Homotopy Group of Spheres, Hopf Fibrations & Villarceau Circles

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May 18, 2022

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Unlike geometry, spheres in topology have been seen as topological invariants, where their structures are defined as topological spaces. Forgetting, the exact notion of geometry, & the impossibility of embedding one into other, the homotopy relates how one sphere of i dimensions can wrap another sphere of n dimensions. Here, depending on the pattern, the relation can be of three types, $i = n$, $i <$ n or $i > n$. Each of them has their affine properties & uniqueness that defines homotopy in the mathematical field of algebraic topology. The most important part of homotopy is the Hopf fibrations where $i > n$ & there a special type of mapping and stereographic projection takes place which can be justified by the relation $S^1 \to S^3 \to S^2$. S^1 is a 1-sphere or a circle which when which exists in the form of points inside the 2-sphere, and the mapping, that transforms, the 3-sphere to the 2-sphere, where each point of 2-sphere acts as a circle in 3-sphere, generates in turn the third homotopy group of the 2-sphere that is,

$$\pi^3(S^2) = \mathbb{Z}, \qquad where\ \mathbb{Z} \in \mathbb{R}$$

If we assume that the stereographic projections that is made by the transform mapping $S^1 \to S^3 \to S^2$ where the third homotopy groups fiber is a 3-dimensional torus of surface area $2\pi R \times 2\pi r$ then along with the 2-circles, the major R and minor r there exists also a pair of circles produced by cutting the torus analytically at a certain angle produces a pair of circles called Villarceau circles where they meet all the latitudinal and longitudinal cross sections of the torus at a point of the minor radius being the locus of the torus where the other 3-circles intersected and passed through.

The n dimensional sphere is known as S^n which is defined geometrically as a set of points $n + 1$ in the Euclidean space E^n with a distance 'unit' from the origin. The $\pi^i(S^n)$ or the homotopy group of degree i denotes a continuous mapping from a S^i sphere to a S^n sphere. What exactly is preserved in this mapping are the equivalence class, a class where the Abelian groups are attached to topological spaces, that in turn generates an 'addition' operation over this classes.

There exists 3 regimes depending upon the fact whether $i = n$, $i <$ n or $i > n$ that I will summarize below;

[1] The trivial homotopy group exists in the mapping of $0 < i <$ n where the mapping maps all of S^i to single points of S^n which can also be termed as continuously deformable in terms of the mapped surface.

[2] For, $i = n$ there exists a degree by which it can be determined that, how many times a sphere is wrapped around itself in the form of the mapping. To ease out, this type of mapping is non-trivial & it can be denoted as an integer $\pi^n(S^n) = \mathbb{z},\ where\ \mathbb{z} \in \mathbb{R}$ where a continuous mapping of every point on the first circle is mapped to the second circle, where, the point on the first circle rotates around it, then the point on the second circle has already moved several times depends upon the degree of the mapping.

[3] The third mapping is the most important mapping where $i > n$ and can be treated as a Hopf fibrations where the mapping occurs in a non-trivial fashions, depending on these dimensions by Adams's theorem as;

$$S^0 \to S^1 \to S^1$$
$$S^1 \to S^3 \to S^2$$
$$S^3 \to S^7 \to S^4$$
$$S^7 \to S^{15} \to S^8$$

Hopf fibration is also known as Hopf bundle or Hopf map which is actually, a fiber bundle constructed over a topological spaces where E, B, F be the elements, the fiber bundle is then denoted by;

$$F \to E \to B$$

Where $p: E \to B$ satisfies the following results;

- If we say E is the total space, where B is the base, F is the fibre then E is the fibration over B with fibre F.
- $p^{-1}(b) \cong F$
- $\forall b \in B\ \exists$ there is a neighborhood U of b, in a way $p^{-1}(U)$ is homeomorphism (surjective, injective, hence bijective) to $U \times F$ via $\delta: U \times F \to p^{-1}(U)$
- We have $p \circ \delta = \pi$ where $\pi: U \times F \to (U)$ is the projection of $U \times F$ into U.

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Since in the long exact sequence $\dots \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \dots$ the group homomorphism $\pi_i(S^1) \to \pi_i(S^3)$ the mapping collapsed as there is only a deformed point from $S^1 \to S^3$ breaks into short exact sequences $0 \to \pi_i(S^3) \to \pi_i(S^2) \to \pi_{i-1}(S^1) \to 0$ which can give to split isomorphisms as;

$$\pi_i(S^2) = \pi_i(S^3) \bigoplus \pi_{i-1}(S^1)$$

As $\pi_{i-1}(S^1)$ vanishes for i in atleast 3 the element $\pi_i(S^2)$ and $\pi_i(S^3)$ are totally isomorphic when i is atleast 3. Thus split by suspension homomorphism occurs.

What has been developed as a fundamental technique and uniquely focused area of research is the computation of positive k for the homotopy group $\pi_{n+k}(S^n)$ where it is independent of n for $n \geq k+2$ & is known as the stable homotopy group of spheres and has been computed upto the maximum value of k as 64. On an existing ring R these groups forms the stable coefficients of extraordinary cohomotopy theory. However, the unstable value has been computed and quite erratic upto the value of $k < 20$. Adams spectral sequence is an important technique for computing stable homotopy groups, while most modern computations came from Jean-Pierre Serre.

To describe the $n-sphere$, it is notable to mention some of its properties for the sake of homotopy, note that, it is a sphere, not a solid ball.

- Any 2-sphere occupies the E^3 space and has the equation form $x_0^2 + x_1^2 + x_3^2 = 1$ with a distance from equal point from the centre as implicit sphere. However, this can be generalizes to spheres of $n - dimensions$ in the equation forms $x_0^2 + x_1^2 + x_3^2 \dots\dots\dots\dots + x_n^2 = 1$.
- Any Disc is a region contained within the circle by the inequality relation $x_0^2 + x_1^2 \leq 1$ which produces the rim (or the circumference) of the circle S^1 having the equation form $x_0^2 + x_1^2 = 1$. Now, think of a balloon, inflated with air, just punctured it and you will get a disc, close the puncture with a drawstring and the repair that you will get is D^2/S^1 generalizes to D^n/S^{n-1} producing S^n. This is called disc with collapsed rim. Take the points lying on the boundary of the Disc and glued it above the rim D^1 in the Northern Hemisphere, while doing the same in the Southern Hemisphere then what will be produced is a $CW\ Complex$.
- The last one is the most important and known as the suspension of the equator. If we take all the points on the equatorial plane and extends it upwards and downwards in NH & SH then, what we get is 2-Hemispheres and for each positive integer n the topological space can be written as $\sum S^n$ with the $n - sphere$ $x_0^2 + x_1^2 + x_3^2 \dots\dots\dots\dots + x_n^2 = 1$ has the equator $x_0^2 + x_1^2 + x_3^2 \dots\dots\dots\dots + x_{n-1}^2 = 1$ with the suspension $\sum S^{n-1}$ produced on S^n.

In terms of open sets, the essence of homotopy group, is that, it preserves continuity relations. One is topologically mapped to other. The first of the fundamental (homotopy) group, is a path connected topological space that is $\pi_1(X)$ where mapping is done from a pointed circle (S^1, s) to the pointed space (X, x) where the map is being bijected from s into x. These maps established under equivalence classes keeping the base point x fixed, where a continuous map has been made called as null homotopic $S^1 \mapsto x$. These classes of maps becomes an 'equator pinch' where one maps the equator in the form of a pointed sphere (here circle) to a point whose both sides are the upper and lower spheres making it look like a 'bouquet of spheres, where the upper and lowe sphere's pointed equator in the middle makes a pinch and completes the map which formulizes as;

$$\bigvee_{i \in I} X_i = \coprod_{i \in I} X_i / \sim$$

This characterizes a wedge sum, which is 'a one point union' between two pointed topological spaces X & Y giving the quotient space of the disjoint union of X & Y by identifying $x_0 \sim y_0 : X \vee Y = (X \coprod Y)/\sim$ where $\sim$ is the equivalence closure relation, with the closure relation $\{(p_i, p_j) : i, j \in I\}$ where its associative and commutative upto homeomorphism.

The pointed $i - sphere$ associated with the i^{th} homotopy group with $\pi_i(X)$ with (S^i, s) is trivial, abelian, finitely generated for the null homotopic classes and for X equal to S^n (for all positive n). A continuous map induces a bijection (or homeomorphism with associated holonomy) with their i^{th} homotopy group isomorphic for all i.

Some associated mappings are:

[1] $\pi_1(S^1) = Z$ where a circle has been wrapped around another circle yielding an integer that can be treated as the winding number of the loop & generates an infinite cyclic loop as windings if not done in the opposite direction can be repeated infinitely.

[2] $\pi_1(S^1) = Z$ where a 2-sphere (3-dimensional sphere) is warped around a 2-sphere.

[3] $\pi_1(S^2) = 0$ where the warping of a circle to a sphere ultimately lasso out and end up in a point as the base dimension or the mapping is done from S^1 to S^2.

[4] $\pi_3(S^2) = Z$ where this mapping is interesting and known as Hopf fibrations where the projection takes place on $S^1 \to S^3 \to S^2$. Lets discuss this in detail below.

Take a complex vector space C^2 & identify its columns $z = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}$ where $z_i \in C$ where the complex projective line $P(C^2) = CP_1$ obtaining from $C^2 \setminus \{0\}$ by factoring out the equivalence relation $z \sim w$ iff $w = \gamma z$ for some $\gamma \in C^{\times} = C \setminus \{0\}$. As this equivalence class is bijective corresponding to 1-dimensional subspaces, therefore, this C^2 can be the projective space, if one restricts to 3-sphere $S^3 \subset R^4 = C^2$ providing $z^{\dagger} z = 1$ and factor out $z \sim w$ iff $w = \gamma z$ for some $S^1 \cong U(1) = \{\gamma \in C, |\gamma| = 1\}$, then using the stereographic maps from CP^1 to the 2-sphere S^2, the complex coordinates are;

$$\alpha \coloneqq \frac{z_2}{z_1} \longmapsto \left(\Re \frac{2\alpha}{1 + |\alpha|^2}, \mathcal{J} \frac{2\alpha}{1 + |\alpha|^2}, \frac{1 - |\alpha|^2}{1 + |\alpha|^2} \right)$$

Wherever $z_1 \neq 0$,

$$\beta \coloneqq \frac{z_2}{z_1} \longmapsto \left(\Re \frac{2\beta}{1 + |\beta|^2}, \mathcal{J} \frac{2\beta}{1 + |\beta|^2}, \frac{1 - |\beta|^2}{1 + |\beta|^2} \right)$$

Wherever $z_1 \neq 0$, a smooth map $\pi: S^3 \to S^2 \subset R^3$ can be expressed as;

$$z \longmapsto (2\Re \bar{z}_1 z_2, 2\mathcal{J} \bar{z}_1 z_2, |\alpha|^2 - |\beta|^2) =: R(z)$$

In CP^1, α, β are the inhomogeneous complex coordinates around $0, \infty$. This is the Hopf map where π is the projection in order of S^3 the principle fiber bundle over the base space $S^2 \cong CP^1$ with the structure group $U(1)$.

"The linkage can be visualized by the stereographic projection of a 3-sphere to its equatorial 3-plane, one obtains one family of Villarceau circles on each torus of a system of nested coaxial coaxial concentric tori orthogonal to the unit 2-sphere in that 3 plane. The linking of the fibres used by Hopf to obtain the homotopy group $\pi^3(S^2) = \mathbb{Z}$"

Alternatively, using Geometric rotations (Quaternions) with the rotation group $SO(3)$ & Spin Group $Spin(3)$ diffeomorphic to the 3-sphere, the spin group acts on S^2 transitively by rotations which makes the Hopf fibration as the 3-sphere, the principle fiber bundle over the 2-sphere. Let's take the quaternionic coordinates $q = \omega + ix + jy + kz$ where $i^2 = j^2 = k^2 = ijk = 1$ are the three complex numbers associated with the coordinates x, y, z. The rotation can be given by;

$$\begin{bmatrix} 1 - 2(y^2 - z^2) & 2(xy - \omega z) & 2(xz - \omega y) \\ 2(xy + \omega z) & 1 + 2(x^2 + z^2) & 2(yz - \omega z) \\ 2(xz - \omega y) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{bmatrix}$$

The above formulae for the projection bundle makes the fixed unit vector along the z axis $(0,0,1)$ rotates to another unit vector given as;

$$2(xz + \omega y), 2(yz - \omega x), 1 - 2(x\text{^}2 + y\text{^}2)$$

Which acts as a continuous function of the coordinates ω, x, y, z which is the image of q on the 2-sphere where the unit vector has been transformed via the z axis, containing the fiber for a given point on S^2. Taking three fixed points on S^2 as a, b, c, the q rotates angle ϑ via q_{ϑ} as;

$$q_{\vartheta} = \cos\vartheta + k \sin\vartheta$$

Which rotates by 2ϑ along the z axis and sweeps a great circle in the fiber S^3 where a, b, c being not the antipode, $(0,0,-1)$, the quaternion forms;

$$q_{(a,b,c)} = \frac{1}{\sqrt{2(1 + c)}} (1 + c - ib + ja)$$

As long as the base point is fixed. This sends to the antipode $(0,0,+1)$ in the composite quaternion form $q_{(a,b,c)} q_{\vartheta}$ which are points on S^3 sphere, as;

$$q_{(a,b,c)}q_{\vartheta} = \frac{1}{\sqrt{2(1+c)}}\big((1+c)\cos(\vartheta), a\sin(\vartheta) - b\cos(\vartheta), a\cos(\vartheta) + \sin(\vartheta), (1+c)\sin(\vartheta)\big)$$

Due to the multiplication of the $q_{(a,b,c)}$ space with its composite way q_{ϑ} there forms a geometric circle where the final fiber $(0,0,-1)$ can be defined $q_{(0,0,-1)}$ equal to i producing,

$$\big((0), (\cos(\vartheta)), (-\sin(\vartheta)), (0)\big)$$

Now, coming to the point of Villarceau circles, the pair circle can be produced if the torus can be cur obliquely at a special angle. Nested with other two circles, the major & the minor circles, the equations can be produced specific to that circle without the loss of generality, keeping the radius of a circle r in the xz plane with the axis of revolution at z the radius being centered at $(R, 0{,}0)$, the existence can be defined;

$$0 = (x-R)^2 + z^2 - r^2$$

Sweeping replaces x by $\sqrt{x^2+y^2}$ and removing the square root produces a quartic equation;

$$0 = (x^2+y^2+z^2+R^2-r^2)^2 - 4R^2(x^2+y^2)$$

The surface that has been swept by the xz place includes a second circle;

$$0 = (x+R)^2 + z^2 - r^2$$

Now, the circle has two common internal tangent lines, with slope at the origin found from the right angle with the hypotenuse R & opposite side r (which has the right angle at the point of tangency), then $\frac{z}{x} = \pm r/\sqrt{R^2-r^2}$ & choosing the plus sign produces a bitangent to the torus;

$$0 = xr - z\sqrt{R^2-r^2}$$

While rotating this frame, all the bitangent planes swept by;

$$0 = xr\cos\varphi + yr\sin\varphi - z\sqrt{R^2-r^2}$$

The intersection of the planes gives a pair of symmetric circles;

$$-r\sin\varphi, r\cos\varphi, 0$$

A more flexible approach has been given by a quartic approach of the torus;

$$0 = (x^2+y^2+z^2+R^2w^2-r^2w^2)^2 - 4R^2w^2(x^2+y^2)$$

And setting w to 0 gives a double point intersection equation at;

$$0 = (x^2+y^2+z^2)^2$$

Coming back to the homotopy group, the framed cobordism of manifolds can be easily expressed as an existing homotopy group $\pi_{n+k}(S^n)$ with its associated framed cobordism group $\Omega_{\mathrm{k}}^{\mathrm{framed}}\left(\mathrm{S}^{\mathrm{n+k}}\right)$ of the holomorphic submanifolds S^{n+k} where every map is homotopic to a differentiable domain map M^k in a 'framed' k-dimensional manifold S^{n+k} as;

$$M^k = f^{-1}(1{,}0{,}0{,}0, \dots\dots, 0) \subset S^{n+k}$$

Like the projection $f: S^n \to S^n$, the Hopf bundle $f: S^3 \to S^2$ can be generated by the embedding of $S^1 \subset S^3$ in a framed way as;

$$\pi_3(S^2) = \Omega_1^{\mathrm{framed}}(\mathrm{S}^3) = \mathrm{Z}$$

For, $\pi_n(S^n)$ or $\pi_{4n-1}(S^{2n})$ with $n \geq 0$ there exists an infinite cyclic group with an abelian group having the p-components torsion for all primes p. For S^n spheres, p-components torsion occurs at $k < 2p-3$ in $\pi_{n+k}(S^n)$, where for S^2 the first p-torsion occurs at $k = 2p - 3 + 1$. The torsion in odd times p for odd dimensional spheres can be given by,

$$\pi_{2m+k}(S^{2m})(p) = \pi_{2m+k-1}(S^{2m-1})(p) \bigoplus \pi_{2m+k}(S^{4m-1})(p)$$

Where p-torsion is defined as, Let M be a module over a ring R, then the value given as;

$$M_p \coloneqq \begin{cases} x \in M \\ x^{p^n} = 1 \\ p - a\ fixed\ prime \\ n \in \mathbb{N} \end{cases}$$

Is the p-torsion submodule of M.

The J-homomorphism can be defined as the image $J: \pi_K(SO(n)) \to \pi_{n+k}(S^n)$ which is a subgroup of $\pi_{n+k}(S^n)$ for $k \geq 2$ where $SO(n)$ is the special orthogonal group in dimension n which preserves the fixed point when operations are needed for composing group transformations. In the stable homotopy range, $n \geq k+2$ the homotopy group $\pi_k(SO(n))$ depends on k(mod 8)known as the Bott periodicity over a interval of 8 with the J-homomorphism given by,

- k is congruent to 0 or 1 (mod 8) if there exists a order 2 cyclic group.
- k being congruent to 2,4,5,6 (mod 8) then its trivial
- k being $4m - 1 \equiv 3$ (mod 4) if the group is cyclic to the order of the denominator $B_{2m}/_{4m}$ where $2m$ is the Bernoulli number.

Through the orthonormal group, the Bott group $O(\infty)$ are periodic to the order $\pi_n(O(\infty)) \cong \pi_{n+8}(O(\infty))$, the first 8 homotopy groups are,

$$\begin{gathered} \pi_0(O(\infty)) \cong \mathbb{Z}_2 \\ \pi_1(O(\infty)) \cong \mathbb{Z}_2 \\ \pi_2(O(\infty)) \cong 0 \\ \pi_3(O(\infty)) \cong \mathbb{Z} \\ \pi_4(O(\infty)) \cong 0 \\ \pi_5(O(\infty)) \cong 0 \\ \pi_6(O(\infty)) \cong 0 \\ \pi_7(O(\infty)) \cong \mathbb{Z} \end{gathered}$$

Given, the ring structure;

$$\pi_*^S = \bigoplus \pi_k^s \ \forall\, k \geq 0$$

Three groups can be categorized as $\begin{cases} (U) - Unitary\ grouop \\ (O) - Orthogonal\ group \\ (Sp) - Symplectic\ group \end{cases}$

The periodic Bott's results are,

$$k = 0{,}1 \begin{cases} \pi_k(U) = \pi_{k+2}(U) \\ \pi_k(O) = \pi_{k+4}(Sp) \\ \pi_k(Sp) = \pi_{k+4}(O) \end{cases}$$

The 2nd & 3rd isomorphisms intertwined to give 8-periodicity results;

$$k = 0{,}1 \begin{cases} \pi_k(O) = \pi_{k+8}(O) \\ \pi_k(Sp) = \pi_{k+4}(Sp) \end{cases}$$

	$\mathbb{S}^1$	$\mathbb{S}^2$	$\mathbb{S}^3$	$\mathbb{S}^4$	$\mathbb{S}^5$	$\mathbb{S}^6$	$\mathbb{S}^7$	$\mathbb{S}^8$	$\mathbb{S}^9$	$\mathbb{S}^{10}$
π_1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
π_2	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0
π_3	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0	0	0
π_4	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0	0	0
π_5	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0	0
π_6	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
π_7	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
π_8	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
π_9	0	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
π_{10}	0	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
π_{11}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
π_{12}	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$
π_{13}	0	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_{24}$

Homotopy groups of spheres as computed upto $\pi_{13}(S^{10})$. *Courtesy:* *Brunerie, G. (2016). On the homotopy groups of spheres in homotopy type theory.* ArXiv, abs/1606.05916.

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