TOPIC: Biostat 778

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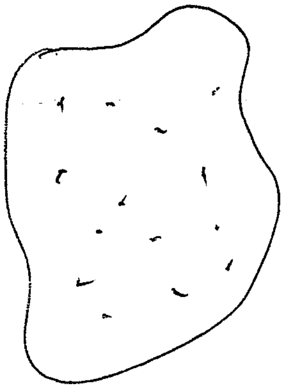

what is meaning of W(s)?

Spatial Modelling

$$Y(s) = \underbrace{\mu(s)}_{\text{fixed effect}} + \underbrace{W(s)}_{\text{latent Gaussian spatial process}} + \underbrace{\varepsilon(s)}_{\text{measurement error? microscale noise?}}$$

meaning?

$$\mu(s) = x^T(s)\beta$$

$$\cancel{W(s) \sim N(0,}$$

$$W = (W(s_1), W(s_2), \ldots, W(s_n)) \sim N(0, \underbrace{\sigma^2}_{\text{sill}} H(\underbrace{\phi}_{\text{range}}))$$

$$\varepsilon(s) \sim N(0, \tau^2)$$

↪ nugget effect

$$H_{ij} = Cov(W(s_i), W(s_j))$$

$$= \rho(s_i - s_j ; \phi) \quad \text{anisotropic}$$

$$= \rho(\|s_i - s_j\|; \phi) \quad \text{isotropic}$$

e.g. $\rho(s_i - s_j, \phi) = \exp(-\phi \|s_i - s_j\|)$

Let $Y = (Y(s_1), Y(s_2), \ldots, Y(s_n))$

$\theta = (\beta, \sigma^2, \phi, \tau^2)$

$$Y \mid \theta, W \sim N(X\beta + W, \tau^2 I)$$

$$W \mid \sigma^2, \phi \sim N(0, \sigma^2 H(\phi))$$

$$\left.\begin{array}{c}\sigma^2\\ \phi\\ \beta\\ \tau^2\end{array}\right. \sim \text{priors}$$

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Bayesian way

$Y | X \sim N(X, \tau^2)$

$X \sim N(0, \sigma^2)$

$X | Y \sim N(BY, (1-B)\sigma^2)$

$\hat\beta | \beta \sim N(\beta, \sigma^2)$

$\beta \sim N(0, \tau^2)$

$\beta | \hat\beta \sim N(c\hat\beta, (1-c)\tau^2)$

$p(\theta|y) \propto p(y|\theta, w)\, p(w|\theta)\, p(\theta)$

$\sigma^2 \sim IG(a_1, b_1)$

$\tau^2 \sim IG(a_2, b_2)$ $\qquad \phi \sim Gamma(a_3, b_3)$

$\beta \sim N(0, D)$ (vague)

~~$\phi \sim IG(e, f)$~~

informative

Compute MLEs

$CDL = p(y|\theta, w)\, p(w|\theta)$

$N(X\beta + W, \tau^2 I)\; N(0, \sigma^2 H(\phi))$

Need

$p(w|\theta, y) \propto p(y|\theta, w)\, p(w|\theta)$

$$CDLL = -\tfrac{1}{2}\log|\tau^2 I| - \tfrac{1}{2}(Y - X\beta - W)^T (Y - X\beta - W)/\tau^2$$
$$- \tfrac{1}{2}\log|\sigma^2 H(\phi)| - \tfrac{1}{2}\underset{1\times n}{W^T}\,\underset{n\times n}{H(\phi)^{-1}}\,\underset{n\times 1}{W}/\sigma^2$$

$$p(w|\theta_{(0)}, y) = N(BY, (I - B)\sigma^2 H(\phi))$$
$$B = \sigma^2 H(\phi)\left[\tau^2 I + \sigma^2 H(\phi)\right]^{-1}$$

Can iterate EM algorithm directly by filling in W and max Q. Estimating β, τ^2 okay but σ^2 and ϕ are harder (ϕ non linear)

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Get posteriors

$$\cancel{Q(\theta \mid \theta_0)} =$$

$$p(y, \theta, w) \propto p(y \mid \overset{\beta,\tau^2}{\theta}, w)\, p(w \mid \overset{\sigma^2,\phi}{\theta})\, \underbrace{p(\theta)}_{p(\sigma^2)p(\tau^2)p(\phi)p(\beta)}$$

informative

w

$$p(\beta \mid \sim) \propto p(y \mid \beta, \tau^2, w)\, p(\beta)$$

$$N(X\beta + w, \tau^2 I)\, N(0, D)$$

$$= N\left(S[X(X^TX)^{-1}X(y-w)], (I-S)D\right)$$

$$S = D[\tau^2 I + D]^{-1}$$

z $y - w$

$X(X^TX)^{-1}X^T(y-w)$

$$p(\tau^2 \mid \sim) \propto p(y \mid \beta, \tau^2, w)\, p(\tau^2)$$

$$= IG\left(a_2 + 1, b_2 + \tfrac{1}{2}(y - X\beta - w)^T(y - X\beta - w)\right)$$

$x^{-(\alpha+1)} e^{-\beta/x}$

$$p(\sigma^2 \mid \sim) \propto p(w \mid \sigma^2, \phi)\, p(\sigma^2)$$

$$IG\left(a_1 + 1, b_1 + \tfrac{1}{2} w^T w\right)$$

$$p(\phi \mid \sim) \propto p(w \mid \sigma^2, \phi)\, p(\phi)$$

$$N(0, \sigma^2 H(\phi))\ \phi^{e-1} e^{-\phi/f}$$

$$|\sigma^2 H(\phi)|^{-1/2} \exp\left(-\frac{1}{2\sigma^2} W^T H(\phi)^{-1} W\right)$$

$$H(\phi) = U(\phi) D(\phi) U(\phi)^T \qquad D(\phi) = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

need Metropolis step

$$\cancel{|\sigma^2|^{n} \left|\prod_{i=1}^{n} \lambda_i\right|^{-1/2}}$$

$$H(\phi)^{-1} = U(\phi) D(\phi)^{-1} U(\phi)^T$$

$$|H(\phi)| = \prod \lambda_i$$

$$\left|\prod_{i=1}^{n} \sigma^2 \lambda_i\right|^{-1/2} \exp\left(-\frac{1}{2\sigma^2} W^T U(\phi) D(\phi)^{-1} U(\phi)^T W\right)$$

$$p(w \mid \sim) \propto p(y \mid \theta, w)\, p(w \mid \theta) \qquad \begin{pmatrix} 1/\lambda_1 & & \\ & \ddots & \\ & & 1/\lambda_n \end{pmatrix}$$

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Prediction at new location.

Suppose we have $X(s_0)$ and we want $Y(s_0)$

↳ ~~Krig~~ (Bayesian) Kriging, interpolation

We want predictive distribution;

$$p(Y(s_0) \mid \underset{\text{obs } Y}{Y}, \underset{\text{obs } X}{X}, X(s_0))$$

$$= \int \underbrace{p(Y_0 \mid Y, X, \theta, X_0)}_{\text{Normal}} \underbrace{p(\theta \mid Y, X)}_{\text{posterior}} \, d\theta$$

$$= \int \left[\int \underbrace{p(Y_0 \mid Y, X, \theta, X_0, W) \, p(W \mid \theta)}_{\text{~~If both normal~~, just use } p(Y_0 \mid Y, X, \theta, X_0)} \, dW \right] p(\theta \mid Y, X) \, d\theta$$

$$= N(X\beta, \tau^2 I + \sigma^2 H(\phi))$$

But if $p(y \mid \theta, w)$ is not normal

(maybe Poisson, Bernoulli) then we are stuck,

need to sample.

$Y \mid \theta \sim N(X\beta, \tau^2 I + \sigma^2 H(\phi))$

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \\ Y_0 \end{pmatrix} \sim N\left(\begin{pmatrix} x_1 - \\ x_2 - \\ \vdots \\ x_n - \\ x_0 - \end{pmatrix} \beta \right)$$

$p(w_0, w \mid \theta)$

$= p(w_0 \mid w; \theta) \, p(w \mid \theta)$

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Y1

$$\begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \\ \hline W_0 \end{pmatrix} \sim N\left(0, \left(\begin{array}{ccc|c} \sigma^2 & & & \\ & \ddots & & \mathrm{cov} \\ & & \sigma^2 & \\ \hline & \mathrm{cov} & & \sigma^2 \end{array}\right)\right)$$

$$W_0 \mid W_1 \sim W_n \sim N(0 + \underset{1\times n}{B}\,\underset{n\times 1}{W},\ (1 - B))$$

$$\begin{pmatrix} W_0 \\ \hline W_1 \\ \vdots \\ W_n \end{pmatrix} \sim N\left(0, \left(\begin{array}{c|ccc} \sigma^2 & & \mathrm{cov}\ R & \\ \hline & \sigma^2 & & \mathrm{cov} \\ \mathrm{cov} & & \ddots & \\ & \mathrm{cov} & & \sigma^2 \end{array}\right)\right)$$

$$W_0 \mid W_1 \sim W_n \sim N\left(\underset{1\times n}{R^T}\,\underset{n\times n}{H(\phi)^{-1}}\,\underset{n\times 1}{W},\ \sigma^2 - R^T H(\phi)^{-1} R\right)$$