Bitswap Analysis

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TODO: Ensure periods at the end of all bullet points/lists are consistent

TODO: Figure out cleaner (more maintainable) solution to math mode spacing

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In this paper, we analyze 3 strategies for a simple 2-player Bitswap infinitely repeated game. We start by defining the system in the most general case, then do an analysis on a system subject to simplifying constraints.

Bitswap is the data exchange protocol for the InterPlanetary File System (IPFS). Our model is meant to reflect this use case of Bitswap as the decision engine implemented by each user in a peer-to-peer distributed file system. In this distributed file system of many users, each user is connect to a set of peers that they trade data with. Every peer has a reputation with every other peer – in other words, for every peer a user has, that user maintains a summary of their interactions with that peer. Then, when deciding how to allocate their resources among their peers at a given time, the user uses these aggregate reputations to provide weights to each of their peers. For example, consider a network of 3 peers, labeled 1, 2, and 3. If peer 2 sends twice as much data to peer 1 as peer 3 sends to peer 1 from time 0 to $t-1$ (and peer 1 sends the same amount of data to both 2 and 3 over that time), then peer 1 might allocate $\frac{2}{3}$ of its bandwidth to peer 2 and $\frac{1}{3}$ to peer 3 at time t.

TODO: necessary to explicitly mention strategies here? trying to stay informal, but might still be a good idea

System

We have a network $\mathcal{N}$ of $|\mathcal{N}|$ users. The terms *users* and *players* will be used somewhat interchangeably, depending on context; the term *peers* is used similarly, but primarily refers to users who are connected (and thus participate as players in the same Bitswap game). Each of the users has a neighborhood of peers, which is the set of users they are connected to. Each pair of peers plays an infinitely repeated Bitswap game. The resource that users have to offer to their peers is bandwidth. We make the following simplifying assumptions about user's bandwidth:

1. All users have the same amount of bandwidth to offer.
2. A single user has the same amount of bandwidth to offer at each time step.

In other words, bandwidth is constant both in peer-space and in time. **TODO**: worth saying it this way, or is 'peer-space' confusing?

We also make the assumption that *all users always have unique data that all of their peers want*. So, whenever a peer plays R, they'll always have some data to send to all of their peers. This, of course, contrasts a more realistic scenario where a peer chooses to reciprocate but simply does not have anything that their peers want at the current time.

Actions and Utility Functions

TODO: ensure lower bound for t is consistently 0 (and not typo'd as 1)

A player has two possible actions: reciprocate (R) or defect (D). The utility function for player i at time t u_i^t:

$$u_i^t = \sum_{j \in \mathcal{N}_i} \delta_{a_j^t R} \; S_j(d_{ji}^t \, , \, \mathbf{d}_j^{-i,t}) \; B \; - \; \delta_{a_i^t R} \; B$$

where

- $\mathcal{N}_i \subseteq \mathcal{N}$ is the neighborhood of user i (i.e. the set of peers i is connected to)
- $a_i^t \in \{R, D\}$ is the action user i takes in round t
- δij is the kronecker delta function
- d_{ji}^t is the reputation of user i as viewed by peer j (also referred to as the *debt ratio* from i to j) in round t
- $\mathbf{d}_j^{-i,t} = (d_{jk}^t \mid \forall k \in \mathcal{N}_j, k \neq i)$ is the vector of debt ratios for all of user j's peers (as viewed by peer j) in round t, *excluding* peer i
- $S_j(d_{ij}^t, \mathbf{d}_j^{-i,t}) \in \{0, 1\}$ is the *strategy function* of user j. This function considers the relative reputation of peer i to the rest of j's peers, and returns a weight for peer i. This weight is used to determine what proportion of j's bandwidth to allocate to peer i in round t.
- $B > 0$ is the (constant) amount of bandwidth that a user has to offer in a given round

The terms *strategy* and *strategy function* are defined as:

- A *strategy* is meant in the standard game-theoretical sense, which is a predetermined set of actions that a user will take in a game (potentially dependent on that user's previous payoffs, the actions of its peers, etc.).

- A *strategy function* is a term used to specify the weighting function that a user uses when running the Bitswap protocol to determine how much bandwidth it wants to allocate to each of its peers whenever it's playing the R strategy.

Putting this all together, we see that the utility of peer i in round t is the total amount of bandwidth that i is allocated by its neighboring peers, minus the amount of bandwidth that i provides to its peers. If i reciprocates, then we say that they provide a total of B bandwidth to the network; otherwise (when i defects), i provides 0 bandwidth in that round.

We can write the debt ratio d_{ij} in terms of the number of bits exchanged between peers i and j:

$$d_{ji}^{t} = \frac{b_{ji}^{t-1}}{b_{ij}^{t-1} \; + \; 1}$$

where b_{ij}^{t-1} is the total number of bits sent from i to j from round 0 through round $t-1$ (so, all rounds prior to round t).

We can define b_{ij}^{t} in terms of b_{ij}^{t} and $\delta_{a_i^t R}$ as follows:

$$b_{ij}^{t} = b_{ij}^{t-1} \; + \; \delta_{a_i^{t-1} R} \; S_i(d_{ij}^{t} \, , \, \mathbf{d}_i^{-j,t}) \; B$$

So, the total number of bits sent from i to j increases by $S_i(d_{ij}^t, \mathbf{d}_i^{-j,t})B$ (the proportion of i's total bandwidth that i allocates to j) if and only if peer i reciprocated in round $t-1$ (i.e., $a_i^{t-1} = R \implies \delta_{a_i^{t-1} R} = 1$).

Now we can write d_{ij}^{t+1} in terms of values from round t.

$$d_{ij}^{t+1} = \frac{b_{ij}^{t} \; + \; \delta_{a_i^t R} \; S_i(d_{ij}^{t} \, , \, \mathbf{d}_i^{-j,t}) \; B}{b_{ji}^{t} \; + \; \delta_{a_j^t R} \; S_j(d_{ji}^{t} \, , \, \mathbf{d}_j^{-i,t}) \; B \; + \; 1}$$

Analysis

For the purposes of this analysis, we make an additional assumption: each user's neighborhood is constant – so any given pair of peers is connected for the entire repeated game. This means that the network topology is static as well.

We now consider consider a specific strategy function that user j uses to weight some peer i:

$$S_j(d_{ji}^{t} \, , \, \mathbf{d}_j^{-i,t}) = \frac{d_{ji}^{t}}{\sum_{k \in \mathcal{N}_j} d_{jk}^{t}}$$

$$u_i^t = \sum_{j \in \mathcal{N}_i} \frac{\delta_{a_j^t R}\ d_{ji}^t}{\sum_{k \in \mathcal{N}_j} d_{jk}^t}\ B\ -\ \delta_{a_i^t R}\ B$$

As this is an infinitely repeated game, we want to be able to calculate the discounted average payoff of player i, U_i.

$$U_i = (1 - \epsilon_i) \sum_{t=1}^{\infty} \epsilon_i^{t-1}\ u_i^t(\mathbf{a}^t)$$

where

- $\epsilon_i \in (0, 1)$ is the *discount factor*, which characterizes how much player i cares about their payoffs in future rounds relative to the current round.
- $\mathbf{a}^t = (a_i^t \mid \forall i \in (1, |\mathcal{N}|))$ is the vector of each player's actions in round t.

Further, rather than d_{ij}^t being an aggregate value over all rounds in $[0, t)$, it will only take the immediately preceding round into account. In other words:

$$b_{ij}^t = \delta_{a_i^t R}\ S_i(d_{ij}^t\ ,\ \mathbf{d}_i^{-j,t})\ B$$

We now consider 3 strategies: tit-for-tat, pavlov, and grim-trigger. For each of these, we'll determine whether the strategy is an subgame-perfect Nash equilibrium (SPNE) for the 2-player infinitely repeated game. While it suffices to show a single case of initial conditions that prove a strategy is not an SPNE, all cases will be shown as they may prove useful in understanding the more complex scenarios. (**TODO**: get others' opinions on whether wording in first half of previous sentence is clear) (**TODO**: be sure to follow up on second half of previous sentence later in analysis)

Note that since each user only has a single peer (the other user), when a player plays R they allocate all of their bandwidth to the other user. (**TODO**: find best place for this statement)

To determine whether a given strategy is an SPNE for the 2-player infinitely repeated (simplified) Bitswap game, we will do the following:

- Consider an initial pair of actions at $t = 0$, (a_1^0, a_2^0).
- Assume player 1 plays the strategy for all rounds.
- Consider two cases:
    1. Player 2 never deviates from the strategy.
    2. Player 2 deviates from the strategy for a single round, at $t = 1$, then plays the strategy in all future rounds.
- Calculate player 2's payoff for the infinitely-repeated game in both cases. If player 2's payoff in case 2 (U_2') is less than or equal to their payoff in case 1 (U_2) for all initial pairs of actions, then the strategy is an SPNE.

Mathematically, a strategy is an SPNE if and only if $U_2' \leq U_2$ for all possible initial actions.

Tit-for-Tat

We start by analyzing the well-studied tit-for-tat (TFT) strategy. A player that uses this strategy always takes the strategy that their peer took in the previous round. So, if player 1 plays action R (D) in round t, then player 2 will play action R in round $t+1$ (D), and vice-versa.

Let's first consider case where the initial pair of actions is (D, D).

Case 1: (D, D)

Here, both players start by playing the D strategy. We'll first consider the case where player 2 does not deviate from TFT. The strategies at each round then follow:

t	0	1	2	3	4	...
a_1^t	D	D	D	D	D	...
a_2^t	D	D	D	D	D	...

Since neither player deviates from TFT, they both continually play their opponent's previous strategy – the initial state is (D, D), so each player repeatedly plays D in this instance.

We can calculate the payoff of player 2 in this case – notice that, since neither player is ever giving or receiving, they payoff at each round is 0.

$$u_2^t = 0 \; \forall \; t \implies U_2 = 0$$

Now we consider this case where player 2 deviates from TFT for 1 round, at $t = 1$. The resulting action sequence is then:

t	0	1	2	3	4	...
a_1^t	D	D	R	D	R	...
a_2^t	D	R	D	R	D	...

In this case, player 2 deviates from TFT for a single round and plays R at $t = 1$ rather than D, then goes back to playing TFT for all rounds after that. Player 1, on the other hand, never deviates from TFT.

We can calculate player 2's discounted average payoff in this second case, U_2'. Note that we consider payoffs starting at $t = 1$, not $t = 0$ – this is because we're calculating the payoff that player 2 would perceive if they deviated from TFT for a single round, and that decision is happening at $t = 1$ in our case. (**TODO**: work on the explanation in this last sentence)

At $t = 1$, player 1 defects and player 2 reciprocates – since there are only two players (**TODO**: any other conditions, e.g. lookbehind length?), player 2 provides all of its bandwidth to player 1 when it plays R. Thus, player 1's payoff is $u_1^1 = B$ and player 2's payoff at $t = 1$ is $u_2^1 = -B$.

At $t = 2$, the players swap strategies (because they're back to playing TFT). So $u_1^2 = -B$ and $u_2^2 = B$.

The game alternates between these two states forever. Thus, the payoff for player 2 in this case is:

$$
\begin{aligned}
U_2' &= -B + \epsilon_2 B - \epsilon_2^2 B + \epsilon_2^3 B - \ldots \\
&= B(-(1 + \epsilon_2^2 + \epsilon_2^4 + \ldots) + \epsilon(1 + \epsilon_2^2 + \epsilon_2^4 + \ldots)) \\
&= B(\epsilon - 1)\frac{1}{1 - \epsilon_2^2} \\
&= -\frac{B}{1 + \epsilon_2}
\end{aligned}
$$

Given U_2 and U_2', we can start to discern whether TFT might be an SPNE for this game. We see that $U_2' < U_2$ – this means that TFT *might* be an SPNE, but we have to verify that this is the case for all other initial conditions as well.

For the rest of the cases, we simply show the action sequences and the discounted average payoff results

Case 2: (D, R)

When player 2 doesn't deviate from TFT:

t	0	1	2	3	4	...
a_1^t	D	R	D	R	D	...
a_2^t	R	D	R	D	R	...

Thus,

$$
\begin{aligned}
U_2 &= B - \epsilon_2 B + \epsilon_2^2 B - \dots \\
&= \frac{B}{1+\epsilon_2}
\end{aligned}
$$

When player 2 does deviate from TFT:

t	0	1	2	3	4	...
a_1^t	D	R	R	R	R	...
a_2^t	R	R	R	R	R	...

Both players give and receive B bandwidth in all rounds $t \in [0, \infty)$, so:

$$
\begin{aligned}
U_2' &= 0 - \epsilon_2 0 + \epsilon_2^2 0 - \dots \\
&= 0
\end{aligned}
$$

Thus, $U_2' < U_2$ in this case.

Case 3: (R, D)

When player 2 doesn't deviate from TFT:

t	0	1	2	3	4	...
a_1^t	R	D	R	D	R	...
a_2^t	D	R	D	R	D	...

Thus,

$$
\begin{aligned}
U_2 &= -B + \epsilon_2 B - \epsilon_2^2 B + \dots \\
&= -\frac{B}{1+\epsilon_2}
\end{aligned}
$$

When player 2 does deviate from TFT:

t	0	1	2	3	4	...
a_1^t	R	D	D	D	D	...
a_2^t	D	D	D	D	D	...

$$\begin{aligned} U_2' &= 0 - \epsilon_2 0 + \epsilon_2^2 0 - \dots \\ &= 0 \end{aligned}$$

We get a differently result in this case, namely $U_2' > U_2$. Therefore, **TFT is not an SPNE for this game**.

Case 4: (R, R)

We now already proven that TFT is not an SPNE. This case gives that result as well.

When player 2 doesn't deviate from TFT:

t	0	1	2	3	4	...
a_1^t	R	R	R	R	R	...
a_2^t	R	R	R	R	R	...

Thus,

$$U_2 = 0$$

When player 2 does deviate from TFT:

t	0	1	2	3	4	...
a_1^t	R	R	D	R	D	...
a_2^t	R	D	R	D	R	...

Thus,

$$U_2' = \frac{B}{1 + \epsilon_2}$$

We again get the result $U_2' > U_2$, which also indicates that TFT is not an SPNE.

Grim-Trigger

Now we consider the well-studied grim trigger (GT) strategy. A player that uses this strategy plays D in all rounds where their peer has previously played D; otherwise, the player plays R. Formally, for the 2-player game, this strategy is characterized by

$$a_i^t = \begin{cases} D & \text{if } D \in (a_j^0, \ldots, a_j^{t-1}) \\ R & \text{otherwise} \end{cases}$$

where $i, j \in \{1, 2\}$ and $i \neq j$.

Case 1: (D, D)

In this first case, both players play D in round $t = 0$. As both users are playing GT, each user defects for all succeeding rounds since their peer played D at some point in the past. Thus, the resulting strategy sequence is:

t	0	1	2	3	4	...
a_1^t	D	D	D	D	D	...
a_2^t	D	D	D	D	D	...

Thus,

$$U_2 = 0$$

When player 2 deviates from GT:

t	0	1	2	3	4	...
a_1^t	D	D	D	D	D	...
a_2^t	D	R	D	D	D	...

In this case, player 1's strategy is still D for all rounds, since 2 played D at $t = 0$. So player 2 ends up giving up B bandwidth at $t = 1$ and never getting anything back, giving:

$$U_2' = -B$$

Thus, $U_2' > U_2$ in this case. Therefore, **GT is not an SPNE for this game**.

Case 2: (D, R)

When player 2 does not deviate from GT:

t	0	1	2	3	4	...
a_1^t	D	R	D	D	D	...
a_2^t	R	D	D	D	D	...

Thus,

$$U_2 = B$$

When player 2 deviates from GT:

t	0	1	2	3	4	...
a_1^t	D	R	R	D	D	...
a_2^t	R	R	D	D	D	...

Thus,

$$\begin{aligned} U_2' &= B - B + \epsilon_2 B \\ &= \epsilon_2 B \end{aligned}$$

In this case, $U_2' < U_2$.

Case 3: (R, D)

When player 2 does not deviate from GT:

t	0	1	2	3	4	...
a_1^t	R	D	D	D	D	...
a_2^t	D	R	D	D	D	...

Thus,

$$U_2 = -B$$

When player 2 deviates from GT:

t	0	1	2	3	4	...
a_1^t	R	D	D	D	D	...
a_2^t	D	D	D	D	D	...

Thus,

$$U_2' = 0$$

In this case, $U_2' < U_2$.

Case 4: (R, R)

When player 2 does not deviate from GT:

t	0	1	2	3	4	...
a_1^t	R	R	R	R	R	...
a_2^t	R	R	R	R	R	...

Thus,

$$U_2 = 0$$

When player 2 deviates from GT:

t	0	1	2	3	4	...
a_1^t	R	R	D	D	D	...
a_2^t	R	D	R	D	D	...

Thus,

$$\begin{aligned} U_2' &= B - \epsilon_2 B \\ &= (1 - \epsilon_2)B \end{aligned}$$

In this case, $U_2' > U_2$, which would be proof that GT is not an SPNE for this game had we not already proven that fact.

Pavlov

Case 1: (D, D)

Case 2: (D, R)

Case 3: (R, D)

Case 4: (R, R)

Discussion

TODO: compare results to that of prisoner's dilemma

TODO: discuss next steps (more peers)