# A trip to Asymptopia

Brian Caffo, Jeff Leek, Roger Peng

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# Asymptotics

- Asymptotics is the term for the behavior of statistics as the sample size (or some other relevant quantity) limits to infinity (or some other relevant number)
- (Asymptopia is my name for the land of asymptotics, where everything works out well and there's no messes. The land of infinite data is nice that way.)
- Asymptotics are incredibly useful for simple statistical inference and approximations
- (Not covered in this class) Asymptotics often lead to nice understanding of procedures
- Asymptotics generally give no assurances about finite sample performance
- The kinds of asymptotics that do are orders of magnitude more difficult to work with
- Asymptotics form the basis for frequency interpretation of probabilities (the long run proportion of times an event occurs)
- To understand asymptotics, we need a very basic understanding of limits

# Numerical limits

- Imagine a sequence
- $a_1 = .9$,
- $a_2 = .99$,
- $a_3 = .999, \ldots$
- Clearly this sequence converges to 1
- Definition of a limit: For any fixed distance we can find a point in the sequence so that the sequence is closer to the limit than that distance from that point on

# Limits of random variables

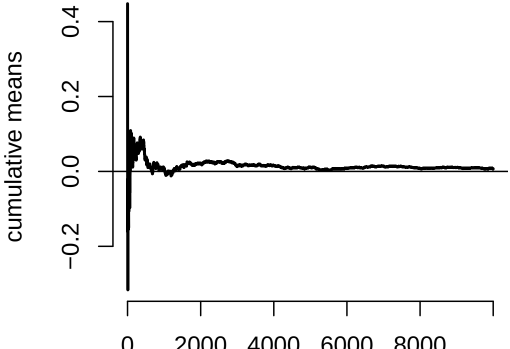
- The problem is harder for random variables
- Consider $\bar{X}_n$ the sample average of the first $n$ of a collection of *iid* observations
- Example $\bar{X}_n$ could be the average of the result of $n$ coin flips (i.e. the sample proportion of heads)
- We say that $\bar{X}_n$ converges in probability to a limit if for any fixed distance the probability of $\bar{X}_n$ being closer (further away) than that distance from the limit converges to one (zero)

# The Law of Large Numbers

- Establishing that a random sequence converges to a limit is hard
- Fortunately, we have a theorem that does all the work for us, called the **Law of Large Numbers**
- The law of large numbers states that if $X_1, \ldots X_n$ are iid from a population with mean $\mu$ and variance $\sigma^2$ then $\bar{X}_n$ converges in probability to $\mu$
- (There are many variations on the LLN; we are using a particularly lazy version, my favorite kind of version)

# Law of large numbers in action

```
n <- 10000; means <- cumsum(rnorm(n)) / (1  : n)
plot(1 : n, means, type = "l", lwd = 2,
     frame = FALSE, ylab = "cumulative means", xlab = "samp
abline(h = 0)
```



# The Central Limit Theorem

- The **Central Limit Theorem** (CLT) is one of the most important theorems in statistics
- For our purposes, the CLT states that the distribution of averages of iid variables, properly normalized, becomes that of a standard normal as the sample size increases
- The CLT applies in an endless variety of settings
- Let $X_1, \dots, X_n$ be a collection of iid random variables with mean $\mu$ and variance $\sigma^2$
- Let $\bar{X}_n$ be their sample average
- Then $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$ has a distribution like that of a standard normal for large $n$.
- Remember the form

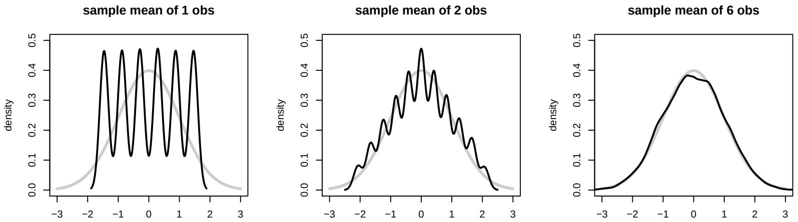
$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{\text{Estimate} - \text{Mean of estimate}}{\text{Std. Err. of estimate}}.$$

- Usually, replacing the standard error by its estimated value doesn't change the CLT

# Example

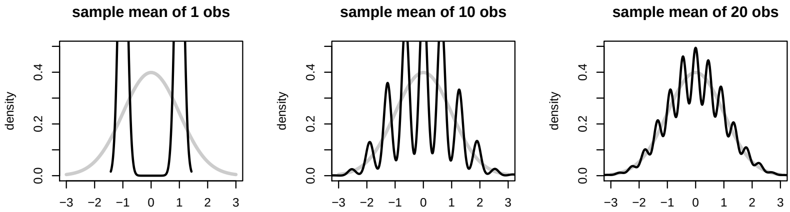
- Simulate a standard normal random variable by rolling $n$ (six sided)
- Let $X_i$ be the outcome for die $i$
- Then note that $\mu = E[X_i] = 3.5$
- $Var(X_i) = 2.92$
- SE $\sqrt{2.92/n} = 1.71/\sqrt{n}$
- Standardized mean
$$\frac{\bar X_n - 3.5}{1.71/\sqrt{n}}$$

# Simulation of mean of $n$ dice



# Coin CLT

- Let $X_i$ be the 0 or 1 result of the $i^{th}$ flip of a possibly unfair coin
- The sample proportion, say $\hat{p}$, is the average of the coin flips
- $E[X_i] = p$ and $Var(X_i) = p(1-p)$
- Standard error of the mean is $\sqrt{p(1-p)/n}$
- Then
$$\frac{\hat{p} - p}{\sqrt{p(1-p)/n}}$$
will be approximately normally distributed



# CLT in practice

- In practice the CLT is mostly useful as an approximation

$$P\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq z\right) \approx \Phi(z).$$

- Recall 1.96 is a good approximation to the $.975^{th}$ quantile of the standard normal
- Consider

$$\begin{aligned} .95 \approx & \quad P\left(-1.96 \leq \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq 1.96\right) \\ = & \quad P\left(\bar{X}_n + 1.96\sigma/\sqrt{n} \geq \mu \geq \bar{X}_n - 1.96\sigma/\sqrt{n}\right), \end{aligned}$$

# Confidence intervals

- Therefore, according to the CLT, the probability that the random interval
$$\bar{X}_n \pm z_{1-\alpha/2}\sigma/\sqrt{n}$$
contains $\mu$ is approximately $100(1-\alpha)\%$, where $z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution
- This is called a $100(1-\alpha)\%$ **confidence interval** for $\mu$
- We can replace the unknown $\sigma$ with $s$

## Give a confidence interval for the average height of sons

in Galton's data

```
library(UsingR);data(father.son); x <- father.son$sheight
```

```
## Loading required package: MASS

## Loading required package: HistData

## Loading required package: Hmisc

## Loading required package: lattice

## Loading required package: survival

## Loading required package: Formula

## Loading required package: ggplot2

##
## Attaching package: 'Hmisc'
```

# Sample proportions

- In the event that each $X_i$ is 0 or 1 with common success probability $p$ then $\sigma^2 = p(1-p)$
- The interval takes the form

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}}$$

- Replacing $p$ by $\hat{p}$ in the standard error results in what is called a Wald confidence interval for $p$
- Also note that $p(1-p) \leq 1/4$ for $0 \leq p \leq 1$
- Let $\alpha = .05$ so that $z_{1-\alpha/2} = 1.96 \approx 2$ then

$$2\sqrt{\frac{p(1-p)}{n}} \leq 2\sqrt{\frac{1}{4n}} = \frac{1}{\sqrt{n}}$$

- Therefore $\hat{p} \pm \frac{1}{\sqrt{n}}$ is a quick CI estimate for $p$

# Example

- Your campaign advisor told you that in a random sample of 100 likely voters, 56 intent to vote for you.
- Can you relax? Do you have this race in the bag?
- Without access to a computer or calculator, how precise is this estimate?
- `1/sqrt(100)=.1` so a back of the envelope calculation gives an approximate 95% interval of `(0.46, 0.66)`
- Not enough for you to relax, better go do more campaigning!
- Rough guidelines, 100 for 1 decimal place, 10,000 for 2, 1,000,000 for 3.

```r
round(1 / sqrt(10 ^ (1 : 6)), 3)
```

```
## [1] 0.316 0.100 0.032 0.010 0.003 0.001
```

# Poisson interval

- A nuclear pump failed 5 times out of 94.32 days, give a 95% confidence interval for the failure rate per day?
- $X \sim \mathit{Poisson}(\lambda t)$.
- Estimate $\hat{\lambda} = X/t$
- $\mathit{Var}(\hat{\lambda}) = \lambda/t$

$$\frac{\hat{\lambda} - \lambda}{\sqrt{\hat{\lambda}/t}} = \frac{X - t\lambda}{\sqrt{X}} \rightarrow N(0, 1)$$

- This isn't the best interval.
- There are better asymptotic intervals.
- You can get an exact CI in this case.

## R code

```
x <- 5; t <- 94.32; lambda <- x / t
round(lambda + c(-1, 1) * qnorm(.975) * sqrt(lambda / t), 3
```

# In the regression class

```r
exp(confint(glm(x ~ 1 + offset(log(t)), family = poisson(li
```

```
## Waiting for profiling to be done...
```

```
##      2.5 %     97.5 %
## 0.01900677 0.11393446
```