Part (B)

Seg-6

Numerical Differentiation

⇒ The process of computing the value of the derivative $\frac{dy}{dx}$ for some particular value of x from the given data when the actual form of the function is unknown, is called numerical differentiation.

SP-24, 23

Au-22

Newton's forward interpolation formula:

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \quad \text{(i)}$$

$$u = \frac{x_0 - x_0}{h} \cdots \text{(ii)}$$

$$\Rightarrow \frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \cdots \quad \text{(iii)}$$

$$\frac{du}{dx} = \frac{1}{h} \cdots \text{(iv)}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \cdots\right]}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{12u^2-36u+22}{4!}\Delta^4 y_0 + \cdots\right]$$

Ex-1 find $\frac{dy}{dx}$, $\frac{d^2y}{d^2x^2}$, $(x=1)$

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$	$\Delta^5 y_0$
1	1					
		7				
2	8		12			
				6		
		19				
3	27		18		0	
						0
		37		6		
4	64		24		0	
				6		
		61				
5	125		30			
		91				
6	216					

$h=1$, $\quad u = \frac{x-x_0}{h} = \frac{1-1}{1} = 0$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \cdots\right]$$

$$= \frac{1}{1}\left[7 + -\frac{1}{\cancel{2}}\times \cancel{12}^{6} + \frac{2}{\cancel{6}}\times\cancel{6}\right]$$

$$= 3$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0\right]$$

$$= \frac{1}{1}\left[12 + (0-1)6\right] = 6$$

Backward interpolation:

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n \cdots \quad \text{---(i)}$$

$$u = \frac{x - x_n}{h} \quad \text{---(ii)}$$

$$\frac{dy}{du} \Rightarrow \nabla y + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n \cdots \quad \text{---(iii)}$$

$$\frac{du}{dx} = \frac{1}{h} \cdots \text{(iv)}$$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \cdots\right]$$

Ex:= $\frac{d\theta}{dt}$, $\frac{d^2\theta}{dt^2}$, $(t = 0.7)$

T	θ	$\nabla\theta$	$\nabla^2\theta$	$\nabla^3\theta$	$\nabla^4\theta$	$\nabla^5\theta$
0.0	0.0					
		0.12				
0.2	0.12		0.24			
		0.36		0.02	0	
0.4	0.48		0.26			0
		0.62		0.02		
0.6	1.10		0.28		0	
		0.90		0.02		
0.8	2.0		0.30			
		1.20				
1.0	3.20					

Here, $x = t = 0.7$, $h = 0.2$

$$u = \frac{t - t_n}{h} = \frac{0.7 - 1.0}{0.2} = -1.5$$

$$\frac{d\theta}{dt} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n\right]$$

$$= \frac{1}{0.2}\left[1.20 + \frac{2(-1.5)+1}{2}\,0.3 + \frac{3(-1.5)^2+6(-1.5)+2}{6}\,0.02\right]$$

$$= \frac{1}{0.2}\left[1.20 - 0.3 \mp 0.0008\right] = 4.5 \text{ radian/sec}$$

$$\frac{d^2\theta}{dt} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n\right]$$

$$= \frac{1}{(0.2)^2}\left[0.3 + (-1.5+1)\,0.02\right] = 7.25 \text{ rad/sec}^2$$

Max - Min Tabulated function.

\# From the table find x, correct to 2 decimal places, for which $y = \max$ & find y.

x	y	Δ	Δ^2	Δ^3	Δ^4
1.2	0.9320				
		0.0316			
1.3	0.9636		-0.0097		
		0.0219		-0.0002	
1.4	0.9855		-0.0099		0.0002
		0.0120		0	
1.5	0.9975		-0.0099		
		0.0021			
1.6	0.9996				

Let $x_0 = 1.2$

For maximum value of y, we have $\frac{dy}{dx} = 0$

Differentiating Newton's forward interpolation formula with respect to u and neglecting terms second differences we get,

$$0 = 0.0316 + \frac{2u-1}{2}(-0.0097)$$

$$\Rightarrow u = 3.8$$

$$\therefore x = x_0 + uh = 1.2 + 3.8 \times 0.1 = 1.58$$

The value 1.58 is closer to $x = 1.6$, hence we use Newton's backward difference formula.

~~[illegible]~~

$$0 = 0.0021 + \frac{2u+1}{2}(-0.0097)$$

$$\therefore u = -0.2.$$

Now,

$$y(1.58) = 0.9996 - 0.2 \times (0.0021) + \frac{(-0.2)(-0.2+1)}{2}(-0.0099)$$

$$= 1.0.$$

The max value occurs at $x = 1.58$

and max value of $y = 1.0 \; (0.999972)$.

SP-24, 23
Au-22

Explain how numerical differentiation can be used to find the max & min values of a tabulated function.

We know, Newton's forward interpolation formula as:

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+3}{3!}\Delta^3 y_0 \cdots\right]$$

We know, max and min values of a function & $y=f(x)$ can be found by $\frac{dy}{dx}=0$ and solution for x.

Now keeping up only up to 2nd difference we have,

$$\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 = 0$$

Solving this for u, we get x as x_0+uh at which y is a max or min. If the value of x is in the for ward then we will use Newton's forward difference formula and if x is in the backward then we will use Newton's Backward difference formula to find max or min value of y.

Au-22

3(b)

If distance x units along the rod is given below for t

(i) the velocity (ii) its accelaration $(t = 0.3s)$

t	x	Δx_0	$\Delta^2 x_0$	$\Delta^3 x_0$	$\Delta^4 x_0$	$\Delta^5 x$	$\Delta^6 x$
0.0	3.013						
		0.149					
0.1	3.162		-0.104				
		0.045		0.216			
0.2	3.207		0.112		-0.484		
		0.157		-0.238		0.773	
0.3	3.364		-0.126		0.319		-1.07
		0.031		0.081		-0.398	
0.4	3.395		-0.045		-0.079		
		-0.014		0.002			
0.5	3.381		-0.043				
		-0.057					
0.6	3.324						

$t = 0.3s$, $h = 0.1$, $u = \frac{t - t_0}{h} = \frac{0.3 - 0.0}{0.1} = 3$

$$\frac{dx}{dt} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \cdots\right]$$

$$= \frac{1}{0.1}\left[0.149 + \frac{2\times 3 - 1}{2} \times (-0.104) + \frac{3(3)^2 - 6\times 3 + 2}{6} \times 0.216\right]$$

$= 0.285$ m/s (unit / sec) $\Rightarrow$ velocity

$$\frac{d^2x}{dt^2} = \frac{1}{h^2}\left(\Delta^2 y_0 + (u-1)\Delta^3 y_0\right)$$

$$= \frac{1}{(0.1)^2}\left(-0.104 + (3-1)\,0.216\right)$$

$= 32.8$ unit / sec^2 $\Rightarrow$ Accelaration