CS170 Final Examination 16 May 2001

NAME (1 pt): ____________________

TA (1 pt): ____________________

Name of Neighbor to your left (1 pt): ____________________

Name of Neighbor to your right (1 pt): ____________________

ANSWERS

This is a closed book, closed calculator, closed computer, closed network, open brain exam, but you are permited a 2 page, double-sided set of notes, large enough to read without a magnifying glass.

You get one point each for filling in the 4 lines at the top of this page. Each other question is marked by the number of points it is worth. You may use the number of points assigned to each problem as a rough estimate for the number of minutes you want to allocate to the problem. The total number of points is 149. (This means that we hope you will have enough time to finish the exam!)

Write all your answers on this exam. If you need scratch paper, ask for it, write your name on each sheet, and attach it when you turn it in (we have a stapler).

Question	Score	Max
Cover		4
1		20
2		20
3		15
4		20
5		15
6		30
7		25
Total		149

1. 2s complement arithmetic. **(20 points.)**

   You are running on a computer with 10-bit 2s complement arithmetic. One integer fits in one 10-bit word.

   (a) **(5 points.)** What is the representable range of integers? I.e. what are the most negative and most positive representable integers? Give you answers in decimal and binary notation.

   ANSWERS. The smallest number is $-2^9 = -512$, and is stored as 1000000000_2. The largest number is $2^9 - 1 = 511$, and is stored as 0111111111_2.

   (b) **(5 points.)** What is the value returned by the following computation? Express your answer as decimal integers. Which operations signal integer overflow?

   -266-246
   300+400
   -305-405
   3*400
   -4*400

   ANSWERS.

   -266-246 yields -512 without overflow
   300+400 yields 700-1024 = -324 with overflow
   -305-405 yields -710+1024 = 314 with overflow
   3*400 yields 1200-1024 = 176 with overflow
   -4*400 yields -1600 + 2*1024 = 448 with overflow

   (c) **(10 points.)** What does the following program print out? All variables are 10-bit 2s complement integers. Justify your answer.

```
$m$ = most negative representable integer
$M$ = most positive representable integer
$count = 0$
for $i = m$ to $M$
    for $j = m$ to $M$
        if $(12 * i = 52 * j - 28)$ then $count = count + 1$
    endfor
endfor
print($count$)
```

   ANSWERS. Since $gcd(12, 1024) = 4$ and $4|(104 * j - 28)$ for any j, the equation $5 * i \equiv 104 * j - 28k \bmod 1024$ has 4 distince solutions i mod 1024, so for each value of i, *count* is incremented by 4, and so *count* is incremented $4 * 1024 = 4096$ times. Finally, $count = 4096 - 4 * 1024 = 0$ is printed.

2. Toy Train. **(20 points)**

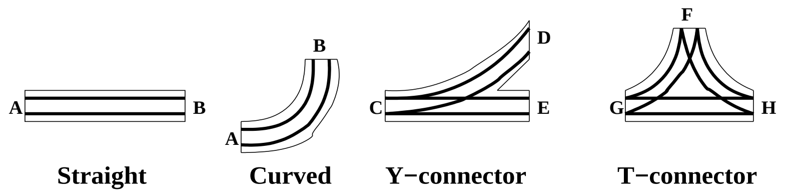
A precocious 4-year-old M likes to play with her toy trains. There are 4 kinds of track pieces that can be connected (for convenience, each place a track piece can connect to another is labeled as in the figure below):

(a) straight pieces (one end is labeled A and the other B),

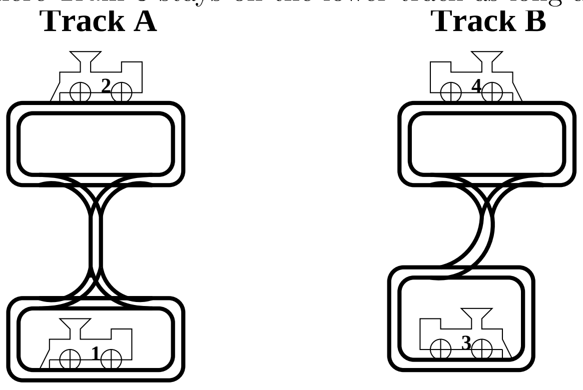
(b) curved pieces (one end is labeled A and the other B),

(c) Y-connectors, where if you enter at C, you can exit either at D or E, and if you enter at D or E, you must exit at C

(d) T-connectors, where no matter where you enter (say F), you can exit at either of the other two places (G or H)



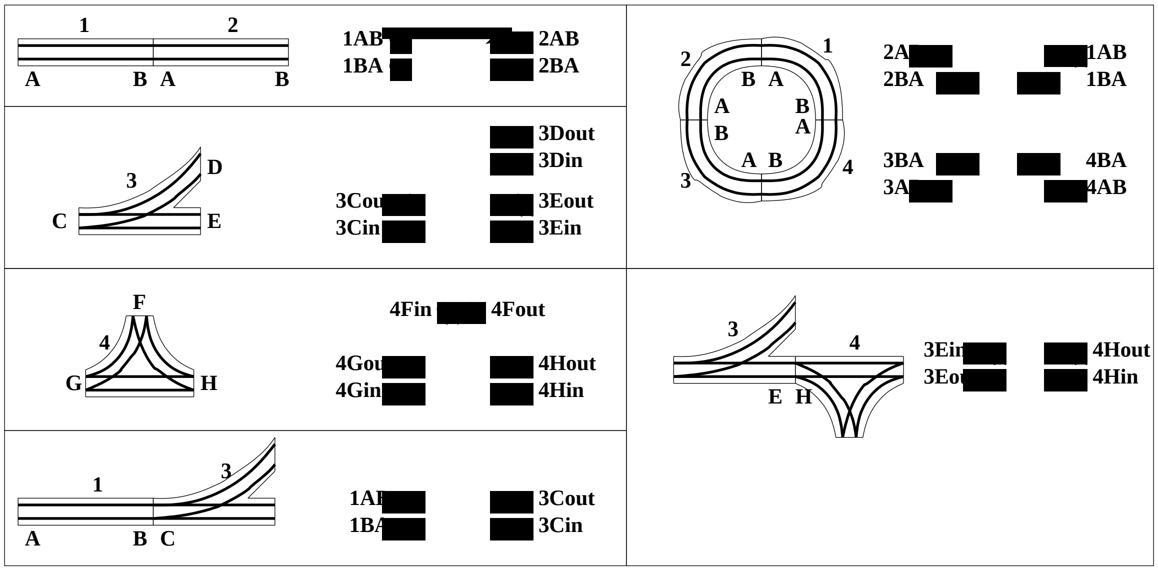
After building lots of different train tracks, M realizes not all are equally fun to play with: On some tracks you can keep moving the train forward as long as you like without running out of track, and on some you can't. On some you can get to any track piece from any other track piece by moving the train in its forward direction. On some you can even get the train to face in either direction on any track piece just by moving forward (see Track A below, where Train 1 can get to the same position as Train 2). In some other tracks you can't even reach some track pieces depending on where you start (see Track B below, where Train 3 stays on the lower track as long as it moves forward).



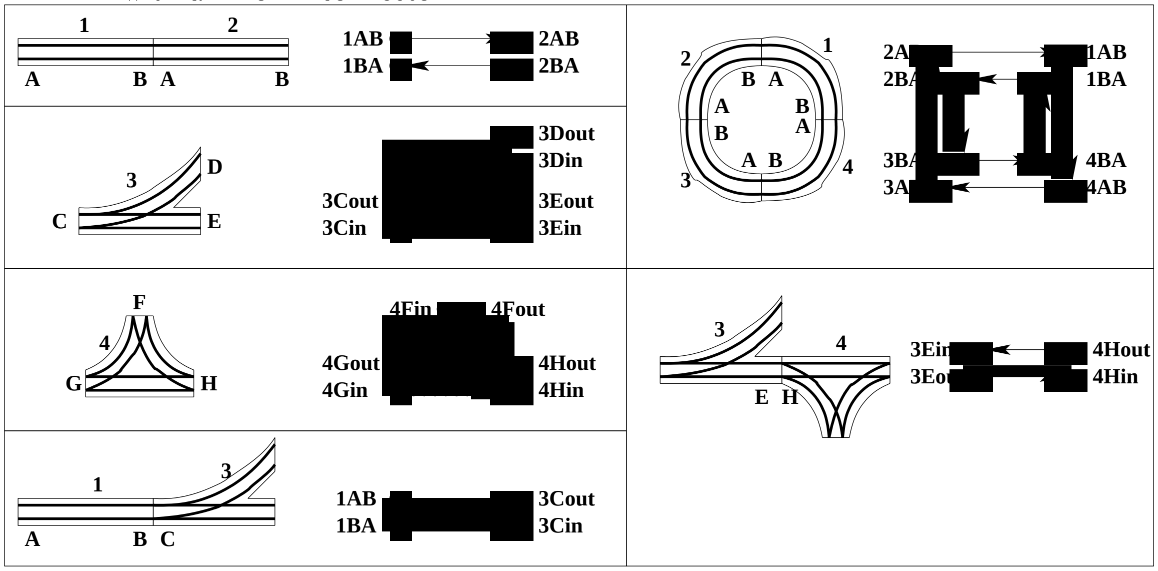
(a) **(5 points.)** Given a train track, we will define a graph G that describes the train track, and how trains can move if they are only allowed to move forward. There will be a pair of vertices to represent each straight or curved piece: one vertex representing a train moving forward from A to B, and the other vertex representing a train moving forward from B to A. (Label these vertices for piece i vertices iAB and iBA, respectively.) Similarly, there will be a pair of vertices for each of the 3 places a Y- or T-connector can connect to another track piece. (For example, two of the vertices for T-connector j are labeled $jFin$ and $jFout$, for trains moving toward the T-connector at F and moving away from the connector at F, respectively.) There will be edges to represent connectors, and to represent how track pieces are joined together. Show how to define these graphs by filling in the missing edges in the figures below (all possible vertices are not shown in each figure; don't worry about edges to vertices that are not shown). One edge is already filled in for you:

the edge from 1AB to 2AB means that a train on piece 1 facing from A to B can move forward to piece 2 and also face from A to B.

If there are a total of s straight or curved pieces and c Y- or T-connectors, give the exact number of vertices n and an upper bound on the number of edges e (in terms of s and c) in your graph G.



ANSWERS. There are $n = 2s + 6c$ vertices and $e \leq 2s + 9c$ edges, since there are at most 2 edges pointing away from the two vertices associated with each straight or curved piece, and at most 9 edges pointing away from the vertices associated with a Y- or T-connector.



(b) **(5 points.)** Briefly describe an efficient algorithm for figuring out whether M can put a train on a chosen straight or curved track piece, and facing in a chosen direction, and move it forward to reach another chosen destination straight or curved track piece facing in a chosen destination direction. For example, Train 3 on Track B above cannot reach the position of Train 4 above moving forward only, but Train 4 can reach the position of Train 3. Similarly, Train 1 on Track A can reach the position of Train 2. Give the complexity in terms of the total number s of straight or curved pieces and the total number c of Y- or T-connectors. Justify your answer briefly.

ANSWERS. The question is whether one can follow the directed edges in the graph from a given starting vertex s to a given destination vertex d. This is answered by doing DFS starting at s and continuing until d is reached, or DFS stops without reaching d. The cost is $O(n + e) = O(s + c)$.

(c) **(5 points.)** Briefly describe an efficient algorithm for figuring out whether M can put a train on a chosen straight or curved track piece and facing in a chosen direction, and then play as long as she wants, i.e. keep moving the train forward without running out of track, provided she makes the correct turns at connectors. Give the complexity in terms of the total number s of straight or curved pieces and the total number c of Y- or T-connectors. Justify your answer briefly.

ANSWERS. The question is whether a cycle is reachable from the starting vertex s. This is answered by doing DFS starting from s and seeing if at least one back edge is encountered (which would mean there is a cycle). One could also compute the DAG of strongly connected components (SCCs) of the graph, with each vertex=SCC labeled by the number of vertices in it, and doing DFS from the SCC containing s to see is a SCC of at least 2 vertices is reachable. The cost is 2 DFSs or $O(n+e) = O(s + c)$.

(d) **(5 points.)** Briefly describe an efficient algorithm for answering the last question when M plays with her eyes closed after putting the train down on a chosen track piece facing in a chosen direction, i.e. may take any possible turn at any connector. Is she guaranteed to be able to keep the train moving forward as long as she likes without running out of track, regardless of what turns she makes at connectors? Give the complexity in terms of the total number s of straight or curved pieces and the total number c of Y- or T-connectors. Justify your answer briefly.

ANSWERS. This is again answered by computing the DAG of strongly connected components (SCCs) of the graph, and seeing if all the sink SCCs (i.e. SCCs which have no outgoing edges in the DAG) reachable from s have at least 2 vertices. The cost is $O(n + e) = O(s + c)$.

(e) **(5 points of extra credit.)** M's little brother N shows up and starts playing, but moves the train either forward or backward as he pleases. Answer question (b) again: Briefly describe an efficient algorithm for figuring out whether N can put a train on a chosen straight or curved track piece, and facing in a chosen direction, and move it forward *or backward* to reach another chosen destination straight or curved track piece facing in a chosen destination direction.

ANSWERS. In place of the directed graph G we use the undirected graph G' gotten from G by "forgetting" the directions on G's edges. The question is whether one can follow the undirected edges in G' from a given starting vertex s to a given destination vertex d. This is again answered by doing DFS starting at s and continuing until d is reached, or DFS stops without reaching d. The cost is $O(n+e) = O(s+c)$.

3. FFT. **15 points.**

(a) (11 points) You want to multiply two degree-n polynomials $p(x) = q(x) \cdot r(x)$ using the FFT. Explain why the following algorithm is wrong and fix it.
We let $q(x) = \sum_{i=0}^{n} q_i x^i$ and $r(x) = \sum_{i=0}^{n} r_i x^i$.

```
Let $qc = [q_0, q_1, ..., q_n]$
Let $rc = [r_0, r_1, ..., r_n]$
$qf = FFT(qc)$
$rf = FFT(rc)$
for $i = 0$ to $n$
    $pf_i = qf_i \cdot rf_i$
$pc = invFFT(pf)$
```

pc contains the desired coefficients of p.

ANSWERS. The degree of the product p is $2n$, so we must compute $2n + 1$ coefficients, whereas the above algorithm only computes $n + 1$ coefficients. To fix it let N be the smallest power of 2 greater than or equal to $2n + 1$, and let $qc' = [q_0, q_1, ..., q_n, 0, ..., 0]$, i.e. append $N - (n + 1)$ zeros to qc to get a vector qc' of length N. Similarly define rc'. Then extract the coefficients of p from the first $2n + 1$ entries of pc' below:

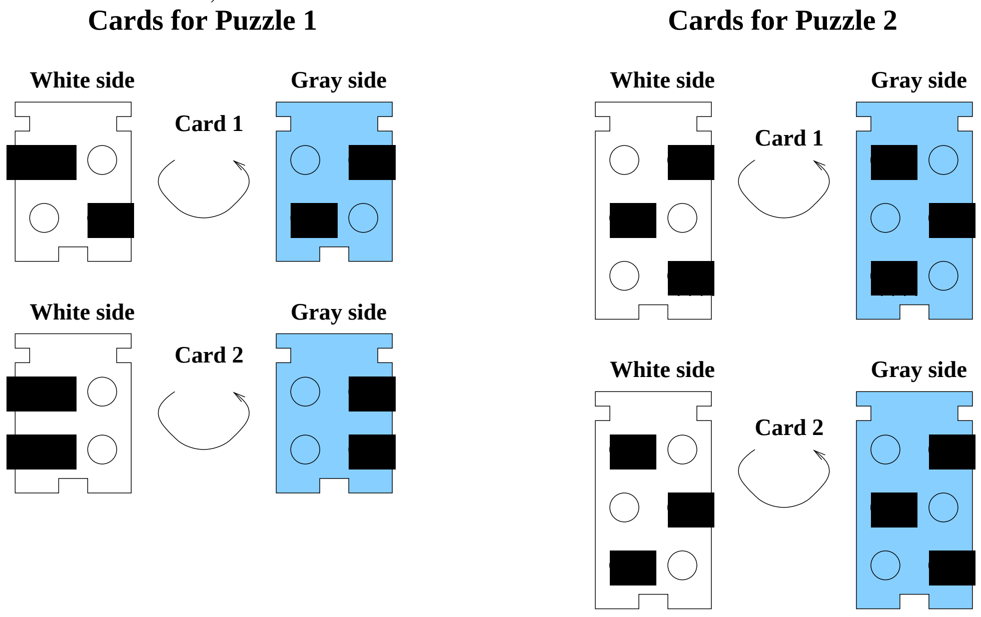
```
$qf' = FFT(qc')$
$rf' = FFT(rc')$
for $i = 0$ to $N - 1$
    $pf'_i = qf'_i \cdot rf'_i$
$pc' = invFFT(pf')$
```

(b) (4 points) Compute the FFT of the vector $[0, 1, 2, 3]$.
ANSWERS. $FFT([0, 1, 2, 3]) = [6, -2 - 2i, -2, -2 + 2i]$.

4. Logic Puzzle. **20 points.**

You are given a box and a collection of N cards as indicated in the following figures. Each card is gray on one side and white on the other side. Because of the the shapes of the cards, which exactly match the shape of the hole in the box, each card will fit in the box in only two possible ways (gray-side-up or white-side-up). Each card contains $2R$ circles lined up in two columns and R rows, each of which may be punched out (so it is a hole) or not. Your puzzle is to see if you can place all the cards in the box (each one either gray-side-up or white-side-up) so as to completely cover the bottom of the box, i.e. each of the $2R$ circle positions is covered by at least one card that has no hole punched out there.

For example, in the cards for Puzzle 1 (left example below), with 2 cards each with 2 rows, it is not possible to put them in the box to cover each circle, but with Puzzle 2 (right example below), with 2 cards each with 3 rows, it is possible. (Punched-out-circles are shown in black).



Your job is to show that solving this puzzle (deciding if the cards can be put in the box to cover each circle position) is NP-complete.

Hint: Show that 3-SAT can be reduced to an instance of solving this puzzle. Recall that 3-SAT is the problem of deciding whether a Boolean expression like the following can be satisfied (made True) by assigning True and False values to the variables x_i:

$$(x_1 \vee \neg x_4 \vee x_9) \wedge (\neg x_2 \vee \neg x_1 \vee x_3) \wedge \cdots \wedge (\neg x_5 \vee x_6 \vee x_7)$$

Try representing each Boolean variable in the 3-SAT problem by a card, and each term (like $(x_1 \vee \neg x_4 \vee x_9)$) by a row in each card.

ANSWERS. The problem is clearly in NP because a witness for the solution is simply a list of colors (gray or white), one for each card, indicating how to put it in the box, after which one checks to see if each circle position is covered.

Suppose there are n Boolean variables $x_1,\ldots,x_n$ and t terms in an instance of 3-SAT. Then we will design a puzzle with $N = n + 1$ cards each with t rows as follows.

We will call a solid (not punched out) circle "T" or "True", and a punched-out circle (hole) "F" or "False".

Let the first n cards be named $C_1, ..., C_n$. We associate row k ($1 \leq k \leq t$) of each card with term k in the 3-SAT problem. There are four cases:

- If variable x_i appears in term k, then row k of card C_i is "TF" (T or solid in the left position of the k-th row of the white side, and F in the right position).
- If variable $\neg x_i$ appears in term k, then row k of card C_i is "FT".
- If both variables x_i and $\neg x_i$ appears in term k, then row k of card C_i is "TT".
- If term k does not depend on variable x_i (or $\neg x_i$) then row k of card i is "FF".

Now stack the cards in the box. If card C_i is white-face-up, let $x_i = T$, and otherwise $x_i = F$. Then by the above construction the k-th circle in the right column is covered if and only if at least one circle on one card in the k-th/right position is T (not punched out), i.e. the k-th term in the 3-SAT problem is true. So the whole right hand column of circle is covered if and only if every term in the 3-SAT problem is true, i.e. the 3-SAT problem is satisfiable.

Similarly, the left column of circles is covered if and only if the opposite assignment of T/F values to each x_i satisfies the 3-SAT problem.

Finally, we need one more card, to select either the right column or left column: this card has F=holes in all the right column, and T = circles in the left column.

5. Collinearity. **15 points.**

   Given n points in the plane, it is easy to determine if any three of them lie in a single straight line, in $O(n^3)$ time: for each set of three points, use the cross product test to see if they are collinear.

   Give a more efficient algorithm for this problem, one that runs in $o(n^3)$ time.

   ANSWERS. Start by choosing the bottommost point p (the one with the least y-coordinate). Break ties by choosing the leftmost one. We will determine, in $O(n \log n)$ time, whether p is collinear with any other two points.

   The idea is to sort all the remaining points by the polar angle relative to p, as we did in homework, or in Graham's Scan for finding the convex hull. This takes $O(n \log n)$ time, and gives a sorted list of point $(p_1, p_2, ..., p_{n-1})$. Now compare each adjacent pair of points (p_1, p_2), (p_2, p_3), ... (p_{n-2}, p_{n-1}), (p_{n-1}, p_1), to see if any pair is collinear with p.

   If we find a collinear set, we are done. Otherwise eliminate p from the list and repeat. The maximum running time is then $O(n^2 \log n)$.

6. Dynamic Programming. **30 points.**

You have decided to make a CD of your favorite songs. You have already picked the songs, but you still need to decide in what order they should appear on the CD. Your task is the compute the "best" possible order for them to appear.

For each pair of songs a and b you've chosen a "compatibility rating," $compat[a, b]$, which is a positive number indicating how good you think b sounds when it's played right after a. (Note that $compat[a, b]$ and $compat[b, a]$ are generally *not* equal). Your job is to use dynamic programming to find an ordering of the songs which maximizes the sum of the compatibilities of consecutive songs. *You do not need to print out the actual ordering, just the maximum total compatibility.*

Let your N favorite songs have identification numbers from 1 to N. Your algorithm should run in $O(N^2 2^N)$ time.

(a) **(5 points)** List the table(s) your algorithm will use, and explain the meaning of each entry.

ANSWERS. The table is $C[S, i]$, where S is a subset of the N songs and i is a member of S.

$C[S, i]$ will equal the maximum total compatibility possible for songs in set S, ending with song i.

(b) **(10 points)** Specify the recursion and the base case(s) used by your algorithm.

ANSWERS.

$$C[S,i] = \begin{cases} 0 & \text{if } |S| = 1 \\ \max_{k \in S \setminus \{i\}} C[S\setminus\{i\}, k] + compat[k,i] & \text{if } |S| > 1 \end{cases}$$

(c) **(10 points)** Implement your algorithm in pseudocode. (Be sure to return the answer at the end.)

ANSWERS.

```
foreach $i$ in $(1..N)$
    $C[\{i\}, i] = 0$
for all $|S|$ from 2 to $N$
    for all $S$ with cardinality $|S|$
        for all $i \in S$
            $C[S, i] = \max_{k \in S \setminus \{i\}} C[S\setminus\{i\}, k] + compat[k, i]$
return $\max_{1 \le i \le N} C[\{1..N\}, i]$
```

(d) **(5 points)** Analyze the running time of your algorithm; justify your answer briefly.

ANSWERS. The loop "foreach i" takes N steps.

The max in the inside the next loop nest runs in time $O(|S|)$. Thus the "for all $i \in S$" loop runs in time $O(|S|^2)$, or $O(N^2)$ for any $|S|$. The two outer loops loop over all possible subsets $|S|$, i.e. 2^N. So the total running time is $O(2^N N^2)$ as desired.

7. Short Answer **(25 points, 5 points each.)**

   Answer in the space allotted. For full credit justify your answer briefly.

   (a) In a disjoint set data structure with path compression and union-by-rank, when can the rank of a representative element differ from the depth of the tree representing its set?

   ANSWERS. Whenever enough FIND operations are performed to lower the number of tree levels, since rank is not updated during path compression.

   (b) Suppose we use optimized Huffman coding to compress a 50,000-symbol file containing 5 different symbols, each of which occurs equally often. What will the output file size be (excluding any header information), measured in the number of bits?

   ANSWERS. 120,000

   (c) Consider the sets of problems P, NP, NP-Complete, and NP-Hard. List all known (proven) containment relationships between pairs of sets, i.e. all relationships of the form $A \subseteq B$. (don't bother with trivial ones like $A \subseteq A$).

   ANSWERS. $P \subseteq NP$, $NP-$Complete $\subseteq NP$, and $NP-$Complete $\subseteq NP-$Hard.

   (d) Consider using RSA encryption with $n = 143 = 11 \cdot 13$. What is wrong with using $e = 3$ as the encryption key? What is the smallest e that would work?

   ANSWERS. $e = 3$ does not work because $gcd(3, (11-1)\cdot(13-1)) = gcd(3, 120) = 3$ instead of 1. $e = 7$ is the smallest integer greater than 1 such that $gcd(e, 120) = 1$.

   (e) Give a simple asymptotic expression (i.e. $\Theta(\cdot)$) for $T(n) = 2T(n/2) + n \log n$, $T(1) = 1$, n a power of 2.

   ANSWERS. $T(n) = \Theta(n \log^2 n)$.