Estimating the largest Lyapunov exponent (LLE) for the Lorenz system (Assignment Sheet 9)

Introduction To Chaos Applied To Systems, Processes And Products (ETSIDI, UPM)

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2026-02-03

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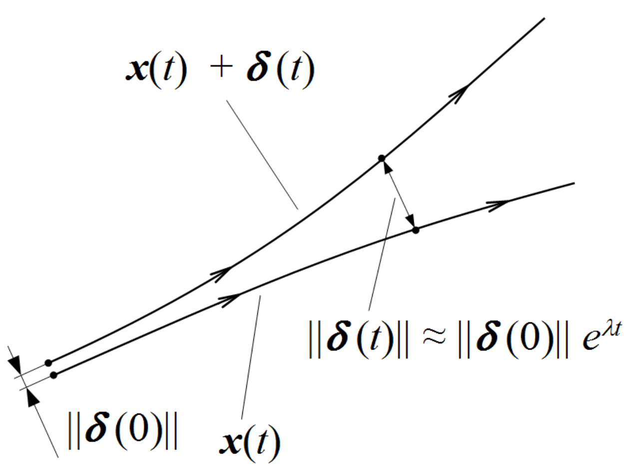
Largest Lyapunov Exponent for 3D flows

The **Largest Lyapunov Exponent (LLE)** quantifies chaos by measuring sensitivity to initial conditions. Here, we'll estimate the LLE for the classical Lorenz attractor, a canonical chaotic system.



Source: Wikipedia – Lyapunov exponent

1. **Load the required packages**

```r
library(ggplot2)
library(deSolve)
```

2. **Define the Lorenz system**

The Lorenz system is defined by:

$$
\begin{aligned}
\dot{x} &= \sigma \cdot (y - x) \\
\dot{y} &= x \cdot (\rho - z) - y \\
\dot{z} &= x \cdot y - \beta \cdot z
\end{aligned}
$$

Lorenz originally used the values $\sigma = 10$, $\rho = 28$, and $\beta = 8/3$, under which the system exhibits chaotic behavior.

```r
lorenz <- function(t, state, parameters) {
  with(as.list(c(state, parameters)), {
    dx <- sigma * (y - x)
    dy <- x * (rho - z) - y
    dz <- x * y - beta * z
    list(c(dx, dy, dz))
  })
}
```

3. **Start with any initial condition in the basin of attraction and iterate until the orbit is on the attractor**.

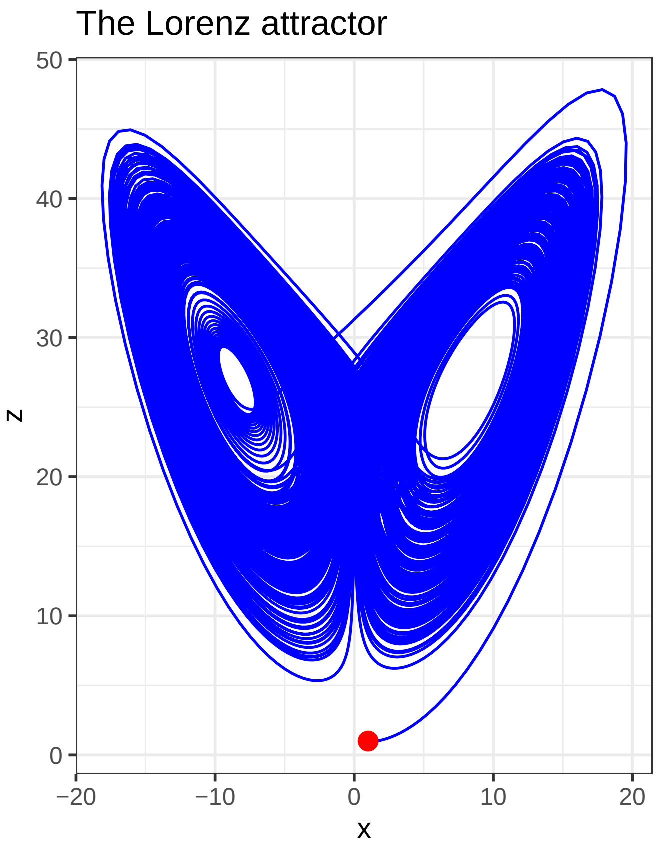
```r
# Parameters
parms <- c(sigma = 10, rho = 28, beta = 8/3)

# Initial conditions and integration settings
state_aux <- c(x = 1, y = 1, z = 1)
dt <- 0.01
total_time <- 200
times <- seq(0, total_time, by = dt)

# Integrate to reach attractor
orbit_aux <- ode(state_aux, times, lorenz, parms, method = "ode45")
orbit_aux_df <- as.data.frame(orbit_aux)
```

4. **Visualize the Lorenz attractor**

```r
ggplot()+
  geom_path(data = orbit_aux_df,
            aes(x = x, y = z), color = "blue")+
  geom_point(data = data.frame(x=state_aux["x"],y=state_aux["y"],z=state_aux["z"]),
             aes(x = x, y = z), color = "red", size = 3)+
  labs(title = "The Lorenz attractor")+
  coord_fixed()+
  theme_bw()
```



5. Select the last point of the previous orbit. It should be in the attractor.

```
# Select final state
state <- orbit_aux[nrow(orbit_aux),c("x","y","z")]
```

6. Select a nearby point (separated by δ_0).

```
# Define initial nearby point separated by small d0
d0 <- 1e-8
state_perturbed <- state + c(d0, 0, 0)
```

7. Advance both orbits one iteration and calculate new separation δ_1.

```
# Advance both states by one iteration (time step dt)
orbit1 <- ode(state, c(0, dt), lorenz, parms, method = "ode45")[2,-1]
orbit2 <- ode(state_perturbed, c(0, dt), lorenz, parms, method = "ode45")[2,-1]

# Calculate new separation d1
d1_vector <- orbit2 - orbit1
d1 <- sqrt(sum(d1_vector^2))
```

8. Evaluate $log|\delta_1/\delta_0|$ in any convenient base.

```r
# Evaluate log(d1/d0)
log(d1/d0)
```

```
## [1] -0.0870614
```

9. Readjust one orbit so its separation is δ_0 in same direction as δ_1.

```r
# Readjust perturbed orbit to separation d0
state <- orbit1
state_perturbed <- orbit1 + (d0 / d1) * d1_vector
```

10. We use a for loop to repeat the previous 3 steps and obtain new values of $\log|\delta_1/\delta_0|$.

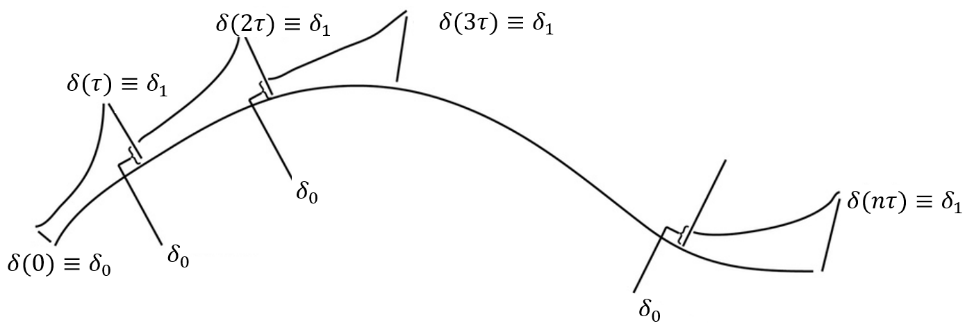


Figure 1: Numerical calculation of the LLE

Source: Numerical calculation of the largest Lyapunov exponent

11. the largest Lyapunov exponent is the average value of $\lambda_1 =< \log|\delta_1/\delta_0| >$.

```r
# Parameters for Lyapunov exponent calculation
n_iter <- 1e4
lyapunov_sum <- 0

# Loop to calculate Lyapunov exponent
for (i in 1:n_iter) {

  # Advance both states by one iteration (time step dt)
  orbit1 <- ode(state, c(0, dt), lorenz, parms, method = "ode45")[2,-1]
  orbit2 <- ode(state_perturbed, c(0, dt), lorenz, parms, method = "ode45")[2,-1]

  # Calculate new separation d1
  d1_vector <- orbit2 - orbit1
  d1 <- sqrt(sum(d1_vector^2))

  # Evaluate log(d1/d0)
  lyapunov_sum <- lyapunov_sum + log(d1/d0)

  # Readjust perturbed orbit to separation d0
```

```
  state <- orbit1
  state_perturbed <- orbit1 + (d0 / d1) * d1_vector
}

# Calculate largest Lyapunov exponent
l1 <- lyapunov_sum / (n_iter * dt)

# Print the result
cat("Largest Lyapunov exponent:", l1, "\n")
```

```
## Largest Lyapunov exponent: 0.9099768
```

Interpretation

- **Positive LLE**: Indicates sensitive dependence on initial conditions and exponential divergence of trajectories (chaos).
- **Larger LLE**: Suggests more sensitive dependence on initial conditions.

```
if (l1 > 0) {
  cat("The system shows chaotic behavior.\n")
} else {
  cat("The system does not show chaotic behavior.\n")
}
```

```
## The system shows chaotic behavior.
```

Further exploration

Try varying the initial conditions and parameters (σ, ρ, β) to explore different dynamical regimes. For instance, estimate the largest Lyapunov exponent of the Lorenz system under the following parameter set: $\sigma = 10$, $\rho = 350$, $\beta = 8/3$