Time Series and Dynamics Econometrics

Assignment 1

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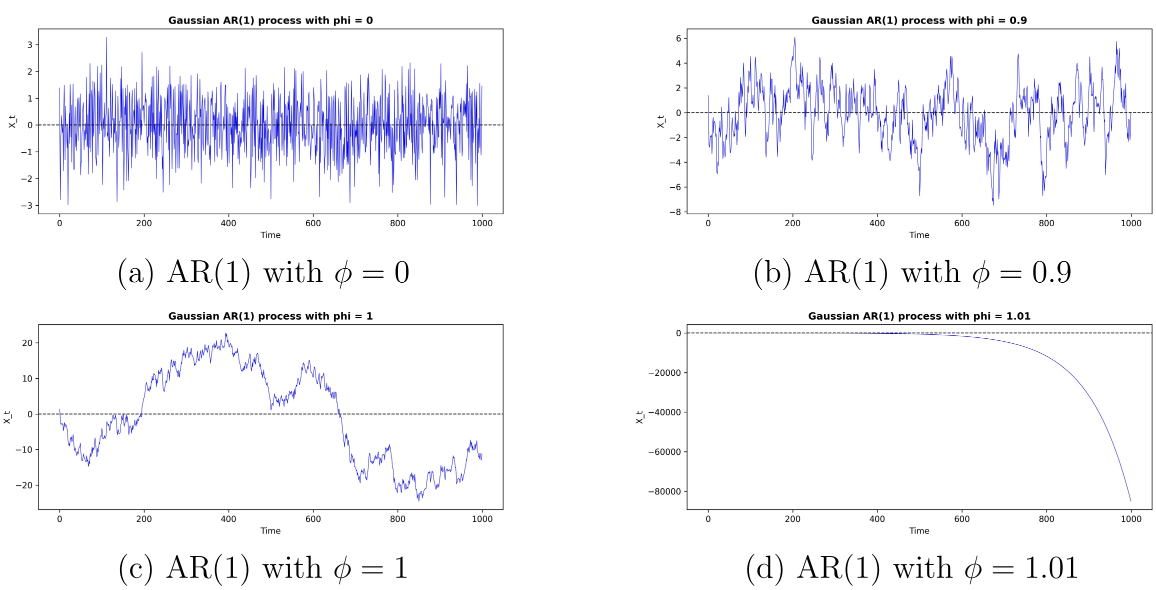
Part I: Introduction to Time Series

Question 1

In Figure 1.1, the autoregressive time series of order 1, AR(1), with $\phi = 0$, is shown, which is just *White Noise*. Since the white noise is $WN \sim NID(0, \sigma^2)$ with $\sigma^2 = 1$, X_t is just equal to the white noise. The values of the process X_t are between $[-3, 3]$.
In Figure 1.2, the autoregressive time series model of order 1, AR(1), with $\phi = 0.9$, is shown, which is a weakly stationary series since $|\phi| < 1$. The values of the process X_t are between $[-6, 6]$. The larger gaps between the curve also suggest that there is some form of dependence among the observations.
In Figure 1.3, the autoregressive time series model of order 1, AR(1), with $\phi = 1$, is shown, which is basically just *Random Walk*. This process is non-stationary.
In Figure 1.4, the autoregressive time series model of order 1, AR(1), with $\phi = 1.01$, is shown, which is obviously an unstable process. The process will not converge as t increases and diverges to infinity.



(a) AR(1) with $\phi = 0$

(b) AR(1) with $\phi = 0.9$

(c) AR(1) with $\phi = 1$

(d) AR(1) with $\phi = 1.01$

Figure 1: Simulated AR(1) processes with different values of ϕ.

Question 2

Since the plot of the data set does not vary a lot and diverges, therefore it is definitely not a *Random Walk* with $\phi = 1$ or a divergence with $\phi = 1.01$. Although the values on the data set are in between 4 and -4, the SACF graph shows the decay of autocorrelation decreases gradually to zero. Therefore, the value of ϕ should not be zero and Figure 1.1 with $\phi = 0$ is not considered to be suitable. As a result, Figure 1.2 is considered to be the most suitable for this data set, that is the AR(1) with $\phi = 0.9$.

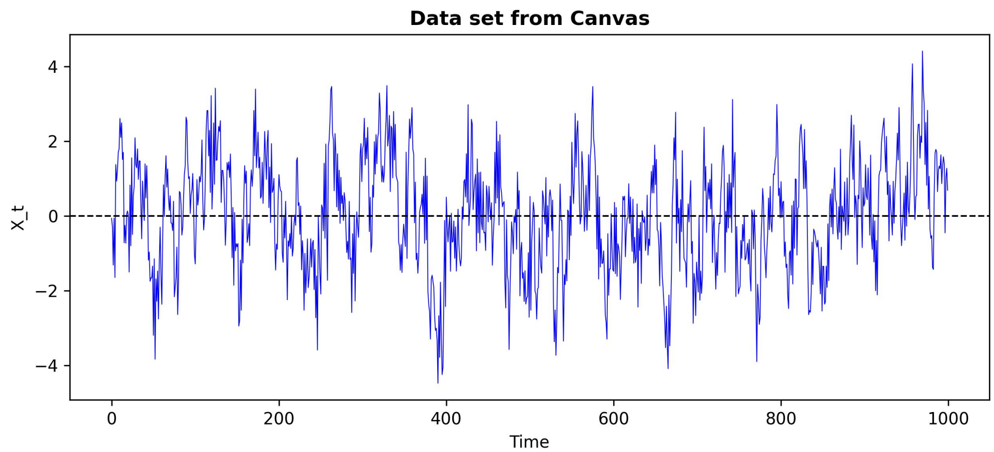


Figure 2.1 Data set from Canvas

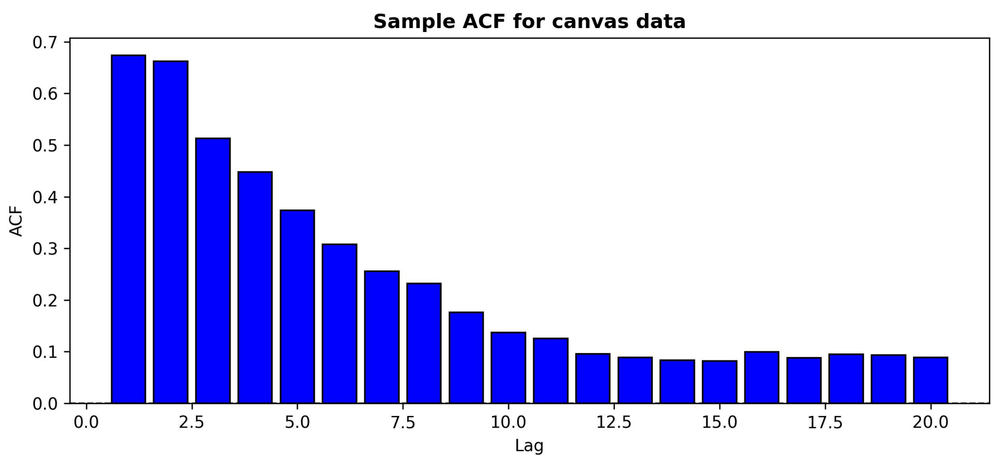


Figure 2.2 The Sample autocorrelation function up to 20 lags

Question 3

The estimate of the AR(1) parameter ϕ is obtained from the sample autocorrelation at lag 1. From SACF we have:

$$\hat{\phi} = \hat{\rho}_X(1) \approx 0.674$$

It means the AR(1) process is weakly stationary, we use this estimate value to compute the model residuals as:

$$\hat{\varepsilon}_t = x_t - \hat{\phi}x_{t-1}, \quad t = 2, ..., T.$$

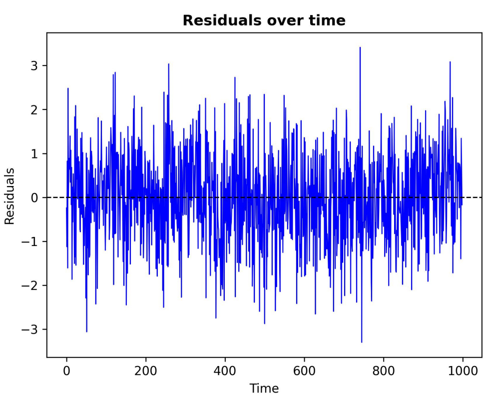
The variance of the residuals is estimated by

$$\hat{\sigma}_\varepsilon^2 \equiv \frac{1}{T-1}\sum_{t=2}^{T}(\hat{\varepsilon}_t - \hat{\mu}_{res})^2 \approx 1.188, \quad \hat{\mu}_{res} \equiv \frac{1}{T-1}\sum_{t=2}^{T}\hat{\varepsilon}_t$$

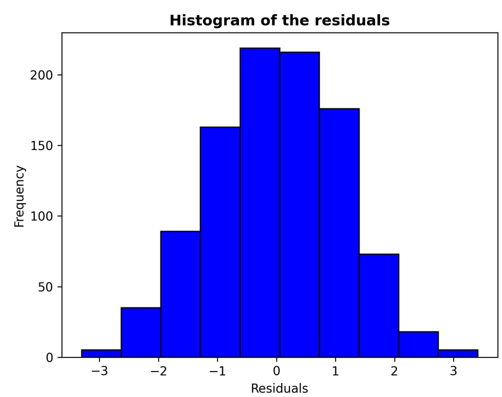
Since the value of ϕ is between -1 and 1, it suggests that the data shows moderate persistence, but not as strong as Random Walk.

Question 4

From Figure 4.1, it is clear that the residuals are randomly distributed with no clear patterns. The histogram in Figure 4.2 reveals that the residuals seem to have a symmetric, normal distribution with a mean at 0. The QQ-Plot in Figure 4.3 shows that the residuals follow a normal distribution. To conclude, the assumption of the AR(1) model that the white noise is a sequence of uncorrelated random variables with zero mean and constant variance. Based on above analysis, the AR(1) model assumption is reasonable.

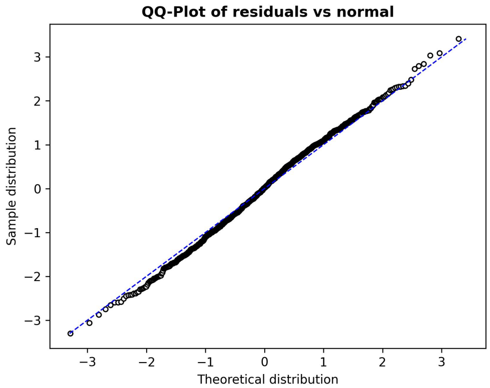


(a) The residuals over time

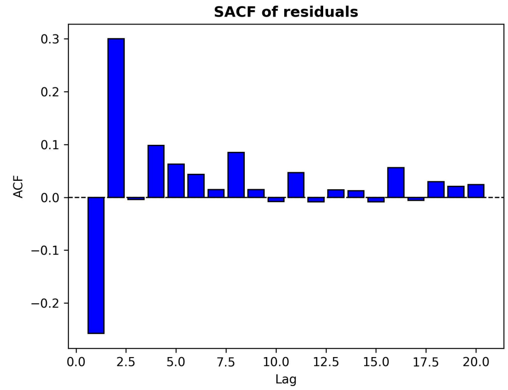


(b) The histogram of residuals

Figure 2: Residual diagnostics (time series and histogram).



(a) The QQ-Plot of residuals



(b) The SACF of residuals

Figure 3: Residual diagnostics (QQ-plot and SACF).

Part II: ARMA Models: Definitions and Stochastic Properties

Question 1

Figure 5.1 and 5.2 show two AR(2) processes with different curves in the simulated paths. For $\phi_1 = 0.4$, $\phi_2 = 0.5$, the process shows moderate fluctuation around 0, and stays within a bounded range. For $\phi_1 = 0.5$, $\phi_2 = 0.5$, the process shows larger fluctuation within a larger range, the shocks die out more quickly. This suggests that the dynamic properties are not determined by the size of the coefficients, but by the location of the roots of the AR polynomial.

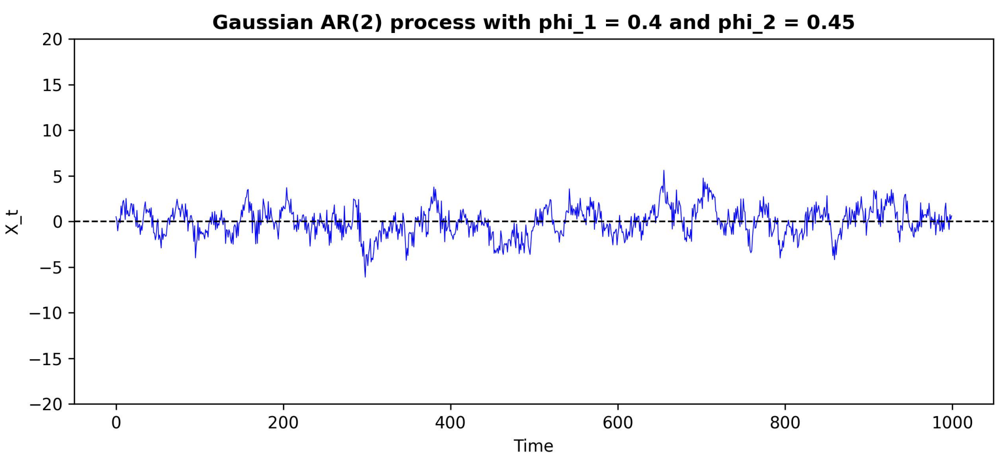


Figure 5.1 The AR(2) process with $\phi_1 = 0.4$ and $\phi_2 = 0.45$

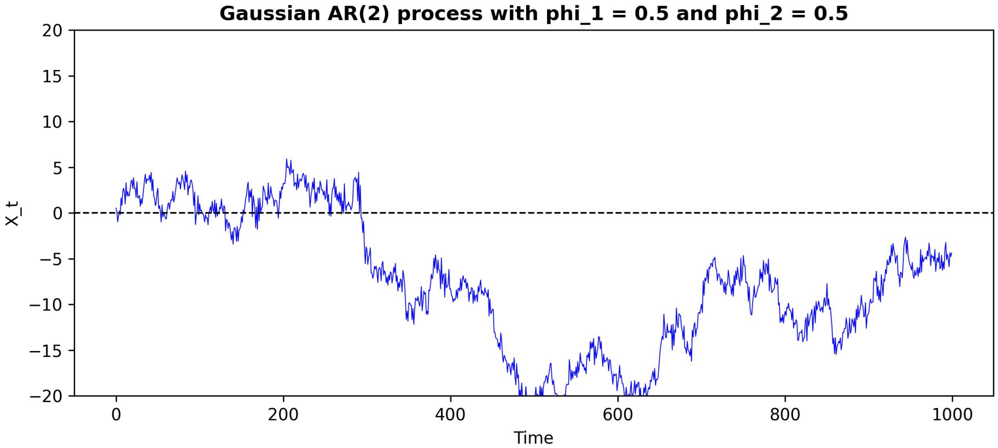


Figure 5.2 The AR(2) process with $\phi_1 = \phi_2 = 0.5$

Question 2

The autoregressive polynomial is defined by :

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2$$

To find the roots, we need to solve the equation:

$$1 - \phi_1 L - \phi_2 L^2 = 0$$

Case 1: $\phi_1 = 0.4$, $\phi_2 = 0.45$

$$L = \frac{0.4 \pm \sqrt{0.4^2 + 4 \cdot 0.45}}{2 \cdot 0.45} = \frac{-0.4 \pm \sqrt{1.96}}{0.9} = \frac{-0.4 \pm 1.4}{0.9}$$
$$L_1 = 10/9 \approx 1.111, \quad L_2 = -2$$

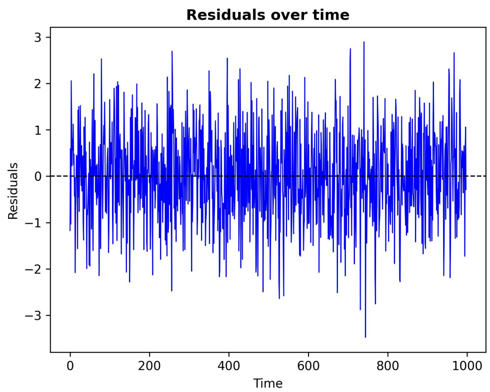
Case 2: $\phi_1 = 0.5$, $\phi_2 = 0.5$

$$L = \frac{-0.5 \pm \sqrt{0.5^2 + 4 \cdot 0.5}}{2 \cdot 0.5} = \frac{-0.5 \pm \sqrt{2.25}}{1} = -0.5 \pm 1.5$$
$$L_1 = 1, \quad L_2 = -2$$

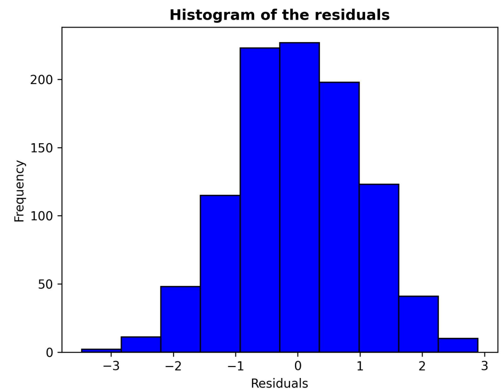
When the roots are less than or equal to 1, it indicates non-stationary or unstable behaviors, this explains why the second process $\phi_1 = 0.5, \phi_2 = 0.5$ shows a larger fluctuation, while the first process has roots slightly bigger than 1, so the first process with $\phi_1 = 0.4, \phi_2 = 0.45$ is stationary, but the growth is slower compared with the other one. It shows more stable over time. The negative root would lead to oscillatory behavior over time for both plots, as the signs alternatively change.

Question 3

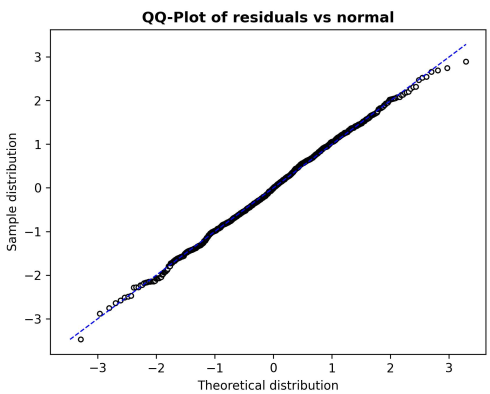
From Figure 7.1, it is clear that the residuals are randomly distributed with no clear patterns. The histogram in Figure 7.2 reveals that the residuals seem to have a symmetric, normal distribution with a mean at 0. The QQ-Plot in Figure 7.3 shows that the residuals follow a normal distribution. To conclude, the assumption of the AR(2) model that the white noise is a sequence of uncorrelated random variables with zero mean and constant variance. The SACF in Figure 7.4 illustrates the values between $[-0.06, 0.04]$, which is much smaller than the range in Figure 4.4 $[-0.3, 0.3]$. The decay of autocorrelation in the AR(2) is much faster than AR(1). Based on above analysis, the AR(2) model assumption is more reasonable than the AR(1) model.



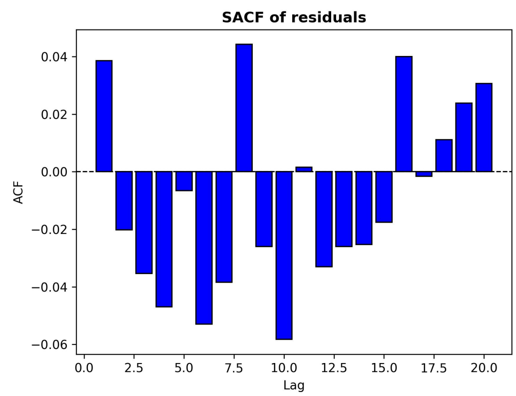
(a) The residuals over time



(b) The histogram of residuals



(c) The QQ-Plot of residuals against normal distribution



(d) The SACF of residuals up to 20 lags

Figure 7 Diagnostics of residuals for AR(2) model

Question 4

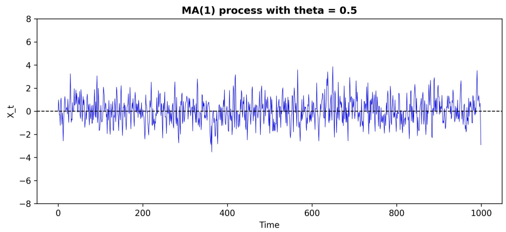


Figure 8.1 MA(1) process with $\theta_1 = 0.5$

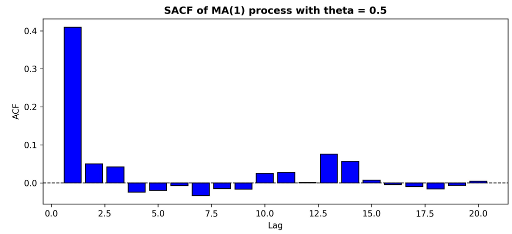


Figure 8.2 SACF of MA(1) with $\theta_1 = 0.5$

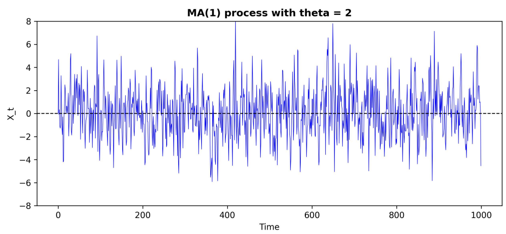


Figure 8.3 MA(1) process with $\theta_1 = 2$

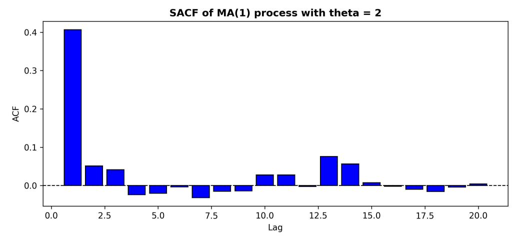


Figure 8.4 SACF of MA(1) with $\theta_1 = 2$

The MA(1) process is defined as

$$X_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad \{\varepsilon_t\} \sim NID(0, 1),$$

Variance:

$$\text{Var}(X_t) = (1 + \theta^2)\sigma_\varepsilon^2.$$

For $\theta = 2$, the variance is 5, and for $\theta = 0.5$, it is 1.25. So the $\theta = 2$ exhibits larger fluctuations.
Autocorrelation function (ACF):

$$\rho(1) = \frac{\theta}{1 + \theta^2}$$

For $\theta = 0.5 : \rho(1) = 0.4$ For $\theta = 2 : \rho(1) = 0.4$ So both their ACF have only lag 1 is nonzero (0.4), and all higher lags are zero. For time series plot, the $\theta = 2$ model has larger amplitude since it has a larger variance. For SACF plot, both have a significant spike at 0.4, and then decay very quickly for other lags, making the ACF plots look similar. This is because $\rho(1)$ of both yield to 4 as the calculation.

For both MA(1) processes with different values of θ, the processes and sample autocorrelation functions (SACF) look very similar. However, the values of X_t are different. X_t are between $[-3, 3]$ for $\theta_1 = 0.5$ while X_t are between $[-6, 6]$ for $\theta_1 = 2$.

Question 5

The ARMA(1,1) model combines the properties of AR(1) and MA(1). AR(1) model only captures persistent dependence and MA(1) model only captures short-lived shocks. ARMA(1,1) can capture both immediate responses to shocks and gradual persistence. The ACF of MA(1) cuts off after lag 1 and the AR(1) decays geometrically. The ARMA(1,1) allows the first-lag correlation decays geometrically (by AR(1) properties) together with independent adjustment (by MA(1) properties). It increases the flexibility of ARMA(1,1) in real observations.