LAST Name ________________ FIRST Name ________________

Lab Time ________________

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 120 minutes to complete. You will be given at least 120 minutes, up to a maximum of 170 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except four double-sided $8.5" \times 11"$ sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 22.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the twenty two numbered pages. If you find a defect in your copy, notify the staff immediately.
- **You will be given a separate document containing formulas and tables.**
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

F-S06.1 (30 Points) Each of the pole-zero diagrams (I)-(VI) on the next two pages belongs to a *causal* LTI system $\mathsf{H} : [\mathbb{R} \to \mathbb{R}] \to [\mathbb{R} \to \mathbb{R}]$ having a rational transfer function $\widehat{H}$ and an impulse response h. Each of the impulse-response plots (a)-(f) in the subsequent pages may correspond to *at most* one of the pole-zero diagrams (I)-(VI). Match each pole-zero diagram with an impulse response plot, or explain why no such match exists. Justify each of your choices succinctly, but clearly and convincingly. Show your work on this page.

(I)

Answer:
Using

$$\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2 + \omega_0^2}$$

poles on the imaginary axis imples that the time domain signal exhibits non-decaying and non-growing oscillatory behavior. In this case $\omega_0 = 6\pi$. This matches **plot (b)**.

(II)

Answer:
The poles are in the left half plane and the system is causal implying that $ROC(h)$ includes the $i\omega$ axis and the system is stable. Furthermore we use

$$e^{-t}\sin(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{(s+1)^2 + \omega_0^2}$$

to determine that $h(t)$ is decaying and oscillatory. Here $\omega_0 = 6\pi$. This matches **plot (e)**.

(III)

Answer:
The poles are in the right half plane and the system is causal implying that $ROC(h)$ does not include the $i\omega$ axis and the system is unstable. Furthermore we use

$$e^{t}\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s-1}{(s-1)^2 + \omega_0^2}$$

to determing that $h(t)$ exhibits an exponentially growing oscillatory behavior. Again $\omega_0 = 6\pi$. This matches **plot (f)**.

(IV)

Answer:
The pole is real and in the right half plane. Furthermore the system is unstable. This implies that $h(t)$ is a growing exponential. This matches **plot (d)**.

(V)

Answer:
There is a double pole at $s = 0$. This suggests $H(s)$ is of the form

$$H(s) = \frac{\alpha}{s^2} \quad \alpha \in \mathbb{R}$$

Thus $h(t)$ is proportional to the integral of the unit step, $u(t)$ (which has Laplace transform $\frac{1}{s}$). This matches **plot (a)**.

(VI)

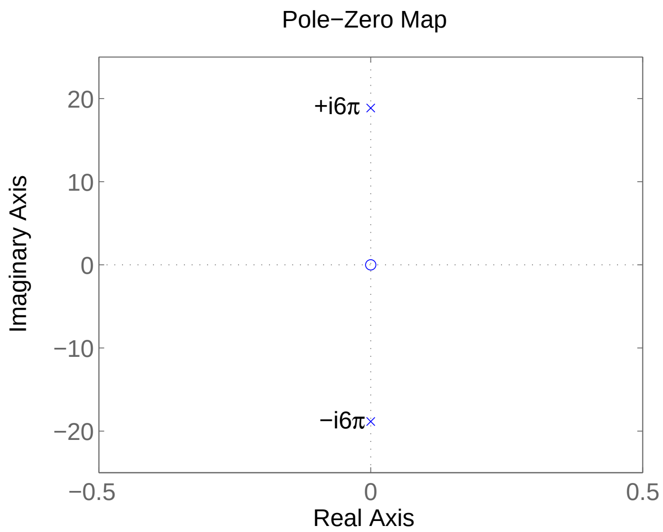
Answer:
Using the scaling by t property:

$$tx(t) \xleftrightarrow{\mathcal{L}} -\frac{d}{ds}\hat{X}(s)$$

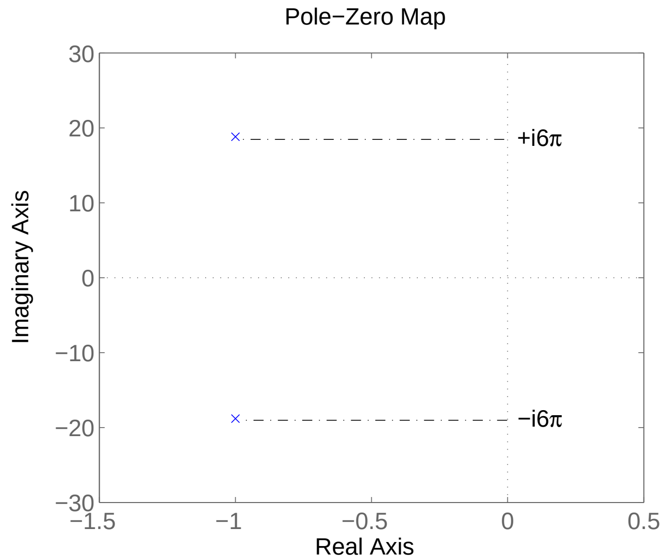
and the transform pair:

$$\sin(\omega_0 t)u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{s^2 + \omega_0^2}$$

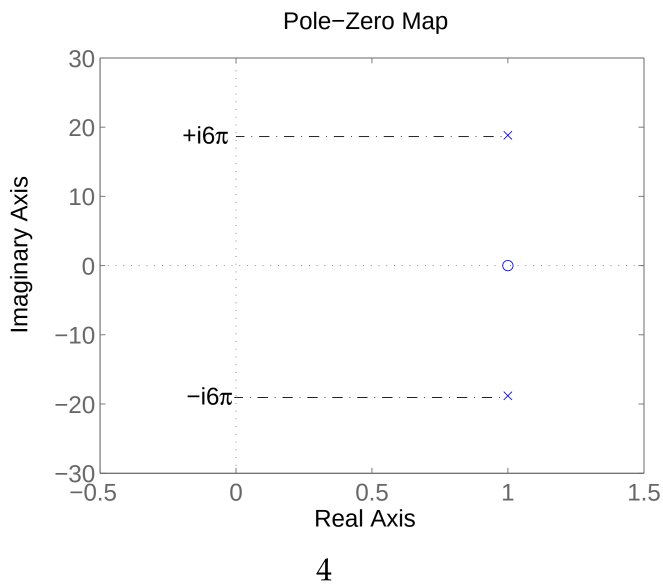
we see that this pole-zero plot corresponds to $t\sin(\omega_0 t)u(t)$, which is a linearly growing sinusoid. This matches **plot (c)**.


Pole−Zero Map
+i6π
−i6π
Imaginary Axis
Real Axis
20
10
0
−10
−20
−0.5
0
0.5

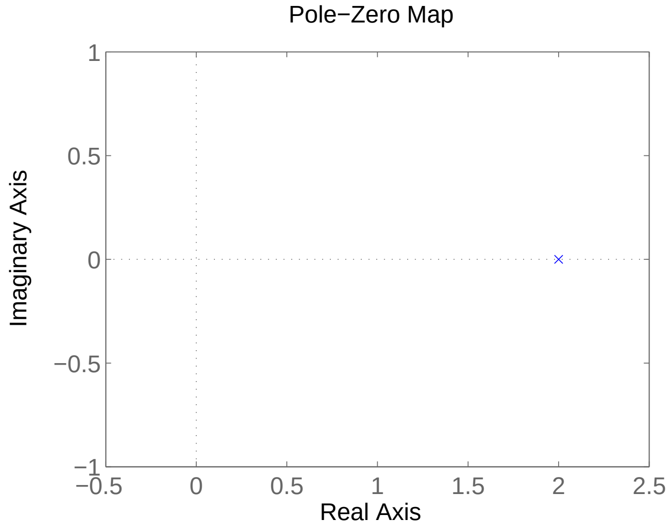
(I)


Pole−Zero Map
+i6π
−i6π
Imaginary Axis
Real Axis
30
20
10
0
−10
−20
−30
−1.5
−1
−0.5
0
0.5

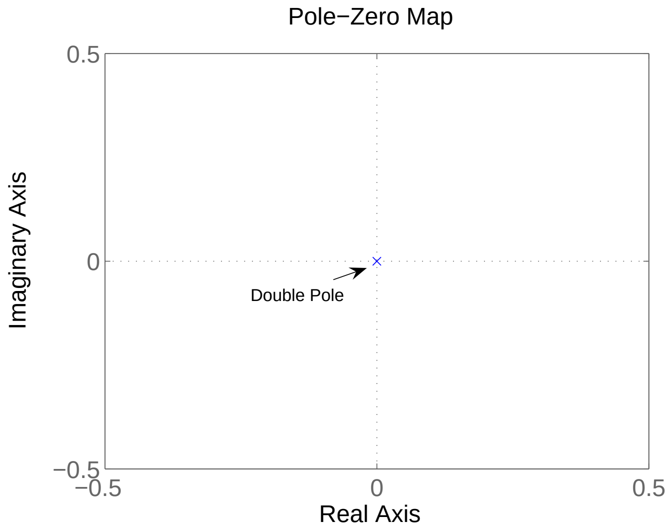
(II)


Pole−Zero Map
+i6π
−i6π
Imaginary Axis
Real Axis
30
20
10
0
−10
−20
−30
−0.5
0
0.5
1
1.5

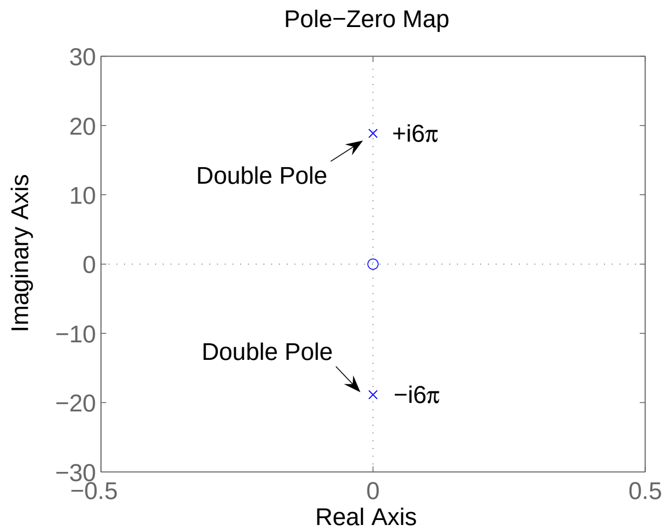
(III)


Pole−Zero Map
Imaginary Axis
Real Axis
1
0.5
0
−0.5
−1
−0.5
0
0.5
1
1.5
2
2.5

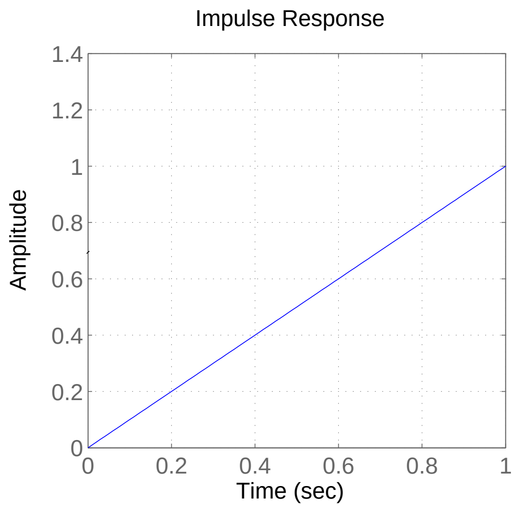
(IV)


Pole−Zero Map
Imaginary Axis
Real Axis
Double Pole
0.5
0
−0.5
−0.5
0
0.5

(V)

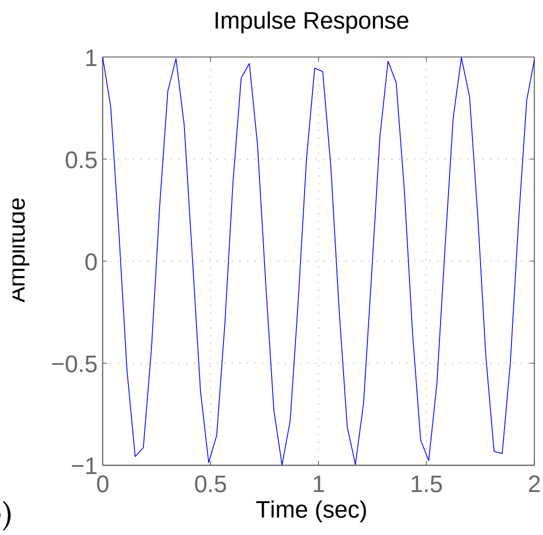

Pole−Zero Map
Imaginary Axis
Real Axis
+i6π
Double Pole
Double Pole
−i6π
30
20
10
0
−10
−20
−30
−0.5
0
0.5

(VI)



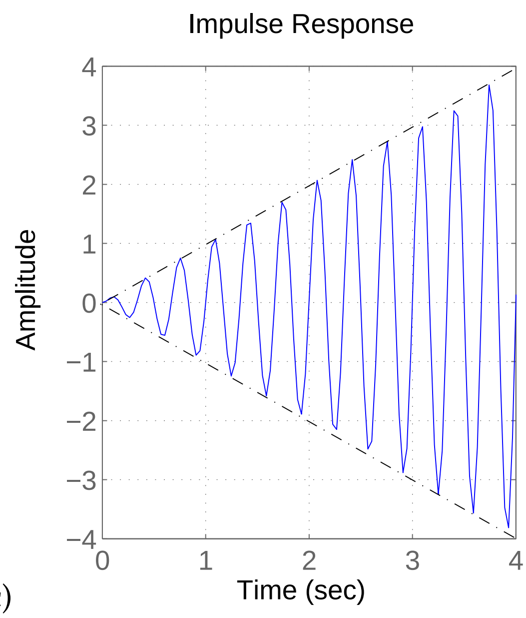
Impulse Response
Amplitude
Time (sec)

(a)



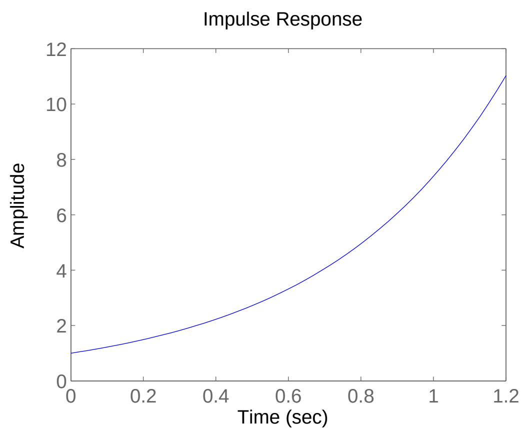
Impulse Response
Amplitude
Time (sec)

(b)

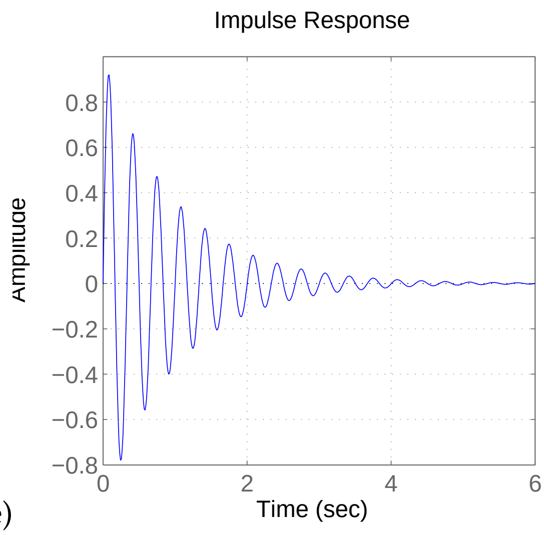


Impulse Response
Amplitude
Time (sec)

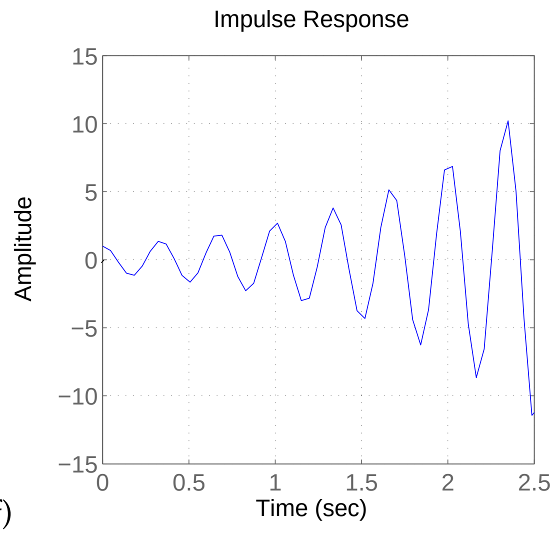
(c)


Impulse Response
Amplitude
Time (sec)
0
2
4
6
8
10
12
0.2
0.4
0.6
0.8
1
1.2

(d)


Impulse Response
Amplitude
Time (sec)
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
2
4
6

(e)


Impulse Response
Amplitude
Time (sec)
15
10
5
0
−5
−10
−15
0.5
1
1.5
2
2.5

(f)

F-S06.2 (40 Points) A signal m phase-modulates a carrier of frequency f_c Hz to produce the signal

$$\forall t, \quad x(t) = \cos(2\pi f_c\, t + m(t)).$$

Suppose $|m(t)| \ll 1$, so this is narrow-band phase modulation (PM).

(a) Find a coherent demodulation scheme to recover the signal m from x. Give an algebraic or block-diagram description of your scheme. Explain why your scheme works.

Answer:

We have

$$x(t) = \cos(2\pi f_c t + m(t)) = \cos 2\pi f_c t \cos m(t) - \sin 2\pi f_c t \sin m(t).$$

Since $|m(t)| << 1$, we use the approximations $\cos m(t) \approx 1, \sin m(t) \approx m(t)$, to get

$$x(t) = \cos 2\pi f_c t - m(t) \sin 2\pi f_c t.$$

The coherent demodulator produces $v(t) = x(t) \sin 2\pi f_c t$ and then passes v through a LPF to obtain w:

$$\begin{aligned}
v(t) &= \cos 2\pi f_c t \sin 2\pi f_c t - m(t)[\sin 2\pi f_c t]^2 \\
&= -\frac{1}{2}m(t) + \frac{1}{2}\sin 4\pi f_c t + \frac{1}{2}\cos 4\pi f_c t \\
&\leftrightarrow -\frac{1}{2}M(f) + \frac{1}{4j}[\delta(f - f_c) - \delta(f + f_c)] \\
&\quad + \frac{1}{4}[M(f - 2f_c) + M(f + 2f_c)], \text{ so} \\
w(t) &= -\frac{1}{2}m(t),
\end{aligned}$$

and $m(t) = -2w(t)$.

(b) Suppose the modulated signal suffers amplitude distortion so that the received signal is y instead of x,

$$\forall t, \quad y(t) = A(t)x(t) = A(t)\cos(2\pi f_c\, t + m(t)),$$

where $1 \le A(t) \le 2$ is the distortion. What signal does your demodulator generate, and how is it related to m?

Answer:

Since $y(t) = Ax(t)$, going through the same approximations as before will give

$$\begin{aligned}
v(t) &= -A(t)\cos 2\pi f_c t \sin 2\pi f_c t - A(t)m(t)[\sin 2\pi f_c t]^2 \\
&= -\frac{1}{2}A(t)m(t) + \frac{1}{2}A(t)\sin 4\pi f_c t + \frac{1}{2}A(t)\cos 4\pi f_c t \\
&\leftrightarrow -\frac{1}{2}M_A(f) + \frac{1}{4j}[A(f-2f_c) + A(f+2f_c)] \\
&\quad +\frac{1}{4}[A(f-f_c) + A(f+f_c)].
\end{aligned}$$

Here $A(f)$ is the FT of $A(t)$ and M_A is the FT of $A(t)m(t)$. The LPF will eliminate the second and third terms above and so the output of the demodulator will be

$$w(t) = -\frac{1}{2}A(t)m(t),$$

which is a distorted version of m.

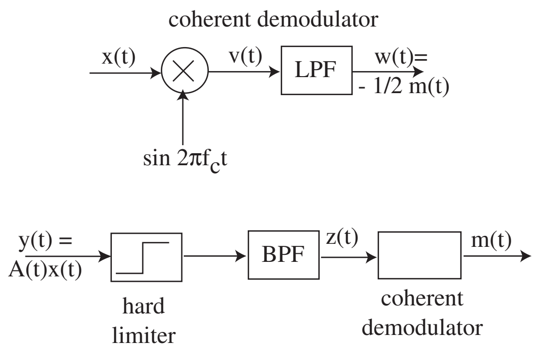


Figure 1: Demodulation schemes for problem 2.

(c) Modify the design of your demodulator so that the effect of the distortion A is eliminated. Remember, you don't know A.

Hint: First, send y through a hard-limiter. A hard-limiter is a memoryless device g whose output is sgn$(y(t))$ when its input is $y(t)$.

Explain why your scheme works.

Answer:
The information about m is contained in the phase of the received signal

$$y(t) = A(t)\cos(2\pi f_c t + m(t)).$$

So we first pass y through a hard limiter to obtain

$$\begin{aligned}
v(t) &= \text{sgn}(y(t)) = \text{sgn}(A(t)\cos(2\pi f_c t + m(t))) \\
&= \text{sgn}(\cos(2\pi f_c t + m(t))).
\end{aligned}$$

We observe that the function

$$w \mapsto \text{sgn}(\cos w)$$

is periodic with period 2π and so it has a Fourier series representation

$$\text{sgn}(\cos w) = \sum_{k=-\infty}^{\infty} A_k e^{jkw}.$$

Substitution of v into this representation gives

$$\begin{aligned} v(t) &= \text{sgn}(\cos(2\pi f_c t + m(t)) \\ &= \sum_{k=-\infty}^{\infty} A_k e^{jk(2\pi f_c t + m(t))}. \end{aligned}$$

So if we pass v through a BPF filter centered at f_c, the result will be the signal z,

$$z(t) = 2ReA_1 e^{(j2\pi f_c t + m(t))} = 2ReA_1 \cos(2\pi f_c t + m(t)),$$

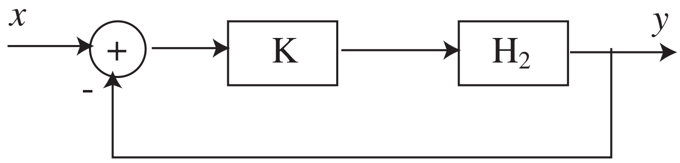
which is the phase-modulated signal without amplitude distortion. We can now demodulate z as before and recover m. Figure 1 gives a block diagram representation of the two schemes.

Alternatively, we can multiply the signal v by $\cos 2\pi f_c t$ and pass the result through a low pass filter. This will recover m as well.

F-S06.3 (40 Points) A plant with transfer function

$$H_2(s) = \frac{s+4}{s\,(s+3)}$$

is arranged in a feedback configuration with a proportional controller K, as shown in the figure below.



(a) What is the closed-loop transfer function?
<u>Answer:</u>
The closed loop transfer function is

$$H(s) = \frac{KH_2(s)}{1+KH_2(s)} = \frac{K(s+4)}{s^2+(K+3)s+4K}.$$

(b) Plot the root locus for $K > 0$ and $K < 0$. Mark the location(s) of the closed-loop poles for $K = 0$ and $K \to \pm\infty$.
<u>Answer:</u>
The poles of the closed loop system are given by the roots of $s^2 + (K+3)s + 4K = 0$, so

$$\begin{aligned} s &= -\frac{K+3}{2} \pm \frac{1}{2}\sqrt{(K+3)^2 - 16K} \\ &= -\frac{K+3}{2} \pm \frac{1}{2}\sqrt{(K-1)(K-9)}, \end{aligned}$$

which gives two real roots for $0 < K < 1$ and $9 < K < \infty$ and a pair of complex conjugate roots for $1 < K < 9$. There are two double real roots for $K = 1, 9$. Figure 2 shows the location for $K = 0, \pm\infty$. The locus for $K < 0$ is shown as dashed lines.

(c) For what values of K is the closed-loop system stable?
<u>Answer:</u>
From the root locus we see that the closed loop system is stable if and only if $K > 0$.

(d) Suppose $K = 1$ and $x(t) = u(t)$, the unit step. Determine $y(t), t \geq 0$. Express y in terms of its transient and steady-state components,

$$y(t) = y_{\text{tr}}(t) + y_{\text{ss}}(t).$$

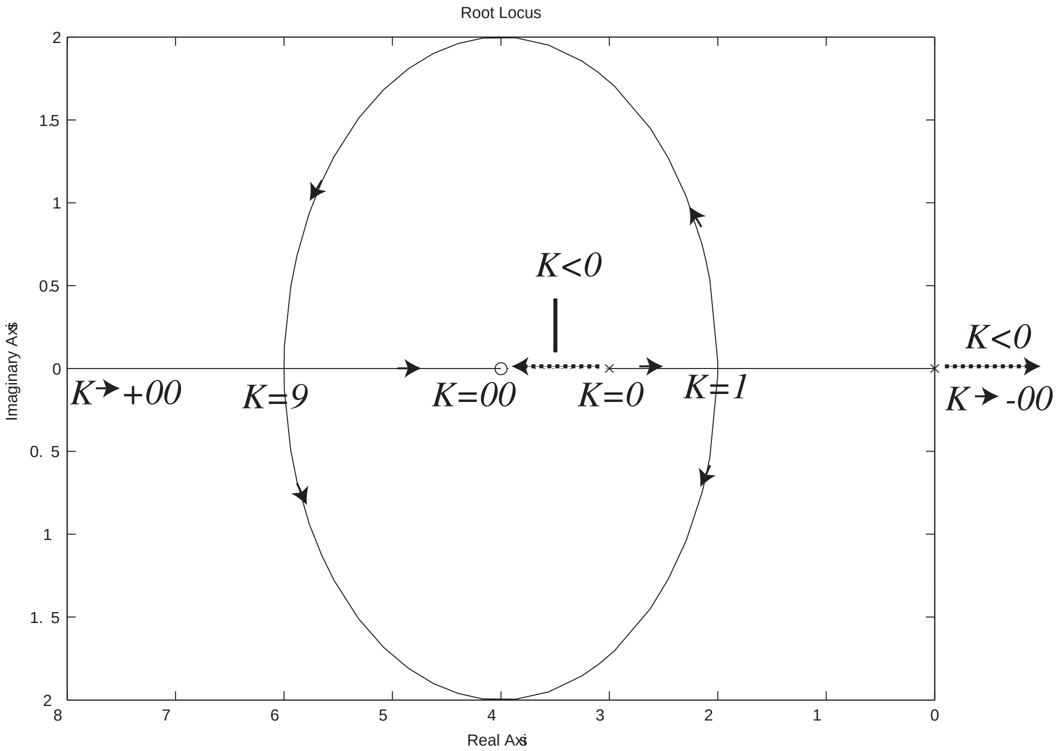


Figure 2: Rootlocus for problem 3

Answer:
If $K = 1$, the closed loop transfer function is $H(s) = \frac{s+4}{s^2+4s+4} = \frac{s+4}{(s+2)^2}$. Since $x(t) = u(t)$, $X(s) = \frac{1}{s}$, hence

$$Y(s) = X(s)H(s) = \frac{1}{s}\frac{s+4}{(s+2)^2} = \frac{1}{s} - \frac{1}{s+2} - \frac{1}{(s+2)^2},$$

and taking inverse LT gives

$$y(t) = u(t) - e^{-2t}u(t) - te^{-2t}u(t),$$

from which

$$y_{tr}(t) = -[1+t]e^{-2t}u(t), \quad y_{ss}(t) = u(t).$$

(e) What is the frequency response of the closed-loop system for $K = 1$?
Answer:
For $K = 1$, the frequency response is

$$H(i\omega) = \frac{4+i\omega}{(2+i\omega)^2}.$$

(f) Suppose $K = 1$ and $x(t) = \sin(2t)\,u(t)$. What is the steady-state response $y_{\mathsf{ss}}(t)$?

Answer:
The steady state response for $x(t) = \sin(2t)u(t)$ is

$$y_{ss}(t) = A\sin(2t + \theta),$$

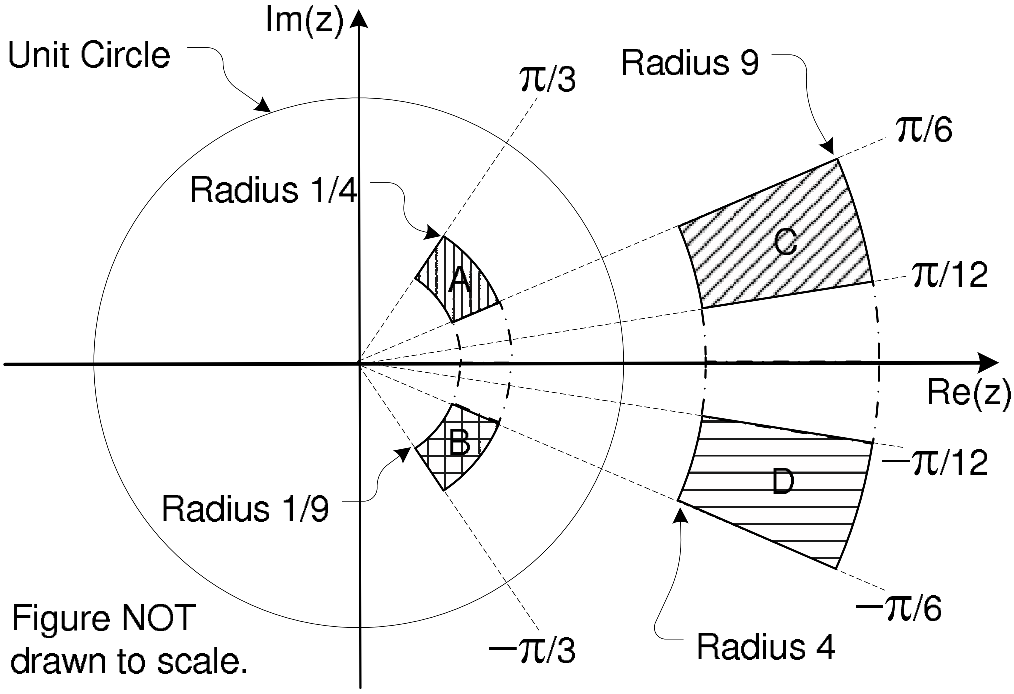
in which $H(i2) = \frac{4+i2}{(2+i2)^2} = Ae^{i\theta}$, so

$$A = |\frac{4+i2}{(2+i2)^2}| = \frac{\sqrt{5}}{2\sqrt{2}}, \quad \theta = \tan^{-1}\frac{2}{4} - 2\tan^{-1}\frac{2}{2} = \tan^{-1}(0.5) - \pi/2.$$

F-S06.4 (40 Points) A *causal* LTI system $\mathsf{H} : [\mathbb{Z} \to \mathbb{R}] \to [\mathbb{Z} \to \mathbb{R}]$ has impulse response h and rational transfer function $\widehat{H}$.

The figure below shows shaded regions where the poles and zeroes of $\widehat{H}$ reside on the complex z-plane. Regions A and B indicate the general locations of the poles, whereas regions C and D indicate the general locations of the zeroes of the system.

The figure is not drawn to scale, but the relative sizes and placements of the angles and radii is correct (e.g., the radius $1/9$ is drawn to be smaller than the radius $1/4$). Each of the radii $1/9$ and $1/4$ indicates the boundary defined by at least one pole (e.g., there is at least one pole at radius $1/4$ and at least one pole at radius $1/9$). The same is true of the radii defining the zero regions.



In this problem, additional LTI systems are introduced whose impulse responses are related to h. You are asked to draw inferences about various properties of those systems, such as the placement of their poles and zeroes, etc.

<u>NOTE</u>: If a pole or zero region of one of the additional systems corresponds to a particular region A, B, C, or D of the system H, you must indicate the correspondence by using the same letter. For example, if a pole cluster of one of the additional systems is due to a mapping of the poles of region A, then you must make that correspondence clear by labeling that cluster with the letter A.

(a) Determine $\mathsf{RoC}(h)$. Is the system H stable?

Answer:

Since H is causal, $\mathsf{RoC}(h) = \left\{z \in \mathbb{C} : |z| > \frac{1}{4}\right\}$. Since $\mathsf{RoC}(h)$ includes the unit circle, H is stable.

(b) An LTI system H_{TR} is characterized by the impulse response h_{TR}, where

$$h_{\mathsf{TR}}(n) = h(-n), \quad \forall n.$$

Provide a well-labeled sketch of the pole and zero regions of the transfer function $\widehat{H}_{\mathsf{TR}}$; determine $\mathsf{RoC}(h_{\mathsf{TR}})$; and explain whether the system H_{TR} is stable.

Answer:

The time-reversal property tells us that

$$h_{\mathsf{TR}}(n) = h(-n) \;\Rightarrow\; \widehat{H}_{\mathsf{TR}}(z) = \widehat{H}\left(\frac{1}{z}\right)$$

Thus, if z_0 is a pole (or zero) of $\widehat{H}$, then $\frac{1}{z_0}$ is a pole (or zero) of $\widehat{H}_{\mathsf{TR}}$, and vice versa. The pole and zero regions for $\widehat{H}_{\mathsf{TR}}$ are shown in Figure 3.

To find $\mathsf{RoC}(h_{\mathsf{TR}})$, we observe that if $h(n)$ is causal, then $h_{\mathsf{TR}}(n) = h(-n)$ is anti-causal, so

$$\mathsf{RoC}(h_{\mathsf{TR}}) = \{z \in \mathbb{C} : |z| < 4\}$$

Therefore, H_{TR} is stable, as expected (since time-reversal should not affect stability).

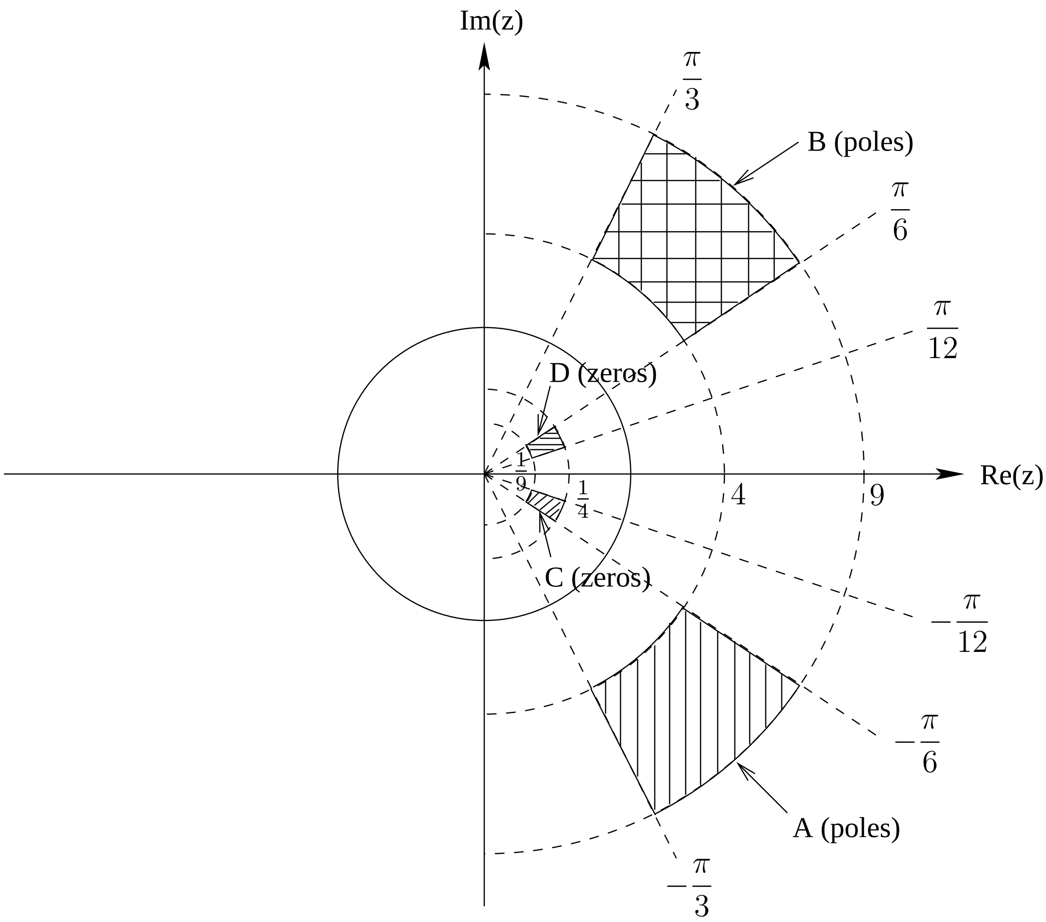


Figure 3: *Pole and zero regions for* $\widehat{H}_{\mathsf{TR}}$.

(c) A system $\mathsf{H}_{\mathsf{INV}}$ is characterized by the transfer function

$$\widehat{H}_{\mathsf{INV}}(z) = \frac{1}{\widehat{H}(z)}\,.$$

Provide a well-labeled sketch of the pole and zero regions of the transfer function $\widehat{H}_{\mathsf{INV}}$. What is $\mathsf{RoC}(h_{\mathsf{INV}})$, if $\mathsf{H}_{\mathsf{INV}}$ is known to be stable?

<u>Answer:</u>

In this case, the poles of $\widehat{H}$ become the zeros of $\widehat{H}_{\mathsf{INV}}$, and the zeros of $\widehat{H}$ become the poles of $\widehat{H}_{\mathsf{INV}}$. The pole and zero regions for $\widehat{H}_{\mathsf{INV}}$ are shown in Figure 4.

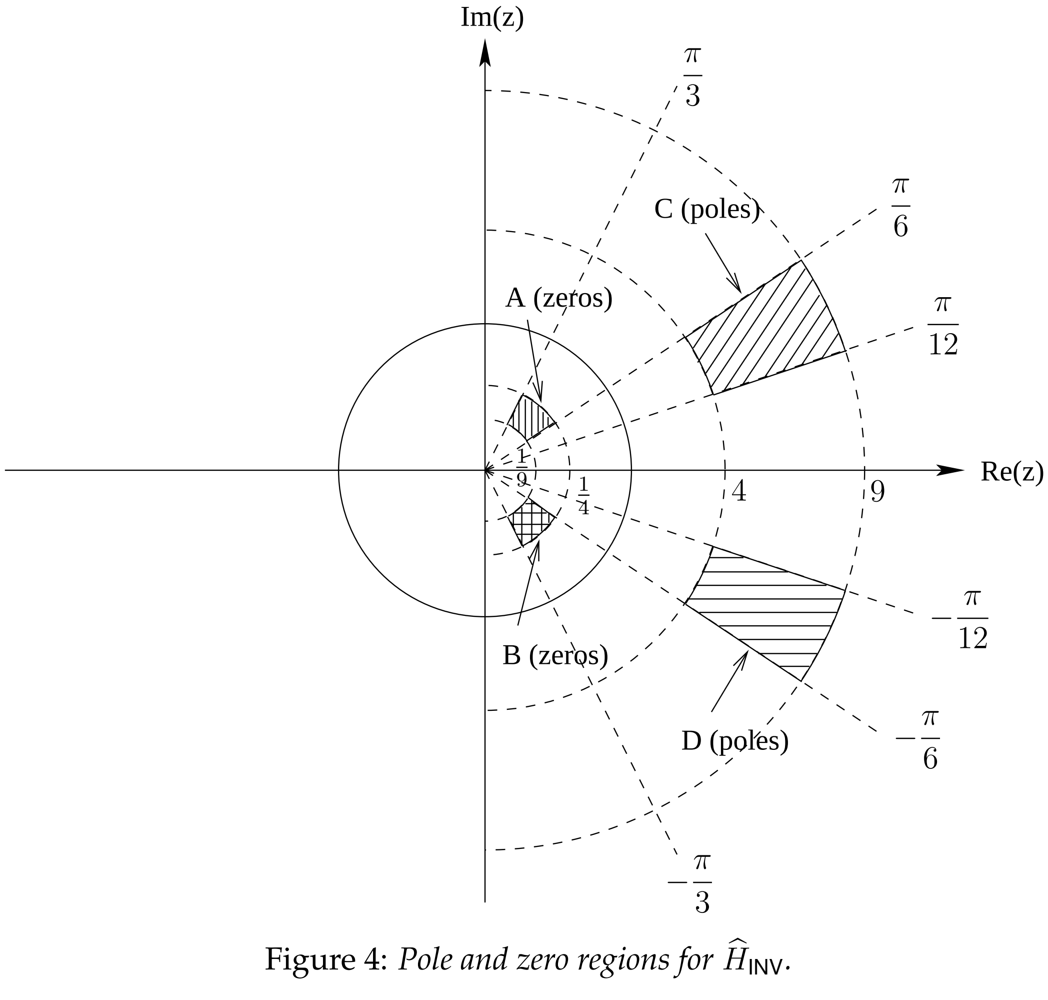


Figure 4: *Pole and zero regions for* $\widehat{H}_{\mathsf{INV}}$.

In order for $\mathsf{H}_{\mathsf{INV}}$ to be stable, we must have

$$\mathsf{RoC}(h_{\mathsf{INV}}) = \{z \in \mathbb{C} : |z| < 4\}$$

(This is anti-causal).

(d) Consider an LTI system G characterized by the impulse response g, where

$$g(n) = (-1)^n \, h(n), \quad \forall n.$$

Select the strongest assertion from the choices below. Explain your choice.

(1) The system G must be stable.
(2) The system G could be stable.
(3) The system G cannot be stable.

<u>Answer:</u>

H stable implies that $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$. In turn, this implies that

$$\sum_{n=-\infty}^{\infty} |g(n)| = \sum_{n=-\infty}^{\infty} |(-1)^n h(n)| < \infty$$

Therefore, G must be stable.

Select the strongest assertion from the choices below. Explain your choice.

(I) The system G must be causal.
(II) The system G could be causal.
(III) The system G cannot be causal.

<u>Answer:</u>

H causal implies that $h(n) = 0$, $\forall n < 0$. Thus, $g(n) = (-1)^n h(n) = 0$, $\forall n < 0$. Thus, G must be causal.

Provide a well-labeled sketch of the pole and zero regions of the transfer function $\widehat{G}$ of G.

<u>Answer:</u>

$$g(n) = (-1)^n h(n) \Rightarrow \widehat{G}(z) = \sum_{n=-\infty}^{\infty} (-1)^n h(n) z^{-n} = \sum_{n=-\infty}^{\infty} h(n)(-z)^{-n}$$

Thus, $\widehat{G}(z) = \widehat{H}(-z)$. If z_0 is a pole (or zero) of $\widehat{H}$, then $-z_0$ is a pole (or zero) of $\widehat{G}$. The pole and zero regions for $\widehat{G}$ are shown in Figure 5. Also note that $\text{RoC}(g) = \left\{z \in \mathbb{C} : |z| > \frac{1}{4}\right\}$.

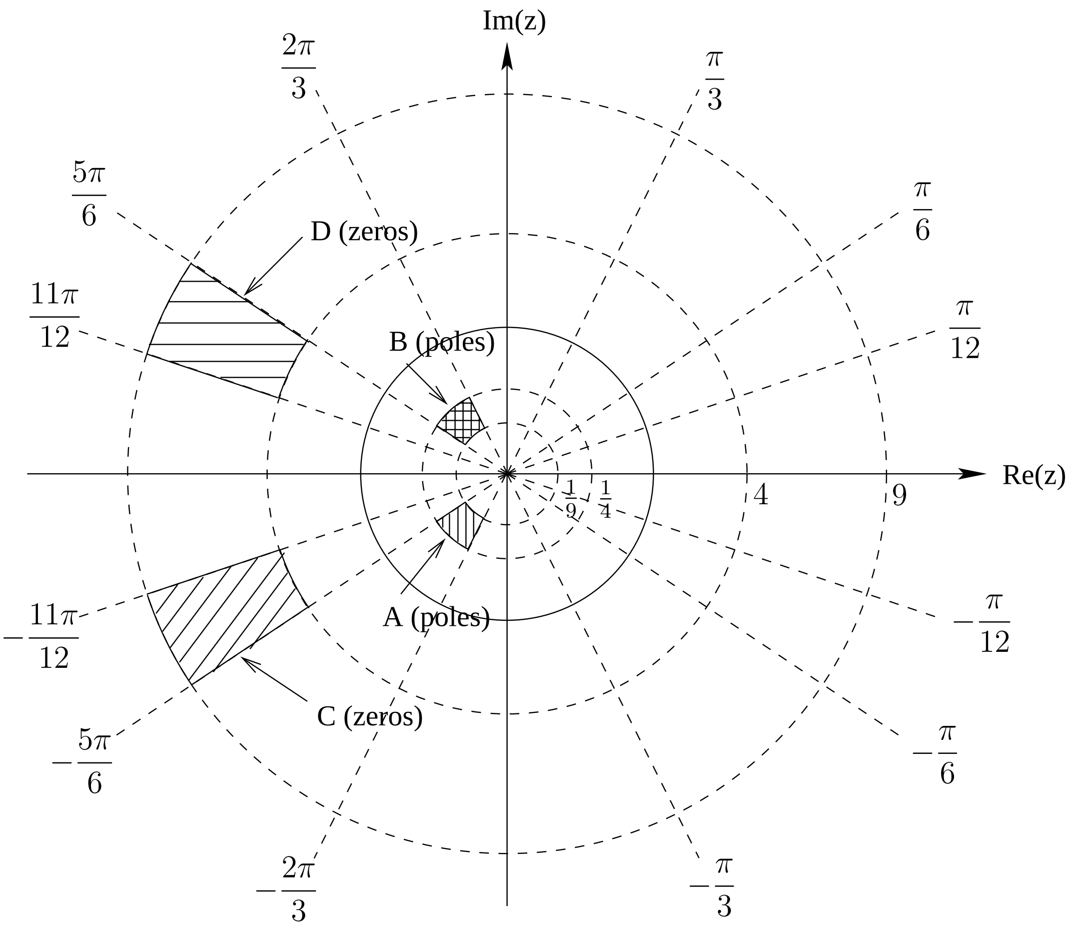


Figure 5: *Pole and zero regions for $\widehat{G}$.*

(e) Consider an LTI system F characterized by the impulse response f, where

$$f(n) = \begin{cases} h\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Select the strongest assertion from the choices below. Explain your choice.

(1) The system F must be stable.

(2) The system F could be stable.

(3) The system F cannot be stable.

Answer:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \Rightarrow \sum_{n=-\infty}^{\infty} |f(n)| < \infty$$

Thus, F is stable.

Select the strongest assertion from the choices below. Explain your choice.

(I) The system F must be causal.

(II) The system F could be causal.

(III) The system F cannot be causal.

Answer:

$$h(n) = 0, \ \forall n < 0 \ \Rightarrow \ f(n) = 0, \ \forall n < 0$$

Thus, F is causal.

Provide a well-labeled sketch of the pole and zero regions of the transfer function $\widehat{F}$ of F.

Answer:

Subsampling by a factor of 2 means that $\widehat{F}(z) = \widehat{H}(z^2)$. If z_0 is a pole (or zero) of H, then $z_0^{\frac{1}{2}}$ is a pole (or zero) of F. Note:

$$z_0 = r_0 e^{i\theta_0} \ \Rightarrow \ z_0^{\frac{1}{2}} = \pm r_0^{\frac{1}{2}} e^{i\frac{\theta_0}{2}}$$

Alternatively,

$$z_0^{\frac{1}{2}} = r_0^{\frac{1}{2}} e^{i\frac{\theta_0}{2}} \text{ or } r_0^{\frac{1}{2}} e^{i(\frac{\theta_0}{2}+\pi)}$$

The pole and zero regions for $\widehat{F}$ are shown in Figure 6 (note that the angles and radii have changed from earlier figures).

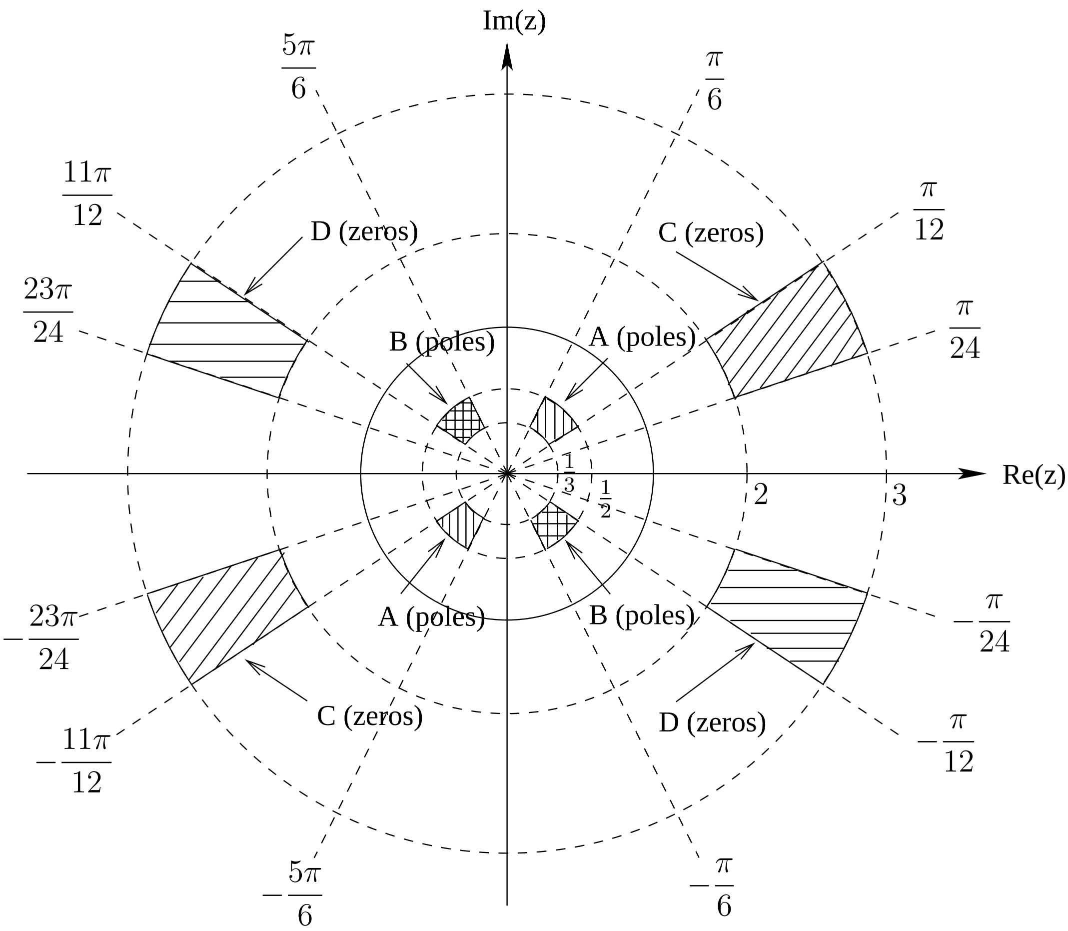


Figure 6: *Pole and zero regions for* $\widehat{F}$.

F-S06.5 (40 Points) Consider a function $h : \mathbb{Z} \to \mathbb{R}$ having Fourier transform H and Z transform $\widehat{H}$. The Z transform $\widehat{H}$ can be written as

$$\widehat{H}(z) = \sum_{n=0}^{N} h(n)\, z^{-n}\,,$$

where N is a positive integer (finite, of course), $h(0) \neq 0$, and $h(N) \neq 0$.

(a) Assume that $\widehat{H}$ does *not* have all its zeros on the unit circle, and that the impulse response h satisfies the following property:

$$\forall n, \quad h(n) = h(N-n)\,.$$

(i) Prove that if z_0 is a zero of $\widehat{H}$, then so is $1/z_0^*$; that is,

$$\widehat{H}(z_0) = 0 \quad \Longrightarrow \quad \widehat{H}(1/z_0^*) = 0\,.$$

This statement means that $\widehat{H}$ has at least one conjugate reciprocal pair of zeroes not residing on the unit circle.

Answer:

From the expression for $\hat{H}$ we note that h is a causal finite-length signal. Also, $h(n) = h(N-n)$ implies that h is symmetric about a midpoint. For example,

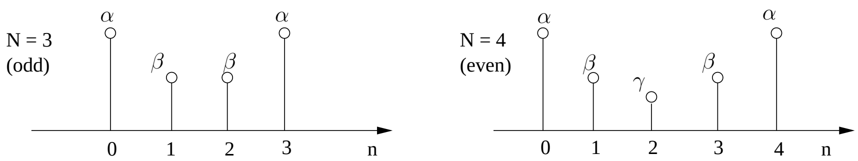


Figure 7: *Example signals for $h(n)$*

Let

$$g(n) = h(-n) \quad (1)$$

and

$$\bar{g}(n) = g(n-N) \quad (2)$$

With these two statements we can then say that

$$\bar{g}(n) = h(-(n-N)) = h(N-n) = h(n) \quad (3)$$

From (1) we get that $\hat{G}(z) = \hat{H}(1/z)$ and from (2) that $\hat{\bar{G}}(z) = z^{-N}\hat{G}(z)$. (3) gives $\hat{H}(z) = \hat{\bar{G}}(z) = z^{-N}\hat{G}(z)$. Putting these together we get

$$\hat{H}(z) = z^{-N}\hat{H}(1/z) = \frac{\hat{H}(1/z)}{Z^N}$$

Thus, if z_0 is a zero of $\hat{H}$ (i.e., $\hat{H}(z_0) = 0$), then $\hat{H}(1/z_0) = 0$. That is, $1/z_0$ is also a zero of $\hat{H}$. Since $h(n) \in \mathbb{R}$, all poles and zeroes appear as complex conjugate pairs. Hence z_0 is a zero $\implies$ z_0^* is a zero $\implies$ $1/z_0^*$ is a zero.

(ii) Assume N is an odd positive integer. Show that the frequency response H can be expressed as follows:

$$\forall \omega, \quad H(\omega) = A(\omega)e^{i\theta(\omega)},$$

where $A(\omega) \in \mathbb{R}$ and $\theta(\omega) = \alpha\omega$, for some constant $\alpha \in \mathbb{R}$. Determine $A(\omega)$ and α (and hence $\theta(\omega)$) in terms of N and the impulse response values $h(n)$.

Answer:

$$\begin{aligned}
H(\omega) &= \sum_{n=0}^{N} h(n)e^{-i\omega n} = (h(0) + h(N)e^{-i\omega N}) + (h(1)e^{-i\omega} + h(N-1)e^{-i\omega(N-1)}) + \\
\cdots & \quad +(h\left(\frac{N-1}{2}\right)e^{-i\omega\frac{N-1}{2}} + h\left(\frac{N+1}{2}\right)e^{-i\omega\frac{N+1}{2}}) \\
&= \sum_{n=0}^{\frac{N-1}{2}} \left[h(n)e^{-i\omega n} + h(N-n)e^{-i\omega(N-n)}\right] \\
&= \sum_{n=0}^{\frac{N-1}{2}} h(n)\left[e^{-i\omega n} + e^{-i\omega(N-n)}\right] \quad \text{(using } h(n) = h(N-n)\text{)} \\
&= \sum_{n=0}^{\frac{N-1}{2}} h(n)\left[e^{-i\omega(n-\frac{N}{2})} + e^{i\omega(n-\frac{N}{2})}\right] e^{-i\omega\frac{N}{2}} \\
&= \left(\sum_{n=0}^{\frac{N-1}{2}} h(n) 2\cos\left[\omega\left(n - \frac{N}{2}\right)\right]\right) e^{-i\omega\frac{N}{2}} \\
\implies & \quad A(\omega) = \sum_{n=0}^{\frac{N-1}{2}} h(n) 2\cos\left[\omega\left(n - \frac{N}{2}\right)\right]
\end{aligned}$$

(iii) Suppose h represents the impulse response of an FIR filter $\mathsf{H} : [\mathbb{Z} \to \mathbb{R}] \to [\mathbb{Z} \to \mathbb{R}]$. Is the following statement true or false? Explain your reasoning succinctly, but clearly and convincingly.

The filter H has a causal and stable inverse $\mathsf{G} : [\mathbb{Z} \to \mathbb{R}] \to [\mathbb{Z} \to \mathbb{R}]$ having transfer function

$$\widehat{G}(z) = \frac{1}{\sum_{n=0}^{N} h(n)\, z^{-n}}\,.$$

Answer:
$\hat{G}(z) = \frac{1}{\hat{H}(z)} \implies$ zeroes of $\hat{H}$ are poles of $\hat{G}$. $\hat{H}$ has a zero not on the unit circle; therefore, it has a zero either inside or outside the unit circle. If the zero is inside, then from (a)(i) we know that $\frac{1}{z_0^*}$—which is outside the unit circle—is also a zero. Similarly, if z_0 is outside, then $\frac{1}{z_0^*}$ is inside.

Upshot: $\hat{G}$ has a pole inside and a pole outside of the unit circle $\implies$ It cannot be *both* causal *and* stable. If it is causal, then it is not stable. If it is stable, it cannot be causal $\implies$ The statement above is *FALSE*.

(b) For this part, suppose $\widehat{H}$ has *no* conjugate reciprocal pair of zeroes; that is,

$$\widehat{H}(z_0) = 0 \quad \implies \quad \widehat{H}(1/z_0^*) \neq 0.$$

Assume, also, that $\widehat{H}$ has no zero on the unit circle.

Consider a finite-length signal $f : \mathbb{Z} \to \mathbb{R}$ that shares these properties with h; that is, the Z transform $\widehat{F}$ of F has no zero on the unit circle and no conjugate reciprocal pair of zeroes (anywhere). Furthermore, F, the DTFT of f, has the same phase as H; that is, if $H(\omega) = |H(\omega)|e^{i\theta_h(\omega)}$ and $F(\omega) = |F(\omega)|e^{i\theta_f(\omega)}$, then

$$\forall \omega, \quad \theta_h(\omega) = \theta_f(\omega).$$

(i) Let $q(n) = (h * f_{\mathsf{TR}})(n)$, where $f_{\mathsf{TR}}(n) = f(-n)$, the time-reversed counterpart of f. Prove that q is an even signal by showing that $Q(\omega) \in \mathbb{R}, \forall \omega$. Determine $\theta_q(\omega) \stackrel{\triangle}{=} \angle Q(\omega)$.
Answer:

$$\begin{aligned} F_{TR}(\omega) &= |F(\omega)|e^{-i\theta_f(\omega)} = F^*(\omega) \\ Q(\omega) &= H(\omega)F_{TR}(\omega) = |H(\omega)|e^{i\theta_h(\omega)}|F(\omega)|e^{-i\theta_f(\omega)} \\ &= |H(\omega)||F(\omega)| \qquad\qquad \text{(since } \theta_h = \theta_f\text{)} \end{aligned}$$

So $Q(\omega) \in \mathbb{R}$.

(ii) Express each of $\widehat{Q}(z)$ and $\widehat{Q}(1/z)$ in terms of the Z transforms $\widehat{H}$ and $\widehat{F}$. Then, use the symmetry of q, as well as what you know about h and f, to prove that

$$\widehat{H}(z_0) = 0 \quad \iff \quad \widehat{F}(z_0) = 0.$$

That is, prove that if z_0 is a zero of $\widehat{H}$, then it is also a zero of $\widehat{F}$, and vice versa.

$$\hat{Q}(z) = \hat{H}(z)F_{TR}(z) = \hat{H}(z)\hat{F}(1/z) \qquad (1)$$

$$q(n) = q(-n) \implies \hat{Q}(z) = \hat{Q}(1/z) \qquad (2)$$

If $\hat{Q}(z_0) = 0$, then $\hat{Q}(z_0^*) = 0$ by conjugate symmetry ($q(n) \in \mathbb{R}$). Furthermore, using (2), if $\hat{Q}(z_0) = 0$, then $\hat{Q}(1/z_0) = 0 \implies \hat{Q}(1/z_0^*) = 0$ (conjugate symmetry again). Lastly note that

$$\hat{Q}(1/z) = \hat{H}(1/z)\hat{F}(z) \tag{3}$$

Putting it together:

$$\hat{H}(z_0) = 0 \implies \hat{Q}(z_0) = 0 \implies \hat{Q}(1/z_0) = 0$$

This means from (3) that $\hat{F}(z_0) = 0$ or $\hat{H}(1/z_0) = 0$. The latter is not possible because $\hat{H}$ does not have reciprocal zeroes, so the former must hold. Thus we've shown that

$$\hat{H}(z_0) = 0 \implies \hat{F}(z_0) = 0$$

The other direction follows similarly:

$$\hat{F}(z_0) = 0 \implies \hat{Q}(1/z_0) = 0 \implies \hat{Q}(z_0) = 0$$

This means from (1) that $\hat{H}(z_0) = 0$ or $\hat{F}(1/z_0) = 0$. Again, the latter is not possible because $\hat{F}$ does not have reciprocal zeroes either. Therefore we've shown that

$$\hat{H}(z_0) = 0 \iff \hat{F}(z_0) = 0$$

(iii) Use the result of Part (b)(ii) to show that

$$f(n) = \beta\, h(n), \quad \exists \beta \in \mathbb{R}.$$

From (b)(ii) we see that h and f have exactly the same zeroes. This means they are finite-length signals of the same length. Using this and phase equality ($\theta_h = \theta_f$) we can write $\hat{F}$ and $\hat{H}$ as follows:

$$\hat{F}(z) = \frac{F_0(z - z_0)(z - z_1)\ldots(z - z_N)}{z^N}$$

$$\hat{H}(z) = \frac{H_0(z - z_0)(z - z_1)\ldots(z - z_N)}{z^N}$$

From this we see that $\hat{F}(z) = \beta\hat{H}(z) \implies f(n) = \beta h(n)$

(iv) Argue that β in Part (b)(iii) is actually a *positive* real number.

$$\begin{aligned} f(n) = \beta h(n) \quad &\implies \quad F(\omega) = \beta H(\omega) \\ &\implies \quad |F(\omega)|e^{i\theta_f(\omega)} = \beta|H(\omega)|e^{i\theta_h(\omega)} \\ &\implies \quad \beta = \frac{|F(\omega)|}{|H(\omega)|} > 0 \qquad\qquad \text{(since } \theta_f(\omega) = \theta_h(\omega)\text{)} \end{aligned}$$

The lesson from this problem is that, under certain circumstances, a signal h can be recovered (to within a positive scale factor β) from *only* the phase of its Fourier transform!!!