EE 120: Signals and Systems
Department of EECS
UC BERKELEY

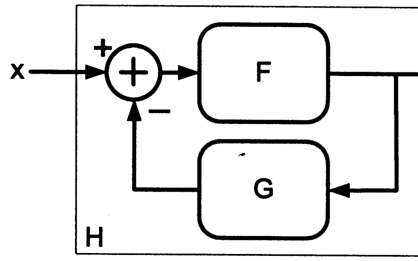
MIDTERM 2
2 May 2007

LAST Name Butterworth FIRST Name Professor

Discussion Time Any time!

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 90 minutes to complete. You will be given at least 90 minutes, up to a maximum of 110 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT2.1 (20 Points) Consider a feedback interconnection of two causal, continuous-time LTI systems, as shown in the figure below.



The transfer function of the *causal* system $\mathsf{F} : [\mathbb{R} \to \mathbb{R}] \to [\mathbb{R} \to \mathbb{R}]$ is

$$\widehat{F}(s) = \frac{1}{s+3}.$$

The transfer function of the *right-sided* system $\mathsf{G} : [\mathbb{R} \to \mathbb{R}] \to [\mathbb{R} \to \mathbb{R}]$ is

$$\widehat{G}(s) = \frac{8 - s^2}{s - 2}.$$

For each part, explain your reasoning succinctly, but clearly and convincingly.

(a) Is the system F BIBO stable? Is the system G BIBO stable?

F has a left-half plane pole @ -3, and it is causal $\Rightarrow$ RoC_F: $-3 < Re(s)$, which includes the $i\omega$-axis $\Rightarrow$ BIBO Stable.

G has a right-half plane pole @ 2 (and another at ∞), and it is right-sided $\Rightarrow$ ROC_G: $2 < Re(s) < \infty$ $\Rightarrow$ Not BIBO Stable

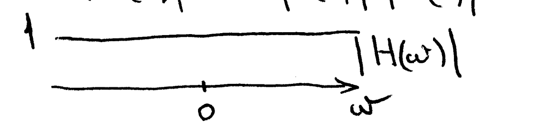
(b) Determine the transfer function $\widehat{H}$ of the closed-loop system $\mathsf{H} : [\mathbb{R} \to \mathbb{R}] \to [\mathbb{R} \to \mathbb{R}]$, and specify its RoC. ALSO, provide a well-labeled plot of the frequency response magnitude $|H(\omega)|$, or explain why H does not have a well-defined frequency response.

$$\hat{H}(s) = \frac{\hat{F}(s)}{1 + \hat{F}(s)\hat{G}(s)} = \frac{\frac{1}{s+3}}{1 + \frac{8-s^2}{(s+3)(s-2)}} = \frac{s-2}{(s+3)(s-2) + 8 - s^2} = \frac{s-2}{\cancel{s^2} + s - 6 + 8 - \cancel{s^2}}$$

$$\Rightarrow \boxed{\hat{H}(s) = \frac{s-2}{s+2}}$$

$\hat{H}$ has the form of an all-pass analog filter. Note that it is stable because it is causal and has a left-half plane pole @ -2 $\Rightarrow$ $H(\omega) = \hat{H}(s)\big|_{s=i\omega}$.

$|H(\omega)| = K = |H(0)| = |\hat{H}(0)| = 1$



MT2.2 (20 Points) Consider a *causal*, discrete-time LTI system whose input-output behavior is in accordance with the difference equation

$$y(n) - y(n-1) - y(n-2) = x(n)\,, \quad \forall n \in \mathbb{Z},$$

where x and y denote the input and output signals, respectively.

It is known that $y(-1) = 0$ and $y(-2) = 1$. The input signal x is simply the unit impulse; that is, $x(n) = \delta(n), \forall n$.

(a) Determine the transfer function $\hat{H}$ of the system, along with its RoC, and specify whether the system is BIBO stable.

Take the Z-transform of both sides of the difference equation: $\hat{Y}(z) - z^{-1}\hat{Y}(z) - z^{-2}\hat{Y}(z) = \hat{X}(z) \Rightarrow$

$\hat{H}(z) = \dfrac{\hat{Y}(z)}{\hat{X}(z)} = \dfrac{1}{1-z^{-1}-z^{-2}} = \dfrac{z^2}{z^2-z-1}$ Poles @ $\dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

ROC_H: $\frac{1+\sqrt{5}}{2} < |z| \Rightarrow$ Not BIBO stable.

(b) Determine $y_{ZSR}(n)$ for $n = 0, 1, 2, \ldots, 5$, where y_{ZSR} is the zero-state response of the system. $y(n) = y(n-1) + y(n-2) + x(n)$ let $x(n) = \delta(n)$ & $y(m) = 0$ $m = -1, -2$

$y_{ZSR}(0) = x(0) = \delta(0) = 1$

$y_{ZSR}(1) = y_{ZSR}(0) + \cancel{y_{ZSR}(-1)} + \cancel{\delta(1)} = 1$

$y_{ZSR}(2) = y_{ZSR}(1) + y_{ZSR}(0) = 2$

$y_{ZSR}(3) = 3$

$y_{ZSR}(4) = 5$

$y_{ZSR}(5) = 8$ ... Fibonacci Sequence

(c) Determine $y_{ZIR}(n)$ for $n = 0, 1, 2, \ldots, 5$, where y_{ZIR} is the zero-input response of the system. Let $x(n) = 0$, and $y(-1) = 0$, $y(-2) = 1$

$y_{ZIR}(0) = y_{ZIR}(-1) + y_{ZIR}(-2) = 1$

$y_{ZIR}(1) = y_{ZIR}(0) + y_{ZIR}(-1) = 1$

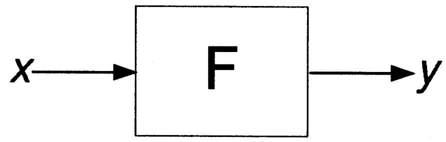
$y_{ZIR}(2) = y_{ZIR}(1) + y_{ZIR}(0) = 2$

$y_{ZIR}(3) = 3$

$y_{ZIR}(4) = 5$

$y_{ZIR}(5) = 8$ ... Fibonacci Sequence.

MT2.3 (65 Points) The figure below depicts a *causal* LTI system known as an N^{th}-order *Butterworth filter* $\mathsf{F} : [\mathbb{R} \to \mathbb{R}] \to [\mathbb{R} \to \mathbb{R}]$, where $N \geq 1$:



For each part, explain your reasoning succinctly, but clearly and convincingly.

(a) The transfer function $\widehat{F}$ of the *first-order* Butterworth filter (i.e., $N = 1$) satisfies the following equation:

$$\widehat{F}(s)\,\widehat{F}(-s) = \frac{1}{1 + \left(\frac{s}{i\omega_c}\right)^2},$$

where ω_c is a positive frequency.

(i) Show that the transfer function of this first-order filter is as depicted below, and specify its corresponding region of convergence:

$$\widehat{F}(s) = \frac{\omega_c}{s + \omega_c}.$$

$$\hat{F}(s)\hat{F}(-s) = \frac{1}{1 - \frac{s^2}{\omega_c^2}} = \frac{\omega_c^2}{\omega_c^2 - s^2} = \frac{\omega_c^2}{(\omega_c - s)(\omega_c + s)} = \underbrace{\frac{\omega_c}{\omega_c - s}}_{\hat{F}(-s)} \underbrace{\frac{\omega_c}{\omega_c + s}}_{\hat{F}(s)}$$

$\hat{F}(s) = \frac{\omega_c}{\omega_c + s}$ gives a causal & stable Butterworth filter.

(ii) For this part only, suppose the input signal applied to the filter is the unit-step function; that is, $x(t) = u(t), \forall t$. Determine the total response $y(t)$, and identify its transient portion $y_{\mathsf{TR}}(t)$.

$$x(t) = u(t) \Rightarrow \hat{X}(s) = \frac{1}{s} \Rightarrow \hat{Y}(s) = \hat{X}(s)\hat{F}(s) = \frac{\omega_c}{s(s+\omega_c)}$$

Partial fraction expansion: $\hat{Y}(s) = \frac{\alpha}{s} + \frac{\beta}{s+\omega_c}$

$$\alpha = \hat{F}(s)\Big|_{s=0} = 1$$

$$\beta = \frac{\omega_c}{s}\Big|_{s=-\omega_c} = -1$$

$$\hat{Y}(s) = \frac{1}{s} - \frac{1}{s+\omega_c} \Rightarrow$$

$$y(t) = u(t) - e^{-\omega_c t}u(t) \Rightarrow \begin{cases} y_{ss}(t) = u(t) \\ y_{TR}(t) = -e^{-\omega_c t}u(t) \end{cases}$$

(b) Throughout this part, assume that F is a *second-order* Butterworth filter (i.e., $N = 2$). Such a filter has transfer function

$$\widehat{F}(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2},$$

where ω_c is a positive frequency.

(i) Determine the linear, constant-coefficient differential equation that governs the input-output behavior of the filter.

$$\frac{\hat{Y}(s)}{\hat{X}(s)} = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2} \implies (s^2 + \sqrt{2}\,\omega_c s + \omega_c^2)\hat{Y}(s) = \omega_c^2 \hat{X}(s)$$

$$\implies \ddot{y}(t) + \sqrt{2}\,\omega_c \dot{y}(t) + \omega_c^2 y(t) = \omega_c^2 x(t)$$

(ii) Provide a well-labeled pole-zero diagram for the filter's transfer function $\widehat{F}$; determine the RoC of F; and explain whether the filter is BIBO stable.

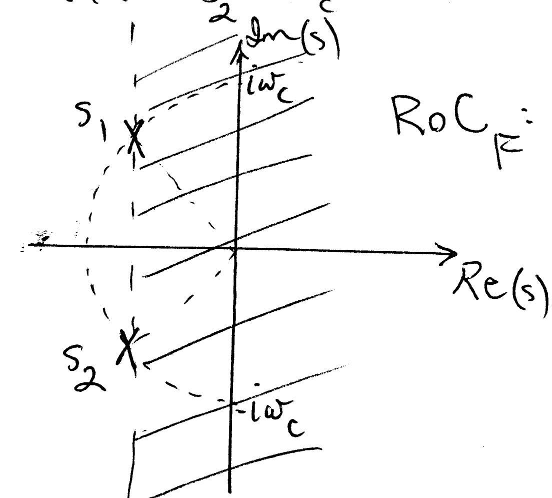
$$\hat{F}(s) = \frac{1}{\left(\frac{s}{\omega_c}\right)^2 + \sqrt{2}\left(\frac{s}{\omega_c}\right) + 1}$$

Poles are the roots of the denominator

$$\frac{s}{\omega_c} = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{-\sqrt{2} \pm i\sqrt{2}}{2} = \begin{cases} \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = e^{i3\pi/4} \\ \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}} = e^{-i3\pi/4} \end{cases} \implies$$

Poles @ $s_1 = \omega_c e^{i3\pi/4}$ and $s_2 = \omega_c e^{-i3\pi/4}$

Filter is BIBO stable



$RoC_F: \frac{-1}{\sqrt{2}} < Re(s)$

(iii) For this part only, suppose the input signal applied to the filter is the unit-step function; that is, $x(t) = u(t), \forall t$. Determine a simple expression for $y_{ss}(t)$, the corresponding steady-state response of the filter.

Filter is stable $\Longrightarrow y_{ss}(t) = \hat{F}(0)\, u(t) = u(t)$

$$y_{ss}(t) = u(t)$$

(iv) Determine an expression for, and provide a well-labeled sketch of, $f(t)$, the impulse response of the second-order Butterworth filter F.
<u>Hint</u>: You may find the following Laplace transform pair useful:

$$e^{-\alpha t}\sin(\omega_0 t)u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}, \quad \text{RoC: } -\alpha < \text{Re}(s).$$

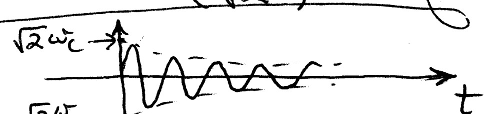
$$\hat{F}(s) = \frac{\omega_c^2}{\left(s+\frac{\omega_c}{\sqrt{2}}\right)^2 + \frac{\omega_c^2}{2}} = \frac{\sqrt{2}\,\omega_c\left(\frac{\omega_c}{\sqrt{2}}\right)}{\left(s+\frac{\omega_c}{\sqrt{2}}\right)^2 + \left(\frac{\omega_c}{\sqrt{2}}\right)^2}$$

$$(s+\alpha)^2 + \omega_0^2 = s^2 + \underbrace{2\alpha}_{\text{let } =\sqrt{2}\omega_c} s + \underbrace{\alpha^2+\omega_0^2}_{=\omega_c^2}$$

$$2\alpha = \sqrt{2}\,\omega_c \Rightarrow \alpha = \frac{\omega_c}{\sqrt{2}}$$

$$\alpha^2+\omega_0^2 = \frac{\omega_c^2}{2} + \omega_0^2 = \omega_c^2 \Rightarrow \omega_0^2 = \frac{\omega_c^2}{2}$$

$$f(t) = \sqrt{2}\,\omega_c\, e^{-\frac{\omega_c t}{\sqrt{2}}} \sin\left(\frac{\omega_c t}{\sqrt{2}}\right) u(t)$$



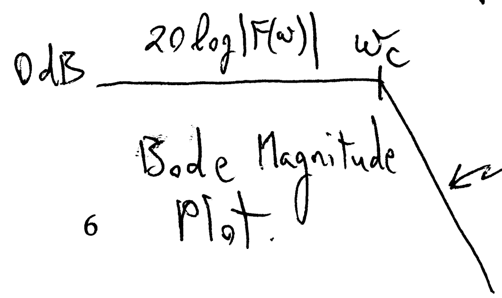
1st zero crossing: $\frac{\omega_c t_0}{\sqrt{2}} = \pi \Rightarrow t_0 = \frac{\sqrt{2}\pi}{\omega_c}$

(v) Explain whether the filter has a well-defined frequency response. If it does, provide a well-labeled Bode plot of its frequency response magnitude $|F(\omega)|$.

F is stable $\Longrightarrow$ It has a well-defined freq response.

$$\hat{F}(s) = \frac{1}{\left(\frac{s}{\omega_c}\right)^2 + \underbrace{\sqrt{2}}_{1/Q}\left(\frac{s}{\omega_c}\right) + 1}$$

$\frac{1}{Q} = \sqrt{2} \Longrightarrow Q = \frac{1}{\sqrt{2}}$ Not high enough to cause peaking.



(c) The transfer function $\widehat{F}$ of the N^{th}-order Butterworth filter satisfies the following equation:

$$\widehat{F}(s)\,\widehat{F}(-s) = \frac{1}{1 + \left(\dfrac{s}{i\omega_c}\right)^{2N}},$$

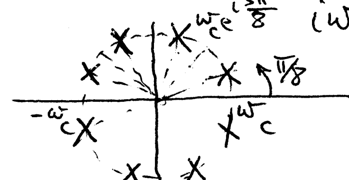
where ω_c is a positive frequency.

(i) How many poles does the product $\widehat{F}(s)\,\widehat{F}(-s)$ have? $2N$ poles, split equally between $\hat{F}(s)$ and $\hat{F}(-s)$.

(ii) Provide a well-labeled pole-zero plot of the product $\widehat{F}(s)\,\widehat{F}(-s)$ if $N = 4$.

$$\left(\frac{s}{i\omega_c}\right)^8 + 1 = 0 \Rightarrow \left(\frac{s_k}{i\omega_c}\right)^8 = -1 = e^{i(2k+1)\pi} \Rightarrow \frac{s_k}{i\omega_c} = e^{i\frac{2k+1}{8}\pi}$$

$$s_k = \omega_c\, e^{i\left(\frac{2k+1}{8}\pi + \frac{\pi}{2}\right)} \qquad k = 0, \dots, 7$$



(iii) Determine an expression for the k^{th} pole, where k is an appropriately-chosen integer.

Hint: You may find at least one of the following expressions useful:

$$1^{\frac{1}{L}} = e^{i\frac{k2\pi}{L}}, \qquad k \in \{0, 1, \dots, L-1\},$$

and

$$(-1)^{\frac{1}{L}} = e^{i\frac{(2k+1)\pi}{L}}, \quad k \in \{0, 1, \dots, L-1\}.$$

$$s_k = \omega_c\, e^{i\left(\frac{2k+1}{2N}\pi + \frac{\pi}{2}\right)} = \omega_c\, e^{i\frac{\pi}{2}\left(\frac{2k+1}{N} + 1\right)}$$

$k = 0, 1, \dots, 2N-1$

Note: There is never a pole on the $i\omega$-axis.

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Discussion Time Any time

Problem	Points	Your Score
Name	10	10
1	20	20
2	20	20
3	65	65
Total	**115**	115