# Tutorial on Qualitative analysis and Bifurcation diagrams

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# I have a model! What now?

**Principle I**

I have equations, so I want solutions!

We will use as example a nice and simple model that we all should care about: the next step in predator–prey (or consumer–resource) models: the **Rosenzweig–MacArthur** model.

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# The Rosenzweig–MacArthur model

- Includes the same ingredients as the Lotka-Volterra equations,
- plus a carrying capacity for the resource,
- and plus a saturation in the predation rate.

$$\frac{dR}{dt} = rR\left(1-\frac{R}{K}\right) - \frac{aRC}{1+ahR}$$
$$\frac{dC}{dt} = \frac{eaRC}{1+ahR} - dC$$

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# Rosenzweig–MacArthur model solutions

- By Principle I, we want the solution of the model.
- We resort to numerical integration, but then. . . which parameters should I use? And which initial conditions?
- For now, let's just guess some parameters, and pick a few different initial conditions.
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- It seems that initial conditions don't matter for the final long-term solution: a fixed point.

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# The phase space flow and the fixed point

- The differential equations define a flow in the phase space: at each point, there's a direction the solution must follow when it goes through that point.
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- The size (the magnitude) of the arrows become small near the fixed point.
- That is, at the fixed point, $\frac{dC}{dt} = 0$ and $\frac{dR}{dt} = 0$.
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# Messing a little with the parameters. . .

- We have seen that predator-prey systems tend to oscillate, but in this case, the long-term solution is stationary.
- Let's, for example, increase the carrying capacity $K$ a little.
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- Now the arrows inside spirals outward, and the flow outside spirals inward, towards a **limit cycle**.
- The fixed point is still there (the arrows' sizes go to zero in the center), but now the solution moves away from it.
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# Varying a parameter systematically: the bifurcation diagram

- Changing the values of parameters haphazardly, it may be hard to see and to synthesize what happens in the system.
- Let's imagine you change the value of a parameter by a very small value:
- the expectation is that the solution changes only a tiny bit.
- But if we sweep a range of values in small steps, we will see a parameter value where the solution attains a new behavior: the bifurcation point.
- Let's do this increasing the resource carrying capacity $K$: what do you expect it is going to happen?

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## The paradox of enrichment

- For very small $K$, the predator is extinct,
- for intermediate $K$, the solution goes to a fixed point (the minimum and maximum of the solution have the same value!)
- and for high $K$, there are oscillations with amplitudes that increase with $K$.
- The "paradox of enrichment" means that boosting the resource population can lead to extinction either of the resource or of the consumer (or both), because the solution passes closer and closer to zero.

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# Seasonal consumer resource dynamics

- In certain situations, we may want to include explicit temporal dependence into our models.
- Seasonality, environmental fluctuations and experimental manipulation are some clear reasons why you would need that.
- Let's see what happens when we take our Rosenzweig-MacArthur model and introduce a seasonal growth rate $r = r_0(1 + \alpha \sin(2\pi t/T))$
- We are going to make a small perturbation ($\alpha$ small), so we shouldn't see much happening.
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# A resonance diagram

- The population oscillates together with the seasonal variation, even though the system with $K = 10$ didn't oscillate – but the amplitude is small.
- We now do a kind of bifurcation diagram, but now what we vary is the **frequency of the external oscillations**.
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- There's a sharp peak in the amplitude around a certain frequency. This is called a **resonance**.
- If we go back to the first plot (with $K = 10$ without seasonal fluctuations), we find that the period of the oscillations in the transient is around 23.
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# What if there are more than 2 equations?

- In that case, the phase space has more than 2 dimensions and doesn't fit into a nice 2-d plot.
- You can still try to plot planes of the phase space, specially ones containing the fixed points of interest.
- Bifurcation diagrams are still very much useful: you don't have to plot all the curves to characterize the solution.

# What if I want to explore a 10-dimensional parameter space?

- Avoid that: you can reduce the number of parameters rescaling your variables (adimensionalizing)
- You can also restrict the values considered based on data and on careful judgement: not every parameter has the same relevance to the outcomes.

But I still have 10 parameters left!

- Well, Good luck!
- You will probably have to sample the space, rather than go through the whole thing. A recommended method is to use so-called Latin Hypercube samples, that uses a random sampling while ensuring a roughly regularly-spaced distribution.

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Thanks for your attention!

All the code for the solutions and plots shown are available at the wiki of the course