A trip to Asymptopia

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Asymptotics

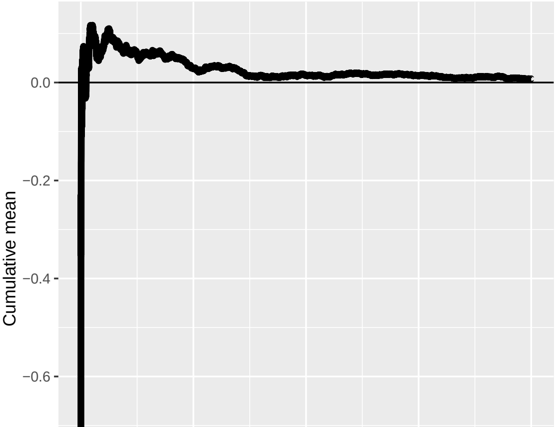
- Asymptotics is the term for the behavior of statistics as the sample size (or some other relevant quantity) limits to infinity (or some other relevant number)
- (Asymptopia is my name for the land of asymptotics, where everything works out well and there's no messes. The land of infinite data is nice that way.)
- Asymptotics are incredibly useful for simple statistical inference and approximations
- (Not covered in this class) Asymptotics often lead to nice understanding of procedures
- Asymptotics generally give no assurances about finite sample performance
- Asymptotics form the basis for frequency interpretation of probabilities (the long run proportion of times an event occurs)

Limits of random variables

- Fortunately, for the sample mean there's a set of powerful results
- These results allow us to talk about the large sample distribution of sample means of a collection of *iid* observations
- The first of these results we intuitively know
- It says that the average limits to what it's estimating, the population mean
- It's called the Law of Large Numbers
- Example $\bar{X}_n$ could be the average of the result of n coin flips (i.e. the sample proportion of heads)
  - As we flip a fair coin over and over, it eventually converges to the true probability of a head The LLN forms the basis of frequency style thinking

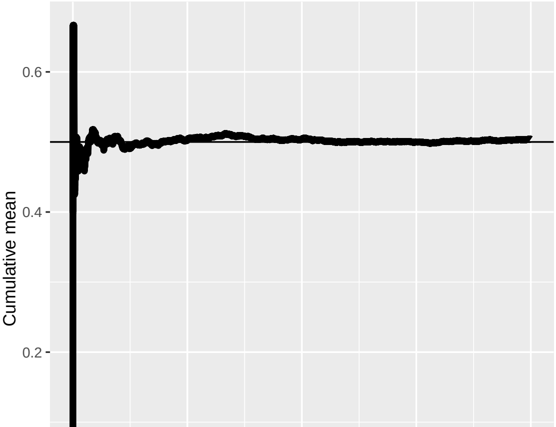
Law of large numbers in action

```
n <- 10000; means <- cumsum(rnorm(n)) / (1  : n); library(g
g <- ggplot(data.frame(x = 1 : n, y = means), aes(x = x, y
g <- g + geom_hline(yintercept = 0) + geom_line(size = 2)
g <- g + labs(x = "Number of obs", y = "Cumulative mean")
g
```



Law of large numbers in action, coin flip

```
means <- cumsum(sample(0 : 1, n , replace = TRUE)) / (1  :
g <- ggplot(data.frame(x = 1 : n, y = means), aes(x = x, y
g <- g + geom_hline(yintercept = 0.5) + geom_line(size = 2)
g <- g + labs(x = "Number of obs", y = "Cumulative mean")
g
```



Discussion

- An estimator is **consistent** if it converges to what you want to estimate
- The LLN says that the sample mean of iid sample is consistent for the population mean
- Typically, good estimators are consistent; it's not too much to ask that if we go to the trouble of collecting an infinite amount of data that we get the right answer
- The sample variance and the sample standard deviation of iid random variables are consistent as well

The Central Limit Theorem

- The **Central Limit Theorem** (CLT) is one of the most important theorems in statistics
- For our purposes, the CLT states that the distribution of averages of iid variables (properly normalized) becomes that of a standard normal as the sample size increases
- The CLT applies in an endless variety of settings
- The result is that

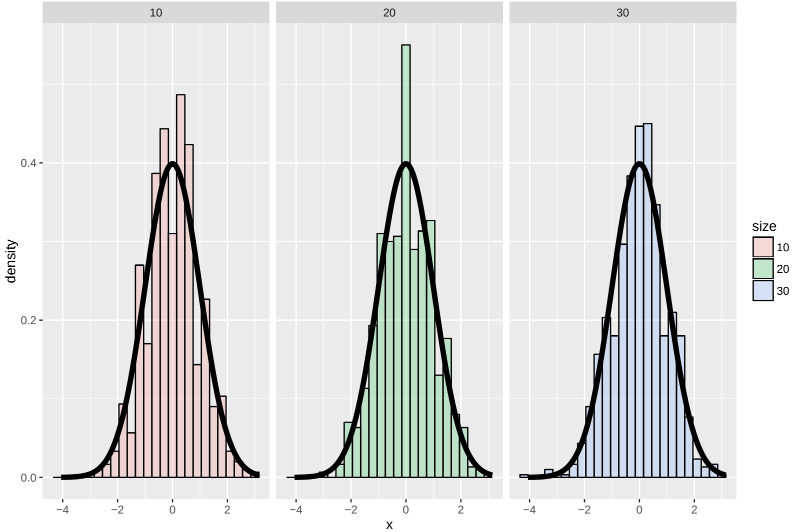
  $$\frac{\bar X_n - \mu}{\sigma/\sqrt{n}} = \frac{\sqrt{n}(\bar X_n - \mu)}{\sigma} = \frac{\text{Estimate} - \text{Mean of estimate}}{\text{Std. Err. of estimate}}$$

  has a distribution like that of a standard normal for large n.
- (Replacing the standard error by its estimated value doesn't change the CLT)
- The useful way to think about the CLT is that $\bar X_n$ is approximately $N(\mu, \sigma^2/n)$

Example

- Simulate a standard normal random variable by rolling n (six sided)
- Let X_i be the outcome for die i
- Then note that $\mu = E[X_i] = 3.5$
- $Var(X_i) = 2.92$
- SE $\sqrt{2.92/n} = 1.71/\sqrt{n}$
- Let's roll n dice, take their mean, subtract off 3.5, and divide by $1.71/\sqrt{n}$ and repeat this over and over

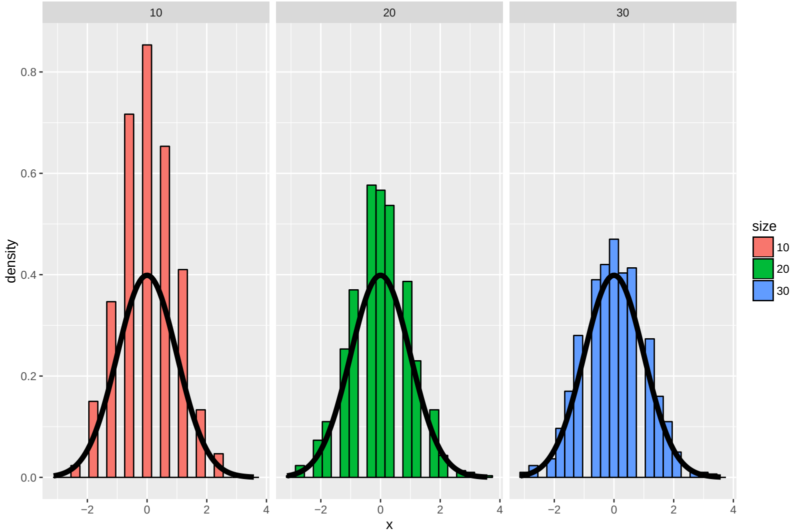
Result of our die rolling experiment



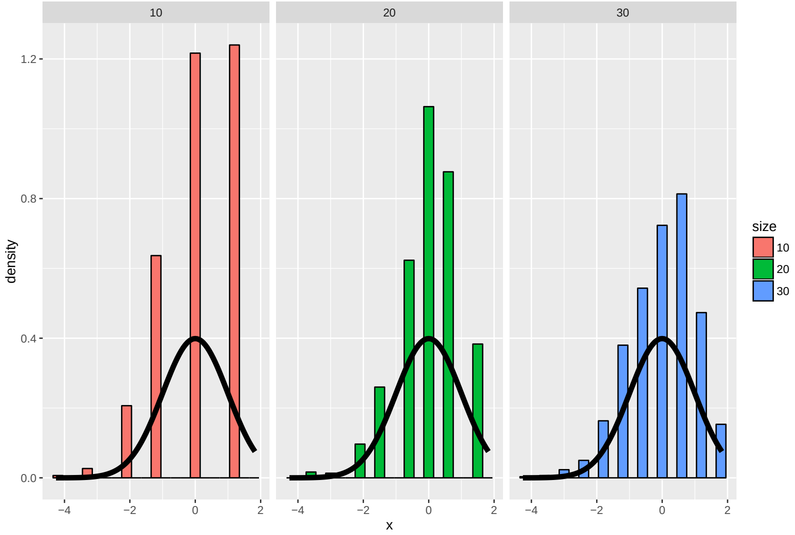
Coin CLT

- Let X_i be the 0 or 1 result of the i^{th} flip of a possibly unfair coin
- The sample proportion, say $\hat{p}$, is the average of the coin flips
- $E[X_i] = p$ and $Var(X_i) = p(1-p)$
- Standard error of the mean is $\sqrt{p(1-p)/n}$
- Then
$$\frac{\hat{p} - p}{\sqrt{p(1-p)/n}}$$
will be approximately normally distributed
- Let's flip a coin n times, take the sample proportion of heads, subtract off .5 and multiply the result by $2\sqrt{n}$ (divide by $1/(2\sqrt{n})$)

Simulation results



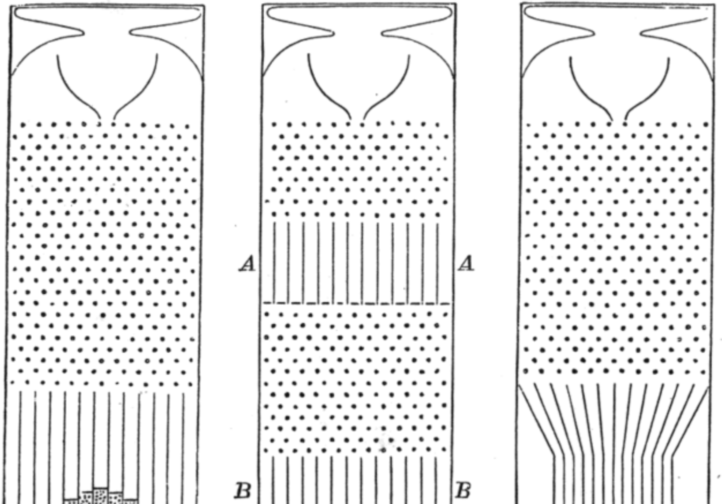
Simulation results, $p = 0.9$



Galton's quincunx

http://en.wikipedia.org/wiki/Bean_machine#mediaviewer/File:Quincunx_(Galton_Box)_-_Galton_1889_diagram.png

FIG.7. FIG.8. FIG.9.



Confidence intervals

- According to the CLT, the sample mean, $\bar{X}$, is approximately normal with mean μ and sd $\sigma / \sqrt{n}$
- $\mu + 2\sigma/\sqrt{n}$ is pretty far out in the tail (only 2.5% of a normal being larger than 2 sds in the tail)
- Similarly, $\mu - 2\sigma/\sqrt{n}$ is pretty far in the left tail (only 2.5% chance of a normal being smaller than 2 sds in the tail)
- So the probability $\bar{X}$ is bigger than $\mu + 2\sigma/\sqrt{n}$ or smaller than $\mu - 2\sigma/\sqrt{n}$ is 5%
  - Or equivalently, the probability of being between these limits is 95%
- The quantity $\bar{X} \pm 2\sigma/\sqrt{n}$ is called a 95% interval for μ
- The 95% refers to the fact that if one were to repeatedly get samples of size n, about 95% of the intervals obtained would contain μ
- The 97.5th quantile is 1.96 (so I rounded to 2 above)
- 90% interval you want (100 - 90) / 2 = 5% in each tail
- So you want the 95th percentile (1.645)

Give a confidence interval for the average height of sons

in Galton's data

```
library(UsingR);data(father.son); x <- father.son$sheight
```

```
## Loading required package: MASS

## Loading required package: HistData

## Loading required package: Hmisc

## Loading required package: lattice

## Loading required package: survival

## Loading required package: Formula

##
## Attaching package: 'Hmisc'

## The following objects are masked from 'package:base':
```

Sample proportions

- In the event that each X_i is 0 or 1 with common success probability p then $\sigma^2 = p(1-p)$
- The interval takes the form

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}}$$

- Replacing p by $\hat{p}$ in the standard error results in what is called a Wald confidence interval for p
- For 95% intervals

$$\hat{p} \pm \frac{1}{\sqrt{n}}$$

  is a quick CI estimate for p

Example

- Your campaign advisor told you that in a random sample of 100 likely voters, 56 intent to vote for you.
- Can you relax? Do you have this race in the bag?
- Without access to a computer or calculator, how precise is this estimate?
- `1/sqrt(100)=0.1` so a back of the envelope calculation gives an approximate 95% interval of `(0.46, 0.66)`
- Not enough for you to relax, better go do more campaigning!
- Rough guidelines, 100 for 1 decimal place, 10,000 for 2, 1,000,000 for 3.

```
round(1 / sqrt(10 ^ (1 : 6)), 3)
```

```
## [1] 0.316 0.100 0.032 0.010 0.003 0.001
```

Binomial interval

```
.56 + c(-1, 1) * qnorm(.975) * sqrt(.56 * .44 / 100)
```

```
## [1] 0.4627099 0.6572901
```

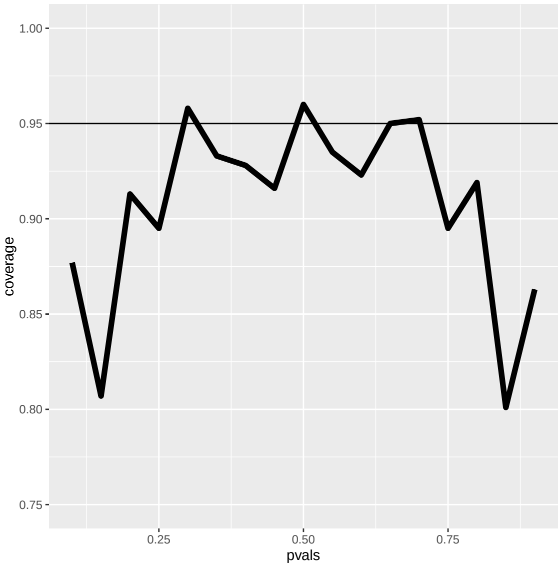
```
binom.test(56, 100)$conf.int
```

```
## [1] 0.4571875 0.6591640
## attr(,"conf.level")
## [1] 0.95
```

Simulation

```
n <- 20; pvals <- seq(.1, .9, by = .05); nosim <- 1000
coverage <- sapply(pvals, function(p){
  phats <- rbinom(nosim, prob = p, size = n) / n
  ll <- phats - qnorm(.975) * sqrt(phats * (1 - phats) / n)
  ul <- phats + qnorm(.975) * sqrt(phats * (1 - phats) / n)
  mean(ll < p & ul > p)
})
```

Plot of the results (not so good)



What's happening?

- n isn't large enough for the CLT to be applicable for many of the values of p
- Quick fix, form the interval with

$$\frac{X+2}{n+4}$$

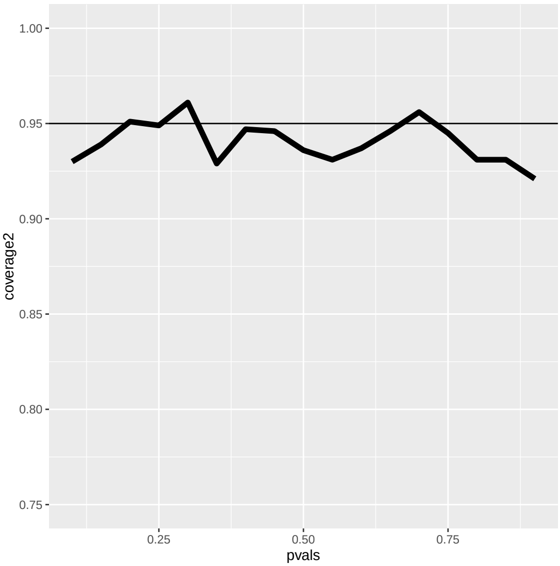
- (Add two successes and failures, Agresti/Coull interval)

Simulation

First let's show that coverage gets better with n

```
n <- 100; pvals <- seq(.1, .9, by = .05); nosim <- 1000
coverage2 <- sapply(pvals, function(p){
  phats <- rbinom(nosim, prob = p, size = n) / n
  ll <- phats - qnorm(.975) * sqrt(phats * (1 - phats) / n)
  ul <- phats + qnorm(.975) * sqrt(phats * (1 - phats) / n)
  mean(ll < p & ul > p)
})
```

Plot of coverage for $n = 100$



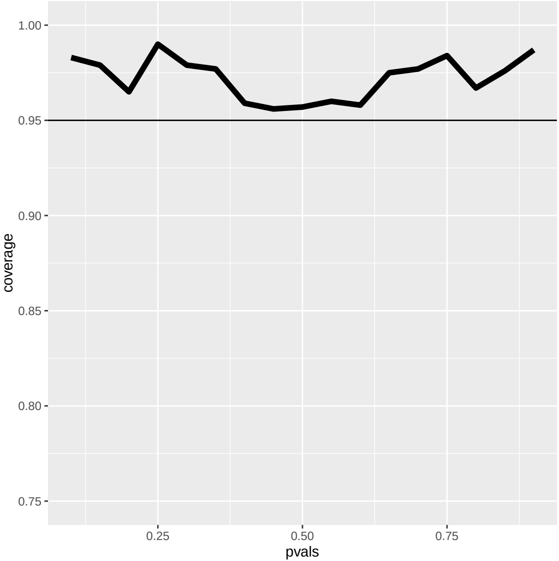
Simulation

Now let's look at $n = 20$ but adding 2 successes and failures

```
n <- 20; pvals <- seq(.1, .9, by = .05); nosim <- 1000
coverage <- sapply(pvals, function(p){
  phats <- (rbinom(nosim, prob = p, size = n) + 2) / (n + 4
  ll <- phats - qnorm(.975) * sqrt(phats * (1 - phats) / n)
  ul <- phats + qnorm(.975) * sqrt(phats * (1 - phats) / n)
  mean(ll < p & ul > p)
})
```

Adding 2 successes and 2 failures

(It's a little conservative)



Poisson interval

- A nuclear pump failed 5 times out of 94.32 days, give a 95% confidence interval for the failure rate per day?
- $X \sim Poisson(\lambda t)$.
- Estimate $\hat{\lambda} = X/t$
- $Var(\hat{\lambda}) = \lambda/t$
- $\hat{\lambda}/t$ is our variance estimate

R code

```r
x <- 5; t <- 94.32; lambda <- x / t
round(lambda + c(-1, 1) * qnorm(.975) * sqrt(lambda / t), 3
```

```
## [1] 0.007 0.099
```

```r
poisson.test(x, T = 94.32)$conf
```

```
## [1] 0.01721254 0.12371005
## attr(,"conf.level")
## [1] 0.95
```

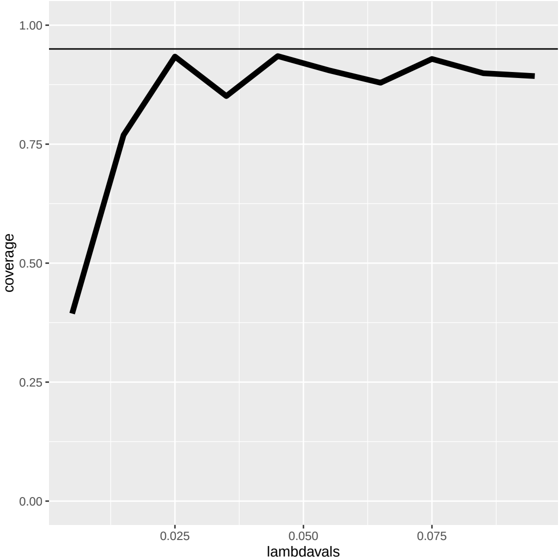
Simulating the Poisson coverage rate

Let's see how this interval performs for lambda values near what we're estimating

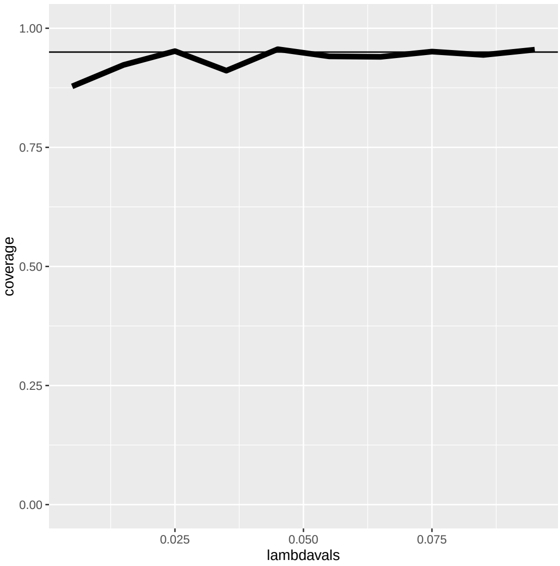
```
lambdavals <- seq(0.005, 0.10, by = .01); nosim <- 1000
t <- 100
coverage <- sapply(lambdavals, function(lambda){
  lhats <- rpois(nosim, lambda = lambda * t) / t
  ll <- lhats - qnorm(.975) * sqrt(lhats / t)
  ul <- lhats + qnorm(.975) * sqrt(lhats / t)
  mean(ll < lambda & ul > lambda)
})
```

Covarage

(Gets really bad for small values of lambda)



What if we increase t to 1000?



Summary

- The LLN states that averages of iid samples converge to the population means that they are estimating
- The CLT states that averages are approximately normal, with distributions
- centered at the population mean
- with standard deviation equal to the standard error of the mean
- CLT gives no guarantee that n is large enough
- Taking the mean and adding and subtracting the relevant normal quantile times the SE yields a confidence interval for the mean
- Adding and subtracting 2 SEs works for 95% intervals
- Confidence intervals get wider as the coverage increases (why?)
- Confidence intervals get narrower with less variability or larger sample sizes
- The Poisson and binomial case have exact intervals that don't require the CLT
- But a quick fix for small sample size binomial calculations is to