Least squares estimation of regression lines

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General least squares for linear equations

Consider again the parent and child height data from Galton

```
## Loading required package: MASS

## Loading required package: HistData

## Loading required package: Hmisc

## Loading required package: lattice

## Loading required package: survival

## Loading required package: Formula

## Loading required package: ggplot2

##
## Attaching package: 'Hmisc'

## The following objects are masked from 'package:base':
```

Fitting the best line

- Let Y_i be the i^{th} child's height and X_i be the i^{th} (average over the pair of) parents' heights.
- Consider finding the best line
- Child's Height $= \beta_0 +$ Parent's Height β_1
- Use least squares

$$\sum_{i=1}^{n} \{Y_i - (\beta_0 + \beta_1 X_i)\}^2$$

Results

- The least squares model fit to the line $Y = \beta_0 + \beta_1 X$ through the data pairs (X_i, Y_i) with Y_i as the outcome obtains the line $Y = \hat\beta_0 + \hat\beta_1 X$ where

$$\hat\beta_1 = Cor(Y, X)\frac{Sd(Y)}{Sd(X)} \quad \hat\beta_0 = \bar Y - \hat\beta_1 \bar X$$

- $\hat\beta_1$ has the units of Y/X, $\hat\beta_0$ has the units of Y.
- The line passes through the point $(\bar X, \bar Y)$
- The slope of the regression line with X as the outcome and Y as the predictor is $Cor(Y, X)Sd(X)/Sd(Y)$.
- The slope is the same one you would get if you centered the data, $(X_i - \bar X, Y_i - \bar Y)$, and did regression through the origin.
- If you normalized the data, $\{\frac{X_i - \bar X}{Sd(X)}, \frac{Y_i - \bar Y}{Sd(Y)}\}$, the slope is $Cor(Y, X)$.

Revisiting Galton's data

Double check our calculations using R

```
y <- galton$child
x <- galton$parent
beta1 <- cor(y, x) *  sd(y) / sd(x)
beta0 <- mean(y) - beta1 * mean(x)
rbind(c(beta0, beta1), coef(lm(y ~ x)))
```

```
##      (Intercept)         x
## [1,]    23.94153 0.6462906
## [2,]    23.94153 0.6462906
```

Revisiting Galton's data

Reversing the outcome/predictor relationship

```r
beta1 <- cor(y, x) *  sd(x) / sd(y)
beta0 <- mean(x) - beta1 * mean(y)
rbind(c(beta0, beta1), coef(lm(x ~ y)))
```

```
##      (Intercept)         y
## [1,]    46.13535 0.3256475
## [2,]    46.13535 0.3256475
```

Revisiting Galton's data

Regression through the origin yields an equivalent slope if you center the data first

```
yc <- y - mean(y)
xc <- x - mean(x)
beta1 <- sum(yc * xc) / sum(xc ^ 2)
c(beta1, coef(lm(y ~ x))[2])
```

```
##                   x
## 0.6462906 0.6462906
```

Revisiting Galton's data

Normalizing variables results in the slope being the correlation

```
yn <- (y - mean(y))/sd(y)
xn <- (x - mean(x))/sd(x)
c(cor(y, x), cor(yn, xn), coef(lm(yn ~ xn))[2])
```

```
##                                xn
## 0.4587624 0.4587624 0.4587624
```