Seg-8

Numerical Solution of ordinary differential eqⁿ

Differential Eqⁿ ⇒ Any eqⁿ that contains an unknown function & its derivative(s) is called DE.

Ordinary dE: ⇒ A DE that involves a single independent variable is called an ordinary DE.

Ex: $\frac{dy}{dx} + 3y = 2$

Partial DE: A DE that involve partial differential co-efficient with respect to more than one independent variable is called partial DE.

Ex: $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz$.

Au-22 (5a)

What do you mean by the order and the degree of a differential eqⁿ, How many initial conditions do we require to solve a 4th order differential eqⁿ?

⇒ The order of DE is the highest derivative that appears in the eqⁿ.

$\frac{dy}{dx} + 3y = 2$ [1st order] [1st degree]

$\left(\frac{dy}{dx}\right)^2 + 2y^2 = 4\left(\frac{dy}{dx}\right) + 2x$ [2nd order] [2nd degree]

The degree of DE is the power (or degree) of the highest order derivative.

We require 4 initial conditions (n=4) to solve 4th order dE.

Linear DE: A DE is called linear if it does not include terms where the dependent variable or its derivatives are multiplied together.

Ex: $y'' + 3y' = 2y + x^2$

This is a 2nd order - linear DE because there are no products of y or its derivatives.

Non Linear DE:

1) $y'' + (y')^2 = 1$, non linear because it includes a product involving y'.

2) $y' = -ay^2$, it includes a product involving y.

Initial Value: When all the conditions are specified at a particular value of the independent variable x, then the problem is called an initial value problem

Boundary Value Problem: If the initial conditions are specified at different values of the independent ~~value~~ variable, then it is called boundary value problem.

General & Exact Solution: A solution with arbitrary constants is called the general solution because it is not unique. When the values of these constants are known and substituted into the general solution, we get a specific solution, known as exact solution.

Taylor Series

*Describe Taylor's series method for the Numerical solution of ordinary DE.

⇒ Taylor's series methods are use to solve initial value problems (IVPs), in ordinary DEs. To solve an IVP, we first need to find the Taylor series expansion of the solution function around the initial point.

Let $y = f(x)$, be a solution of the eqn

$$\frac{dy}{dx} = f(x, y) \quad \text{where } y(x_0) = y_0$$

Expanding the Taylor's series about the point x_0, we get

$$f(x) = f(x_0) + \frac{x - x_0}{1!} f'(x_0) + \frac{x - x_0}{2!} f''(x_0) + \cdots$$

$$y = f(x) = y_0 + \frac{x - x_0}{1!} y_0' + \frac{x - x_0}{2!} y_0'' + \cdots$$

Putting $x = x_1 = x_0 + h$, we get

$$\boxed{y_1 = f(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0'''} \cdots$$

where $h = x_1 - x_0$,

Similarly, $y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$

The above formula is known as Taylor's Series method.

Ex-10.1

Solve $\frac{dy}{dx} = x+y$, $y(1)=0$. numerically up to $x=1.2$ with $h=0.1$

$[h = x_{p} - x_0]$

$y(x_0) = y_0 \quad y(1)=0$

$\Rightarrow x_0 = 1,\ y_0 = 0$

$\frac{dy}{dx} = y_0' = x + y = x_0 + y_0$

$= 1 + 0 = 1$

$y_0'' = 1 + y_0' = 1+1 = 2$

$y_0''' = 0 + y_0'' = 2$

$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0'''$

$= 0 + 0.1 \times 1 + \frac{(0.1)^2}{\cancel{2}} \times \cancel{2} + \frac{(0.1)^3}{\cancel{6}_3} \times \cancel{2}$

$= 0.11$

$x_1 = x_0 + h$

$= 1 + 0.1 = 1.1$

$\therefore x_1 = 1.1,\ y_1 = 0.11$

$y_1' = x_1 + y_1 = 1.1 + 0.11 = 1.21$

$y_1'' = 1 + y_1' = 1 + 1.21 = 2.21$

$y_1''' = 0 + y_1'' = 2.21$

$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1'''$

$= 0.11 + (0.1 \times 1.21) + \frac{(0.1)^2}{2} \times 2.21 + \frac{(0.1)^3}{6} \times 2.21$

$= 0.2314 + 0.0111$

$= 0.232$

$\therefore x_2 = x_1 + h = 1.1 + 0.1$

$= 1.2$

$\therefore y(1.2) = 0.232$ (A)

10.2

$\frac{dy}{dx} = 1 + xy$ .... $y=1, x=0$ Compute $y(0.1)$ correct to 4 places of decimal using T.S. Method.

$\Rightarrow$ $y(x_0) = y_0$, $x_0 = 0$, $y_0 = 1$

$\frac{d^2y}{dx^2} = 0 + \left[x\frac{\partial y}{\partial x} + y\frac{\partial y}{\partial x}\right]$

$= x \cdot \frac{\partial y}{\partial x} + y$

$= 0 + 1 = 1$

$y_0' = 1 + xy$

$= 1 + 0 \times 1 = 1$

$\frac{d^2y}{dx^2} = y'' = 0 + \left(x\frac{dy}{dx} + y\frac{dy}{dx}\right)$

$= x\frac{dy}{dx} + y = 0 \times \frac{dy}{dx} + 1 = 1$.

$\frac{d^3y}{dx^3} = y''' = x \cdot \frac{dy}{dx} + y$

$= x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{dx}{dx} + \frac{dy}{dx}$

$= x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}$

$= 0 + 2 \times 1 = 2$

$\frac{d^4y}{dx^4} = y'''' = x\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} = 0 + 3 = 3$.

$\therefore y_1 = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0''''$

$= 1 + (0.1)1 + \frac{(0.1)^2}{2} \times 1 + \frac{(0.1)^3}{3!}2 + \frac{(0.1)^4}{4!}3$

$= 1.1053425$

$\therefore y(0.1) = 1.1053425$ (Ans)

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Euler's Method

Describe Euler's Method for the Numerical solution of ordinary DE.

⇒ It is a numerical method for approximating the solutions of Ordinary DE.

Consider the taylor's series method,

$$y(x) = y(x_0) + \frac{x-x_0}{1!} y'(x_0) + \frac{x-x_0}{2!} y''(x_0) + \cdots + \frac{(x-x_0)}{n!} y^n(x_0)$$

Taking the first two terms,

$$y(x) = y(x_0) + (x-x_0) y'(x_0)$$

Given the differential eqn,

$$y'(x_0) = f(x,y) \quad \text{with} \quad y(x_0) = y_0$$

We have, $y'(x_0) = f(x_0, y_0)$

$$\therefore y(x) = y(x_0) + (x-x_0) f(x_0, y_0)$$

Then the value of $y(x)$ at $x = x_1$ is given by

$$y(x_1) = y(x_0) + (x_1 - x_0) f(x_0, y_0)$$

Letting $h = x_1 - x_0$, we obtain,

$$y_1 = y_0 + h f(x_0, y_0)$$

In general form,

$$y_{i+1} = y_i + h f(x_i, y_i)$$

This is the Euler's method and can be used recursively to evaluate $y_1, y_2 \ldots$ of $y(x_1), y(x_2) \ldots$ starting from the initial condition $y_0 = y(x_0)$

Q. Solve $\frac{dy}{dx} = 1 - y$, with the initial conditions $x=0, y=0$ using Euler's method and tabulated the solutions at $x = 0.1, 0.2, 0.3$.

Here, $f(x,y) = 1-y$

$h = 0.1, \quad x_0 = 0, \quad y_0 = 0$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$x_2 = 0.2$

$x_3 = 0.3$.

Formula:

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$\therefore \quad y_1 = y_0 + 0.1 f(x_0, y_0)$$

$$= 0 + 0.1 \times (1 - 0)$$

$$= 0.1$$

$$y_2 = y_1 + (0.1) f(x_1, y_1)$$

$$= 0.1 + (0.1)(1 - 0.1)$$

$$= 0.19$$

$$y_3 = y_2 + (0.1) f(x_2, y_2)$$

$$= 0.19 + (0.1)(1 - 0.19)$$

$$= 0.271$$

10.5 $\frac{dy}{dx} = x^3 + y$, $y(0) = 1$, compute $y(0.02)$ ..., $h = 0.01$

$$\frac{dy}{dx} = x^3 + y.$$

We have,

$$f(x,y) = x^3 + y$$

$$x_0 = 0, \quad y_0 = 1, \quad h = 0.01$$

$$\therefore x_1 = x_0 + h = 0 + 0.01 = 0.01$$

$$x_2 = x_0 + 2h = 0 + 2 \times 0.01 = 0.02$$

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$\therefore y_1 = y_0 + (0.01)(x_0^3 + y_0)$$

$$= 1 + (0.01)(0 + 1)$$

$$= 1.01$$

$$y_2 = y_1 + h\, f(x_1^3 + y_1)$$

$$= 1.01 + (0.01)\{(0.01)^3 + 1.01\}$$

$$= 1.02$$

$$\therefore y(0.02) = 1.0201$$

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Heun's Method

Describe Heun's method:

In Euler's method, the slope at the beginning of the interval is used to extrapolate y_i to y_{i+1} over the entire interval. Thus,

$$y_{i+1} = y_i + m_1 h. \quad [\text{where } m_1 \text{ is the slope at } (x_i, y_i)]$$

$$y_{i+1} = y_i + m_2 h \quad [m_2 \text{ is the slope at } (x_{i+1}, y(x_{i+1}))]$$

A third approach is to use a line whose slope is the avg of the slopes at the end points of the interval. Then.

$$y_{i+1} = y_i + \frac{m_1 + m_2}{2} h$$

This approach is known as Heun's Method.

Now,

Given the eq^n,

$$y'(x) = f(x, y)$$

$$m_1 = y'(x_i) = f(x_i, y_i)$$

$$m_2 = y'(x_{i+1}) = f(x_{i+1}, y_{i+1})$$

Therefore, $m = \dfrac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2}$.

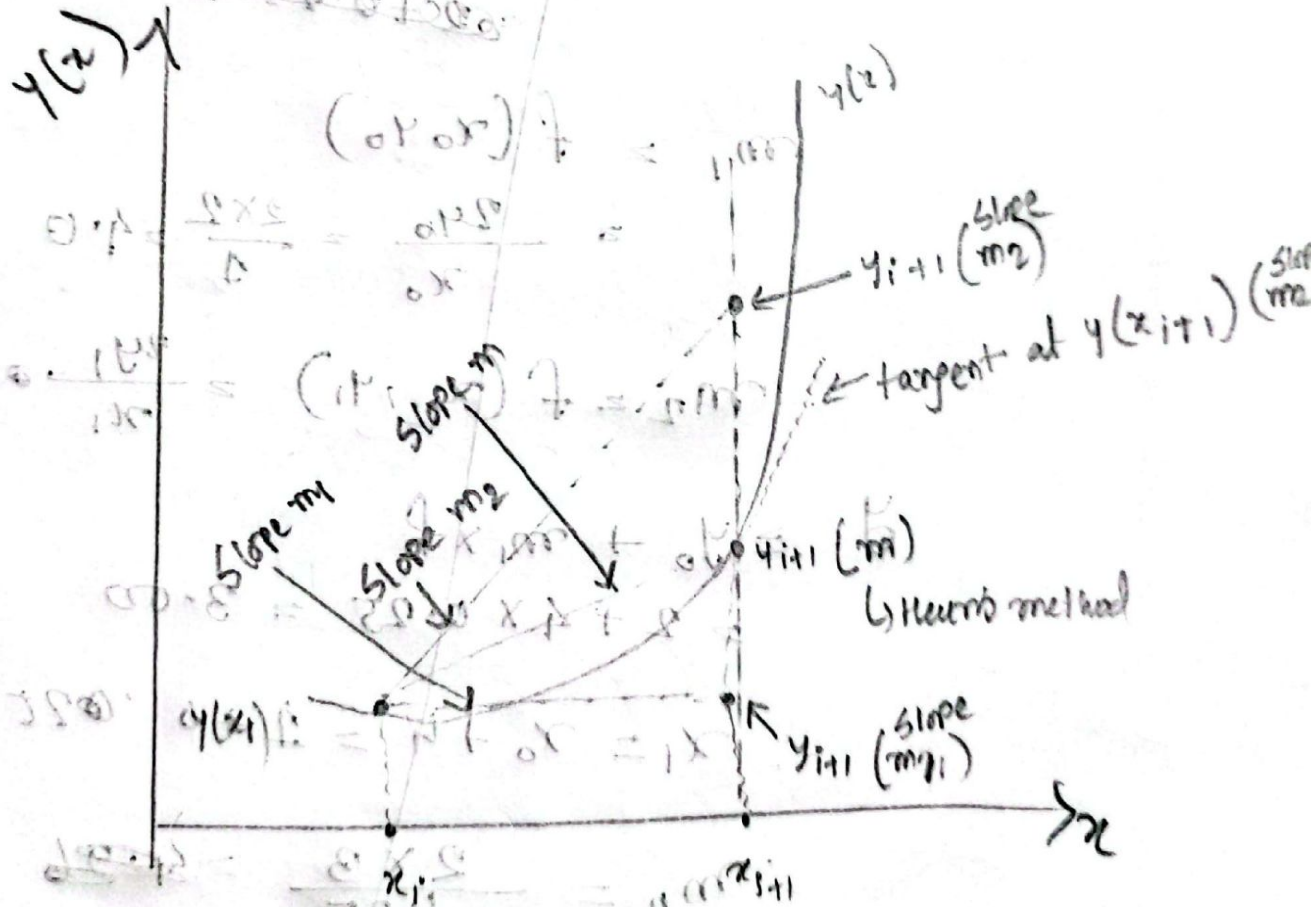
$$\boxed{\therefore y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right]} \quad \text{---(i)}$$

Now,

$$y_{i+1} = y_i + h f(x_i; y_i) \rightarrow \text{by Euler's formula}$$

$\therefore$ The Heun's formula becomes,

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1} + y_i + h f(x_i; y_i))\right]$$



Balagurusamy 10.6

$y'(x) = -\frac{2y}{x}$ with $y(1) = 2$, $h = 0.25$

estimate $y(2)$. using Heun's method.

$y' = f(x,y) = \frac{2y}{x}$

$x_0 = 1,\ y_0 = 2,\ h = 0.25$

1st iteration $m_1 = f(x_0, y_0) = \frac{2y_0}{x_0} = \frac{2\times 2}{1} = 4.0$

$y_e(1.25) = y_0 + h \times m_1 = 2 + 0.25\times 4 = 3.0$

$\Rightarrow x_1 = x_0 + h = 1 + 0.25 = 1.25$

$m_2 = f(x_1, y_1) = \frac{2y_1}{x_1} = \frac{2\times 3}{1.25} = 4.8$

$\therefore y_1(1.25) = 2 + \frac{0.25}{2}(4.0 + 4.8) = 3.1$

2nd

$m_1 = \frac{2\times 3.1}{1.25} = \cancel{6}\ 4.96$

$y_e(1.5) = 3.1 + 0.25\times 4.96 = 4.34$

$x_2 = x_1 + h = 1.25 + 0.25 = 1.5$

$m_2 = \frac{2\times 4.34}{1.5} = 4.8$

$\therefore y_2(1.5) = 3.1 + \frac{4.96 + 4.8}{2}\times 0.25 = 4.44$

3rd

$m_1 = \frac{\cancel{4.96}\ 2\times 4.44}{1.5} = 5.92$

$\therefore y_e(1.75) = 4.44 + 0.25\times 5.92 = 5.92$

$x_3 = x_2 + h = 1.5 + 0.25$, $m_2 = \frac{2\times 5.92}{1.75} = 6.77$

$= 1.75$

$$\therefore y_3(1.75) = 4.44 + \frac{5.92+6.77}{2} \times 0.25$$
$$= 6.03$$

4th

$$m_1 = \frac{2\times 6.03}{1.75} = 6.89$$

$$\therefore y_e(2) = 6.03 + 0.25 \times 6.89 = 7.75$$

$$x_3 = x_2 + h = 1.75 + 0.25 = 2$$

$$\therefore m_2 = \frac{2\times 7.75}{2} = 7.75$$

$$\therefore y(2.0) = 6.03 + \frac{0.25}{2}(6.89 + 7.75)$$
$$= 7.86$$

(4)

Fourth order Runge Kutta

$m_1 = f(x_i, y_i)$

$m_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_1 h}{2}\right)$

$m_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{m_2 h}{2}\right)$

$m_4 = f\left(x_i + h, y_i + m_3 h\right)$

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$$\boxed{y_{i+1} = y_i + \left(\frac{m_1 + 2m_2 + 3m_3 + 4m_4}{6}\right) h}$$

To estimate $y(0.4)$ when.

\# $y'(x) = x^2 + y^2$ with $y(0) = 0$, $h = 0.2$

$\Rightarrow$ $x_0 = 0$, $y_0 = 0$, $h = 0.2$. $\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$

$m_1 = f(x_0, y_0) = x^2 + y^2 = 0$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right) = f(0.1, 0) = (0.1)^2 = 0.01$

$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}\right) = f(0.1, 0.001) = (0.1)^2 + (0.001)^2 = 0.01$

$m_4 = f(x_0 + h, y_0 + m_3 h) = f(0.2, 0.002) = (0.2)^2 + (0.002)^2 = 0.04$

$$\therefore y(0.2) = 0 + \frac{0 + 2\times 0.01 + 3\times 0.01 + 0.04}{6} \times 0.2$$

$$= 0.003$$

Iteration-2

$f(x^2+y^2)$

$x_1 = 0.2$, $x_2 = x_1 + h$ $\quad h = 0.2$

$y_1 = 0.003$ $\quad = 0.2 + 0.2$

$= 0.4$

$$\therefore m_1 = f(x_i, y_i) = (0.2)^2 + (0.003)^2 = 0.04$$

$$m_2 = f\left(x_i + \frac{h}{2}, y_1 + \frac{m_1 h}{2}\right)$$

$$= f\left(\left(0.2 + \frac{0.2}{2}\right), \left(0.003 + \frac{0.04 \times 0.2}{2}\right)\right)$$

$$= f(0.3, 0.007)$$

$$= (0.3)^2 + (0.007)^2 = 0.09$$

$$m_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_2 h}{2}\right)$$

$$= f\left(0.3, 0.003 + \frac{0.09 \times 0.2}{2}\right)$$

$$= f(0.3, 0.012) = (0.3)^2 + (0.012)^2 = 0.09$$

$$m_4 = f(x_1 + h, y_1 + m_3 h)$$

$$= f(0.2 + 0.2, 0.003 + 0.09 \times 0.2)$$

$= f(0.4, 0.0213)$

$= (0.4)^2 + (0.0213)^2 = 0.16$

$\therefore y(0.4) = 0.003 + \frac{0.04 + 2\times 0.09 + 3\times 0.09 + 0.16}{6} \times 0.2$

$= 0.024767$

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$y(0.2)$, $\frac{dy}{dx} = x + y$, $y(0) = 1$, $h = 0.1$

1st Iteration

$x_0 = 0$, $y_0 = 1$, $h = 0.1$ $\qquad x_1 = x_0 + h = 0 + 0.1 = 0.1$

$m_1 = f(x_0, y_0) = f(0, 1) = 0 + 1 = 1$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right) = f\left(\frac{0.1}{2}, 1 + \frac{1\times 0.1}{2}\right)$

$= f(0.05, 1.05) = (0.05) + (1.05) = 1.1$

$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}\right) = f\left(0.05, 1 + \frac{1.1 \times 0.1}{2}\right)$

$= f(0.05, 1.055) = (0.05) + (1.05) = 1.1$

$m_4 = f(x_0 + h, y_0 + m_3 h) = f(0.1, 1 + 1.1 \times 0.1)$

$= f(0.1, 1.11) = (0.1) + (1.11) = 1.2$

$\therefore y_1(0.1) = 1 + \frac{1 + 2\times 1.1 + 3\times 1.1 + 1.2}{6} \times 0.1$

$= 1.13$

nd Iteration

$x+y$

$x_1 = 0.1 \qquad h = 0.1$

$y_1 = 1.13, \qquad \therefore x_2 = x_1 + h = 0.2.$

$m_1 = f(x_1, y_1) = 0.1 + 1.13 = 1.23$

$m_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_1 h}{2}\right)$

$= f\left(0.1 + \frac{0.2}{2}, 1.13 + \frac{1.23 \times 0.2}{2}\right)$

$= f(0.15, 1.19) = 1.34$

$m_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_2 h}{2}\right)$

$= f\left(0.1 + \frac{0.1}{2}, 1.13 + \frac{1.34 \times 0.1}{2}\right)$

$= f(0.15, 0.197) = 1.35$

$m_4 = f(x_1 + h, y_1 + m_3 h)$

$= f(0.1 + 0.1, 1.13 + 1.035 \times 0.1)$

$= f(0.2, 1.265) = 1.47$

$\therefore y(0.2) = 1.13 + \frac{1.23 + 2 \times 1.34 + 3 \times 1.35 + 1.47}{6} \times 0.1$

$= 1.2872$

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$\frac{dy}{dx} = x^2 + 2y^2$, interval $0 < x \le 0.4$, $h = 0.2$
$y(0) = 1$

1st Iteration $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$

$m_1 = f(x_0, y_0) = f(0,1) = 0^2 + 2(1)^2 = 2$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right) = f(0.1, 1.2) = 2.89$

$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_2 h}{2}\right) = f(0.1, 1.289) = 3.33$.

$m_4 = f(x_0 + h, y_0 + m_3 h) = f(0.2, 1.666) = 0.59$

$\therefore y(0.2) = 1.6$

2nd

$x_1 = 0.2$, $y = 1.6$, $h = 0.2$

$\therefore x_2 = x_1 + h = 0.4$

$m_1 = f(x_1, y_1) = f(0.2, 1.6) = 5.16$

$m_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_1 h}{2}\right) = f(0.3, 2.116) = 9.045$

$m_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_2 h}{2}\right) = f(0.3, 2.505) = 12.64$

$m_4 = f(x_p + h, y_p + m_3 h) = f(0.4, 4.128) = 34.24$

$\therefore y(0.4) = 4.8$. (an)

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$\frac{dy}{dx} = y - x$, where $y(0) = 2$, find $y(0.1)$ and $y(0.2)$ by Euler's up-to two decimal places.

$x_0 = 0$, $y_0 = 2$, $h = 0.1$,

$x_1 = 0.1$, $x_2 = 0.2$.

$y_1 = y_0 + hf(x_0, y_0)$

$= 2 + 0.1\, f(0, 2)$

$= 2 + 0.1 \times (2 - 0) = 2.2$

$y_2 = y_1 + hf(x_1, y_1)$

$= 2.2 + 0.1 \times f(0.1, 2.2)$

$= 2.2 + 0.1 \times (2.2 - 0.1)$.

$= 2.41$

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Advantages and limitations of

(1) Taylor:

Ad
1) Provides precise solutions
2) Applicable to a wide range of function and equations.

dis:
1) It has computational complexity
2) Convergence issues

(ii) Euler:

Ad
1) Easy to implement
2) Low computational cost

Dis:
1) Limited Accuracy
2) Global error accumulation.

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Major Problem of taylor series and Heun's Method.

⇒ **Taylor**
1) Computational Complexity.
2) Convergence issues.

Heun's
1) Intermediate Computation Requirements
2) Limited Stability.