exp(-lambda2*t)))/(lambda2-lambda1)
n3= n0*(lambda1*(1-np.exp(-lambda2*t))-lambda2*(1-np.exp(-lambda1*t)))/(lambda1-lambda2)
plt.plot(t,n1, label="$N$ of $^{139}Cs$")
plt.plot(t,n2, label="$N$ of $^{139}Ba$")
plt.plot(t,n3, label="$N$ of $^{139}La$")
plt.xlabel("t [hours]")
plt.ylabel("N(t)")
plt.grid()
plt.legend()
plt.show()


#####################################
#Activity
#####################################
#1Curie
n0=(37e6)/lambda1 #renormalised for the plot of the activity over the 12 hour
print(n0)
a_1= lambda1*n0*np.exp(-lambda1*t)
a_2 = lambda2*n0*(lambda1*(np.exp(-lambda1*t)-np.exp(-lambda2*t)))/(lambda2-lambda1)
plt.plot(t,a_1, label="$\mathcal{A}$ of $^{139}Cs$")
plt.plot(t,a_2, label="$\mathcal{A}$ of $^{139}Ba$")
plt.xlabel("t [hours]")
plt.ylabel("Bq")
plt.grid()
plt.legend()
plt.show()
########
#e)
print("max activity: ",np.max(a_2))
print("after ", np.where(a_2 == np.max(a_2)))

#######
#d)
#diff=abs(a_2 - a_1)
# I set 2 because the 2 functions meet themself for the first time before the 2 hours

diff=abs(a_2[:2000] - a_1[:2000])
m=np.min(diff)
print("The 2 activities are equally large: ", m)
print("after ", np.where(diff == m))
```