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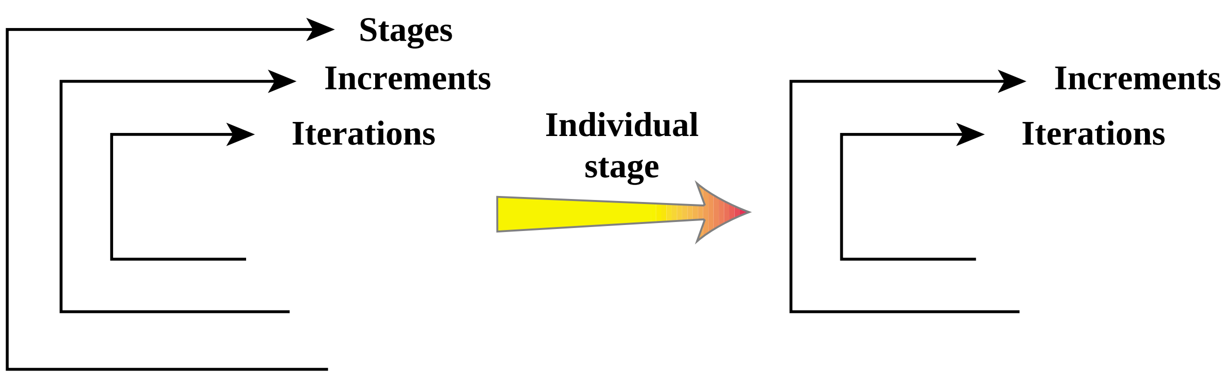
Overview of Solution Methods

Nonlinear Structural Analysis is a Multilevel Continuation Process



Study Focus Restriction

In this and subsequent Chapters we will stay within an individual stage, which need not be identified.

Consequently we will focus on the two innermost solution levels: increments and iterations.

These are also called predictor and corrective levels, especially in literature dealing with numerical solution of ODEs

Classification (Advancing over an Individual Stage)

Purely incremental methods

- **Also called predictor-only methods**
- **Corrective level is missing**
- **Popular in path dependent problems, e.g., plasticity**

Incremental-iterative methods

- **Also called predictor-corrector methods**
- **Corrective phase aims to eliminate drift error**
- **Popular in path independent problems**

Basic Description and Notation (1)

To advance the solution, the stage is broken up into incremental steps, or increments by short.

If necessary, increments will be identified by subscript n For example the state vector after the n-th increment is $\mathbf{u}_n$

The state vector before any increment (initial state or stage start) is $\mathbf{u}_0$

Over each incremental step the state vector u and staging parameter λ undergo finite changes denoted by

$$\Delta\mathbf{u}_n \quad \Delta\lambda_n$$

respectively.

Basic Description and Notation (2)

Iteration steps will be usually identified by the superscript k
For example $\{ \mathbf{u}_n^k , \lambda_n^k \}$ **may denote the solution after the kth iteration of the nth step, whereas** $\{\mathbf{u}_n^0, \lambda_n^0\}$ **is the predicted solution before starting the corrective process.**

Iterative changes in $\Delta\mathbf{u}$ **and** $\Delta\lambda$ **are often shortened to** $\mathbf{d}$ **and** η **, resp.**

Certain "argument omission" abbreviations will be used throughout the exposition to reduce clutter. If $\{ \mathbf{u}_n , \lambda_n \}$ **is a state-control pair computed after n incremental step we denote**

$$\boxed{\mathbf{r}_n = \mathbf{r}(\mathbf{u}_n, \lambda_n) \qquad \mathbf{K}_n = \mathbf{K}(\mathbf{u}_n, \lambda_n) \qquad \mathbf{q}_n = \mathbf{q}(\mathbf{u}_n, \lambda_n)} \quad \text{etc}$$

Similarly if the state control pair is $\{ \mathbf{u}_n^k , \lambda_n^k \}$ **after** k **iterations carried out at the** n**th increment, we denote**

$$\boxed{\mathbf{r}_n^k = \mathbf{r}(\mathbf{u}_n^k, \lambda_n^k)} \quad \text{etc}$$

and likewise for other quantities

Decisions Must Be Made in Three Continuation Ingredients

Increment control: **how far to advance**

Predictor: **advancing from last solution**

Corrector: **eliminating or reducing the drift error**

Increment Control

Performing the *n*th increment step

$$\Delta \mathbf{u}_n = \mathbf{u}_{n+1} - \mathbf{u}_n \qquad \Delta\lambda_n = \lambda_{n+1} - \lambda_n$$

Stepsize constraint

$$c(\Delta \mathbf{u}_n,\, \Delta\lambda_n) = 0$$

Rate form of stepsize constraint

$$\mathbf{a}^T\, \dot{\mathbf{u}} + g\, \dot{\lambda} = 0,$$

in which

$$\mathbf{a}^T = \frac{\partial c}{\partial \mathbf{u}}, \qquad g = \frac{\partial c}{\partial \lambda}.$$

Predictor

Predictor gives the increments

$$\boxed{\Delta\mathbf{u}_n^0, \quad \Delta\lambda_n^0,}$$

Simplest predictor is Forward Euler applied to the stiffness rate equation

$$\boxed{\dot{\mathbf{r}} = \mathbf{0}, \qquad \text{or} \quad \mathbf{K}\dot{\mathbf{u}} = \mathbf{q}\,\dot{\lambda}.}$$

For a prescribed λ increment, it gives

$$\boxed{\Delta\mathbf{u}_n^0 = \mathbf{K}_n^{-1}\,\mathbf{q}_n\,\Delta\lambda_n^0 = \mathbf{v}_n\,\Delta\lambda_n^0}$$

Corrector

Inserting the predicted values in the residual gives the drift error

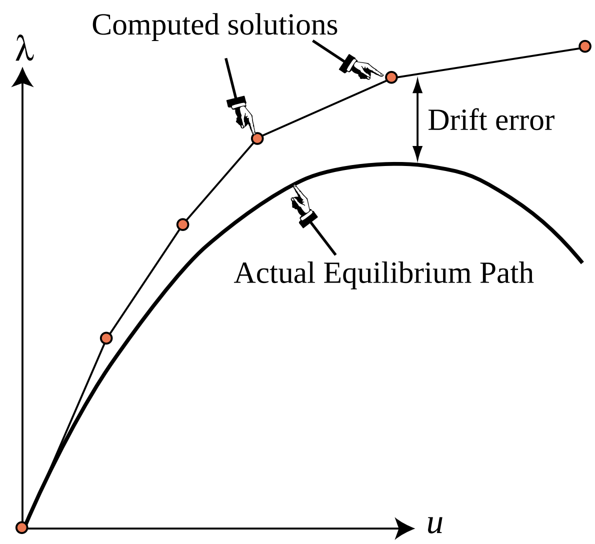
$$\mathbf{r}_n^0 = \mathbf{r}\big(\mathbf{u}_n + \Delta\mathbf{u}_n^0, \lambda_n + \Delta\lambda_n^0\big) \neq \mathbf{0}.$$

A corrective process generates a series of updates (*k* = iteration index)

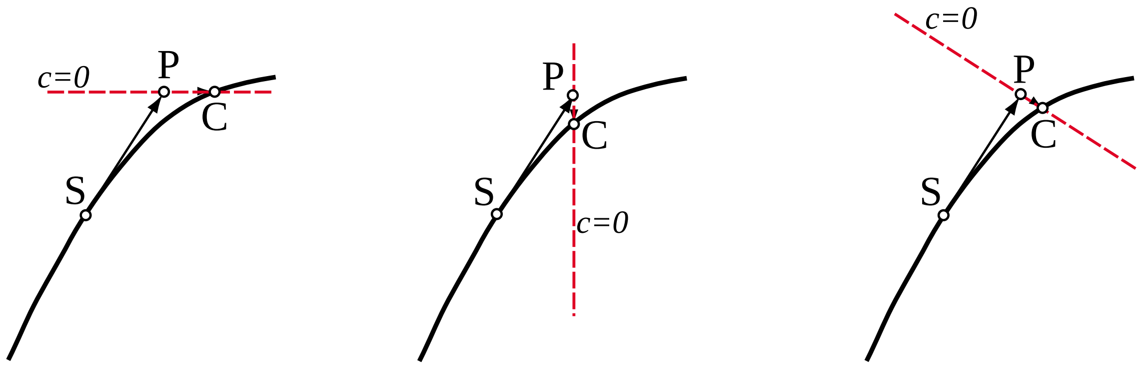
$$\Delta\mathbf{u}^k \quad \Delta\lambda^k$$

that hopefully make the total solution converge towards equilibrium thus eliminating the drift error

Drift Error in Predictor-Only Continuation Process



Stepsize Control Constraint Types

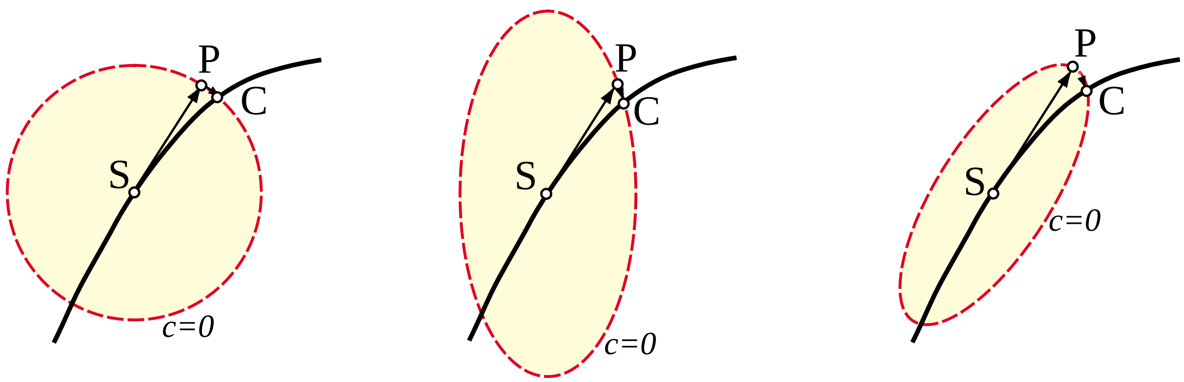


Stage parameter control (aka load control if λ is a load factor)

State control

Arclength control

Stepsize Increment Control Types (2)



Hyperspherical control | **Global hyperelliptic control** | **Local hyperelliptic control**

Predictor With Stage Parameter (aka Load) Control

$$c(\Delta u_n,\, \Delta\lambda_n) = \Delta\lambda_n - \ell_n = 0$$

$$\Delta\mathbf{u}_n^0 = \mathbf{v}_n\, \ell_n \qquad \Delta\lambda_n^0 = \ell_n$$

Predictor With Arclength Control

$$c(\Delta u_n, \Delta\lambda_n) = |\Delta s_n| - \ell_n = \frac{1}{f_n}\left|\mathbf{v}_n^T \Delta\mathbf{u}_n + \Delta\lambda_n\right| - \ell_n = 0$$

$$f_n = +\sqrt{1 + \mathbf{v}_n^T \mathbf{v}_n}$$

$$\Delta\lambda_n^0 = \frac{\ell_n f_n}{\pm(\mathbf{v}_n^T \mathbf{v}_n + 1)} = \frac{\ell_n}{\pm\sqrt{\mathbf{v}_n^T \mathbf{v}_n + 1}} = \pm\frac{\ell_n}{f_n}$$

$$\Delta\mathbf{u}_n^0 = \pm\frac{\mathbf{v}_n \ell_n}{f_n}$$

Traversing Equilibrium Path in Positive Sense

Criterion: positive external work over increment

$$\boxed{\Delta W = \mathbf{q}_n^T \, \Delta\mathbf{u}_n^0 = \mathbf{q}_n^T \, \mathbf{v}_n \, \Delta\lambda_n > 0.}$$

Generally effective, but fails if

$$\boxed{\mathbf{q}^T \mathbf{v} = 0}$$

E.g., bifurcation points or places where incremental velocity vector v vanishes

Stage Parameter Control

Constraint:

$$\boxed{\Delta\lambda_n = \ell_n,}$$

Rate form:

$$\boxed{\mathbf{a} = \mathbf{0}, \qquad g = 1.}$$

State Control (aka Displacement Control)

Constraint:

$$c(\Delta\mathbf{u}_n) \equiv (\Delta\mathbf{u}_n^T \Delta\mathbf{u}_n)^2 - \ell_n\, u^2 = 0$$

Rate form:

$$\mathbf{a}^T = 2\Delta\mathbf{u}_n \qquad g = 0$$

Arclength Control

Constraint:

$$\boxed{c(\Delta u_n, \Delta\lambda_n) = |\Delta s_n| - \ell_n = \frac{1}{f_n}\left|\mathbf{v}_n^T \Delta\mathbf{u}_n + \Delta\lambda_n\right| - \ell_n = 0}$$

$$f_n = +\sqrt{1 + \mathbf{v}_n^T \mathbf{v}_n}.$$

Rate form:

$$\boxed{\mathbf{a}^T = \mathbf{v}_n / f_n \qquad g = 1/f_n}$$