Математическая статистика
и её приложения
Домашняя работа №1

№1.

Дано:

$X_1, \dots, X_n$ – н.с.в. $\in B(p)$, $p \in (0;1)$

$\hat{\theta}_1 = \frac{1}{n-1} \cdot T(X_1, \dots, X_n)$ и $\hat{\theta}_2 = \frac{1}{n} \cdot T(X_1, \dots, X_n)$, где

$T(X_1, \dots, X_n) = \sum_{k=1}^{n} (X_k - \bar{X})^2$, $\bar{X} = \frac{1}{n} \cdot \sum_{k=1}^{n} X_k$

1) • $\sphericalangle$ $\mathbb{E}(X_i^2) = 0^2 \cdot \mathbb{P}(X_i = 0) + 1^2 \cdot \mathbb{P}(X_i = 1) = p$

• $\sphericalangle$ $\mathbb{E}(\bar{X}^2) = \mathbb{E}\left(\frac{1}{n^2} \cdot (X_1 + \dots + X_n)^2\right) = \frac{1}{n^2} \cdot \mathbb{E}\left((X_1 + \dots + X_n)^2\right) =$

$= \frac{1}{n^2} \cdot \mathbb{E}\left(X_1^2 + \dots + X_n^2 + 2 \cdot (X_1 X_2 + X_1 X_3 + \dots + X_{n-1} \cdot X_n)\right) = \left\{\underset{\text{т.к. они независ.}}{\mathbb{E}(X_i X_j) = p^2}\right\} =$

$= \frac{1}{n^2}\left(np + \cancel{2} \cdot \frac{n(n-1)}{\cancel{2}} p^2\right) = \frac{1}{n}\left(p + (n-1)p^2\right)$

• $\sphericalangle$ $\mathrm{var}(X_i) = \mathbb{E}\left[(X_i - \mathbb{E}(X_i))^2\right] = \mathbb{E}[X_i^2] - 2 \cdot \mathbb{E}\left[X_i \cdot \mathbb{E}(X_i)\right] +$

$+ \left(\mathbb{E}[X_i]\right)^2 = p^2 - 2p^2 + p = p - p^2$

• $\sphericalangle$ $\mathbb{E}(\theta_1) = \frac{1}{n-1} \cdot \mathbb{E}\left[\sum_{k=1}^{n} (X_k - \bar{X})^2\right] = \frac{1}{n-1} \cdot \mathbb{E}\left[X_1^2 + \dots + X_n^2 + n \cdot \bar{X}^2 -\right.$

$\left.- 2 \cdot \bar{X}(X_1 + \dots + X_n)\right] = \frac{1}{n-1} \cdot \left(np + p + (n-1)p^2 - 2np^2\right) =$

$= \frac{1}{n-1} \cdot \left(p(n+1) - p^2(n+1)\right) = \frac{n+1}{n-1}\left(p - p^2\right)$ ▨

№2

$p_\theta(x) = \lambda \cdot e^{-\lambda(x-\mu)} \cdot \mathbb{I}(x \geq \mu)$, $\lambda, \mu > 0$

ML: 1) $L_{\vec{\theta}}(\vec{x}) = \prod_{k=1}^{n} p_\theta(x_k) = \prod_{k=1}^{n} \left(\lambda \cdot e^{-\lambda(x_k - \mu)} \cdot \mathbb{I}(x_k \geq \mu)\right) =$

$= \lambda^n \cdot e^{-\lambda(x_1 + \dots + x_n - n\mu)} \cdot \mathbb{I}\left(x_k \geq \mu\right)^{k \in \overline{1,\dots,n}} \longrightarrow \max_{(\mu;\lambda) \in (0;+\infty) \times (0;+\infty)}$

2) • Как мы видим, чем больше μ, тем больше наша ф-я правдоподобия => берём макс. возможное μ

• Чтобы наши индикаторы не обнулились, самое большое μ, кот-е мы можем взять – это $\min\{x_1, \dots, x_n\}$

Итого, $\boxed{\hat{\mu}_{ML} = \min\{x_1, \dots, x_n\}}$

3) $\measuredangle$ $\log L_{\vec{\theta}}(\vec{x}) = \log(\lambda^n \cdot e^{-\lambda(x_1 + \dots + x_n - n\mu)}) = n \cdot \log\lambda + \lambda n \mu -$
$- \lambda(x_1 + \dots + x_n)$

$$\frac{\partial \log L_{\vec{\theta}}(\vec{x})}{\partial \lambda} = \frac{n}{\lambda} + n\mu - (x_1 + \dots + x_n) = 0 \Rightarrow$$

$$\Rightarrow \hat{\lambda}_{ML} = \frac{n}{x_1 + \dots + x_n - n\mu} = \boxed{\frac{1}{\bar{X} - \hat{\mu}_{ML}}}$$

MM:
$$\begin{cases} \mathbb{E}[X_1]\big|_{\hat{\lambda}, \hat{\mu}} = \frac{1}{n} \cdot \sum\limits_{k=1}^{n} X_k \\ \mathbb{E}[X_1^2]\big|_{\hat{\lambda}, \hat{\mu}} = \frac{1}{n} \cdot \sum\limits_{k=1}^{n} X_k^2 \end{cases}$$

1) $\mathbb{E}[X_1] = \int\limits_{\mu}^{+\infty} x \cdot \lambda \cdot e^{-\lambda(x-\mu)} dx = -\frac{(\lambda x + 1) \cdot e^{\lambda(\mu - x)}}{\lambda} \Big|_{\mu}^{+\infty} = \frac{\lambda\mu + 1}{\lambda}$

2) $\mathbb{E}[X_1^2] = \int\limits_{\mu}^{+\infty} x^2 \lambda \cdot e^{-\lambda(x-\mu)} dx = -\frac{(\lambda^2 x^2 + 2\lambda x + 2) \cdot e^{\lambda(\mu - x)}}{\lambda^2} \Big|_{\mu}^{+\infty} = \frac{(\lambda\mu + 1)^2 + 1}{\lambda^2}$

3) $$\begin{cases} \frac{\lambda\mu + 1}{\lambda} = \bar{X} \quad (1) \\ \frac{(\lambda\mu + 1)^2 + 1}{\lambda^2} = \overline{X^2} \quad (2) \end{cases}$$

$(2) - (1)^2$: $\frac{1}{\lambda^2} = \overline{X^2} - (\bar{X})^2 \Rightarrow \boxed{\hat{\lambda}_{MM} = \frac{1}{\sqrt{\overline{X^2} - (\bar{X})^2}}}$

4) $\hat{\mu}_{MM} = \frac{\bar{X} \cdot \hat{\lambda}_{MM} - 1}{\hat{\lambda}_{MM}} = \boxed{\bar{X} - \sqrt{\overline{X^2} - (\bar{X})^2}}$

№3

$$p(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot x^{\alpha - 1} e^{-\beta x} \cdot \mathbb{I}\{x > 0\}, \alpha, \beta > 0$$

1) $\exists \alpha = 2 \Rightarrow p_\beta(x) = \beta^2 x \cdot e^{-\beta x} \cdot \mathbb{I}\{x > 0\}$

ЭС: $p(x) = h(x) \cdot e^{\eta \cdot T(x) - A(\eta)}$

$\Longrightarrow \begin{cases} h(x) = x \cdot \mathbb{I}\{x > 0\} \\ \eta = \beta \\ T(x) = -x \\ A(\eta) = -2 \cdot \log(\eta) \end{cases}$

№3 (продолжение)

$$\hat{\eta}_{ML} = \hat{\eta}_{MM} = (A')^{-1}\left(\frac{1}{n}\cdot\sum_{k=1}^{n} T(X_k)\right) \ominus$$

$$\measuredangle\ A'(\eta) = -\frac{2}{\eta}\ ;\ (A')^{-1}(y) = -\frac{2}{y}$$

$$\ominus\ -\frac{2}{\frac{1}{n}\sum_{k=1}^{n}(-X_k)} = \frac{2}{\frac{1}{n}\sum_{k=1}^{n}X_k} = \boxed{\frac{2}{\overline{X}}}$$

2) a) $p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\cdot x^{\alpha-1}\cdot e^{-\beta x}\cdot \mathbb{I}\{x>0\} = \frac{\mathbb{I}\{x>0\}}{x}\cdot e^{-\beta x+\alpha\cdot\log x+\alpha\cdot\log\beta-\log\Gamma(\alpha)} \Rightarrow$

$$\Rightarrow \begin{cases} \eta = (\alpha;\beta) \\ T(x) = (\log x; -x) \\ A(\eta) = -\alpha\cdot\log\beta + \log[\Gamma(\alpha)] \end{cases} \blacksquare$$