Nonlinear Dynamics and Chaos

Dedicated to Henri Poincare

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1 The Basics

1.1 Introduction L1

- Time evolution operators are equations that describe a change of state over time (can be shown by $\frac{dx}{dt}$, x_{n+1} or $A \to B$) (I presume).
  - If the time evolution operator **does not** depend on time then it is referred to as an **autonomous dynamical system**.
- The number of variables in dynamical system determines the systems 'dimension'.
- Dynamical equivalence occurs when one can map one system to another smoothly.
- The Feigenbaum constant (4.669) appears in all period doubling cascades.
- **Orbit** or **Trajectory** is a solution $x(t)$ of an ODE system.

1.2 Deriving ODE's L2

- From **first principle** (Temporal derivative)
  - $\dot{x} = \lim_{\tau \to 0} \dfrac{x(t+\tau) - x(t)}{\tau}$
- **Mass Action law** for when there are populations of 'agents'.
  - Conditions: Process affect agents independently and at random; number of agents is large enough to describe them as continuous variables; the agents have no 'memory' of previous events.
  - If $A \xrightarrow{r} B$ then $\dot{A} = -rA$ and $\dot{B} = rA$
  - If $A + B \xrightarrow{r} C$ then $\dot{A} = \dot{B} = -rAB$ and $C = -rAB$
  - $2A + 3B \xrightarrow{r} A + C$ then multiply reactions to get 'functional form' A^2B^3 then compute net gain for all agents to get $\dot{A} = -rA^2B^3$, $\dot{B} = -3rA^2B^3$ and $\dot{C} = rA^2B^3$
  - For a system just add the individual solutions of the reactions together.

1.3 Existance and Uniqueness L3

- Picard-Lindelöf (Cauchy-Lipschitz) theorem, given $\dot{x} = f(x,t)$.
  - If $\dot{x} = f(x,t)$ is continuous in time t and $\frac{\partial \dot{x}}{\partial x} = f'(x,t)$ is continuous (Lipschitz condition) within an interval, then a solution to the initial value problem exists and the solution is unique in that interval.
  - Important: Further implications of this is that *trajectories* are **dense** in the system and **do not branch or cross**

2 Stability Analysis and Bifurcations

2.1 Linear Stability Analysis L4

- Conservation law - higher degree systems can sometimes be simplified to a lower degree system if there is a conserved quantity.
- Long term behaviour is the behaviour of a system after infinite time. 1-dimensional ODE's can only have one type of long term behaviour, called 'stationary states' (off to infinity, oscillate or converge)
- **Steady state -** the variables in a system stop changing and the system remains in that state forever.
  - $\boxed{\dot{x} = f(x) = 0}$
  - $x^* = 0$ is considered the 'trivial steady state'.
- **Linear Stability Analysis**
  - (Uses **local asymptotic stability**) $\rightarrow x = x^* + \delta$ (perturbation) then $\dot{x} = \dot{\delta} = f(x^* + \delta) = f(x^*) + f'(x^*)\delta + O(\delta^2) = f'(x^*)\delta$, if $x^* = 0$, this ode $\dot{\delta} = f'(x^*)\delta$ implies the following bit of dodge proof tbh:
  - $\boxed{f'(x^*) < 0 \rightarrow \text{steady state stable (attractor)}}$ and $\boxed{f'(x^*) > 0 \rightarrow \text{steady state unstable (repellor)}}$
  - In a bifurcation diagram solid lines represent stable **branches** and dotted unstable.
- **Basin of Attraction** is the set of all *initial states* from which trajectories approach the attractor. If basin of attraction is the whole state space (all possible values of the variables) then the system is **globally stable**.
- **Basin stability** the size of the maximum perturbations after which the system will return to the attractor.

2.2 Solving maps (multi-dim) and stability L5

- The solution of a 1-dim linear map of form $x_{i+1} = ax_i$ is $x_i = a^i x_0$.
- The solution of a n-dim linear map $\mathbf{x}_{i+1} = \mathbf{J}\mathbf{x}_i$ is $\mathbf{x}_i = \mathbf{J}^i\mathbf{x}_0 = \sum_n c_n(\lambda)^i \mathbf{v}_n$, where $\mathbf{v}_n$ and λ_n are the n-th eigenvector and eigenvalues and c_n are scalar constants such that $\sum_n c_n\mathbf{v}_n = \mathbf{x}_0$ (separationsansatz)
  - $\mathbf{J}$ is considered a **propagator** that moves the system in time.
- From problem sheet: Stability criterion for fixed points of maps ($x_i = f(x_i)$): The eigenvalues of the Jacobian must satisfy the condition $|\lambda| < 1$

[**Nullclines -** these are just the solution curves to $\dot{x_i} = 0$, given a system of differential equations. When you are finding the fixed point of a system of ODE's you are finding the intersectioin of these nullclines, e.g. $\dot{x} = f(x) = x^2 + y = 0$, $f(y) = \dot{y} = -x = 0$, the nullclines are $f(x) = 0$ and $f(y) = 0$ and the fixed point are at $f(x) = f(y) = 0$].

2.3 Solving linear ODE's (multi-dim) and stability L6

- The solution of an n-dim linear ODE $\dot{\mathbf{x}} = \mathbf{J}\mathbf{x}$ is $\mathbf{x}(t) = \sum_n c_n e^{\lambda_n t}\mathbf{v}_n$, where c_n can be found from $\sum_n c_n\mathbf{v}_n = \mathbf{x}(0)$
- **Stability**
  - From Taylor expansion of perturbation again we get $\dot{\delta} = \mathbf{J}\delta$
  - If **ALL** eigenvalues of the Jacobian are Re(**negative**) then the steady steady state is stable. Otherwise it is unstable.

- If eigenvalues are complex then this implies oscillations are present in the solution (always a pair). The imaginary part determines the frequency of the oscillations. Real part determines the approach/departure from the steady state. In the long run the eigenvalue with the largest real part wins. (L7 - plane example)

2.4 Bifurcations L8

- Classification of steady states (2D)
  - 2 Re(neg). Eig. $\rightarrow$ **node**, 2 Re(pos). Eig. $\rightarrow$ **anti-node**
  - 1 Re(neg), 1 Re(pos) Eig. $\rightarrow$ **saddle**
  - 2 complex Re(neg) Eig. $\rightarrow$ **focus**, 2 complex Re(pos) Eig. $\rightarrow$ **anti-focus**
  - Spirals and nodes are considered *qualitatively* (topologically) the same.
- Bifurcations
  - A bifurcation point is a **parameter value** where there is a *qualitative* change in the systems dynamics (like suddenly oscillating etc...). Essentially it tells us that if we use this particular parameter value the system does something different.
  - Changes to the phase portrait are called bifurcations.
  - Bifurcations are **local** if they are confined close to the steady state.
  - **codimension - n** means we can find the bifurcation by tuning exactly n parameters. Local codimension-1 are the simplest bifurctions.

Fold bifurcation

- Normal form: $\dot{x} = a \pm x^2$
- Steady states: $x^*_{1,2} = \pm\sqrt{a}$
- Jacobian eigenvalue: $\lambda = f'(x^*) = \pm 2x^* = \pm 2\sqrt{a}$. Note, $\lambda_1 = 2\sqrt{a} = \lambda_2 = -2\sqrt{a}$ only when $a = 0$, which is where the two steady states collide and annihilate (the place where the eigenvalues meet).
- Bifurcation at $a = 0$
- It occurs when a saddle collides with a node or anti-node.
- If two steady states collide and both have an eigenvalue of opposite sign then they will collide and annihilate to form a fold bifurcation.

Transcritical bifurcation

- Normal form: $\dot{x} = ax \pm x^2$
- Steady states: $x^*_1 = 0, x^*_2 = a$
- Jacobian eigenvalue: $\lambda = a \pm 2x^*$. Note, $\lambda_1 = a = \lambda_2 = a - 2a = -a$ only when $a = 0$. The eigenvalues meet at $a = 0$ and then each eigenvalue traces out the path of the other. In other words, two steady states (stationary states) exchange their stability.
- Bifurcation at $a = 0$

Pitchfork bifurcation

- Normal form: Subcritical $\dot{x} = ax + x^3$, Supercritical $\dot{x} = ax - x^3$. Notice that something like $f(x) = \dot{x} = x \pm \sin x$ will also have a local pitchfork bifurcation because its Taylor expansion has a cubic.

- If we add an x^5 term to the normal form, i.e. $f(x) = ax - x^3 + x^5$, we get a pitchfork bifurcation and two saddle-node (fold) bifurcations, which together create a jump bifurcation, i.e. hysteresis.

- Fold and Transcritical bifurcations can only occur when an eigenvalue becomes zero (when changing the parameter).
- In higher dimensions to see if one of the eigenvalues of a steady state becomes zero one can use the fact that $|J| = det(J) = \lambda_1\lambda_2..\lambda_n$ will become zero.

2.5 Normal Forms L9

- Two dynamical systems (say $\dot{x} = g(x)$ and $\dot{y} = h(y)$ are considered **topologically equivalent** if a mapping exists such that $x = v(y)$ and $y = w(x)$ and u, v are continuous functions that do not 'invert time'.
- Two systems are **locally topologically equivalent** if they are topologically equivalent in the same neighbourhood.

A fold bifurcation requires all of the following conditions. We can use them to prove that a fold bifurcation exists:

Steady State condition: $f(x^*, a^*) = 0$

Bifurcation Condition: $f'(x^*, a^*) = f_x(x^*, a^*) = 0$.

Tranversality condition: $f_a(x^*, a^*) \neq 0$

Nondegeneracy condition: $f_{xx}(x^*, a^*) \neq 0$

A transcritical bifurcation requires the following conditions:

Steady State condition: $f(x^*, a^*) = 0$

Bifurcation Condition 1: $f'(x^*, a^*) = f_x(x^*, a^*) = 0$.

Bifurcation Condition 2: $f'(x^*, a^*) = f_a(x^*, a^*) = 0$.

Bifurcation Condition 3: $f'(x^*, a^*) = f_{aa}(x^*, a^*) = 0$.

Nondegeneracy condition: $f_{xx}(x^*, a^*) \neq 0$

Tranversality condition: $f_{ax}(x^*, a^*) \neq 0$

Hopf Bifurcation (2-dim system)

- When a complex pair of eigenvalues crosses the imaginary axis a Hopf bifurcation occurs.
- Normal norm: $\dot{x} = ax - y \pm (x^2 + y^2)x$, $\dot{y} = ay + x \pm (x^2 + y^2)y$
- In normal polar form $\dot{r} = r(a \pm r^2)$ (like transcritical form exept with cubic), $\dot{\phi} = 1$. **Supercritical Hopf** is of the form $\dot{r} = r(a - r^2)$ and **Subcritical Hopf** is of the form $\dot{r} = r(a + r^2)$
- Steady state at $x = y = 0$
- Bifurcation at $a = 0$
- Identify a Hopf bifurcation by calculating the eigenvalues of the Jacobian and then setting the real part to zero and seeing whether there exists a complex part (i.e. the root is negative). This means there are complex eigenvalues crossing the real axis, which implies Hopf.
- If the Lyapunov coefficient is negative then this implies a supercritical Hopf.

- For a phase diagram for a supercritical Hopf, for parameter values smaller than the parameter value at which the Hopf bifurcation occurs (i.e. a limit cycle is formed) its a spiral inwards, beyond the parameter value, the phase diagram has a limit cycle and spiral going to it and outside of it going outwards.

3 Manifolds, Coarse-graining, Higher Co-dimension Bifurcations

3.1 Manifolds, Separatrices L10

- **Phase Space**: The set of all possible values that the variables can take. If there are n variables the phase space is n-dimensional.
- **Parameter space**: The set of all possible values that the parameters can take. n parameters corresponds to an n-dimensional parameter space.
- **Time Evolution operator** ϕ_t moves a state forward in time.
- **Invariant set**: A set that does not change in time: $I = \phi_t I$. Examples are stationary states, cycles, stable manifolds, etc... A **Closed Invariant Set** is a trajectory that remains close to the invariant set I think.
- **Manifold**; A set that is locally smooth. A stable manifold corresponds to curve given by the eigenvector going towards a steady state. The opposite applies for an unstable manifold.
  - Dimensionality: The stable manifold has a dimensionality equal to the number of negative real part eigenvalues and the unstable manifold has a dimensionality equal to the number of positive real part eigenvalues.
  - If there is a zero eigenvalue then there exists a so-called "center" manifold. (Occurs at the bifurcation point in a fold bifurcation.)
- **Separatrices** are stables manifolds that separate different basins of attraction, i.e. the "flow" behaves differently when you cross a stable manifold that is a **separatrix**.

3.2 Coarse-graining

- A method of model reduction, in particular for slow-fast systems.
- Identify the slow system as the one that is being multiplied by a parameter and the fast system as the one without a parameter. Set the parameter to zero and find what steady state the system now approaches to. Call this the **slow manifold**. Not entirely sure about the rest, do a few examples.

Review the following with handwritten notes

3.3 Homoclinic and Heteroclinic bifurcations L14

- The concept of **genericity** (being general): The principle of generalising similarly behaving systems of ODE's, and then using this genericity to classify them. For example, a zero eigenvalue 'generally' (generically) indicates a fold bifurcation, unless higher derivatives indicate a transcritical or pitchfork bifurcation. The **degenerate** cases are exceptions to the generic cases.
- The **centre manifold** is the manifold in which the changes associated with the bifurcation occurs. Apparently it is tangential to the space formed by the eigenvectors and eigenvalues caused by the bifurcation, but don't understand this.
- **Some Rules of Thumb**
  - Smooth parameter changes implies the dynamics changes smoothly too.

  - Bifurcation usually cause local changes not global changes (however Homoclinic and Heteroclinic bifurcations are not local).
  - If manifolds collide they have to match up correctly.

- **Homoclinic Bifurcation**
  - A homoclinic bifurcation occurs when a cycle collides with a saddle and they form a **homoclinic** connection briefly. A homoclinic connection is where one of the unstable manifolds moves round and connects back to the stable manifold of a saddle to form a single looped manifold.

- **Heteroclinic Bifurcation**
  - When two opposite saddle's meet/collide and briefly form what I assume is a heteroclinic connection, whereby the unstable manifold becomes the stable manifold of the other saddle.

3.4 Bifurcations of higher codimension (more parameters). L15

- Essentially, bifurcations of lower variable dimension and parameter dimension will be exactly the same for higher dimensional variable and parameter space.
- Bifurcations of higher codimension (higher parameter dimension needed for bifurcation conditions) are essentially bifurcations of bifurcations. Think of changing one parameter and then another at a bifurcation point.
- Recall that codimension is the number of parameters needed to find the bifurcation. If there are two parameters that can be formulated as one, then its still just a 1-codimension bifurcation.

Cusp bifurcation

- Normal form: $\dot{x} = a + bx \pm x^3$.
- There are two folds along $x = \sqrt{b/3}$ and $x = -\sqrt{b/3}$
- The equation of the curve that describes all the points of a and b on which the fold bifurcation lies is $27a^2 + 10a^3$ or something like that. Its the equation of a semi-cubic parabola (looks live a "v")
- A cusp bifurcation point/ cusp singularity occurs at the origin.
- A cusp bifurcation implies **hysteresis** occurs, which means there will be a "catastrophic jump" to a different stable state. This is due to a region of **bistability**.
- **Generically** a cusp bifurcation **is identifiable when two fold bifurcations collide and annihilate eachother**.

Takens-Bogdanov Bifurcation

- Occurs when a Hopf bifurcation ends tangentially on a fold bifurcations (more than one is also implied here).
- A homoclinic bifurcation is always present in a Takens-Bogdanov bifurcation, so having a TB bifurcation 'proves' that the homoclinic bifurcation exists.
- Apparently marked by having two double zero eigenvalues of the Jacobian.

- Other bifurcations, such as the Gavrilov-Guckenheimer bifurcation, Double Hopf, Bautin bifurcations are only briefly mentioned so won't go into detail unless I see a question on it.

4 Bifurcation of maps and cycles

4.1 Stoboscopic maps and Poincare maps L16

Both systems are used for orbits that are periodic.

- Stroboscopic map (takes slices through time) (also a particular kind of Poincare map)
  - Given the solutions to an ODE system, the solution can be expressed in terms of a map, which essentially just plots the solution at a certain interval. This is very useful to identify oscillations in a solution. E.g. $\dot{x} = ax \to x(t) = x_0 e^{at} \to x(t+1) = x_0 e^{a(t+1)} = e^a e^{at} = x_0 e^a x(t) \to x_{i+1} = x_0 e^a x_i$. a is a Lyapunov exponent. A factor τ can be added to vary the time step, e.g. $x_{i+1} = e^{a\tau} x_i$.
  - A non-autonomous system, i.e. $\dot{x} = f(x,t)$ can turned into a autonomous system by making t a space variable, i.e. now $\dot{x} = f(x), \dot{t} = 1$, which makes every slice in 'time' the same as a slice in space so a Stroboscopic map is in this case just a Poincare map.
- Poincare Map (takes a slice through space)
  - A Poincare section is an $N-1$ dimensional manifold (surface/line etc.) in the solution space of a N dimensional system. So if we have a 3-dim system like Lorentz attractor, the the Poincare section is a plane.
  - Loops/oscillations (limit cycles) appear as fixed points on a Poincare section. Tori show up as cycles.
  - The Poincare map of a Rössler attractor can be approximated by a logistic map (apparently)
  - We can use the Poincare map when system slightly changes and it is easier to model it as a change in the map then it is to develop new system of ODE's. We can then work out steady states, for example, by setting $x_i = f(x_i)$.

4.2 Bifurcations of Maps

- Essentially, if $|\lambda| < 1$ then the steady state is stable for a particular value, if $|\lambda| > 1$ then its unstable, but if its $|\lambda| = 1$ then we have the following three cases:
- **Fold bifurcation** $\to f'(x) = \lambda = 1$ (passes through postive real part of unit circle)
  - In a Poincare map it shows to limit cycles (one stable, one unstable) colliding to form a spiral.
- **Neimark-sacker Bifurcation** $\to f'(x) = \lambda_{1,2} = e^{\pm} j\theta = \cos\theta + j\sin\theta$ (crosses through complex part of unit circle)
  - In a Poincare map, a torus is born around an unstable cycle
- **Flip bifurcation** $\to f'(x) = \lambda = -1$ (negative real part of unit circle)
  - Alternates between two stable fixed points on the Poincare map.
- Beware that a transcritical bifurcation can also occur if you have two eigenvalues going through $\lambda = 1$ but in two different directions.

5 Chaos

5.1 Transition to Chaos L18

- **Dissipative** systems have fewer *conservation laws* than variables. This means that orbits with different initial conditions move closer together in time.
- **Conservative** systems have the same number of *conservation laws* as variables.
- **Properties of chaotic attractors** by Robert L. Davaney: Need own notes here!

- Sensitive to initial conditions
- Topological mixing
- Dense periodic orbits
- There are an infinite number of periodic and non-periodic orbits
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- **Strange attractors** are non-smooth geometrical fractals.
- **Butterfly effect** example of a choatic system.
- **Identifying Chaos in Attractors using Lyapunov exponents.**
  - Eigenvalues of the Jacobian matrix are so-called **local Lyapunov exponents**. They determine how infinitesimally close trajectories behave. $\lambda > 0 \to$ trajectories diverge, $\lambda < 0 \to$ trajectories converge, $\lambda = 0$ trajectories stay at a constant distance.
  - **Global Lyapunov exponents** are found by integrating the local Lyapunov exponents along the attractor.

Type	Number of $\mathrm{Re}(\lambda) > 0$	Number of $\mathrm{Re}(\lambda) = 0$	Number of $\mathrm{Re}(\lambda) < 0$	Result
Steady state	0	0	n	n steady states
Limit cycle	0	1	$n-1$	a limit cycle is formed
2-dim torus	0	2	$n-2$	A 2-dim torus is formed
3-dim torus	0	3	$n-3$	A 3-dim torus is formed
chaos	$m \geq 1$	1	$n-1-m$	We have chaos

  - The largest eigenvalue (magnitude) must be negative real for us to be considering attractors.

- Hopf-Landau Idea - the transition to chaos happens through an infinite series of Hopf-like bifurcations.
- **Shilnikov Chaos**: A homoclinic bifurcation leads to the formation of a chaotic attractor if we have a *saddle-focus* (pos. real. eigv. (λ_1) + two neg. real complex. eigv. pair $(\lambda_{2,3})$ and the ratio (**saddle index** υ) between the real components of neg. complex eigenvalues and positive eigenvalue is less than one, i.e. $\upsilon = \frac{|Re(\lambda_{2,3})|}{Re(\lambda_1)} < 1$. I suppose it just shows that the positive eigenvalue is more 'dominant' resulting in more chaotic behaviour.

5.2 Period doubling cascade L19

- Essentially this lecture just shows that after 3 period doublings of the logistic map the system becomes chaotic.
- Every time a bifurcation occurs and the fixed points double, the distance between these bifurcations decreases by Feigenbaum's constant 4.669

5.3 Symbolic Dynamics L20

- Piecewise defined maps
  - Tent map $\to x_{n+1} = \begin{cases} 2x_n & 0 \leq x_n \leq 0.5 \\ 2 - 2x_n & 0.5 \leq x \leq 1 \end{cases}$
  - Doubling map $\to x_{n+1} \begin{cases} 2x_n & 0 \leq x_n \leq 0.5 \\ 2x_n - 1 & 0.5 \leq x \leq 1 \end{cases}$
- **Symbolic Labels** you divide the map in two and label all $x_n < 0.5$ (in this case) as 0 and $x \geq 0$ as 1.

- **Itinerary**: A recorded 'journey' of the symbol sequence for an initial condition (and parameter, I presume), i.e. you record the the values the iterated map generates and set them to 1 or 0 depending on where they land in the specified interval e.g. $a = S_D(x_0) = S_D(0.1) = \{x_0, x_1, x_2, ..\} = \{0.1, 0.2, 0.7, 0.3, 0.6, etc...\} = .00101...$ Need notes for examples

- **Itinerary evolution** using a shift operator $\sigma(a)$ just means you can move along the itinerary, i.e. move dot right and neglect everything to the left. E.g. $a = .011$ at time n then $\sigma(a) = .11$ at time $n + 1$.

- You can use symbolic dynamics to prove that the piecewise maps above are chaotic, but won't go into this here unless you see a question on it.

Lyapunov functions L21

- We can use Lyapunov functions to prove **global** stability of the system. No longer to do with Chaos. We have only looked at small pertubations (using Taylor expansion) to show local stability sofar.

- **If there exists a Lyapunov function then there is global stability**
  - $S : \mathbb{R}^n \to \mathbb{R}$ is a Lyaponov function if:
    * S is continously differentiable.
    * $S(x^*) = 0$, i.e. zero at the steady state.
    * $S(x) > 0 \quad \forall x \neq x^*$
    * $\dot{S}(x) < 0 \quad \forall x \neq x^*$