

The algorithm is very similar to the basic Gale-Shapley algorithm from the text. At any point in time, a student is either “committed” to a hospital or “free.” A hospital either has available positions, or it is “full.” The algorithm is the following:

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While some hospital  $h_i$  has available positions
   $h_i$  offers a position to the next student  $s_j$  on its preference list
  if  $s_j$  is free then
     $s_j$  accepts the offer
  else ( $s_j$  is already committed to a hospital  $h_k$ )
    if  $s_j$  prefers  $h_k$  to  $h_i$  then
       $s_j$  remains committed to  $h_k$ 
    else  $s_j$  becomes committed to  $h_i$ 
      the number of available positions at  $h_k$  increases by one.
      the number of available positions at  $h_i$  decreases by one.

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The algorithm terminates in  $O(mn)$  steps because each hospital offers a positions to a student at most once, and in each iteration, some hospital offers a position to some student.

Suppose there are  $p_i > 0$  positions available at hospital  $h_i$ . The algorithm terminates with an assignment in which all available positions are filled, because any hospital that did not fill all its positions must have offered one to every student; but then, all these students would be committed to some hospital, which contradicts our assumption that  $\sum_{i=1}^m p_i < n$ .

Finally, we want to argue that the assignment is stable. For the first kind of instability, suppose there are students  $s$  and  $s'$ , and a hospital  $h$  as above. If  $h$  prefers  $s'$  to  $s$ , then  $h$  would have offered a position to  $s'$  before it offered one to  $s$ ; from then on,  $s'$  would have a position at *some* hospital, and hence would not be free at the end — a contradiction.

For the second kind of instability, suppose that  $(h_i, s_j)$  is a pair that causes instability. Then  $h_i$  must have offered a position to  $s_j$ , for otherwise it has  $p_i$  residents all of whom it prefers to  $s_j$ . Moreover,  $s_j$  must have rejected  $h_i$  in favor of some  $h_k$  which he/she preferred; and  $s_j$  must therefore be committed to some  $h_\ell$  (possibly different from  $h_k$ ) which he/she also prefers to  $h_i$ .

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<sup>1</sup>ex304.339.892