Engineering Physics 2

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1 Kinematics

Kinematics is the study of classical mechanics which describes the motion of **points**, **rigid bodies** (objects) and **systems of bodies** (groups of objects) without consideration of the causes of motion.

1.1 Particle kinematics

1.1.1 Position

- Use **right-handed Cartesian** coordinate system.
- $\mathbf{r}(t) = (x(t), y(t), z(t))^T$ (temporal) or $\mathbf{r}(s) = (x(s), y(s), z(s))^T$ (arc-length)
- Recall that for circles: Arclengh $s = R\theta = R\omega t$

1.1.2 Velocity

- $\boxed{\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = \dot{\mathbf{r}}(t)}$
- Velocity is always **tangent** to the path of the particle.

1.1.3 Acceleration

- $\boxed{\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = \ddot{\mathbf{r}}(t)}$
- Acceleration has **tangential** AND **normal** components to the path of the particle: $\mathbf{a}(t) = \mathbf{a_t}(t) + \mathbf{a_n}(t)$
- The normal component points towards the centre of the best approximating circular arc.
- To find the tangential component, find $\mathbf{e_t}$ from the fact that velocity is always tangential. So $\mathbf{e_t} = \frac{\mathbf{v}}{|\mathbf{v}|}$ and then project $\mathbf{a}$ onto $\mathbf{e_t}$ to find $\mathbf{a_t} = \mathbf{a_t} \cdot \mathbf{e_t}$. Note, for circular motion this is zero.
- Apparently from geometry it can be shown that $\frac{d\mathbf{e_t}(s)}{ds} = \frac{1}{\rho}\mathbf{e_n}$ and hence $\boxed{\mathbf{a_n} = \frac{|v|^2}{\rho}}$, where ρ is the radius of curvature, i.e. the radius of the best approximating circular arc.

1.1.4 Parametrisation of path

- Temporal - $\mathbf{r}(t)$
- Arclength
  - Think of it as 'distance' instead of the more normally used 'displacement'
  - $s(t)$ is the length of path the particle has travelled since $t = 0$ which implies $s(0) = 0$
  - Hence speed $\dot{s}(t) = |\mathbf{v}(t)|$
  - Thus, arclength (distance) as a function of time is calculated as $s(t) = \int_0^t |\mathbf{v}(\tau)| d\tau$
  - Finally to find $r(s)$ we invert $s(t)$ to $t(s)$ and then substitute that into $r(t)$, i.e. $r(s) = r(t(s))$

1.1.5 Helical Motion

- $\boxed{\mathbf{r}(t) = (x_0 + Rcos(wt), y_0 + Rsinwt, z_0 + v_z t)}$

1.2 Rigid Body kinematics

Summary: A rigid body only has **one angular velocity** and **one angular acceleration**. For **planar motion** there is a point P with instantaneous zero velocity.

1.2.1 Position and Parametrisation

- **(Derivative of a vector of constant length)** The distance of two material points on a rigid body is, by definition, constant, i.e. $|r_{AB}| = const.$, which implies $\mathbf{r}_{AB}^2 = const.$. Thus you can show that $\dot{\mathbf{r}_{AB}} = \mathbf{w} \times \mathbf{r_{AB}}$, since $\dot{\mathbf{r}_{AB}}$ and $\mathbf{r_{AB}}$ are orthogonal to each other. Note, because they are orthogonal there will always be an angular velocity vector orthogonal to both of them.

1.2.2 Velocity and Angular velocity

- Consequently since $\boxed{\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{AB}, \mathbf{v_B} = \mathbf{v_A} + \boldsymbol{\omega} \times \mathbf{r}_{AB}}$

1.2.3 Zero Velocity Point

- $\boxed{0 = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{AP}, \text{ now } \textbf{multiplying by } \boldsymbol{\omega}, 0 = \mathbf{w} \times \mathbf{v}_A - w^2 \mathbf{r}_{AP}}$. And thus finally the vector from A to the zero velocity point P is $\boxed{\mathbf{r}_{AP} = \frac{\boldsymbol{\omega} \times \mathbf{v_A}}{\omega^2}}$
- You can also find the zero velocity point from the intersection of the lines going perpendicular to each velocity points on the body.

1.2.4 Acceleration and Angular Acceleration

- Differentiate $\mathbf{v_B} = \mathbf{v_A} + \boldsymbol{\omega} \times \mathbf{r}_{AB}$ to get $\boxed{\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\varepsilon} \times \mathbf{r}_{AB} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{AB})}$ for 3D and 2D and $\boxed{\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\varepsilon} \times \mathbf{r}_{AB} - \omega^2 \mathbf{r}_{AB}}$ in 2D.
- Note, if a point on a body is moving along a wall/flat surface then the acceleration is only tangential because $\rho \to \infty$

Point mass

- Arclength - $s(t) = \int_0^t |\mathbf{v}(\tau)| d\tau$ IF $\mathbf{v}(t) \neq 0$. (Note, for a circle $s = R\omega t$)
- Acceleration has both tangential and normal components - $\ddot{\mathbf{r}}(t) = \mathbf{a}(t) = \mathbf{a}_t(t) + \mathbf{a}_n(t)$, $\boxed{|\mathbf{a}_t| = \frac{d|\mathbf{v}|}{dt}}$, $\boxed{|\mathbf{a}_n| = \frac{|\mathbf{v}|^2}{\rho}}$, where ρ is the <u>radius of curvature</u> (the radius of the circle that best fits the path at a point in time). To find the radius of curvature of point (p) travelling in space, whether attached to a rigid body or not, you can use $\rho = \frac{|\mathbf{a}_p|}{|\mathbf{v}_p|^2}$ iff $\mathbf{a}_p \perp \mathbf{v_p}$ (because then $\mathbf{a}_p$ naturally equals $\mathbf{a}_{Normal}$)
- $|\mathbf{v}(t)| = \dot{s}(t)$ (speed it the derivative of distance - two scalars)
- Velocity is always tangent to the path of the particle $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$
- Useful trick to find tangent and normal vector $\mathbf{e}_t = \frac{\mathbf{v}}{|\mathbf{v}|} \Rightarrow \mathbf{a}_t = (\mathbf{e}_t \cdot \mathbf{a})\mathbf{e}_t$

2 Energy Methods: Lagrangian Mechanics

2.1 Work, Power and Energy for Particles

2.1.1 Work

- $\boxed{W_{12} = \int_{t_1}^{t_2} \mathbf{F}(t) \cdot \mathbf{v}(t) dt \text{ or } W_{12} = \int_{s_1}^{s_2} \mathbf{F}(s) \mathbf{ds}}$ (arc-length parametrisation)

2.1.2 Power

- The rate at which work is done J/s. $\boxed{P = \mathbf{F} \cdot \mathbf{v}}$

2.1.3 Potential Energy

- A force has a potential if the work done is independent of the path (i.e. the field is conservative - $curl\mathbf{f} = 0$). This means $W_{12} = \int_{t_1}^{t_2} \mathbf{F}(t) \cdot \mathbf{v}(t) dt = U(t_2) - U(t_1)$ (Note this is the work done on the system not by the system).
- In 1D all forces that are independent of velocity (e.g. F = kx (dependent on position)) have potential.
- In 2D and 3D a force has a potential if is independent of velocity (change of position) and $curl\mathbf{F}(\mathbf{r}) = 0$.
- Force is the negative gradient of the potential $\mathbf{F} = -gradU(\mathbf{r})$

2.1.4 Kinetic Energy

- Recall, kinetic energy is the energy required to accelerate a particle from rest to velocity $\mathbf{v}$
- $T = \frac{1}{2} m |\mathbf{v}|^2$

2.1.5 Work-Energy theorem

- Theorem: $T_2 - T_1 = W_{12}$
- Differential form of theorem: $\boxed{\dot{T} = P}$. Note, it only works for **single** degree of freedom systems.

2.2 Lagrangian Mechanics, 1 DoF

2.2.1 Generalised coordinates

- A generalised coordinate is any variable that describes the configuration of a mechanism uniquely. Denoted by q.

2.2.2 Lagrange's equation

- $\boxed{L = T - U, \ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0}$

2.2.3 Equilibrium

- An equilibrium position is when there is **no motion**, i.e. zero kinetic energy and constant potential energy
- $\boxed{\frac{\partial L}{\partial q} = 0}$. Denoted by q_0
- $\boxed{\text{Stable if } K > 0 \text{ and unstable if } K < 0}$

2.2.4 Linearisation/Differential equation

- Only valid if the motion does not go far from equilibrium $q = q_0 + \bar{q}$
- $\boxed{M\ddot{\bar{q}} + K\bar{q} = 0, \text{ where } M = \frac{\partial^2 L}{\partial \dot{q}^2}|_{\dot{q}=0} \text{ and } K = -\frac{\partial^2 L}{\partial q^2}|_{q=q_0}}$

2.2.5 Natural frequency and stability

  - Note, $M > 0$ always
  - System is $\boxed{\text{stable if } K > 0 \text{ and unstable if } K < 0 \text{ and if } K = 0 \text{ we cannot know}}$.
  - If $K > 0$ then we define $\boxed{\omega_n = \sqrt{K/M}}$ and the solution becomes $q(t) = Acos\omega_n t + Bsin\omega_n t$

2.3 Lagrangian Mechanics, higher DoF

2.3.1 Lagrangian Equation

- $\boxed{L = T - U,\ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0}$

2.3.2 Equilibrium

- $\boxed{\frac{\partial L}{\partial q_j} = 0}$. Denoted by q_j^*
- Equilibrium is stable if K is positive definite, i.e. all eigenvalues are positive and unstable if all eigenvalues are negative. Inclusive if the eigenvalues are of different sign.

2.3.3 Degrees of freedom

- The degrees of freedom (N) of a mechanical system is the minimum number of generalised coordinates needed to describe the system.
- **Holonomic constraint** - "A constraint is **holonomic** if it can be expressed in terms of the particles positions: $f(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N, t) = 0$

2.3.4 Linearisation

- Linear equation of motion (around equilibrium) is: $\mathbf{M}\ddot{\bar{\mathbf{q}}} + (\mathbf{G}\dot{\bar{\mathbf{q}}}) + \mathbf{K}\bar{\mathbf{q}} = \mathbf{0}$, where $\boxed{\mathbf{M}_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j},\ \mathbf{K}_{ij} = -\frac{\partial^2 L}{\partial q_i \partial q_i}}$ and ($\mathbf{G} = \frac{\partial^2 L}{\partial \dot{q}_i \partial q_i}$)
  - $\mathbf{M}, \mathbf{K}$ are symmetric Hessian matrices.
  - $\mathbf{M}$ is positive definite, which means **all its eigenvectors** are greater than zero.
  - $\mathbf{K}$ can be positive, negative or indefinite - the stability depends on it.
- Solution to the matrix ODE: $\bar{\mathbf{q}} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \end{bmatrix} = \sum_{i=1}^{N} \mathbf{c}_i (A_i cos\omega_{ni} t + B_i sin\omega_{ni} t)$. Therefore, there if there are N degrees of freedom, then there are N generalised coordinates, N natural frequencies ω_n and N mode shape vectors $\mathbf{c}$. To find ω_n and $\mathbf{c}$ is equivalent to finding eigenvalues and eigenvectors of $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{c} = 0$

2.3.5 Natural frequency

- There is one natural frequency for each mode shape - solve $\boxed{det((\mathbf{K} - \omega^2\mathbf{M}) = 0}$ to find $w_n's$

2.3.6 Vibration modes

- Solve for $\mathbf{c}$ from equation $\boxed{(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{c} = 0}$. Always sketch what it represents, and be aware of the order of generalised coordinates you are using! The moment you calculate the mass and stiffness matrix, the order is determined.

2.3.7 Stability

- $\boxed{\text{System is stable if } \mathbf{K} \text{ is positive definite or } w^2 > 0}$ (all eigenvalues are positive)
- Furthermore, it is also stable if $\mathbf{c}^T(\mathbf{K}\mathbf{c}) > 0$.

2.3.8 Forces without potential

- One can extend Lagrange's equation to include 'generalised' forces **without a potential**, like friction (so far all the forces have been integrated into the potential energy). Then:
  - Lagrange's equation becomes: $\boxed{L = T - U,\ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_j}$, where:
  - $\boxed{Q_j = \sum_{i=1}^n \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_{i=1}^n \mathbf{F}_i \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}}$ (notice the dot ;)
    - Note, you can also use this for generalised moments - $Q = \mathbf{M} \cdot \frac{\partial \theta}{\partial q} = \mathbf{M} \cdot \frac{\partial \omega}{\partial \dot{q}}$ (use omega instead of theta, because it is hard to describe theta's direction).
  - $\mathbf{F}_i$ is the force acting on the i-th particle in the direction of generalised coordinate j and $\mathbf{r}_i$ and $\mathbf{v}_i$ are the position and velocity, respectively, of the PARTICLE (you will have to use rigid body dynamics here).

3 Dynamics in 2D - Rigid body dynamics and the Lagrange equation

3.1 Centre of Mass and Moment of Inertia

3.1.1 Centre of Mass

- Recall that $\rho = m/V$ and hence $m = \int\int\int_V \rho(x, y, z)dV$ or equivalent for area (2D) or along a line (1D) and in other coordinate systems.
- $$\boxed{\mathbf{r}_C = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{m}\left(\int\int\int_V \mathbf{r}\rho dV\right)}$$
- If the mass is evenly distributed in an object then the c.o.m. is equivalent to the geometric centre. $\boxed{\mathbf{r_c} = \frac{\sum \mathbf{r}A}{\sum A}}$ - for empty spaces A is considered negative (so you'll get a minus above and below the fraction). Note it's just assuming $\rho = 1$.
- If force is applied to the c.o.m of an object the object will move in the direction of the force **without** any rotation.

3.1.2 Moment of Inertia

- $\boxed{I_A = \int\int_A |\mathbf{r}_{Ap}|^2 \rho dA}$ (scalar pointing in z-direction to plane)
- Some worth knowing off the top of your head - uniform rectangle with sides a, b: $I_{ZC} = \frac{1}{12}m(a^2 + b^2)$; uniform disc of mass m and radius R: $I_{ZC} = \frac{1}{2}mR^2$

- For a composite body $I_P = \sum_i I_{Ci} + \sum m_i R_{CPi}^2$ and if uniform you can just use areas instead of masses - $I_P = \sum_i I_{Ci} + \sum A_i R_{CPi}^2$ - essentially get all the inertia's of the individual bodies from the same point P (using parallel axis theorem) and add them up. (If they are empty spaces - subtract them.)

3.1.3 Kinetic energy

- $\boxed{T = \frac{1}{2}m|\mathbf{v}_C|^2 + \frac{1}{2}I_C|\boldsymbol{\omega}|^2}$ when you are dealing with the **centre of mass** of a rigid body.
- $\boxed{T = \frac{1}{2}I_A|\boldsymbol{\omega}|^2}$ when you are dealing with a **stationary point** - because $\mathbf{v}_A = 0$ by definition.

3.2 Parallel Axis Theorem and important examples

3.2.1 Parallel Axis Theorem

- $\boxed{I_A = I_C + m|\mathbf{r}_{AC}|^2}$ Only works from the centre of mass!.

3.2.2 Potential Energy

- You can only calculate potential from the centre of mass. E.g. under gravity $\boxed{U = mgh_C = -m(\mathbf{g} \cdot \mathbf{r_c})}$ (comes from $W = -\Delta U$).
- **Good Definition of Potential**: The potential energy of an object is the energy **difference** between the potential energy of the object at a given position **relative** to the potential it has at a certain reference point.

3.2.3 Friction

- We still use the 300 year old Coloumb-Amontons model which states that in the case of $\boxed{\textbf{slip } F_f = -\mu N}$ and that it opposes the direction of motion, where N is the normal force between surfaces, F_f is the friction force and μ is the coefficient of friction.
- In the case of $\boxed{\textbf{stick} \text{ we know that } F_f \in [\mu N, \mu N]}$ and that **the surfaces have zero relative velocity**. Draw diagram here:

4 Dynamics in 3D

4.1 Newton's laws

1. In an inertial frame an object either remains at rest or continues to move at a constant velocity, unless an external force is applied to it.
   - $\mathbf{F} = 0 \Leftrightarrow \mathbf{v} = const.$ || $\mathbf{M_Q} = 0 \Leftrightarrow \mathbf{H_Q} = const.$, where Q is a **stationary point (A)** or **centre of mass (C)**.
2. In an inertial frame
   - $\dot{\mathbf{p}} = \mathbf{F}$, where $\mathbf{p} = m\mathbf{v}$ || $\dot{\mathbf{H}}_\mathbf{Q} = \mathbf{M_Q}$, where $\mathbf{H_Q} = \mathbf{r}_{QA} \times \mathbf{p}_A$
3. When two bodies interact, the force exerted by the first body on the second is exactly equal and opposite to the force exerted by the second body on the first.
   - $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ || $\mathbf{M}_{Q_{ij}} = -\mathbf{M}_{Q_{ji}}$

4.1.1 Angular Momentum for a rigid body

- $\boxed{\mathbf{H}_Q = \mathbf{I}_Q\boldsymbol{\omega}}$ ($Q \equiv C$ or $Q \equiv A$), where $\mathbf{I}_Q = \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & I_{yz} & I_z \end{pmatrix} = \int\int\int_V \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \rho dV$
- $\boxed{\dot{\mathbf{H}}_Q = \mathbf{I}_Q\boldsymbol{\varepsilon} + \boldsymbol{\omega} \times (\mathbf{I}_Q\boldsymbol{\omega}) = \mathbf{M}_Q}$ Furthermore, $\boxed{\dot{\mathbf{H}}_Q = \sum_{i=1}^{N} \mathbf{M}_{Q_i}^{ext}}$. Remember this is only true for $Q = C$ (c.o.m) or Q is stationary!

4.1.2 Moment of Inertia tensor

- Parallel axis theorem (ONLY from c.o.m): $\boxed{\mathbf{I}_A = \mathbf{I}_C + m(\mathbf{r}_{AC}^2\mathbf{I}_{dentity} - \mathbf{r}_{AC} \otimes \mathbf{r}_{AC})}$, where *otimes* is like the cartesian product - not quite - but similar in the sense that you create a matrix from two vectors. Another formulation of above is $\mathbf{I}_A = \mathbf{I}_C + m\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$
- Moment of inertia for a uniform solid cylinder: $\boxed{\mathbf{I}_C = \begin{pmatrix} \frac{1}{4}mR^2 + \frac{1}{12}mh^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 + \frac{1}{12}mh^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{pmatrix}}$

4.1.3 Equations of Motion

- Recall that $Q = A, C$
- $\boxed{m\mathbf{a}_C = \sum \mathbf{F}_C}$
- $\boxed{\dot{\mathbf{H}}_Q = \sum \mathbf{M}_Q}$

4.1.4 Kinetic Energy

- $\boxed{T = \frac{1}{2}m|\mathbf{v}_Q|^2 + \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbf{I}_A\boldsymbol{\omega})}$

4.1.5 Parallel Axis Theorem 3D

- The Parallel axis theorem transforms the moment of inertia tensor from the centre of mass C to another point A.
  - $\boxed{\mathbf{I}_A = \mathbf{I}_C + m(\mathbf{r}_{AC}^2\mathbf{I}_d - \mathbf{r}_{AC} \otimes \mathbf{r}_{AC})}$

4.2 Extra: Rotating frames of reference

- Let T be the rotating frame with origin O' relative to the ground frame G with origin O with angular velocity ω (meant as vector). Then:
  - $\boxed{\frac{d}{dt}\mathbf{x}|_G = \frac{d}{dt}\mathbf{x}|_T + \omega_T \times \mathbf{x}}$
  - Then for example the velocity of a point in ground frame, given the velocity of the point in the rotating frame is ${}^G\mathbf{v}_A =^T \mathbf{v}_A + \omega_{frame} \times^T \mathbf{r}_{0'A}$ which should not be confused with two points on a rigid body where ${}^G\mathbf{v}_A =^G \mathbf{v}_B + \omega_{body} \times^G \mathbf{r}_{BA}$ relative to the ground frame.
  - **Angular velocities can be added** even if they're in different reference frames: ${}^{\mathrm{G}}\boldsymbol{\omega} = {}^{\mathrm{G}}\boldsymbol{\omega}_1 + {}^{\mathrm{T}}\boldsymbol{\omega}_2$ (Note, that $\boldsymbol{\omega}_2$ is rotating with $\boldsymbol{\omega}_1$
  - Angular acceleration therefore is: ${}^G\boldsymbol{\epsilon} =^T \frac{d}{dt}\boldsymbol{\omega} + \boldsymbol{\omega}_T \times \boldsymbol{\omega} = {}^{\mathrm{G}}\boldsymbol{\omega}_1 \times {}^{\mathrm{T}}\boldsymbol{\omega}_2$