Supplementary Note: Detailed Computations

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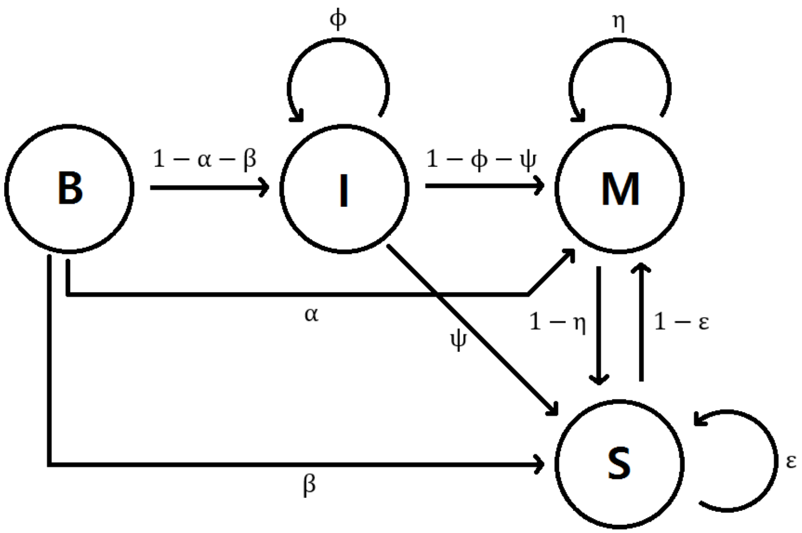
- K: number of cores
- T: number of discretized age indices in the stack
- N_i: number of discretized depth indices in the core i
- $\{\mathrm{A}_t\}_{t=1}^{\mathrm{T}}$: ages assigned to age indices in the stack
- $\{\mu_t\}_{t=1}^{\mathrm{T}}$: means of $\delta^{18}\mathrm{O}$ concentrations assigned to age indices in the stack
- $\{\sigma_t\}_{t=1}^{\mathrm{T}}$: standard deviations of $\delta^{18}\mathrm{O}$ concentrations assigned to age indices in the stack
- $\left\{d_n^{(i)}\right\}_{n=1}^{\mathrm{N}_i}$: depths assigned to depth indices in the core i
- $\left\{v_n^{(i)}\right\}_{n=1}^{\mathrm{N}_i}$: $\delta^{18}\mathrm{O}$ assigned to depth indices in the core i, if it exists; -1 otherwise.
- $\left\{\xi_{n,t}^{(i)}\right\}_{n=1,t=1}^{\mathrm{N}_i,\mathrm{T}}$: empirical distribution of ages from the radiocarbon data assigned to depth indices in the core i, if it exists; $\xi_{n,t}^{(i)} \equiv 1/\mathrm{T}$ otherwise.
- $\left\{\mathrm{C}_n^{(i)}\right\}_{n=1}^{\mathrm{N}_i}$: (hidden) ages assigned to depth indices in the core i
- R_i: average sedimentation rate in the core i
- $p\left(v_n^{(i)} \middle| \mathrm{C}_n^{(i)} = \mathrm{A}_t, \xi_{n,1:\mathrm{T}}^{(i)}\right) \triangleq \begin{cases} \mathrm{N}\left(v_n^{(i)} \middle| \tau^{(i)} + \mu_t, \sigma_t\right) \cdot \xi_{n,t}^{(i)}, & v_n^{(i)} \neq -1 \\ \xi_{n,t}^{(i)}, & \text{otherwise} \end{cases}$
- $\alpha_i, \beta_i, \phi_i, \psi_i, \eta_i, \varepsilon_i, \tau_i$: core parameters for the core i, where
  (Note that the right model is valid under the assumption that we have the reference stack long enough to cover whole parts to which the core depths are supposed to be aligned. If it is not expected, an additional state for reflecting core depths supposed not to be aligned.)



1. Forward Algorithm:

1.1. Initialization:

$$\mathbf{F}_{\boldsymbol{i}}(\mathbf{2},\mathbf{1},\mathbf{1}) \triangleq p\left(v_{1:2}^{(i)}, \mathrm{C}_1^{(i)} = \mathrm{A}_1, \mathrm{C}_2^{(i)} = \mathrm{A}_1 \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_2^{(i)} \middle| \mathrm{C}_2^{(i)} = \mathrm{A}_s, \xi_{2,1:\mathrm{T}}^{(i)}\right) \cdot p\left(v_1^{(i)} \middle| \mathrm{C}_1^{(i)} = \mathrm{A}_1, \xi_{1,1:\mathrm{T}}^{(i)}\right) \cdot p\left(\mathrm{C}_2^{(i)} = \mathrm{A}_1, \mathrm{C}_1^{(i)} = \mathrm{A}_1 \middle| d_1^{(i)}, d_2^{(i)}\right)$$

$$= \boldsymbol{p}\left(\boldsymbol{v}_2^{(i)} \middle| \mathbf{C}_2^{(i)} = \mathbf{A}_s, \boldsymbol{\xi}_{2,1:\mathrm{T}}^{(i)}\right) \cdot \boldsymbol{p}\left(\boldsymbol{v}_1^{(i)} \middle| \mathbf{C}_1^{(i)} = \mathbf{A}_1, \boldsymbol{\xi}_{1,1:\mathrm{T}}^{(i)}\right) \cdot \mathbf{1}_{\left\{\mathbf{R}_i\left|\boldsymbol{d}_2^{(i)} - \boldsymbol{d}_1^{(i)}\right| < 1\right\}} \cdot \boldsymbol{\beta}_{\boldsymbol{i}}$$

1.1.1. For each $s \in \{2, \cdots, \mathrm{T}\}$,

$$\mathbf{F}_{\boldsymbol{i}}(\mathbf{2},\mathbf{1},\boldsymbol{s}) \triangleq p\left(v_{1:2}^{(i)}, \mathrm{C}_1^{(i)} = \mathrm{A}_1, \mathrm{C}_2^{(i)} = \mathrm{A}_s \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_2^{(i)} \middle| \mathrm{C}_2^{(i)} = \mathrm{A}_s, \xi_{2,1:\mathrm{T}}^{(i)}\right) \cdot p\left(v_1^{(i)} \middle| \mathrm{C}_1^{(i)} = \mathrm{A}_1, \xi_{1,1:\mathrm{T}}^{(i)}\right) \cdot p\left(\mathrm{C}_2^{(i)} = \mathrm{A}_s, \mathrm{C}_1^{(i)} = \mathrm{A}_1 \middle| d_1^{(i)}, d_2^{(i)}\right)$$

$$= \boldsymbol{p}\left(\boldsymbol{v}_2^{(i)} \middle| \mathbf{C}_2^{(i)} = \mathbf{A}_s, \boldsymbol{\xi}_{2,1:\mathrm{T}}^{(i)}\right) \cdot \boldsymbol{p}\left(\boldsymbol{v}_1^{(i)} \middle| \mathbf{C}_1^{(i)} = \mathbf{A}_1, \boldsymbol{\xi}_{1,1:\mathrm{T}}^{(i)}\right) \cdot \boldsymbol{\rho}\left(\frac{\mathbf{A}_s - \mathbf{A}_1}{\mathbf{R}_i\left|\boldsymbol{d}_2^{(i)} - \boldsymbol{d}_1^{(i)}\right|}\right) \cdot \boldsymbol{\alpha}_{\boldsymbol{i}}$$

1.1.2. For each $s \in \{2,3, \cdots, \mathrm{T}\}$,

$$\mathbf{F}_{\boldsymbol{i}}(\mathbf{2},\boldsymbol{s},\boldsymbol{s}) \triangleq p\left(v_{1:2}^{(i)}, \mathrm{C}_1^{(i)} = \mathrm{A}_s, \mathrm{C}_2^{(i)} = \mathrm{A}_s \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_2^{(i)} \middle| \mathrm{C}_2^{(i)} = \mathrm{A}_s, \xi_{2,1:\mathrm{T}}^{(i)}\right) \cdot p\left(v_1^{(i)} \middle| \mathrm{C}_1^{(i)} = \mathrm{A}_s, \xi_{1,1:\mathrm{T}}^{(i)}\right) \cdot p\left(\mathrm{C}_2^{(i)} = \mathrm{A}_s, \mathrm{C}_1^{(i)} = \mathrm{A}_s \middle| d_1^{(i)}, d_2^{(i)}\right)$$

$$= \boldsymbol{p}\left(\boldsymbol{v}_2^{(i)} \middle| \mathbf{C}_2^{(i)} = \mathbf{A}_s, \boldsymbol{\xi}_{2,1:\mathrm{T}}^{(i)}\right) \cdot \boldsymbol{p}\left(\boldsymbol{v}_1^{(i)} \middle| \mathbf{C}_1^{(i)} = \mathbf{A}_s, \boldsymbol{\xi}_{1,1:\mathrm{T}}^{(i)}\right) \cdot \mathbf{1}_{\left\{\mathbf{R}_i\left|\boldsymbol{d}_2^{(i)} - \boldsymbol{d}_1^{(i)}\right| < 1\right\}} \cdot (\mathbf{1} - \boldsymbol{\alpha}_{\boldsymbol{i}} - \boldsymbol{\beta}_{\boldsymbol{i}}) \cdot \boldsymbol{\phi}_{\boldsymbol{i}}^{\boldsymbol{s}-\mathbf{1}} \cdot \boldsymbol{\psi}_{\boldsymbol{i}}$$

1.1.3. For each $s, t \in \{2,3, \cdots, \mathrm{T}\}$ such that $s > t$,

$$\mathbf{F}_{\boldsymbol{i}}(\mathbf{2},\boldsymbol{s},\boldsymbol{t}) \triangleq p\left(v_{1:2}^{(i)}, \mathrm{C}_1^{(i)} = \mathrm{A}_t, \mathrm{C}_2^{(i)} = \mathrm{A}_s \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_2^{(i)} \middle| \mathrm{C}_2^{(i)} = \mathrm{A}_s, \xi_{2,1:\mathrm{T}}^{(i)}\right) \cdot p\left(v_1^{(i)} \middle| \mathrm{C}_1^{(i)} = \mathrm{A}_t, \xi_{1,1:\mathrm{T}}^{(i)}\right) \cdot p\left(\mathrm{C}_2^{(i)} = \mathrm{A}_s, \mathrm{C}_1^{(i)} = \mathrm{A}_t \middle| d_1^{(i)}, d_2^{(i)}\right)$$

$$= \boldsymbol{p}\left(\boldsymbol{v}_2^{(i)} \middle| \mathbf{C}_2^{(i)} = \mathbf{A}_s, \boldsymbol{\xi}_{2,1:\mathrm{T}}^{(i)}\right) \cdot \boldsymbol{p}\left(\boldsymbol{v}_1^{(i)} \middle| \mathbf{C}_1^{(i)} = \mathbf{A}_t, \boldsymbol{\xi}_{1,1:\mathrm{T}}^{(i)}\right) \cdot \boldsymbol{\rho}\left(\frac{\mathbf{A}_s - \mathbf{A}_t}{\mathbf{R}_i\left|\boldsymbol{d}_2^{(i)} - \boldsymbol{d}_1^{(i)}\right|}\right) \cdot (\mathbf{1} - \boldsymbol{\alpha}_{\boldsymbol{i}} - \boldsymbol{\beta}_{\boldsymbol{i}}) \cdot \boldsymbol{\phi}_{\boldsymbol{i}}^{\boldsymbol{t}-\mathbf{1}} \cdot (\mathbf{1} - \boldsymbol{\phi}_{\boldsymbol{i}} - \boldsymbol{\psi}_{\boldsymbol{i}})$$

1.2. Iterations: for each $n \in \{3,4,\cdots,\mathrm{N}_i\}$,

1.2.1. For each $s \in \{2,3,\cdots,\mathrm{T}\}$,

$$\mathbf{F}_{\boldsymbol{i}}(\boldsymbol{n},\boldsymbol{s},\boldsymbol{s}) \triangleq p\left(v_{1:n}^{(i)}, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_s, \mathrm{C}_n^{(i)}=\mathrm{A}_s \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_{1:n}^{(i)}, \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_s, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_s, \mathrm{C}_n^{(i)}=\mathrm{A}_s \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right) + \sum_{u<s} p\left(v_{1:n}^{(i)}, \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_u, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_s, \mathrm{C}_n^{(i)}=\mathrm{A}_s \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_n^{(i)} \middle| \mathrm{C}_n^{(i)}=\mathrm{A}_s, \xi_{n,1:\mathrm{T}}^{(i)}\right) \cdot p\left(\mathrm{C}_n^{(i)}=\mathrm{A}_s \middle| \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_s, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_s, d_{n-2}^{(i)}, d_{n-1}^{(i)}, d_n^{(i)}\right) \cdot p\left(v_{1:n-1}^{(i)}, \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_s, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_s \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$+\sum_{u<s}\left[p\left(v_n^{(i)} \middle| \mathrm{C}_n^{(i)}=\mathrm{A}_s, \xi_{n,1:\mathrm{T}}^{(i)}\right) \cdot p\left(\mathrm{C}_n^{(i)}=\mathrm{A}_s \middle| \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_u, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_s, d_{n-2}^{(i)}, d_{n-1}^{(i)}, d_n^{(i)}\right) \cdot p\left(v_{1:n-1}^{(i)}, \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_u, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_s \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)\right]$$

$$= \boldsymbol{p}\left(\boldsymbol{v}_{\boldsymbol{n}}^{(\boldsymbol{i})} \middle| \mathbf{C}_{\boldsymbol{n}}^{(\boldsymbol{i})}=\mathbf{A}_{\boldsymbol{s}}, \boldsymbol{\xi}_{\boldsymbol{n},\mathbf{1}:\mathbf{T}}^{(\boldsymbol{i})}\right) \cdot \mathbf{1}_{\left\{\mathbf{R}_i\left|d_n^{(i)}-d_{n-1}^{(i)}\right|<1\right\}}\left[\boldsymbol{\varepsilon}_{\boldsymbol{i}} \cdot \mathbf{F}_{\boldsymbol{i}}(\boldsymbol{n}-\mathbf{1},\boldsymbol{s},\boldsymbol{s}) + (\mathbf{1}-\boldsymbol{\eta}_{\boldsymbol{i}})\sum_{\boldsymbol{u}<\boldsymbol{s}} \mathbf{F}_{\boldsymbol{i}}(\boldsymbol{n}-\mathbf{1},\boldsymbol{u},\boldsymbol{s})\right]$$

1.2.2. For each $s,t \in \{2,3,\cdots,\mathrm{T}\}$ such that $s>t$,

$$\mathbf{F}_{\boldsymbol{i}}(\boldsymbol{n},\boldsymbol{t},\boldsymbol{s}) \triangleq p\left(v_{1:n}^{(i)}, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_t, \mathrm{C}_n^{(i)}=\mathrm{A}_s \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_{1:n}^{(i)}, \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_t, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_t, \mathrm{C}_n^{(i)}=\mathrm{A}_s \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right) + \sum_{u<t} p\left(v_{1:n}^{(i)}, \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_u, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_t, \mathrm{C}_n^{(i)}=\mathrm{A}_s \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_n^{(i)} \middle| \mathrm{C}_n^{(i)}=\mathrm{A}_s, \xi_{n,1:\mathrm{T}}^{(i)}\right) \cdot p\left(\mathrm{C}_n^{(i)}=\mathrm{A}_s \middle| \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_t, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_t, d_{n-2}^{(i)}, d_{n-1}^{(i)}, d_n^{(i)}\right) \cdot p\left(v_{1:n-1}^{(i)}, \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_t, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_t \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$+\sum_{u<t}\left[p\left(v_n^{(i)} \middle| \mathrm{C}_n^{(i)}=\mathrm{A}_s, \xi_{n,1:\mathrm{T}}^{(i)}\right) \cdot p\left(\mathrm{C}_n^{(i)}=\mathrm{A}_s \middle| \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_u, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_t, d_{n-2}^{(i)}, d_{n-1}^{(i)}, d_n^{(i)}\right) \cdot p\left(v_{1:n-1}^{(i)}, \mathrm{C}_{n-2}^{(i)}=\mathrm{A}_u, \mathrm{C}_{n-1}^{(i)}=\mathrm{A}_t \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)\right]$$

$$= \boldsymbol{p}\left(\boldsymbol{v}_{\boldsymbol{n}}^{(\boldsymbol{i})} \middle| \mathbf{C}_{\boldsymbol{n}}^{(\boldsymbol{i})}=\mathbf{A}_{\boldsymbol{s}}, \boldsymbol{\xi}_{\boldsymbol{n},\mathbf{1}:\mathbf{T}}^{(\boldsymbol{i})}\right)\left[(\mathbf{1}-\boldsymbol{\varepsilon}_{\boldsymbol{i}}) \cdot \boldsymbol{\rho}\left(\frac{\mathbf{A}_{\boldsymbol{s}}-\mathbf{A}_{\boldsymbol{t}}}{\mathbf{R}_i\left|\boldsymbol{d}_{\boldsymbol{n}}^{(i)}-\boldsymbol{d}_{\boldsymbol{n}-\mathbf{1}}^{(i)}\right|}\right) \cdot \mathbf{F}_{\boldsymbol{i}}(\boldsymbol{n}-\mathbf{1},\boldsymbol{t},\boldsymbol{t}) + \boldsymbol{\eta}_{\boldsymbol{i}}\sum_{\boldsymbol{u}<\boldsymbol{s}}\left[\boldsymbol{\rho}\left(\frac{\mathbf{A}_{\boldsymbol{t}}-\mathbf{A}_{\boldsymbol{u}}}{\mathbf{R}_i\left|\boldsymbol{d}_{\boldsymbol{n}-\mathbf{1}}^{(i)}-\boldsymbol{d}_{\boldsymbol{n}-\mathbf{2}}^{(i)}\right|}, \frac{\mathbf{A}_{\boldsymbol{s}}-\mathbf{A}_{\boldsymbol{t}}}{\mathbf{R}_i\left|\boldsymbol{d}_{\boldsymbol{n}}^{(i)}-\boldsymbol{d}_{\boldsymbol{n}-\mathbf{1}}^{(i)}\right|}\right) \cdot \mathbf{F}_{\boldsymbol{i}}(\boldsymbol{n}-\mathbf{1},\boldsymbol{u},\boldsymbol{s})\right]\right]$$

2. Backward Sampling:

2.1. Initialization: for each $t_{\mathrm{N}_i-1}^{(i),(l)}, t_{\mathrm{N}_i}^{(i),(l)} \in \{1,2,3,\cdots,\mathrm{T}\}$,

$$\mathbb{P}\left(\mathbf{C}_{\mathrm{N}_i-\mathbf{1}}^{(\boldsymbol{i}),(\boldsymbol{l})} = \mathbf{A}_{t_{\mathrm{N}_i-1}^{(i),(l)}}, \mathbf{C}_{\mathrm{N}_i}^{(\boldsymbol{i}),(\boldsymbol{l})} = \mathbf{A}_{t_{\mathrm{N}_i}^{(i),(l)}} \middle| \boldsymbol{v}_{\mathbf{1}:\mathrm{N}_i}^{(\boldsymbol{i})}, \boldsymbol{d}_{\mathbf{1}:\mathrm{N}_i}^{(\boldsymbol{i})}, \boldsymbol{\xi}_{\mathbf{1}:\mathrm{N}_i,\mathbf{1}:\mathbf{T}}^{(\boldsymbol{i})}\right) \propto p\left(v_{1:\mathrm{N}_i}^{(i)}, \mathrm{C}_{\mathrm{N}_i-1}^{(i),(l)} = \mathrm{A}_{t_{\mathrm{N}_i-1}^{(i),(l)}}, \mathrm{C}_{\mathrm{N}_i}^{(i),(l)} = \mathrm{A}_{t_{\mathrm{N}_i}^{(i),(l)}} \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= \mathbf{E}_{\boldsymbol{i}}\left(\mathbf{N}_{\boldsymbol{i}}, \boldsymbol{t}_{\mathbf{N}_{\boldsymbol{i}}-\mathbf{1}}^{(\boldsymbol{i}),(\boldsymbol{l})}, \boldsymbol{t}_{\mathbf{N}_{\boldsymbol{i}}}^{(\boldsymbol{i}),(\boldsymbol{l})}\right)$$

2.2. Iterations: for each $n \in \{m-2, m-3, \cdots, 1\}$ and $t_n^{(i),(l)} \in \left\{1,2,3,\cdots,t_{n+1}^{(i),(l)}\right\}$,

2.2.1. $t_{n+1}^{(i),(l)} = t_{n+2}^{(i),(l)}$:

2.2.1.1. $t_n^{(i),(l)} = t_{n+1}^{(i),(l)}$:

$$\mathbb{P}\left(\mathbf{C}_{\boldsymbol{n}}^{(i),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n}}^{(i),(l)}} \middle| \mathbf{C}_{\boldsymbol{n}+\mathbf{1}}^{(i),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n}+\mathbf{1}}^{(i),(l)}}, \mathbf{C}_{\boldsymbol{n}+\mathbf{2}}^{(i),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n}+\mathbf{2}}^{(i),(l)}}, \boldsymbol{v}_{\mathbf{1}:\boldsymbol{n}+\mathbf{2}}^{(\boldsymbol{i})}, \boldsymbol{d}_{\mathbf{1}:\mathrm{N}_i}^{(\boldsymbol{i})}, \boldsymbol{\xi}_{\mathbf{1}:\mathrm{N}_i,\mathbf{1}:\mathbf{T}}^{(\boldsymbol{i})}\right) \propto \mathbb{P}\left(\mathrm{C}_n^{(i),(l)} = \mathrm{A}_{t_n^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, v_{1:n+2}^{(i)} \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_{n+2}^{(i)} \middle| \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, \xi_{n+2,1:\mathrm{T}}^{(i)}\right) p\left(\mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}} \middle| \mathrm{C}_n^{(i),(l)} = \mathrm{A}_{t_n^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, d_n^{(i)}, d_{n+1}^{(i)}, d_{n+2}^{(i)}\right) p\left(\mathrm{C}_n^{(i),(l)} = \mathrm{A}_{t_n^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, v_{1:n+1}^{(i)} \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_{n+2}^{(i)} \middle| \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, \xi_{n+2,1:\mathrm{T}}^{(i)}\right) \cdot 1_{\left\{\mathrm{R}_i \left| d_{n+2}^{(i)} - d_{n+1}^{(i)} \right| < 1\right\}} \cdot \mathrm{F}_i\left(n+1, t_n^{(i),(l)}, t_{n+1}^{(i),(l)}\right) \cdot \varepsilon_i$$

$$\propto \mathbf{F}_{\boldsymbol{i}}\left(\boldsymbol{n}+\mathbf{1}, \boldsymbol{t}_{\boldsymbol{n}}^{(i),(l)}, \boldsymbol{t}_{\boldsymbol{n}+\mathbf{1}}^{(i),(l)}\right) \cdot \boldsymbol{\varepsilon}_{\boldsymbol{i}}$$

2.2.1.2. $t_n^{(i),(l)} < t_{n+1}^{(i),(l)}$:

$$\mathbb{P}\left(\mathbf{C}_{\boldsymbol{n}}^{(i),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n}}^{(i),(l)}} \middle| \mathbf{C}_{\boldsymbol{n}+\mathbf{1}}^{(i),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n}+\mathbf{1}}^{(i),(l)}}, \mathbf{C}_{\boldsymbol{n}+\mathbf{2}}^{(i),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n}+\mathbf{2}}^{(i),(l)}}, \boldsymbol{v}_{\mathbf{1}:\boldsymbol{n}+\mathbf{2}}^{(\boldsymbol{i})}, \boldsymbol{d}_{\mathbf{1}:\mathrm{N}_i}^{(\boldsymbol{i})}, \boldsymbol{\xi}_{\mathbf{1}:\mathrm{N}_i,\mathbf{1}:\mathbf{T}}^{(\boldsymbol{i})}\right) \propto \mathbb{P}\left(\mathrm{C}_n^{(i),(l)} = \mathrm{A}_{t_n^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, v_{1:n+2}^{(i)} \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_{n+2}^{(i)} \middle| \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, \xi_{n+2,1:\mathrm{T}}^{(i)}\right) p\left(\mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}} \middle| \mathrm{C}_n^{(i),(l)} = \mathrm{A}_{t_n^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, d_n^{(i)}, d_{n+1}^{(i)}, d_{n+2}^{(i)}\right) p\left(\mathrm{C}_n^{(i),(l)} = \mathrm{A}_{t_n^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, v_{1:n+1}^{(i)} \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_{n+2}^{(i)} \middle| \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, \xi_{n+2,1:\mathrm{T}}^{(i)}\right) \cdot 1_{\left\{\mathrm{R}_i \left| d_{n+2}^{(i)} - d_{n+1}^{(i)} \right| < 1\right\}} \cdot \mathrm{F}_i\left(n+1, t_n^{(i),(l)}, t_{n+1}^{(i),(l)}\right) \cdot (1 - \eta_i)$$

$$\propto \mathbf{F}_{\boldsymbol{i}}\left(\boldsymbol{n}+\mathbf{1}, \boldsymbol{t}_{\boldsymbol{n}}^{(i),(l)}, \boldsymbol{t}_{\boldsymbol{n}+\mathbf{1}}^{(i),(l)}\right) \cdot (\mathbf{1} - \boldsymbol{\eta}_{\boldsymbol{i}})$$

2.2.2. $t_{n+1}^{(i),(l)} < t_{n+2}^{(i),(l)}$:

2.2.2.1. $t_{n}^{(i),(l)} = t_{n+1}^{(i),(l)}$:

$$\mathbb{P}\left(\mathbf{C}_{\boldsymbol{n}}^{(\boldsymbol{i}),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n}}^{(i),(l)}} \middle| \mathbf{C}_{\boldsymbol{n+1}}^{(\boldsymbol{i}),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n+1}}^{(i),(l)}}, \mathbf{C}_{\boldsymbol{n+2}}^{(\boldsymbol{i}),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n+2}}^{(i),(l)}}, \boldsymbol{v}_{\mathbf{1}:\boldsymbol{n+2}}^{(\boldsymbol{i})}, \boldsymbol{d}_{\mathbf{1}:\mathrm{N}_i}^{(\boldsymbol{i})}, \boldsymbol{\xi}_{\mathbf{1}:\mathbf{N}_{\boldsymbol{i}},\mathbf{1}:\mathbf{T}}^{(\boldsymbol{i})}\right) \propto \mathbb{P}\left(\mathrm{C}_{n}^{(i),(l)} = \mathrm{A}_{t_{n}^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, v_{1:n+2}^{(i)} \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_{n+2}^{(i)} \middle| \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, \xi_{n+2,1:\mathrm{T}}^{(i)}\right) p\left(\mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}} \middle| \mathrm{C}_{n}^{(i),(l)} = \mathrm{A}_{t_{n}^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, d_{n}^{(i)}, d_{n+1}^{(i)}, d_{n+2}^{(i)}\right) p\left(\mathrm{C}_{n}^{(i),(l)} = \mathrm{A}_{t_{n}^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, v_{1:n+1}^{(i)} \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_{n+2}^{(i)} \middle| \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, \xi_{n+2,1:\mathrm{T}}^{(i)}\right) \cdot \rho\left(\frac{\mathrm{A}_{t_{n+2}^{(i),(l)}} - \mathrm{A}_{t_{n+1}^{(i),(l)}}}{\mathrm{R}_i\left|d_{n+2}^{(i)} - d_{n+1}^{(i)}\right|}\right) \cdot \mathrm{F}_i\left(n+1, t_{n}^{(i),(l)}, t_{n+1}^{(i),(l)}\right) \cdot (1 - \varepsilon_i)$$

$$\propto \mathbf{F}_{\boldsymbol{i}}\left(\boldsymbol{n} + \mathbf{1}, \boldsymbol{t}_{\boldsymbol{n}}^{(\boldsymbol{i}),(\boldsymbol{l})}, \boldsymbol{t}_{\boldsymbol{n+1}}^{(\boldsymbol{i}),(\boldsymbol{l})}\right) \cdot (\mathbf{1} - \boldsymbol{\varepsilon}_{\boldsymbol{i}})$$

2.2.2.2. $t_{n}^{(i),(l)} < t_{n+1}^{(i),(l)}$:

$$\mathbb{P}\left(\mathbf{C}_{\boldsymbol{n}}^{(\boldsymbol{i}),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n}}^{(i),(l)}} \middle| \mathbf{C}_{\boldsymbol{n+1}}^{(\boldsymbol{i}),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n+1}}^{(i),(l)}}, \mathbf{C}_{\boldsymbol{n+2}}^{(\boldsymbol{i}),(\boldsymbol{l})} = \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n+2}}^{(i),(l)}}, \boldsymbol{v}_{\mathbf{1}:\boldsymbol{n+2}}^{(\boldsymbol{i})}, \boldsymbol{d}_{\mathbf{1}:\mathrm{N}_i}^{(\boldsymbol{i})}, \boldsymbol{\xi}_{\mathbf{1}:\mathbf{N}_{\boldsymbol{i}},\mathbf{1}:\mathbf{T}}^{(\boldsymbol{i})}\right) \propto \mathbb{P}\left(\mathrm{C}_{n}^{(i),(l)} = \mathrm{A}_{t_{n}^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, v_{1:n+2}^{(i)} \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_{n+2}^{(i)} \middle| \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, \xi_{n+2,1:\mathrm{T}}^{(i)}\right) p\left(\mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}} \middle| \mathrm{C}_{n}^{(i),(l)} = \mathrm{A}_{t_{n}^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, d_{n}^{(i)}, d_{n+1}^{(i)}, d_{n+2}^{(i)}\right) p\left(\mathrm{C}_{n}^{(i),(l)} = \mathrm{A}_{t_{n}^{(i),(l)}}, \mathrm{C}_{n+1}^{(i),(l)} = \mathrm{A}_{t_{n+1}^{(i),(l)}}, v_{1:n+1}^{(i)} \middle| d_{1:\mathrm{N}_i}^{(i)}, \xi_{1:\mathrm{N}_i,1:\mathrm{T}}^{(i)}\right)$$

$$= p\left(v_{n+2}^{(i)} \middle| \mathrm{C}_{n+2}^{(i),(l)} = \mathrm{A}_{t_{n+2}^{(i),(l)}}, \xi_{n+2,1:\mathrm{T}}^{(i)}\right) \cdot \rho\left(\frac{\mathrm{A}_{t_{n+1}^{(i),(l)}} - \mathrm{A}_{t_{n}^{(i),(l)}}}{\mathrm{R}_i\left|d_{n+1}^{(i)} - d_{n}^{(i)}\right|}, \frac{\mathrm{A}_{t_{n+2}^{(i),(l)}} - \mathrm{A}_{t_{n+1}^{(i),(l)}}}{\mathrm{R}_i\left|d_{n+2}^{(i)} - d_{n+1}^{(i)}\right|}\right) \cdot \mathrm{F}_i\left(n+1, t_{n}^{(i),(l)}, t_{n+1}^{(i),(l)}\right) \cdot \eta_i$$

$$\propto \boldsymbol{\rho}\left(\frac{\mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n+1}}^{(i),(l)}} - \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n}}^{(i),(l)}}}{\mathbf{R}_{\boldsymbol{i}}\left|\boldsymbol{d}_{\boldsymbol{n+1}}^{(\boldsymbol{i})} - \boldsymbol{d}_{\boldsymbol{n}}^{(\boldsymbol{i})}\right|}, \frac{\mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n+2}}^{(i),(l)}} - \mathbf{A}_{\boldsymbol{t}_{\boldsymbol{n+1}}^{(i),(l)}}}{\mathbf{R}_{\boldsymbol{i}}\left|\boldsymbol{d}_{\boldsymbol{n+2}}^{(\boldsymbol{i})} - \boldsymbol{d}_{\boldsymbol{n+1}}^{(\boldsymbol{i})}\right|}\right) \cdot \mathbf{F}_{\boldsymbol{i}}\left(\boldsymbol{n} + \mathbf{1}, \boldsymbol{t}_{\boldsymbol{n}}^{(\boldsymbol{i}),(\boldsymbol{l})}, \boldsymbol{t}_{\boldsymbol{n+1}}^{(\boldsymbol{i}),(\boldsymbol{l})}\right) \cdot \boldsymbol{\eta}_{\boldsymbol{i}}$$

2.3. Termination: iterate the above sampling steps to get $t_{\mathrm{N}_i}^{(i),(l)}, t_{\mathrm{N}_i-1}^{(i),(l)}, \cdots, t_{1}^{(i),(l)}$ recursively until possible.

2.3.1. If $\mathrm{F}_i\left(n+1, t, t_{n+1}^{(i),(l)}\right) \equiv 0$ for some $n \geq 1$, then stop and discard the obtained sample and go to the first step to get it again.

2.3.2. After getting $\mathrm{L}^{(i)}$ samples $\left\{t_{1}^{(i),(l)}, t_{2}^{(i),(l)}, \cdots, t_{\mathrm{N}_i}^{(i),(l)}\right\}_{l=1}^{\mathrm{L}^{(i)}}$, stop and proceed to the next step.

3. Updating core parameters by ML estimators:

Suppose that we get $\mathrm{L}^{(i)}$ samples for the core i by running the above backward sampling algorithm.

Let:

$$\mathrm{A} \triangleq \#\text{ of samples where } \tilde{t}_1^{(i),(l)} = 1 \text{ and } \tilde{t}_2^{(i),(l)} \neq 1$$

$$\mathrm{B} \triangleq \#\text{ of samples where } \tilde{t}_1^{(i),(l)} = 1 \text{ and } \tilde{t}_2^{(i),(l)} = 1$$

$$\mathrm{C} \triangleq \#\text{ of samples where } \tilde{t}_1^{(i),(l)} \neq 1$$

$$\mathrm{D} \triangleq \#\text{ of samples where } \tilde{t}_1^{(i),(l)} \neq 1 \text{ and } \tilde{t}_2^{(i),(l)} \neq 1 \text{ and } \mathrm{A}_{\tilde{t}_1^{(i),(l)}} = \mathrm{A}_{\tilde{t}_2^{(i),(l)}}$$

$$\mathrm{E} \triangleq \#\text{ of samples where } \tilde{t}_1^{(i),(l)} \neq 1 \text{ and } \tilde{t}_2^{(i),(l)} \neq 1 \text{ and } \mathrm{A}_{\tilde{t}_1^{(i),(l)}} \neq \mathrm{A}_{\tilde{t}_2^{(i),(l)}}$$

$$\mathrm{F} \triangleq \#\text{ of samples where } \mathrm{A}_{\tilde{t}_n^{(i),(l)}} = \mathrm{A}_{\tilde{t}_{n+1}^{(i),(l)}} = \mathrm{A}_{\tilde{t}_{n+2}^{(i),(l)}}$$

$$\mathrm{G} \triangleq \#\text{ of samples where } \mathrm{A}_{\tilde{t}_n^{(i),(l)}} = \mathrm{A}_{\tilde{t}_{n+1}^{(i),(l)}} \neq \mathrm{A}_{\tilde{t}_{n+2}^{(i),(l)}}$$

$$\mathrm{H} \triangleq \#\text{ of samples where } \mathrm{A}_{\tilde{t}_n^{(i),(l)}} \neq \mathrm{A}_{\tilde{t}_{n+1}^{(i),(l)}} = \mathrm{A}_{\tilde{t}_{n+2}^{(i),(l)}}$$

$$\mathrm{I} \triangleq \#\text{ of samples where } \mathrm{A}_{\tilde{t}_n^{(i),(l)}} \neq \mathrm{A}_{\tilde{t}_{n+1}^{(i),(l)}} \neq \mathrm{A}_{\tilde{t}_{n+2}^{(i),(l)}}$$

Then, with weak Beta priors for the stability,

$$\hat{\boldsymbol{\alpha}}_i \triangleq \frac{\mathbf{A}+\mathbf{1}}{\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{3}}, \qquad \hat{\boldsymbol{\beta}}_i \triangleq \frac{\mathbf{B}+\mathbf{1}}{\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{3}}$$

$$\hat{\boldsymbol{\phi}}_i \triangleq \frac{\sum_{l=1}^{\mathbf{L}^{(i)}} \tilde{\boldsymbol{t}}_1^{(i),(l)} - \mathbf{L}^{(i)} + \mathbf{1}}{\sum_{l=1}^{\mathbf{L}^{(i)}} \tilde{\boldsymbol{t}}_1^{(i),(l)} - \mathbf{L}^{(i)} + \mathbf{D}+\mathbf{E}+\mathbf{3}}, \qquad \hat{\boldsymbol{\psi}}_i \triangleq \frac{\mathbf{D}+\mathbf{1}}{\sum_{l=1}^{\mathbf{L}^{(i)}} \tilde{\boldsymbol{t}}_1^{(i),(l)} - \mathbf{L}^{(i)} + \mathbf{D}+\mathbf{E}+\mathbf{3}}$$

$$\hat{\boldsymbol{\eta}}_i \triangleq \frac{\mathbf{I}+\mathbf{1}}{\mathbf{I}+\mathbf{H}+\mathbf{2}}, \qquad \hat{\boldsymbol{\varepsilon}}_i \triangleq \frac{\mathbf{F}+\mathbf{1}}{\mathbf{F}+\mathbf{G}+\mathbf{2}}$$

$$\hat{\mathbf{R}}_i \triangleq \frac{\mathbf{1}}{\mathbf{L}^{(i)}} \sum_{l=1}^{\mathbf{L}^{(i)}} \frac{\mathbf{A}_{\tilde{t}_{\mathrm{N}_i}^{(i),(l)}} - \mathbf{A}_{\tilde{t}_1^{(i),(l)}}}{d_{\mathrm{N}_i} - d_1}, \qquad \hat{\boldsymbol{\tau}}_i \triangleq \frac{\sum_{l=1}^{\mathbf{L}^{(i)}} \sum_{n=1}^{\mathbf{N}_i} \left(\boldsymbol{v}_n^{(i)} - \tilde{\boldsymbol{\mu}}_{\tilde{t}_n^{(i),(l)}}\right)}{\mathbf{L}^{(i)} \mathbf{N}_i}$$

4. Updating stack parameters by ML estimators:

If we are only interested in aligning each core to the stack, it is not needed to read this section; if we are interested in constructing the new stack from the data as well, it is also required to learn means and standard deviations $\{\mu_t\}_{t=1}^{T}$ and $\{\sigma_t\}_{t=1}^{T}$. An EM algorithm can learn both the alignments and stack parameters simultaneously: to be specific, it first aligns each core to the stack by generating samples, and then updates the stack parameters by using samples.

$$\widehat{\boldsymbol{\mu}}_t \triangleq \begin{cases} \dfrac{\sum_{i=1}^{K}\sum_{l=1}^{L^{(i)}}\sum_{n=1}^{N_i}\left[\left(\boldsymbol{v}_n^{(i)}-\hat{\boldsymbol{\tau}}_i\right)\mathbf{1}_{\left\{\tilde{t}_n^{(i),(l)}=t\right\}}\right]}{\sum_{i=1}^{K}\sum_{l=1}^{L^{(i)}}\sum_{n=1}^{N_i}\mathbf{1}_{\left\{\tilde{t}_n^{(i),(l)}=t\right\}}}, & \displaystyle\sum_{i=1}^{K}\sum_{l=1}^{L^{(i)}}\sum_{n=1}^{N_i}\mathbf{1}_{\left\{\tilde{t}_n^{(i),(l)}=t\right\}} > 0 \\ \boldsymbol{\mu}_t, & \textbf{otherwise} \end{cases}$$

$$\widehat{\boldsymbol{\sigma}}_t \triangleq \begin{cases} \sqrt{\dfrac{\sum_{i=1}^{K}\sum_{l=1}^{L^{(i)}}\sum_{n=1}^{N_i}\left[\left(\boldsymbol{v}_n^{(i)}-\hat{\boldsymbol{\tau}}_i-\widehat{\boldsymbol{\mu}}_t\right)^2\mathbf{1}_{\left\{\tilde{t}_n^{(i),(l)}=t\right\}}\right]}{\sum_{i=1}^{K}\sum_{l=1}^{L^{(i)}}\sum_{n=1}^{N_i}\mathbf{1}_{\left\{\tilde{t}_n^{(i),(l)}=t\right\}}}}, & \displaystyle\sum_{i=1}^{K}\sum_{l=1}^{L^{(i)}}\sum_{n=1}^{N_i}\mathbf{1}_{\left\{\tilde{t}_n^{(i),(l)}=t\right\}} > 0 \\ \boldsymbol{\sigma}_t, & \textbf{otherwise} \end{cases}$$

, where $\{\tilde{\mu}_t\}_{t=1}^{T}$ are the updated means in the previous iteration.

Note that EM algorithms are quite sensitive to the initial conditions, which means that it is, in general, not possible to see good inference if we do not use "proper" initializations. One suggestion to deal with this defect is initializing stack parameters by using any existing stacks (for example, LR04) and core parameters by using "proper" values, for example, initializing $1 \approx \eta_i \gg \varepsilon_i \approx 0$, R_i by initial guess on the range of ages of the core, and τ_i by the difference between mean of $v_n^{(i)}$ and $\hat{\mu}_t$. Then, EM algorithms will "refine" our initial guesses on the parameters to the "real" ones iteratively from the given data.