

*Zero-weight-cycle* is in  $\mathcal{NP}$  because we can exhibit a cycle in  $G$ , and it can be checked that the sum of the edge weights on this cycle are equal to 0.

We now show that *Subset Sum*  $\leq_P$  *Zero-weight-cycle*. We are given numbers  $w_1, \dots, w_n$ , and we want to know if there is a subset that adds up to exactly  $W$ . We construct an instance of *Zero-weight-cycle* in which the graph has nodes  $0, 1, 2, \dots, n$ , and an edge  $(i, j)$  for all pairs  $i < j$ . The weight of edge  $(i, j)$  is equal to  $w_j$ . Finally, there is an edge  $(n, 0)$  of weight  $-W$ .

We claim that there is a subset that adds up to exactly  $W$  if and only if  $G$  has a zero-weight cycle. If there is such a subset  $S$ , then we define a cycle that starts at 0, goes through the nodes whose indices are in  $S$ , and then returns to 0 on the edge  $(n, 0)$ . The weight of  $-W$  on the edge  $(n, 0)$  precisely cancels the sum of the other edge weights. Conversely, all cycles in  $G$  must use the edge  $(n, 0)$ , and so if there is a zero-weight cycle, then the other edges must exactly cancel  $-W$  — in other words, their indices must form a set that adds up to exactly  $W$ .

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<sup>1</sup>ex642.498.819