Midterm 1: Solutions

1. When the input to a causal LTI system is:

$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}2^n u[-n-1],$$

the z-transform of the output is:

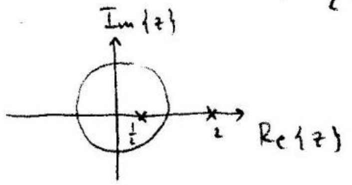
$$Y(z) = \frac{1+z^{-1}}{(1-z^{-1})(1+\frac{1}{2}z^{-1})(1-2z^{-1})}.$$

a) (10 points) Find the z-transform of $x[n]$.

b) (10 points) Determine the region of convergence for $Y(z)$.

c) (10 points) Find the impulse response $h[n]$.

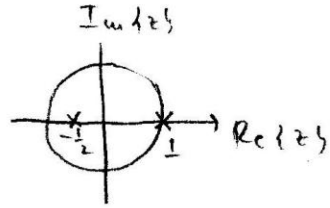
a) $$X(z) = -\frac{1}{3}\cdot\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{4}{3}\frac{1}{1-2z^{-1}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$



Since X is double-sided sequence

$ROC_X: \frac{1}{2} < |z| < 2$

b) $$H(z) = \frac{Y(z)}{X(z)} = \frac{(1-\frac{1}{2}z^{-1})(1+z^{-1})}{(1+\frac{1}{2}z^{-1})(1-z^{-1})}$$



Since the system is causal $\Rightarrow ROC_H: |z| > 1$

$Y(z) = H(z)\cdot X(z) \Rightarrow ROC_Y: |z| > \frac{1}{2}, |z| > 1, |z| < 2$

$\Rightarrow ROC_Y: 1 < |z| < 2$

c) $$H(z) = \frac{(1-\frac{1}{2}z^{-1})(1+z^{-1})}{(1+\frac{1}{2}z^{-1})(1-z^{-1})} = \frac{1-\frac{1}{2}z^{-1}+z^{-1}-\frac{1}{2}z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}} = \frac{1+\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}}{1-\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}}$$

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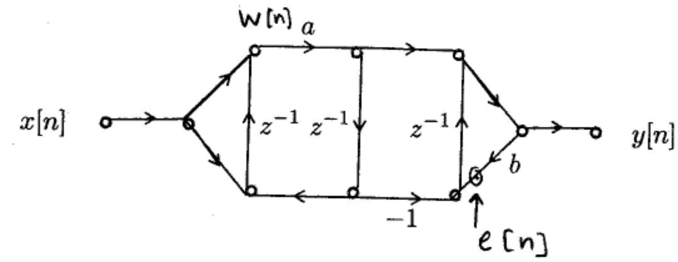
$$= 1 + \frac{z^{-1}}{(1+\frac{1}{2}z^{-1})(1-z^{-1})} = 1 + \frac{2}{3}\frac{1}{1-z^{-1}} - \frac{2}{3}\frac{1}{1+\frac{1}{2}z^{-1}}$$

$$\Rightarrow \boxed{h[n] = \delta[n] + \frac{2}{3}u[n] - \frac{2}{3}\left(-\frac{1}{2}\right)^n u[n]}$$

2. a) (15 points) Find the transfer function implemented by the flow diagram:



b) (5 points) Determine the ranges of the parameters a and b that guarantee BIBO stability.

c) (10 points) Assume that this system is implemented with $(B+1)$-bit two's complement fixed point arithmetic, and the products are rounded to $(B+1)$ bits. Ignoring the quantization noise due to the coefficient a, write an expression for the output noise variance introduced by the coefficient b.

a) $w[n] = x[n] + x[n-1] + a\,w[n-2]$

$y[n] = a\,w[n] + \left(-a\,w[n-2] + b\,y[n-1]\right)$

$W(z) = X(z) + z^{-1}X(z) + az^{-2}W(z)$

$W(z)(1-az^{-2}) = X(z)(1+z^{-1}) \implies W(z) = \dfrac{1+z^{-1}}{1-az^{-2}} \cdot X(z)$

$Y(z) = aW(z) - az^{-2}W(z) + bz^{-1}Y(z)$

$Y(z)(1-bz^{-1}) = W(z)(a-az^{-2}) \implies Y(z) = \dfrac{a(1-z^{-2})}{1-bz^{-1}} \cdot W(z)$

$\implies Y(z) = \dfrac{a(1-z^{-2})(1+z^{-1})}{(1-az^{-2})(1-bz^{-1})} \cdot X(z)$

$\implies \boxed{H(z) = \dfrac{a(1-z^{-2})(1+z^{-1})}{(1-az^{-2})(1-bz^{-1})}}$

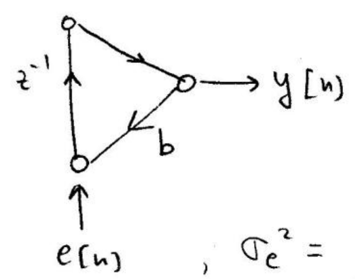
b) 3 poles: $z_1 = \sqrt{a}$, $z_2 = -\sqrt{a}$, $z_3 = b$

Since the system is causal $\Rightarrow$ ROC: $|z| > |\sqrt{a}|$
$|z| > |b|$

Therefore, BIBO stability is guaranteed when

$|a| < 1$, $|b| < 1$

c) Since we set input $x[n]$ to 0, the system reduces to:



$$\Rightarrow \frac{Y(z)}{E(z)} = \frac{z^{-1}}{1 - bz^{-1}} = H_b(z)$$

$$\Rightarrow h_b[n] = b^{n-1} u[n-1]$$

$$\sigma_e^2 = \frac{2^{-2B}}{12}$$

$$\Rightarrow \sigma_y^2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |H_b(e^{j\omega})|^2 \right) \cdot \sigma_e^2 = \left(\sum_n |h_b[n]|^2 \right) \cdot \sigma_e^2$$

$$= \sigma_e^2 \sum_{n=1}^{\infty} |b^{n-1}|^2 = \sigma_e^2 \sum_{n=0}^{\infty} |b|^{2n} = {}^{5}\sigma_e^2 \cdot \frac{1}{1-|b|^2}$$

3. Consider the length-8 sequence:

$$x[n] = \{-3,\ 5,\ 4,\ -1,\ -9,\ -6,\ -8,\ 2\}$$

and let $X[k]$ be its 8-point DFT.

a) (10 points) Evaluate $\sum_{k=0}^{7}(-1)^k X[k]$.

b) (10 points) Evaluate $\sum_{k=0}^{7}|X[k]|^2$.

c) (10 points) Determine the DFT of $X[n]$; that is, the sequence obtained by applying the DFT twice to $x[n]$.

d) (10 points) Find the length-4 sequence $w[n]$ whose 4-point DFT $W[k]$ is given by:

$$W[k] = X[2k] \quad k = 0, 1, 2, 3.$$

a) $x[n] = \frac{1}{8}\sum_{k=0}^{7} X[k] W_8^{-nk}$ $\qquad W_8^{-4k} = e^{j\cdot\frac{8\pi k}{8}} = (-1)^k$

$\Rightarrow x[4] = \frac{1}{8}\sum_{k=0}^{7}(-1)^k \cdot X[k] \Rightarrow \sum_{k=0}^{7}(-1)^k \cdot X[k] = 8\cdot x[4] = -72$

b) Parseval's theorem: $\sum_{n=0}^{7}|x[n]|^2 = \frac{1}{8}\sum_{k=0}^{7}|X[k]|^2$

$\Rightarrow \sum_{k=0}^{7}|X[k]|^2 = 8\cdot\sum_{n=0}^{7}|x[n]|^2 = 1888$

c) $DFT(DFT(x[n])) = 8\cdot[x((-k))_8)]$

$= \{-24, 16, -64, -48, -72, -8, 32, 40\}$

$W[k] = X[2k] \qquad k = 0, 1, 2, 3$

$$X[2k] = \sum_{n=0}^{7} x[n] \, W_8^{2kn}$$

$$= \sum_{n=0}^{3} x[n] \cdot W_4^{kn} + \sum_{n=4}^{7} x[n] \, W_4^{kn}$$

$$= \sum_{n=0}^{3} x[n] \, W_4^{kn} + \sum_{n=0}^{3} x[n+4] \, W_4^{k(n-4)}$$

$$= \sum_{n=0}^{3} x[n] \, W_4^{kn} + \sum_{n=0}^{3} x[n+4] \, W_4^{kn}$$

$$= \sum_{n=0}^{3} \left(x[n] + x[n+4] \right) \cdot W_4^{kn}$$

$\Rightarrow \; w[n] = x[n] + x[n+4] \qquad$ for $\qquad n = 0, 1, 2, 3$

$w[n] = \{ -12, -1, -4, 1 \}$