1. Let $\boldsymbol{v} = \boldsymbol{i} - \lambda\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{w} = \lambda\boldsymbol{i} - 4\boldsymbol{j} + \mu\boldsymbol{k}$, where λ and μ are real numbers.

2 pts (a) Calculate $\|\boldsymbol{v}\|$ in terms of λ and μ.

5 pts (b) Calculate $\boldsymbol{v} \cdot \boldsymbol{w}$ and $\boldsymbol{v} \times \boldsymbol{w}$ in terms of λ and μ.

5 pts (c) Use your answer to part (b) to find the values of λ and μ such that $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel. (Assume λ and μ are ***positive*** real numbers.)

Solution

(a) $\|\boldsymbol{v}\| = \sqrt{1^2 + (-\lambda)^2 + 3^2} = \sqrt{10 + \lambda^2}$

(b) We have the following.

$$\boldsymbol{v} \cdot \boldsymbol{w} = (1)(\lambda) + (-\lambda)(-4) + (3)(\mu) = 5\lambda + 3\mu$$

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -\lambda & 3 \\ \lambda & -4 & \mu \end{vmatrix} = (-\lambda\mu + 12)\boldsymbol{i} + (3\lambda - \mu)\boldsymbol{j} + (-4 + \lambda^2)\boldsymbol{k}$$

(c) The vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel if and only if $\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{0}$. Hence we must solve the following simultaneous system of equations.

$$\begin{aligned} -\lambda\mu + 12 &= 0 \\ 3\lambda - \mu &= 0 \\ -4 + \lambda^2 &= 0 \end{aligned}$$

The only positive solution to the third equation is $\lambda = 2$, whence $\mu = 6$ from the second equation. We may then verify that these values of λ and μ satisfy the first equation. Hence $\lambda = 2$ and $\mu = 6$.

2. The lines ℓ_1 and ℓ_2 are given by the following parametrizations.

$$\begin{aligned} \ell_1 &: \quad \boldsymbol{r}_1(t) = \langle -2, -1, 4 \rangle + t\,\langle -5, 5, 1 \rangle \\ \ell_2 &: \quad \boldsymbol{r}_2(t) = \langle 0, -10, 10 \rangle + t\,\langle 3, 4, -7 \rangle \end{aligned}$$

6 pts (a) Show that ℓ_1 and ℓ_2 intersect and find the point of intersection. Is this point also a collision point? Explain.

6 pts (b) Find an equation of the plane $\mathcal{P}$ that contains both ℓ_1 and ℓ_2.

Solution

(a) We seek two parameter values t_1 and t_2 such that $\boldsymbol{r}_1(t_1) = \boldsymbol{r}_2(t_2)$. Hence we must solve the following simultaneous system of equations.

$$\begin{aligned} -2 - 5t_1 &= 3t_2 \\ -1 + 5t_1 &= -10 + 4t_2 \\ 4 + t_1 &= 10 - 7t_2 \end{aligned}$$

Adding the first two equations gives $-3 = -10 + 7t_2$, whence $t_2 = 1$. We then find from the first equation that $t_1 = -(3t_2 + 2)/5 = -1$. (This gives a solution for t_1

and t_2 to the first pair of equations only. We must then verify explicitly that this pair satisfies the final equation; otherwise the lines would not intersect.)

The point of intersection is the tip of $\boldsymbol{r}_2(t_2) = \langle 3, -6, 3\rangle$, or the point $(3, -6, 3)$. This is not a collision point since the parameter values t_1 and t_2 are not equal.

(b) Let $\boldsymbol{v}_1 = \langle -5, 5, 1\rangle$ and $\boldsymbol{v}_2 = \langle 3, 4, -7\rangle$ be the direction vectors for ℓ_1 and ℓ_2, respectively. Then a vector normal to the plane $\mathcal{P}$ is $\boldsymbol{n} = \boldsymbol{v}_1 \times \boldsymbol{v}_2$.

$$\boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -5 & 5 & 1 \\ 3 & 4 & -7 \end{vmatrix} = -39\boldsymbol{i} - 32\boldsymbol{j} - 35\boldsymbol{k}$$

A point in the plane $\mathcal{P}$ is $(3, -6, 3)$. Hence an equation for the plane is

$$-39(x-3) - 32(y+6) - 35(z-3) = 0$$

3. Consider the curve $\mathcal{C}$ with parametrization

$$\boldsymbol{r}(t) = (t^2 - 3)\boldsymbol{i} + (3t^2 + 5)\boldsymbol{j} + \tfrac{2}{3}t^3\boldsymbol{k} \quad , \quad t \geq 0$$

6 pts (a) Find the length of $\mathcal{C}$ over the interval $0 \leq t \leq 1$.

6 pts (b) Find the curvature of $\mathcal{C}$ at the point $\boldsymbol{r}(1)$. You may use the formula

$$\kappa(t) = \frac{\|\boldsymbol{r}''(t) \times \boldsymbol{r}'(t)\|}{\|\boldsymbol{r}'(t)\|^3}$$

Solution

(a) We have the following.

$$\boldsymbol{r}'(t) = 2t\boldsymbol{i} + 6t\boldsymbol{j} + 2t^2\boldsymbol{k}$$
$$\|\boldsymbol{r}'(t)\| = \sqrt{(2t)^2 + (6t)^2 + (2t^2)^2} = \sqrt{40t^2 + 4t^4} = 2t\sqrt{10 + t^2}$$

(Note that $\sqrt{t^2} = t$ since $t \geq 0$.) The arc length is thus given by the following integral.

$$s = \int_0^1 2t\sqrt{10 + t^2}\, dt = \frac{2}{3}(10 + t^2)^{3/2}\Big|_{t=0}^{t=1} = \frac{2}{3}(11^{3/2} - 10^{3/2})$$

(b) First note that $\boldsymbol{r}''(t) = 2\boldsymbol{i} + 6\boldsymbol{j} + 4t\boldsymbol{k}$. Hence $\boldsymbol{r}'(1) = 2\boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{r}''(1) = 2\boldsymbol{i} + 6\boldsymbol{j} + 4\boldsymbol{k}$. We now have the following calculations.

$$\boldsymbol{r}''(1) \times \boldsymbol{r}'(1) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 6 & 2 \\ 2 & 6 & 4 \end{vmatrix} = 12\boldsymbol{i} - 4\boldsymbol{j}$$
$$\|\boldsymbol{r}''(1) \times \boldsymbol{r}'(1)\| = \sqrt{12^2 + (-4)^2} = \sqrt{160}$$
$$\|r'(1)\|^3 = (\sqrt{2^2 + 6^2 + 2^2})^3 = 44^{3/2}$$

Hence the curvature at the desired point is $\kappa(1) = \frac{\sqrt{160}}{44^{3/2}}$.

4. *Assume that the positive x-axis points East and the positive y-axis points North.* Suppose you are hiking on a terrain modeled by the equation $z = \sqrt{3}xy - 2x^2 - 1$ and you are standing at the point $(1, \sqrt{3}, 0)$.

5 pts (a) Determine the angle of inclination you would encounter if you headed due West.

5 pts (b) Determine the steepest slope you could encounter from your position and the compass direction measured in degrees anticlockwise from East that you would head to realize this steepest slope.

4 pts (c) In what direction should you head to encounter no change in elevation? Give your answer as an angle measured in degrees anticlockwise from East.

Solution

(a) Let $P = (1, \sqrt{3})$. First we calculate the gradient at P.

$$\nabla f = \left\langle \sqrt{3}y - 4x, \sqrt{3}x \right\rangle \Longrightarrow \nabla f_P = \left\langle -1, \sqrt{3} \right\rangle$$

A unit vector pointing in the West direction is $\boldsymbol{u} = \langle -1, 0 \rangle$. Hence the slope of the terrain at P in the West direction is

$$D_{\boldsymbol{u}} f(P) = \nabla f_P \cdot \boldsymbol{u} = \left\langle -1, \sqrt{3} \right\rangle \cdot \langle -1, 0 \rangle = 1$$

This slope corresponds to an angle of inclination of $\psi = \tan^{-1}(1) = \frac{\pi}{4}$, or 45 degrees.

(b) The steepest possible slope at P is $\|\nabla f_P\| = \sqrt{1^2 + \sqrt{3}^2} = 2$. This slope is realized by heading in the direction of ∇f_P from P. Note that this means one should walk in a direction corresponding to 1 unit in the negative x-direction and $\sqrt{3}$ units in the positive y-direction. The corresponding compass angle is an angle θ in the second quadrant such that $\tan(\theta) = \sqrt{3}$. Hence the compass angle is 120 degrees.

Note: The solution above is for the slope of steepest *ascent*. Since the word "steepest" here is ambiguous, it can also reasonably be interpreted as steepest slope of *descent*. In that case, the steepest slope is -2 and the compass angle is 180 degrees from the direction of steepest ascent, or 300 degrees.

(c) No change in elevation is realized if one walks perpendicular to the direction of ∇f_P from P. Hence no change in elevation is realized at a compass direction of either 30 degrees or 210 degrees.

10 pts **5.** Calculate the following limit or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy}{3x^2 + 2y^2} \right)$$

Solution

Consider the limit along the lines $y = mx$ for an arbitrary value of m.

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx}} \left(\frac{xy}{3x^2 + 2y^2} \right) = \lim_{x\to 0} \left(\frac{mx^2}{3x^2 + 2m^2x^2} \right) = \frac{m}{3 + 2m^2}$$

Since this limit depends on m, the limit along lines through the origin has a value that depends on the particular line along which we calculate the limit. Hence the original limit does not exist.

10 pts **6.** Find all points on the graph of $z = xy^3 + 10y^{-1} + 12$ where the vector $\boldsymbol{n} = \langle 16, -7, 2 \rangle$ is normal to the tangent plane.

Solution

Let (a, b, c) be the unknown point of tangency and let $f(x, y) = xy^3 + 10y^{-1} + 12$

Since $\boldsymbol{n}$ is normal to the tangent plane, the tangent plane must be described by an equation of the form

$$16x - 7y + 2z = d$$

for some constant d. This is equivalent to

$$z = -8x + \frac{7}{2}y + \tilde{d}$$

for some constant $\tilde{d}$. In this form of the equation for the tangent plane, the coefficients of x and y are simply the partial derivatives of f evaluated at (a, b). So first we calculate the partial derivatives of f.

$$f_x(a, b) = b^3 \quad , \quad f_y(a, b) = 3ab^2 - 10b^{-2}$$

Then we set each of these partial derivative equal to the corresponding coefficients in our tangent plane. Hence we have the following simultaneous system of equations.

$$b^3 = -8$$
$$3ab^2 - 10b^{-2} = \frac{7}{2}$$

The first equation immediately gives $b = -2$. Substituting $b = -2$ into the second equation then gives $a = \frac{1}{2}$. Note that $f(a, b) = 3$, whence the point of tangency is $(-2, \frac{1}{2}, 3)$.

12 pts **7.** Let r, s, and t be independent parameters and suppose x, y, and z are given by

$$x = 2r - 3s + t$$
$$y = 5r + 2s - 6t$$
$$z = -r + s$$

Let $w = f(x, y, z)$ where f is an arbitrary differentiable function. Calculate the sum

$$A(r, s, t) = \frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} + \frac{\partial w}{\partial t}$$

Write your answer as a function of r, s, and t. Simplify as much as possible.
(Since f is arbitrary, your answer may still contain the symbol f or related symbols. But you must write your answer as a function of r, s, and t.)

Solution
Let $\boldsymbol{R} = \langle 2r - 3s + t, 5r + 2s - 6t, -r + s \rangle$. Then the chain rule gives us the following.

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \boldsymbol{\nabla} f \cdot \frac{\partial \boldsymbol{R}}{\partial r} = \langle f_x, f_y, f_z \rangle \cdot \langle 2, 5, -1 \rangle = 2f_x + 5f_y - f_z \\
\frac{\partial w}{\partial s} &= \boldsymbol{\nabla} f \cdot \frac{\partial \boldsymbol{R}}{\partial s} = \langle f_x, f_y, f_z \rangle \cdot \langle -3, 2, 1 \rangle = -3f_x + 2f_y + f_z \\
\frac{\partial w}{\partial t} &= \boldsymbol{\nabla} f \cdot \frac{\partial \boldsymbol{R}}{\partial t} = \langle f_x, f_y, f_z \rangle \cdot \langle 1, 6, 0 \rangle = f_x + 6f_y
\end{aligned}$$

Summing these expressions gives $A = f_y(x, y, z)$. Hence, as a function of r, s, and t, the quantity A is given by

$$A(r, s, t) = f_y(2r - 3s + t, 5r + 2s - 6t, -r + s)$$

8. Let $f(x, y) = x^2 + y^2 - xy - 6x$

9 pts (a) Find the critical point of f and the corresponding critical value. Then classify it as a local minimum, local maximum, or neither (saddle).

Let S be the square $\{(x, y) : 0 \leq x \leq 6, 0 \leq y \leq 6\}$.

9 pts (b) Find the minimum and maximum values of f on each of the four edges of S. Then determine the global extreme values of f on S. *Fill in the table below as you work.*

edge of S	bottom edge	right edge	top edge	left edge
minimum value of f				
maximum value of f				

Solution

(a) The critical point of f must satisfy the simultaneous system of equations $f_x = f_y = 0$.

$$\begin{aligned}
f_x &= 2x - y - 6 = 0 \\
f_y &= 2y - x = 0
\end{aligned}$$

The second equation immediately gives $x = 2y$. Substituting $x = 2y$ into the first equation gives $3y - 6 = 0$, or $y = 2$. Hence the critical point is $(4, 2)$ and the critical value is $f(4, 2) = -12$.

To classify the critical point, we examine the disciminant function.

$$D(x, y) = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

Since both $D(4, 2) > 0$ and $f_x(4, 2) > 0$, the critical value is a local minimum.

(b) For each edge, we examine f restricted to that edge, and then we determine the minimum and maximum values using elementary calculus.

(Recall from Calculus I that to find the maximum value of a function $h(u)$ on the interval $u \in [a, b]$, we first find the critical numbers of h by solving $h'(u) = 0$ or determining where $h'(u)$ does not exist. Extreme values must then either occur at a critical number or at either of the endpoints $u = a$ or $u = b$.)

bottom: The bottom edge is described by $y = 0$ with $0 \le x \le 6$. Hence we must find the extreme values of

$$g_1(x) = f(x, 0) = x^2 - 6x$$

on the interval $[0, 6]$. Observe that $g_1'(x) = 2x-6$, hence the only critical number is $x = 3$. The candidate extreme values for g_1 are $g_1(0) = 0$, $g_1(3) = -9$, and $g_1(6) = 0$. Hence the minimum value is -9 and the maximum value is 0.

right: The right edge is described by $x = 6$ with $0 \le y \le 6$. Hence we must find the extreme values of

$$g_2(y) = f(0, y) = y^2 - 6y$$

on the interval $[0, 6]$. This is the same problem we solved for the bottom edge. Hence the minimum value is -9 and the maximum value is 0.

top: The top edge is described by $y = 6$ with $0 \le x \le 6$. Hence we must find the extreme values of

$$g_3(x) = f(x, 6) = x^2 - 12x + 36 = (x - 6)^2$$

on the interval $[0, 6]$. By inspection we find that the minimum value is $g_3(6) = 0$ and the maximum value is $g_3(0) = 36$.

left: The left edge is described by $x = 0$ with $0 \le y \le 6$. Hence we must find the extreme values of

$$g_4(y) = f(0, y) = y^2$$

on the interval $[0, 6]$. By inspection we find that the minimum value is $g_4(0) = 0$ and the maximum value is $g_4(6) = 36$.

These results are summarized in the table below.

edge of S	bottom edge	right edge	top edge	left edge
minimum value of f	-9	-9	0	0
maximum value of f	0	0	36	36

To determine the global extreme values of f on S, recall from part (a), that f has a local minimum value of -12 in the interior of S. Hence the global minimum value of f on S is -12 and the global maximum value of f on S is 36.