MIDTERM 1 SOLUTIONS

1. Consider the LTI system:

$$y[n] - 1.1y[n-1] + 0.3y[n-2] = x[n].$$

a) (10 points) Write the transfer function and determine if the system is BIBO stable.

b) (15 points) Use z-tranforms to find the output $y[n]$ when $x[n] = (0.6)^n u[n]$ where $u[n]$ is the unit step function.

Now consider the median filter:

$$y[n] = \text{median}\{x[n], x[n-1], x[n-2], x[n-3], x[n-4]\}.$$

c) (10 points) Find the impulse and step responses, and determine if this filter is BIBO stable.

a) Take Z-Transform.

$$Y(z) - 1.1z^{-1}Y(z) + 0.3z^{-2}Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-1.1z^{-1}+0.3z^{-2}} = \frac{1}{(1-0.5z^{-1})(1-0.6z^{-1})} \quad +5$$

System is causal. Poles @ $z = 0.5, 0.6$ +2

ROC extends outward from $|z|=0.6$, includes unit circle $\Rightarrow$ BIBO stable +3

b) $$x[n] = (0.6)^n u[n] \overset{Z}{\longleftrightarrow} X(z) = \frac{1}{1-0.6z^{-1}}$$

+5 $$Y(z) = H(z)X(z) = \frac{1}{(1-0.5z^{-1})(1-0.6z^{-1})} \cdot \frac{1}{1-0.6z^{-1}} = \frac{1}{(1-0.5z^{-1})(1-0.6z^{-1})^2}$$

+5 Use PFE to simplify:

(or residue) $$Y(z) = \frac{1}{(1-0.5z^{-1})(1-0.6z^{-1})^2} = \frac{A}{(1-0.5z^{-1})} + \frac{B}{(1-0.6z^{-1})} + \frac{C}{(1-0.6z^{-1})^2}$$

$$A = \left.\frac{1}{(1-0.6z^{-1})^2}\right|_{z=0.5} = 25 \; ; \; C = \left.\frac{1}{1-0.5z^{-1}}\right|_{z=0.6} = 6 \; ;$$

$$1 = A(1-0.6z^{-1})^2 + B(1-0.5z^{-1})(1-0.6z^{-1}) + C(1-0.5\ldots$$

$$1 = A(1-1.2z^{-1}+0.36z^{-2}) + B(1-1.1z^{-1}+0.3z^{-2}) + C(1-0.5z\ldots$$

$$z^2 = A(z^2-1.2z+0.36) + B(z^2-1.1z+0.3) + C(z^2-0.5z\ldots$$

take derivative twice

$$1 = A + B + C$$

$$B = 1 - 25 - 6 = -30$$

$$Y(z) = \frac{25}{(1-0.5z^{-1})} - \frac{30}{(1-0.6z^{-1})} + \frac{6}{(1-0.6z^{-1})^2}$$

+5 $$\overset{Z^{-1}}{\longleftrightarrow} y[n] = 25(0.5)^n u[n] - 30(0.6)^n u[n] + 10(n+1)(0.6)^{n+1} u[n+1]$$

(+2 if any attempt at Z^{-1} transform)

note: $$\frac{6}{(1-0.6z^{-1})^2} = 10z\frac{0.6z^{-1}}{(1-0.6z^{-1})^2} \overset{Z^{-1}}{\longleftrightarrow} 10(n+1)(0.6)^{n+1} u[n+1]$$

w

Additional workspace for Problem 1

C. $y[n] = \text{median}\{x[n], x[n-1], x[n-2], x[n-3], x[n-4]\}$

Impulse response:

$h[n] = \text{median}\{\delta[n], \delta[n-1], \delta[n-2], \delta[n-3], \delta[n-4]\}$ +3

$= \text{median}\{1, 0, 0, 0, 0\} = 0$ +1

Step response:

$a[n] = \text{median}\{u[n], u[n-1], u[n-2], u[n-3], u[n-4]\}$ +3

$a[0] = \text{median}\{1, 0, 0, 0, 0\} = 0$

$a[1] = \text{median}\{1, 1, 0, 0, 0\} = 0$ +1

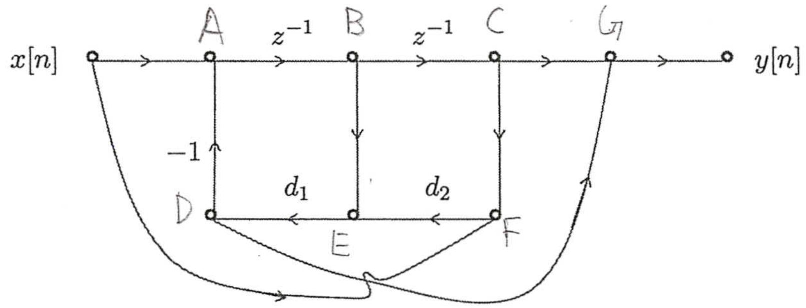
$a[2] = \text{median}\{1, 1, 1, 0, 0\} = 1$

$= 1$

BIBO stable because the max value of $y[n]$ is bounded by the maximum value of $x[n]$.

+2

2. a) (15 points) Find the transfer function implemented by the flow diagram:



b) (15 points) Assume that this system is implemented with $(B+1)$-bit two's complement fixed point arithmetic, and the products are rounded to $(B+1)$ bits. Draw a linear noise model and write an expression for the output noise variance.

a) Find the node equations:

$A = X - D$

$B = z^{-1} A$

$C = z^{-1} B$

$D = d_1 E$

$E = B + d_2 F$

$F = C + X$

$G = C + D$

+10 correct eq.

Solve for nodes other [illegible]:

$B = z^{-1}(X - d_1(B + d_2(z^{-1}B + X)))$

$B = z^{-1}X - d_1 z^{-1}B - z^{-2} d_1 d_2 B - z^{-1} d_1 d_2 X$

$B + d_1 z^{-1} B + z^{-2} d_1 d_2 B = z^{-1}X - z^{-1} d_1 d_2 X$

$$B = \frac{(z^{-1} - z^{-1} d_1 d_2)}{(1 + d_1 z^{-1} + z^{-2} d_1 d_2)} X$$

$$C = \frac{z^{-1}(z^{-1} - z^{-1} d_1 d_2)}{(1 + d_1 z^{-1} + z^{-2} d_1 d_2)} X$$

Solve for Y:

$Y = G = C + D = C + X - z^{-2} C \ldots$

$$Y = \left(\frac{(1 - z^{-2})(z^{-2} - z^{-2} d_1 d_2)}{(1 + d_1 z^{-1} + z^{-2} d_1 d_2)} + 1\right) X$$

$$= \left(\frac{z^{-2} - z^{-2} d_1 d_2 - 1 + d_1 d_2 + 1 + d_1 z^{-1} + z^{-2} d_1 d_2}{(1 + d_1 z^{-1} + z^{-2} d_1 d_2)}\right)$$

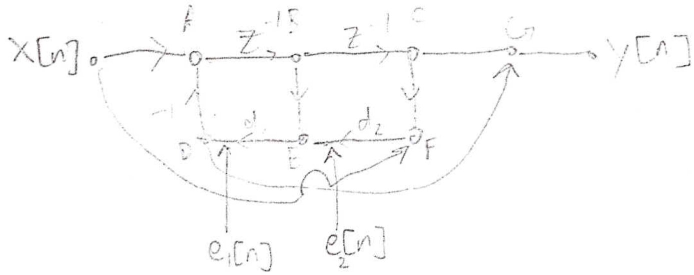
$$Y = \frac{z^{-2} + d_1 d_2 + d_1 z^{-1})}{(1 + d_1 z^{-1} + z^{-2} d_1 d_2)} X$$

$$H(z) = \frac{Y(z)}{X(z)} = \left(\frac{d_1 d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_1 d_2 z^{-2}}\right)$$

+2 attempt to solve

+3 if correct

b)



Correct error placement
+3 for each
-1 for extra

Note: no error after (−1) branch because it is only changing sign, not magnitude.

Output noise variance:

Combine errors to node D: $e[n] = e_1[n] + d_1 e_2[n]$

$$H_{DY} = \frac{Y(z)}{D(z)} = \frac{\left(\frac{d_1 d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_1 d_2 z^{-2}}\right)}{\left(\frac{d_1 d_2 + d_1 z^{-1} + z^{-2} d_1 d_2}{1 + d_1 z^{-1} + d_1 d_2 z^{-2}}\right)} = \left(\frac{d_1 d_2 + d_1 z^{-1} + z^{-2}}{d_1 d_2 + d_1 z^{-1} + d_1 d_2 z^{-2}}\right)$$

$$\sigma_y^2 = \sigma_e^2 \sum_{n=-\infty}^{\infty} |h(n)|^2$$

Parseval's Theorem: $\sum_{n=-\infty}^{\infty} |h(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega$

$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$

~~+2~~ if σ_e^2
+5 for writing eqn. for σ_y^2

+ 2 for solving correctly

3. The following are the first 7 samples of a 12-point DFT $X[k]$ of a length-12 real sequence $x[n]$:

$$X[k] = \{11,\ 8-2j,\ 1-12j,\ 6+3j,\ -3+2j,\ 2+j,\ 15\} \quad 0 \le k \le 6.$$

a) (10 points) Determine the remaining 5 samples of $X[k]$.

b) (15 points) Evaluate the following values: i) $x[0]$, ii) $x[6]$, iii) $\sum_{n=0}^{11} x[n]$.

c) (10 points) The 12-point DFT of another length-12 sequence $y[n]$ is given by:

$$Y[k] = \begin{cases} 12 & \text{if } k = 0 \\ 0 & \text{if } 1 \le k \le 11. \end{cases}$$

Calculate the 12-point circular convolution of $x[n]$ and $y[n]$.

a. X[n] is a real sequence
Therefore X[K] is even periodic +4
$X[k] = \{2-j, -3-2j, 6-3j, 1+12j, 8+2j\}$ $7 \le k \le 11$ +6

b. $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}$ $\quad W_N = e^{-j\frac{2\pi}{N}}$ +3

i) $x[0] = \frac{1}{12}\sum_{k=0}^{11} X[k] = \frac{54}{12} = 4.5$ +4

ii) $x[6] = \frac{1}{12}\sum_{k=0}^{11} (-1)^n X[k] = \frac{1}{12}[11 - 8+2j + 1-12j - 6-3j - 3+2j - 2-j + 15 - 2+j - 3-2j - 6+3j + 1+12j - 8-2j]$
$= -\frac{10}{12} = -\frac{5}{6}$ +4

iii) $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$ +4

$X[0] = \sum_{n=0}^{11} x[n] = 11$

c. $X[K]Y[K] = \begin{cases} 132 & \text{if } k=0 \\ 0 & \text{if } 1 \le k \le 11 \end{cases}$ +2 +4

taking the IDFT +2 +2
$x[n] = \frac{1}{12}\sum_{k=0}^{11} X[k] W_N^{-kn} = \frac{1}{12}X[0] = 11$ $\quad 0 \le n \le N-1$