(a) Assume graph $G = (V, E)$ has vertex set V and edge set E. Create a bipartite graph with the two sides both corresponding to V, that is, for each node $v \in V$ we add two nodes v_{in} and w_{out} to the bipartite graph. For each undirected edge $e = (v, w) \in G$ we add the edge (v_{out}, w_{in}) to the bipartite graph G'. We claim that cycle covers in G are in one-to-one correspondence with perfect matchings in G'. This is true as a set of edges forms a cycle cover if and only if it contains exactly one edge entering and exactly one edge leaving each vertex v. Hence we can find a cycle cover in $O(mn)$ time by finding a perfect matching in G'.

(b) Clearly the Cycle Cover Problem is in NP, as given the set of edges that form a cycle cover, it is easy to check if they form a cycle cover of G. We will prove that Cycle Cover with at most 3 edges in each cycle is NP-complete by a reduction from *3-Dimensional Matching*. Consider an instance of the *3-Dimensional Matching* Problem, given by disjoint sets X, Y, and Z, each of size n; and a set $T \subseteq X \times Y \times Z$ of ordered triples. We create an instance of the cycle cover problem as follows. Let $S = X \cup Y \cup Z$ denote the set of nodes. First add 3 nodes with a triangle connecting them for each triple $t \in T$. These will be the only triangles used in our construction. Let the 3 nodes in the triangle correspond to the 3 elements of the set t, but node that if a node $s \in S$ is covered by multiple triples, than we add separate nodes corresponding to v for each triple. For a node $s \in S$ let A_s be the set of these nodes corresponding to node v. We will also add a set of B_v additional nodes corresponding to v with $|B_v| = |A_v| - 1$, and add edges between all nodes in the sets A_v and B_v in both direction. This finishes the construction of the graph G.

Now we claim that there is a perfect matching in the *3-Dimensional Matching* Problem if and only if G has a Cycle Cover with at most 3 edges in each cycle. First assume we have perfect 3D matching. The matching corresponds to a cycle cover as follows. For each triple t in the 3D matching, select the corresponding triangle. This covers exactly one node in each set A_s for each $s \in S$. Now cover the remaining nodes by cycles of length 2 going between the sets A_s and B_s.

Finally, we need to show that if G has a Cycle Cover with at most 3 edges in each cycle, than there is a perfect 3D matching in the *3-Dimensional Matching* Problem. So see this note that the elements of the sets B_s can only be covered by 2 cycles, as no 3 cycle passes through these nodes. Any set of 2 cycles covering the node $\cup_s B_s$ leaves one node uncovered from each set A_s. The union of the set $\cup_s A_s$ consists of node-disjoint triangles corresponding to the triples in T, so these remaining nodes can only be covered if they correspond to triples in T.

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