FLUID EQUATIONS

General knowledge:

$$p = \rho RT$$

For a fixed temperature $pV = constant$

Fluid Statics:

- Things that should be very clear. Pressure increases as you go down and decreases as you go up. So if you're using hydrostatic equation, make ρgh term negative or positive correspondingly.

- The pressure at any two points on a horizontal plane is the same as long as they are connected by a line wholly in the fluid. Hydrostatic equation $\frac{\partial p}{\partial z} = -\rho g \Rightarrow p + \rho gh = constant$

- Gauge pressure $= p - p_a$ Any pressure compared to atmospheric pressure $p_a = 101,300Pa$. 1 bar = 100,000 = 10^5 Pa.

- Barometer. Used to measure atmospheric pressure. The 'vacuum' that is created is not quite a vacuum, instead it has pressure p_v. If it is a vacuum then pressure would be zero. $p_a = p_v + \rho_{Hg} gh$

- Manometer. Measures difference in static pressure. Equate pressures at the same height of different fluid densities.

- Specific gravity is the density of a fluid relative to water. E.g. $S.G. = 0.8 = 800\frac{kg}{m^3}$

- Velocity of a raindrop through air. Work out forces on the raindrop. These are Drag, Buoyancy and Weight. For spherical drop, the buoyancy would be $B = \frac{4}{3}\pi r^3 \rho_a g$, where ρ_a is density of the air being displaced. Drag is given by drag equation and weight would simply be $W = \frac{4}{3}\pi r^3 \rho_w g$, where ρ_w is the density of the rain drop. Then simply equate forces $D + B = W$.

- Forces on submerged objects -like dams. Break up the forces into vertical and horizontal. Remember fluid is static so forces must balance.

  - $F_{vertical}$ is the equal to the effective weight of the fluid above.$F_v = \rho \times Vol \times g = m \times g$ or simply $F_v = W_{above}$
  - Vertical force of air (p_a) is simply pressure times area of surface $F_{v,air} = pA$
  - For the horizontal force F_h imagine the pressure acting on a vertical surface. The horizontal force is equal to the pressure at the centre of gravity of the projected shape times the area of that projected shape. $F_h = \iint_A p dA = \rho g h_{C.G} A$, where $h_{C.G}$ is the height of centre of gravity $h_{C.G} = \frac{1}{A}\int h dA$, for a rectangle $h_{C.G}$ would be half the height.

- Total force is then given by $F = \sqrt{(F_v)^2 + (F_h)^2}$
- The horizontal force acts through the centre of pressure and not the centre of gravity. The line of action of the force is $h_{C.P} = \frac{I_{xx}}{h_{C.G}A} + h_{C.H}$, I_{xx} for a rectangle is $\frac{bH^3}{12}$, b being the base and H the height.

$$F_H = \iint\limits_A p dA = \rho g h_{C.G} A \tag{1}$$

$$h_{C.P} - h_{C.H} = \frac{I_{xx}}{h_{C.G}A} \tag{2}$$

Fluid Behaviour and Flow Similarity:

- Definitions:
  - Pathline. The path traced by an element of fluid over an interval of time
  - Streakline. A trace joining the instantaneous positions of fluid elements which have passed a given point. Shown by smoke or dye injection.
  - Streamline A curve drawn from the flow field at an instant in time, whose tangent at any point is in the direction of the velocity vector at that point. Hence, there is no flow perpendicular to a streamline. (The closer the streamlines the faster the velocity of the particles.
  - The above are identical in steady flow.
  - Stagnation point On dividing streamline (DS). Velocity is zero, so static pressure p = total pressure P_0. For potential flow this means there is no normal flow so $V_r = 0$
  - Viscosity means flow will be zero at the surface of a body. The boundary layer thickness (δ) is defined as the height above the body where the velocity in 99 percent of the freestream value.
  - Turbulent flow. Unsteady, random flow with eddies of varying sizes and frequencies. Boundary layer growth faster for turbulent flow. (so more exponential looking curve on velocity profile)
  - Favourable pressure gradient - flow is accelerating (velocity increasing and pressure dropping)
  - Adverse pressure gradient - flow is decelerating (velocity decreasing and pressure increasing). Sharp expansions lead to this.
  - Good to know. Laminar flow separates much earlier than turbulent flow. The wider the wake the higher the drag. The earlier the separation the lower the pressure. Skin friction drag much higher than form drag for streamlined bodies. Cavitation can occur when the pressure in a liquid drops below vapour pressure, it causes erosion problems in propellers.

- Total Pressure $P_0 = p + \frac{1}{2}\rho V^2 + \rho g z = constant$
- Mass flow rate (Continuity) $\frac{dm_1}{dt} = \frac{dm_2}{dt} \rightarrow \rho A_1 V_1 = \rho A_2 V_2 \Rightarrow A_1 V_1 = A_2 V_2$
  - Valid for viscous and inviscid flow
- Reynold's number (L is reference length, v is viscous force) $:= Re = \frac{\rho V L}{\mu} = \frac{VL}{v} = \frac{intertia force}{viscous force}$ Low Re implies viscous forces are important to the flow. High Re implies there are significant viscous forces confined to small region near body. Transition Re - flow becomes turbulent. Decelerating flow (adverse pressure gradient), turbulent flow and increased surface roughness lowers the Reynolds number of a body.
  - For high Reynolds number flow over bluff bodies:
    * Subcritical ($Re \approx 4 \times 10^2$ to 3×10^5) - early laminar separation
    * Critical (($Re \approx 3 \times 10^5$ to 3×10^6) - laminar separation bubble (i.e. two separation points on each side of the body) + delayed turbulent separation.
    * Supercritical ($Re > 3 \times 10^6$) - turbulent separation with constant pressure at the rear.
- Lift and Drag. Let shear stress $= \tau = (\tau_x, \tau_y)$ then:
  - $D = \int \tau_x dx - \int p n_x dy$
  - $L = \int \tau_y dy - \int p n_y dx$
  - where p is the pressure and n_x and n_y are the normal vectors of where the pressure is acting.
  - $C_D = \frac{D}{\frac{1}{2}\rho V_\infty^2 A_D}$
  - $C_L = \frac{D}{\frac{1}{2}\rho V_\infty^2 A_L}$
- Galilean Transformation - Static pressure and temperature remain unchanged.
- Two flows over two different bodies are similar if:
  - Streamlines are the same
  - Bodies are geometrically similar
  - Flow quantities are the same throughout the flow field.
  - Matching similarity parameters.
- Euler Number (Pressure Coefficient, Cavitation Coefficient) $C_P = \frac{\Delta p}{\frac{1}{2}\rho V^2}$
- Mach Number $M = \frac{V}{a}$, where a is a compressibility factor. For $M < 0.3$ in air, the air can be considered incompressible. For very high M inertia effects dominate.

- In Wind Tunnel testing it is difficult to match both Re and M at model scale. Either don't bother by working with very low speeds (Re dominates, M can be neglected) or working with very high speeds (Re can be neglected). You can also change the fluid or do test in a pressurised wind tunnel.

1-D Flow:

- Quasi-one-dimensional Approximation. For 1D flow, we assume **average velocity** across velocity profile, that **curvature** is small and **area variation** gradual.

- Acceleration of Fluid particle. For steady flow, $a_s = \frac{\partial V}{\partial s} V$, where a_s is the acceleration along a streamline. For steady a streamline curve of radius r_s, there a a normal acceleration to the flow $a_n = \frac{V^2}{r_s}$

- Euler's Equation (from conservation of linear momentum) $:= dp + \rho V dV + \rho g dz = 0$. This is valid for inviscid, steady, quasi 1D, compressible and incompressible flow. It can only be integrated if the variation of density with pressure is known.

- **Bernoulli's equation**. Assuming density to be constant, Euler's equation can be integrated to give: $p + \frac{1}{2}\rho V^2 + \rho g z = constant$. It is valid for steady, inviscid, constant-density flow along a streamline.

- Conservation of cross stream momentum. Don't fully understand, but the derived equation is $\frac{\partial p}{\partial n} + \rho g \frac{\partial z}{\partial n} = \rho \frac{V^2}{r_s}$ Assuming steady flow, otherwise valid for compressible, incompressible, viscous, inviscid steady flow. You can use this equation to derive $\frac{\partial p}{\partial z} = -\rho g$.

- For cross-stream pressure variation on a boundary layer, $\frac{\partial p}{\partial n} = 0$, where p is the pressure and n is the local normal to each streamline. Equation not valid for boundary layer separation or streamlines with large curvature.

- The Coanda effect - The reason jets of fluid remain attached to convex bodies. The reason is that the pressure at the body surface is lower than the atmospheric pressure at the free surface. So there is a net force on the body.

- Flow through sharp-edged orifice. Use Bernoulli's equation between any point in the container and the vena contracta. At the vena contracta, the cross sectional area is minimum, the streamlines are parallel and the pressure is equal to the atmospheric pressure p_a. So the equation $p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2$ becomes $\rho g h = \frac{1}{2}\rho V_2^2$, because $p_1 = p_a + \rho g(h - z_1)$ (pressure from the surface) and $p_2 = p_a$ (pressure at vena contracta) and $\rho g z_2 = 0$. Also V_1^2 becomes 0, if the area of the container is large. For a gas, we can neglect hydrostatic terms and from $p = p_a + \frac{1}{2}\rho V_{jet}^2$ we can get $V_jet = \sqrt{\frac{2(p-p_a)}{\rho}}$

- Volume flow rate $Q = A_1V_1 = A_2V_2 = C_d\sqrt{2gh} \times A_{orifice}$, where C_d for a sharp edged circular orifice is around 0.6 to 0.66.

- Pitot tube. Basically a way of measuring total pressure. For steady incompressible flow brought to rest $P_0 = p_1 + \frac{1}{2}\rho V_1^2 = p_2$, where p_2 is the pressure within the tube. This means the pitot tube measurement is static ($= p_a + \rho gh$ if you are dealing with fluids at certain heights) plus dynamic pressure. The static holes must be placed far enough away from the pitot opening to allow static pressure to recover.

- Convergent/Divergent Flow Passage Apply Bernoulli between two points and continuity. Pressure is lower at smaller area and velocity higher. (Makes sense). Velocity is independent of viscous effects, but total pressure will drop due to the static pressure losses turning to entropy. If expansion is too rapid, flow separation will occur.

- Venturi-meter. (Convergent - divergent duct). Used to calculate velocity by taking pressure at two points with cross sectional area. Apply Bernoulli and continuity at same horizontal height.

- Vertical Jet of Water. Use Bernoulli from the point where the water leaves the nozzle to the desired point, because the static pressure will be the same, namely p_a. Note, a jet going upwards will increase in diameter and a jet going downwards will decrease in diameter.

- Suction Device. Set up Bernoulli equations between various points and apply continuity. If another tube is connected to the main tube (like a manometer) the pressure above the tube in the main tube becomes $p_1 = p_a - \rho_m g\Delta h$, where Δh is the change in height of manometer fluid.

- Wind Tunnel. Same principles apply, accept that you cannot use Bernoulli beyond the propeller because energy is lost. Power output of fan $= \dot{E}_{fan} = \frac{1}{2}\dot{m}V_{point}^2$, remember that $\dot{m} = \rho AV$

- Siphon. Set $z = 0$ at the bottom of the siphon. Apply continuity and Bernoulli from top of siphon to bottom.

- Good to know. Far upstream velocity is considered to be zero. Static pressure is constant throughout the width of a tube if the flow is parallel (uniform) and steady. If you are using average velocity and not the velocity along a streamline then you have to include Δp_{loss} term to take pressure losses due to viscous effects into consideration.

Control Volume Analysis:

- Draw a control volume. Then consider the rate at which: mass, linear momentum, angular momentum and energy; enter and leave the CV. Use these to derive forces and moments of the system. Analysis is restricted to steady, incompressible flows.

- Conservation of mass. $\dot{m_{CV}} = \dot{m_{in}} - \dot{m_{out}} = \rho_{in}V_{in}A_{in} - \rho_{out}V_{out}A_{out}$. Inlet and outlet areas must be measured perpendicular to the velocities. For steady problems $\dot{m_{CV}} = 0$, so for steady incompressible (constant density) flow, $V_{in}A_{in} = V_{out}A_{out}$ (Conservation of flow volume).

- Conservation of steady flow linear momentum. The change in momentum of the CV is dependent on the change in velocity, therefore $\delta\underline{M} = (\delta m_{in}\underline{V_{in}} - \delta m_{out}\underline{V_{out}}0$, so for example in the x-direction for steady flow $(\delta m_{in} = \delta m_{out} = \delta m)$ this would be $\delta M_x = \delta m(V_{x,in} - V_{x,out})$. To find the force exerted BY the CV we divide by δt so $\underline{F_{tot}} = \dot{\underline{M}} = \dot{m}(\underline{V_{in}} - \underline{V_{out}})$. For the total force ON a CV $\underline{F_{tot}} = -\dot{\underline{M}} = \dot{m}(\underline{V_{out}} - \underline{V_{in}})$. The components in each direction are shown below:

$$f_{tot_x} = \dot{M}_x = \dot{m}(V_{x,out} - V_{x,in})$$
$$f_{tot_y} = \dot{M}_y = \dot{m}(V_{y,out} - V_{y,in})$$
$$f_{tot_z} = \dot{M}_z = \dot{m}(V_{z,out} - V_{z,in})$$

  The net forces on the system are

  - Pressure forces acting normally to the CV boundary. At constant pressure this integrates to zero.
  - Viscous shear forces. The act tangentially to the CV boundary.
  - Solid objects within the CV.
  - Gravitational forces are integrated over the whole CV.

- Conservation of Energy. First law of thermodynamics states $\delta E = \delta Q - \delta W$ (Energy = Heat - Work done by CV). $\delta W = \delta W_s + \delta W_v + \delta W_p$, Meaning the total work done by the CV is composed of shaft work, viscous work, and pressure work. Also $\delta E = e_{out}\delta m_{out} - e_{in}\delta m_{in}$, where $e = \frac{E}{m}$. You also need to know that the work done by the pressure $\delta W_p = \frac{p_{out}\delta m_{out}}{\rho_{out}} - \frac{p_{in}\delta m_{in}}{\rho_{in}}$ (Pretty cool that Work done = pressure times mass divided by density). Oh and that $e = \hat{e} + \frac{1}{2}V^2 + gz)$, meaning the energy per unit mass consists of intermolecular energy, kinetic and gravitational potential energy. After a fancy derivation you can use this to show the following result (without viscous work $\delta W_v = 0$ and steady incompressible flow $\delta Q = 0$ (adiabatic flow)):

$$p_i + \frac{1}{2}\rho V_i^2 + \rho g z_i = p_e + \frac{1}{2}\rho V_e^2 + \rho g z_e + \rho w_s$$

  where ρw_s is the shaft work done by the CV. They normally write this as ΔP_{loss}. The loss coefficient is defined as $\frac{\Delta p_{loss}}{\frac{1}{2}\rho V_1^2} = (1 - \frac{A_1}{A_2})^2$

- Abrupt enlargement/contraction You have to apply Bernoulli between two points (one in the contraction and one in the enlargement) $p_1 + \frac{1}{2}\rho V_1^2 +$

$\rho g z_1 = p_2 + \frac{1}{2}\rho V_{e2}^2 + \rho g z_2 + \Delta P_{loss}$. Then you have to apply steady momentum equation, which in this case is $P_1 A_1 + P_1(A_2 - A_1) - P_2 A_2 = \rho A_1 V_1 (V_2 - V_1)$. Finally using continuity, you can show that $\frac{\Delta p_{loss}}{\frac{1}{2}\rho V_1^2} = (1 - \frac{A_1}{A_2})^2 = k$.

- The point is to know that energy is converted into shaft work, viscosity (friction) and heat.
- When dealing with flow surrounded by atmospheric pressure hitting a surface, you can use the Steady Flow Momentum Equation $F = \rho A V^2$ to find the reaction force of the surface acting on the water.
- Worked examples, see extra sheet.
- The propeller and turbine. Create two control volumes between four sections of the slipstream boundary. To work out all the equations you can apply continuity, Bernoulli and force balance, but you have to remember the following: The disc is considered so thin that $A_2 = A_3 = A_d$ and $V_2 = V_3 = V_d$ and you can only apply Bernoulli between section 1 and 2 and 3 and 4. If working with atmospheric slipstream boundary leave out all pressure forces (since they all cancel). Key results are:

$$V_d = \frac{1}{2}(V_1 + V_4)$$
$$F = \rho A_d V_d (V_4 - V_1) = (p_3 - p_2) A_d = \frac{1}{2}\rho(V_4^2 - V_1^2)$$

- Key results for propeller
  - If rotor is stationary (e.g. helicopter hovering), then $V_1 = 0$ You can then use this to work out F and V_d ($V_d = \frac{1}{2}V_4$)
  - For a forward moving propeller $V_1 = v$
    * $V_d = \frac{1}{2}(V_4 + v)$
    * $F = \frac{1}{2}\rho A_d (V_4^2 - v^2)$ etc...
    * Efficiency $= \eta = \frac{P_{out}}{P_{in}} = \frac{F V_d}{F V_1} = \frac{2V_1}{V_4 + V_1}$
    * 'inflow factor' $= a = \frac{V_d - v}{v}$
- Key results for a turbine
  - Efficiency $= \eta = \frac{P_{out}}{P_{in}} = \frac{P_{disc}}{P_{wind}} = \frac{-F V_d}{\frac{1}{2}\dot{m} V_1^2} = -\frac{\rho A_d V_d^2 (V_4 - V_1)}{\frac{1}{2}\rho A_d V_d^3} = \frac{(V_4 + V_1)(V_1^2 - V_4^2)}{2V_1^3}$
  - Max efficiency at $\frac{V_4}{V_1} = 1/3$ and min efficiency at $\frac{V_4}{V_1} = -1$. Found by setting $\frac{\partial \eta}{\partial V_4} = 0$

Potential Flow:

Derivation comes from fundamental principles of conservation of mass, conservation of momentum and conservation of energy.

Main Assumptions:

- Irrotational Flow
- Flow along a stream-line
- Incompressible Flow
- Viscous effect negligible (Inviscid)
- Steady

Equations for a streamline are:

$$\frac{dy}{dx} = \frac{u}{v} \tag{3}$$
$$\psi(x, y) = constant. \tag{4}$$

D'Alembert's paradox states that the drag for any body placed in an inviscid, incompressible 2D flow will be zero, irrespective of circulation.

Stream function v Potential Flow function.

- ψ can be used for rotational flow, while ϕ only implies irrotationality
- ψ is only defined for 2D flow, while ϕ exists for 3D flow.
- $\Delta\psi$ gives volume flow rate and $\rho\Delta\psi$ gives mass flow rate
- Side notes:
  - They are orthogonal functions
  - $\nabla^2\psi = \nabla^2\phi = 0$

For the **superposition of a source and a sink** the stream-function can be found at any point P as $\psi = \frac{\Lambda}{2\pi}\theta_1 - \frac{\Lambda}{2\pi}\theta_2$, where θ_1 and θ_2 are angles from the horizontal to the point.
Stagnation points. Differentiate stream or potential function to find velocities. Then set velocities to zero and solve for x, y or θ, r depending on which coordinate system you are working with. To find velocities on the 'dividing stream line' (the body) simply plug in the values for x, y or θ, r into the stream function and you should get a constant, because remember $\psi(x, y) = const.$ (It's a parametric equation and the constant determines which streamline you are investigating).
Max width. Use Cartesian form and set $x \to \infty$. Then solve for y. Answer gives height - Remember to double if asked for width.
Solving for Stream function ψ and Potential Flow function ϕ. For

Cartesian or Polar velocities, partially differentiate and then solve for constants by integrating. For example, in polar coordinates say $V_r = \frac{\Lambda}{2\pi r}$ and $V_\theta = 0$, then do the following:

$$V_r = \frac{\partial \phi}{\partial r} = \frac{\Lambda}{2\pi r} \tag{5}$$

$$\phi = \frac{\Lambda}{2\pi} \ln r + f(\theta) \tag{6}$$

$$V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{1}{r}\frac{\partial f(\theta)}{\partial \theta} = 0 \tag{7}$$

$$f(\theta) = const. \tag{8}$$

$$\phi = \frac{\Lambda}{2\pi} \ln r + const. \tag{9}$$

Do exactly the same with the given partial differential equations for the stream function.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{10}$$

$$C_P = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{V^2}{U_\infty^2} \tag{11}$$

Circulation - Circulation is zero around a curve if the flow is irrotational everywhere within the curve. Circulation is present in irrotational potential flow because of vortices.

- **Lift**
  - The force acting vertically down on a body is $dF = pds$. If this body is circular then $ds = Rd\theta$ (because it is the arc length), so $dF = pRd\theta$
  - Lift is opposite to the downward force, so you have to take the opposite and integrate over the area you are looking to find the lift for. For the whole circle this would be, $l = -\int_0^{2\pi} p(\theta)Rsin(\theta)d\theta$
  - Kutta-Joukowski Theorem says $l = \rho_\infty U_\infty \Gamma$
  - Apparently the stagnation point for a lifting cylinder is at $\theta = 270$ degrees.