EE 120: Signals and Systems MIDTERM 1
Department of EECS 27 October 2011
UC Berkeley

LAST Name ______________ FIRST Name ______________

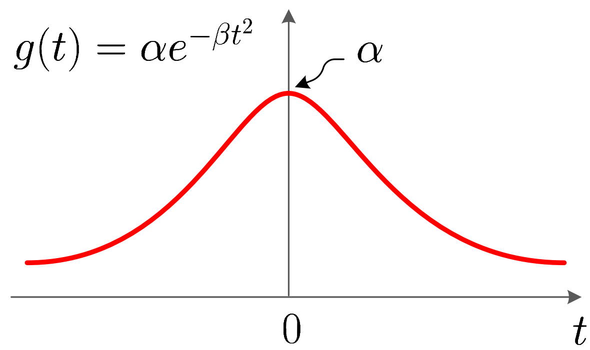
Discussion Time ______________

- **(10 Points)** Print your name and discussion time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 90 minutes to complete. You will be given at least 90 minutes, up to a maximum of 110 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided 8.5" $\times$ 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT1.1 (50 Points) The Gaussian function g, described generically below, plays an important role in science, engineering, and mathematics:

$$\forall t \in \mathbb{R}, \quad g(t) = \alpha e^{-\beta t^2},$$

where $\alpha, \beta > 0$. The figure below shows a qualitative plot of g.



We know certain facts about the Gaussian function:

1. The area under the curve for g is determined by the normalization parameter α and the decay parameter β as follows:

$$\int_{-\infty}^{+\infty} g(t)dt = \int_{-\infty}^{+\infty} \alpha e^{-\beta t^2} dt = \alpha\sqrt{\frac{\pi}{\beta}}.$$

2. The Gaussian function is the unique solution to the following differential equation:

$$\dot{g}(t) + 2\beta t g(t) = 0, \quad \text{where } g(0) = \alpha.$$

   This is because the derivative of the Gaussian is

$$\dot{g}(t) = -2\beta t \underbrace{\alpha e^{-\beta t^2}}_{g(t)}.$$

In what follows, you may or may not find it useful to know the time-differentiation property of the CTFT:

$$\begin{aligned} g(t) &\overset{\mathcal{F}}{\longleftrightarrow} G(\omega) \\ \dot{g}(t) &\overset{\mathcal{F}}{\longleftrightarrow} i\omega G(\omega). \end{aligned}$$

(a) (35 Points) In this part, you will determine the Fourier transform $G(\omega)$ of the Gaussian function, as well as its relationship to the time-domain characteristics of $g(t)$.

(i) (20 Points) Show that $G(\omega)$, defined by the CTFT analysis equation

$$\forall \omega \in \mathbb{R}, \quad G(\omega) = \int_{-\infty}^{+\infty} g(t) e^{-i\omega t} dt,$$

satisfies the differential equation

$$\frac{dG(\omega)}{d\omega} + 2\widehat{\beta}\,\omega G(\omega) = 0, \quad \text{and } G(0) = \widehat{\alpha},$$

for some parameters $\widehat{\alpha}, \widehat{\beta} > 0$. Determine $\widehat{\alpha}$, $\widehat{\beta}$, and a reasonably simple expression for $G(\omega)$. You must express everything in terms of the parameters α and β that appear in the expression for $g(t)$.

If, by sound reasoning, you arrive at an expression for $G(\omega)$ that is written correctly in terms of $\widehat{\alpha}$ and $\widehat{\beta}$, you will receive partial credit even if you fail to express $\widehat{\alpha}$ and $\widehat{\beta}$ in terms of α and β.

What is the most striking observation that you can make about $G(\omega)$ in terms of the type of function that it represents?

(ii) (15 Points) Provide well-labeled, but otherwise qualitative, plots of $g(t)$ and $G(\omega)$ for $\beta = \beta_1$ and $\beta = \beta_2$, where $0 < \beta_1 < \beta_2$. Describe how the shapes of the plots for $g(t)$ and $G(\omega)$ change as β increases.

How, if at all, is this a manifestation of the time-frequency uncertainty principle?

Based on your findings, determine the truth or falsehood of the following assertion:

> **True or false?** A signal that is <u>not</u> time-limited must be band-limited, and, conversely, a signal that is <u>not</u> band-limited must be time-limited.

Be succinct, but clear and convincing, in your reasoning.

(b) (15 Points) The following diagram illustrates a CT-LTI system G.

$$x \longrightarrow \boxed{\;G\;} \longrightarrow y$$

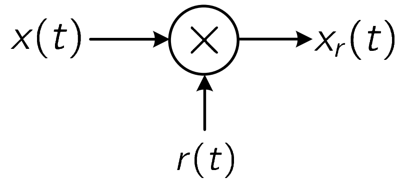
The impulse response of the system is the following particular Gaussian function:

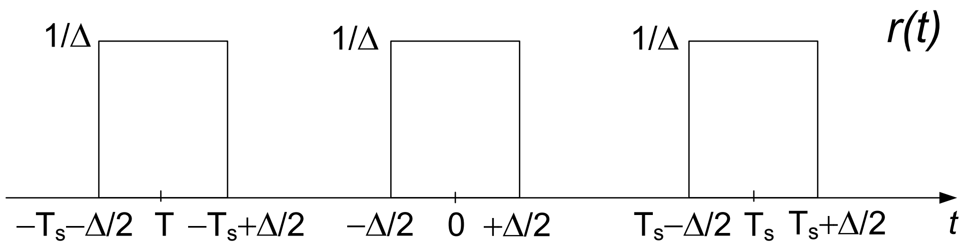
$$\forall t \in \mathbb{R}, \quad g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2},$$

where $\sigma > 0$ is called *standard deviation*, a measure of the spread of the Gaussian.

Determine a reasonably simple expression for, and provide a well-labeled qualitative plot of, the *step response* of the system (i.e., the response of the system if $x(t) = u(t)$, where u is the unit step). Your expression for $y(t)$ must be in terms of the function g.

MT1.2 (45 Points) A continuous-time signal x is modulated by a uniform train r of square pulses, as shown below.





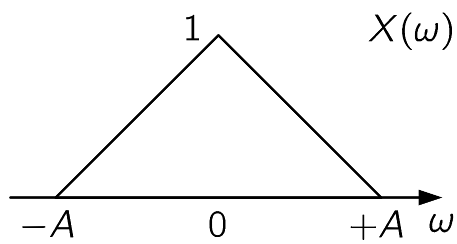
The parameter T_s (seconds) denotes the fundamental period of the pulse train, and it corresponds to a fundamental frequency of $\omega_s = 2\pi/T_s$ radians per second.

(a) (15 Points) Show that the coefficients Q_k in the complex-exponential Fourier series expansion of r are given by

$$Q_k = \frac{1}{T_s}\,\mathrm{sinc}\left(\frac{k\Delta}{T_s}\right),$$

where $\mathrm{sinc}(\alpha) \triangleq \dfrac{\sin(\pi\alpha)}{\pi\alpha}$.

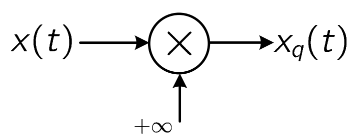
(b) (10 Points) The spectrum of the input signal x is as depicted by the figure below:



Suppose $\omega_s = 2A$ and $\Delta = T_s/2$. Provide a well-labeled plot of $X_r(\omega)$, the spectrum of the signal x_r produced by the pulse modulation of x. Your plot should include $|\omega| \leq 5\omega_s$.

(c) (10 Points) For the values of T_s and Δ specified in part (b), state whether it's possible to recover x from x_r. If so, how would you recover x?

(d) (10 Points) Compare the pulse train modulation of x, which produced the signal x_r, with the impulse train sampling of x as shown below, which produces the signal x_q.



$$q(t) = \sum_{\ell=-\infty}^{+\infty} \delta(t - \ell T_s)$$

Clearly, in impulse train sampling we retain less of the signal x (only values at integer multiples of T_s), compared with pulse train modulation, in which we retain segments of width Δ centered at integer multiples of T_s.

Do the additional segments of the signal x retained by pulse train modulation offer any advantage insofar as the rate ω_s goes? In other words, can ω_s in pulse train modulation be *less* than the Nyquist rate that constrains impulse train sampling?

MT1.3 (20 Points) A continuous-time signal z is described by $z(t) = x(t)y(t)$, for all t, where x and y are periodic with period p.

(a) (10 Points) Show that the complex-exponential Fourier series coefficients of z are given by

$$Z_n = \sum_{k\in\mathbb{Z}} X_k Y_{n-k},$$

where X_k and Y_k denote the corresponding coefficients for x and y, respectively.

(b) (10 Points) Let $y(t) = x^*(t)$, and prove Parseval's identity

$$\frac{1}{p}\int_{\langle p\rangle} |x(t)|^2 dt = \sum_{k\in\mathbb{Z}} |X_k|^2.$$

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Problem	Points	Your Score
Name	10	
1	50	
2	45	
3	20	
Total	**125**	