Control 2

Dedicated to Thomas Alva Edison

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Closed loop vs Open

1. CL control system is able to reduce system **sensitivity**.
2. In CL system the feedback parameters can often be adjusted to yield the desired result.
3. A CL system has a small steady state small steady-state error, while an OP system will have a 'significant' steady state error.
4. OL systems are more unstable
5. OL systems are cheaper.

1 Introduction

- Assume all systems are linear time-invariant systems (LTI) (i.e. they scale linearly and are independent of time) and that they are also SISO system (single-input single-output systems).
- OL systems examples - toasters, gas ovens, record turntable
- CL systems examples - guided missiles, car and driver
- Automatic control - control without humans
- AIM: To make the output equal the input demand and control how it gets from the input to the output.
- $\boxed{ERROR = INPUT - OUTPUT}$

2 System Modeling

2.1 Standard forms - always put your ODE's into standard form

- 1st-order system standard form of ODE: $\boxed{\tau}\dot{x} + \boxed{x} = Ky$, where τ is the **time constant** and K is the **steady state gain**. IN L. domain $X(s) = \frac{KY(s)}{\tau s+1}$
- 2nd-order system standard form of ODE: $\boxed{\frac{1}{\omega_n^2}}\ddot{x} + \boxed{\frac{2\zeta}{\omega_n}}\dot{x} + \boxed{x} = Ky$, where ζ is the **damping coefficient** and ω_n is the **natural frequency** of the system IN L. domain $X(s) = \frac{Aw_n}{\boxed{s^2 + 2\zeta\omega_n s + \omega_n^2}}$
- TRANSFER FUNCTION = OUTPUT/INPUT (in s-domain). For simple closed loop system $C.L.T.F = \frac{G(s)}{1+G(s)H(s)}$, since Error $E(s) = R(s)[input] - C(s)H(s)[output * loopgain]$ and $C(s)[output] = G(s)E(s)[error * forwardpathgain]$.
- Generally the C.L.T.F can be written as $\frac{something}{1+K(O.L)}$ and the open loop transfer function O.L.T.F is the output if you where to cut the feedback line just before summation: so $O.L = G(s)$ for unity feedback, but $O.L = G(s)H(s)$ for non unity feedback.

2.2 Basic electronics

- Remember voltage flows in opposite direction to current.
- **Resistors** Ohm's law $V = iR$ IN L.T. $V_R(s) = RI(s)$
- **Capacitors** $i = C\frac{dV}{dt} = C\dot{V}$ IN L.T. $V_C(s) = \frac{1}{SC}I(s)s$
- **Inductors** $V = L\frac{di}{dt} = L\dot{i}$ or $i = \frac{1}{L}\int V dt$ IN L.T. $V_L(s) = LI(s)s$
- KCL (Kirchoff's first law) - currents entering a junction equal currents leaving.
- KVL (Kirchhoff's second law) - sum of voltages in a loop add to zero
- See little note for more info

2.3 DC Motor

- Governing Equations: $\boxed{F = BilN = K_T i}$, $Torque = Fr = K_T ir = J\dot{\omega} \rightarrow \boxed{i = J\dot{\omega}/K_T}$,
- Emf of motor $= NBl\omega r = K_e\omega$
- Then you'll be given some circuit to work out governing equations

2.4 Water Tank

- Governing equations: change of mass = rate of input (q1) - rate of output (q2), mass = density*volume, where volume is change in height times cross-sectional area (sca) (A(h1 - h2) = $\dot{h}$A).
- And finally key thing: Flow rate output is proportional to height and if the output of tank goes into another tank then it is proportional to the difference of the two tank water heights.

2.5 Simplifying circuits

- To get to unity feedback simply solve $\frac{G(s)}{1+G(s)H(s)} = \frac{T(s)}{1+T(s)}$ to find T(s). Note this equation only holds for a gain of $K = 1$. The systems will be behave differently for different values of K.

2.6 Block diagram representation

- Remember: $\boxed{output/input = transferfunction}$ so a line entering a box is the input, the box is the transfer function and output is the line leaving the box.
- If there is a summation in the line then add whatever is coming into the line.
- If the line splits or there is a junction the line carries the information to both lines.

3 Initial Analysis of Transient Response

[If a linear approximation can be made to the system (i.e. the ODE's) then Laplace transforms can be used in analysis]

- Roots of characteristic equation (C.E.) (denominator of L.T of ODE) are called 'poles' and roots of nominator are called 'zeros'.

3.1 Solving ODE's and stability

- If poles of denominator of transfer function are negative real then solution to ode is stable (i.e. it converges).
- Use: $L[1] = \frac{1}{s}$, $L[t^n] = \frac{n!}{s^{n+1}}$ $(L^{-1}[\frac{1}{s^{n+1}}] = \frac{t^n}{n!}$, $L[e^{-at}] = \frac{1}{s+a}$ if $Re(s+a) > 0$, $L[cos(\omega t)] = \frac{s}{s^2+\omega^2}$, $L[sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$
- Find **time response** by taken inverse L.T.
- Check convergence with **FINAL VALUE THEOREM**: $\boxed{\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)}$, where $Y(s)$ is the transfer function here.

3.2 Second order Time response

- Poles of C.E. in standard form (see X(s) in 2.1.) are $\boxed{s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}}$
- Useful equations for s-plane drawing: $\boxed{\zeta = cos(\theta)}$ (θ measured from negative half upwards and only true for $0 < \zeta < 1$), $\boxed{\omega_n = |s_1| = |s_2|}$, $\boxed{\omega_n\sqrt{1-\zeta^2} = \frac{\pi}{T_p}}$, $\boxed{\zeta\omega_n = \frac{4}{T_s}}$
- **DAMPING** - $\zeta > 1$ over-damped, $\zeta < 1$ under-damped, $\zeta = 1$ critically damped (roots real and repeated), $\zeta = 0$ undamped.

3.3 Forced and Natural Response

- Time response consists of two components: forced response - from input stimulus and natural response - from system dynamics.

3.4 Sensitivity

- Definition: Sensitivity of the transfer function (or something else like steady state error) $T(s)$ with respect to $G(s)$ is $S_G^T = \frac{\frac{\Delta T(s)}{T(s)}}{\frac{\Delta G(s)}{G(s)}}$ in the limit $S_G^T = \frac{G}{T}\frac{\delta T}{\delta G}$ (derivative)
- The sensitivity can be measured with respect to any parameter, not just G, e.g. $\boxed{S_a^T = \frac{a}{T}\frac{\delta T}{\delta a}}$

3.5 Transient Response

- Def: The response of a system as a function of time. C.L system has a better response.
- Tests: impulse $\delta(t)$ $\Delta(s) = 1$; step(position) $u(t) = P$ $U(s) = P/s$; ramp(vel.) $v(t) = Pt$ $V(s) = P/s^2$; parabola(acc.) $a(t) = Pt^2$ $A(s) = 2P/s^3$
  - Might see this formulation $5u(t) \to 5/s$, or $5t^2u(t) = 5/s^3$

3.6 Steady-state error

- YOU must bring the system into a unity gain feedback system to apply this.
- Recall for a O.L system $E(s) = R(s) - C(s) = [R(s) - R(s)G(s)] = [1 - G(s)]R(s)$ and recall that for a C.L. system $E(s) = R(s) - C(s)H(s) = R(s) - (E(s)G(s))H(s) = R(s) - G(s)E(s)$ (assuming H(s) = 1, unity feedback system) then $E(s) = \frac{1}{1+G(s)}R(s)$
- The steady state error, e_{ss}, is given by final value theorem $e(\infty) = \lim_{s\to 0} sE(s)$, where E(s) will be some function of R(s) (step input, impulse etc.)
- $G(0)$ is called the DC gain and is normally > 1. Therefore $e(\infty)_{C.L} < e(\infty)_{O.L}$

3.7 Feedback and Disturbance

- Disturbance signal can be minimised/eliminated with a closed loop system by increasing the transfer function (I assume gain is meant by increase) before disturbance.

4 Definitions of Pole/Zero Position for Transient Response

4.1 Pole position and transient response

- -ve and complex poles correspond to a decaying sinewave (e.g. $e^{-at}sin(\omega t)$) $\rightarrow$ Decay rate $\tau = 1/a$ and frequency of oscillation $= w$
- Generally real part tells us about rate of decay and imaginary about frequency.

4.2 Effect of Additional Poles and Zeros

- A system with additional poles and zeros can be approximated by a second order system if:
  - They have **2** dominant poles
  - All other poles can be neglected if they are more than five times further from the origin then the dominant poles
- Additional zeros act as gain factors if they are far enough from the dominant poles.

4.3 Standard Performance Measures

- Rise time: Measure of swiftness of response
- Peak time: Measure of swiftness of response - $T_p = \frac{\pi}{w_n\sqrt{1-\zeta^2}}$ (0 - 100% used for underdamped systems and 0 - 90% for overdamped systems)
- Overshoot: (max - final)/final - $\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \Rightarrow \boxed{\zeta = \frac{-ln(\%OS/100)}{\sqrt{\pi^2 + ln^2(\%OS/100)}}}$
- Settling time: time until output is within 2% of final value - for second order system this is obtained at 4τ. Hence because decay is given by $e^{-\zeta w_n}$ $\boxed{T_s = \frac{4}{\zeta w_n}}$

5 Closing the loop

5.1 Stable systems

- bounded input and bounded in magnitude output produce stable system. Poles in LH plane (bounded output) produce stable systems.

5.2 Routh- Hurwitz Stability criterion

- You can only apply the Routh- Hurwitz algorithm if all coefficients of the characteristic equation are of the **same sign** and **non-zero**. If either of these fail, the system is automatically unstable.
- Criterion for stability: "There can be no change in sign in the first column coefficients of the Routh array". "The number of sign changes (e.g. + - + = 2 sign changes) corresponds to the number of poles in the R.H plane".
- There are two unusual cases you need to be aware of:

- If there is a zero in the first column but no other zeros in that row. Simply substitute this with a small positive real number ϵ and continue as you would otherwise. But make sure to take the limit as $\epsilon \to 0$ at the end.
  - If there is a zero and its entire row contains zeros then you have to differentiate the row above that row (remembering that the coefficients belong to the "s"'s in a zig-zag fasion). Finally you must also check the stability of the equation you just differentiated.

- To make calculations quicker when you want to find s for a certain gain K instead of solving the entire characteristic equation you can use one of the 'auxiliary' equations. (I think).
- If you get 'system is stable' if $K > 2$ then this means that the system is marginally stable at $K = 2$.
- See extra notes on how to perform algorithm

5.3 System type number

- Remember: First convert into a unity gain feedback system. For the next bit, let the transfer function in this unity feedback system be called 'G(s)'.
- Then G(s) is of the form $G(s) = \frac{K\prod(s+z_i)}{s^N\prod(s+p_k)}$. The type number is given by N. Then following holds:
  - For a type 0 system:
    * $e_{ss} = \frac{A}{1+G(0)} = \frac{A}{1+K_p}$ for a STEP INPUT (A/s) (K_p is the position constant)
    * $e_{ss} \to \infty$ for Ramp (A/s^2) and Acceleration (A/s^3) inputs
  - For a type 1 system:
    * $e_{ss} = 0$ for step input
    * $e_{ss} = \frac{A\prod p_i}{K\prod z_i} = \frac{A}{K_v}$ for a RAMP INPUT (A/s) (K_v is the velocity constant)
    * $e_{ss} \to \infty$ for Acceleration input.
  - For type 2 system:
    * $e_{ss} = 0$ for step and ramp input
    * $e_{ss} = \frac{A\prod p_i}{K\prod z_i} = \frac{A}{K_a}$ for an ACCELERATION INPUT (A/s)
  - For types of $N \geq 3$, $e_{ss} = 0$
- Side note - a system only has a steady-state error if the system is stable $\to$ check this with Routh-Hurwitz array.

6 Root locus techniques

6.1 Introduction

- The root-locus is a plot of the poles of the characteristic equation of the **closed loop system**
- Basically all it does is plot the the solution to the equation $\boxed{1 + KG(s)H(s) = 0(= 1 + K\frac{Z(s)}{P(s)} = P(s) + KZ(s) = 0)}$ (if unity gain then H = 1). Use this property, also written $|KGH| = 1$ to find K for any s on the root locus.
- A very important property of the root-locus is that for a pole/zero to be on the root locus, all angles from the original roots and zeros to that pole/zero must add up to 180 degrees, i.e. $\boxed{arg[KGH] = 180°}$

6.2 Root locus Construction Rules

1. There are n lines corresponding to the number of poles or zeros of the open-loop t.f. depending on which has more - 3 poles, 2 zeros corresponds to 3 lines.
2. As K goes from 0 to infinity the roots move from the open-loop poles to the open-loop zeros.
3. Complex roots always come in pairs - so once the rootlocus becomes complex there will be a perfect reflection on around the x-axis.
4. At no time will a root locus of a pole cross over its on path
5. This one is a bit odd, but useful - The portion of the **real axis** to the left of an odd number of open loop poles and zeros are part of the the loci (counting from right to left).
6. Lines break out and into the real axis at 90°. (Note this occurs at the maximum value of **real axis** gain K.)
7. If there are not enough poles and zeros to make pairs then the extra lines go to infinity. (Poles GO to infinity, and zeros COME from infinity - odd).
8. Lines go to or come from infinity along asymptotes.
   - Number of asymptotes n_a = no. of poles - no. of zeros
   - Angle of asymptotes: $\phi_a = \frac{\pm 180 \pm n360}{n_a}$
   - Centroid of asymptotes: $\alpha = \frac{\sum Re(Poles) - \sum Re(Zeros)}{n_a}$
9. If there are at least 2 lines going off to infinity then the sum of the roots is constant.
10. Angles of departure: Use $\sum \phi_z - \sum \phi_p = 180$ to find the angle - note once you have one, you can reflect it if the roots are complex (using rule 3).
11. Imaginary axis crossover: Compute Routh-Array for K and find the value of K where K is marginally stable (if K ¿ 2, then K is marginally stable at K = 2). Then use the closed loop C.E. or an auxiliary equation and solve for s. Answer should be complex.
12. Break in/away from real axis: Two options, either use this equation $Z(s)\frac{dP(s)}{ds} - P(s)\frac{dZ(s)}{ds} = 0$ and solve for s or find the value max of gain K for the real axis breakaway points.

Beware of double poles! I think they give each other 90 degree angles not zero angles.

7 Controller Design Rules

7.1 Control design via root locus

- Transient response is improved my the addition of a forward path zero (lead design)
- Steady state error is improved my the addition of a forward path pole (lag design)

7.2 Ideal Integrator Compensator (PI Controller)

- General form: $\frac{s+a}{s}$
- A pole is placed at the origin (and zero close by to ensure transient response is not affected), to increase the system type number and thereby reduce e_{ss} to zero.

7.3 Lag Compensator

- General form: $G_c(s) = \frac{(s+z_c)}{(s+p_c)}$, $|z_c| > |p_c|$, i.e the pole is closer to the origin.

1. Find type of system
2. Find $K_{p,v,a}$ (depending on type of system) by finding $KG(0)$
3. Use appropriate e_{ss} formula
4. Change e_{ss} to what is desired and rearrange equation to find new $K\prime_{p,v,a}$
5. Finally use the fact that $K\prime_{p,v,a}/K_{p,v,a} = z_c/p_c$ to find z_c after arbitrarily (as in you decide where) placing p_c close to the origin (e.g $p_c = -0.01$)
6. Finally find the gain K!

7.4 Ideal Derivative Compensator (PD Controller)

- General form: $(s + z_c)$, where z_c is found using the second root locus criterion $\phi_{z_c} + \sum \phi_z - \sum \phi_p = 180$ from the desired 2nd order poles. It's pretty genius - basically it's saying "give me the transient response you want, i.e. damping ratio, settling time etc., and i'll find those points in the s-plane and I'll make the current root-locus go through that point so that the second criterion (180 criterion) holds by adding a zero".
- Cheeky note: I think its like a lead compensator, except the pole is place at negative infinity, so its angular contribution is basically zero.

7.5 Lead Compensator

- General form: $G_c(s) = \frac{(s+z_c)}{(s+p_c)}$, $|p_c| > |z_c|$,

1. Find design points, i.e. the desired points in the s-plane that produce the required design requirements - use section 4.3 (Standard Performance Measures for this)
2. Arbitrarily chose where to position the compensator zero z_c.
3. Then find angular contribution ϕ_{p_c} needed from p_c so the design point is on the root locus.
4. Use trigonometry (tan function) to then find the position of the pole p_c.
5. Finally find the gain K.

- Design depends on using the second order approximation so make sure to check this!

Side note: Lag compensation can also be used to change transient response (if the desired response is to the right of the current root-locus (since it requires pole nearer to the origin than a zero). Also you can use $\sum$ angle of poles - $\sum$ angle of zeros = 180 to check second criteria.

8 Frequency Response Techniques using Bode Plots

8.1 Introduction

- Frequency response is the steady-state response for sine waves at all values of frequency (ω)
- For the following $G(S)$ represents the open-loop transfer function (in practise it could be $G(s)H(S)$...
- Magnitude := $|G(s)| = \frac{|Z(s)|}{|P(s)|} \Rightarrow$ GAIN := $20log_{10}(|G(s)|)$
- Phase: $\phi = arg(G(jw)) = tan^{-1}\left(\frac{Im(G(jw))}{Re(G(jw))}\right) = arg\left(\frac{Z(jw)}{P(jw)}\right) = tan^{-1}\left(\frac{Im(Z(jw))}{Re(Z(jw))}\right) - tan^{-1}\left(\frac{Im(P(jw))}{Re(P(jw))}\right)$

8.2 Frequency Response Methods

- Because a logarithmic scale is used on ω, responses can be added when poles and zeros are added to the system.
- Easy to see effect of individual poles and zeros.

8.3 Drawing Bode Diagrams

- Constant gain K
  - Gain = $20log_{10}(K)$
  - Phase = 0
- Pole/Zero(s) at origin ($(jw)^{\pm n}$)
  - Gain = $-20nlog(\omega)$ (P), $+20nlog(\omega)$ (Z) [goes through $w = 1$ and slope is $\pm 20dB/dec$ depending on whether its a pole or zero]
  - Phase = $-n90°$ (P), $+n90°$ (Z)
- Pole/Zero(s) on real axis ($(1 + \tau w)^{\pm 1}$)
  - Gain: For sketching its 0 until $\omega = 1/\tau$ where it then slopes down (or up) at $\pm 20dB/dec$. In reality it 0 until $\omega = 1/\tau$ where it is $-3dB$ and then it slopes at $\pm 20dB/dec$
  - Phase: $\tan(\tau w)$ $[0 \rightarrow -45°(atw = 1/\tau \rightarrow -90]$
- Poles/Zero(s) on the complex plane [must convert into this form: $\left(1 - \left(\frac{w}{w_n}\right)^2 + 2\zeta\left(\frac{w}{w_n}\right)\right)^{\pm 1}$]
  - Gain: Equation is nasty so just use: [0 $\rightarrow$ until $w = w_n$ where it starts going downwards at $-40dB/dec$]
  - Phase = $-tan^{-1}\left(\frac{2\zeta\frac{w}{w_n}}{1 - \frac{w}{w_n}^2}\right)$ $[0 \rightarrow -90$ (at $w = w_n$) $\rightarrow -180°]$

8.4 Measures of Stability (and other features)

- Gain Margin - margin of gain from zero (downwards being positive) at a phase of 180 degrees.
- Phase Margin - margin of phase from 180 degrees (upwards being positive) at a gain of 0.
- Gain crossover frequency - where the gain crosses the 0 gain axis
- Bandwidth - the frequency at which the gain reaches $-3dB$
- DC gain - I think its where the gain crosses the y axis (gain at very low frequencies)

9 System Identification

- Opposite of bode plots - requires practise!
- If there are 'resonant' peaks in the gain plot then it is likely that there are complex poles in the transfer function that produced that plot.

9.1 Extra little things

- Damping is caused by a reduction in energy, so any real physical system is resistors etc... will reduce the systems energy and thereby cause damping $\rightarrow$ so system cannot go unstable. For instability you need an increase in energy.