Control Theory

For Pontryagin

November 18, 2020

1 Essentials

- In general any physical system can be represented as a set of first order ODEs expressing the rate of change of the states as a function of the current state values: $\dot{x}_1 = f_1(x_1, x_2, x_3, ...)$, $\dot{x}_2 = f_2(x_1, x_2, x_3, ...)$, ...
- Define the **system output** y and is function of the system states we want to control: $y = h(x_1, x_2, ...)$.
- A control system is characterised by a set of states, inputs and outputs. The most general form is:

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = h(\mathbf{x}, \mathbf{u}),$$

  where $\mathbf{x} \in \mathbb{R}^{n\times 1}$, $\mathbf{u} \in \mathbb{R}^{m\times 1}$, $\mathbf{y} \in \mathbb{R}^{p\times 1}$.
- *Time-invariant, linear* systems take the form: $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$, where the matrix dimensions are:

$$A \Rightarrow n \times n, B \Rightarrow n \times m, C \Rightarrow p \times n, D \Rightarrow p \times m,$$

  where n is the number of states, p is the number of outputs and m is the number of control inputs. By time-invariant we mean that the system matrices don't change in time. (These are coupled dynamical systems).

2 Essential Physics

- Dampers: $-c\dot{y}$ (always opposes motion)
- Springs: $\pm ky$ (can both aid or hinder motion).
- In a tank, volume = cross-sectional-area $\times$ height.

3 Linearisation

- Assume a non-linear system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$ that is in equilibrium at $\dot{\mathbf{x}}^* = f(\mathbf{x}^*, \mathbf{u}^*) = \mathbf{0}$. Then the matrices become Jacobians: $A = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}, B = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}, C = \frac{\partial \mathbf{h}}{\partial \mathbf{x}}, D = \frac{\partial \mathbf{h}}{\partial \mathbf{u}}$ ($\rightarrow x_1...x_n$, $\downarrow f_1...f_n$ to form matrices).
- Useful Properties: we can treat matrices similar to scalars when dealing with exponentials:
  - $e^{At} = I + At + A^2t^2/2! + ...$
  - $\frac{de^{At}}{dt} = Ae^{At} = e^{At}A$ (commutative)
  - e^{At} is invertable ($[e^{At}]^{-1} = e^{-At}$ if A has full rank $\Rightarrow$ all rows are linearly independent $det(A) \neq 0$ if A is a square matrix.

3.1 General Solution of a Linear System

- **Open-loop (uncontrolled)** $\Rightarrow \mathbf{u} = 0$
  - Given $\mathbf{x}(t_0) = \mathbf{x}_0$, then $\mathbf{x}(t) = e^{A(t-t_0)}\mathbf{x}_0$

- **Closed-loop (controlled)** $\Rightarrow \mathbf{u} \neq 0$
  - Given $\mathbf{x}(t_0) = \mathbf{x}_0$, then $\mathbf{x}(t) = e^{A(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} e^{A(t-\hat{t})} B\mathbf{u}(\hat{t}) d\hat{t}$
- Solving the issue of computing e^{At}
  - We wish to diagonalize, since if A is a diagonal matrix $A = diag\{\lambda_1, \lambda_2, ...\}$ then $e^{At} = diag\{e^{\lambda_1 t}, e^{\lambda_2 t}, ...\}$
    * Solution: $e^{At} = Te^{Dt}T^{-1}$ where $D = diag\{\lambda_1, \lambda_2, ...\}$ is the diagonal matrix containing A's eigenvalues, and $T = [\mathbf{v}_1, \mathbf{v}_2, ...]$ is the matrix of A's eigenvectors corresponding to the eigenvalues.
- Using the above we can create a change of coordinates directly to the system and then the solution can be used directly (so no need to do the above).
  - Let $\mathbf{z}(t) = T^{-1}\mathbf{x}(t)$, then the system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0$ can be written as $\dot{\mathbf{z}} = D\mathbf{z} + \hat{B}\mathbf{u}$, $\mathbf{z}(0) = T^{-1}\mathbf{x}(0)$, where $\hat{B} = T^{-1}B$.
  - Then the solution becomes: $z_i(t) = e^{\lambda_i} z_{i0} + \int_{t_0}^{t} e^{\lambda_i(t-\hat{t})}\hat{B}\mathbf{u}(\hat{t})d\hat{t}$. Notice that we are assuming $t_0 = 0$.

4 Stability

- Characterising how a system reacts to small perturbations from a given equilibrium state. One of the main goals of control is to stabilise unstable systems.
- Given a system $\dot{\mathbf{x}} = f(\mathbf{x}), \mathbf{x}(0) = \mathbf{x}_0$ that has an equilibrium point at $\mathbf{x} = \mathbf{x}^*$, i.e. $f(\mathbf{x}^*) = 0 = \dot{\mathbf{x}}^*$, then the system is considered:
  - Stable: If $\forall \epsilon > 0$ there exists a $\delta > 0$ such that if $||\mathbf{x}_0 - \mathbf{x}^*|| < \delta$ then $||\mathbf{x}(t) - \mathbf{x}|| < \epsilon, \forall t \geq 0$.
  - Asymptotically stable: Needs to be stable and $\lim_{t \Rightarrow \infty} \mathbf{x}(t) = \mathbf{x}^*$
  - Else it's considered unstable.
- We can show for a linear (unforced) system $\dot{\mathbf{x}} = A\mathbf{x}, \mathbf{x}(0) = \mathbf{x}_0$. the system is asymptotically stable iff all the eigenvalues of A have negative real parts, i.e. $Re\{\lambda_i\} < 0 \iff \lambda_i \in \mathbb{C}^-$
- BIBO (bounded-input-bounded-output) Stability: The system response is bounded in magnitude when the system is subjected to a bounded input. A system that is asymptotically stable is also BIBO-stable, but not vice versa. Specifically, if the linear system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0$ is asymptotically stable when $\mathbf{u} = 0$ then the system is guaranteed to be BIBO-stable when $\mathbf{u} \neq 0$.
- Lyapunov Indirect Method: The idea is to prove that the total 'energy' of the system is decreasing along the system trajectories. For non/linear systems $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \mathbf{x}(0) = \mathbf{x}_0$ the equilibrium $\mathbf{x}^*$ is a stable equilibrium of the system if there exists a 'Lyapunov function' $V(\mathbf{x})$ (scalar field: $V : \mathbb{R}^n \to \mathbb{R}$) that satisfies the following conditions:
  - $V(\mathbf{x}^*) = 0$ (no energy at equilibrium point)
  - $V(\mathbf{x}) > 0 \forall \mathbf{x} \neq \mathbf{x}^*$ (positive)
  - V is continuous and differentiable.
  - $\frac{d}{dt}V(\mathbf{x}(t)) < 0$ (decreasing function/negative slope, energy goes down as time increases).
- **Quadratic Lyapunov Functions** (simplest type of Lyapunov function): $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x} = x_1^2 P_1 + x_2^2 P_2$, where $P = P^T > 0$ (symmetric, and definite positive (all elements along diagonal > 0)).
  - For linear systems $\dot{\mathbf{x}} = A\mathbf{x}$ for any $Q > 0$ if there exists $P = P^T > 0$ solving the **Lyapunov equation** $\Rightarrow$ $A^T P + PA = -Q$ then the $Re\{\lambda_i\} < 0\ \forall i$ and the vice versa is also true.
  - Equilibrium $\mathbf{x}^*$ is **locally** asymptotically stable iff the eigenvalues of the Jacobian matrix are negative real. To show global stability we need to use Lyapunov functions.

5 Controllability

- A linear time invariant (LTI) system of the form $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0$ is (completely) controllable if there exists a (piece-wise continuous) control input ($\mathbf{u}(t)$) which can drive the system from *any* initial state $\mathbf{x}_0$ to any other desired state $\mathbf{x}_1$ in finite time t_1.
- Controllability can be assessed by experiments or direct inspection.
- The system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0$ is controllable iff the controllability matrix $\mathcal{C}$ defined as:

$$\mathcal{C} = [B \;\; AB \;\; .... \;\; A^{n-1}B]$$

has full rank. If the number of control inputs $m = 1$, (i.e. $B \Rightarrow n \times 1$), then $\mathcal{C}$ is a square matrix and has full rank if $det(\mathcal{C}) \neq 0$. Can use 'ctrb(A,B)' and 'rank' in Matlab.

- Full rank means that all rows $\boldsymbol{r}_1, \boldsymbol{r}_2, ...$ in a matrix are linearly independent. The rows are linearly independent iff $c_1\boldsymbol{r}_1 + c_2\boldsymbol{r}_2 + ... = 0$, with $c_1 = c_2 = c_3 = ... = 0$ being the only solution.
- If using a diagonalized system the controllability $\mathcal{C}_z = T^{-1}\mathcal{C}$, where T is the matrix of eigenvectors from above. The controllability properties are identical (if the diagonalized system is un/controllable then so is the undiagonalized system and vice versa). Now, $\mathcal{C}_\ddagger = B^*W$ where $B^* = diag\{\hat{B}\} = diag\{T^{-1}B\}$, and W is the so-called Van der Mond matrix, which always has full rank. Thus we can immediately see if a system is uncontrollable if $det(B^*) = 0$, which means we just need to check if one of the $\hat{b}_i = \beta_i = 0$.

$$\beta_i = (T^{-1}B)_i = 0 \Rightarrow \textit{system uncontrollable}$$

6 Observability

- This is about determining whether the states of a given system can be estimated by measuring (observing) its inputs and outputs. If a system is *observable*, then this means that the system states x can be estimated by measuring the system input u and system output y.
- Formally, the LTI system, $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0$ producing $\mathbf{y} = C\mathbf{x}$, is (completely) observable if knowledge of $\mathbf{u}(t)$ and $\mathbf{y}(t)$ on some finite interval, $0 \leq t \leq t_1$, is sufficient to determine the initial state $\mathbf{x}_0$ uniquely. (Note that $C^{-1}\mathbf{y} \neq \mathbf{x}$ because C is not a square matrix.)
- The LTI system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{y} = C\mathbf{x}$ is observable iff the observability matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full rank. Note that if the number of outputs $p = 1$, then the observability matrix $\mathcal{O}$ is square and has full rank iff $det(\mathcal{O}) \neq 0$.

- Alternatively, observability can also be explained in terms of the modes of the system $\hat{C} = CT = (\gamma_1, \gamma_2, ..., \gamma_n)$. If a $\gamma_i = 0$, then the system is not completely observable.

7 Feedback Control

- Feedback is essential for control, thus we choose $\mathbf{u} = f(\mathbf{x}, t)$, i.e. we make the control input to the system a function of the system states.

7.1 State-Feedback Controller (SFC)

- We are able to observe and control the states $\mathbf{x}$ directly. Naturally, we can only use this type of control if the system is controllable. Consider LIT the system for the following: $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0$.
- We can thus choose $\mathbf{u}(t) = -K\mathbf{x}(t)$ (proportional control, K is the gain matrix), this then implies that $\dot{\mathbf{x}} = (A-BK)\mathbf{x}$, and we can thus adjust ('tune') K so that the eigenvalues of $(A-BK)$ are as we desire, thus controlling the system. An unstable equilibrium can thus become a stable one when the control is added, 'switched on'.
- As long as the system is controllable (i.e. $det(\mathcal{C}) = 0$), then we can always find a K that places eigenvalues of $(A - BK)$ in any desired location.

7.1.1 Tuning gain matrix K

1. **Direct Inspection**: Find the characteristic equation of $\hat{A} = (A - BK)$ and then compare it to the characteristic equation of the eigenvalues that you wish the system to have (the desired characteristic polynomial). If there are parameters in matrix A you can use the quadratic formula (for simple cases), or the Routh-Hurwitz Criterion method (change of sign) to determine what the gain parameters need to be.

2. **Ackermann's Formula**: (Great for systems too large for direct inspection), 'acker' and 'place' in Matlab. Simply, it says that the gain matrix can be found immediately by computing:

$$K = \underbrace{[0\,0\ldots1]}_{n\times1}\ \underbrace{\mathcal{C}^{-1}}_{n\times(n\cdot m)}\ q_d(A),$$

   where $q_d(A) = A^n + \alpha_1 A^{n-1} + ... + \alpha_{n-1}A + I\alpha_n$ is the desired characteristic polynomial where instead of eigenvalues λ you have the matrix A instead.

7.2 Output-Feedback Controller (OFC)

- We use this when we cannot observe the state vector $\mathbf{x}$. Instead we estimate it by measuring the system input $\mathbf{u}$ and system output $\mathbf{y}$. This type of control only works is the system is observable. All we know is $A, B, \mathbf{u}, C, \mathbf{y}$. Again consider the linear system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0$:
- The idea is that we design an **observer** (linear observer equation) (Kalman Filter):

$$\mathbf{u} = -K\hat{\mathbf{x}},$$

$$\dot{\hat{\mathbf{x}}} - A\hat{\mathbf{x}} + B\mathbf{u} + \underbrace{M}_{n\times p}(\mathbf{y} - C\hat{x}), \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0.$$

   We can then define an 'estimation error' $\mathbf{e}(t) = \hat{\mathbf{x}} - \hat{x} \Rightarrow \dot{\mathbf{e}}(t) = (A - MC)\mathbf{e}(t), \mathbf{e}(0) = \hat{\mathbf{x}}(0) - \mathbf{x}(0)$. For this error to go to zero (and $\hat{\mathbf{x}} \to \mathbf{x}$), as per usual, the eigenvalues of $A - MC$ need to be negative real. Note that the estimate must be computed sufficiently (1 degree of magnitude) faster that the system (plant) dynamics.

7.3 Translating speeds into eigenv. positions

- No oscillations $\Rightarrow$ eigenvalues are real.
- Settling time $t_s \approx 5\tau$, where τ is the **system time constant** and if the eigenvalues are real and different from one another, then $\tau \approx \max\{-\frac{1}{\lambda_1}, -\frac{1}{\lambda_2}\}$. Use this to find an upper bound for the eigenvalues.

8 Optimal Control

- The idea now is not only control a system, but to make sure that it is being controlled in the most efficient and optimal way. An optimal control problem would, for example, be getting a rocket from A to B with minimum fuel usage.
- More mathematically, optimal control aims to find $\mathbf{u}$ such that $\min_{\mathbf{u}} J(\mathbf{x}, \mathbf{u})$, the cost function, or **performance index**, is minimised.
- Naturally, the mathematics of optimal control would lead to choosing $\mathbf{u} = 0$. However, $\mathbf{u} \neq 0$, otherwise there would be not control and nothing would happen.

8.1 Minimum time optimal control problem

- The minimum-time problem is to find a $\mathbf{u}(t)$ such that the system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$ is driven to a desired final (terminal) condition $\mathbf{x}(t_1) = \mathbf{x}_1$ in the *shortest possible time*, i.e.

$$\min_{\mathbf{u}} J := \min_{\mathbf{u}} \int_0^{t_1} F(\mathbf{x}, \mathbf{u}, t)dt \Rightarrow \text{often } (\min_{\mathbf{u}} \int_0^{t_1} 1dt = min_{\mathbf{u}} t_1)$$

8.2 Minimum Control Effort

- Same as above, except this time we try to attain $\mathbf{x}_1$ while minimising the *overall control effort.* Same again as above, but now we usually set $F(\mathbf{x}, \mathbf{u}, t) = \mathbf{u}^T R\mathbf{u} = r_1 u_1^2 + r_2 u_2^2 + r_3 u_3^2...$, where $R = diag\{r_1, r_2, r_3, ...\}$ is a positive definite, symmetric matrix. Thus:

$$\min_{\mathbf{u}} \int_0^{t_1} F(\mathbf{x}, \mathbf{u}, t)dt = \min_{\mathbf{u}} \int_0^{t_1} (\mathbf{u}^T R\mathbf{u})dt$$

8.3 Linear Regulator Problem

- Given a model of the system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$, we want to find the control inputs $\mathbf{u}$ such that $\mathbf{x}(t)$ is taken as closely as possible to some reference values $\mathbf{x}^*$.Then we find a $\mathbf{u}$ such that:

$$\min_{\mathbf{u}} \int_0^{t_1} F(\mathbf{x}, \mathbf{u}, t)dt = \min_{\mathbf{u}} \int_0^{t_1} ((\mathbf{x} - \mathbf{x}^*)^T Q(\mathbf{x} - \mathbf{x}^*))dt,$$

  where Q is a positive definite symmetric matrix of the form $Q = diag\{q_1, q_2, ...q_n\}$.
- The regulator problem can also be used if we want the system state $\mathbf{x}(t)$ to approach a desired signal $\mathbf{r}(t)$. In that case we choose:

$$\min_{\mathbf{u}} \int_0^{t_1} F(\mathbf{x}, \mathbf{u}, t)dt = \min_{\mathbf{u}} \int_0^{t_1} ((\mathbf{x} - \mathbf{r})^T Q(\mathbf{x} - \mathbf{r}) + \mathbf{u}^T R\mathbf{u})dt.$$

  We often just assume that R is the identity matrix in this case.

8.4 Model Predictive Control - Terminal Problem

- This is about minimising the 'tracking' error at final time t_1 (an extra criterion that must be met at t_1, like a certain speed or something). So we now try to solve:

$$\min_{\mathbf{u}} \int_0^{t_1} ((\mathbf{x}(t) - \mathbf{r}(t))^T Q(\mathbf{x}(t) - \mathbf{r}(t)) + \mathbf{u}(t)^T R\mathbf{u}(t))dt + (\mathbf{x}(t_1) - \mathbf{r}(t_1))^T W(\mathbf{x}(t_1) - \mathbf{r}(t_1),$$

  where W is again a positive definite, symmetric matrix. The model predictive control is like the combined problem of all of the above.

8.5 Generalising the above - The Variational Problem

- Given the system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t), \mathbf{x}(0) = \mathbf{x}_0$, the problem is to find a the control input $\mathbf{u}(t)$ that minimises the performance index:
$$J = \int_0^{t_1} F(\mathbf{x}, \mathbf{u}, t)dt + \Phi(\mathbf{x}(t_1))$$

8.6 The solution to above - Pontryagin Minimum Principle

- Potryagin and his research team proposed the first methodology to solve the general problem proposed above.
- The idea is to characterise how J varies with $\mathbf{u}$, since $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$ and $J(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t)$, where λ is a Lagrangian multiplier introduced into J:
$$J(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) = \int_0^{t_1} [H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) + \dot{\boldsymbol{\lambda}}^T \mathbf{x}]dt + \boldsymbol{\lambda}^T(0)\mathbf{x}(0) - \boldsymbol{\lambda}^T(t_1)\mathbf{x}(t_1) + \Phi(\mathbf{x}(t_1)),$$
where $H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) = F(\mathbf{x}, \mathbf{u}, t) + \boldsymbol{\lambda} f(\mathbf{x}, \mathbf{u}, t)$.
- From the equation above, if we look at the variational effects on J, i.e. δJ, we find that:
$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}}, \boldsymbol{\lambda}(t_1) = \frac{\partial \Phi}{\partial \mathbf{x}}|_{t=t_1}; \frac{\partial H}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial H}{\partial x_1} & \frac{\partial H}{\partial x_2} & \cdots & \frac{\partial H}{\partial x_n} \end{bmatrix}, \frac{\partial \Phi}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \Phi}{\partial x_1} & \frac{\partial \Phi}{\partial x_2} & \cdots & \frac{\partial \Phi}{\partial x_n} \end{bmatrix}$$
$$\delta J = 0 \Longleftrightarrow \frac{\partial H}{\partial \mathbf{u}} = 0; \frac{\partial H}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial H}{\partial u_1} & \frac{\partial H}{\partial u_2} & \cdots & \frac{\partial H}{\partial u_n} \end{bmatrix}$$
,
which implies that $\mathbf{u} = \mathbf{u}(\mathbf{x}, \boldsymbol{\lambda}, t)$
- Note that Pontryagin's method might not produce a feedback controller - in which case more constaints can be added.
- Further note that the solution to the optimal control problem does not require an upper or lower bound (so you could end up getting stuff like 'use an infinite amount of energy to minimise the time', which is unrealistic), so we just add them in manually. We thus use only **admissible control inputs**, i.e. $|u_i| \leq M_i$ or $M_1 \leq u_i \leq M_2$.
- The above is only worth doing if:
  - If the terminal conditions on the system states $\mathbf{x}(t_1) = x_1$ are free/unfixed (we only minimise $\Phi(\mathbf{x}(t_1))$) [Don't entirely get this]
  - The final time t_1 is known and fixed.
- There are two special cases where things need to be done slightly differently:
  - **Fixed terminal conditions ($\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$ given)**
    * I believe we just don't need an initial condition on $\boldsymbol{\lambda}$, sin
  - **Free terminal time (t_1 not fixed)**
    * We just set $F(\mathbf{x}, \mathbf{u}, t) = 1$ and compute $\frac{\partial H}{\partial \mathbf{u}} = 0$ and $\frac{\partial H}{\partial t} = 0 \Rightarrow H = H(t)$ is constant with respect to time.

8.7 Bang-Bang control

- Including $\mathbf{u}^T R\mathbf{u}$ term in J will guarantee boundedness, but will not impose any precise constraints for upper and lower bounds. In practise the control is often constrained by satisfy so-called 'saturation inequalities' (same as admissible above): $-M_i \leq u_i \leq M_i$.
- These inequalities are generally non-linear and general analytic solutions may be impossible. A class of treatable problems is where the system equations are linear.
- Given a linear system, and constraints on the control inputs $\mathbf{u}$, and our aim is to try to find t_1, given $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$ is known (free terminal time): $H = F + \boldsymbol{\lambda}^T f = 1 + \boldsymbol{\lambda}^T A\mathbf{x} + \boldsymbol{\lambda}^T B\mathbf{u}$.

- Now, we can't just compute $\frac{\partial H}{\partial \mathbf{u}} = 0$ because $\mathbf{u}$ is constrained, so what we do is, given $\mathbf{u}$ is only found in the $\boldsymbol{\lambda}^T B\mathbf{u}$ term, we simply set $\mathbf{u}(t) = -M$ (lower constraint) if $\boldsymbol{\lambda}(t)^T B > 0$ and $\mathbf{u}(t) = +M$ if $\boldsymbol{\lambda}(t)^T B < 0$ (upper constraint) so as to render minimum the Hamiltonian H [not sure I get this logic entirely].
- Thus the control inputs constantly get switched from max to min (hence bang-bang, on-off).
- This is still not as it is currently described a feedback-controller.

8.8 Linear Quadratic Problem (LQP) (Feedback Contol)

- The idea now is to use Pontryagins Minimum Principle to synthesise 'linear' feedback controllers.
- Given an LTI system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0$ the LQP is to find the feedback controller that minimises the 'quadratic' performance index:

$$J = \int_0^{t_1} F(\mathbf{x}, \mathbf{u}, t)dt + \Phi(\mathbf{x}(t_1)) = \frac{1}{2}\int_0^{t_1} (\mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T\mathbf{u})dt + \frac{1}{2}\mathbf{x}^T(t_1)W\mathbf{x}(t_1)$$

- Such a feedback control input can be found analytically and has the form: $\mathbf{u} = -B^T K(t)x(t)$ [if $m = 1$ I believe (otherwise it would be $\mathbf{u} = -K(t)x(t)B$ I think). Very similar to SFC $\mathbf{u} = -Kx(t)$. (Note that K is a function of time now, however). See derivation in slides.
- To find $K(t)$ we use the so-called **Riccati equation** (see derivation in notes):

$$\dot{K(t)} + K(t)A + A^T K(t) - KBB^T K + Q = 0,$$

  where $K(t_1) = W$ is the initial condition.

  - Generally this equation needs to be solved numerically. **Existance theorem**: *if $W \geq 0$ then the Riccati Equation has a unique, symmetrix solution $K(t) \geq 0$ and moreover K is bounded for all t*, however, although a solution exists, there is no general solution to the problem (as of now).

8.9 Linear Quadratic Regulator (LQR) $t_1 \to \infty$

- It can be shown that when $t_1 \to \infty$ we have: $\lim_{t_1\to\infty} K(t) = const. = \bar{K}$.
- Given the LTI system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \mathbf{y} = h(\mathbf{x}, \mathbf{u})$ the problem is to find the optimal feedback controller $\mathbf{u}(t)$ such that $\lim_{t\to\infty} y(t) = y_{ref}$, e.g. $y_{ref} = 0$, while minimising the performance index:

$$J = \frac{1}{2}\int_0^{\infty} (\mathbf{y}^T\mathbf{y} + \mathbf{u}^T\mathbf{u})dt = \int_0^{\infty} (\mathbf{x}^T C^T C\mathbf{x} + \mathbf{u}^T\mathbf{u}) = \int_0^{\infty} (\mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T\mathbf{u})$$

- Then (assuming observability and controllability) the solution is $\mathbf{u} = -B^T\bar{K}\mathbf{x}$
- K can be found from the **algebraic Riccatir equation** ($\dot{K} = 0$): $\bar{K}A + A^T\bar{K} - \bar{K}BB^T\bar{K} + Q = 0$. Matlab 'lgr'.
- Can easily see if the system is stable by substituting in $\mathbf{u} = -B^T\bar{K}\mathbf{x}$ into $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{x}(0) = \mathbf{x}_0$ to give: $\dot{\mathbf{x}} = (A - BB^T\bar{K})\mathbf{x}$