

Heat Transfer

Dedicated to Fourier

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1 Essentials

- Heat Transfer: transfer of thermal energy from one (hot) body/atom to another (cold) body/atom (in WATTS, so its really a form of POWER = ENERGY/TIME). Find **rate of Heat transfer** just means find heat transfer Q . They seem to be used interchangeably.
- Conduction: Inter-molecular action (intra-body action), mainly in solids. **Fourier's Law:** $Q = -kA \frac{dT}{dx}$
- Convection: Movement of heated fluid (inter-body action), **Newton's cooling law:** $Q = -hA\Delta T$. If you know the surface temperature, e.g T_b , and the surface area A_s and the surrounding fluid temperature T_∞ and h , then you can find the max (optimal) heat transfer as $Q = -hA_s(T_\infty - T_b)$.
- Radiation: Electromagnetic waves: **Stefan-Boltzmann's Law:** $Q = \sigma AT^4$
- Terminology: Laminar (parallel flow lines, low Reynolds number), Turbulent (eddies, mixing, high Reynolds number 5×10^5 I think), Steady (does not change with time), Developed (does not change with position), Homogeneous (same in all positions $k_x(x, y) = \text{const.}$), Isotropic (same in all direction $k_x = k_y$), Adiabatic (no heat transfer to the surroundings), Incompressible (density does not change, $\nabla \cdot V = 0$), Bounded flow (flow limited by boundaries), Newtonian Fluid (shear stress proportional to velocity gradient $\tau = \mu \frac{du}{dy}$)
- Numbers:
 - Fourier number $Fo = \frac{\alpha dt}{L^2} = \frac{\alpha t}{L^2}$ is the ratio of diffusive transport rate to storage rate. (dimensionless)
– used for finite difference simplification. For numerical stability $Fo \leq \frac{1}{2}$ and lumped capacitance.
 - Biot number $Bi = \frac{hL}{k} = \frac{hV}{kA_s} = \frac{hdx}{k}$ is the ratio of conductive heat resistance to convective heat transfer in an object. For $Bi < 0.1$ we have solids that are 'thermally thin', i.e. the object is at a constant temperature throughout.
 - Reynold number (Inertial/viscous forces $Re = \frac{\rho u x}{\mu}$) (describes flow)
 - Prandl number (viscous/thermal diffusion $Pr = \frac{\mu C_p}{k}$) (describes fluid). $Pr = 1, \delta_H = \delta_T$ for most gases. $Pr < 1, \delta_T > \delta_H$ for liquid metals, and $Pr > 1, \delta_T < \delta_H$ for oils.
 - Nusselt number (Convective/conductive heat transfer $Nu = \frac{hL}{k}$) (describes heat transfer)
- Constants:
 - Thermal conductivity $k[W/(mK)]$
 - Specific heat capacity at constant pressure $C_p[J/(kgK)]$
 - Density $\rho[kg/m^3]$
 - Thermal diffusivity $\alpha = \frac{k}{\rho C_p}[m^2/s]$
 - Heat transfer coefficient $h[W/(m^2K)]$ (lower for gases). It's not actually a constant, it depends on fluid properties, flow structure (turbulent or laminar), **velocity of flow**, and geometry of surface. I think generally $h > 0$.
 - Dynamic viscosity μ

2 Conduction

- Sign convention: Q is positive in direction of heat flow (i.e. when going from hot to cold). **Remember to always set x in the direction going from hot to cold.**
- Conduction depends on geometry, material properties, boundary conditions, non-steady conditions, heat generation in material.
- **Total Heat transfer** is the heat generated times the volume of the body $Q_g = q_g V$

2.1 Fourier's law

- For 1D flow through a planar wall: $Q = -kA \frac{dT}{dx}$. For flow through a planar composite wall: $Q = -UAdT$, where $\frac{1}{U} = \frac{x_{12}}{k_{12}} + \frac{x_{23}}{k_{23}} + \dots$ (Combine thicknesses (Δx) and thermal conductivities into a single overall heat transfer coefficient U).
- Fourier's law can always be applied.

2.2 The General Conduction Equation

In Cartesian coordinates:

$$\underbrace{\frac{\partial}{\partial x} \left(k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial T}{\partial z} \right)}_{\text{Conduction}} + \underbrace{q_g}_{\text{int. gen.}} = \underbrace{\rho C_p \frac{\partial T}{\partial t}}_{\text{int. storage}}$$

In cylindrical coordinates:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(k_r r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \phi} \left(k_\phi \frac{\partial T}{\partial \phi} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial T}{\partial z} \right) + q_g = \rho C_p \frac{\partial T}{\partial t}$$

Spherical coordinates:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(k_r r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left(k_\phi \frac{\partial T}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(k_\theta \sin \theta \frac{\partial T}{\partial \theta} \right) + q_g = \rho C_p \frac{\partial T}{\partial t}$$

2.2.1 Derivation of Conduction Equation in Cartesian coordinates

- (Aside: Total heat flow through an object $Q = -\sum_{i=1}^n k \delta A_i \left(\frac{\delta T}{\delta n} \right)_i$, where n is normal to the isothermals in the object.)
- 1) Create a 3D 'control volume' of a solid (so no change in mechanical energy or work done and only thermal energy needs to be considered.)
- 2) Then $Q_{i+di} = Q_i + \frac{\partial Q_i}{\partial i} di$, $i \in \{x, y, z\}$, $Q_{generated} = q_g dx dy dz$, $Q_{stored} = \rho C_p \frac{\partial T}{\partial t} dx dy dz$ (essentially just internal energy (stored energy) over time)
- 3) Apply energy balance $Q_{in} + Q_{generated} = Q_{stored} + Q_{out}$
- 4) Finally use Fourier's law to find the $\frac{\partial Q_i}{\partial i}$ terms by differentiating, e.g. $\rightarrow \frac{\partial Q_x}{\partial x} = -\frac{\partial}{\partial x} (k_x \frac{\partial T}{\partial x}) dy dz$ to arrive at the GCE.

2.3 Solving the GCE

- We require *two* boundary condition for each dimension x, y, z and *one* initial condition ($t = 0$).
 - Dirichlet - const. surface temperature $T(0, t) = T_{surface}$
 - Neumann - const. surface heat flux $-k \frac{\partial T}{\partial x}|_{x=L} = q_{surface}$ or $\frac{\partial T}{\partial x}|_{x=L} = 0$ for adiabatic (insulators) (from Fourier, since $Q = 0$), notice how there is no k here. If there is a maximum allowable temperature in an insulator (say T_i) then you can have then you have two boundary conditions: $\frac{\partial T}{\partial x}|_{x=L} = 0$ because its insulator and $T(L) = T_i$ because the temperature is known.

- Robin - convection surface condition (convection = conduction) $-k \frac{\partial T}{\partial x}|_{x=0} = -h[T(\infty, t) - T(0, t)]$
- Use separation of variable technique with dummy variable $\theta = \frac{T-T_1}{T_2-T_1}$ to simplify things.

2.3.1 Conduction Assumptions and Consequent Simplifications

- Homogeneous and Isotropic $\rightarrow k = \text{const.} \rightarrow k \nabla^2 T + q_g = \rho C_p \frac{\partial T}{\partial t} \rightarrow \nabla^2 T + \frac{q_g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$
- No internal heat generation $\rightarrow q_g = 0$
- The process is in steady state $\rightarrow \frac{\partial T}{\partial t} = 0 \rightarrow \frac{\partial}{\partial x} \left(k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial T}{\partial z} \right) + q_g = 0$
- Number of dimensions $\rightarrow y, z$ may not be relevant.

2.4 Numerical Methods

- Used mainly for 2D or 3D cases, where an analytic solution is too hard to find.

2.4.1 Finite difference approximations in space

- 1) Create a grid with lengths dx and dy in the x-y directions. If square then $dx = dy = a$.
- 2) Use Taylor expansion around an origin $(x = 0, y = 0) = T_{i,j}$ for both x, y directions. E.g. $T(x) = T_0 + \frac{dT}{dx}|_{x=0} \frac{x}{1!} + \frac{d^2T}{dx^2}|_{x=0} \frac{x^2}{2!} \dots$ and the same goes for y .
- 3) Then substitute in $x = a$ or $y = a$ (or if not square then $x = dx, y = dy$) to find $T_{i,j+1}, T_{i,j-1}, T_{i+1,j}, T_{i-1,j}$ in terms of the second order derivative we are trying to get rid of. For first order derivatives try and get them to be given in terms of T_{i+1} and T_{i-1} , e.g. $\frac{dT}{dx} = \frac{T_{i+1} - T_{i-1}}{2dx}$
- 4) You should end up with a simple averaging equation, i.e the centre node temperature is an average of its surrounding node temperatures (classic Laplacian): $T_{i,j} = \frac{1}{4}(T_{i,j+1} + T_{i,j-1} + T_{i+1,j} + T_{i-1,j})$
- For cylindrical GCE use Taylor series in the same way just for r . For a simple 1D case $GCE = \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = 0 \rightarrow \frac{\partial T}{\partial r} + r \frac{\partial^2 T}{\partial r^2} = 0$ (product rule) and after finite difference approximation: $T_i = \frac{\Delta r}{4r} (T_{i+1} - T_{i-1}) + \frac{1}{2} (T_{i+1} + T_{i-1})$

Methods:

- Take an initial guess for unknown node temperatures. Then repeatedly go through the grid always updating the initially unknown temperatures using surrounding nodes (inc. you don't know) until they converge. The difference between the **Jacobi method** and **Gauss-seidel method** lies entirely in when an 'update' is made.
- The Jacobi method updates by considering the previous update for all unknown temperatures, can be represented by a table i.e. 1.[0, 0, 0], 2.[1, 3, 3], 3.[4, 5, 6]...
- Gauss-Seidel updates every time an unknown node temperature changes, e.g. 1.[0, 0, 0], 2.[1, 0, 0], 3.[1, 5, 0], 4.[1, 5, 6], 5.[4, 5, 6]
- There is another method, called 'Successive over-relaxation method' but don't think it is examinable [check though].
- Note **corner nodes** are considered to be the average temperature of two connecting sides.
- Heat transfer through the side of a meshgrid works as follows: Use fourier's law for each node on the side you are calculating the heat transfer through; k is often unknown so just keep it in the equation, $A = dx \times 1 (= dy)$, where $1 = dz$ (a unit of) (imagine the grid going into the plane), dT/dy or dT/dx (depending on side) $= (node_w - node_{in})/dy$. If its a corner node, then $A = A/2$. The area goes from a half a node spacing to another half a node spacing for each node, which is why this is the case for corner nodes. Finally then compute the sum of the heat transfers to get the total through the wall.

2.4.2 Finite difference approximations in time

- We need this for a when a problem is not in steady state, i.e. $\frac{\partial T}{\partial t} \neq 0$
- *Forward difference (explicit)* (Euler method) approximates $T_{i,j}^{t+1} = T_{i,j}^t + \frac{\partial T}{\partial t}|_t \frac{dt}{1!} + \dots$ (Use this to then find $\frac{\partial T}{\partial t}$).
 - Explicit implies the temperature of interest appear only once and all temperatures at time t are known temperatures.
- For example if the GCE is $\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ the using finite difference we get $\frac{1}{\alpha} \frac{T_{i,j}^{t+1} - T_{i,j}^t}{dt} = \frac{(T_{i+1,j}^t + T_{i-1,j}^t - 2T_{i,j}^t)}{dx^2}$, which gives: $T_{i,j} = Fo(T_{i+1,j}^t + T_{i-1,j}^t) + (1 + 2Fo)T_{i,j}^t$ as the iterative method, where $Fo = \frac{\alpha dt}{dx^2}$.
 - For stability (i.e. for it to iteratively converge we require $Fo \leq 0.5$ in this case). In 2D it would be $Fo \leq 0.25$.
 - **Generally, for numerical stability we require the coefficient of $T_{i,j}^t$ to be ≥ 0 .**
- *Backward difference (implicit)* (Backward Euler). Apply same idea only use $T_{i,j}^{t+1} = T_{i,j}^t - \frac{\partial T}{\partial t}|_t \frac{dt}{1!} + \dots$ to get $T_{i,j}^{t+1} = \frac{T_{i,j}^t + Fo(T_{i+1,j}^{t+1} + T_{i-1,j}^{t+1})}{(1 + 2Fo)}$
 - Implicit implies temperature of interest depends on other unknown temperatures at new time step

2.4.3 Dealing with boundary conditions

(For when boundary conditions aren't Dirichlet)

- Neumann boundary condition $\frac{\partial T}{\partial x}|_{i,j} = -\frac{q_{surface}}{k}$. (If heat is coming in then $q_{surface} > 0$). Use this information in the Taylor expansion around a different point dx away. Then insert this into PDE and rearrange.
- Robin (Convective) conditions. This time $\frac{\partial T}{\partial x}|_{i,j} = \frac{-h}{k}(T_\infty - T_{i,j})$, then just do the same as above. You can use simplification constant $Bi = \frac{h dx}{k}$.
- **Adiabatic:** Implies nodes outside the surface are equal to those inside, if R_2 is at the boundary then $T_{R_2+dr} = T_{R_2-dr}$.

2.4.4 Lumped Capacitance - These are the really fucking hard looking questions

- If thermal conductivity $k \gg h$ heat transfer coefficient, then the material can be considered isothermal. If the Biot number is less than 0.1, $Bi = \frac{hL}{k} < 0.1$, then one can apply the lumped capacitance method.
- The lumped capacity energy equation: $hA_s(T_\infty - T) = mC_p \frac{dT}{dt}$.
- Solve using substitution $T - T_\infty = \theta$ and separate variables and integrate and use IC $\theta = \theta_0$ at $t = 0$ to get $\theta = \theta_0 e^{-\frac{hA_s}{mC_p}t} = \theta_0 e^{-BiFo} = \theta_0 e^{-t/\tau}$, where $\tau = \frac{mC_p}{hA_s}$, $Bi = \frac{hL}{k}$, $Fo = \frac{\alpha t}{L^2} = \frac{k}{\rho C_p L^2}$.
- To reach say 99% of th final temperature then $\frac{\theta_0 - \theta}{\theta_0} = 0.99 \rightarrow \frac{\theta}{\theta_0} = 0.01$.

2.5 Heat Transfer through Fins

- Background: Fins are used to increase heat transfer from a surface to the surrounding fluid by *increasing* surface area. They are used for heat sinks and heat exchangers.

$$\frac{d^2 T}{dx^2} + \left(\frac{1}{A_c} \frac{dA_c}{dx} \right) \frac{dT}{dx} - \left(\frac{1}{A_c} \frac{h dA_s}{k dx} \right) (T - T_\infty) = 0$$

2.5.1 Derivation of General Fin Equation

- 1) Make a sketch of a slice of fin and add heat transfers Q_x coming in and out $Q_{x+dx} = Q_x + \frac{dQ_x}{dx} dx$ and dQ_{conv} (normal to surface) and dx and surface area $A_s(x)$ and cross-sectional area $A_c(x)$.
- 2) Using Fourier's law $Q_x = -kA_c \frac{dT}{dx}$ and $Q_{x+dx} = -\frac{d}{dx}(kA_c \frac{dT}{dx})dx$ and ofc $dQ_{conv} = -hdA_s(T_\infty - T)$
- 3) Use conservation of energy $Q_x - Q_{x+dx} = dQ_{conv}$
- Finally sub in, use product rule and rearrange to get the General Fin Equation: $\frac{d^2 T}{dx^2} + \left(\frac{1}{A_c} \frac{dA_c}{dx} \right) \frac{dT}{dx} - \left(\frac{1}{A_c} \frac{h}{k} \frac{dA_s}{dx} \right) (T - T_\infty) = 0$
- Assumptions: Homogeneous, isotropic, steady state, no internal heat generation, 1D, uniform convection.
- Alternative derivation: Use GCE, and set $q_g = \frac{-h}{A_c} \frac{dA_s}{dx} (T - T_\infty)$ which comes from $dQ_{conv} = -hd \underbrace{A_s}_{\text{surface area}} (T_\infty - T) = -q_g \underbrace{A_c dx}_{\text{volume}}$

2.5.2 Prismatic fins

- Cross-sectional area is constant $\rightarrow \frac{dA_c}{dx} = 0$
- Surface area is perimeter times length $A_s = Px$
- So under prismatic conditions the GFE becomes $\frac{d^2 T}{dx^2} - \frac{hP}{kA_c} (T - T_\infty) = 0$ (2).

2.5.3 Solving the GFE for prismatic fins

- Solve the GFE above (2) using a substitution $\theta(x) = T(x) - T_\infty$ and use arbitrary constant $m^2 = \frac{hP}{kA_c}$ to get $\frac{d^2 \theta}{dx^2} - m\theta = 0$ which has the solution $\theta(x) = C_1 e^{mx} + C_2 e^{-mx} = D_1 \sinh(mx) + D_2 \cosh(mx)$.

Boundary conditions

- BC1: Known base temperature, T_b , implies that $\theta(0) = T_b - T_\infty = \theta_b$
- BC2:
 - Infinitely long fin tip: L (length of fin) $\rightarrow \infty$, implies $T_L = T_\infty \rightarrow \theta_L = 0$
 - * constants become $C_1 = 0, C_2 = \theta_b$ (use limit $L \rightarrow \infty$) and thus $\theta(x) = \theta_b e^{-mx}$
 - * Due to steady conditions, heat transfer out of fin (and through) must be equal to heat transfer into fin base by conduction. Thus applying Fourier's law at the base $Q_b = -kA_c \frac{dT}{dx}|_{x=0} = kA_c m \theta_b e^{-mx}|_{x=0} = kA_c m \theta_b$.
 - * An infinite fin actually implies that the tip will be adiabatic (no heat transfer because it is at the same temperature as the surrounding fluid). This allows us to find the minimum length at which the infinite length assumption is correct to e.g. 1%. Just compute $0.99mkA_c \theta_b \leq mkA_c \theta_b \tanh(mL)$ (which is the heat flow of the adiabatic case).
 - Adiabatic fin tip: No heat transfer at the tip ($x = L$) so $Q_L = 0 \rightarrow \frac{d\theta}{dx}|_{x=L} = 0$
 - * $\theta(x) = \theta_b \frac{1}{\cosh(mL)} \sinh(m(L-x))$ and thus heat transfer becomes $Q_b = mkA_c \theta_b \tanh(mL)$
 - Known fin tip temperature: At $x = L, T = T_L$ and thus $\theta(L) = T_L - T_\infty = \theta_L$
 - * $\theta(x) = \frac{\theta_L \sinh(mx) + \theta_b \sinh(m(L-x))}{\sinh(mL)}$ and $Q_b = kA_c m \frac{\theta_b \cosh(mL) - \theta_L}{\sinh(mL)}$
 - Convection at fin tip: At $x = L, Q = Q_{conv}$ which implies that $hA_c[T_L - T_\infty] = -kA_c \frac{dT}{dx}|_{x=L} \rightarrow h\theta_L = -k \frac{d\theta}{dx}|_{x=L}$
 - * Gets nasty: $\theta(x) = \theta_b \frac{\cosh(m(L-x)) + (\frac{h}{mK}) \sinh(m(L-x))}{\cosh(mL) + (\frac{h}{mK}) \sinh(mL)}$ and $Q_b = \theta_b mkA_c \frac{\sinh(mL) + (\frac{h}{mK}) \cosh(mL)}{\cosh(mL) + (\frac{h}{mK}) \sinh(mL)}$

2.5.4 Fin performance

- Performance depends on material properties (k), construction (machined or stuck), geometry of fins, position of fins and boundary conditions.
- **Fin effectiveness:** Ratio of heat transfer through fin to heat transfer without fin. $\epsilon_{fin} = \frac{Q_{actualfin}}{Q_{nofin}} = \frac{Q_b}{\underbrace{hA_c\theta_b}_{\text{convection if fin wasn't there}}}$ describes how much *more* heat is transferred using a fin. To justify using a particular kind of fin, fin effectiveness should be ≥ 1 , usually ≥ 2 .
- Example for infinitely long fin $\epsilon_{fin} = \frac{mkA_c\theta_b}{hA_c\theta_b} = \sqrt{\frac{kP}{hA_c}}$ so can improve efficiency by increasing thermal conductivity (better material) increasing ratio of perimeter to cross-sectional area and placing fins on gas rather than on liquid (lowers h).
- **Fin efficiency:** Ratio of actual heat transfer through fin to max heat transfer if the entire fin was at the base temperature. $\eta_{fin} = \frac{Q_{actualfin}}{Q_{max}} = \frac{Q_b}{\underbrace{hA_s\theta_b}_{\text{convection of whole surface of fin}}}$
- Example: For a fin with an adiabatic tip: $\eta_{fin} = \frac{\tanh(mL)}{mL}$ which implies that $\eta_{fin} \rightarrow 1$ when $L \rightarrow 0$ and $\eta_{fin} \rightarrow 0$ when $L \rightarrow \infty$
- Multiple fins: N fins, A_s is the surface area of 1 fin and A_e is the exposed surface area between fins on the base. $A_t = NA_s + A_e$ is the total surface area of fins and base.
- **Total surface efficiency** $\eta_O = \frac{Q_{total}}{Q_{maxtotal}} = \frac{Q_{actualfin} + Q_{base}}{Q_{hA_t\theta_b}} = \frac{N\eta_{fin}hA_s\theta_b + hA_e\theta_b}{hA_t\theta_b} = 1 - \frac{NA_s}{A_t}(1 - \eta_{fin})$, where Q_{total} is the actual heat transferred through base and fins and $Q_{maxtotal}$ is the max heat transfer.

3 Convection

- Free convection - motion of fluid is *only* driven by temperature gradients.
- Forced convection - fluid is driven by its bulk motion through pumps or fans.

3.1 Boundary layers

- Prandtl proposed splitting up the flow over a surface into two regions; the boundary layer ('no slip' at surface, velocity and temperature vary) and free stream (velocity and temperature are constant).
- Expression for the heat transfer coefficient h : $q_{conv} = -h(T_\infty - T_w) = q_{cond} = -k\frac{\partial T}{\partial y}|_{y=0}$ at surface, where T_w is the wall temperature, so $h = \frac{k\frac{\partial T}{\partial y}|_{y=0}}{(T_\infty - T_w)}$. The average heat transfer coefficient is computed as $\bar{h} = \frac{1}{L} \int_0^L h_x dx$. If you have a laminar and turbulent flow, split the integral into $\bar{h} = \frac{1}{L} [\int_0^{x_{cr}} h_{laminar} dx + \int_{x_{cr}}^L h_{turbulent} dx]$, find x_{cr} using Reynold's number. To find the total heat transfer (or total flux) use Newton's law with the average heat transfer coefficient, i.e. $Q = -\bar{h}A_s(T_\infty - T_w)$

3.2 Analysis procedure

- 1) Find or assume a velocity profile, 2) Find or assume a temperature profile, 3) Determine Q using Newton's cooling law or Fourier's law, 4) Calculate h_x and (\bar{h}) average heat transfer coefficients, usually written as a Nusselt number $Nu = \frac{hL}{k}$.

3.3 External flows

- External flows are unbounded flows. The thickness of the boundary layer develops to a maximum. Shear stress is calculated using velocity gradient at surface, while heat transfer is calculated using temperature gradient at surface. Examples include, flow over aerofoils, past building, over vehicles etc.

3.3.1 Hydraulic boundary layer

- Hydraulic boundary layers: u_s (free stream velocity $\rightarrow y \geq \delta_H$ then $\frac{\partial u}{\partial y} = 0$), δ_H (hydraulic boundary layer thickness at $x = 0$, $\delta_H = 0$), developed flow occurs when the velocity profiles are no longer changing.
- As the velocity profile/gradient changes, so does shear stress:
 - Local shear stress (decreases as boundary layer develops): $\tau_x = \mu \frac{\partial u}{\partial y}|_{y=0}$
 - Total shear stress on whole surface: $\tau = \int_{x=0}^{x=L} \tau_x dx$.
 - Average shear stress on unit length of surface: $\bar{\tau} = \frac{\tau}{L}$.

3.4 Thermal boundary layer

- T_s (free stream temperature, at $y \geq \delta_T \rightarrow \frac{\partial T}{\partial y} = 0$), T_w (Temperature of wall/surface), δ_T (Boundary layer thickness)
- As temperature gradient changes so does *heat flux* q :
 - Local heat flow at surface (where the velocity is zero): $q_x [W/m^2] = -h_x(T_s - T_w) = -k \frac{\partial T}{\partial y}|_{y=0}$. Heat flow decreases as thermal boundary layer develops.
 - Total heat flow: $q = \int_{x=0}^{x=L} q_x dx$ [W/unit depth] and $q = -\bar{h}L(T_s - T_w)$
 - Average heat flow $\bar{q} = \frac{q}{L}$.

3.5 Turbulence

- Turbulent flow develops when $Re_x = \frac{\rho u_s x}{\mu} \geq 5 \times 10^5$.
- Critical from leading edge is thus: $x_{cr} = \frac{\mu}{\rho u_s} 5 \times 10^5$
- Gives steeper velocity and temperature gradients (flatter profiles) so higher shear stress and heat flow (and boundary layer is thicker).

3.6 Bounded flow

- Bounded flow - flow is confined in a certain space. E.g flow through tubes.
- Constant mass flow rate: $\dot{m} = \rho u A_c$
- Use polar coordinates instead, so use r radius and z for the length of the tube.

3.6.1 Hydraulic Boundary Layer

- u_s (free stream velocity), u_c (centre-line velocity, note $u_0 > u_c$ and thus at $r = 0 \rightarrow u(r) = u_c \rightarrow \frac{\partial u}{\partial r} = 0$ [MISTAKE IN NOTES?]), δ_H only goes up to the centreline where it then meets another, oppositely symmetrical δ_H .
- $\tau_z = \mu \frac{\partial u}{\partial r}|_{r=R}$, $\tau = \int_{z=0}^{z=L} \tau_z$

3.6.2 Thermal Boundary Layer

- T_s (free stream temperature), T_{cl} (temperature at centreline), average temperature of fluid will increase along the tube.
- Temperature profile depends on boundary conditions.
- $q_z [W/m^2] = -h_z(T_0 - T_w) = -k \frac{\partial T}{\partial r}|_{r=R}$, $q = \int_{x=0}^{x=L} q_z dx$, $\bar{h} = \frac{1}{L} \int_0^L h_z dz$

3.7 Analysing Boundary Layers

- Integral Equations give information about the *overall* behaviour of the whole boundary layer, while Navier Stokes gives detailed information about a small element in the boundary layer.

3.7.1 Equations of Motion

- There are multiple ways to derive the equations of motion. You can use a ‘bottom up’ approach or a ‘top down’ one using the Navier Stokes equations.

3.8 Navier Stokes Momentum Equations - finds velocity

- The Navier Stokes equations describe all the forces acting on a fluid. They are too hard to solve without simplifications. Solutions for forced convection (with negligible viscous dissipation) usually take the form $Nu = f(Re, Pr)$.
- **Simplifications:**
 - For incompressible fluids $\rightarrow \nabla \cdot V = 0$
 - For steady flow $\rightarrow \frac{\partial}{\partial t} = 0$ (this does not mean $\frac{D}{Dt} = 0$!)
 - For zero or negligible body forces $\rightarrow F_x = F_y = F_z = 0$.
 - For external (unbounded) flows \rightarrow pressure negligible $\rightarrow \frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0$ **except** for in the flow of fluid travel so $\frac{\partial P}{\partial x} \neq 0$. [Ignored in Q12?]
 - Fully developed: u only changes in y not in x .
 - 1D $\rightarrow v = 0, w = 0$.
 - Flow similarity also exists when the Reynold number is the same for the two bodies of different sizes.

3.9 Navier Stokes Energy Equation - finds temperature

$$\rho \frac{De}{dt} + P \nabla \cdot V = q_g + \phi + \underbrace{\frac{\partial}{\partial x} \left(k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial T}{\partial z} \right)}_{\text{Conduction}}$$

where $e = C_p T$ is the internal energy, ϕ is the viscous dissipation and $P \nabla \cdot V$ is the reversible conversion from mechanical energy to thermal energy. Note that:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}, \nabla \cdot V = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- **Simplifications:**
 - For incompressible fluids $\rightarrow \nabla \cdot V = 0$
 - For steady flow $\rightarrow \frac{\partial}{\partial t} = 0$ (this does not mean $\frac{D}{Dt} = 0$!)
 - For relatively low speed flow $\rightarrow \phi = 0$
 - For zero flow (like in solids) $u, v, w = 0, P \nabla \cdot V = 0, \phi = 0$ (get GCE doing this)
 -
 - For zero or negligible body forces $\rightarrow F_x = F_y = F_z = 0$.
 - For external (unbounded) flows \rightarrow pressure negligible $\rightarrow \frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0$ **except** for in the flow of fluid travel so $\frac{\partial P}{\partial x} \neq 0$.
 - Fully developed: u only changes in y not in x .
 - 1D $\rightarrow v = 0, w = 0$.

3.10 Similarity

- Velocity and temperature profiles are identical if Prandtl number $Pr = 1$. $Pr = 1, \delta_H = \delta_T$ for most gases. $Pr < 1, \delta_T > \delta_H$ for liquid metals, and $Pr > 1, \delta_T < \delta_H$ for oils.
- When $Pr = 1$, heat transfer can be found if shear stress is known: Use $\frac{u(y)}{u_s} = \frac{T(y)-T_w}{T_s-T_w} = \frac{\theta(y)}{\theta_s}$, then differentiate w.r.t to y . Substitute in Fourier's law and shear stress to get

$$\frac{1}{u_s} \frac{\tau_x}{\mu} = \frac{1}{\theta_s} \frac{q_x}{k}$$

- For similarity the Reynolds $\frac{\rho u_s L}{\mu}$ must be constant, if the Prandtl $\frac{\mu C_p}{k}$ is constant too then, then the Nusselt number $\frac{hL}{k}$ must be constant too!. Ideally we want an equation of the form $Nu = f(Re, Pr)$ so we can find the heat transfer coefficient h to then find the heat flux q .

3.11 Integral Equations

- Integral Equation of Momentum - needs velocity profile, finds $\delta_H(x)$ and τ_x or the other way round. Unknowns are δ_H and u .

$$\underbrace{\mu \frac{\partial u}{\partial y}}_{\text{shear stress}}|_{y=0} = \rho u_s^2 \frac{d}{dx} \int_0^{\delta_H} \frac{u}{u_s} \left(1 - \frac{u}{u_s}\right) dy$$

- Integral Energy (Enthalpy) Equation - needs temperature and velocity profile, finds δ_T and heat flow q_x or the other way round. Unknowns are δ_T and T .

$$\alpha \frac{\partial T}{\partial y}|_{y=0} = \theta_s u_s \frac{d}{dx} \int_0^{\delta_T} \frac{u}{u_s} \left(1 - \frac{\theta}{\theta_s}\right) dy$$

3.11.1 Assumptions:

- Unbounded flow \rightarrow negligible pressure gradients (except in direction of flow I presume).
- 1D flow in x -direction.
- Only considers dominant terms (?)
- Control volume larger than boundary layers $dy > \delta_H, \delta_T$.
- Steady flow, constant fluid properties, newtonian fluid.

3.11.2 Derivation of Integral Equation of Momentum

- Use conservation of momentum to get net momentum out of control volume: IN: $\int_A^B \rho u^2 dy + u_s \frac{\partial}{\partial x} \int_A^B \rho u dy dx$, OUT: $\int_A^B \rho u^2 dy + \frac{\partial}{\partial x} \int_A^B \rho u^2 dy dx$, NET OUT: $\rho \frac{\partial}{\partial x} \int_A^B (u^2 - uu_s) dy dx$
- Net momentum out = shear force at wall surface: $\tau_w dx = -\mu \frac{\partial u}{\partial y}|_{y=0} = \rho \frac{\partial}{\partial x} \int_A^B (u^2 - uu_s) dy dx$
- Split integral to go from 0 to δ_H , because outside this boundary layer $u = u_s$ the other integral goes to zero.

3.11.3 Derivation of Integral Energy (Enthalpy) Equation

- Recall that Enthalpy is the internal energy + the product of pressure and volume in a thermodynamic system. Here enthalpy $e = C_p t$
- Use conservation of enthalpy: IN: $C_p \int_A^B \rho u T dy + C_p T_s \frac{\partial}{\partial x} \int_A^B \rho u dy dx$, Enthalpy OUT: $C_p \int_A^B \rho u T dy + C_p \frac{\partial}{\partial x} \int_A^B \rho T u dy dx$, NET OUT: $\rho C_p \frac{\partial}{\partial x} (uT - uT_s) dy dx$
- Net enthalpy out = conduction at wall surfaces: $-k \frac{\partial T}{\partial y} |_{y=0} dx = \rho C_p \frac{\partial}{\partial x} \int (uT - uT_s) dy dx$
- Rearrange and split integral to go from 0 to δ_T , because outside this boundary layer $T = T_s$ the other integral goes to zero and use the substitutions $\theta = T - T_w$, $\theta_s = T_s - T_w$ and $\alpha = \frac{k}{\rho C_p}$ to get the form shown above.

3.11.4 Solution method for Integral Equations - Laminar Flow

- Postulate velocity profile and write δ_H in terms of u or $\frac{u}{u_s}$. Note these are not the equations of the boundary layers. Bit confusing why it's cubic, because it implies u grows as y increases? Anyway, just remember that the profile is orthogonal at the boundary layer.
 - Linear $u = a + by$
 - Polynomial $u = a + by + cy^2 + dy^3$
 - Trigonometric $u = a \sin(by + c)$,
 - Power Law (Blasius) (turbulent flows) $u = a + by^{1/7}$
 - Boundary conditions: $y = 0 \rightarrow u = 0$ (no slip condition), $y = 0 \rightarrow \frac{\partial^2 u}{\partial y^2} = 0$, $y = \delta_H \rightarrow u = u_s \rightarrow \frac{\partial u}{\partial y} = 0$
- Solve Integral equation of momentum to find δ_H .
 - 1) Insert δ_H by evaluating limits of integration.
 - 2) You might end up with an ODE so use product rule $\frac{d}{dx}(u^2) = 2u \frac{du}{dx}$ to get a nicer ODE / or just solve it directly, tbh probably easier. Then solve the ODE to find δ_H .
 - 3) Then try to see if you can find any Reynold numbers $Re = \rho u x / \mu$
- Find local shear stress τ_x Total $\rightarrow [N/m]$, local $\rightarrow [N/m^2]$
 - Use $\tau_x = \mu \frac{\partial u}{\partial y} |_{y=0}$ and sub in δ_H .
 - You can then also find total shear stress $\tau = 2L\tau_L = \int_0^L \tau_x dx$ and average shear stress $\bar{\tau} = \frac{\tau}{L} = 2\tau_L$
- Stop here and FIRST calculate the Prandtl number. Skip to 5 and see what to do. Postulate temperature profile and find θ in terms of δ_T
 - Exact same option of profiles as for the velocity profile.
 - Boundary conditions: $y = 0 \rightarrow \theta = 0 \rightarrow \frac{\partial^2 \theta}{\partial y^2} = 0$, $y = \delta_T \rightarrow \theta = \theta_s \rightarrow \frac{\partial \theta}{\partial y} = 0$
- Solve integral energy equation to find δ_T
 - You'll probably end up with an ODE in terms of δ_T and δ_H . This is where you need to know what the Prandtl number is: If its 1 (gases) then all $\frac{\delta_T}{\delta_H} = 1$, if its a liquid metal or an oil you need to see which terms dominate.
 - If $Pr \ll 1$, $\delta_T > \delta_H$ (plug flow, liquid metals like liquid sodium) then the best thing to do is solve the integral energy equation again but this time using the fact that $u = u_s$. Means you don't have to work with δ_H .

- If $Pr \gg 1$, $\delta_H > \delta_T$ only use dominant terms i.e. get rid of all terms $\neq \frac{\delta_T}{\delta_H}$ and use substitution $\Delta = \frac{\delta_T}{\delta_H}$ (note if the temperature and velocity profile are the same then $\Delta \neq f(x)$ and can be ignored when solving the ODE).

6. Find local heat flux q_x

- If $Pr = 1$ use similarity idea from above, just use this equation to find heat flux $\frac{1}{u_s} \frac{\tau_x}{\mu} = \frac{1}{\theta_s} \frac{q_x}{k}$, i.e.

$$q_x = \frac{\theta_s k}{u_s \mu} \tau_x.$$

- If $Pr \ll 1$ or $Pr \gg 1$ use $q_x = -k \frac{\partial \theta}{\partial y}|_{y=0}$
- Total heat flux $q = 2Lq_L = \int_0^L q_x dx$ and average heat flux $\bar{q} = \frac{q}{L} = 2q_L$

7. Find local heat transfer coefficient h_x and $Nu_x = f(Re_x, Pr)$

- If $Pr \ll 1$ use the equation $q_x = -h_x \theta_s$ and cleverly arrange everything to get $Nu_x = f(Re_x, Pr)$

Summary: laminar flow

Velocity Boundary Layer	Thermal Boundary Layer
Integral equation of momentum $\mu \frac{\partial u}{\partial y} \Big _{y=0} = \rho u_s^2 \frac{d}{dx} \int_0^{\delta_H} \frac{u}{u_s} \left(1 - \frac{u}{u_s}\right) dy$	Integral energy equation $k \frac{\partial T}{\partial y} \Big _{y=0} = \rho C_p \theta_s u_s \frac{d}{dx} \int_0^{\delta_T} \frac{u}{u_s} \left(1 - \frac{\theta}{\theta_s}\right) dy$
Velocity profile (polynomial) $\frac{u}{u_s} = \frac{3}{2} \frac{y}{\delta_H} - \frac{1}{2} \left(\frac{y}{\delta_H}\right)^3$	Temperature profile (polynomial) $\frac{\theta}{\theta_s} = \frac{3}{2} \frac{y}{\delta_T} - \frac{1}{2} \left(\frac{y}{\delta_T}\right)^3$
Layer thickness $\delta_H = 4.64x(Re_x)^{-1/2}$	Layer thickness (polynomial profile) $\begin{aligned} Pr \ll 1 & \quad \delta_T = \sqrt{\frac{8\alpha x}{u_s}} \\ Pr = 1 & \quad \delta_T = \delta_H \\ Pr \gg 1 & \quad \delta_T = 4.53x(Re_x)^{-1/2}(Pr)^{-1/3} \end{aligned}$
Local shear stress $\tau_x = 0.323\rho u_s^2(Re_x)^{-1/2}$	Local heat flow (polynomial profile) $\begin{aligned} Pr \ll 1 & \quad q_x = -0.530\theta_s \frac{k}{x}(Re_x)^{1/2}(Pr)^{1/2} \\ Pr = 1 & \quad q_x = \frac{\theta_s}{u_s} \frac{k \tau_w}{\mu} \\ Pr \gg 1 & \quad q_x = -0.33\theta_s \frac{k}{x}(Re_x)^{1/2}(Pr)^{1/3} \end{aligned}$
Total drag $\tau = \int \tau_x dx$	Total heat flow $q = \int q_x dx$

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3.12 Special Cases - where integral equations don't work - Have to simplify Navier Stokes instead L6

3.12.1 Internal (bounded) flows

- Aside: Developed region occurs at $L_{hydraulic} = 0.05ReD$ and $L_{thermal} = 0.05RePrD$ for laminar flows. $L_{hydraulic} = L_{turbulent} \approx 10D$ for turbulent flows.
- Use same notation as above in bounded flows, include mean velocity $u_m = \frac{1}{A_c} \int u dA_c$ (A_c is the cross-sectional area) and mean ('bulk') temperature $T_m = \frac{1}{u_m A_c} \int u T dA_c$.

- For internal flows we expect $Nu = \text{const.}$. There is a little proof showing this using substitution $\theta = \frac{T_w - T}{T_w - T_m}$ and fact that $\frac{\partial \theta}{\partial r} \neq f(z)$ (z is along the tube) and then using conservation of energy $k \frac{\partial T}{\partial r} = -h(T_m - T_w)$ to find $Nu = hL/k = \text{const.}$.

3.12.2 Turbulent flows

- Have momentum in y direction now too. Shear stress depends on viscosity and eddy motion $\tau = \tau_e + \mu \frac{\partial \bar{u}}{\partial y}|_{y=0}$ and velocity profile is given by Blasius' power law ($u = ay^{1/7} + b$). Thermal conductivity is a function of both turbulent and laminar values.
- Velocity of an element of fluid varies with time but the time-averaged velocity is constant $\bar{u} = \text{const.}$. Turbulent velocity in x direction is now $u = \bar{u} + u'$, where $u' = u_e$ the instantaneous fluctuation (eddy velocity). Same in y -direction $v = \bar{v} + v'$. $\int u' dt = 0$, but not entirely sure why (p27, l6)
- Total shear stress: $\tau = \rho(\epsilon + \nu) \frac{\partial \bar{u}}{\partial y}$, where ϵ is a function of turbulence ($= 0$ for laminar flow) and kinematic viscosity $\nu = \frac{\mu}{\rho}$
- Total heat flow: $q = \rho C_p(\epsilon_H + \alpha) \frac{\partial \bar{T}}{\partial y}$, where $\bar{T} = \text{const.} T - T'$ (T' eddy temperature?). Or in terms of thermal conductivities $q = (k_t + k) \frac{\partial \bar{T}}{\partial y}$, where $k_t = \rho C_p \epsilon_H$. ϵ_H is also just a function of turbulence.

3.12.3 High Speed flows

- For high speeds (e.g $M > 0.5$) viscous dissipation (extreme aerodynamic heating) cannot be ignored so we would need to keep the ϕ term in the NS equation.
- For high speed flow past an insulated surface, if energy conversion is reversible we can say that kinetic energy = internal energy, i.e. $\frac{1}{2}u_s^2 = C_p(T_0 - T_s)$, where T_0 is the stagnation temperature, and since $T_0 = T_s + \frac{u_s^2}{2C_p} \rightarrow T_0 > T_s$. Recall T_s is the stream temperature.
- **Adiabatic surfaces** (insulated surfaces) $\frac{dT}{dy}|_{y=0} = 0$ and $T = T_{as}$. Note if energy conversion not reversible then $T_{as} < T_0$, otherwise if reversible $T_{as} = T_0$. $q = -h(T_s - T_w)(?)$.
- **Recovery factor** $r = \frac{T_{as} - T_s}{T_0 - T_s}$ (for non-reversible conversion, $r < 1$). Empirically for laminar flows $r \approx Pr^{1/2}$ and for turbulent flow $r \approx Pr^{1/3}$ (so turbulent flow is less reversible).
- Non adiabatic constant temperature surfaces: heat transfer driven by $T_{as} - T_w$ instead of $T_s - T_w$, so $q = -h(T_{as} - T_w)$
- So heat transfer can be found 1) using the standard formula $q = -k \frac{dT}{dy}|_{y=0}$ and $\frac{dT}{dy}$ can be found from NS energy equation. 2) using the above formula $q = -h(T_{as} - T_w)$ (Find T_0 from energy conservation from above and T_{as} from recovery factor, h can be found from integral energy equation or from $Nu = f(Re, Pr)$ relations.