

Conditional Probability

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Conditional probability, motivation

- ▶ The probability of getting a one when rolling a (standard) die is usually assumed to be one sixth
- ▶ Suppose you were given the extra information that the die roll was an odd number (hence 1, 3 or 5)
- ▶ *conditional on this new information*, the probability of a one is now one third

Conditional probability, definition

- ▶ Let B be an event so that $P(B) > 0$
- ▶ Then the conditional probability of an event A given that B has occurred is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- ▶ Notice that if A and B are independent (defined later in the lecture), then

$$P(A \mid B) = \frac{P(A)P(B)}{P(B)} = P(A)$$

Example

- ▶ Consider our die roll example
- ▶ $B = \{1, 3, 5\}$
- ▶ $A = \{1\}$

$$P(\text{one given that roll is odd}) = P(A \mid B)$$

$$= \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A)}{P(B)}$$

$$= \frac{1/6}{3/6} = \frac{1}{3}$$

Bayes' rule

Baye's rule allows us to reverse the conditioning set provided that we know some marginal probabilities

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid B^c)P(B^c)}.$$

Diagnostic tests

- ▶ Let $+$ and $-$ be the events that the result of a diagnostic test is positive or negative respectively
- ▶ Let D and D^c be the event that the subject of the test has or does not have the disease respectively
- ▶ The **sensitivity** is the probability that the test is positive given that the subject actually has the disease, $P(+ \mid D)$
- ▶ The **specificity** is the probability that the test is negative given that the subject does not have the disease, $P(- \mid D^c)$

More definitions

- ▶ The **positive predictive value** is the probability that the subject has the disease given that the test is positive, $P(D \mid +)$
- ▶ The **negative predictive value** is the probability that the subject does not have the disease given that the test is negative, $P(D^c \mid -)$
- ▶ The **prevalence of the disease** is the marginal probability of disease, $P(D)$

More definitions

- ▶ The **diagnostic likelihood ratio of a positive test**, labeled DLR_+ , is $P(+ | D)/P(+ | D^c)$, which is the

$$sensitivity / (1 - specificity)$$

- ▶ The **diagnostic likelihood ratio of a negative test**, labeled DLR_- , is $P(- | D)/P(- | D^c)$, which is the

$$(1 - sensitivity) / specificity$$

Example

- ▶ A study comparing the efficacy of HIV tests, reports on an experiment which concluded that HIV antibody tests have a sensitivity of 99.7% and a specificity of 98.5%
- ▶ Suppose that a subject, from a population with a .1% prevalence of HIV, receives a positive test result. What is the positive predictive value?
- ▶ Mathematically, we want $P(D \mid +)$ given the sensitivity, $P(+ \mid D) = .997$, the specificity, $P(- \mid D^c) = .985$, and the prevalence $P(D) = .001$

Using Bayes' formula

$$\begin{aligned}P(D \mid +) &= \frac{P(+ \mid D)P(D)}{P(+ \mid D)P(D) + P(+ \mid D^c)P(D^c)} \\&= \frac{P(+ \mid D)P(D)}{P(+ \mid D)P(D) + \{1 - P(- \mid D^c)\}\{1 - P(D)\}} \\&= \frac{.997 \times .001}{.997 \times .001 + .015 \times .999} \\&= .062\end{aligned}$$

- ▶ In this population a positive test result only suggests a 6% probability that the subject has the disease
- ▶ (The positive predictive value is 6% for this test)

More on this example

- ▶ The low positive predictive value is due to low prevalence of disease and the somewhat modest specificity
- ▶ Suppose it was known that the subject was an intravenous drug user and routinely had intercourse with an HIV infected partner
- ▶ Notice that the evidence implied by a positive test result does not change because of the prevalence of disease in the subject's population, only our interpretation of that evidence changes

Likelihood ratios

- ▶ Using Bayes rule, we have

$$P(D \mid +) = \frac{P(+ \mid D)P(D)}{P(+ \mid D)P(D) + P(+ \mid D^c)P(D^c)}$$

and

$$P(D^c \mid +) = \frac{P(+ \mid D^c)P(D^c)}{P(+ \mid D)P(D) + P(+ \mid D^c)P(D^c)}.$$

Likelihood ratios

- ▶ Therefore

$$\frac{P(D \mid +)}{P(D^c \mid +)} = \frac{P(+ \mid D)}{P(+ \mid D^c)} \times \frac{P(D)}{P(D^c)}$$

ie

post-test odds of $D = DLR_+ \times$ pre-test odds of D

- ▶ Similarly, DLR_- relates the decrease in the odds of the disease after a negative test result to the odds of disease prior to the test.

HIV example revisited

- ▶ Suppose a subject has a positive HIV test
- ▶ $DLR_+ = .997 / (1 - .985) \approx 66$
- ▶ The result of the positive test is that the odds of disease is now 66 times the pretest odds
- ▶ Or, equivalently, the hypothesis of disease is 66 times more supported by the data than the hypothesis of no disease

HIV example revisited

- ▶ Suppose that a subject has a negative test result
- ▶ $DLR_- = (1 - .997)/.985 \approx .003$
- ▶ Therefore, the post-test odds of disease is now .3% of the pretest odds given the negative test.
- ▶ Or, the hypothesis of disease is supported .003 times that of the hypothesis of absence of disease given the negative test result

Independence

- ▶ Two events A and B are **independent** if

$$P(A \cap B) = P(A)P(B)$$

- ▶ Equivalently if $P(A | B) = P(A)$
- ▶ Two random variables, X and Y are independent if for any two sets A and B

$$P([X \in A] \cap [Y \in B]) = P(X \in A)P(Y \in B)$$

- ▶ If A is independent of B then
- ▶ A^c is independent of B
- ▶ A is independent of B^c
- ▶ A^c is independent of B^c

Example

- ▶ What is the probability of getting two consecutive heads?
- ▶ $A = \{\text{Head on flip 1}\} \sim P(A) = .5$
- ▶ $B = \{\text{Head on flip 2}\} \sim P(B) = .5$
- ▶ $A \cap B = \{\text{Head on flips 1 and 2}\}$
- ▶ $P(A \cap B) = P(A)P(B) = .5 \times .5 = .25$

Example

- ▶ Volume 309 of Science reports on a physician who was on trial for expert testimony in a criminal trial
- ▶ Based on an estimated prevalence of sudden infant death syndrome of 1 out of 8,543, the physician testified that that the probability of a mother having two children with SIDS was $\left(\frac{1}{8,543}\right)^2$
- ▶ The mother on trial was convicted of murder

Example: continued

- ▶ Relevant to this discussion, the principal mistake was to *assume* that the events of having SIDs within a family are independent
- ▶ That is, $P(A_1 \cap A_2)$ is not necessarily equal to $P(A_1)P(A_2)$
- ▶ Biological processes that have a believed genetic or familiar environmental component, of course, tend to be dependent within families
- ▶ (There are many other statistical points of discussion for this case.)

IID random variables

- ▶ Random variables are said to be iid if they are independent and identically distributed
- ▶ Independent: statistically unrelated from one and another
- ▶ Identically distributed: all having been drawn from the same population distribution
- ▶ iid random variables are the default model for random samples
- ▶ Many of the important theories of statistics are founded on assuming that variables are iid
- ▶ Assuming a random sample and iid will be the default starting point of inference for this class