Continuum Mathematics

Dedicated to Augustin-Louis Cauchy

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1 Solid Mechanics

Basics:

- **Strain**: The deformation of a body - in its simplest form $\epsilon = \frac{\Delta L}{L}$
- **Stress**: Forces that cause strain (force per cross-sectional area) - in its simplest form $\sigma = F/A$.
- Prismatic bar - uniform cross-section. Isotropic - same material properties in all directions.
- Material constants - Young's modulus ($E = \frac{\sigma}{\epsilon} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$), Poisson ratio ($v = \frac{\lambda}{2(\lambda+\mu)}$, Bulk modulus ($K = \lambda + \frac{2}{3}\mu$), shear modulus ($G = \mu$). Any two material constants determine the stress-strain relationship (because they define the Lame constants λ, μ).
- Hooke's law: $\sigma = \epsilon * E$ ($F = kx$)

1.1 Strain

- Derive using a deformation $\mathbf{u}(\mathbf{x}_a)$, i.e. $\delta\mathbf{y}_i = \delta\mathbf{x}_i + \mathbf{u}(\mathbf{x}_i)$. Then use a taylor expansion and rearrange to get $\delta\mathbf{y} \approx \delta\mathbf{x} + \delta\mathbf{x}\nabla\mathbf{u}(\mathbf{x}_i = (I+M)\delta\mathbf{x}$, where I is the identity and M is the Jacobian of $\mathbf{u}$. We only care about the amount of deformation so we need $|\delta\mathbf{y}|^2 - |\delta\mathbf{x}|^2 = \delta\mathbf{x}^T 2\boldsymbol{\epsilon}\delta\mathbf{x}$, where the **strain matrix/tensor** $\boldsymbol{\epsilon} = \frac{1}{2}(M^T + M + M^T M) \to \epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{3}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j})$.
- If we assume the deformations **u are small**, then we get the **Engineering strain** $e_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \approx \epsilon_{ij}$.
- The strain tensor does not change under translation or rotation of a rigid body (makes sense, if you move or rotate a deformed body it remains deformed in the same way). A tensor, generally, is invariant under non-intertial coordinate transformations - hence its called a strain tensor not a strain matrix. The strain tensore is also symmetric, i.e. $\boldsymbol{\epsilon}^T = \boldsymbol{\epsilon}$.
- **Principle strains** are the eigenvalues of the strain tensor. They are the largest *normal* strains you can find in the material.
- **Principle directions** of strain are the directions of the largest strains in the body and are found by finding the eigenvectors of the strain tensor.**The eigenvectors of the strain tensor are the same as those of the stress tensor** (i.e. principle directions are the same - makes sense, a body is going to deform most in the direction that it is stretched in).

1.2 Stress

- A strained (deformed) body is under stress and any slice in the body will have *Normal stresses* (stress forces normal ($\mathbf{n}$) (perpendicular) to cutting plane) and *Shear stress* (stress forces in the cutting plane). Thus we define a stress tensor (Cauchy stress tensor) $\boldsymbol{\sigma}$, so we can represent the forces as $\mathbf{f} = \boldsymbol{\sigma}A\mathbf{n}$, where the diagonals components of $\boldsymbol{\sigma}$ represent the normal stresses and the off-diagonals the shear stresses. We now wish to relate this to the strain tensor:

- Extending the linear relationship observed by Hooke's law, and *assuming the material is isotropic* then we can express the strain tensor as $\boldsymbol{\sigma} = 2\mu\boldsymbol{\epsilon} + \lambda Tr(\boldsymbol{\epsilon})I$ or $\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\sum_{k=1}^{3}\epsilon_{kk}$, where μ, λ are **Lamé Constants**, δ_{ij} is the Kronecker delta.

- **Principle stresses** are the eigenvalues of the stress tensor (call them s_1, s_2, s_3 - note these are normal stresses, not shear stresses) and **principle directions** are the eigenvectors.

1.3 Mohr's Circle

- Mohr's Circle relates normal stress to shear stress. We can use it to work out the maximum shear stress. Assuming that $s_1 < s_2 < s_3$ then:
  - In 2D max shear stress is $\frac{s_2 - s_1}{2}$
  - In 3D max shear stress is $\frac{s_3 - s_1}{2}$
  - I presume you can also find the max shear stress between two eigenvectors, not just the largest and the smallest...

1.4 Torsion

- Analagous to shear stress: Combine moment with stress tensor and a twist force matrix... don't need to know for exam apparently.

- It's useful stuff for power transmission - when you need rotating energy sources.

1.5 Beam Theory

- The longitudinal (horizontal) behaviour of a thin beam with small displacements can be modelled using the wave equation $Cu_{xx} = u_{tt}$, where x goes sideways along the beam and u is the longitudinal displacement of the beam.

- For **Transverse (Vertical) behaviour** we get the famous Euler-Bernulli Beam equation:

$$Tw_{xx} - EIw_{xxxx} - \rho g = \rho w_{tt},$$

  where w is the transverse (vertical) displacement, w_x is the angle formed between beam and vertical displacement, $-EIw_{xx} = M$ is the moment, $-IEw_{xxx} = N$ is the shear, E is Young's modulus, I is the area moment of inertia, T is the tensile force (max pulling stress before beam breaks), x is the horizontal displacement and t is time.

- Boundary conditions:
  - Clamped end at e.g. $x = 0 \rightarrow$ no vertical displacement and no angle $\rightarrow w(0,t) = 0, w_x(0,t) = 0$.
  - Pinned end at e.g. $x = 0 \rightarrow$ no vertical displacement, no moments $\rightarrow w(0,t) = 0, w_{xx}(0,t) = 0$
  - Free end at e.g. $x = L \rightarrow$ no moments, no shear $\rightarrow w_{xx}(L,t) = 0, w_{xxx} = 0$. If there is a point force P on the free end the shear gets set equal to it, i.e. $-EIw_{xxx} = -P$

- It is common for $T \approx 0$ (breaks almost immediately when getting pulled - like wood , I presume) $\rightarrow$ can use separation of variables to solve the resulting PDE, you'll get a mixture of $sins, cos, cosh, sinh$ in your solution for w.

- For the static case, just ignore the w_{tt} term. If $T = 0$ we just get $-EIw_{xxxx} +$ [extra forces] $= 0$. The extra forces are either $-\rho g$ (force caused by bar under its own weight) or $-P\delta(x - L)$, where L here is **not** the end of the bar - otherwise we'd use it in the boundary condition.

- Note the area moment of inertia $I(x)$ is a function of x and so can change the thickness of the beam changes. You'll need *compatibility conditions (CC)* if this is the case. CC's tell us where w, w_x, w_{xx}, w_{xxx} continous from left to right, i.e. e.g. $w(L/2)_{left} = w(L/2)_{right}$

- Properties of the dirac delta δ: 1) $\int \delta(x-L)dx = H(x)$, $\int H(x-L)dx = (x-L)H(x-L)$, $\int(x-L)H(x-L)dx = \frac{(x-L)^2}{2}H(x-L)$ and so on...

- Sign conventions: x always goes from left to right, if there is point load at $x = 0$ (free end) then P is positive going down.

- If you have two clamped ends and point load in the middle, split the problem in half and use bending moment of half the point load e.g. $xP/2$ instead of [extra forces], e.e. beam equation become $-EIw_{xxxx} + xP/2 = 0$ and BC's are $w(0) = 0, w_x(L/2) = 0(?)$ $-EIw_{xxx} = -P/2$

2 Complex Variables

Fundamentals:

- $i = \sqrt{-1}$, Conjugate: $i^* = -i$, so e.g. $\cos{(a+ib)}^* = \cos{(a-ib)}$, Forms: $z = x + iy = re^{i\theta} = r(\cos\theta + \sin\theta)$, Modulus: $|z| = \sqrt{zz^*} = \sqrt{a^2+b^2}$

- Principle Argument **A**rg(z) is the first argument angle, e.g. $Arg(-1) = \pi$, while $arg(-1) = \pi, 3\pi, 5\pi, ...$ So $arg(z) = Arg(z) + 2\pi n$

- Euler's formula: $e^{ix} = \cos x + i\sin x$, thus $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$. i inscribes wave behaviour!

- de Moivre's theorem: $(e^x)^n = \cos nx + i\sin nx$

- Without i we get exponential decay/growth: $e^{ix} = \cos x + \sinh x \to \cos x = \frac{e^x+e^{-x}}{2}$ and $\sin x = \frac{e^x-e^{-x}}{2}$

- **Taylor series**: $f(z)_{z=c} = \sum_{n=1}^{\infty}(z-c)^n \frac{1}{n!}\frac{d^n f(z)}{dz^n}|_{z=c}$

- **Transcendental equations** $\to \infty$ roots.

2.1 Fluid Mechanics

Idea: We wish to describe the flow of irrotational ($\nabla \times \mathbf{v} = 0$) and incompressible ($\nabla \cdot \mathbf{v} = 0$) fluid around an object, using not a velocity vector field but rather continuous functions (streamlines). To do this we need to find the potential field using a complex mapping, or if we have the potential field, we can find the streamlines.

- **Flow velocity**: $\mathbf{v} = (u, v) = u + iv$

- **Complex Potential** $f(x+iy) = \psi(x,y) + i\phi(x,y)$, where ψ is the **velocity potential** ($\phi = const.$ are equi-potentials) and ϕ is the **stream function** where $\phi = Im(f) = const.$ are **streamlines**. Because ψ is the velocity potential of $\mathbf{v}$ then $\mathbf{v} = \nabla\psi$.
  - Thus $\mathbf{v} = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\right) = \nabla\psi$
  - Since the fluid is incompressible, this implies that $\nabla \cdot \mathbf{v} = 0$ which implies that $\nabla^2\psi = 0$ (satisfies Laplace's equation)
  - Since the mapping $g(z) = f$ from $z = x + iy$ to $f = \psi(x,y) + \text{‘}\phi(x,y)$ is **conformal** and can thus be expressed as an infinite Taylor series it must satisfy the Cauchy-Riemann Equations which implies that:
    * $\boxed{\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y} \text{ and } \frac{\partial\psi}{\partial y} = \frac{\partial-\phi}{\partial x}}$ so $\mathbf{v}$ also $= \left(\frac{\partial\phi}{\partial y}, -\frac{\partial\phi}{\partial x}\right)$ and $\nabla^2\phi$ also $= 0$ (satisfies Laplace)
    * **AND** that $\left(\frac{df(z)}{dz}\right)^* = \mathbf{v} = u + iv$

2.1.1 Some known Complex Potentials

Be wary of Transcendental/many-to-one equations: i.e. $e^{i\theta} = e^{i(\theta+2\pi n)}$ or $\sin(\theta) = \sin(\theta + 2\pi n)$, $n \in Z$

- Uniform flow: $f(z) = az$, where $a, z \in \mathcal{C}$
- Flow from a source: $f(z) = alog(z)$, $z \in \mathcal{C}$ and i) $a \in Re$ then we have uniform flow (**straight lines**), ii) $a \in C$, i.e. $a = c + id$, i.e. with real and complex components, then we have **spirals** (vortex) and iii) if $a \in Im$ then we get **rings** protruding from the source.
- Uniform flow around circular obstruction: $f(z) = a\left(z + \frac{b^2}{z}\right)$ (Joukowski mapping) for $a, b \in \mathcal{R}]$. At $z = \pm b$, $\mathbf{v} = 0$ (stagnation points). On $|\mathbf{v}| = b$ we trace out the circular obstacle.
- Flow around a corner. $f = \frac{1}{2}az^2$ with restricted domain to one corner and $a \in \mathcal{R}$

2.1.2 Mappings

This is sort of the key idea behind why it useful to use complex numbers for fluid flow, because essentially what we are doing is taking a real points in (x, y) and then describing their flow as 'streamlines' found by a mapping into the complex plane (and then taking its imaginary part ϕ). But it also of interest to see the entire mapping $g(z)$ not just looking at the real or imaginary parts.

- Parameterise a 'curve' in the complex x-y plane, using a parameter like t and then apply the mapping to the parameterised curve (and domain) to find a new 'curve' in the new u-v plane.
- e.g. mapping $f(z) = z^2$ smooths corners. The Joukowski mapping $f(z) = a(z + b^2/z)$ maps a circule into an aerofoil.
- Elements:
  - let $z, c \in C$, then $f(z) = z + c$ is a *translation* and $f(z) = zc$ is a *rigid rotation and a magnification.* These transformations are all **conformal** because they *preserve angles.* Note z^n is not a conformal mapping, because the angle at $z = 0$ is not conserved.
- A **Taylor series** $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is a series of instructions about how $f(z)$ translates (z_0), stretches (n), magnifies $(|a_n|)$ and translates $(arg(a_n))$ the complex plane. **A function f is conformal except at points where** $\frac{df(z)}{dz} = 0$. This also means $f(z)$ obeys the Cauchy-Riemann Equations. If $f(z)$ depends on z^* then it does not obey the CR Equations.
  - Note, the CR equations must apply for f to have a unique derivative.

2.2 Taylor's theorem

If $f(z)$ is differentiable on domain D then:

- $f(z)$ satisfies the CR-Equations, all higher derivatives exist in D, $f(z)$ is equal to its Taylor Series for all $z \in D$ given by,
  - $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, wher $a_n = \frac{1}{n!}\frac{d^n f(z)}{dz^n}|_{z=z_0}$
- This means the Taylor Series of $f(z)$ always converges to the correct value, irrespective of what number you expand about.
- **Radius of Convergence** is the distance from some point z_0 to some other point z^* where $f(z)$ is not analytic (z^* is a singularity), i.e. the Taylor series converges for all $|z - z_0| < |z_0 - z^*|$.

If this is the case then $f(z)$ is **analytic/holomorphic/regular on D**. If D is the entire complex plane then $f(z)$ is considered **entire**. Examples of entire functions include $e^z, sin(z), cos(z), cosh(z), ...$

2.3 Complex Integrals

- Specify beginning and end of integral as an *integration contour* C in the Complex Plane. The contour/path has an orientation! A contour is *smooth* if there exists a paramaterisation $z(t)$, which has a continuous derivative $z'(t)$ at all points on C.
- If the integral exists (converges) for all contours in a domain D then we say $f(z)$ is *integrable on D*
- **Parameterizing the integral**:
  - Use $dz = z'(t)dt$ to find parametric form of integral, i.e. $\int_C = \int_{t_1}^{t_2} f(z(t))z'(t)dt$, which brings us back to an integral over real t. The integral is *independent* of the parameterisation of C and oriented.
- Properties:
  - Linearity (can split sums in integral)
  - Addition of contours (can split paths)
  - Orientation $\int_{-C} = -\int_C$ (can flip path orientation and sign)
  - Closed contour $\rightarrow \oint$ and endpoints are the same, i.e. C is a loop.

2.4 The fundamental theorem of complex contour integration

- If f is analytic in a 'simply connected' domain D then a contour integral depends only on its end points, i.e. if $C \subset D$ is a path from z_1 to z_2 then:

$$\int_C f(z)dz = F(z_2) - F(z_1),$$

  where $\frac{dF}{dz} = f$.

2.5 Cauchy's Theorem

- If C is a closed loop inside D in which f is analytic then:

$$\oint_C f(z)dz = F(z_1) - F(z_1) = 0$$

  (which is just a special case of the **fundamental theorem of complex contour integration**).
- C does not need to be circle for this to be true, as along as f is analytic in the domain.

2.6 Singularities

Singularities are places where $f(z)$ or one of its derivatives are *infinite*. A function $f(z)$ cannot be written as a Taylor series around a singularity. There are three main types:

- **Isolated singularities**: $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$, but there exists an ϵ such that f is finite everywhere on the annulus $0 < |z - z_0| < \epsilon$, i.e. the function is finite in the neighbourhood around the singularity. We can classify isolated singularities as follows, the singularity z_0 is:
  - A *pole of order* n if $\lim_{z\rightarrow z_0}(z - z_0)^n f(z) = L \neq 0$, where n is determined by the lowest order term of the Laurent series. It is a **simple pole** if $n = 1$ (GET BACK TO THIS ONE)
  - a *removable pole* of f if $\lim_{z\rightarrow z_0} f(z)$ exists.
  - an *essential singularity* of f if: $\lim_{z\rightarrow z_0}(z - z_0)^n f(z)$ does not exist for any n.
- **Singularity at Infinity**: $|f(z)| \rightarrow \infty$ as $z \rightarrow \infty$. Equivalently $|f(\frac{1}{z})| \rightarrow \infty$ as $z \rightarrow 0$.
- **Branch points**: The point at which two Riemann Surfaces split. E.g. if $f(z) = z^{1/2}$ the branch point is $z = 0$ (and z at infinity), as this splits the two surfaces $f_1(z) = |z|^{1/2}e^{i\theta/2}$ and $f_2(z) = |z|^{1/2}e^{i(\theta+2\pi n)/2} = -|z|^{1/2}e^{i\theta/2}$

2.7 Cauchy's Integral Formula

Aside - integrals that do not vanish (equal zero) have singularities and have **simple poles**.

- Cauchy's Integral formula relates the **residue** $f(z_0)$ to a complex integral of the following form. Given the f is analytic, except at the singularities z_0, inside a circle with boundary C (oriented counter clockwise), then:
$$\frac{d^m}{dz^m} f(z)|_{z=z_0} = \frac{m!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{m+1}} dz,$$
or as sometimes useful to see in the simple case:
$$\oint_C \frac{f(z)}{z-z_0} = 2\pi i f(z_0),$$
which implies that only $\frac{f(z)}{(z-z_0)^1}$ type functions (i.e. simple poles) in the integral actually have residues! Integrals of $\frac{f(z)}{(z-z_0)^n}$, $n \neq 1$ will always be zero!!

- This provides a formula for $a_n = \frac{1}{n!} \frac{d^m}{dz^m} f(z)|_{z=z_0} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{m+1}} dz$ for the Taylor series (i.e. the Taylor series now works for negative n).

2.8 Cauchy's Residue Theorem

- Let C be an anticlockwise oriented circle and let g be analytic except at the poles $z_n, n = 1, 2, ..., N$ where it has residues $Res(g, z_n)$ ($= f(z_n)$ if $g(z) = \frac{f(z)}{(z-z_1)(z-z_2)...}$):
$$\oint_C g(z)dz = 2\pi i \sum_{n=1}^{N} Res(g, z_n) \left(= 2\pi i \sum_{n=1}^{N} f(z_n) \right).$$

- There are different ways to find $Res(g, z_n)$:
  - Use Cauchy's integral formula, i.e. reformulate the integral so $g(z) = \frac{f(z)}{(z-z_1)(z-z_2)...}$) (like above).
  - Use L'Hopital's Rule: If z_n is a pole of order m then $Res(g, z_n) = \lim_{z \to z_n} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}((z-z_n)^m f(z))$
  - Use Laurent Series (which are just Taylor series from $-\infty$ to $+\infty$ that use the complex integral a_n form shown above. Only the a_{-1} will contribute to a *residue*, so $Res(g, z_n) = a_{-1} = \frac{1}{2\pi i} \oint_C f(z)dz$

Essentially, to evaluate an integral 1) find the number of singularities within the closed contour you are integrating about and 2) Find the residue, using either of the three methods mentioned. Bam dats it.

2.9 Laurent Series

Like an extended Taylor series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{m+1}} dz$$

- We use Laurent Series when expanding around singularities, and Taylor Series when expanding around non-singularities.

2.10 Real Integrals

Integrals of the form $\int_0^{2\pi} f(sin(\theta), cos(\theta))d\theta$

- 1) Use unit circle parameterisation $z = e^{i\theta}$ and substitute $sin(\theta) = \frac{1}{2i}(z - \frac{1}{z})/ \ cos(\theta) = \frac{1}{2}(z + \frac{1}{z})$ 2) Then rearrange to get integral into Cauchy Integral form if possible (else have to use L'Hopital or Laurent), 3) Find residues 4) sum residues and multiply by $2\pi i$. Done.

Integrals of the form $\int_{-\infty}^{\infty} f(t)cos(kt)dt$ or $\int_{-\infty}^{\infty} f(t)sin(kt)dt$

- Jordan's Lemma: $\displaystyle\int_{-\infty}^{\infty} f(t)e^{ikt}dt = \oint_C f(z)e^{ikz}dz = 2\pi i \times \begin{pmatrix} \text{sum of residues in} \\ \text{upper half plane} \end{pmatrix}$
- Thus:
  - $\displaystyle\int_{-\infty}^{\infty} f(t)cos(kt)dt = Re\oint_C f(z)e^{ikz}dz = -2\pi \times Im\begin{pmatrix} \text{sum of residues in} \\ \text{upper half plane} \end{pmatrix}$
  - $\displaystyle\int_{-\infty}^{\infty} f(t)sin(kt)dt = Im\oint_C f(z)e^{ikz}dz = 2\pi \times Re\begin{pmatrix} \text{sum of residues in} \\ \text{upper half plane} \end{pmatrix}$
  - Note, this is just the same as doing $Re/Im\left(2\pi i \times \begin{matrix} \text{sum of residues in} \\ \text{upper half plane} \end{matrix}\right)$ for cos/sin.

Inverse Laplace Transforms:

- Forward transform: $F(s) = \int_0^{\infty} e^{st}f(t)dt$
- Inverse transform $f(t) = \frac{1}{2\pi i}\oint_C F(s)e^{st}e^{st}ds = 2\pi i \times \begin{pmatrix} \text{sum of residues in} \\ \text{left half plane (inc. Im axis)} \end{pmatrix}$

3 Partial Differential Equations

3.1 Intro to PDE's L1

- Dependent variable must be dependent on at least two independent variables.
- Classification:
  - Order: order of highest *partial* derivative.
  - Number of independent variables: 'PDE in n variables".
  - Constant vs Variable (change with independent variables) coefficients.
  - Homogenous: All terms contain the dependent variable or its derivatives.
  - Linear: If the dependent variable and its derivatives appear in a linear fashion.
  - Semi-Linear: The highest derivatives of the dependent variable appear in a linear fashion **and** coefficients do not dependent on the dependent variable. $a(x,y)u_{xx} + b(x,y)u_{yy} = c(x,y,u,u_x,u_y)$
  - Quasi-linear: Highest derivatives appear in a linear fashion: $a(x,y,u,u_x,u_y)u_{xx}+b(x,y,u,u_x,u_y)u_{yy} = c(x,y,u,u_x,u_y)$ (e.g. u_{xx}^2 is not quasi-linear)
  - Fully non-linear: Highest derivatives appear in a non-linear fashion, e.g. $u_{xx}u_{yy}$.
- Classification of 2nd order PDEs:
  - Given a PDE of the form $Au_{xx} + Bu_{xy} + Cu_{yy} + \text{lower order terms} = 0$: Then $\alpha = B^2 - 4AC$.
    * If $\alpha < 0$, then the PDE is **elliptic** (e.g. Laplace $u_{xx} + u_{yy} = 0$)
    * If $\alpha = 0$ then the PDE is **parabolic** (e.g. Heat Equation $u_t = u_{xx}$)
    * If $\alpha > 0$ then the PDE is **hyperbolic** (e.g. Wave Equation $u_{tt} = u_{xx}$)

3.2 Method of Characteristics L2, L3, (L10 for examples)

3.2.1 First order quasi-linear PDE's

For *first order quasi-linear* PDE's $a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$

- Idea: Construct solution plane: $[u_x, u_y, -1] \cdot [a,b,c] = 0$, then use parameter (t) that generates curves that follow tangent plane given by $[a,b,c]$ so $\frac{dx}{dt} = a(x(t),y(t),u(t))$, $\frac{dy}{dt} = b(x(t),y(t),u(t))$ and $\frac{du}{dt} = c(x(t),y(t),u(t))$

- The solutions to these equations are called **characteristic curves** and their projection on the $x-y$ plane are called **characteristic projections**.
- Solutions to the ODE's above are not unique (because of integration constant) so we need to define initial/boundary data.
  - Typically define a curve Γ in x, y, u - space using another parameter (s), $s \in [s_0, s_1]$. So the initial x, y, u values are $x = x_0(s)(= x(s, t = 0)), y = y_0(s), u = u_0(s)$. Thus the solution corresponds to finding the surface $u(x, y)$ that passes through Γ.
- **Cauchy Data**: This implies that the PDE will have a unique analytic solution near Γ (Cauchy-Kowaleski theorem). Cauchy Data must satisfy $a\frac{dy_0}{ds} - b\frac{dx_0}{ds} \neq 0 \forall s \in [s_0, s_1]$. This also means $\begin{vmatrix} \frac{dx}{ds} & \frac{dx}{dt} \\ \frac{dy}{ds} & \frac{dy}{dt} \end{vmatrix} = \frac{dx}{ds}\frac{dy}{dt} - \frac{dx}{dt}\frac{dy}{ds} \neq 0, \infty$

. Use this to define the **domain of definition**, i.e. everywhere where Cauchy data exists (**L4**).

- So the domain of definition can also be affected (instead of just where the Jacobian becomes zero, like above) by *finite time blow-up* which is when the solution of the PDE $u(x, y) = ...$ blow's up for certain values of x, y.
- The crossing of characteristic projections show where the domain of definition ends.

3.2.2 Extending this to systems of PDE's (L5 - Don't fully understand)

- Convert second order PDE into system of 1st order PDE using e.g. $\frac{\partial u}{\partial x} = p$ and $\frac{\partial u}{\partial y} = q$.
- Put the system into this form: $\mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y = \mathbf{c}$ (bc is an n by 1 matrix, while $\mathbf{A}, \mathbf{B}$ are n by n matrices).
- $det(\mathbf{B} - \lambda\mathbf{A}) = 0$ gives the characteristic directions apparently. You need to find some vector $\mathbf{l}$, but I am not sure how...

3.2.3 Second order semi-linear PDE's L6

For second order semi-linear PDE's of the form: $a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} = f(x, y, u_x, u_y, u)$ (let $2b = B$), which can also be written into two first order PDE's of the form $\mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y = \mathbf{f}$

- We define a characteristic that does not uniquely define the second order derivatives as: $a\left(\frac{dy}{dt}\right)^2 - 2b\frac{dx}{dt}\frac{dy}{dt} + c\left(\frac{dx}{dt}\right)^2 = 0$, such that $\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \frac{B \pm \sqrt{B^2 - 4ac}}{2a}$
  - If the PDE is elliptic, i.e. $b^2 - ac = B^2 - 4ac < 0$, this implies $\frac{dy}{dx} = C \pm Di$
    - Choose $\xi + \eta i = const.$
    - **Canonical/Normal Form**: $\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = G(\xi, \eta, u, u_\xi, u_\eta)$
  - If the PDE is parabolic, i.e. $b^2 - ac = 0$, then $\frac{dy}{dx} = C$
    - Find ξ from const. of solution of $\frac{dy}{dx}$ and just set $\eta = const.$ (as long η and ξ are not parallel choose whatever)
    - **Canonical/Normal/Standard Form**: $\frac{\partial^2 u}{\partial \xi^2} = G(\xi, \eta, u, u_\xi, u_\eta)$
  - If the PDE is hyperbolic, i.e. $b^- ac > 0$, then $\frac{dy}{dx} = C \pm D =$
    - Solve $\frac{dy}{dx}$ and rearrange to find constants and then set this equal to ξ and η. e.g. $x - a_0 y = const. = \xi, x + a_0 y = const. = \eta$
    - **Canonical/Normal Form**: $\frac{\partial u}{\partial \xi \partial \eta} = G(\xi, \eta, u, u_\xi, u_\eta)$
    - If $G(\xi, \eta)$ only then the solution is of the form: $u(\xi, \eta) = \int\int G(\xi, \eta) d\xi d\eta + \phi(\eta) + \phi(\xi)$ (which essentially gives D'Alembert's solution L7).
- Use coordinate transforms $\frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}$ and the same for y $\frac{\partial}{\partial y} = \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}$ to find $u_{xx} = \frac{\partial}{\partial x}[u_x], u_{yy}, u_x$ etc... Beware if $\frac{\partial}{\partial x} = f(x)$ or $\frac{\partial}{\partial y} = f(y)$ - you'll have to use the product rule!.

3.3 Wave Equations L7, L8

- Resolve tensions vertically and horizontally in a string, make small angle approximations *i.e.* $\cos(\alpha) \approx 1$ and $\tan(\alpha) = sin(\alpha)/1 = \frac{du}{dx}$. Finally take limits to get $u_{tt} = c^2 u_{xx}$ (hyperbolic).
- Domain of dependence and range of influence ??
- Wave Eq. in 2D $u_{tt} = c^2(u_{xx} + u_{yy}) = c^2\nabla \cdot \nabla u = c^2\nabla^2 u$. Notice the Laplacian. Note that $\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$ in polar coordinates so the wave eq. becomes $u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta})$
- We can find **standing waves** using the separation of variables method.
- You can get the Bessel Equations from the Helmoltz equation.
- **Travelling wave Ansatz**: $u(x,t) = Ae^{i[kx-wt]}$ where:
  - A is the amplitude
  - k is the **wavenumber**, related to wavelength (λ) by $k = 2\pi/\lambda$. Waves with different wavelengths or wavenumbers travel at different phase speeds.
  - $Re(w)$ is the *angular* frequency - w can be complex. The $Re(w) > 0$ for a travelling wave.
  - **Phase speed** (speed of individual wave, carrier wave) $c_p = \frac{Re(w)}{k}$.
  - **Dispersion Relation** - links the *frequency* w and *wavenumber* k of a travelling wave with an equation. Find this by substituting the ansatz into the given wave equation.
  - A wave packet containing multiple frequencies will disperse.
  - Non-zero $Im(w)$ implies intial wave profiles are attenuated/reduced.
  - **Group velocity**: Speed of the *envelope* $c_g = \frac{dw}{dk}$

3.4 Diffusion Equations L11, L12

- Derivation: Rod with heat transferring through it. Then in a little section $[x, x+\Delta x]$, the net change in heat in this section is the sum net flux of heat across the boundaries and the total heat generated inside the section.
- Use the mean value theorem: 'There exists a $\xi \in [a,b]$ such that $\int_a^b g(x)dx = g(\xi)[b-a]$, i.e the area under the curve is the same as a rectangle with the width of the integral and an unknown height - makes sense.' to get rid of integrals in the derivation and get $u_t = \alpha^2 u_{xx} + F(x,t)$ (This actually a reaction-diffusion equation because of the source term, without it, it becomes: $\boxed{u_t = \alpha^2 u_{xx}}$
- Can also derive using random walkers and finite difference methods.
- You can solve it using **Separation of variables** with a negative $-k^2$ constant. Recall that: $\ddot{X} + k^2 X = 0 \to A = A\cos(kt) + B\sin(kt)$, while $\ddot{X} - k^2 X = 0 \to A = A\cosh(kt) + B\sinh(kt)$ and $\dot{X} + k^2 X = 0 \to Ae^{-k^2 t}$ (L11)
- You can also solve the diffusion equation using Fourier Transforms, you'll need to know how to use convolution as well. The solution will be of the form: $u(x,t) = \frac{1}{\sqrt{4\pi\alpha^2 t}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\alpha^2 t}} f(y)dy$.
  - We can often extract **moments** (e.g. mean and variance, because the solution is essentially gaussian) without solving the PDE by finding:
  - For the mean: $\mu = \int_{-\infty}^{\infty} xu(x,t)dx$ just multiply the PDE by x and integrate (might have to do integration by parts)
  - For the variance: $\mu = \int_{-\infty}^{\infty} x^2 u(x,t)dx$ just multiply the PDE by x^2 and integrate (might have to do integration by parts)

3.5 Reaction-diffusion Equations, Non-homogenous PDE's L13

- Standard form $u_t = \alpha^2 u_{xx} + f(u, x, t)$ (RD PDE). Useful for biology, chemistry, physics - e.g. Turing patterns.
- **Eigenfunction expansion method** for solving these kinds of problems. Two step procedure:
  - 1) Decompose the source term and find $f_n(t)$ and $X_n(x)$: $f(x,t) = f_1(t)X_1(x) + f_2(t)X_2(x) + ... = \sum_{n=1}^{\infty} f_n(t)X_n(x)$ and the aim is to find the solutions $u_n = T_n(t)X_n(t)$ such that the final solution $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$, where:
    * $X_n(x)$ are the eigenvectors of the Sturm-Liouville system, which essentially just means the solution for $X_n(e.g = sin(n\pi x/L)$ from the homogeneous separations of variables method where $u(x,t) = X(x)T(t)$.
    * Find $f_n(t)$ by multiplying $f_n(x,t) = f_n(t)X_n(x)$ by $sin(m\pi x/L)$ and integrate from 0 to L. The $\sin(m...)\sin(n...)$ will integrate to $0 \forall m = n$ and $L/2, m = n$. Thus $f_n(t) = \frac{2}{L}\int_0^L \sin(\frac{n\pi x}{L} dx$ (Just Fourier really).
  - 2) Find solution components (recall $u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} \sin(n\pi x/L)T_n(t)$: The RD PDE can be written now as $\sum_{n=1}^{\infty}[T_n' + \alpha^2(n^2\pi^2/L^2)T_n - f_n(t)]X_n$ which gives the ODE: $\boxed{T_n' + \alpha^2(\frac{n\pi}{L})^2 T_n = f_n(t)}$ with the IC $T_n(0) = \frac{2}{L}\int_0^L u(x,0)\sin(n\pi x/L)dx$
  - So essentially find f_n using integral with X_n then solve ode for T_n and find $T_n(0)$ using other integral to find the coefficient for each $T_n(t)$ solution and then do this for all n.

3.6 Transformation techniques L14

- For Reaction-diffusion equations you could use $u(x,t) = e^{-\beta t}w(x,t)$ to potential transform it into a diffusion equation.
- **Travelling wave solutions** (a wave that travels at constant speed without vanishing: $u(x,t) = U(z) = U(x - ct) \rightarrow \frac{du}{dt} = \frac{dU}{dz}\frac{dz}{dt}$ and $\frac{du}{dx} = \frac{dU}{dx}\frac{dz}{dx}$. $c \geq 0$ for meaningful solutions, otherwise $u(x,t) = 0$ Which will result in a system of first order ODEs. Use this method for the KdV equation for example because it is known to have solitary travelling wave (soliton) solutions.

3.7 Nonlinear Dynamic Analysis L14

Use this to analyse the system of first order ODE's that you get from having used a transformation on the PDE.

- Find steady states (nullclines) from $\dot{x}, \dot{y} = 0$.
- Find Jacobian and its eigenvalues and sub in steady states to find their **stability**. For stability, note that (in 2D) $Trace(J) = \lambda_1 + \lambda_2 < 0$ and $Det(J) = \lambda_1\lambda_2 ... > 0$
- Remark - phase plane trajectories are solutions to $\frac{dx}{dy}$.
- Visualising this analysis (L15):
  - Find nullclines and mark where they intersect with a ring.
  - Know the types of equilibria (stable/unstable node, saddle, stable/unstable spiral).

3.8 Turing patterns L16

- To see *patterns* we need some wave numbers (k) which give an unstable eigenvalue and so we require Det(J) ¡ 0 for some k. Where this occurs is a Turing bifurcation I think, i.e. where $Det(J) = 0$.

3.9 Elliptic Equations L17

- Have no characteristics and therefore no information propagation, so they are suited for static/steady state problems.
- $\nabla^2 u >, =, < 0$ represent that u is smaller, equal to, or larger than its average surrounding values (convex, flat, concave). This is relevant for both parabolic and hyperbolic equations as laplaces equation is entailed within them.
- Boundary conditions:
  - Dirichlet (first kind) - value of dependent variable is specified on the boundary. E.g. $\nabla^2 u = 0$ with $u(1, \theta) = sin(\theta), \theta \in [0, 2\pi]$
  - Neumann (second kind) - outward normal derivative is specified on the boundary. E.g. $\nabla^2 u = 0$ with $u_r(1, \theta) = sin(\theta), \theta \in [0, 2\pi]$
  - Robin (third kind) - mixture of the first two kinds. E.g. $u_n = h(g(n) - u)$ like when conduction and convection meet.
  - Solve homogeneous elliptic equations using **separation of variables** - but beware that the sign of the constant k^2 depends on the boundary conditions.

3.10 Cauchy-Euler Equation L18

- The Cauchy-Euler Equation creeps up when working with Laplacian equations in polar coordinates. I.e. once you use separation of variables and get your ODE's one of them might be in the form $rR'' + rR' - kR = 0$.
- The general form of the C-E equation is $a_n r^n \frac{d^n R}{dr^n} + a_{n-1} r^{n-1} \frac{d^{n-1} R}{dr^{n-1}} + ... a_0 R = 0$.
- Use substitution $t = ln(r)$, i.e. $R(r) = \phi(ln(r)) = \phi(t)$ and then solve the new ODE using its characteristic polynomial (eng. maths 1) and finally reverse the substitution. You should get solutions of the form: $R(r) = c_1 r^{\lambda_1} + c_2 r^{\lambda_2}$ or for a repeated root $R(r) = c_1 r^{\lambda_1} + c_2 ln(r) r^{\lambda_2}$