6.8 Laplace Transform: General Formulas

Formula	Name, Comments	Sec.
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	6.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	6.1
$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$ $\mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t)$	s-Shifting (First Shifting Theorem)	6.1
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Differentiation of Function Integration of Function	6.2
$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\,d\tau$ $= \int_0^t f(t - \tau)g(\tau)\,d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution	6.5
$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)u(t - a)$	t-Shifting (Second Shifting Theorem)	6.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}$	Differentiation of Transform Integration of Transform	6.6
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}}\int_0^p e^{-st} f(t)\,dt$	f Periodic with Period p	6.4 Project 16

6.9 Table of Laplace Transforms

For more extensive tables, see Ref.* [A9] *in Appendix 1.

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
1	$1/s$	1	6.1
2	$1/s^2$	t	
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$	
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	
7	$\dfrac{1}{s-a}$	e^{at}	6.1
8	$\dfrac{1}{(s-a)^2}$	te^{at}	
9	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\dfrac{1}{(n-1)!}t^{n-1}e^{at}$	
10	$\dfrac{1}{(s-a)^k} \quad (k > 0)$	$\dfrac{1}{\Gamma(k)}t^{k-1}e^{at}$	
11	$\dfrac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{a-b}(e^{at} - e^{bt})$	
12	$\dfrac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\dfrac{1}{a-b}(ae^{at} - be^{bt})$	
13	$\dfrac{1}{s^2+\omega^2}$	$\dfrac{1}{\omega}\sin \omega t$	6.1
14	$\dfrac{s}{s^2+\omega^2}$	$\cos \omega t$	
15	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sinh at$	
16	$\dfrac{s}{s^2-a^2}$	$\cosh at$	
17	$\dfrac{1}{(s-a)^2+\omega^2}$	$\dfrac{1}{\omega}e^{at}\sinh \omega t$	
18	$\dfrac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos \omega t$	
19	$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1}{\omega^2}(1 - \cos \omega t)$	6.2
20	$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{1}{\omega^3}(\omega t - \sin \omega t)$	

(continued)

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
21	$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	6.6
22	$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega}\sin \omega t$	
23	$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	
24	$\dfrac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\dfrac{1}{b^2-a^2}(\cos at - \cos bt)$	
25	$\dfrac{1}{s^4+4k^4}$	$\dfrac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$	
26	$\dfrac{s}{s^4+4k^4}$	$\dfrac{1}{2k^2}\sin kt \sinh kt$	
27	$\dfrac{1}{s^4-k^4}$	$\dfrac{1}{2k^3}(\sinh kt - \sin kt)$	
28	$\dfrac{s}{s^4-k^4}$	$\dfrac{1}{2k^2}(\cosh kt - \cos kt)$	
29	$\sqrt{s-a} - \sqrt{s-b}$	$\dfrac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$	
30	$\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\dfrac{a-b}{2}t\right)$	*I* 5.5
31	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	*J* 5.4
32	$\dfrac{s}{(s-a)^{3/2}}$	$\dfrac{1}{\sqrt{\pi t}}e^{at}(1+2at)$	
33	$\dfrac{1}{(s^2-a^2)^k} \quad (k > 0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$	*I* 5.5
34	e^{-as}/s	$u(t-a)$	6.3
35	e^{-as}	$\delta(t-a)$	6.4
36	$\dfrac{1}{s}e^{-k/s}$	$J_0(2\sqrt{kt})$	*J* 5.4
37	$\dfrac{1}{\sqrt{s}}e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
38	$\dfrac{1}{s^{3/2}}e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}} \quad (k > 0)$	$\dfrac{k}{2\sqrt{\pi t^3}}e^{-k^2/4t}$	

(*continued*)

Table of Laplace Transforms (*continued*)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
40	$\frac{1}{s}\ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$	γ 5.5
41	$\ln\frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt} - e^{at})$	
42	$\ln\frac{s^2+\omega^2}{s^2}$	$\frac{2}{t}(1 - \cos \omega t)$	6.6
43	$\ln\frac{s^2-a^2}{s^2}$	$\frac{2}{t}(1 - \cosh at)$	
44	$\arctan\frac{\omega}{s}$	$\frac{1}{t}\sin \omega t$	
45	$\frac{1}{s}\operatorname{arccot} s$	$\mathrm{Si}(t)$	App. A3.1

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathscr{L}(f)$

	$f(t)$	$\mathscr{L}(f)$
1	1	$1/s$
2	t	$1/s^2$
3	t^2	$2!/s^3$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\dfrac{1}{s-a}$

	$f(t)$	$\mathscr{L}(f)$
7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
11	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
12	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$