Some Common Distributions

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May 19, 2016

The Bernoulli distribution

- The **Bernoulli distribution** arises as the result of a binary outcome
- Bernoulli random variables take (only) the values 1 and 0 with probabilities of (say) p and $1 - p$ respectively
- The PMF for a Bernoulli random variable X is

$$P(X = x) = p^x(1 - p)^{1-x}$$

- The mean of a Bernoulli random variable is p and the variance is $p(1 - p)$
- If we let X be a Bernoulli random variable, it is typical to call $X = 1$ as a "success" and $X = 0$ as a "failure"

Binomial trials

- The *binomial random variables* are obtained as the sum of iid Bernoulli trials
- In specific, let $X_1, \ldots, X_n$ be iid Bernoulli(p); then $X = \sum_{i=1}^n X_i$ is a binomial random variable
- The binomial mass function is

$$P(X = x) = \left(\begin{array}{c} n \\ x \end{array} \right) p^x (1-p)^{n-x}$$

for $x = 0, \ldots, n$

Choose

- Recall that the notation
$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$
(read "n choose x") counts the number of ways of selecting x items out of n without replacement disregarding the order of the items

$$\binom{n}{0} = \binom{n}{n} = 1$$

Example

- Suppose a friend has 8 children (oh my!), 7 of which are girls and none are twins
- If each gender has an independent 50% probability for each birth, what's the probability of getting 7 or more girls out of 8 births?

$$\binom{8}{7} .5^7(1-.5)^1 + \binom{8}{8} .5^8(1-.5)^0 \approx 0.04$$

```
choose(8, 7) * .5 ^ 8 + choose(8, 8) * .5 ^ 8
```

```
## [1] 0.03515625
```

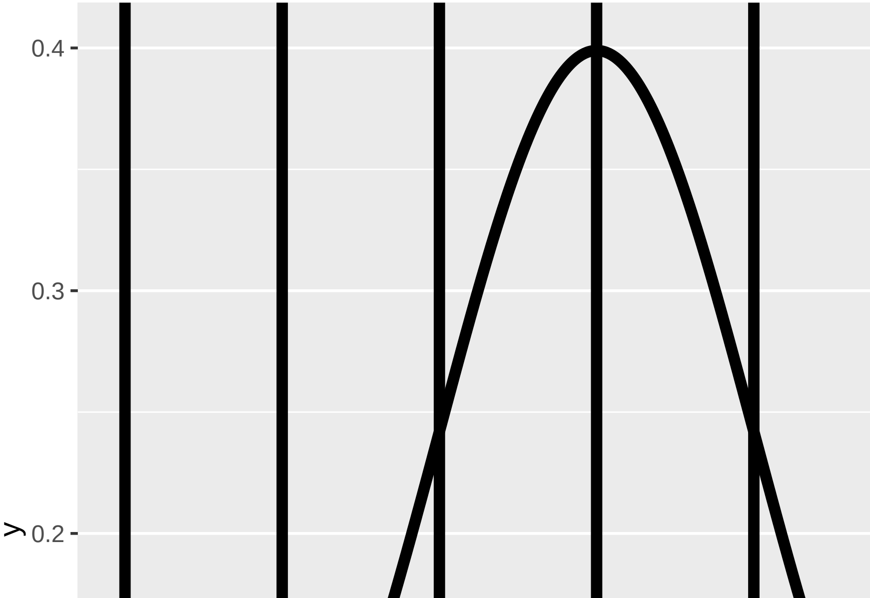
```
pbinom(6, size = 8, prob = .5, lower.tail = FALSE)
```

```
## [1] 0.03515625
```

The normal distribution

- A random variable is said to follow a **normal** or **Gaussian** distribution with mean μ and variance σ^2 if the associated density is
$$(2\pi\sigma^2)^{-1/2} e^{-(x-\mu)^2/2\sigma^2}$$
If X a RV with this density then $E[X] = \mu$ and $Var(X) = \sigma^2$
- We write $X \sim \mathrm{N}(\mu, \sigma^2)$
- When $\mu = 0$ and $\sigma = 1$ the resulting distribution is called **the standard normal distribution**
- Standard normal RVs are often labeled Z

The standard normal distribution with reference lines



Facts about the normal density

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

If Z is standard normal

$$X = \mu + \sigma Z \sim \mathrm{N}(\mu, \sigma^2)$$

More facts about the normal density

1. Approximately 68%, 95% and 99% of the normal density lies within 1, 2 and 3 standard deviations from the mean, respectively
2. -1.28, -1.645, -1.96 and -2.33 are the 10^{th}, 5^{th}, 2.5^{th} and 1^{st} percentiles of the standard normal distribution respectively
3. By symmetry, 1.28, 1.645, 1.96 and 2.33 are the 90^{th}, 95^{th}, 97.5^{th} and 99^{th} percentiles of the standard normal distribution respectively

Question

- What is the 95^{th} percentile of a $N(\mu, \sigma^2)$ distribution?
- Quick answer in R `qnorm(.95, mean = mu, sd = sd)`
- Or, because you have the standard normal quantiles memorized and you know that 1.645 is the 95th percentile you know that the answer has to be

$$\mu + \sigma 1.645$$

- (In general $\mu + \sigma z_0$ where z_0 is the appropriate standard normal quantile)

Question

- What is the probability that a $N(\mu, \sigma^2)$ RV is larger than x?

Example

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It's not very likely, 1,160 is 2.8 standard deviations from the mean

```r
pnorm(1160, mean = 1020, sd = 50, lower.tail = FALSE)
```

```
## [1] 0.00255513
```

```r
pnorm(2.8, lower.tail = FALSE)
```

```
## [1] 0.00255513
```

Example

Assume that the number of daily ad clicks for a company is (approximately) normally distributed with a mean of 1020 and a standard deviation of 50. What number of daily ad clicks would represent the one where 75% of days have fewer clicks (assuming days are independent and identically distributed)?

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```r
qnorm(0.75, mean = 1020, sd = 50)
```

```
## [1] 1053.724
```

The Poisson distribution

- Used to model counts
- The Poisson mass function is
$$P(X = x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$
for $x = 0, 1, \ldots$
- The mean of this distribution is λ
- The variance of this distribution is λ
- Notice that x ranges from 0 to ∞

Some uses for the Poisson distribution

- Modeling count data
- Modeling event-time or survival data
- Modeling contingency tables
- Approximating binomials when n is large and p is small

Rates and Poisson random variables

- Poisson random variables are used to model rates
- $X \sim Poisson(\lambda t)$ where
- $\lambda = E[X/t]$ is the expected count per unit of time
- t is the total monitoring time

Example

The number of people that show up at a bus stop is Poisson with a mean of 2.5 per hour.

If watching the bus stop for 4 hours, what is the probability that 3 or fewer people show up for the whole time?

```
ppois(3, lambda = 2.5 * 4)
```

```
## [1] 0.01033605
```

Poisson approximation to the binomial

- When n is large and p is small the Poisson distribution is an accurate approximation to the binomial distribution
- Notation
- $X \sim \text{Binomial}(n, p)$
- $\lambda = np$
- n gets large
- p gets small

Example, Poisson approximation to the binomial

We flip a coin with success probablity 0.01 five hundred times.

What's the probability of 2 or fewer successes?

```
pbinom(2, size = 500, prob = .01)
```

```
## [1] 0.1233858
```

```
ppois(2, lambda=500 * .01)
```

```
## [1] 0.124652
```