Generalized linear models, binary data

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Key ideas

- Frequently we care about outcomes that have two values
- Alive/dead
- Win/loss
- Success/Failure
- etc
- Called binary, Bernoulli or 0/1 outcomes
- Collection of exchangeable binary outcomes for the same covariate data are called binomial outcomes.

Example Baltimore Ravens win/loss

Ravens Data

```
download.file("https://dl.dropboxusercontent.com/u/7710864/
                , destfile="./data/ravensData.rda",method="cu
load("./data/ravensData.rda")
head(ravensData)
```

```
##   ravenWinNum ravenWin ravenScore opponentScore
## 1           1        W         24             9
## 2           1        W         38            35
## 3           1        W         28            13
## 4           1        W         34            31
## 5           1        W         44            13
## 6           0        L         23            24
```

Linear regression

$$RW_i = b_0 + b_1 RS_i + e_i$$

RW_i - 1 if a Ravens win, 0 if not

RS_i - Number of points Ravens scored

b_0 - probability of a Ravens win if they score 0 points

b_1 - increase in probability of a Ravens win for each additional point

e_i - residual variation due

Linear regression in R

```
lmRavens <- lm(ravensData$ravenWinNum ~ ravensData$ravenSco
summary(lmRavens)$coef
```

```
##                             Estimate  Std. Error  t value
## (Intercept)               0.28503172 0.256643165 1.110615 0.
## ravensData$ravenScore 0.01589917 0.009058997 1.755069 0.
```

Odds

Binary Outcome 0/1

$$RW_i$$

Probability (0,1)

$$\mathrm{Pr(RW_i|RS_i, b_0, b_1)}$$

Odds $(0, \infty)$

$$\frac{\mathrm{Pr(RW_i|RS_i, b_0, b_1)}}{1 - \mathrm{Pr(RW_i|RS_i, b_0, b_1)}}$$

Log odds $(-\infty, \infty)$

$$\log\left(\frac{\mathrm{Pr(RW_i|RS_i, b_0, b_1)}}{1 - \mathrm{Pr(RW_i|RS_i, b_0, b_1)}}\right)$$

Linear vs. logistic regression

Linear

$$RW_i = b_0 + b_1 RS_i + e_i$$

or

$$E[RW_i | RS_i, b_0, b_1] = b_0 + b_1 RS_i$$

Logistic

$$\mathrm{Pr(RW_i|RS_i, b_0, b_1)} = \frac{\exp(\mathrm{b_0 + b_1 RS_i})}{1 + \exp(\mathrm{b_0 + b_1 RS_i})}$$

or

$$\log\left(\frac{\mathrm{Pr(RW_i|RS_i, b_0, b_1)}}{1 - \mathrm{Pr(RW_i|RS_i, b_0, b_1)}}\right) = b_0 + b_1 RS_i$$

Interpreting Logistic Regression

$$\log\left(\frac{\mathrm{Pr(RW_i|RS_i, b_0, b_1)}}{\mathrm{1 - Pr(RW_i|RS_i, b_0, b_1)}}\right) = b_0 + b_1 RS_i$$

b_0 - Log odds of a Ravens win if they score zero points

b_1 - Log odds ratio of win probability for each point scored (compared to zero points)

$\exp(b_1)$ - Odds ratio of win probability for each point scored (compared to zero points)

Odds

- Imagine that you are playing a game where you flip a coin with success probability p.
- If it comes up heads, you win X. If it comes up tails, you lose Y.
- What should we set X and Y for the game to be fair?

$$E[earnings] = Xp - Y(1-p) = 0$$

- Implies
$$\frac{Y}{X} = \frac{p}{1-p}$$
- The odds can be said as "How much should you be willing to pay for a p probability of winning a dollar?"
  - (If $p > 0.5$ you have to pay more if you lose than you get if you win.)
  - (If $p < 0.5$ you have to pay less if you lose than you get if you win.)

Visualizing fitting logistic regression curves

```
x <- seq(-10, 10, length = 1000)
manipulate(
    plot(x, exp(beta0 + beta1 * x) / (1 + exp(beta0 + beta
        type = "l", lwd = 3, frame = FALSE),
    beta1 = slider(-2, 2, step = .1, initial = 2),
    beta0 = slider(-2, 2, step = .1, initial = 0)
    )
```

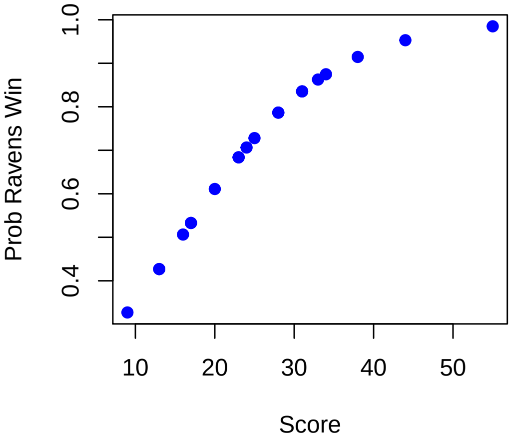
Ravens logistic regression

```
logRegRavens <- glm(ravensData$ravenWinNum ~ ravensData$rav
summary(logRegRavens)
```

```
##
## Call:
## glm(formula = ravensData$ravenWinNum ~ ravensData$ravenS
##     family = "binomial")
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -1.7575  -1.0999   0.5305   0.8060   1.4947
##
## Coefficients:
##                       Estimate Std. Error z value Pr(>|z
## (Intercept)           -1.68001    1.55412  -1.081     0.
## ravensData$ravenScore  0.10658    0.06674   1.597     0.
##
## (Dispersion parameter for binomial family taken to be 1)
```

Ravens fitted values

```
plot(ravensData$ravenScore,logRegRavens$fitted,pch=19,col='
```



Odds ratios and confidence intervals

```r
exp(logRegRavens$coeff)
```

```
##              (Intercept) ravensData$ravenScore
##                0.1863724             1.1124694
```

```r
exp(confint(logRegRavens))
```

```
## Waiting for profiling to be done...
```

```
##                             2.5 %   97.5 %
## (Intercept)           0.005674966 3.106384
## ravensData$ravenScore 0.996229662 1.303304
```

ANOVA for logistic regression

```
anova(logRegRavens,test="Chisq")
```

```
## Analysis of Deviance Table
##
## Model: binomial, link: logit
##
## Response: ravensData$ravenWinNum
##
## Terms added sequentially (first to last)
##
##
##                        Df Deviance Resid. Df Resid. Dev I
## NULL                                      19     24.435
## ravensData$ravenScore  1   3.5398         18     20.895
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.
```

Interpreting Odds Ratios

- Not probabilities
- Odds ratio of 1 = no difference in odds
- Log odds ratio of 0 = no difference in odds
- Odds ratio < 0.5 or > 2 commonly a “moderate effect”
- Relative risk $\frac{\Pr(\mathrm{RW_i} | \mathrm{RS_i}=10)}{\Pr(\mathrm{RW_i} | \mathrm{RS_i}=0)}$ often easier to interpret, harder to estimate
- For small probabilities RR $\approx$ OR but **they are not the same**!

Wikipedia on Odds Ratio

Further resources

- Wikipedia on Logistic Regression
- Logistic regression and glms in R
- Brian Caffo's lecture notes on: Simpson's paradox, Case-control studies
- Open Intro Chapter on Logistic Regression