Likelihood

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Likelihood

- A common and fruitful approach to statistics is to assume that the data arises from a family of distributions indexed by a parameter that represents a useful summary of the distribution
- The **likelihood** of a collection of data is the joint density evaluated as a function of the parameters with the data fixed
- Likelihood analysis of data uses the likelihood to perform inference regarding the unknown parameter

Likelihood

Given a statistical probability mass function or density, say $f(x, \theta)$, where θ is an unknown parameter, the **likelihood** is f viewed as a function of θ for a fixed, observed value of x.

Interpretations of likelihoods

The likelihood has the following properties:

1. Ratios of likelihood values measure the relative evidence of one value of the unknown parameter to another.
2. Given a statistical model and observed data, all of the relevant information contained in the data regarding the unknown parameter is contained in the likelihood.
3. If $\{X_i\}$ are independent random variables, then their likelihoods multiply. That is, the likelihood of the parameters given all of the X_i is simply the product of the individual likelihoods.

Example

- Suppose that we flip a coin with success probability θ
- Recall that the mass function for x

$$f(x, \theta) = \theta^x (1 - \theta)^{1-x} \quad \text{for} \quad \theta \in [0, 1].$$

where x is either 0 (Tails) or 1 (Heads)
- Suppose that the result is a head
- The likelihood is

$$\mathcal{L}(\theta, 1) = \theta^1 (1 - \theta)^{1-1} = \theta \quad \text{for} \quad \theta \in [0, 1].$$

- Therefore, $\mathcal{L}(.5, 1)/\mathcal{L}(.25, 1) = 2$,
- There is twice as much evidence supporting the hypothesis that $\theta = .5$ to the hypothesis that $\theta = .25$

Example continued

- Suppose now that we flip our coin from the previous example 4 times and get the sequence 1, 0, 1, 1
- The likelihood is:

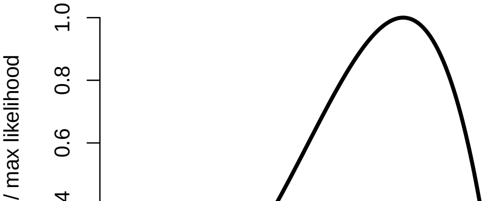
$$\begin{aligned}
\mathcal{L}(\theta, 1, 0, 1, 1) = & \quad \theta^1(1-\theta)^{1-1}\theta^0(1-\theta)^{1-0} \\
\times & \quad \theta^1(1-\theta)^{1-1}\theta^1(1-\theta)^{1-1} \\
= & \quad \theta^3(1-\theta)^1
\end{aligned}$$

- This likelihood only depends on the total number of heads and the total number of tails; we might write $\mathcal{L}(\theta, 1, 3)$ for shorthand
- Now consider $\mathcal{L}(.5, 1, 3)/\mathcal{L}(.25, 1, 3) = 5.33$
- There is over five times as much evidence supporting the hypothesis that $\theta = .5$ over that $\theta = .25$

Plotting likelihoods

- Generally, we want to consider all the values of θ between 0 and 1
- A **likelihood plot** displays θ by $\mathcal{L}(\theta, x)$
- Because the likelihood measures *relative evidence*, dividing the curve by its maximum value (or any other value for that matter) does not change its interpretation

```r
pvals <- seq(0, 1, length = 1000)
plot(pvals, dbinom(3, 4, pvals) / dbinom(3, 4, 3/4), type =
```



Maximum likelihood

- The value of θ where the curve reaches its maximum has a special meaning
- It is the value of θ that is most well supported by the data
- This point is called the **maximum likelihood estimate** (or MLE) of θ

$$MLE = \text{argmax}_{\theta}\mathcal{L}(\theta, x).$$

- Another interpretation of the MLE is that it is the value of θ that would make the data that we observed most probable

Some results

- $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ the MLE of μ is $\bar{X}$ and the ML of σ^2 is the biased sample variance estimate.
- If $X_1, \dots, X_n \overset{iid}{\sim} \mathit{Bernoulli}(p)$ then the MLE of p is $\bar{X}$ (the sample proportion of 1s).
- If $X_i \overset{iid}{\sim} \mathit{Binomial}(n_i, p)$ then the MLE of p is $\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n n_i}$ (the sample proportion of 1s).
- If $X \overset{iid}{\sim} \mathit{Poisson}(\lambda t)$ then the MLE of λ is X/t.
- If $X_i \overset{iid}{\sim} \mathit{Poisson}(\lambda t_i)$ then the MLE of λ is $\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n t_i}$

Example

- You saw 5 failure events per 94 days of monitoring a nuclear pump.
- Assuming Poisson, plot the likelihood

```
lambda <- seq(0, .2, length = 1000)
likelihood <- dpois(5, 94 * lambda) / dpois(5, 5)
plot(lambda, likelihood, frame = FALSE, lwd = 3, type = "l'
lines(rep(5/94, 2), 0 : 1, col = "red", lwd = 3)
lines(range(lambda[likelihood > 1/16]), rep(1/16, 2), lwd =
lines(range(lambda[likelihood > 1/8]), rep(1/8, 2), lwd = 2
```