Book (Ch-9) Seg-8 (Numerical Integration)

1(a) What do you mean by Numerical Integration?

Numerical Integration: Numerical integration is the process of finding the numerical value of a definite integral.

Au-23/SP-24

$$I = \int_a^b f(x)\,dx$$

Here, $y = f(x)$

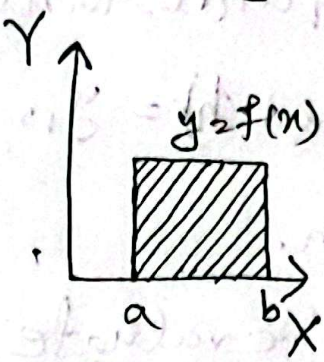

Fig: $y = f(x)$

SP-23/Au-22/SP-22

4) a) When do we need to use a numerical method instead of analytical method for integration?

Ans: Numerical methods are used when analytical integration is impractical, such as ~~for non ele~~ when the function is too complex, involves experimental data rather than a mathematical expression.

Q4(b) / 4(a)

Derive the general quadrature formula using Newton's forward difference formula.

Au-23 / SP-24 / SP-23 / SP-22

Solve:

Let, $y = f(x)$ is a function whose values are known at equidistant values of x i.e.

$f(x_0) = y_0$, $f(x_0 + h) = y_1$, $f(x_0 + 2h) = y_2$, ...,

$f(x_0 + nh) = y_n$.

We have to evaluate, $I = \int_a^b f(x)\,dx$

To evaluate, I, we have to replace $f(x)$ by a suitable interpolation formula. Let the interval $[a, b]$ be divided into n subintervals with the division points

$a = x_0 < x_0 + h < \dots \dots\ x_0 + nh = b$ where h is the width of each subinterval.

Approximating, $f(x)$ by Newton's forward interpolation formula,

$$I = \int_a^b f(x)\,dx$$

$$= \int_{x_0}^{x_0+nh} \left(y_0 + u\Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \cdots \right) dx$$

since, $u = \frac{x - x_0}{h}$ | where, $h = \frac{b-a}{n}$, $a = x_0$, $b = x_0 + nh$

For lower boundary, when, $x = x_0$

$$u = \frac{x_0 - x_0}{h} = 0$$

$\therefore u = 0$

For upper boundary when,

$$x = x_0 + nh$$

$$u = \frac{x_0 + nh - x_0}{h}$$

$\therefore u = n$

And,

$$u = \frac{x - x_0}{h}$$

$$\Rightarrow uh = x - x_0$$

$$\Rightarrow x = x_0 + uh$$

$$\Rightarrow \frac{d}{du}(x) = \frac{d}{du}(x_0 + uh)$$

$$\Rightarrow \frac{dx}{du} = 0 + h$$

$$\therefore dx = h\,du$$

Hence,

$$I = h\int_0^n \left(y_0 + u\Delta y_0 + \frac{\overset{u(u-1)}{u^2-u}}{2}\Delta^2 y_0 + \frac{\overset{u(u-1)(u-2)}{u^3-3u^2+2u}}{6}\Delta^3 y_0 + \cdots\right)du$$

$$= h\left[u y_0 + \frac{u^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{u^4}{4} - \frac{3u^3}{3} + 2\cdot\frac{u^2}{2}\right)\Delta^3 y_0 + \cdots\right]_0^n$$

$$= h\left[\left\{n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - \frac{3n^3}{3} + n^2\right)\Delta^3 y_0 + \cdots\right\} - 0\right]$$

$$\therefore I = h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(n^3/3 - n^2/2\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots\right]$$

∴ This equation is called the general quadrature formula or general gauss Legendre quadrature formula.

Formula:

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int \cancel{x}\, dx = x$$

Trapezoidal Rule

Au-22

(i) Derive the Trapezoidal Rule for Integration Using Newton's forward difference Formula.

Solve:
According to the General Quadrature formula, We have,

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= \int_{x_0}^{x_0+nh} \left(y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \right) dx$$

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots \right]$$

Substituting $n=1$ in the general quadrature formula and neglecting all differences greater than the first, we get,

$$I_1 = \int_{x_0}^{x_0+h} f(x)\,dx$$

$$= h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= 2h\left[2y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{h}{2}\left[2y_0 + y_1 - y_0\right]$$

$$= \frac{h}{2}\left[y_0 + y_1\right]$$

$$I_2 = \int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}(y_1 + y_2)$$

$$I_3 = \int_{x_0+2h}^{x_0+3h} f(x)\,dx = \frac{h}{2}(y_2 + y_3)$$

$$I_n = \int_{x_0(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Combining all these expressions, we obtain,

$$I = I_1 + I_2 + I_3 + \dots + I_n$$

$$= \frac{h}{2}(y_0 + y_1) + h/2\,(y_1 + y_2) + \dots + h/2\,(y_{n-1} + y_n)$$

$$= \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \dots y_{n-1}) + y_n\right]$$

↳ Formula

Example: Calculate the value $\int_0^1 \frac{x}{1+x}\,dx$ correct up to 3 significant figures taking six intervals by trapezoidal Rule. [illegible]

Solve:

Given that, $f(x) = \dfrac{x}{1+x}$

$n = 6,\ a = 0,\ b = 1$

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

NT: Table (0-1) পর্যন্ত boundary হবে point পরের ৬ ঘর নিবো.

x	0	1/6	2/6	3/6	4/6	5/6	6/6
$y=f(x)$	0	0.14286	0.25	0.33333	0.4	0.45454	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

We know,

Using Trapezoidal Rule,

$$I = \frac{h}{2}\left[y_0 + y_6 + 2\left(y_1 + y_2 + y_3 + y_4 + y_5\right)\right]$$

$$= \frac{1/6}{2}\left[0 + 0.5 + 2\left(0.14286 + 0.25 + 0.33333 + 0.4 + 0.45454\right)\right]$$

$$= 0.30512$$

$$\therefore I = 0.305$$

correct to 3 significant figures.

p-22, Au-2

Calculate the value $\int_0 \frac{1}{1+x} dx$ correct up to 3 significant figures taking six intervals by trapezoidal Rule.

Solve:

Given that, $f(x) = \frac{1}{1+x}$

$n = 6, \quad a = 0, \quad b = 1$

$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

x	0	1/6	2/6	3/6	4/6	5/6	6/6
$f(x)=y$	1	0.85714	0.75	0.66667	0.6	0.54545	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

We know,

$$I = \frac{h}{2}\left[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1/6}{2}\left[1 + 0.5 + 2(0.85714 + 0.75 + 0.66667 + 0.6 + 0.54545)\right]$$

$= \cancel{0.69488}\ 0.77821$

$\therefore I = \cancel{0.694}\ 0.778$

Correct to 3 significant figures.

Au-21

$\int_0^{45} \cos x \, dx$ by Trapezoidal Rule. Here upper and lower is in degree. The interval will be divided into 3 strips.

Solve: Given that,

$$f(x) = \cos x$$

$$a = 0^\circ, \quad b = 45^\circ, \quad n = 3$$

$$h = \frac{b-a}{n} = \frac{45^\circ - 0^\circ}{3} = 15^\circ$$

x	0°	15°	30°	45°
$y = f(x)$	1	0.96592	0.86602	0.70711
	y_0	y_1	y_2	y_3

We know,

$$I = \frac{h}{2}\left[y_0 + y_3 + 2(y_1 + y_2)\right]$$

$$= \frac{15}{2}\left[1 + 0.70711 + 2(0.96592 + 0.8660\right.$$

$$= 40.282425$$

Example - 9.3

Find the value of $\int_1^5 \log_{10} x\,dx$, taking 8 subintervals correct to ~~four~~ four decimal places by Trapezoidal Rule.

Solve: Given that,

$$f(x) = \log_{10} x \qquad h = \frac{b-a}{n}$$

$$a = 1,\ b = 5,\ n = 8 \qquad = \frac{5-1}{8} = 0.5$$

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	0	0.17609	0.30102	0.39794	0.47712	0.54406	0.60205	0.65321	0.69897
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

We know,

$$I = \frac{h}{2}\left[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{0.5}{2}\left[0 + 0.69897 + 2(0.17609 + 0.30102 + 0.54406 + 0.60205 + 0.65321)\right]$$

$$= 1.75050$$

$$\therefore I = 1.7505$$

SP-24 / Au-23

4 (c) Calculate the value $\int_0^1 \sqrt{1-x^3}\, dx$ correct up to 3 significant figures taking six intervals by trapezoidal rule.

Solve:

Here, $f(x) = \sqrt{1-x^3}$

$a = 0$, $b = 1$, $n = 6$ $h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

x	0	1/6	2/6	3/6	4/6	5/6	6/6	
$f(x)$	1	0.99	0.98	0.35	0.84	0.65	0	
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	

We know,

$$I = \frac{h}{2}\left[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.5}{2}\left[1 + 0 + 2(0.99 + 0.98 + 0.35 + 0.84 + 0.65)\right]$$

$= 2.155$

$\therefore$ Correct to 3 significant figures.

CamScanner

SP-22

The velocity of a train which starts from rest is given by the following table, the time being recorded in minutes from the start and the speed in Km/hr.

t (minutes)	2	4	6	8	10	12	14	16	18	20
v (Km/hr)	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0

Estimate approximately the total distance run in 20 minutes using

i) Trapezoidal Rule.

ii) Simpson's 1/3 Rule.

Solve:

$$v = \frac{ds}{dt}$$

$$ds = v.dt$$

$$\int ds = \int v.dt$$

$$S = \int_0^{20} v.dt$$

The train starts from rest

the velocity $v=0$ when $t=0$

t	0	2	4	6	8	10	12	14	16	18	20
v	0	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

$$h = \frac{2}{60} = \frac{1}{30} \text{ hrs}$$

Using Trapezoidal Rules:

$$I = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\right]$$

$$= \frac{1/30}{2}\left[(0+0) + 2(16 + 28.8 + 40 + 46.4 + 51.2 + 32.0 + 17.6 + 8 + 3.2) + 0\right]$$

$$= \frac{1}{60} \times 486.4$$

$$= 8.106$$

Using Simpson's 1/3 rule:

$$I = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{1/30}{3}\left[(0+0) + 4(16 + 24.0 + 51.2 + 17.6 + 3.2) + 2(28.8 + 46.4 + 32.0 + 8)\right]$$

$$= \frac{1}{90} \times 742.4$$

$$= 8.25 \text{ Km}$$

∴ The distance by the train in 20 mins = 8.25 Km

SP-23

1) c) A river is 80 unit wide. The depth d at a distance x unit from one bank is given by the following table.

	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Find the area of cross-section of the river using i) Trapezoidal Rule ii) Simpson's Rule

Solve:

$$\int_0^{80} f(x)\,dx$$

$h = 10,$

Using Trapezoidal Rule:

$$I = \frac{h}{2}\left[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{10}{2}\left[0 + 3 + 2(4 + 7 + 9 + 12 + 15 + 14 + 8 + 3)\right.$$

$$= \cancel{5[3}\ 5 \times 147$$

$$= 735 \text{ units}^2$$

Using Simpson's 1/3 rule:

$$I = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{10}{3}\left[(0+3) + 4(4+9+15+8) + 2(7+12+14)\right]$$

$$= 710 \text{ units}^2$$

Simpson's Rule (ONE-THIRD)

Q. Derive the Simpson's 1/3 rule for integration using Newton's forward difference formula.

Solve: According to the general Quadrature formula, we have,

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \cdots\cdots\right]$$

Substituting $n=2$ in this formula and neglecting the third and other higher order differences, we get,

$$I_1 = \int_{x_0}^{x_0+2h} f(x)\,dx$$

$$= h\left[2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - \frac{4}{2}\right)\Delta^2 y_0\right]\cdots$$

$$\frac{8}{6} - \frac{4}{4} \qquad \frac{16-12}{12} = \frac{4}{12} = \frac{1}{3}$$

$$= h\left[2y_0 + 2\Delta y_0 + \frac{1}{3}\Delta^2 y_0\right]$$

$\Delta y_0 = y_1 - y_0$

$\Delta^2 y_0 = y_2 - 2y_1 + y_0$

$$= \frac{h}{3}\left[6y_0 + 6\Delta y_0 + \Delta^2 y_0\right]$$

$$= \frac{h}{3}\left[6y_0 + 6(y_1 - y_0) + y_2 - 2y_1 + y_0\right]$$

$$= \frac{h}{3}\left[6y_0 + 6y_1 - 6y_0 + y_2 - 2y_1 + y_0\right]$$

$$I_1 = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly,

$$I_2 = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$$I_3 = \frac{h}{3}\left[y_4 + 4y_5 + y_6\right]$$

$\vdots$

$$I_{n/2} = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right].$$

Combining all these expression,

$$I = I_1 + I_2 + \cdots + I_{n/2}$$

$$= \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + y_{n-2})\right]$$

$\hookrightarrow$ formula

Example 9.4 Au-22

Find the value $\int_0^{0.6} e^x \, dx$; taking $n=6$ correct to five significant figures by Simpson's one-third rule.

Solve: We have, $f(x) = e^x$

$a = 0$, $b = 0.6$, $n = 6$

$$h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	1.00000	1.10517	1.22140	1.34986	1.49182	1.64872	1.82212
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

The Simpson's rule is,

$$I = \frac{h}{3}\left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{0.1}{3}\left[1 + 1.82212 + 4(1.10517 + 1.34986 + 1.64872) + 2(1.22140 + 1.49182)\right]$$

$$= \frac{0.1}{3}\left[2.82212 + 4(4.10375) + 2(2.71322)\right]$$

$$I = 0.82212$$

Simpson's 3/8 Rule

$$I_1 = \int_{x_0}^{x_0+3h} f(x)\,dx$$

$$= h\left[3y_0 + \frac{9}{2}\Delta y_0 + \left(\frac{27}{3} - \frac{9}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{6}\right]$$

$$= h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \frac{3}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

$$I_2 = \int_{x_0+3h}^{x_0+6h} f(x)\,dx$$

$$= \frac{3}{8}h\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

$$I_3 = \frac{3}{8}h\left(y_6 + 3y_7 + 3y_8 + y_9\right)$$

$$I_{n/3} = \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3}{8}h\left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)$$

Combining all these expressions,

$$I = I_1 + I_2 + I_3 + \cdots + I_{n/2}$$

$$= \frac{3h}{8}\Big[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \cdots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\Big]$$

$$= \frac{3h}{8}\Big[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})\Big]$$

SP-24

Evalute $\int_0^{\pi/2} e^{\sin x}\, dx$, correct to four significant figures by Simpson's three-eight rule.

Solve:

Here,

$x_0 = 0,\ x_1 = \frac{\pi}{6},\ x_2 = \frac{\pi}{3},\ x_3 = \frac{\pi}{2}$

$h = \frac{\pi}{6}$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = e^{\sin x}$	1	1.64872	2.36320	2.71828
	y_0	y_1	y_2	y_3

We know,

$$I = \frac{3h}{8}\left[(y_0 + y_3) + 3(y_1 + y_2)\right]$$

$$= \frac{3 \times \frac{\pi}{6}}{8}\left[(1 + 2.71828 + 3(1.64872 + 2.36320))\right]$$

$$= \frac{\pi}{16}\left[(3.71828 + 12.03576)\right]$$

$=$

Example - 9.6

A tank in discharging water through an orfice at a depth of x meter below the surface of the water whose area is $A m^2$. The following are the values of x for the corresponding values of A.

A	1.257	1.39	1.52	1.65	1.809	1.962	2.123	2.295	2.462	2.650	2.827
x	1.50	1.65	1.80	1.95	2.10	2.25	2.40	2.55	2.70	2.85	3.00

Using the formula (0.018) $T = \int_{1.5}^{3.0} \frac{A}{\sqrt{x}}\, dx$, calculate T the time in seconds ~~for~~ the level of the water to drop from 3.0 m to 1.5 m above the orfice.

Solve:

$h = 0.15$

$f(x) = \frac{A}{\sqrt{x}}$

x	1.50	1.65	1.80	1.95	2.10	2.25	2.40	2.55	2.70	2.85	3.00
$y = \frac{A}{\sqrt{x}}$	1.025	1.082	1.132	1.182	1.249	1.308	1.375	1.498	1.598	1.572	1.632
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

Using Simpson's rule,

$$I = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{0.15}{3}\left[(1.025 + 1.632) + 4(1.081 + 1.182 + 1.308 + 1.438 + 1.571) + 2(1.132 + 1.249 + 1.375 + 1.498)\right]$$

$$= 1.9743$$

Using this formula

$$(0.018)\,T = \int_{1.5}^{3} \frac{A}{\sqrt{x}}\,dx$$

$$(0.018)\,T = 1.9743$$

$$T = \frac{1.9743}{0.018}$$

$$= 110 \text{ sec}$$

$$\therefore T = 110 \text{ sec}$$