Model based prediction

Jeffrey Leek

May 18, 2016

Basic idea

1. Assume the data follow a probabilistic model
2. Use Bayes' theorem to identify optimal classifiers

Pros:

- Can take advantage of structure of the data
- May be computationally convenient
- Are reasonably accurate on real problems

Cons:

- Make additional assumptions about the data
- When the model is incorrect you may get reduced accuracy

Model based approach

1. Our goal is to build parametric model for conditional distribution $P(Y = k|X = x)$
2. A typical approach is to apply [Bayes theorem](`http://en.wikipedia.org/wiki/Bayes'_theorem`):

$$Pr(Y = k|X = x) = \frac{Pr(X = x|Y = k)Pr(Y = k)}{\sum_{\ell=1}^{K} Pr(X = x|Y = \ell)Pr(Y = \ell)}$$

$$Pr(Y = k|X = x) = \frac{f_k(x)\pi_k}{\sum_{\ell=1}^{K} f_\ell(x)\pi_\ell}$$

3. Typically prior probabilities π_k are set in advance.
4. A common choice for $f_k(x) = \frac{1}{\sigma_k\sqrt{2\pi}} e^{-\frac{(x-\mu_k)^2}{\sigma_k^2}}$, a Gaussian distribution
5. Estimate the parameters (μ_k, σ_k^2) from the data.
6. Classify to the class with the highest value of $P(Y = k|X = x)$

Classifying using the model

A range of models use this approach

- Linear discriminant analysis assumes $f_k(x)$ is multivariate Gaussian with same covariances
- Quadratic discrimant analysis assumes $f_k(x)$ is multivariate Gaussian with different covariances
- Model based prediction assumes more complicated versions for the covariance matrix
- Naive Bayes assumes independence between features for model building

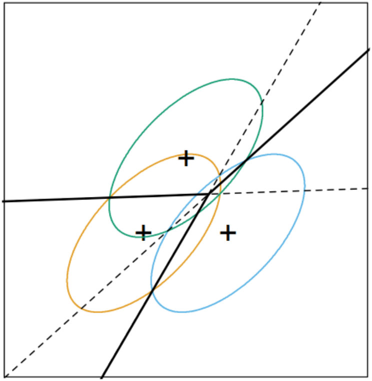
`http://statweb.stanford.edu/~tibs/ElemStatLearn/`

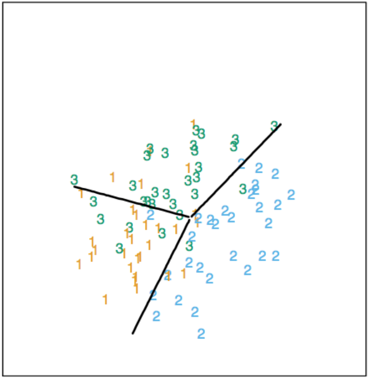
Why linear discriminant analysis?

$$
\begin{aligned}
&log \frac{Pr(Y = k|X = x)}{Pr(Y = j|X = x)} \\
&= log \frac{f_k(x)}{f_j(x)} + log \frac{\pi_k}{\pi_j} \\
&= log \frac{\pi_k}{\pi_j} - \frac{1}{2}(\mu_k + \mu_j)^T \Sigma^{-1} (\mu_k + \mu_j) \\
&+ x^T \Sigma^{-1} (\mu_k - \mu_j)
\end{aligned}
$$

`http://statweb.stanford.edu/~tibs/ElemStatLearn/`

Decision boundaries





Discriminant function

$$\delta_k(x) = x^T \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k \Sigma^{-1} \mu_k + log(\mu_k)$$

- Decide on class based on $\hat{Y}(x) = argmax_k \delta_k(x)$
- We usually estimate parameters with maximum likelihood

Naive Bayes

Suppose we have many predictors, we would want to model:
$P(Y = k|X_1, \dots, X_m)$

We could use Bayes Theorem to get:

$$
\begin{aligned}
P(Y = k|X_1, \dots, X_m) &= \frac{\pi_k P(X_1, \dots, X_m|Y = k)}{\sum_{\ell=1}^{K} P(X_1, \dots, X_m|Y = k)\pi_\ell} \\
&\propto \pi_k P(X_1, \dots, X_m|Y = k)
\end{aligned}
$$

This can be written:

$$
\begin{aligned}
P(X_1, \dots, X_m, Y = k) &= \pi_k P(X_1|Y = k)P(X_2, \dots, X_m|X_1, Y = k) \\
&= \pi_k P(X_1|Y = k)P(X_2|X_1, Y = k)P(X_3, \dots, X_m|X_1, X_2, Y = k) \\
&= \pi_k P(X_1|Y = k)P(X_2|X_1, Y = k) \dots P(X_m|X_1 \dots, X_{m-1}, Y = k)
\end{aligned}
$$

We could make an assumption to write this:

$$\propto \pi_k P(X_1|Y = k)P(X_2|Y = k) \dots P(X_m|Y = k)$$

Example: Iris Data

```r
data(iris); library(ggplot2)
names(iris)
```

```
## [1] "Sepal.Length" "Sepal.Width"  "Petal.Length" "Petal.
## [5] "Species"
```

```r
table(iris$Species)
```

```
##
##     setosa versicolor  virginica
##         50         50         50
```

Create training and test sets

```r
library(caret)
```

```
## Loading required package: lattice
```

```r
inTrain <- createDataPartition(y=iris$Species,
                               p=0.7, list=FALSE)
training <- iris[inTrain,]
testing <- iris[-inTrain,]
dim(training); dim(testing)
```

```
## [1] 105   5
```

```
## [1] 45  5
```

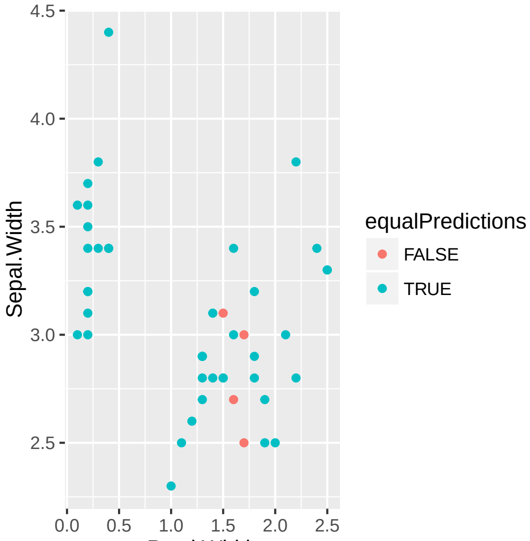
Build predictions

```r
library(MASS); library(klaR)
modlda = train(Species ~ .,data=training,method="lda")
modnb = train(Species ~ ., data=training,method="nb")
```

```
## Warning in data.row.names(row.names, rowsi, i): some row
## 4,5,6,12,14,16,18,19,20,21,22,24,27,30,31,34,39,41,51,53
## --> row.names NOT used

## Warning in data.row.names(row.names, rowsi, i): some row
## 4,5,6,12,14,16,18,19,20,21,22,24,27,30,31,34,39,41,51,53
## --> row.names NOT used

## Warning in data.row.names(row.names, rowsi, i): some row
## 4,5,8,11,15,16,18,20,24,34,35,36,41,43,46,50,53,54,58,59
## --> row.names NOT used

## Warning in data.row.names(row.names, rowsi, i): some row
## 4,5,8,11,15,16,18,20,24,34,35,36,41,43,46,50,53,54,58,59
```

Comparison of results

```
equalPredictions = (plda==pnb)
qplot(Petal.Width,Sepal.Width,colour=equalPredictions,data=
```



Notes and further reading

- Introduction to statistical learning
- Elements of Statistical Learning
- Model based clustering
- Linear Discriminant Analysis
- Quadratic Discriminant Analysis