We claim that such a graph G has a tree decomposition $(T, \{V_t\})$ in which each piece V_t corresponds uniquely to an internal triangular face of G. We prove this by induction on the number of nodes in G.

Choose any internal edge $e = (u, v)$ of G; deleting u and v produces two components A and B. Let G_1 be the subgraph induced on $A \cup \{u, v\}$ and G_2 the subgraph induced on $B \cup \{u, v\}$. By induction, there are tree decompositions $(T_1, \{X_t\})$ and $(T_2, \{Y_t\}$ of G_1 and G_2 respectively in which the pieces correspond uniquely to internal faces. Thus there are nodes $t_1 \in T_1$ and $t_2 \in T_2$ that correspond to the faces containing the edge (u, v). If we let T denote the tree obtained by adding an edge (t_1, t_2) to $T_1 \cup T_2$, then $(T, \{X_t\} \cup \{Y_t\})$ is a tree decomposition having the desired properties.

[1] ex203.262.545