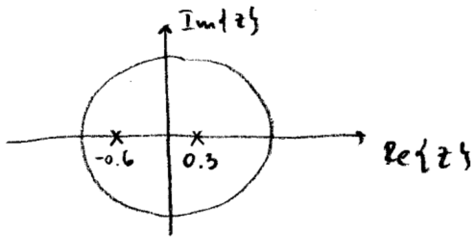
Midterm 1: Solutions

1. (30 points) The z-transform of a sequence $x[n]$ is given by:

$$X(z) = \frac{3z^2 + 0.1z + 0.87}{(z+0.6)(z-0.3)}.$$

Specify all possible regions of convergence, and find the respective inverse z-transforms.

poles at: -0.6 , 0.3



$ROC_x^{(1)}$: $|z| > 0.6$

$ROC_x^{(2)}$: $0.3 < |z| < 0.6$

$ROC_x^{(3)}$: $|z| < 0.3$

$ROC_x^{(4)} = \emptyset$

$$X(z) = 3 + \frac{-0.8z + 1.41}{(z+0.6)(z-0.3)} = 3 - \frac{2.1}{z+0.6} + \frac{1.3}{z-0.3}$$

$$= 3 - 2.1z^{-1} \cdot \frac{z}{z+0.6} + 1.3z^{-1} \cdot \frac{z}{z-0.3}$$

$$h^{(1)}[n] = 3\delta[n] - 2.1\,(-0.6)^{n-1}u[n-1] + 1.3 \cdot (0.3)^{n-1}u[n-1]$$

$$h^{(2)}[n] = 3\delta[n] + 2.1\,(-0.6)^{n-1}u[-n] + 1.3\,(0.3)^{n-1}u[n-1]$$

$$h^{(3)}[n] = 3\delta[n] + 2.1\,(-0.6)^{n-1}u[-n] - 1.3 \cdot (0.3)^{n-1}u[-n]$$

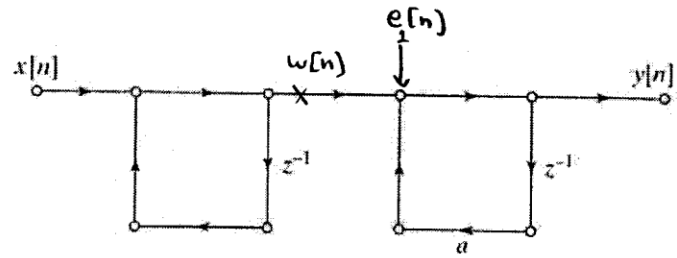
$h^{4}[n] \to$ does not exist

2. The parameter a in the flow diagrams below is a real number in the interval $(0, 1)$.

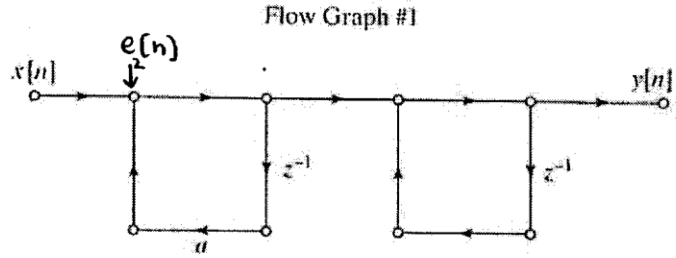
a) (10 points) Calculate the transfer function implemented by these flow diagrams and determine its BIBO stability.

b) (10 points) What condition must the input sequence $x[n]$ satisfy so that the output $y[n]$ remains bounded.

c) (10 points) Assuming that these systems are implemented with two's complement fixed point arithmetic with B magnitude bits, calculate the output noise variance for the Flow Graph #1. Which of the two flow graphs would have a larger output noise variance?



Flow Graph #1



Flow Graph #2

a)

Flow graph #1: $w[n] = x[n] + w[n-1] \Rightarrow W(z)(1-z^{-1}) = X(z)$

$y[n] = w[n] + a\,y[n-1] \Rightarrow Y(z)(1-az^{-1}) = W(z)$

$\Rightarrow Y(z)(1-az^{-1})(1-z^{-1}) = X(z)$

$$H_1(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1-az^{-1})(1-z^{-1})}$$

Similarly for the Flow graph #2 : $H_2(z) = \dfrac{1}{(1-az^{-1})(1-z^{-1})}$

$H_1(z) = H_2(z)$

Since both systems are causal , and $a \in (0,1)$

$\Rightarrow$ $ROC_{H_1} = ROC_{H_2}$. $|z| > 1$ $\Rightarrow$ H_1, H_2 are <u>not stable</u>

b) $H_1(z) = H_2(z) = \dfrac{1}{(1-az^{-1})(1-z^{-1})}$

$Y(z) = H_1(z) \cdot X(z)$

In this system instability comes from the integrator block : $\dfrac{1}{1-z^{-1}}$. Hence, any summable $x[n]$ would work i.e. $\sum_n x[n] < \infty$. To see this, observe a system :

$$w(n) \longrightarrow \boxed{\frac{1}{1-z^{-1}}} \longrightarrow y(n)$$

$$\begin{aligned} y[n] &= w(n) + y[n-1] \\ &= w[n] + w(n-1) + y(n-2) \\ &= w[n] + w(n-1) + w(n-2) + y(n-3) \\ &= w(n) + w(n-1) + w(n-2) + w(n-3) + \cdots \end{aligned}$$

Hence, $y(n)$ is bounded as long as $\sum_n w(n) < \infty$.

c) $\sigma_e^2 = \dfrac{2^{-2B}}{12}$

$$e_1[n] \longrightarrow \boxed{H_{e_1}(z)} \longrightarrow y[n]$$

$$H_{e_1}(z) = \frac{1}{1 - a z^{-1}} \quad \Rightarrow \quad h_{e_1}[n] = a^n u[n]$$

$$\sigma_{f_1}^2 = \sigma_e^2 \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_{e_1}(e^{j\omega})|^2 \, d\omega = \frac{2^{-2B}}{12} \sum_{n=0}^{\infty} |h_{e_1}(n)|^2$$

$$= \frac{2^{-2B}}{12} \cdot \sum_{n=0}^{\infty} |a|^{2n} = \frac{2^{-2B}}{12} \cdot \frac{1}{1 - |a|^2}$$

$$e_2(n) \longrightarrow \boxed{H_{e_2}(z)} \longrightarrow y[n]$$

$$H_{e_2}(z) = \frac{1}{(1 - a z^{-1})(1 - z^{-1})}$$

Flow graph #2 has larger output noise variance because $H_{e_2}(z)$ is a cascade of two systems: $H_{e_2}(z)$ and $H_{e_2}'(z) = \frac{1}{1 - z^{-1}}$. $H_{e_2}'(z)$ introduces infinite output noise power to the system since $h_{e_2}'[n] = u(n)$ and $\sum_{n=-\infty}^{\infty} |h_{e_2}'[n]|^2 \to \infty$

3. The following values from the 8-point DFT of a length-8 real sequence $x[n]$ are known:

$X[0] = 3,\ X[2] = 0.5 - 4.5j,\ X[4] = 5,\ X[5] = 3.5 + 3.5j,\ X[7] = -2.5 - 7j.$

a) (10 points) Evaluate $x[0]$ and $x[4]$.

b) (10 points) Find the 8-point DFT of the circular convolution:

$$x[n] \circledast_8 \delta[n-1]$$

where $\delta[n]$ is the unit impulse.

c) (10 points) Find the 4-point DFT of the length-4 sequence $w[n]$ given by:

$$w[n] = x[n] + x[n+4] \quad n = 0, 1, 2, 3.$$

d) (10 points) Find the 8-point DCT2 coefficients $X^{c2}[0]$, $X^{c2}[2]$, $X^{c2}[4]$, $X^{c2}[6]$.

a) $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-nk}$

$x[0] = \frac{1}{8} \cdot \sum_{k=0}^{7} X[k] = \frac{11}{8}$

$x(n)$ is real, hence $X[k] = X^*[(-k)_8]$

$\Rightarrow X[1] = X^*[7] = -2.5 + 7j$

$X[3] = X^*[5] = 3.5 - 3.5j$

$X[6] = X^*[2] = 0.5 + 4.5j$

b) $y[n] = x[n] \circledast_8 \delta[n-1]$

$\Rightarrow Y[k] = X[k] \cdot W_8^k$

$Y[0] = 3$, $Y[1] = (-2.5+7j) \cdot W_8$, $Y[2] = 4.5 + 0.5j$

$Y[3] = (3.5-3.5j) W_8^3$, $Y[4] = -5$, $Y[5] = (3.5+3.5j) W_8^5$

$Y[6] = 4.5 - 0.5j$, $Y[7] = (-2.5-7j) W_8^7$;

c) $w[n] = x[n] + x[n+4]$, $n = 0, 1, 2, 3$

$\Rightarrow W[k] = X[2k]$, $k = 0, 1, 2, 3$ (sampling in freq. domain is causing aliasing in time domain).

$W[0] = 3$, $W[1] = 0.5 - 4.5j$, $W[2] = 5$, $W[3] = 0.5 + 4.5j$.

d) $$X^{c_2}[k] = 2 \cdot Re\left\{ \tilde{X}[k]\, e^{-j \cdot \frac{\pi k}{2N}} \right\} \quad , \quad N = 8$$

$X^{c_2} \downarrow$ 8 pt $\qquad \tilde{X}[k] \downarrow$ 16 pt

$\tilde{X}[k]$ is DFT transform of $\quad \hat{x}[n] = \begin{cases} x(n) & , n = 0, \dots, N-1 \\ 0 & , \text{otherwise} \end{cases}$

$$\tilde{X}[k] = \sum_{n=0}^{7} x[n] \cdot W_{16}^{nk} = \sum_{n=0}^{7} x[n]\, W_8^{nk/2} = \begin{cases} X\left[\frac{k}{2}\right] & , k \text{ even} \\ ? & k \text{ odd} \end{cases}$$

$$X^{c_2}[0] = 2 \cdot Re\{ \tilde{X}[0] \} = 2 Re\{ X[0] \} = 6$$

$$X^{c_2}[2] = 2 Re\left\{ \tilde{X}[2] \cdot e^{-j \cdot \frac{2\pi}{16}} \right\} = 2 Re\left\{ X[1] e^{-j \cdot \frac{\pi}{8}} \right\} = 0.7382$$

$$X^{c_2}[4] = 2 Re\left\{ \tilde{X}[4]\, e^{-j \cdot \frac{4\pi}{16}} \right\} = 2 Re\left\{ X[2]\, e^{-j \cdot \frac{\pi}{4}} \right\} = -5.6569$$

$$X^{c_2}[6] = 2 Re\left\{ \tilde{X}[3]\, e^{-j \cdot \frac{6\pi}{16}} \right\} = 2 Re\left\{ X[3] \cdot e^{-j \cdot \frac{3\pi}{8}} \right\} = -3.7884$$