EE 120: Signals and Systems MIDTERM 1

Department of EECS 22 September 2011

UC Berkeley

LAST Name ter Nautch FIRST Name Phil

Discussion Time Don't you know?

- **(10 Points)** Print your name and discussion time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 90 minutes to complete. You will be given at least 90 minutes, up to a maximum of 110 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5" $\times$ 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT1.1 (30 Points) (a) (15 Points) The impulse response of a causal, linear, time-invariant (LTI) system G is given by

$$g(t) = A\left[e^{-at} + te^{-at}\right]u(t), \quad = g_1(t) + g_2(t)$$

where u is the continuous-time unit step, A is a non-zero scalar, and $\mathrm{Re}(a) > 0$.

Determine a reasonably simple expression for $G(\omega)$, the frequency response of the system. Your final answer should be in terms of A and a.

You may or may not find it helpful to know that

$$\int u\,dv = uv - \int v\,du$$

and

$$\int_0^{+\infty} e^{-(a+i\omega)t}\,dt = \frac{1}{i\omega + a}, \text{ if } \mathrm{Re}(a) > 0).$$

$$g_1(t) = Ae^{-at}u(t) \Rightarrow G_1(\omega) = A\int_0^\infty e^{-(a+i\omega)t}dt = \frac{A}{i\omega+a}$$

$\mathrm{Re}(a) > 0$

$$G_1(\omega) = \int_{-\infty}^{\infty} g_1(t)e^{-i\omega t}dt \Rightarrow \frac{dG_1(\omega)}{d\omega} = -i\int_{-\infty}^{\infty} t g_1(t)e^{-i\omega t}dt$$

$$\Rightarrow \int_{-\infty}^{\infty} \underbrace{t g_1(t)}_{g_2(t)} e^{-i\omega t}dt = i\frac{dG_1(\omega)}{d\omega} \Longrightarrow$$

But the left-hand side is simply $G_2(\omega)$ $\Rightarrow$

$$G_2(\omega) = i\frac{dG_1(\omega)}{d\omega} = iA\frac{-i}{(i\omega+a)^2} = \frac{A}{(i\omega+a)^2}$$

$$\Rightarrow G(\omega) = G_1(\omega) + G_2(\omega) = A\,\frac{i\omega+a+1}{(i\omega+a)^2}$$

$$\Rightarrow G(\omega) = A\,\frac{i\omega+(a+1)}{(i\omega+a)^2}$$

Alternative Method: $G_2(\omega) = A\int_0^\infty te^{-(a+i\omega)t}dt$

$u = t \Rightarrow du = dt$

$dv = e^{-(a+i\omega)t}dt \Rightarrow v = \dfrac{-e^{-(a+i\omega)t}}{i\omega+a}$

$$G_2(\omega) = \cancelto{0}{A\cdot\frac{-te^{-(a+i\omega)t}}{i\omega+a}\Big|_0^\infty} + \frac{A}{i\omega+a}\int_0^\infty e^{-(a+i\omega)t}dt = \frac{A}{(i\omega+a)^2}$$

The rest is the same as above.

(b) (15 Points) The input-output behavior of a causal LTI system $\mathbf{H}$ is described the following linear, constant-coefficient differential equation (LCCDE):

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m},$$

where x and y denote the input and output, respectively, and a_0 is non-zero. It's understood that the system is initially at rest (otherwise, it wouldn't be LTI)

(i) (10 Points) Determine a reasonably simple expression for $H(\omega)$, the frequency response of the system. Your expression should be in terms of the coefficients a_n and b_m. Keep $i\omega$ intact (e.g., resist the temptation to write, say, $(i\omega)^2$ as $-\omega^2$).

Let $x(t) = e^{i\omega t}$. Then $y(t) = H(\omega)e^{i\omega t}$. The derivatives of x & y are $\frac{d^m x(t)}{dt^m} = (i\omega)^m e^{i\omega t}$ and $\frac{d^n y(t)}{dt^n} = (i\omega)^n H(\omega) e^{i\omega t}$. Plug these into the LCCDE to obtain:

$$\sum_{n=0}^{N} a_n (i\omega)^n H(\omega) \cancel{e^{i\omega t}} = \sum_{m=0}^{M} b_m (i\omega)^m \cancel{e^{i\omega t}} \Rightarrow$$

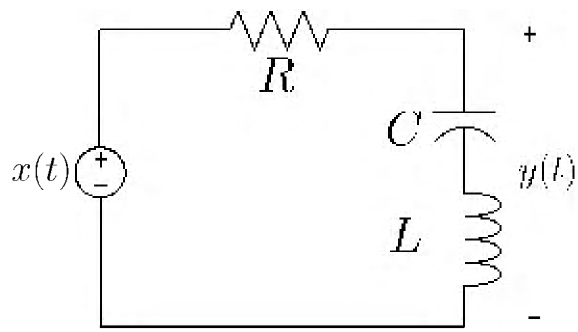
$$H(\omega) = \frac{\sum_{m=0}^{M} b_m (i\omega)^m}{\sum_{n=0}^{N} a_n (i\omega)^n}$$

(ii) (5 Points) True or false? Every causal, LTI system described by an LCCDE has a frequency response that is rational in terms of $i\omega$. That is, you can write the frequency response as the ratio of a polynomial in $i\omega$ over another polynomial in $i\omega$. Explain your answer.

True. We imposed no restriction on the LTI system other than that it be causal and characterized by an LCCDE. The presence of the LCCDE coefficients in the numerator and denominator polynomials of $H(\omega)$ is transparent, and should be noted.

MT1.2 (50 Points) Consider a causal, BIBO stable, LTI system G consisting of a resistive-capacitive-inductive (RLC) circuit, as the figure below depicts. The system's input is the voltage source x and its output is the combined voltage across the capacitor C and inductor L—both indicated on the circuit diagram.

Everything is initially at rest, so $y(0) = 0$ and $\dot{y}(0) = 0$. This means that there is no initial voltage across the capacitor (i.e., it's uncharged) and no current through the inductor.



The circuit is governed by the linear, constant-coefficient differential equation

$$LC\ddot{y}(t) + RC\dot{y}(t) + y(t) = LC\ddot{x}(t) + x(t),$$

where R, L, and C denote the resistance, the inductance, and the capacitance, respectively. Throughout this problem, assume that the units of R, L, and C are chosen to be consistent with those of the driving voltage x and the output voltage y.

(a) (15 Points) Show that the frequency response of the system is of the form

$$G(\omega) = \frac{(i\omega)^2 + \omega_0^2}{(i\omega)^2 + 2\zeta\,(i\omega) + \omega_0^2}, \qquad \forall \omega \in \mathbb{R}.$$

Determine ζ and ω_0 in terms of the circuit parameters R, L, and C.

Let $x(t) = e^{i\omega t}$, so $y(t) = G(\omega)e^{i\omega t}$. Differentiating, we have $\dot{x}(t) = (i\omega)e^{i\omega t}$, $\ddot{x}(t) = (i\omega)^2 e^{i\omega t}$, $\dot{y}(t) = (i\omega)H(\omega)e^{i\omega t}$, and $\ddot{y}(t) = (i\omega)^2 H(\omega) e^{i\omega t}$

Plug these into the differential equation:

$$LC(i\omega)^2 H(\omega)\cancel{e^{i\omega t}} + RC(i\omega)H(\omega)\cancel{e^{i\omega t}} + H(\omega)\cancel{e^{i\omega t}} = LC(i\omega)^2\cancel{e^{i\omega t}} + \cancel{e^{i\omega t}}\,1 \Rightarrow$$

$$H(\omega) = \frac{LC(i\omega)^2 + 1}{LC(i\omega)^2 + RC(i\omega) + 1} = \frac{(i\omega)^2 + 1/LC}{(i\omega)^2 + \frac{R}{L}(i\omega) + \frac{1}{LC}}$$

$\omega_0^2 = \frac{1}{LC} \Rightarrow \omega_0 = \sqrt{1/LC}$

$2\zeta = \frac{R}{L} \Rightarrow \zeta = \frac{R}{2L}$

Throughout parts (b) & (c), assume $\zeta < \omega_0$ (this guarantees complex roots for $s^2+2\zeta s+\omega_0$. To facilitate certain useful approximations, you're asked to assume more strictly that $\zeta \ll \omega_0$.

(b) (35 Points) We can express the frequency response as

$$G(\omega) = \frac{(i\omega + i\omega_0)(i\omega - i\omega_0)}{(i\omega - s_1)(i\omega - s_2)},$$

where s_1 and s_2 are the roots of the polynomial $s^2 + 2\zeta s + \omega_0^2$.

(i) (15 Points) Determine s_1 and s_2 in terms of ζ and ω_0. How are s_1 and s_2 related? Furthermore, determine reasonable approximations for s_1 and s_2, if $\zeta \ll \omega_0$ ~~and $1 < \omega_0$~~.

$$s^2+2\zeta s+\omega_0^2 = 0 \implies s = -\zeta \pm \sqrt{\zeta^2-\omega_0^2} = -\zeta \pm i\omega_0\sqrt{1-(\zeta/\omega_0)^2}$$

$$s_1 = -\zeta + i\omega_0\sqrt{1-(\zeta/\omega_0)^2} \qquad s_2 = -\zeta - i\omega_0\sqrt{1-(\zeta/\omega_0)^2}$$

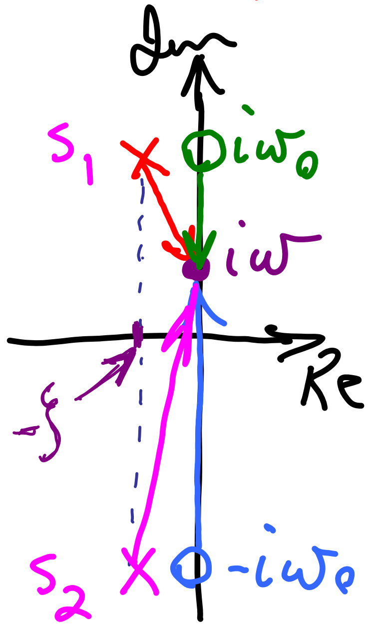
For $\zeta \ll \omega_0$, we have $s_1 \approx -\zeta + i\omega_0$ and $s_2 \approx -\zeta - i\omega_0$

$$s^2+2\zeta s+\omega_0^2 \approx (s+\zeta-i\omega_0)(s+\zeta+i\omega_0)$$

(ii) (20 Points) Provide a well-labeled, approximate sketch of the magnitude response plot $|G(\omega)|$, if $\zeta \ll \omega_0$ ~~and $1 < \omega_0$~~, *and* determine the output signal values $y(t)$ corresponding to the input $x(t) = A + \cos(\omega_0 t)$, for some non-zero constant A.

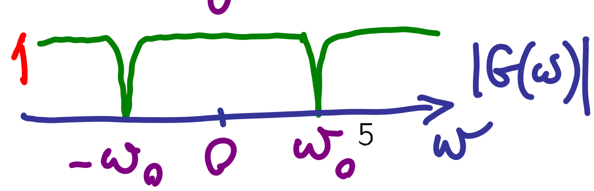
Use a geometric approach to plot $|G(\omega)|$.

$$|G(\omega)| = \frac{|i\omega - i\omega_0|\,|i\omega - (-i\omega_0)|}{|i\omega - \underbrace{(-\zeta+i\omega_0)}_{s_1}|\,|i\omega - \underbrace{(-\zeta-i\omega_0)}_{s_2}|}$$



- $G(\pm\omega_0) = 0$
- For frequencies NOT in a small neighborhood of $\pm\omega_0$, we know $i\omega - i\omega_0 \approx i\omega - s_1$ and $i\omega + i\omega_0 \approx i\omega - s_2$, so $G(\omega) \approx 1$.

Clearly, we have a notch filter.



If $x(t) = A + \cos(\omega_0 t)$

$y(t) \approx A$

MT1.3 (35 Points) Consider the periodic continuous-time signal x described by

$$x(t) = \sum_{n=-\infty}^{+\infty} (-1)^n \delta(t - nT),$$

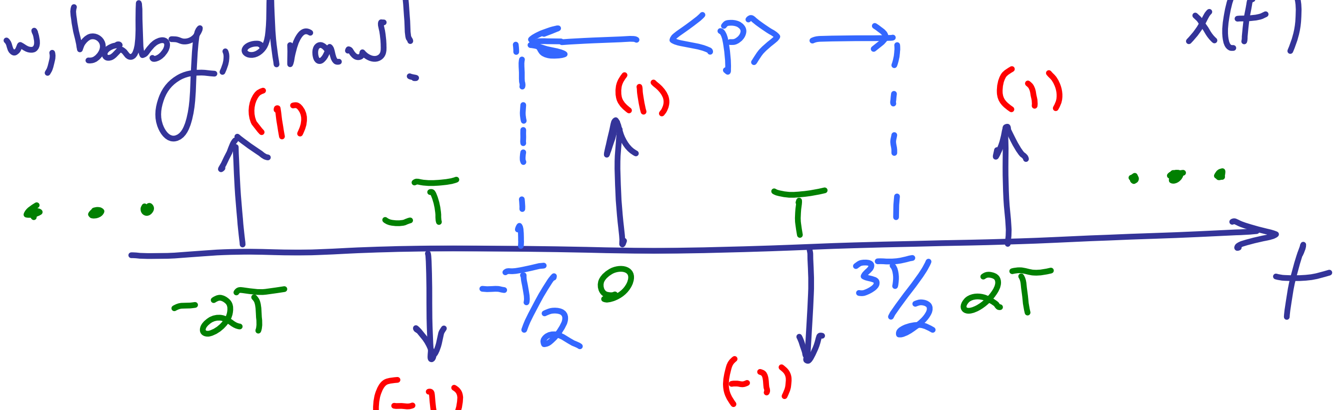
where $T > 0$. Determine the complex exponential continuous-time Fourier series coefficients for the signal x. Be sure to determine the fundamental period and fundamental frequency for the signal.

Potentially useful formulas:

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k \, e^{ik\omega_0 t}$$

$$X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) \, e^{-ik\omega_0 t} dt.$$

Draw, baby, draw!



Fundamental period: $p = 2T$.

Fundamental frequency: $\omega_0 = \frac{2\pi}{p} = \frac{2\pi}{2T} = \frac{\pi}{T}$

$$X_k = \frac{1}{2T} \int_{-T/2}^{3T/2} x(t) e^{-ik\omega_0 t} dt = \frac{1}{2T} \int_{-T/2}^{3T/2} [\delta(t) - \delta(t-T)] e^{-ik\omega_0 t} dt$$

$$\Rightarrow X_k = \frac{1}{2T}\left(1 - e^{-ik\omega_0 T}\right) = \frac{1}{2T}\left(1 - e^{-ik\pi}\right) \Rightarrow X_k = \frac{1 - (-1)^k}{2T}$$

But $\omega_0 = \frac{\pi}{T} \Rightarrow \omega_0 T = \pi$

$$\Rightarrow X_k = \begin{cases} 0 & k \text{ even} \\ \frac{1}{T} & k \text{ odd} \end{cases}$$

Compare w/ the non-alternating impulse train.

LAST Name Ter Nautch FIRST Name Phil

Discussion Time Don't you know?

Problem	Points	Your Score
Name	10	10
1	30	30
2	50	50
3	35	35
Total	~~115~~ 125	125