EE40 Spring 2011 Midterm 2 Solutions

Problem 1

This problem is identical to Example 7.14 in the textbook.

The majority of you wrote a differential equation:

$$\frac{v_o}{R_2} = C\frac{dV}{dt}$$

This is correct, but V is the voltage across the capacitor, which is not v_o! How do we find the voltage across the capacitor?

Lets take an intuitive approach. The initial current flowing through R_2 is:

$$i_{R2}(0) = \frac{v_o(0)}{R_2}$$

Since no current can flow into the input of an op-amp, the current through R_1 is the same as the current through R_2 (this is true for all t). From this, we can calculate the voltage across R_1.

$$V_{R1} = i_{R2} \cdot R_1$$

Because the negative terminal of the op-amp is 0V, the voltage across the capacitor is:

$$V_C = -V_{R1} = -i_{R2} \cdot R_1$$

At t=0,

$$V_C(0) = -\frac{v_o(0)}{R_2} \cdot R_1$$

This is the initial condition voltage across the capacitor. Again, because of the virtual-short rule, R_2 does not affect the capacitor's discharging. Thus, C discharging through R_1 determines the current as a function of time.

$$i(t) = i_0 \cdot e^{-\frac{t}{R1 \cdot C}}$$

Finally, we have:

$$v_o(t) = i_0 R_{2.} \cdot e^{-\frac{t}{R1 \cdot C}} = v_o(0) \cdot e^{-\frac{t}{R1 \cdot C}}$$

Many of you had $v_o(0) \cdot e^{-\frac{t}{(R1+R2) \cdot C}}$ or $v_o(0) \cdot e^{-\frac{t}{R2 \cdot C}}$. We gave very few points for these answers because all the solutions that arrived at these answers were seriously flawed or wild guesses.

The numeric answers for part b are:

Version	**Answer**
1	$v_o(157ms) = 10.1mV$
2	$v_o(249ms) = 51.6mV$
3	$v_o(175ms) = 2.35V$

Problem 2
This problem is an extension of P9/1. Almost all of you got this one.

$$V_C = i_L \cdot R_L$$
$$E_C = \frac{1}{2} C V_C^2$$

$$r = \frac{\frac{1}{2} C {V_{C,f}}^2}{\frac{1}{2} C {V_{C,i}}^2} = \left(\frac{i_{L,f}}{i_{L,i}} \right)^2$$
$$V_C = i_{L,f} \cdot R_L$$

To get the time, we solve the discharging equation:
$$V(t) = V_{C,i} \cdot e^{-\frac{t}{RC}}$$
$$t = -RC \cdot \ln\left(\frac{V_{C,f}}{V_{C,i}} \right)$$

Version	**r**	**V_c**	**T**
1	0.348	4.72V	84.1s
2	0.212	3.22V	211s
3	0.563	5.25V	76.6s

Problem 3
You knew this was coming! This one was copied from homework and last year's midterm and part of the optional HW11 (problem 12).

a)
For V_{dd} = 1V, $f_{s,max}$ = 400MHz.
$$T = \frac{M}{f_{s,\max}}$$
$$E = \frac{1}{2} N \cdot C_{in} \cdot V_{dd}^2 \cdot M \cdot \eta = \frac{1}{2} N \cdot C_{in} \cdot V_{dd}^2 \cdot f_s \cdot T \cdot \eta$$

Version	**T**	**E**
1	12.5ms	40.7mJ
2	22.5ms	25.4mJ
3	7.5ms	6.91mJ

b)

Using, the expression $E = \frac{1}{2} N \cdot C_{in} \cdot V_{dd}^2 \cdot M \cdot \eta$, solve for V_{dd}.

$$V_{dd,x} = \sqrt{\frac{2E}{N \cdot C_{in} \cdot M \cdot \eta}}$$

$$f_x = 400MHz \cdot V_{dd}^2$$

Version	**f_x**	**$V_{dd,x}$**
1	6.55MHz	128mV
2	7.22MHz	134mV
3	39.9MHz	316mV

c)
Some of you are still uncomfortable dealing with multiple processors.
$\frac{1}{2} N \cdot C_{in} \cdot V_{dd}^2 \cdot f_s \cdot T \cdot \eta$ becomes $E = \frac{1}{2} N \cdot C_{in} \cdot V_{dd}^2 \cdot f_s \cdot T \cdot \eta \cdot P$, where f_s is the frequency of each processor and T is the total operating time. My personal preference is to calculate V_{dd} first and then back-calculate f_s and P.

$$E = \frac{1}{2} N \cdot C_{in} \cdot V_{dd}^2 \cdot M \cdot \eta \qquad \text{(this is independent of the number of processors)}$$

$$V_{dd} = \sqrt{\frac{2E}{N \cdot C_{in} \cdot M \cdot \eta}}$$

$$f_s = 400MHz \cdot V_{dd}^2$$

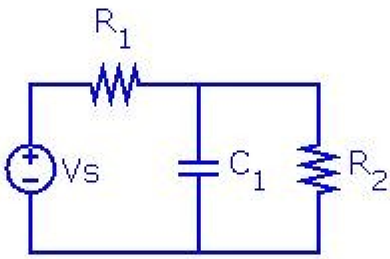
$$T = \frac{M}{f_s \cdot P} \Rightarrow P = \frac{M}{T \cdot f_s}$$

Version	**P**	**f_s**
1	100	68.7MHz
2	182	59.9MHz
3	21	399MHz

Problem 4

This was a pretty difficult problem and a lot of you had trouble with it. Some of you noticed that the circuit was similar to the 555 audio synthesizer you built in lab. The key difference however, is that C_1 does not charge through R_1 and R_2 in series when the switch is closed. We're interested in the steady state solution so lets divide it into 2 phases.

Phase 1 (switch closed):



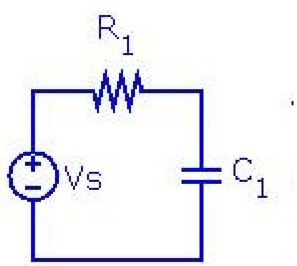
You analyzed this circuit on the sample midterm. I won't solve the differential equation here (please refer to HW11 solutions if you're having trouble with this) and just write the answer.

$$V = \frac{R_2}{R_1 + R_2} V_S + \left(0.7 - \frac{R_2}{R_1 + R_2} \right) V_S e^{-\frac{t}{(R1 \| R2)C}}$$

We solve for $V = 0.3V_S$:

$$T_1 = -(R1 \| R2)C \cdot \ln \left(\frac{0.3V_S - \frac{R_2}{R_1 + R_2} V_S}{\left(0.7 - \frac{R_2}{R_1 + R_2} \right)} \right)$$

Phase 2 (switch open):
This one is much easier to solve as it doesn't involve the parallel resistors.



We charge the capacitor from $0.3V_S$ to $0.7V_S$.

$$V = V_S + \left(0.3V_S - V_S \right) e^{-\frac{t}{R1C}}$$

Solve for t when $V = 0.7V_S$:

$$T_2 = -R_1 C \cdot \ln \left(\frac{0.7V_S - V_S}{0.3V_S - V_S} \right)$$

We want $T_1 + T_2 = T$, so:

$$-R_1 C\cdot\ln\left(\frac{0.7V_S - V_S}{0.3V_S - V_S}\right)+\left(-(R1\,||\,R2)C\cdot\ln\left(\frac{0.3V_S - \frac{R_2}{R_1+R_2}V_S}{\left(0.7-\frac{R_2}{R_1+R_2}\right)}\right)\right)=T$$

$$C=\frac{T}{-R_1\ln\left(\frac{0.7V_S - V_S}{0.3V_S - V_S}\right)+\left(-(R1\,||\,R2)\ln\left(\frac{0.3V_S - \frac{R_2}{R_1+R_2}V_S}{\left(0.7-\frac{R_2}{R_1+R_2}\right)}\right)\right)}$$

Version	C
1	528pF
2	331pF
3	528pF

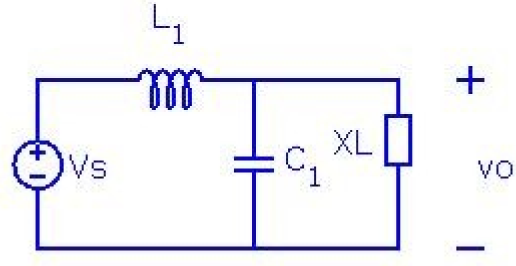
(Woah, two versions randomly have the same answer!)

Problem 5

This problem was assigned in P10 (problem 8). The circuit is called a buck converter and used to efficiently lower the input voltage. Is similar to the boost converter (used to increase the input voltage) you analyzed in lab and it was covered in a review session. The components for and boost converters are a little differently but the concepts are pretty much the same. I know the way we usually explain these converters is a little fuzzy since we assume that v_o is some value and go from there. How do we know v_o settles to some value? Come to our office hours and we'll be happy to explain everything intuitively but for now, let's just accept this.

Again, we have two phases.

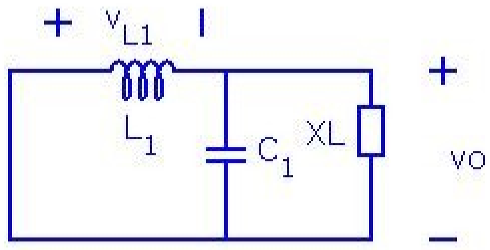
Phase 1 (switch closed):



The diode acts as a open circuit since $v_D < 0$. The current through L_1 increases because $V_S > v_o$. Remember that the change in current in an inductor is the integral of the voltage across it. Since we're integrating a constant voltage, we can simply multiply V_L by T.

$$\Delta i_{L,1}=\frac{1}{L}\cdot T_1\left(V_S - v_o\right)$$

Phase 1 (switch open):



Now the diode acts as a short. The current through the inductor decreases because the V_{L1} is negative.

$$\Delta i_{L,2} = -\frac{1}{L} \cdot T_2 v_o$$

Since the circuit is in steady state, the change in current during the two phases must cancel each other out. Therefore, we have:

$$\frac{1}{L} T_2 v_o = \frac{1}{L} \cdot T_1 \left(V_S - v_o \right) \Rightarrow T_2 v_o = T_1 \left(V_S - v_o \right) \qquad (1)$$

Also, we know:

$$T_1 + T_2 = T \qquad (2)$$

Solve the two equations to get T_1 and T_2:

$$T_1 = \frac{T \cdot v_o}{V_S}$$

$$T_2 = \frac{T(V_S - v_o)}{V_S}$$

We can plug one of these values back into the Δi equations to find the necessary L:

$$L_1 = \frac{T_1 \left(V_S - v_o \right)}{\Delta i_{L,1}}$$

Version	**T_1**	**T_2**	**L_1**
1	1.58us	2.62us	1.54mH
2	0.982us	1.72us	799uH
3	0.8us	1.6us	941uH

Problem 6

This problem tests your ability to make the right approximations. It's the same as Problem 3 in HW8, except R_2 is now replaced with a current source (which seemed to throw some of you off!). The scores for this one were generally pretty low but it seems like most of you ran out of time so you couldn't calculate the capacitor value.

a)
Some of you wrongly assumed that steady state implies no voltage across the inductor (and thus $v_{L,min} = V_{dd}$). Here, we have a switch opening and closing periodically so the inductor voltage is not 0.

We assume that the current through the inductor is $\frac{V_{dd}}{R_1}$ when the I_L goes high. Because inductors resist change in current, I_L draws current "backwards" through R_1, which creates a large negative voltage.

Applying KCL, we have:

$$I_{R1} = I_{L1} - I_L = \frac{V_{dd}}{R_1} - I_L$$

$$v_L = I_{R1}R_1 = R_1 \cdot \left(\frac{V_{dd}}{R_1} - I_L\right)$$

Version	$v_{L,min}$
1	-4.26V
2	-84.2V
3	-37.0V

We gave this solution full credit. However, it's actually pretty far off. The flaw lies in our assumption the current through the inductor is $\frac{V_{dd}}{R_1}$ when I_L goes high, which is only valid if $\frac{L}{R} << t$, which is not true in this case. Here's how to solve it "properly."

Let $I_{L,f1}$ and $L_{L,f2}$ be the current through the inductor at the end of phase 1 and phase 2, respectively. We write down the inductor exponential charging/discharging equations for each phase.

Phase 1 ($I_L = 0$):

$$I_{L1}(t) = \frac{V_{dd}}{R_1} + \left(I_{L,f2} - \frac{V_{dd}}{R_1}\right)e^{-\frac{t}{L/R}}$$

$$\Rightarrow \frac{V_{dd}}{R_1} + \left(I_{L,f2} - \frac{V_{dd}}{R_1}\right)e^{-\frac{0.9T}{L/R}} = I_{L,f1} \qquad (1)$$

Phase 2 ($I_L = I_{L,max}$):

$$I_{L1}(t) = \frac{V_{dd}}{R_1} + I_{L,\max} + \left(I_{L,f1} - \frac{V_{dd}}{R_1} - I_{L,\max} \right) e^{-\frac{0.1T}{L/R}}$$

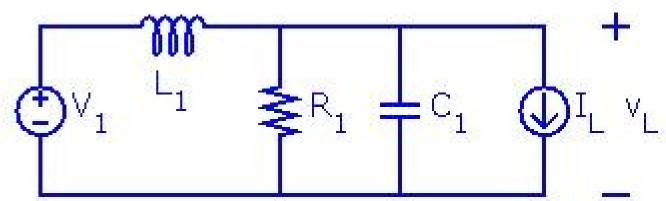
$$\Rightarrow \frac{V_{dd}}{R_1} + I_{L,\max} + \left(I_{L,f1} - \frac{V_{dd}}{R_1} - I_{L,\max} \right) e^{-\frac{0.1T}{L/R}} = I_{L,f2} \qquad (2)$$

With these expressions, we can solve for $I_{L,f1}$ and $L_{L,f2}$. The algebra gets quite messy so it's much easier to do it numerically. The correct answers are:

Version	$\mathbf{v_{L,min}}$
1	-3.73V
2	-82.1V
3	-34.5V

b)

So we need to add something that can deliver lots of current when needed and thus keep v_L stable. Almost all of you remembered that putting a capacitor in parallel with R_1 does the job. During phase 2, I_L can draw current from the capacitor instead of R_1.



During this time, the change in the charge across the capacitor is:

$\Delta Q = C \cdot \Delta V$

We assume that L_1 continues supplying current to R_1 and I_L solely draws current from C_1.

$\Delta V = 0.1 \cdot V_{dd}$

$\Delta Q = I_{L,\max} \cdot \frac{0.1}{f_s}$

$C = I_{L,\max} \cdot \frac{0.1}{f_s} \cdot \frac{1}{0.1 V_{dd}}$

Some of you suggested putting an inductor in series with R_1 or I_L. This doesn't work because the instantaneous change in current will generate an infinite voltage, which is definitely not what we want.