

We show that *Competitive 3-SAT* \leq_P *Geography on a Graph* by encoding the Boolean formula Φ over which the players are competing as a directed graph G . The construction we use is depicted in the accompanying figure.

For each variable x_i , we create a “diamond” consisting of vertices labeled $a_i, x_i, \overline{x_i}, b_i$ with directed edges as in the figure. The starting node for the game is a_1 , and there is an edge (a_{i+1}, b_i) for each $i < n$. Thus, in the initial moves of the game, Player 1 essentially chooses a side of the i^{th} diamond, for each odd value of i , while Player 2 chooses a side of the i^{th} diamond for each even value of i . At the end of this process, the current node is b_n , and it is Player 2’s turn to move.

The next two moves of the game will now determine which player wins. Below b_n , we create a node c_i , $i = 1, \dots, k$, representing the k clauses of Φ . Out of each node c_i , we construct edges to nodes associated with terms occurring in clause i , as follows: if x_j occurs in clause i , then we add the edge (c_i, x_j) ; and if $\overline{x_j}$ occurs in clause i , then we add the edge $(c_i, \overline{x_j})$. These edges model the following construction: Player 2 chooses a clause of Φ , and Player 1 must then choose a term in this clause that has been set to T by the earlier play. If he can do this for any clause, then he must have had a winning strategy for the *Competitive 3-SAT* instance; otherwise, Player 2 must have had a winning strategy. We make this concrete in the following claim.

(1) *Player 1 has a forced win in the Competitive 3-SAT instance if and only if Player 1 has a forced win from a_1 in the Geography on a Graph instance on G .*

Proof. First suppose that Player 1 has a forced win in the *Competitive 3-SAT* instance. Then he applies this strategy in the *Geography on a Graph* instance as follows. Each time Player 2 moves to a node x_j (respectively $\overline{x_j}$), Player 1 interprets this as setting $x_j = F$ (respectively $x_j = T$), i.e., the true value is the one that is left blank for later use. When Player 1 has to decide whether to move to node x_i or $\overline{x_i}$, he consults his *Competitive 3-SAT* strategy. If it calls for him to set $x_i = T$, then he moves to $\overline{x_i}$; if it calls for him to set $x_i = F$, then he moves to x_i .

Now consider the final two moves of the game. Player 2 chooses the node c_i . We know, since Player 1’s choices forced a win in the *Competitive 3-SAT* instance, that there is a term t in c_i that evaluates to T . This means that in the earlier moves, someone chose the node labeled with the negation of t , but no one chose the node labeled t . Consequently, Player 1 can take the edge (c_i, t) , and then Player 2 will not be able to make a move.

To show the converse direction, suppose that Player 1 has a forced win in the *Geography on a Graph* instance. Then we can convert this to a *Competitive 3-SAT* strategy in a very similar way: moves to x_i (respectively $\overline{x_i}$) in *Geography on a Graph* correspond to setting $x_i = F$ (respectively $x_i = T$) in the *Competitive 3-SAT* instance. Now, because Player 1 can force a win in *Geography on a Graph*, he must have a legal move for every choice of a node c_i by Player 2. This means that in every clause of the *Competitive 3-SAT* instance, there is some term that evaluates to T — so Player 1 has won the *Competitive 3-SAT* instance. ■

¹ex554.747.411