

*Galactic Shortest Path* is in NP: given a path  $P$  in a graph, we can add up the lengths and risks of its edges, and compare them to the given bounds  $L$  and  $R$ .

*Galactic Shortest Path* involves adding numbers, so we naturally consider reducing from the *Subset Sum* problem. Specifically, we'll prove that  $\text{Subset Sum} \leq_P \text{Galactic Shortest Path}$ .

Thus, consider an instance of *Subset Sum*, specified by numbers  $w_1, \dots, w_n$  and a bound  $W$ ; we want to know if there is a subset  $S$  of these numbers that add up to exactly  $W$ . *Galactic Shortest Path* looks somewhat different on the surface, since we have *two kinds* of numbers (lengths and risks), and we are only given *upper bounds* on their sums. However, we can use the fact that we also have an underlying graph structure. In particular, by defining a simple type of graph, we can encode the idea of choosing a subset of numbers.

We define the following instance of *Galactic Shortest Path*. The graph  $G$  has a nodes  $v_0, v_1, \dots, v_n$ . There are two edges from  $v_{i-1}$  to  $v_i$ , for each  $1 \leq i \leq n$ ; we'll name them  $e_i$  and  $e'_i$ . (If one wants to work with a graph containing no parallel edges, we can add extra nodes that subdivide these edges into two; but the construction turns out the same in any case.)

Now, any path from  $v_0$  to  $v_n$  in this graph  $G$  goes through edge one from each pair  $\{e_i, e'_i\}$ . This is very useful, since it corresponds to making  $n$  independent binary choices — much like the binary choices one has in *Subset Sum*. In particular, choosing  $e_i$  will represent putting  $w_i$  into our set  $S$ , and  $e'_i$  will represent leaving it out.

Here's a final observation. Let  $W_0 = \sum_{i=1}^n w_i$  — the sum of all the numbers. Then a subset  $S$  adds up to  $W$  if and only if its complement adds up to  $W_0 - W$ .

We give  $e_i$  a length of  $w_i$  and a risk of 0; we give  $e'_i$  a length of 0 and a risk of  $w_i$ . We set the bound  $L = W$ , and  $R = W_0 - W$ . We now claim: there is a solution to the *Subset Sum* instance if and only if there is a valid path in  $G$ . For if there is a set  $S$  adding up to  $W$ , then in  $G$  we use the edges  $e_i$  for  $i \in S$ , and  $e'_j$  for  $j \notin S$ . This path has length  $W$  and risk  $W_0 - W$ , so it meets the given bounds. Conversely, if there is a path  $P$  meeting the given bounds, then consider the set  $S = \{w_i : e_i \in P\}$ .  $S$  adds up to at most  $W$  and its complement adds up to at most  $W_0 - W$ . But since the two sets together add up to exactly  $W_0$ , it must be that  $S$  adds up to exactly  $W$  and its complement to exactly  $W_0 - W$ . Thus,  $S$  is valid solution to the *Subset Sum* instance.

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