ARE LOG TRANSFORMED?

Introduction

In this page, we will discuss how to interpret a regression model when some variables in the model have been log transformed. The example data can be downloaded here (https://stats.idre.ucla.edu/wp-content/uploads/2016/02/lgtrans.csv) (the file is in .csv format). The variables in the data set are writing, reading, and math scores (**write**, **read** and **math**), the log transformed writing (**lgwrite**) and log transformed math scores (**lgmath**) and **female**. For these examples, we have taken the natural log (ln). All the examples are done in Stata, but they can be easily generated in any statistical package. In the examples below, the variable **write** or its log transformed version will be used as the outcome variable. The examples are used for illustrative purposes and are not intended to make substantive sense. Here is a table of different types of means for variable **write**.

Variable	Type	Obs	Mean	[95% Conf.	Interval]
write	Arithmetic	200	52.775	51.45332	54.09668
	Geometric	200	51.8496	50.46854	53.26845
	Harmonic	200	50.84403	49.40262	52.37208

Outcome variable is log transformed

Very often, a linear relationship is hypothesized between a log transformed outcome variable and a group of predictor variables. Written mathematically, the relationship follows the equation

$$\log(y_i) = \beta_0 + \beta_1 x_{1i} + \cdots + \beta_k x_{ki} + e_i,$$

where y is the outcome variable and $x_1, \cdots, x_k$ are the predictor variables. In other words, we assume that $\log(\mathbf{y}) - \mathbf{X}^T\beta$ is normally distributed, (or $\mathbf{y}$ is log-normal conditional on all the covariates). Since this is just an ordinary least squares regression, we can easily interpret a regression coefficient, say β_1, as the expected change in log of y with respect to a one-unit increase in x_1 holding all other variables at any fixed value, assuming that x_1 enters the model only as a main effect. But what if we want to know what happens to the outcome variable y itself for a one-unit increase in x_1? The natural way to do this is to interpret the exponentiated regression coefficients, $\exp(\beta)$, since exponentiation is the inverse of logarithm function.

Let's start with the intercept-only model.

lgwrite	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
intercept	3.948347	.0136905	288.40	0.000	3.92135	3.975344

$\log(\mathbf{write}) = \beta_0 = 3.95.$

We can say that 3.95 is the unconditional expected mean of log of **write**. Therefore the exponentiated value is $\exp(3.948347) = 51.85$. This is the geometric mean of **write**. The emphasis here is that it is the geometric mean instead of the arithmetic mean. OLS regression of the original variable y is used to to estimate the expected arithmetic mean and OLS regression of the log transformed outcome variable is to estimated the expected geometric mean of the original variable.

Now let's move on to a model with a single binary predictor variable.

lgwrite	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
female	.1032614	.0265669	3.89	0.000	.050871	.1556518
intercept	3.89207	.0196128	198.45	0.000	3.853393	3.930747

$$\begin{aligned}\log(\mathbf{write}) &= \beta_0 + \beta_1 \times \mathbf{female} \\ &= 3.89 + .10 \times \mathbf{female}.\end{aligned}$$

Before diving into the interpretation of these parameters, let's get the means of our dependent variable, **write**, by gender.

```
males

    Variable |      Type         Obs        Mean    [95% Conf. Interval]
-------------+--------------------------------------------------------
       write | Arithmetic         91    50.12088     47.97473    52.26703
             |  Geometric         91    49.01222      46.8497    51.27457
             |   Harmonic         91    47.85388      45.6903    50.23255
-------------------------------------------------------------------------

females

    Variable |      Type         Obs        Mean    [95% Conf. Interval]
-------------+--------------------------------------------------------
       write | Arithmetic        109    54.99083     53.44658    56.53507
             |  Geometric        109    54.34383     52.73513     56.0016
             |   Harmonic        109    53.64236     51.96389    55.43289
-------------------------------------------------------------------------
```

Now we can map the parameter estimates to the geometric means for the two groups. The intercept of 3.89 is the log of geometric mean of **write** when **female** $= 0$, i.e., for males. Therefore, the exponentiated value of it is the geometric mean for the male group: $\exp(3.892) = 49.01$. What can we say about the coefficient for **female**? In the log scale, it is the difference in the expected geometric means of the log of **write** between the female students and male students. In the original scale of the variable **write**, it is the ratio of the geometric mean of **write** for female students over the geometric mean of **write** for male students, $\exp(.1032614) = 54.34383/49.01222 = 1.11$. In terms of percent change, we can say that switching from male students to female students, we expect to see about 11% increase in the geometric mean of writing scores.

Last, let's look at a model with multiple predictor variables.

```
------------------------------------------------------------------------------
     lgwrite |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      female |    .114718   .0195341     5.87   0.000      .076194     .153242
        read |   .0066305   .0012689     5.23   0.000     .0041281    .0091329
        math |   .0076792   .0013873     5.54   0.000     .0049432    .0104152
   intercept |   3.135243   .0598109    52.42   0.000     3.017287    3.253198
------------------------------------------------------------------------------
```

$$\begin{aligned}\log(\mathbf{write}) &= \beta_0 + \beta_1 \times \mathbf{female} + \beta_2 \times \mathbf{read} + \beta_3 \times \mathbf{math} \\ &= 3.135 + .115 \times \mathbf{female} + .0066 \times \mathbf{read} + .0077 \times \mathbf{math}.\end{aligned}$$

The exponentiated coefficient $\exp(\beta_1)$ for **female** is the ratio of the expected geometric mean for the female students group over the expected geometric mean for the male students group, when **read** and **math** are held at some fixed value. Of course, the expected geometric means for the male and female students group will be different for different values of **read** and **math**. However, their ratio is a constant: $\exp(\beta_1)$. In our example, $\exp(\beta_1) = \exp(.114718) \approx 1.12$. We can say that writing scores will be 12% higher for the female students than for the male students. For the variable **read**, we can say that for a one-unit increase in **read**, we expect to see about a 0.7% increase in writing score, since $\exp(.0066305) = 1.006653 \approx 1.007$. For a ten-unit increase in **read** , we expect to see about a 6.9% increase in writing score, since $\exp(.0066305 \times 10) = 1.0685526 \approx 1.069$.

The intercept becomes less interesting when the predictor variables are not centered and are continuous. In this particular model, the intercept is the expected mean for $\log(\mathbf{write})$ for male ($\mathbf{female} = 0$) when **read** and **math** are equal to zero.

In summary, when the outcome variable is log transformed, it is natural to interpret the exponentiated regression coefficients. These values correspond to changes in the ratio of the expected geometric means of the original outcome variable.

Some (not all) predictor variables are log transformed

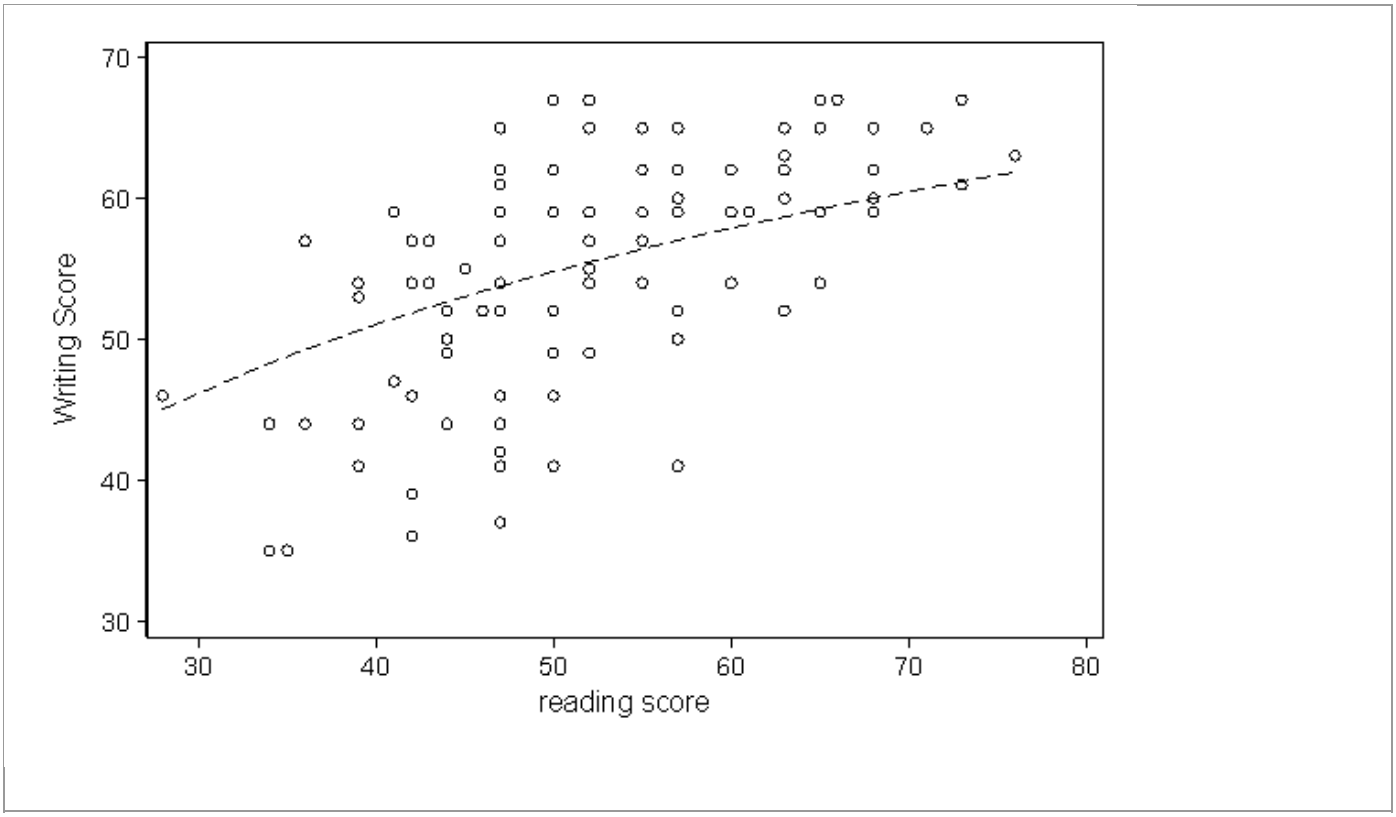
Occasionally, we also have some predictor variables being log transformed. In this section, we will take a look at an example where some predictor variables are log-transformed, but the outcome variable is in its original scale.

```
------------------------------------------------------------------------------
       write |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      female |   5.388777   .9307948     5.79   0.000     3.553118    7.224436
      lgmath |   20.94097   3.430907     6.10   0.000     14.17473     27.7072
      lgread |   16.85218   3.063376     5.50   0.000     10.81076    22.89359
   intercept |  -99.16397   10.80406    -9.18   0.000    -120.4711   -77.85685
------------------------------------------------------------------------------
```

Written in equation, we have

$$\begin{aligned}\mathbf{write} &= \beta_0 + \beta_1 \times \mathbf{female} + \beta_2 \times \log(\mathbf{math}) + \beta_3 \times \log(\mathbf{read}) \\ &= -99.164 + 5.389 \times \mathbf{female} + 20.941 \times \log(\mathbf{math}) + 16.852 \times \log(\mathbf{read}).\end{aligned}$$

Since this is an OLS regression, the interpretation of the regression coefficients for the non-transformed variables are unchanged from an OLS regression without any transformed variables. For example, the expected mean difference in writing scores between the female and male students is about 5.4 points, holding the other predictor variables constant. On the other hand, due to the log transformation, the estimated effects of $\mathbf{math}$ and $\mathbf{read}$ are no longer linear, even though the effect of $\log(\mathbf{math})$ and $\log(\mathbf{read})$ are linear. The plot below shows the curve of predicted values against the reading scores for the female students group holding math score constant.



How do we interpret the coefficient of 16.852 for the variable of log of reading score? Let's take two values of reading score, r_1 and r_2. The expected mean difference in writing score at r_1 and r_2, holding the other predictor variables constant, is $\mathbf{write}(r_2) - \mathbf{write}(r_1) = \beta_3 \times [\log(r_2) - \log(r_1)] = \beta_3 \times [\log(r_2/r_1)]$. This means that as long as the percent increase in $\mathbf{read}$ (the predictor variable) is fixed, we will see the same difference in writing score, regardless where the baseline reading score is. For example, we can say that for a 10% increase in reading score, the difference in the expected mean writing scores will be always $\beta_3 \times \log(1.10) = 16.85218 \times \log(1.1) \approx 1.61$.

Note:

Recalling the Taylor expansion of the function $f(x) = \log(1+x)$ around $x_0 = 0$, we have $\log(1+x) = x + \mathcal{O}(x^2)$. Therefore, for a small change in the predictor variable we can approximate the difference in the expected mean of the dependent variable by multiplying the coefficient by the change in the

predictor variable. In our example we can say that for a 1% increase in reading score, the difference in the expected mean writing scores will be approximately $\beta_3 \times 0.01 = 16.85218 \times 0.01 = .1685218$. If we use the log, the exact value will be $\beta_3 \times \log(1.01) = 16.85218 \times \log(1.01) = .1676848$.

Both the outcome variable and some predictor variables are log transformed

What happens when both the outcome variable and predictor variables are log transformed? We can combine the two previously described situations into one. Here is an example of such a model.

```
------------------------------------------------------------------------------
     lgwrite |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      female |   .1142399   .0194712     5.87   0.000       .07584    .1526399
      lgmath |   .4085369   .0720791     5.67   0.000     .2663866    .5506872
        read |   .0066086   .0012561     5.26   0.000     .0041313    .0090859
   intercept |   1.928101   .2469391     7.81   0.000     1.441102    2.415099
------------------------------------------------------------------------------
```

Written as an equation, we can describe the model:

$$\begin{aligned}\log(\mathbf{write}) &= \beta_0 + \beta_1 \times \mathbf{female} + \beta_2 \times \log(\mathbf{math}) + \beta_3 \times \mathbf{read} \\ &= 1.928101 + .1142399 \times \mathbf{female} + .4085369 \times \log(\mathbf{math}) + .0066086 \times \mathbf{read}.\end{aligned}$$

For variables that are not transformed, such as **female**, its exponentiated coefficient is the ratio of the geometric mean for the female to the geometric mean for the male students group. For example, in our example, we can say that the expected percent increase in geometric mean from male student group to female student group is about 12% holding other variables constant, since $\exp(.1142399) \approx 1.12$. For reading score, we can say that for a one-unit increase in reading score, we expected to see about 0.7% of increase in the geometric mean of writing score, since $\exp(.0066086) = 1.007$.

Now, let's focus on the effect of **math**. Take two values of **math**, m_1 and m_2, and hold the other predictor variables at any fixed value. The equation above yields

$$\log(\mathbf{write}(m_2)) - \log(\mathbf{write}(m_1)) = \beta_2 \times [\log(m_2) - \log(m_1)]$$

It can be simplified to $\log[\mathbf{write}(m_2)/\mathbf{write}(m_1)) = \beta_2 \times [\log(m_2/m_1)]$, leading to

$$\frac{\mathbf{write}(m_2)}{\mathbf{write}(m_1)} = \left(\frac{m_2}{m_1}\right)^{\beta_2}$$

This tells us that as long as the ratio of the two math scores, m_2/m_1 stays the same, the expected ratio of the outcome variable, $\mathbf{write}$, stays the same. For example, we can say that for any 10% increase in $\mathbf{math}$ score, the expected ratio of the writing score will be $(1.10)^{\beta_2} = (1.10)^{.4085369} = 1.0397057$. In other words, we expect about 4% increase in writing score when math score increases by 10%.

Note:

Here also we can use an approximation method. Since, $(1+x)^a \approx 1 + ax$ for a small value of $|a|x$, therefore for a small change in the predictor variable we can approximate the expected ratio of the of the dependent variable by multiplying the coefficient by the ratio of the change in the predictor variable. For example, we can say that for any 1% increase in $\mathbf{math}$ score, the expected ratio of the writing score is approximately $1 + .01 \times \beta_2 = 1 + .01 \times .4085369 = 1.004085$. The exact value will be $(1.01)^{\beta_2} = (1.01)^{.4085369} = 1.004073$.

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