LP Modeling Using Julia/JuMP

15.S60, IAP 2014

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Our Goal

- In OR-related research, chances are you'll need to solve a linear or mixed integer program at some point.
- In this class, we're going to learn how to model LPs and MIPs in Julia, solve them, and do interesting things with them.
- We're going to use a new programming language called **Julia** and a package for Julia called **JuMP** (**Julia for Mathematical Programming**).

Background of Julia and JuMP

- Julia:
  - "high-level, high-performance, open-source dynamic language for technical computing"
  - Dynamic nature allows rapid development, but Julia is also *fast*
    - Within factor of 2 of C/C++
  - See Lubin and Dunning (2013) (http://arxiv.org/abs/1312.1431) for comprehensive experiments
- JuMP:
  - Algebraic Modeling Language (AML) for Julia developed by community of OR researchers (https://github.com/JuliaOpt/JuMP.jl)

Outline

- Julia basics
- JuMP basics
- Facility location exercises
- Network revenue management exercises

References

- For further information on Julia:
  - Speed tutorial: http://learnxinyminutes.com/docs/julia/
  - Doc: http://docs.julialang.org/en/latest/manual/

- For further information on JuMP:
  - Doc: https://jump.readthedocs.org/en/release-0.2/jump.html
  - Issues: https://github.com/JuliaOpt/JuMP.jl/issues

- You may find it useful to have these open as we go through the class.

Getting started with Julia

- Open up a terminal window, and type in
  - `julia`

Basic stuff

- Simple arithmetic:
  - 3 + 7
  - 4.8 / 9.3
  - 11^5
- Comparisons:
  - 12 <= 5
  - !(5 == 8)
  - 5 != 8
  - (2014 % 4) == 0
- Strings:
  - “Peanut butter and jelly are friends”
  - “Fred has $(25 % 2) cat”

Variables

- Variables are easy:
  - x = 5
  - y = 12
  - z = x – y
  - z
  - w
- Everything has a type:
  - typeof(z)   #Int64
  - typeof(z + 1.5) #Float64
  - typeof(z >= 0)   #Bool

Arrays

- Arrays are for sequences of values, indexed from 1 to n:
  - v = [1, 2, 3]
  - v(1)   # you'll get an error – different from MATLAB
  - v[1]   # access elements using [ ]; starts from 1
  - v[0]   # gives an error!
  - v[end]   # last element

- 2D arrays are made the same way as in MATLAB:
  - A = [1 1 1; -1 -1 -2; 0 0 0]
  - B = [0.1 0.5 0.4; 0.2 0.3 0.5; 0.8 0.1 0.1]
  - B[2,3]
  - A + B
  - B^100
  - B[1,:]

Arrays continued

- Modifying arrays:
  - a = [1, 2, 3]
  - a[end+1] = 4   # won’t work!
- Push/pop commands:
  - push!(a, 4) # append 4 to end of a
  - pop!(a) # remove last element of a; 4
  - a   # a is back to being [1, 2, 3]
  - z = [ ]
  - push!(z, 5) # what happens?
  - z = Int64[ ]
  - push!(z,5) # how about now?

Exercise 1a

- Create the arrays A, x and b as follows:
  - A = rand(10,3); x = rand(3,1); b = rand(10,1)
- Write an expression that evaluates to
  - **true** if the inner product of the fourth row of A and x is greater than the fourth element of b; and
  - **false** otherwise.
- What do you find?

Exercise 1a’s lesson

- Julia is much stricter with arrays than MATLAB!
- For example:
  - A = rand(10,3); x = rand(3,1); b = rand(10,1)
  - A[4,:] * x – b[4]
  - A[4,:] * x – b[4] < 0   # what happens?
  - (A[4,:] * x – b[4])[1] < 0 # how about now?

- Yet another trap!
  - z = [1:1:3] # what is it?
  - z[:]   # what do you get?
  - z[:,:] # how about now?

Initializing arrays

- List comprehensions:
  - oddNumbers = [2*i - 1 for i in 1:10]
  - evenNumbers = [j + 1 for j in oddNumbers]
- Can build matrices this way:
  - A = [i + j for i in 1:5, j in 1:5]

Tuples

- Tuples are “fixed” arrays:
  - t = (1, 2, 3)
- Can access them like arrays, but can’t change them:
  - t[1]   # gives back 1
  - t[1] = 5 # error!
- Tuples can be initialized in other ways:
  - x = tuple(1, 2, 3)
  - y = [1:5]
  - z = tuple(y...)

Dictionaries

- Dictionaries are like arrays, except they can be indexed by arbitrary objects:
  - mydict = Dict()   # makes an empty dictionary.
  - mydict[“Cats”] = “are alright”   # add key-value pairs…
  - mydict[“Dogs”] = “are awesome!”
  - myotherdict = [“King’s Landing”=> 1, “Winterfell”=> 2, “Qarth”=>3]
  - myotherdict[“Pyke”]   # gives an error
- Keys of dictionary:
  - keys(mydict)
- Values of dictionary:
  - values(mydict)

Conditionals and loops

- If-then-else:
  - x = 2
  - if x > 6
    ```
    if x > 6
        x = x + 20
    elseif (x < 5)
        x = factorial(x)
    else
        println("Not changing x")
    end
    ```
- for loops:
  - for k =1:5
    ```
    for k =1:5
        println( k^2 )
    end
    ```
- while loops:
  - while x < 2000
    ```
    while x < 2000
        print("*")
        x = 2 * x
    end
    ```

Functions

- Functions are defined using the keyword function:
  - ```
    function convexcomb(x,y, theta)
        (1 – theta) * x + theta * y
    end
    ```
- Functions by default return the last statement, but you can also use return:
  - ```
    function convexcomb2(x, y, theta)
        if (theta < 0 || theta > 1)
            println("Bad theta!!!")
        else
            return (1 – theta) * x + theta * y
        end
    end
    ```

Exercise 1b

- Write a function that takes a matrix A, vectors x and b, and returns a vector containing the row indices for which Ax >= b does not hold.
- There are many ways to do this!

- Hints:
  - .< performs element-wise comparisons
  - “find” function returns indices of non-zero elements for Int64 or Float64 arrays, true elements for Bit arrays

Scripts in Julia

- Julia scripts are plain text files with the extension ".jl".
- In a terminal window, you can run one (say script.jl) by passing it as a command line argument to Julia – e.g.,
  - julia script.jl

Scripts continued

- Within a session, you can use the keyword ‘using’ to load everything in a script.
- E.g. suppose test.jl was:
  - function myFunction(x, y)
            return gamma(x)*gamma(y)/gamma(x+y)
    end
  - println(“My function loaded!”)
- In Julia session:
  - using test.jl
  - myFunction(1,2)
  - myFunction(0.5,0.5)

Time to JuMP!

Why an AML?

- Algorithms like simplex work with an optimization problem that is expressed in the form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Functions like linprog take A, b, c explicitly as inputs, solve this problem and return an optimal x.

- While it is possible to represent the data A, b, c explicitly for any given model, doing the conversion – from a mathematical description, to solver-ready form – can be quite laborious.

Why an AML? (continued)

- An AML allows you to specify an optimization problem in a compact way that you, as a human being, can understand.
  - Optimization variables are represented by programming variables.
  - Constraints are built with expressions resemble.
  - Throughout, we can use all of the tools available to us from programming languages – conditionals, loops, functions, etc. – to build our model.

- Once the model is built in this way, the AML takes care of converting your description to solver-ready data (A, b, c).

Let’s model a simple LP

- Run the following:
  - using JuMP

  - m = Model()
  - @defVar(m, x >= 0)
  - @defVar(m, 0 <= y <= 3)

  - @addConstraint(m, x + y <= 3)
  - @addConstraint(m, 5x + 3y <= 11)
  - @setObjective(m, Max, 1x + 4y)

  - print(m)

- What did we just model?

Now solve it!

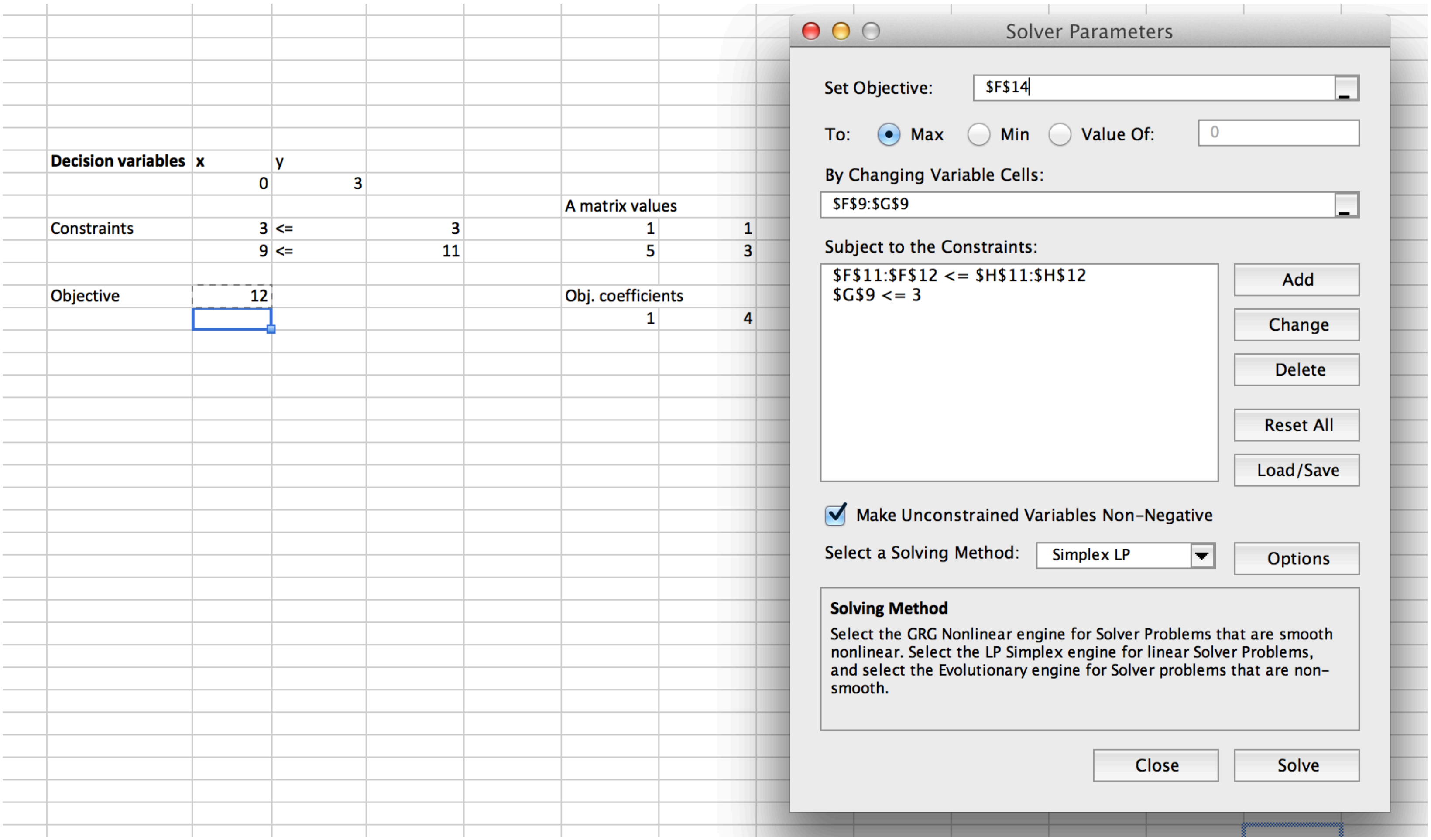
- Run the following:
  - status = solve(m)

  - println("Z = ", getObjectiveValue(m))
  - println("x = ", getValue(x))
  - println("y = ", getValue(y))

Without an AML, in MATLAB

- In MATLAB, with linprog:
  - c = [-1; -4];
  - A = [1 1; 5 3];
  - b = [3; 11];
  - x = linprog(c, A, b, [], [], [0;0], [inf; 3])

Without an AML, in Excel



Building more general models

- More general ways to build things:
  - m2 = Model()
  - @defVar(m2, x[1:10, 1:2] >= 0)
  - @addConstraint(m2, sum{ x[i,j] , i =1:10, j=1:2} <= 1)

  - primes = [2, 3, 5, 7]
  - @addConstraint(m2, sum{x[i,j] , i in primes, j = 1:2} >=0.2)

  - @addConstraint(m2, sum{x[i,j], i =1:10, j =1:2; i + j <= 5} >= 0.05)

  - myTest(i,j) = i + j <= 5   # what did we do here?
  - @addConstraint(m2, sum{x[i,j], i = 1:10, j=1:2; myTest(i,j)} >= 0.05)

Solver options

- The Model() constructor accepts a specification of a solver – e.g.,
  - using JuMP, Gurobi
  - m = Model(solver = GurobiSolver())

- The solver constructor (GurobiSolver()) accepts parameters. Parameter names/values follow the same naming/meaning as within the solver – e.g.,
  - m = Model(solver = GurobiSolver(Method = 2, Crossover = 0))
  - m = Model(solver = GurobiSolver(Presolve = 0))

Facility Location

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{M} \sum_{j=1}^{N} d_{ij} x_{ij} \\
\text{subject to} \quad & x_{ij} \le y_j, \quad \forall\, i = 1, \ldots, M,\ j = 1, \ldots, N, \\
& \sum_{j=1}^{N} x_{ij} = 1, \quad \forall\, i = 1, \ldots, M, \\
& \sum_{j=1}^{N} y_j \le K, \\
& 0 \le x_{ij} \le 1, \quad \forall\, i = 1, \ldots, M,\ j = 1, \ldots, N, \\
& y_j \in \{0, 1\}, \quad \forall\, j = 1, \ldots, N.
\end{aligned}
$$

Without an AML...

- In MATLAB, using linprog?
  - Think about how you would need to keep track of indices of many rows and columns of A matrix will have double indices (rows because of x_ij <= y_j, columns because of x_ij).
  - This can be nasty to debug (trust me).

- In Excel, using Solver?
  - Could potentially handle it – have large plots of spreadsheet space where rows and columns correspond to i and j.
  - But what if you have new data? What if the dimensions of the problem change?

Exercise 2

- Suppose that we live in a 1D world, where customers and facilities are located on a line as follows:
  - customerLocs = [3, 7, 9, 10, 12, 15, 18, 20]
  - facilityLocs = [1, 5, 10, 12, 24]
  - d_ij = absolute value distance between customer i and facility j

- Model and solve this problem in Julia using JuMP! (Set K = 3.)

- HINT: to make a variable binary, use “Bin” in @defVar:
  - @defVar(m, z[1:5], Bin)

Exercise 3

- In MIPs, it is sometimes useful to know not only the best solution, but also the second best, third best and so on.
- Modify your code so that, after the first solve, the model is solved another three times.
- Each time, add a constraint that cuts off the current facility decision, re-solve the model and output the objective value.

Exercise 3 Hint

- Suppose our current solution is

$$\{\bar{y}_j\}_{j=1}^{N}$$

- Suppose

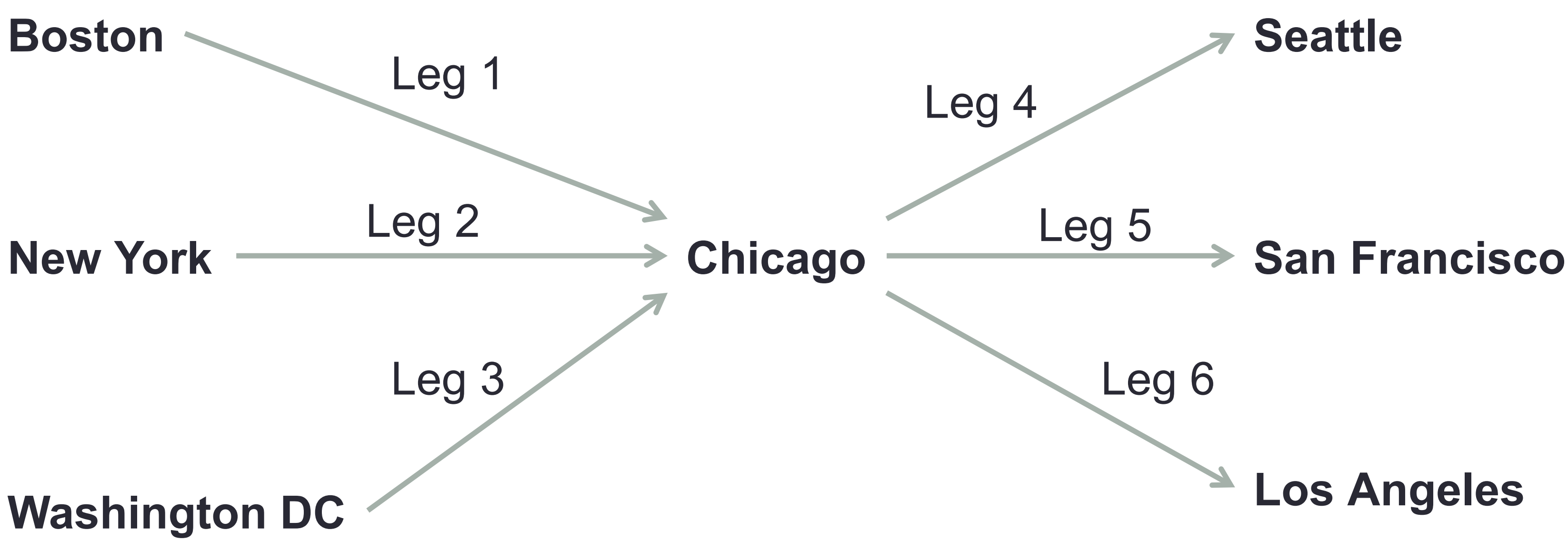
$$J_1 = \{j \mid \bar{y}_j = 1\} \qquad J_0 = \{j \mid \bar{y}_j = 0\}$$

- Then the following constraint cuts off the current solution:

$$\sum_{j \in J_1} y_j + \sum_{j \in J_0} (1 - y_j) \leq N - 1$$

Network revenue management

- Suppose we are running an airline that operates the following network:



Network revenue management - II

- We sell fares that may use capacity on one or two of the links in the network.
  - E.g.: we may sell Boston -> Chicago, Chicago -> LA, or Boston -> LA *via* Chicago.

- For now, we'd like to decide how many requests of each fare to accept, so as to maximize our revenue, subject to:
  - How much capacity is on each flight.
  - How many requests we expect of each fare, given that we are selling over T days.

LO Formulation

$$A_{\ell,f} = \begin{cases} 1 & \text{if fare } f \text{ uses link/leg } \ell, \\ 0 & \text{otherwise.} \end{cases}$$

$$R_f = \text{revenue from fare } f$$

$$p_f = \begin{array}{l} \text{probability of daily request} \\ \text{being of fare } f \end{array}$$

$$T = \text{length of horizon}$$

LO Formulation

$$
\begin{aligned}
\text{maximize} \quad & \sum_{f \in \mathcal{F}} r_f x_f \\
\text{subject to} \quad & \sum_{f \in \mathcal{F}} A_{\ell,f} x_f \leq c_\ell, \quad \forall \ell \in \mathcal{L}, \\
& 0 \leq x_f \leq T \cdot p_f, \quad \forall f \in \mathcal{F}.
\end{aligned}
$$

Data

- We have the following data:
  - legs = [1, 2, 3, 4, 5, 6]
  - fareLegs = [(1), (2), (3), (4), (5), (6), (1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)]
  - fareProbabilities = [0.06, 0.096, 0.046, 0.073, 0.159, 0.067, 0.043, 0.019, 0.112, 0.075, 0.031, 0.044, 0.012, 0.0210, 0.1130]
  - fareRevenues = [40, 30, 30, 10, 40, 10, 190, 80, 90, 70, 60, 190, 60, 50, 10]
  - legCapacities = [20, 20, 20, 20, 20, 20]
  - T = 100

Exercise 4

- Formulate the LO problem in Julia and JuMP, and solve it. What is the optimal revenue?

- Hint: what happens when you try this:
  - z = (1, 2, 3) # what is the type of z?
  - 2 in z   # what happens?
  - 4 in z   # what happens?

Exercise 5

- Create a function to solve the problem, that takes
  - legs, fareLegs, fareProbabilities, fareRevenues, legCapacities, T
- as inputs.
- You may use Ex5_NRM_Function_test.jl to test it.

Constraint references

- Like variables, constraints can have a “name” – in JuMP, this is a *constraint reference*.
- Constraint references are created using the @defConstrRef macro.
- The output of @addConstraint can then be assigned to a constraint reference.
- Example:
  - m = Model(:Max)
  - @defVar(m, x[1:4] >= 0)
  - @defConstrRef myCons[1:4]
  - for i =1:4
        myCons[i] = @addConstraint(m, x[i] <= 10)

Dual information

- JuMP also provides a function, getDual, for obtaining dual information.
- If cRef is a constraint reference, then getDual(cRef) returns the shadow price of the constraint.
- If var is a variable, then getDual(var) returns the reduced cost of the variable.

Exercise 6

- Modify the function from Exercise 5 to return the shadow prices of the leg constraints in an array, as well as the optimal objective.
- You may use Ex6_NRM_DualFunction_test.jl to test it.

Online NRM

- In reality, the problem is an *online* problem.
- On each day, there is a request for a fare. We must decide whether to accept it or reject it.
  - Accepting a fare now gives us revenue, but uses capacity that could be used on more lucrative fares later.
  - Rejecting a fare keeps our capacity intact, but we do not earn any revenue.

- This is a notoriously difficult stochastic control problem.

Bid-price control

- Let

$$\lambda_\ell(\vec{c}, t) = \begin{array}{l} \text{dual price of leg } \ell\text{'s capacity constraint,} \\ \text{when the remaining capacity is } \vec{c} \\ \text{and } t \text{ days remain.} \end{array}$$

- In BPC, the action to take (when there is sufficient capacity) is given by comparing the dual prices to the revenue of the fare:

$$\pi_t(f) = \begin{cases} \text{Accept} & \text{if } \sum_{\ell \in \mathcal{L}} A_{\ell,f} \lambda_\ell(\vec{c}, t) \leq r_f, \\ \text{Reject} & \text{otherwise.} \end{cases}$$

Exercise 7

- Ex7_NRM_SimulateControl.jl contains two simulation functions – one for BPC and one for a greedy policy. The BPC function is missing two lines of code that you need to fill in.
- To test it, use Ex7_NRM_SimulateControl_test.jl.

- How does BPC compare to the greedy policy?

Thank you for listening!

- For further information on Julia:
  - Speed tutorial: http://learnxinyminutes.com/docs/julia/
  - Doc: http://docs.julialang.org/en/latest/manual/

- For further information on JuMP:
  - Doc: https://jump.readthedocs.org/en/release-0.2/jump.html
  - Issues: https://github.com/JuliaOpt/JuMP.jl/issues

- If you find JuMP useful, please cite it as
  - M. Lubin and I. Dunning, “Computing in Operations Research using Julia”, under review.