Delay and Stochastic Differential Equations

For Laplace

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1 Stochastic Differential Equations

L1: Intro

- Stochastic systems are ones where the outcome is not deterministic, rather it has uncertainty embedded into it. The dynamics of the system has uncertainties and is known only probabilistically.
- One models a stochastic system either through a trajectory description or a probability description.
- **Constructing SDE**:
  - **Multiplicative:** Multiply dependent terms by noise term $L(t)$, e.g. $\ddot{x} = L_1(t)\dot{x} + L_2 x + h(t)$. where h(t) is a deterministic function.
  - **Additive noise:** Simply add noise term to end of ODE, e.g. $\ddot{x} = x + L(t)$.
  - **THE LANGEVIN EQUATION** (just a particular case of additive noise): $\ddot{x} = f(\dot{x}, x, t) + L(t)$
- **Noise Properties**
  - $\langle ... \rangle$ means the average over noise realisations. A realisation is particular instance of what the noise could look like over time. It distributes over all terms in an equation, like inside an integral.
  - Usually **average** $\langle L(t) \rangle = 0$ and **correlation** $\langle L(t)L(t') \rangle = 2\sigma^2 g(t-t')$, where $sigma^2$ is the **covariance noise strength** (usally the variance of a Gaussian distribution) and $g(t)$ is the **noise correlation** (basis form is $e^{-|t-t'|/\tau}$ (**Coloured Noise (Gaussian)**), where τ is the correlation time.
  - **White Noise (Gaussian too):** $\langle L(t) \rangle = 0$ and $\langle L(t)L(t') \rangle = 2D\delta(t-t')$, where D is the diffusion constant (if x is a space variable) and comes from $\sigma \to \infty$ and $\tau \to 0$. δ is the Dirac delta function.

- For the Langevin Equation write $\ddot{x}$ as $\dot{v}$ and solve first order ODE. The **average** velocity is then found by solving for v and then computing $\langle v(t)\rangle$. The **variance** (or mean square deviation) is computed as $\langle v^2(t)\rangle - \langle v(t)\rangle^2$, so you'll have to work out what v^2 is and use the correlation average.

L2: Details for solving Langevin Equations

- Note that when integrating the noise term $L(t)$ w.r.t. time it becomes $\displaystyle\int_0^t dsL(s)$.
- For basic $\dot{v} = L(t)$, $\langle L(t)\rangle = 0$, $\langle L(t)L(t')\rangle = e^{-|t-t'|/\tau}$, then use substitution $y = t' - s$ so the limits become $-s$ and $t-s$. Then split the integral into $\int_{-s}^{0} dye^{y/\tau} + \int_0^{t-s} dye^{-y/\tau}$, to get rid of the absolute value.
- **Brownian Motion:** $\dot{v} = -\gamma v(t) + L(t)$. Use Laplace transform.

Laplace Transform

$\mathcal{L}\{x(t)\} = \displaystyle\int_0^\infty dtx(t)e^{-\epsilon t} = \tilde{x}(\epsilon)$. For forward transform: $\dot{v}(t)$ becomes $\epsilon\tilde{v}(\epsilon) - v(0)$. $\ddot{v}(t)$ becomes $\epsilon^2\tilde{v}(\epsilon) - \epsilon v(0) - \dot{v}(0)$ Noise term becomes $\tilde{L}(\epsilon)$. For inverse transform:

- $\frac{1}{\epsilon+\gamma}$ becomes $e^{-\gamma t}$. For the noise term $L(t) = \int_0^t dse^{-\gamma(t-s)}L(s)$ (Because of convolution property - multiplication in laplace domain implies convolution in time domain).
- $\mathcal{L}^{-1}\left\{\frac{1}{(\epsilon+\gamma)^2}\right\} = te^{-\gamma t}$.
- $\mathcal{L}^{-1}\{1\} = \frac{1}{\epsilon}$.
- $\mathcal{L}^{-1}\{\frac{s+a}{(s+a)+b^2}\} = e^{-at}\cos(bt)$, $\mathcal{L}^{-1}\{\frac{b}{(s+a)+b^2}\} = e^{-at}\sin(bt)$

L3: Continued examples

- Aside: e^{-Real} (positive under square-root) results in an overdamped system and e^{-Im} (negative under square-root) results in an underdamped system.
- If you have a complicated double function as the noise term, just treat them together as one when Laplace transforming.
- Can use this fact: $\epsilon\tilde{f}(\epsilon) = \frac{df}{dt} + f(0)$ to find the inverse in some cases (Lecture 4).

L4: First Order Langevin Equation

- Has the form: $\boxed{\dot{x}(t) = -\gamma h(t)x(t) + L(t) \text{ with } \langle L(t)\rangle = 0 \text{ and } \langle L(t)L(t')\rangle = 2D\delta(t-t')}$ and $x(0) = x_0$ of course.

- Has the general solution: $\boxed{x(t) = x_0 G(t,0) + \int_0^t ds G(t,s)L(s), \text{ where } G(t,s) = e^{-\gamma \int_s^t dt' h(t')}}$. Note, $x_0 G(t,0)$ is the same thing you get if you solve the differential equation for $\langle x(t) \rangle$.

- Lecture also goes into solving a system of ODE's. Requires skill in integration by parts on integrals within integrals. Sub one solution into the other ODE.

- Furthermore, it goes into multiplicative cases. If possible try to divide $\dot{x}$ by x to use the substitution $y(t) = ln(x(t))$. Then solve the integral and substitute back in x. To take the assemble average $\langle x(t) \rangle$, you'll have to use the following principle: $\boxed{\langle e^z \rangle = e^{\frac{\langle z^2 \rangle}{2}}.}$

- Note that $(\int_0^t dt)^2 = \int_0^t dt \int_0^{t'} dt'$.

- If you see a tanh, cosh ODE, try using $\frac{d \sinh(y(t))}{dt} = \dot{y}(t) \cosh(y(t))$ substitution.

L5: Potentials, Wiener Processes, Ito and Stratanovich convention:

- $\frac{dW(t)}{dt} = L(t)$, where $W(t)$ is the Wiener process.

- A Wiener process satisfies the following three conditions (note, $t_2 > t_1$):
  - $\Delta W = W(t_2) - W(t_1)$. ΔW is normally distributed with mean 0 and variance $t_2 - t_1$.
  - ΔW_{t_2} and ΔW_{t-1} are independent of each other, provided $t_2 \neq t_1$.
  - $W(t)$ has a discontinuous derivative everywhere $\rightarrow L(t)$ is discontinuous.

  % $[W(t_2) - W(t_1)]^2 = t_2 - t_1$, i.e. the variance.

- From the general form: $\frac{dx}{dt} = f(x,t) + g(x,t)L(t) \rightarrow dx = f(x,t)dt + g(x,t)W(t)$. If you integrate this, you get an issue with how to integrate the discontinuous integral over $g(x,t)W(t)$. This is where the Ito $t^* = t$ and Stratonovich ($t^* = t + T/2$) conventions come in. The Stratonovich convention follows the standard, deterministic rules, of Calculus, whereas Ito's method requires a correction term for the chain rule. This is given by Ito's formula: $du = u'dx + \frac{1}{2}u''(dx)^2$.

- This is where the lecture moves on from stochastic differential equations to probabilistic differential equations. This is only possible for Markov systems, i.e. systems where the past does not influence the presence (like the Wiener process).

- **From SDE's to the Probabilistic Fokker-Plank Equation**
  - Given the equation form: $\boxed{\frac{dx}{dt} = f(x,t) + g(x,t)L(t), \langle L(t) \rangle = 0, \langle L(t)L(t') \rangle = c\delta(t-t')}$ and $x(0) = x_0$ as per usual.
  - **Stratonovich convention:**

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\left[f(x,t) + \frac{1}{2}\frac{\partial g(x,t)}{\partial x}g(x,t)\right]P(x,t)\right\} + \frac{c}{2}\frac{\partial^2}{\partial x^2}\left[g^2(x,t)P(x,t)\right]$$

  - **Ito Convention:** (for when $g(x,t) = const.$ or $g(x,t) = g(t)$)

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[f(x,t)P(x,t)\right] + \frac{c}{2}\frac{\partial^2}{\partial x^2}\left[g^2(x,t)P(x,t)\right]$$

- Aside: The Ito convention is used more in Finance as it does not require knowledge of the future. The Stratonovich convention is used more in Physics, where SDE's typically arise from continuous time processes.

L6: Reflective Boundary Condition, Steady state and Diffusion Equation

- Reflective Boundary Case: $\frac{\partial P(x,t)}{\partial x}|_{x=0} = 0$. 'If both boundaries are reflective then const. $= 0$'.Use to find the first constant of integration.
- If Steady state, set $\frac{\partial P(x,t)}{\partial t} = 0$. Then go straight to $-fP = \frac{1}{2}\frac{\partial}{\partial}[g^2 P] = const.$ (if Ito convention). This can then be arranged to $\frac{f(x)}{g^2(x)}P(x) - \frac{1}{2}\frac{d[g^2(x)P(x)}{dx} = 0$ (if const. $= 0$). Then go straight to the solution for $P(x)$, which is $P(x) = \frac{C}{g^2(x)}e^{2\int dx \frac{f(x)}{g^2(x)}}$. Constant C depends on the other boundary condition.
- Diffusion equation (obtained from $\dot{x} = L(t)$) using Ito/Stratonich convention. Solve using Fourier transform (because of space dependence).

L7: Diffusion Equation:

- In Fourier space the diffusion equation $\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2}$ becomes $\frac{d\hat{P}(k,t)}{dt} = -Dk^2\hat{P}(k,t)$. Notice that only x gets changed to k in Fourier space. This has solution $\hat{P}(k,t) = e^{-Dk^2 t}\hat{P}(k,0)$, where $\hat{P}(k,0)$ is the initial condition. Note the use of 'hat' instead of 'tilde' for Fourier.
- If $P(x,0) = \delta(x - x_0)$ then $\hat{P}(k,0) = e^{ikx_0}$. Note that $\int f(x)\delta(x - x_0)dx = f(x_0)$. Furthermore, given this initial condition, the solution becomes $P(x,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{(x-x_0)^2}{4Dt}}$ (after using convolution and Gaussian Fourier Identity and delta integral). To show that this integrates to 1, use substitution $z = \frac{x-x_0}{\sqrt{4Dt}}$ and the fact that $\int_{-\infty}^{\infty} e^{-z^2}dz = \sqrt{\pi}$.
- The n-th moment is given by $\langle x^n \rangle = \int_{-\infty}^{\infty} x^n P(x,t)dx$

Fourier Transform

$$\mathcal{F}\{f(x)\} = \hat{F}(k) = \int_{-\infty}^{\infty} dxe^{-ikx}f(x) \text{ and } \mathcal{F}^{-1}\{\hat{F}(k)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{ikx}\hat{F}(k)$$

- $\mathcal{F}\{f'(x)\} = ik\hat{f}(k), \mathcal{F}\{f''(x)\} = (ik)^2\hat{f}(k)$ and so on...
- Fourier Convolution Theorem:
  $\mathcal{F}\{\int_{-\infty}^{\infty} dx' g(x - x')h(x')\} = \mathcal{F}\{\int_{-\infty}^{\infty} dx' g(x')h(x - x')\} = \hat{G}(k)\hat{H}(k)$
- Gaussian Fourier Identity: $\boxed{\mathcal{F}\{\sqrt{\frac{\alpha}{\pi}}e^{-\alpha x^2}\} = e^{-\frac{k^2}{4\alpha}}}$
- $\mathcal{F}\{\frac{d[xf(x)]}{dx}\} = -k\frac{d\hat{f}(k)}{dk}$

L8: Method of Images, Boundary conditions

- Probability of finding a particle in interval $[0, a]$ is $\int_0^a dx P(x,t) = \frac{1}{2}[erf(\frac{-x_0}{\sqrt{4Dt}}) + erf(\frac{a-x_0}{\sqrt{4Dt}})]$, where $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x dz e^{-z^2}$.
- Special case: If $f(x,t) = f(x)$ and $g(x,t) = 1$ in $\dot{x} = f(x,t) + g(x,t)L(t)$, then you can write this equation in terms $f(x)'s$ potential $\frac{-U(x)}{dx}$. The resulting Fokker-Plank can be simplified and the steady state system for when $t \to \infty$ is rewritten in terms of $P(x, t \to \infty) = Q(x)$. If const. $= 0$, this you can solve the equation for $Q(x)$ giving $Q(x) = Ae^{\frac{-U(x)}{D}}$. To have unit area $A = 1/(\int_{-\infty}^{\infty} dx e^{\frac{-U(x)}{D}})$.
- Natural boundary conditions: $lim_{x \to \pm\infty} P(x,t) = 0$. Think bellshape I suppose.
- **Method of Images**
  - If I have understood it correctly, the method of images is a quick way of obtaining the solution of a bounded problem given the knowledge of the unbounded solution.
  - It only works when the process without the boundary (the unbounded solution) is translationally invariant around the boundary point.
  - Given an initial condition $Q(x, 0) = \delta(x - x_0)$ and a boundary condition $Q(0, t) = 0$ (absorbing), the method of images can be used to give $\boxed{Q(x,t) = P_{x_0}(x,t) - P_{-x_0}(x,t)}$, (**(-) for absorbing boundary**), where $P(x,t)$ is the unbounded solution to the problem.
  - Given initial condition $Q(x, 0) = \delta(x - x_0)$, and reflective boundary condition: $\frac{dQ}{dx}|_{x=0} = 0$. Then $\boxed{Q(x,t) = P_{x_0}(x,t) + P_{-x_0}(x,t)}$, (**(+) for reflecting boundary**)
  - Note, $Q(x,t)$ is (or should be) normalised to 1.

L9: **Ohrstein-Uhlenbeck Process** - Combines a lot of the course so likely to be examined.

- Process form: $\boxed{\dot{x} = -\gamma x + L(t)}$, $\langle, \langle L(t)L(t') \rangle = 2D\delta(t - t'), x(0) = x_0$.
- You can show that $\frac{d\langle x \rangle}{dt} = -\gamma \langle x \rangle$ and $\frac{d\langle x^2 \rangle}{dt} = -2\gamma \langle x^2 \rangle + 2D$ (using by adding/subtracting $\frac{D}{\gamma} - \frac{D}{\gamma}$ to the differential of $\langle x^2 \rangle$)
- Next, find the Fokker-Plank equation and Fourier transform it. Keep $\frac{d}{dx}[xPx]$ together so that you can Fourier transform the equation.
  - To solve the differential equation in Fourier space, use the Ansatz: $\boxed{\hat{P}(k,t) = e^{-ikA(t) - Dk^2 B(t)}}$
  - Since $P(\hat{k}, 0) = e^{-ikx_0}$, $A(0) = x_0$ and $B(0) = 0$. Plug the ansatz into the ODe and find the differential equations $A(\dot{t}) = -\gamma A(t)$ and $B(\dot{t}) = -2\gamma B(t) + 1$. Solve $B(t)$ using the first order Langevin equation from above (or laplace transform).
  - The solution in Fourier space becomes $\hat{P}(k,t) = e^{-ikx_0 e^{-\gamma t} - \frac{-Dk^2}{2\gamma}(1 - e^{-2\gamma t})}$
  - For the inversion, split $\hat{P}(k,t)$ into two functions and use convolution. The first will be $\delta(x - x_0 e^{-\gamma t})$ function, so in the convolution it becomes $\delta(x' - (x - x_0 e^{-\gamma t}))$. For the second function you need to use the Gaussian Fourier Identity.
  - $P(x, t \to 0) = \delta(x - x_0)$ and $P(x, t \to \infty) = \frac{1}{\sqrt{2\pi D/\gamma}} e^{\frac{-x^2}{2\pi D/\gamma}}$

2 Delay Differential Equations

L1: Introduction:

- DDEs are differential equations where the past exerts its influence on the present, whereas for ODEs the present determines the future.
- Time lag or delay is given by the symbol τ. There can be multiple in an equation and they can be time-dependent e.g. $x(t-\tau(t))$ or state-dependent e.g. $x(t-\tau(x(t)))$
- Classifications: *retarded type* implies present depends on past and present, i.e. $\dot{x}$, $x(t)$ and a $x(t-\tau)$ are present; *neutral type* implies past and present depend on past and present i.e. $\dot{x}$, $x(t\dot{-}\tau)$ $x(t)$ and a $x(t-\tau)$ are present;*advanced type* implies the past depends on the present, i.e. $x(t\dot{-}\tau)$ $x(t)$ and a $x(t-\tau)$ are present.
- Initial condition needs to be a function for $t \in [-\tau, 0]$ in order to solve the DDE. So we set $x(t) = \theta(t), t \in [\tau, 0]$, where $\theta(t)$ is usually just a constant. It does not matter if $\theta(t)$ actually solves the DDE.

L2: Differences to ODEs

- DDE $\dot{x}(t) = -x(t-\tau)$ has oscillatory solution $x(t) = c_1 \cos t + c_2 \sin t$.
- Smooth initial data does not imply smooth solution, there can be jumps, especially when going backwards in time.
- A DDE might have no solution or infinite solutions. Uniqueness, that ODEs have, does not apply to DDEs.
- You can show that DDEs don't have uniqueness using the three equations $\dot{x}(t) = 2y(t)$, $\dot{y} = -z(t) + x(t-1), \dot{z}(t) = 2y(t-1)$. This results in all solutions being on a plane, so any point outside this plane cannot be reached. Whereas for ODEs one can always find an initial condition (x_0, t_0) such that the system passes through point (x_1, t_1)
- Beware of Taylor expansions! They aren't always accurate.

2.1 Method of Steps

L3: Method of Steps:

- Forms the basis of all numerical methods to solve DDE's.
- Idea: Simply compute solutions for intervals. Solve for the next interval by using the information in the previous interval.
- Not particularly useful for identifying bifurcations.

2.2 D-subdivision Method

L4: D-subdivision method intro

- Recall that the stability of a system of ODEs is given by the real part of the eigenvalues, that are determined by the roots of the **characteristic equation**, $D(\lambda) = \det(\lambda I - A) = 0$, where $\dot{x}(t) = A(t)x(t)$ and I is the identity matrix.
- For DDEs, the resulting characteristic equation is not polynomial ($D(\lambda)$) but exponential ($EP(\lambda)$), which has an infinite number of roots/eigenvalues.
- Use the Ansatz $x(t) = Ce^{\lambda}t$ to find the characteristic equation.
- Since changes in behaviour of a system (bifurcation) are of interest, and these occurs when the eigenvalues cross the imaginary axis, the subdivision method finds the parameter values that cause eigenvalues to do exactly this and the direction they are travelling in when they cross this axis.
- First: set $\lambda = i\omega$ (to be on the imaginary axis) and find the **D-curves** given by $R(\omega) = 0$ and $S(\omega) = 0$, where $R(\omega) = Real(EP(i\omega))$ and $S(\omega) = Im(EP(i\omega))$, where $EP(\lambda)$ is the exponential characteristic equation.

L5: Hayes equation $\dot{x} = ax(t) + bx(t-1)$ - examining the trivial solution $x(t) = 0$.

- First perturb around the trivial solution/ equilibria: Let $x(t) = 0 + Ce^{\lambda t}$, C and λ can be complex.
- For $b = 0$, $\dot{x} = ax(t)$ so the solution is stable for $a < 0$ and unstable for $a > 0$.
- When $a = 0$, $\dot{x} = bx(t-1)$, you get the following characteristic equation: $EP(\lambda) = \lambda + b^{-\lambda} = 0$.
- There are an infinite number of eigenvalues to the left of a vertical line in the complex plane and finite number to the right. Each complex eigenvalue has a complex conjugate associated with it.
- D-curves: $R(\omega) = b\cos(\omega) = 0$, $S(\omega) = \omega + b\sin(\omega)$.
- **The D-curves are where the eigenvalues cross the imaginary axis**.

L6: Direction of Eigenvalue movement

- The direction of eigenvalue movement on the imaginary axis is determined by $Dir = Re(\frac{d\lambda}{db})|_{\gamma=0}$, where $\lambda = \gamma + i\omega$, and b is the parameter interest.
- If $Dir > 0$, then we're going from stable to unstable and if $Dir < 0$ we're going from unstable to stable.

- To find $\frac{d\lambda}{db}$ use implicit differentiation.

L7: Continuing on Hayes Equation and begin of delayed oscillators

- For eigenvalue movement, if you are considering multiple parameters then take the partial derivative.
- Lectures goes in depth into different cases for eigenvalues crossing imaginary axis.
- **Delayed Oscillators**:
  - These are of the form: $\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) = b_0x(t-\tau)$.
    * Where $b_0 = P$ is the delayed 'P (proportional) Controller'. If $P = b_0 = 0$, then you get a damped oscillator.

L8: Delayed Oscillators

- Considers the 'simplest' form of a delayed oscillator: $\ddot{x}(t) + a_0x(t) = b_0x(t-\tau)$ $(a_1 = 0)$.
- Gives amazing triangular stability parameter space diagram.
- Aside: Note, that when two eigenvalues cross the imaginary axis, as if mostly the case (except when $\omega = 0$), then the system goes through a Hopf bifurcation.

L9: Continuation of Delayed Oscillators

- At the apex of the triangles in the stability parameter diagram, where two D-curves intersect, **four** eigenvalues cross the imaginary axis - this is a double-Hopf bifurcation or Hopf-Hopf.
  - The reason for a double hopf is that two eigenvalues are crossing from stability to instability, and vice versa.
  - An invariant Torus is formed instead of a limit cycle.
- Only at the first apex, where $w = 0$ (so one eigenvalue crosses the axis) for one D-curve and where $w \neq 0$) (two eigenvalues) for another D-curve, do we get a fold-Hopf bifurcation.
- For the full delayed osciallator equation $a_0, b_0, a_1 \neq 0$, you get a similar stability diagram, but the triangles are smoothed out and the stability regions are connected.
- If a $k_d\dot{x}(t-\tau)$ term is added to the delayed oscillator, then k_d is referred to as the 'derivative gain'.
- We can use delay terms to stablilize an unstable system, such as the inverted pendulum: add derivative control. The critical value is $a_0 = -6g/l$, if its too negative, i.e. l is too small, the system cannot be stabilized.

L10: Continuation with Delayed Oscillators and Multiple Delays, Laplace Transform

- Acceleration control can also be added as a delay term $k_a\ddot{x}(t-\tau)$
- **Distributed Delay**
  - E.g. the Cushing equation $\dot{x}(t) = ax(t) + b\int_0^\sigma w(s)x(t-s)ds$, where σ is the distributed delay and $w(s)$ is the kernel (weighting), it can come in various forms:
    * $w(s) = \delta(s-\tau)$, Dirac delta corresponds to discrete delay case.
    * $w(s) = \frac{a^{n+1}}{n!}s^n e^{-\alpha s}$, the Gamma distribution $(\int_0^\infty w(s)ds = 1$. The mean or average delay is given by $\frac{n+1}{\alpha}$.
    * $w(s) = 1$ corresponds to uniformly distributed delay.
- You can also use Laplace transforms to solve DDEs.
  1. Multiply both sides of the DDE by e^{-st} and integrate from zero to infinity. Delay terms become $e^{-s\tau}X(s)$ in L-space.
  2. Rearrange and try and invert. Only works for linear problems (think so does the sub-division method too though).
- Lastly the lecture looks at some computational methods, but these are not examinable.

Method of Steps

1. Remember to sub in $t-\tau$ into the equation for the previous interval. That's because you need to evaluate the gradient at $t-\tau$ to be in the previous interval.
2. If you are going backwards in time use substitution $t' = t - \tau$ and find the gradient $\dot{x}$ in the interval of t so that you are only left with the delay terms.

D-subdivision method

1. Given a DDE $\dot{x} = f(x(t-\tau), x)$ or whatever, use the Ansatz $x(t) = a + e^{\lambda t}$, where a is an equilibrium point, and find the exponential characteristic equation $EP(\lambda) = D(\lambda)$ (interchangeably in his notes).

2. Substitute in $\lambda = i\omega$, set the characteristic equation to zero and find the real and complex parts of this new equation.

3. The D-curves are the real and imaginary part, which should now only be a function of ω (index on the imaginary axis) and parameters present in the characteristic equation: This results in $R(\omega, a) = 0$ and $S(\omega, a) = 0$.

4. Next, solve for the parameter and ω values that make this true. For each b value not equal to zero you should get two values for ω indicating that two conjugate eigenvalues are crossing the imaginary axis. Note that ω cannot be complex, it must be a real number.

5. When solving $\frac{d\gamma}{db}$ or whatever it is, to find the eigenvalue movement, note that ω and γ both might depend on parameter b, which is important for when you are trying to find the derivative. Furthermore, you can use the real and imaginary parts separately and then combine them.

Tips and tricks:

- Sometimes it might be easier to eliminate the $e^{-}\lambda\tau$ by using the characteristic equation, instead of sin/cos expanding and then finding real and imaginary parts for eigenvalue direction crossings.

- Use knowledge of previous parts of questions to simplify things.

- Taylor expansion of delay terms: $x(t-\tau) = x(t) - \tau\dot{x}(t) + \frac{1}{2}\tau^2\ddot{x}(t)... = (-1)^n \frac{1}{n!}\tau^n x^{(n)}(t)$