Final Physics Homework
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Question 1

(a)

I_R^C = Moment of inertia of the rhombus about its centre of mass C.
I_T^C = Moment of inertia of one of the triangles formed by quartering the rhombus.

$$I_R^C = 4I_T^C \tag{1}$$

Finding I_T^C (using the triangle in the upper fourth of the rhombus):

$$I_T^C = \iint_R r^2 dm = \iint_R x^2 + y^2 \rho dx dy$$
$$\rho = \frac{m_T}{A_T} = \frac{\frac{m_R}{4}}{\frac{1}{2}\frac{d}{2}\frac{\sqrt{3}d}{2}} = \frac{2m_R}{\sqrt{3}d}$$

limits of region R:

$$0 \leq x \leq \frac{d}{2} - \frac{y}{\sqrt{3}}$$
$$0 \leq y \leq \frac{\sqrt{3}d^2}{2}$$

Performing the integration:

$$\begin{aligned}
I_T^C &= \rho \int_0^{\frac{\sqrt{3}d}{2}} \int_0^{\frac{d}{2}-\frac{y}{\sqrt{3}}} x^2 + y^2 dx dy \\
&= \rho \int_0^{\frac{\sqrt{3}d}{2}} \left[\frac{x^3}{3} + y^2 x\right]_0^{\frac{d}{2}-\frac{y}{\sqrt{3}}} dy \\
&= \rho \int_0^{\frac{\sqrt{3}d}{2}} \left[\frac{(\frac{d}{2}-\frac{y}{\sqrt{3}})^3}{3} + y^2(\frac{d}{2}-\frac{y}{\sqrt{3}})\right] dy \\
&= \rho \int_0^{\frac{\sqrt{3}d}{2}} \left[\frac{(\frac{d}{2}-\frac{y}{\sqrt{3}})^3}{3} + y^2\frac{d}{2} - \frac{y^3}{\sqrt{3}}\right] dy \\
&= \rho \left[\frac{(-\sqrt{3})(\frac{d}{2}-\frac{y}{\sqrt{3}})^4}{(3)(4)} + y^3\frac{d}{(2)(3)} - \frac{y^4}{\sqrt{3}(4)}\right]_0^{\frac{\sqrt{3}d}{2}} \\
&= \frac{2m_R}{\sqrt{3}d^2}\left[0 + \frac{d^4 3^{3/2}}{(8)(2)(3)} - \frac{d^4 3^2}{(4)(\sqrt{3})(16)} + \frac{(\sqrt{3})(\frac{d}{2})^4}{(3)(4)} + 0 + 0\right] \\
&= \frac{2m_R d^4}{\sqrt{3}d^2(16)}\left[\frac{3^{3/2}}{(3)} - \frac{3^2}{(4)(\sqrt{3})} + \frac{\sqrt{3}}{(12)}\right] \\
&= \frac{m_R d^2}{\sqrt{3}(8)}\left[\sqrt{3} - \frac{3\sqrt{3}}{(4)} + \frac{\sqrt{3}}{(12)}\right] = \frac{m_R d^2}{\sqrt{3}(8)}\left[\frac{\sqrt{3}}{(3)}\right] = \frac{m_R d^2}{24}
\end{aligned}$$

Finally substituting this into equation (1) we get $I_R^C = \frac{m_R d^2}{6}$

(b)

(i)

I chose the generalised coordinates to be θ and y, where θ represents the angle from A to the centre of mass C relative to a horizontal line and y to represent the vertical displacement of the centre of mass from the horizontal line at A.

Finding the potential energy:

$$U = mgy + \frac{1}{2}k(y - lsin\theta)^2 + \frac{1}{2}k(y + lsin\theta)^2$$
$$U = mgy + ky^2 + kl^2sin\theta^2$$

Finding the kinetic energy and determining $\frac{1}{2}mv_{cm}^2$:

$$\underline{r_{cm}} = \begin{pmatrix} lcos\theta \\ y \\ 0 \end{pmatrix} \Rightarrow \underline{v_{cm}} = \begin{pmatrix} -lsin\theta\dot{\theta} \\ \dot{y} \\ 0 \end{pmatrix} \Rightarrow v_{cm}^2 = l^2sin^2\theta\dot{\theta}^2 + \dot{y}^2$$

$$\Rightarrow T = \frac{1}{2}m(l^2sin^2\theta\dot{\theta}^2 + \dot{y}^2) + \frac{1}{2}I_{cm}\dot{\theta}^2$$

(ii)

Equilibrium:

$$L = T - U$$
$$L = \frac{1}{2}m(l^2sin^2\theta\dot{\theta}^2 + \dot{y}^2) + \frac{1}{2}I_{cm}\dot{\theta}^2 - mgy - ky^2 - kl^2sin\theta^2$$

$$\frac{\partial L}{\partial y} = -mg - 2ky = 0$$
$$\Rightarrow y = -\frac{mg}{2k}$$
$$\frac{\partial L}{\partial \theta} = -2kl^2sin\theta cos\theta$$
$$\Rightarrow \theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \pi$$

General mass matrix

$$M_{11} = \frac{\partial^2 L}{\partial \dot{y}^2} = \frac{\partial}{\partial \dot{y}}(m\dot{y}) = m$$
$$M_{12} = \frac{\partial^2 L}{\partial \dot{y}\partial\dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}(m\dot{y}) = 0$$
$$M_{21} = \frac{\partial^2 L}{\partial \dot{\theta}\partial\dot{y}} = \frac{\partial}{\partial \dot{y}}\left(ml^2sin^2\theta\dot{\theta} + I_{cm}\dot{\theta}\right) = 0$$
$$M_{22} = \frac{\partial^2 L}{\partial \dot{\theta}\partial\dot{y}} = \frac{\partial}{\partial \dot{y}}\left(ml^2sin^2\theta\dot{\theta} + I_{cm}\dot{\theta}\right) = ml^2sin^2\theta + I_{cm}$$
$$\Rightarrow \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & ml^2sin^2\theta + I_{cm} \end{bmatrix}$$

General stiffness matrix

$$K_{11} = -\frac{\partial^2 L}{\partial y^2} = -\frac{\partial}{\partial y}(-mg - 2ky) = 2k$$

$$K_{12} = -\frac{\partial^2 L}{\partial y \partial \theta} = -\frac{\partial}{\partial \theta}(-mg - 2ky) = 0$$

$$K_{21} = -\frac{\partial^2 L}{\partial \theta \partial y} = -\frac{\partial^2 L}{\partial y}\left(-2kl^2 sin\theta cos\theta\right) = 0$$

$$K_{22} = -\frac{\partial^2 L}{\partial \theta^2} = -\frac{\partial^2 L}{\partial \theta}\left(-2kl^2 sin\theta cos\theta\right) = -\frac{\partial^2 L}{\partial \theta}\left(-kl^2 sin2\theta\right) = 2kl^2 cos2\theta$$

$$\Rightarrow \mathbf{K} = \begin{bmatrix} 2k & 0 \\ 0 & 2kl^2 cos2\theta \end{bmatrix}$$

(iii)

I will calculate the natural frequencies for each vibration mode for each equilibrium respectively. Note $y = \frac{-mg}{2k}$ for all equilibria.
At $\theta = 0$:

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & I_{cm} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 2k & 0 \\ 0 & 2kl^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow det(\mathbf{K} - w^2\mathbf{M})$$

$$\Rightarrow det \begin{bmatrix} 2 - 2w^2 & 0 \\ 0 & \frac{1}{2} - w^2 \end{bmatrix} = (2 - 2w^2)(\frac{1}{2} - w^2) = 0$$

$$\Rightarrow w_1^2 = 1$$

$$\Rightarrow w_2^2 = \frac{1}{2}$$

Vibration mode for w_1:

$$\Rightarrow \begin{bmatrix} 2 - 2 & 0 \\ 0 & \frac{1}{2} - 1 \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

$$\Rightarrow \text{let } C_{11} = \alpha \tag{4}$$

$$\Rightarrow (0)\alpha + (-\frac{1}{2})C_{12} = 0 \Rightarrow C_{12} = 0 \tag{5}$$

$$\Rightarrow \underline{C_1} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{6}$$

This mode describes the rhombus moving up and down without any rotation.
Vibration mode for w_2:

$$\Rightarrow \begin{bmatrix} 2 - 1 & 0 \\ 0 & \frac{1}{2} - \frac{1}{2} \end{bmatrix} \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

$$\Rightarrow \text{let } C_{22} = \alpha \tag{9}$$

$$\Rightarrow (1)C_{21} + (0)\alpha = 0 \Rightarrow C_{21} = 0 \tag{10}$$

$$\Rightarrow \underline{C_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{11}$$

This mode describes the rhombus rotating around its centre of mass while not moving up or down.

For $\theta = \frac{\pi}{2}$

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & ml^2 + I_{cm} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 2k & 0 \\ 0 & -2kl^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$
$$\Rightarrow det(\mathbf{K} - w^2\mathbf{M})$$
$$\Rightarrow det \begin{bmatrix} 2 - 2w^2 & 0 \\ 0 & -\frac{1}{2} - \frac{3w^2}{2} \end{bmatrix} = (2 - 2w^2)(-\frac{1}{2} - \frac{3w^2}{2}) = 0$$
$$\Rightarrow w_1^2 = 1$$
$$\Rightarrow w_2^2 = -\frac{1}{3}$$

Vibration mode for w_1:

$$\Rightarrow \begin{bmatrix} 2 - 2 & 0 \\ 0 & \frac{1}{2} - \frac{3}{2} \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{13}$$
$$\Rightarrow \text{let } C_{11} = \alpha \tag{14}$$
$$\Rightarrow (0)\alpha + (-1)C_{12} = 0 \Rightarrow C_{12} = 0 \tag{15}$$
$$\Rightarrow \underline{C_1} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{16}$$

This mode again describes the rhombus moving up and down without any rotation.
Vibration mode for w_2:

$$\Rightarrow \begin{bmatrix} 2 + \frac{2}{3} & 0 \\ 0 & -\frac{1}{2} + \frac{(3)(1)}{(2)(3)} \end{bmatrix} \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$
$$\Rightarrow \begin{bmatrix} \frac{8}{3} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{18}$$
$$\Rightarrow \text{let } C_{22} = \alpha \tag{19}$$
$$\Rightarrow (\frac{8}{3})C_{21} + (0)\alpha = 0 \Rightarrow C_{21} = 0 \tag{20}$$
$$\Rightarrow \underline{C_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{21}$$

This mode again describes the rhombus rotating around its centre of mass while not moving up or down.

For the equilibria $\theta = \pi$ and $\theta = \frac{3\pi}{2}$ we get the exact same results as shown above, due to symmetry.

Question 2

(a)

I believe the mechanism <u>can</u> have 2 degrees of freedom, because the mechanism's freedom depends on the friction. Since we do not know what the magnitude of the friction is, we have to describe the system using two generalised coordinates, so it can have 2 DOF.

(b)

The generalised coordinates I have chosen are x, the displacement of the mechanism down the slope and θ, the angle by which the wheel rotates.

$$T = \underbrace{\frac{1}{2}m\dot{x}^2}_{\textbf{block}} + \underbrace{\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_{cm}\dot{\theta}^2}_{\textbf{disc}}$$
$$T = m\dot{x}^2 + \frac{1}{4}mR^2\dot{\theta}^2$$
$$U = \underbrace{-mgsin\beta x}_{\textbf{block}}\underbrace{-mgsin\beta x}_{\textbf{disc}}$$
$$U = -2mgsin\beta x$$

(c)

Friction is not included in the potential energy, because it is not conservative. Let A be the point of contact between the disk and the slope.

$$Q_x = F_f \cdot \frac{\partial \mathbf{v}_A}{\partial \dot{x}}$$
$$Q_\theta = F_f \cdot \frac{\partial \mathbf{v}_A}{\partial \dot{\theta}}$$

Finding V_A

$$\mathbf{v}_A = \mathbf{v}_C + \omega \times \mathbf{r}_{CA}$$
$$\mathbf{v}_A = \begin{pmatrix} \dot{x} \\ 0 \\ 0 \end{pmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta} \\ 0 & -R & 0 \end{vmatrix} = \begin{pmatrix} \dot{x} + \dot{\theta}R \\ 0 \\ 0 \end{pmatrix}$$

Finding the generalised forces Q_x and Q_θ

$$Q_x = \begin{pmatrix} F_f \\ 0 \\ 0 \end{pmatrix} \cdot \frac{\partial}{\partial \dot{x}} \begin{pmatrix} \dot{x} + \dot{\theta}R \\ 0 \\ 0 \end{pmatrix} = F_f$$
$$Q_\theta = \begin{pmatrix} F_f \\ 0 \\ 0 \end{pmatrix} \cdot \frac{\partial}{\partial \dot{\theta}} \begin{pmatrix} \dot{x} + \dot{\theta}R \\ 0 \\ 0 \end{pmatrix} = RF_f$$

From free body diagram, $F_f = -\mu_2 mgcos\beta sign(\dot{x} + R\dot{\theta})$, so:

$$Q_x = -\mu_2 mgcos\beta sign(\dot{x} + R\dot{\theta})$$
$$Q_\theta = -R\mu_2 mgcos\beta sign(\dot{x} + R\dot{\theta})$$

(d)

$$L = T - U$$
$$L = m\dot{x}^2 + \frac{1}{4}mR^2\dot{\theta}^2 + 2mgsin\beta x$$

Equation of motion for x

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = Q_x$$

$$\frac{\partial L}{\partial \dot{x}} = 2m\dot{x} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 2m\ddot{x}$$

$$\frac{\partial L}{\partial x} = 2mgsin\beta$$

$$\Rightarrow (1) \qquad 2m\ddot{x} - 2mgsin\beta = F_f$$

Equation of motion for θ

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}mR^2\dot{\theta} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}mR^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow (2) \qquad \frac{1}{2}mR^2\ddot{\theta} = RF_f$$

Determining the minimum value of the friction coefficient μ_2 so that the disc does <u>not</u> slip:

If the disc sticks then there is a zero velocity point at v_A, so:

$$\dot{x} + R\dot{\theta} = 0 \Rightarrow \ddot{x} + R\ddot{\theta} = 0 \Rightarrow \ddot{\theta} = -\frac{\ddot{x}}{R}$$

Substituting for $\ddot{\theta}$ into equation of motion (2)

$$\frac{1}{2}mR^2\left(-\frac{\ddot{x}}{R}\right) = RF_f$$

$$\Rightarrow (3) \qquad -\frac{1}{2}m\ddot{x} = F_x$$

Solving for $\ddot{x}$ using e.o.m (1).

$$2m\ddot{x} - 2mgsin\beta = -\frac{1}{2}m\ddot{x}$$

$$\Rightarrow \ddot{x} = \frac{4}{5}gsin\beta$$

Substituting this into equation (3) to find μ_2

$$-\frac{1}{2}m\left(\frac{4}{5}gsin\beta\right) = F_f$$

$$|F_f| \leq \mu_2 mgcos\beta$$

$$\Rightarrow \frac{2}{5}mgsin\beta \leq \mu_2 mgcos\beta$$

$$\Rightarrow \frac{2}{5}tan\beta \leq \mu_2$$

which is the minimum value the friction coefficient can have for the mechanism not to slip.