EE 120: Signals and Systems MIDTERM 1
Department of EECS 1 October 2008
UC BERKELEY

LAST Name Philter FIRST Name Fir

Discussion Time 365/7/24

- **(10 Points)** Print your name and discussion time in legible, block lettering above AND on the last page where the grading table appears.

- This exam should take up to 90 minutes to complete. You will be given at least 90 minutes, up to a maximum of 110 minutes, to work on the exam.

- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided $8.5" \times 11"$ sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.

- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.

- Please write neatly and legibly, because *if we can't read it, we can't grade it.*

- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*

- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.

- We hope you do a *fantastic* job on this exam.

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

MT1.1 (30 Points) The following discrete-time systems F, G, H should be treated mutually independently; properties that hold for one system *cannot* be assumed to hold for the others.

For each part, explain your reasoning succinctly, but clearly and convincingly.

(a) A discrete-time system $\mathsf{F} : [\mathbb{Z} \to \mathbb{C}] \to [\mathbb{Z} \to \mathbb{C}]$ produces the output signal y described by

$$y(n) = \left(\frac{1}{2}\right)^n, \quad \forall n,$$

in response to the input signal x characterized by

$$x(n) = \left(\frac{1}{2}\right)^n u(n), \quad \forall n.$$

Select the strongest true assertion from the list below.

(i) The system must be BIBO stable.

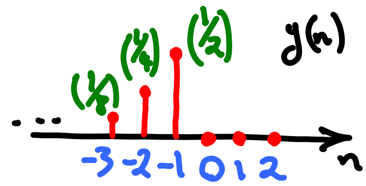
(ii) The system could be BIBO stable, but does not have to be.

(iii) The system cannot be BIBO stable.

The input x is bounded; namely, $|x(n)| \leq 1$, $\forall n$. However, the output y grows exponentially, without bound, as $n \longrightarrow -\infty$. Therefore, the system <u>cannot</u> be BIBO stable.

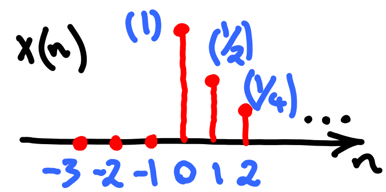
(b) A *time-invariant* discrete-time system $\mathsf{G} : [\mathbb{Z} \to \mathbb{C}] \to [\mathbb{Z} \to \mathbb{C}]$ produces the output signal y described by

$$y(n) = 2^n u(-n-1), \forall n\,,$$



in response to the input signal x characterized by

$$x(n) = \left(\frac{1}{2}\right)^n u(n), \quad \forall n.$$

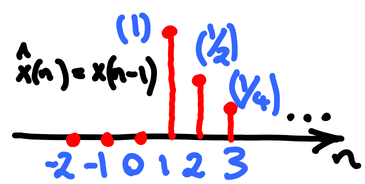


Select the strongest true assertion from the list below.

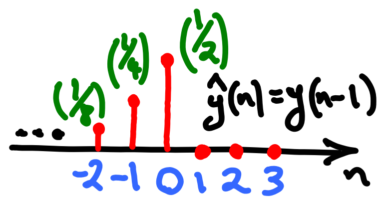
(i) The system must be causal.

(ii) The system could be causal, but does not have to be.

(iii) The system cannot be causal.



The system is known to be time invariant. Hence, if $\hat{x}$ described by $\hat{x}(n) = x(n-1)$ is applied to G, the corresponding output $\hat{y}$ is $\hat{y}(n) = y(n-1)$. Note that $x(n) = 0 = \hat{x}(n)$ $\forall n \leq -1$, but $\hat{y}(-1) = y(-2) = 1/4$, whereas $y(-1) = \frac{1}{2} \neq \hat{y}(-1)$. So the system cannot be causal.



(c) A discrete-time system $\mathsf{H} : [\mathbb{Z} \to \mathbb{C}] \to [\mathbb{Z} \to \mathbb{C}]$ produces an output signal y described by $y = \mathrm{Re}(x)$, in response to every input signal x. Select the strongest true assertion from the list below.

(i) The system must be linear.

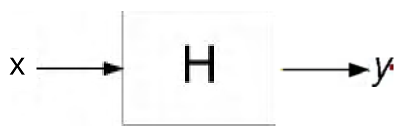
(ii) The system could be linear, but does not have to be.

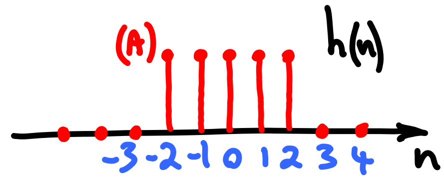
(iii) The system cannot be linear.

The system satisfies the additivity property. That is, if $y_1 = \mathrm{Re}(x_1)$ and $y_2 = \mathrm{Re}(x_2)$, then the output y corresponding to the input $x = x_1 + x_2$ is $y = \mathrm{Re}(x) = \mathrm{Re}(x_1 + x_2) = \mathrm{Re}(x_1) + \mathrm{Re}(x_2)$, so $y = y_1 + y_2$. However, the system fails the scaling property. In particular, if $x = x_R + i x_I$, then $y = \mathrm{Re}(x) = x_R$. However, if the input $\hat{x} = i x$ is applied, we know $\hat{x} = i(x_R + i x_I) = -x_I + i x_R \Rightarrow$ $\hat{y} = \mathrm{Re}(\hat{x}) = -x_I$, which is not $i x_R$.

MT1.2 (45 Points) Consider a discrete-time FIR filter $\mathsf{H} : [\mathbb{Z} \to \mathbb{R}] \to [\mathbb{Z} \to \mathbb{R}]$ having impulse response h and frequency response H.

h has region of support that is 2K+1 samples long





Suppose the impulse response h is a finite-length rectangular pulse, described by

$$h(n) = \begin{cases} A & |n| \leq K \\ 0 & |n| > K\,, \end{cases}$$

where $A > 0$ and $K \in \{1, 2, 3, \ldots\}$.

(a) Determine (in terms of A, K, or both) a reasonably simple expression for the frequency response $H(\omega)$.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n} = A\sum_{n=-K}^{K} e^{-i\omega n} \overset{\text{if } \omega \neq 0}{=} A\,\frac{e^{-i\omega(K+1)} - e^{i\omega K}}{e^{-i\omega} - 1} = \frac{A\cancel{e^{-i\omega/2}}}{\cancel{e^{-i\omega/2}}}\,\frac{e^{-i\omega(K+\frac{1}{2})} - e^{i\omega(K+\frac{1}{2})}}{e^{-i\omega/2} - e^{i\omega/2}}$$

$$\boxed{H(\omega) = A\,\frac{\sin[\omega(K+\frac{1}{2})]}{\sin(\omega/2)}}$$

You could also look this up in the Tables provided

If $\omega = 0 \Rightarrow H(0) = \sum_{n=-K}^{K} A = (2K+1)A$. It turns out, this is exactly $\lim_{\omega \to 0} H(\omega)$ in the expression above.

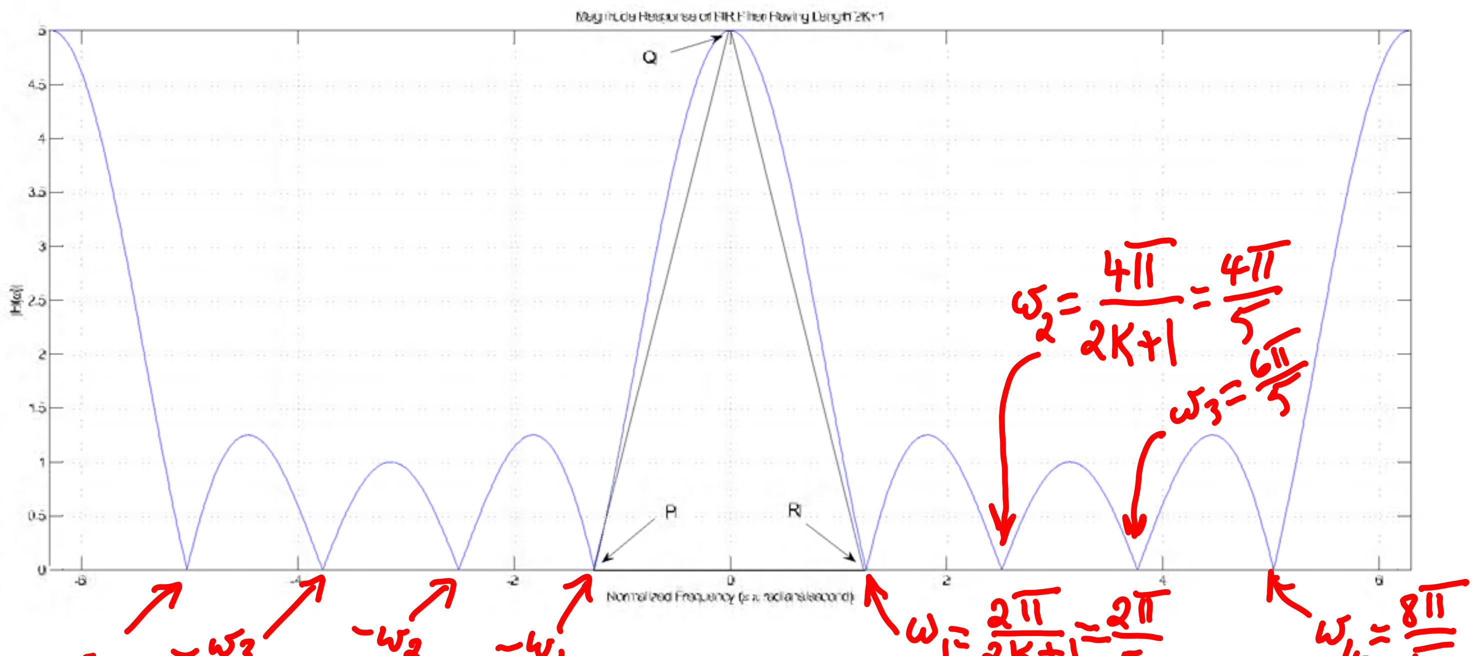
(b) Determine (in terms of A, K, or both) a simple expression for $\int_{\langle 2\pi \rangle} H(\omega)\, d\omega$, the area under the filter's frequency response curve.

Evaluate the DTFT synthesis equation $h(n) = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} H(\omega)\, e^{+i\omega n}\, d\omega$ at $n = 0$.

$$h(0) = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} H(\omega)\, d\omega \Rightarrow \int_{\langle 2\pi \rangle} H(\omega)\, d\omega = 2\pi\, h(0)$$

So $$\boxed{\int_{\langle 2\pi \rangle} H(\omega)\, d\omega = 2\pi A}$$

(c) The figure below depicts $|H(\omega)|$, the magnitude of the filter's frequency response, for a particular value of A and a particular value of K. The plot shows the region $-2\pi < \omega < 2\pi$.



(i) Determine (in terms of A, K, or both) the frequencies at which $|H(\omega)| = 0$. Then compute the particular numerical values of A and K that produced the magnitude response plot above. Be sure to label on the plot the values of all the frequencies at which $H(\omega) = 0$.

$H(\omega)=0$ for $\omega(K+1/2)=\pi l$ $\quad l \in \mathbb{Z}-\{0\}$. In other words, $\omega_l = \frac{\pi l}{K+1/2} = \frac{2\pi l}{2K+1}$ $\quad l \in \mathbb{Z}-\{0\}$. As long as $l \in \{1,2,\ldots,2K\}$, we're assured that $\sin(\omega/2) \neq 0$. The first $\omega > 0$ at which the denominator of $H(\omega)$ is zero is $\omega = 2\pi$. The plot shows four zero crossings between $\omega=0$ & $\omega = 2\pi$. Hence, $2K=4 \Rightarrow \underline{K=2}$. $H(0)=(2K+1)A = 5A = 5 \Rightarrow \underline{A=1}$

(ii) Determine (in terms of A, K, or both) the area of the triangle PQR inscribed within the main lobe of $|H(\omega)|$, and compare your answer with what you found in part (b) above.

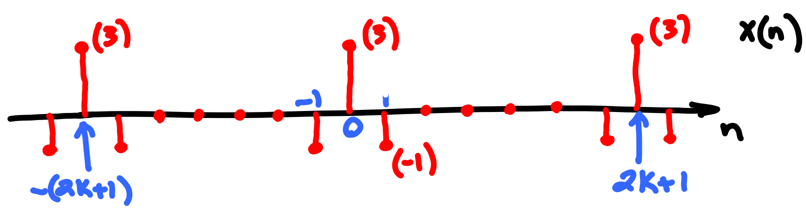
The first zero crossing frequencies on either side of $\omega=0$ are $\omega_1 = \frac{2\pi}{2K+1}$ and $\omega_{-1} = \frac{-2\pi}{2K+1}$.

So the area of the triangle is the height $(2K+1)A$, times half the length of the base $\frac{2\pi}{2K+1}$, i.e., $\text{Area}(\Delta_{PQR}) = (2K+1)A\,\frac{2\pi}{2K+1} \Longrightarrow \text{Area}(\Delta_{PQR}) = 2\pi A$, which is $\int_{\langle 2\pi\rangle} |H(\omega)|\,d\omega$.

(iii) Suppose the input signal x is defined as the periodic extension of a finite-duration signal x_0 as follows:

$$x_0(n) = -\delta(n+1) + 3\,\delta(n) - \delta(n-1) \qquad \text{and} \qquad x(n) = \sum_{\ell=-\infty}^{+\infty} x_0\Big(n-(2K+1)\ell\Big)$$

Determine the response of the filter to the input signal x. If you're confident in the numerical values you found in part (c)(i), you may express your answer here numerically. Otherwise, you may express your answer in terms of A, K, or both.



The signal x is periodic with period $2K+1$ and fundamental frequency $\omega_0 = \frac{2\pi}{2K+1}$. It has a DFS expansion $x(n) = \sum_{\ell=0}^{2K} X_\ell e^{i\ell\omega_0 n} = \sum_{\ell=0}^{2K} X_\ell e^{i\ell\frac{2\pi}{2K+1}n}$

But $\omega_0, 2\omega_0, \ldots, 2K\omega_0$ are all frequencies at which $H(\omega)=0$. So the only portion of x that passes through the filter is $X_0 e^{i0\cdot\omega_0 n} = X_0$. The output $y(n) = H(0)\,X_0 e^{i0n} = (2K+1)A\,X_0$.

$$X_0 = \frac{1}{2K+1}\sum_{\ell=0}^{2K} x(n) = \frac{(2K+1)A}{2K+1} = A \Rightarrow y(n) = (2K+1)A^2$$

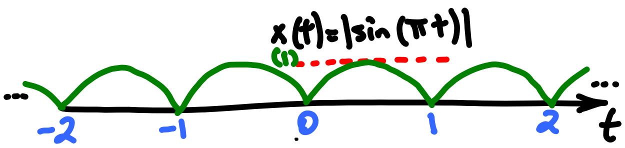
MT1.3 (30 Points) Parts (a) and (b) of this problem are mutually independent and may be tackled in either order.

(a) A full-wave rectified sine wave x is described by $x(t) = |\sin(\pi t)|, \quad \forall t.$

(i) Determine the complex exponential Fourier series coefficients of x.
Note: You may find the following integral useful:

$$\int_a^b e^{\lambda t}\, dt = \frac{e^{b\lambda} - e^{a\lambda}}{\lambda}.$$

(ii) Use your result in Part (a)(i) to prove the identity



$$\sum_{k=-\infty}^{+\infty} \frac{(-1)^k}{1-4k^2} = \frac{\pi}{2}.$$

The signal x has period $p=1$ and fundamental frequency $\omega_0 = \frac{2\pi}{1} = 2\pi$. Its Fourier series coefficients are

$$X_k = \frac{1}{1}\int_0^1 \sin(\pi t) e^{ik2\pi t}\, dt$$

$$= \frac{1}{2i}\left[\int_0^1 e^{i\pi(2k+1)t}\, dt - \int_0^1 e^{i\pi(2k-1)t}\, dt\right]$$

$$= \frac{1}{2i}\left[\frac{e^{i\pi(2k+1)} - 1}{i\pi(2k+1)} - \frac{e^{i\pi(2k-1)} - 1}{i\pi(2k-1)}\right]$$

$$= \frac{1}{2i}\left[\frac{-2}{i\pi(2k+1)} + \frac{2}{i\pi(2k-1)}\right] = \frac{-2}{2\pi}\left(\frac{-1}{2k+1} + \frac{1}{2k-1}\right)$$

$$= \frac{-1}{\pi}\,\frac{1 - \cancel{2k} + \cancel{2k} + 1}{4k^2 - 1} = \frac{2}{\pi}\,\frac{1}{1-4k^2}$$

$$x(1/2) = 1 = \frac{2}{\pi}\sum_{k=-\infty}^{\infty} \frac{e^{ik2\pi(1/2)}}{1-4k^2} \Rightarrow \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{1-4k^2} = \frac{\pi}{2}$$

(b) A continuous-time signal x_{CT}, characterized by $x_{\mathsf{CT}}(t) = \cos(\omega_0 t)$, $\forall t$, is sampled every T seconds to produce a discrete-time signal x_{DT} described by $x_{\mathsf{DT}}(n) = x_{\mathsf{CT}}(nT)$, $\forall n$.

Determine a condition on the sampling period T (in terms of ω_0) to guarantee that x_{DT} is periodic.

$$x_{DT}(n) = x_{CT}(nT) = \cos(\underbrace{\omega_0 T}_{\Omega_0} n) = \cos(\Omega_0 n)$$

The DT signal x_{DT} is periodic if, and only if, $\Omega_0 = \frac{L}{M}\pi$ where $\frac{L}{M}$ is a rational number. Hence, x_{DT} is periodic if, and only if,

$$\omega_0 T = \frac{L}{M}\pi \iff \underline{T = \frac{L}{M}\frac{\pi}{\omega_0}}$$

You may use the blank space below for scratch work. Nothing written below this line on this page will be considered in evaluating your work.

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Discussion Time 365/24/7

Problem Name	Points	Your Score
	10	10
1	30	30
2	45	45
3	30	30
Total	**115**	115