Advanced Nonlinear Dynamics and Chaos

For Lorenz the Attractor

November 18, 2020

1 Prerequisite Knowledge L1

- Eigenvalues, Eigenvectors of Matrices, Taylor series in multiple dimensions, linearisation at an equilibrium, local bifurcations and their eigenvalues ($\lambda = 0 \implies$ fold, transcritical or pitchfork, and $\lambda = \pm i\omega \implies$ Hopf for ODEs and $\mu = -1 \implies$ period-doubling for maps).
  - Check for nearby behaviour around the imaginary axis too.
- Remember: Find Jacobian, then eigenvalues of Jacobian, then eigenvectors.
- **Skill: Linearising around equilibrium**: Simply use a change of variables $x = eq_1 + u$, $y = eq_2 + v$, where eq are the equilibrium points.

2 Structural Stability, Local Bifurcations, and Normal Forms L2-L7

Dynamical Systems

Given state space X, time t, evolution operator ϕ and initial conditions $\mathbf{x}_0$, a dynamical system is the transformation or map from $\mathbf{x}_0 \in X$ to some $\mathbf{x}_t \in X$ at time t. Mathematically: $\mathbf{x}_t = \phi^t \mathbf{x}_0$

- Autonomous $\implies$ that if $\dot{\mathbf{x}} = f(\mathbf{x}, \alpha)$ then f is not explicitly a function of time t.
- Smooth $\implies$ continuously differentiable everywhere in the state space.
- Higher order ODEs should be written as lowest order ODEs as possible, and non-autonomous ODEs can be written as autonomous ODEs by letting $x_i = \dot{t} = 1$. Sinusoidal forcing can be accommodated by making $\theta = \omega t \mod 2\pi$ or $\theta = t \mod 2\pi/\omega$. Same applies to maps.

- Poincaré map allows one to go from flow of ODE to discrete-time maps. Take Poincaré section S through state space of dimension less and define return map $P : S \to S : x \to P(x)$. Stroboscopic Poincaré map for forced Duffing could be $\theta = 0 \mod 2\pi$ or for Lorenz equation $z = const$.

Phase Portrait

A phase portrait of a dynamical system is a partitioning of state space into orbits/trajectories: $\{x \in X : x = \phi^t x_0 : t \in \mathbb{R}\}$

Topological Equivalence

Two dynamical systems are considered to be topologically equivalent if there exists a homeomorphism (continuous transformation with a continuous inverse) mapping the phase portrait of one system onto the phase portrait of the other. Apparently there is an explicit continuous transformation that can take a linear system at a stable focus (complex) to a stable node (real).

- You can find the homeomorphic transformation by solving the ODEs and trying to find their connection.

L3: Linearisation and Bifurcations: 2005 Q2a; 2006 Q2;

Structural Stability

A dynamical system is said to be structurally stable if it is topologically equivalent to any ϵ-perturbation (small perturbation) *within the same class of dynamical systems* (i.e. a smooth autonomous ODE must be topologically equivalenet to all systems of the form $\dot{\mathbf{x}} = f(\mathbf{x}, \alpha) + \epsilon g(\mathbf{x})$ where g is any smooth function of the same dimension as f and ϵ is small a number). Simple definition: A system $\dot{x} = f(x, a)$ is structurally stable if all small perturbations to f leads to systems that are topologically equivalent. The failure of structural stability defines a bifurcation point. A necessary condition for structural stability near an equilibrium is that the system is hyperbolic (no eigenvalues on imaginary axis).

- Linear systems: Recall $\dot{\mathbf{x}} = A\mathbf{x} \implies \mathbf{x}(t) = \sum_{i=1}^{n} a_i \mathbf{v}_i e^{\lambda_i t}$. **Asymptotically stable** means all eigenvalues of A have real part < 0. **Structural stability is very different to stability!** Stable and unstable nodes (= real eig) or foci (= complex eig) can be structurally stable while not being stable.
- Saddle focus 3D $\to \lambda_1 < 0, \lambda_{2,3} = \mu \pm i\omega, \mu > 0 \implies$ 'saddle ratio' $= \delta = \mu/|\lambda_1|$ (Shilnikov homoclinic bifurcation occurs at $\delta < 1$, this is a global bifurcation that generates chaos)

Invariant Set

An invariant set of a dynamical system is a subset of phase space $A \subset X$ that is invariant under the dynamics: given $x \in A$ then $\phi_t(x) \in A \ \forall t \in \mathbb{R}$. For two nonlinear ODEs to be equivalent they must have the same number and types of invariant sets.

Attractor

A closed invariant set is called an attractor if it is asymptotically stable in time: that is there exists a $U \supset A$ such that $\phi^t(U) \in U$ for all t and $\phi^t(U) \to A$, as $t \to \infty$. Attractors include, equilibria (steady-states, stationary points, fixed points), Periodic orbits (limit cycles), Tori (quasi (irregular) -periodic orbits), strange attractors (chaotic attractors).

- How to study equilibria: change co-ordinate system so that equilibrium is at the origin, throw away non-linear terms, classify linear system based on eigenvalues.

Hyperbolic Equilibria

An equilibrium point is said to be hyperbolic if its linearisation has no eigenvalues with zero real part, i.e. nothing lies on the imaginary axis.

The Hartman-Grobman Theorem

In a sufficiently small neighbourhood (how small?) of a hyperbolic equilibrium $\mathbf{x}_e$, *non-linear* dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$ is topologically equivalent to its linearisation $\dot{\mathbf{x}} = Df(\mathbf{x}_e)\mathbf{x}$. (where $Df(\mathbf{x}_e)$ is the Jacobian derivative evaluated at the equilibrium x_e). Corollary: If all eigenvalues of $DF(\mathbf{x}_e)$ are negative real, then the equilibrium point is stable. The theorem shows that hyperbolic equilibria are structurally stable. Non-hyperbolic equilibria are structurally unstable.

- Putting linear systems into diagonal form: given $\dot{\mathbf{x}} = A\mathbf{x}$, and $\boxed{V = (\mathbf{v}_1|\mathbf{v}_2|...)}$ (matrix of eigenvalues of A), then $\Lambda = V^{-1}AV$ is the diagonal matrix, and if you let $\mathbf{x} = V\mathbf{y}$ the system $\dot{\mathbf{x}} = A\mathbf{x}$, can be written in diagonal form as $\dot{\mathbf{y}} = \Lambda\mathbf{y}$

2008 Q1;

L4: Local Bifurcations + Codim 1 Normal forms

Bifurcation L4

A system $\dot{\mathbf{x}} = f(\mathbf{x}, \alpha)$ is said to undergo a bifurcation at parameter value $\alpha = \alpha_0$ if for any (small) neighbourhood of $\alpha_0 \in \mathbb{R}^p$ there is an α-value containing dynamics that are not topologically equivalent to those at α_0. A dynamical system at a bifurcation point is structurally unstable, however the reverse is not true. A true bifurcation occurs when parameter α 'unfolds' the degeneracy at $\alpha = \alpha_0$. A **bifurcation diagram** plots the invariant sets (equilibria) of a dynamical system against a single or multiple bifurcation parameters, indicating stability. A **local bifurcation** is a bifurcation that can be analysed purely in terms of a change in the linearisation around a single invariant set or attractor (caused by changes in the stability properties of an invariant set) or topological changes to the phase portrait in the neighbourhood of an equilibrium. A **global bifurcation** cannot be analysed like this (all other bifurcations are called global), topological changes to the entire phase portrait.

Codimension

The codimension of a bifurcation is the difference between the dimension of parameter space and the corresponding bifurcation set. Alternatively, codimension is the number of parameters we need to add in a generic way to 'unfold' bifurcation. The minimum number of parameters needed to create the bifurcation. Folds and Hopf are codim 1 bifurcations.

Normal Form (near a non-structurally stable equilibrium)

Normal forms (or universal unfolding, or versal unfolding) is a simplified system that only contains the **resonant** nonlinear terms (such that there exists parameter values corresponding to all possible structurally stable phase portraits that are a small perturbation away from the degenerate phase portrait at the bifurcation point). Resonant terms cannot be removed by a near identity transform. A topological normal form is a the same as a normal form except that additionally its phase portraits are topologically equivalent to any system undergoing the bifurcation in a generic way.

- Fold normal form: $\dot{x} = \alpha + sx^2$. Hopf normal form: $\dot{x} = \alpha x - \omega y + lx(x^2 + y^2)$, $\dot{y} = \alpha y + \omega x + ly(x^2 + y^2)$. Or $\dot{r} = \alpha r + lr^3, \dot{\theta} = \omega$. Parameter α is the key parameter that causes the bifurcation. The sign of Lyapunov coefficient l determines whether it is a sub (pos) or super-critical (neg). These are always considered around an equilibrium point.

- Transcritical bifurcation $\dot{x} = \alpha x + sx^2$, Pitchfork $\dot{x} = \alpha x + sx^3$ (sub/super-critical depending on sign of s). The pitchfork is a special case of a **cusp** codim 2 bifurcation with normal form $\dot{x} = b + ax + sx^3$.

- Codim 2 Bifurcations include Takens-Bogdanov (double zero eigenvalues, not chaotic), Gavrilov-Guckenheimer (fold + Hopf, i.e. eigs are 0 and $\pm i\omega$, chaotic), Hopf-Hopf (eigs

$\pm i\omega_1$, eigs $\pm i\omega_2$, chaotic).

Sketching Hopf: 2019 Q2;
Computing Normal form and Lyapunov coefficients L5: Specifically for Hopf Bifurcation: Skill: finding complex tranformation, lyapunov coefficients and removing nonlinear terms: 2007 Q3 d; 2011 Q2);

- I think the appropriate coefficient for the Hopf Bifurcation: Given $\dot{x} = \lambda x + \sum_{k=2}^{N} g_k x^k, x \in \mathbb{R}$, then $p_k = \frac{-g_k}{\lambda(k-1)}$
- You can transform a system that undergoes a hopf bifurcation in complex form which allows one to easily determine the Lyapunov coefficient l, and see if its a subcritical $(l > 0)$ (unstable periodic orbit) or super critical $(l < 0)$ (stable orbit) [safe one].
- **Skill: Liapunov coefficinets and complex transform for Hopf**: 2008 Q3 b); 2019 Q2 c)
  - **Old method**: Find an eigenvector at hopf bifurcation eigenvalues, e.g. $\lambda = \pm i$. Next, find the eigenvector $\mathbf{v} = (\bar{p_1}, \bar{p_2})$ of the negative eigenvalue. Then $z = \bar{p_1}x + \bar{p_2}y$.
  - **New Method:** Take positive eigenvalue and find its eigenvector w. Then construct the matrix $W = (w|\bar{w})$, and compute W^{-1}. Then use $(x, y)^T = W(z, \bar{z})^T$ and $(z, \bar{z})^T = W^{-1}(x, y)^T$ to find the necessary equations.
  - Next, find $\dot{z}$ (e.g. $\dot{z} = i\dot{x} + \dot{y}$ and sub in the equations, and use the transformations to get $\dot{z}$ in terms of z and $\bar{z}$ only.
  - Next compare this to the formula $\dot{z} = i\omega z + \sum_{n+m\geq 2} g_{nm} z^n \bar{z}^m$ to find the coefficients. Note that $i\omega z$ is where the bifurcation happens. So it shouldn't cancel when you expand.
  - Finally use the formula $l = Re(\frac{1}{2\omega}(ig_{20}g_{11} + g_{21}))$ to find the Liapunov coefficient. If $l > 0$ this implies a sub critical hopf (according to 2007 answers).
- **Skill: removing terms: Near Identity transform**
  - The aim is to find a parameter value that will get rid of a term in a new transformed state.
  - Take the equation at hand e.g. $\dot{x} = ...$ then set $x = y + p_k y^3$ or whatever term you want to remove.
  - Differentiate $\dot{x} = \dot{y} + 3p_k y^2 \dot{y}$ and then sub in the $\dot{x}$ equation and x values, so everything is in terms of y. Next, use a Taylor expansion for $\frac{1}{1+x} = 1 - x...$ and find the coefficient for the term you want to get rid of (here y^3). Then set this coefficient to zero and rearrange to find the value for p_k.
  - If you get the divide by zero error that indicates that you cannot remove this term. These are the **resonant terms**.

- Hopf Bifurcation topological normal form: $\dot{z} = (\alpha + i\omega z) + lz|z|^2$

General theory for normal form L6: Go through example sheet

- This is also bloody hard, you'll have to do questions to understand it.

Resonant terms

One cannot remove a term $G_{ia}y_1^{a_1}...y_m^{a_m}$ if the indices a_j satisfy the resonance condition: $\lambda_i = \sum_{j=1}^{m} a_j \lambda_j$. The system $\dot{\mathbf{x}} = f(\mathbf{x})$ is re-written in diagonal form $\dot{\mathbf{x}} = f(\mathbf{x}) = \boldsymbol{\lambda}\mathbf{x} + G(\mathbf{x})$, so $\boldsymbol{\lambda}$ are the eigenvalues of the Jacobian $Df(0)$ (equilibrium is at $\mathbf{x} = 0$). Vector $\boldsymbol{a}$ acts as some sort of indexer, which I don't fully understand. I think the key purpose of this is to find out if the eigenvalues of a system can be written as a sum of each other. See example in L6.

- Resonance illustrates that the Hartman-Grobman Theorem only provides a continuous linearising transformation not an arbitrary smooth differentiable one.

Invariant Manifold L6

Given a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ and an $m < n$ dimensional manifold M, we say that M is invariant under the flow generated by f, if M is composed of solution trajectories. Supposing that M can locally be written as $\mathbf{x} = h(\boldsymbol{y})$, where $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^n$. The for a trajectory $x(t) \in M$ we have the **invariance condition** $\dot{\mathbf{x}} = Dh(\mathbf{y})\dot{\mathbf{y}} = f(h(y))$.

Stable and Unstable Manifolds L6

Given a stable and unstable manifold meeting at a saddle, a stable Manifold $W^s(\mathbf{x}_0)$ is the set of all solutions/trajectories that tend to the saddle as $t \to \infty$. The unstable manifold $W^u(\mathbf{x}_0)$ is the set of solutions that tend to the saddle as $t \to -\infty$. The stable and unstable manifolds are tangent to the eigenspaces E^s and E^u, respectively, near the saddle $\mathbf{x}_0$. Stable/unstable **Eigenspaces** are made of the eigenvectors corresponding to positive/negative eigenvalues.

- For a non-degenerate Hopf bifurcation we require that the coefficient of $|z|^2z$ be non-zero. Furthermore $f_\lambda \neq 0$.

L5,L6 Questions to do: 2005 1c,d;

L7: Centre Manifold: 2005 Q2b,c; 2006 Q1 a-c); 2007 a-c); 2008 Q1 a);

Centre Manifold L7

The centre manifold $W^c(\mathbf{x}_0)$ of a non-hyperbolic equilibrium (i.e. at the bifurcation point) is an n_c dimensional invariant set that is locally tangent to the centre eigenspace E^c (which must be made of the eigenvectors corresponding to the eigenvalues on the imaginary axis). It consists of solution trajectories that are neither attracted to or repulsed by the stable and unstable manifolds. Note that the dimensions of the stable, unstable and centre eigenspaces make up the dimensions of the vector field f, i.e. the dimension of the system (number of dependent variables): $(n_s + n_u + n_c = n)$.

Centre Manifold Theorm L7

Don't fully understand this, do question on it. I think the idea is that it allows you to write the centre manifold as a function.

Finding the centre manifold:

- Seek transformation $v = h(u, \beta)$. u is the other parameter after the transformation $(u, v) = V \cdot (x, y), \beta$ is the parameter. If only quadratic terms in u and β are required then $v = h_{20}u^2 + h_{11}u\beta + h_{02}\beta^2$. If more terms are needed at them like so: $h_{30}u^3 h_{21}u^2\beta$, etc... like the binomial distribution. h_{ij}, where you have u^i and β^j.
- Find derivative $\dot{v} = (2h_{20}u + h_{11}\beta)\dot{u}$. This has to satisfy the ODEs, so substitute in equations for $\dot{v}$ and $\dot{u}$.
- Rearrange and substitute in $v = h(u, \beta)$ (only keep up to quadratic terms). Then compare coefficients to find h's. The comparing is rather specific, it compares the variable and parameters behind each h_{ij} term, so don't compare $O(x)$ with $O(\beta x)$ for example.
- Finally sub in $v = h(u, \beta)$ into the $\dot{u}$ equation to get flow on manifold or whatever is asked for.
- You can then have transcritical, pitchforks, folds on the manifold which you can sketch.

Parmaterised Centre Manifold means: Given a parameterised system $\dot{x} = f(x, \alpha)$, we add $\dot{\alpha} = 0$, and then compute the centre manifold of the extended system.

3 Periodic Orbits, Quasi Periodic Motion and Synchronisation L8-L11

L8: Periodic Orbits and Their Bifurcations: 2006 Q3a,b-;

- For maps $(x_{n+1} = f(x_n, \alpha))$ eigenvalues are sometimes referred to as (floquet) multipliers.

Stability is implied by multipliers inside the unit circle $|\lambda_i| < 1$. Hyperbolic fixed points implies **no** $|\lambda_i| = 1$.

- According to 2008 Q4 paper, a pitchfork,fold,transcritical bifurcation could occur if the floquet multiplier $\lambda = 1$. This is computed by solving the **linearised** equation and then seeing if $x(2\pi) = x(0)$. If this is true then a pitchfork bifurcation has occured. For period doubling $\lambda = -1$, so you need to check if $x(2\pi) = -x(0)$.

- Fold occurs when multiplier/eigenvalue $\lambda = +1$. Normal form is $x_{n+1} = \alpha + x + x^2$.

- For complex multipliers $\lambda = e^{\pm i\theta}$ (notice that this on the unit circle) then a torus (Neimark-Sacker) bifurcation occurs. Normal form: $z_{n+1} = (1+\alpha)e^{i\theta}z_n + lz_n|z_n|^2$. It creates an invariant circle for maps and a torus for ODEs provided $\theta \neq 2\pi/q$, $q = 1, 2, 3, 4$.

- Period doubling or flip bifurcation occurs when $\lambda = -1$. $x_{n+1} = -(1+\alpha)x_n - x_n^3$ leads to Feigenbaum period doubling cascade ($\delta = \lim_{n\to\infty} \frac{a_{n-1}-a_{n-2}}{a_n - a_{n-1}} = 4.669\,201\,609\ldots$).

- **Floquet Multipliers**
  - Essentially these are just eigenvalues for determining stability of NON-LINEAR systems that have oscillations.
  - Floquet Multipliers are used to determine the stability of nonlinear oscillatory behaviour. They are the eigenvalues of a **monodromy matrix** M that gives orbital stability for the periodic solution of a system. **The monodromy matrix is a matrix that describes the transformation of one perturbation to another (on a Poincare section) after every cycle**. If the absolute values of *all apart from one at unity* Floquet multipliers are less than 1 (unity) then the periodic solution is stable. The $n-1$ Floquet multipliers can also be found from the associated Poincare map and one trivial Floquet multiplier associated with the eigenvector $Y = \dot{z}$, where $z(t)$ is the periodic orbit with period T of the system of ODEs $\dot{\mathbf{y}} = g(\mathbf{y})$
  - The monodromy matrix is determined by linearising around the periodic orbit $z(t)$ and finding the *linear variational equations* $\dot{Y} = Df(z(t))Y$, for $0 \leq t < T$. $\mathbf{Y}(T)$ Apparently $M = Y(T)$.

Skill: Finding Floquet Multipliers: 2006 Q4 d; 2007 Q3 c); 2011 Q3 a,b;

1. Turn the system into an autonomous system. See if the equations are uncoupled. When linearising, linearise around the equilibrium point given in the question, e.g. if $r = 1$ is the equilibria, then use $p = 1 + r$ as your substitution.

2. If there is sinusoidal forcing (see lecture 1) then linearise the system about $x_1 = t \mod 2\pi/w$, where w comes from $\cos(wt)$ or $\sin(wt)$. The period then becomes $T = 2\pi/w$.

3. Then find the eigenvalues of the linearised matrix (presumeably this is the monodromy matrix).

4. The Floquet multipliers are then $e^{eig(M)*T}$. Note, there should be three for a 3 dim system. If they have different signs, then this represents a saddle I believe.

- Quasi-Periodic Motion: A type of motion produced by a dynamical system that has two or more incommensurable frequencies, like on an invariant torus. Incommensurable means that for two numbers p, q, p/q is not rational. So e.g. if f_1/f_2 is irrational we get quasi-periodic frequencies. Otherwise if f_1/f_2 is rational we get *phase-locked* motion.
- To sketch phase/mode locking motion of the form $p : q$ where $f_1/f_2 = p/q$, then create a 2D plane and cross the horizontal axes p times and the vertical axes q times.

- Floquet multiplier of one corresponds to finding $x(0) = x(2\pi)$? 2008 Q4 b);

Going from torus to circle map: Poincare section, then a rotation angle or arclength, which then maps onto itself. 2011 Q3c) L9: Resonance, and Resonance (Arnold) Tongues, Mathieu Equation: 2005 Q4a; 2008 Q4c-e));

- Mathieu Equation: $\ddot{x} + \delta\dot{x} + (\beta + \alpha\cos(t))x = 0$. Has a pitchfork ($x(2\pi) = x(0)$) and period doubling ($x(2\pi) = -x(0)$) bifurcations.
- Resonance tongues plot α against β.

Mode locking resonance on a Torus

This arises when there are two independent frequencies plus nonlinear coupling. This can occur for a system with a natural frequency ω_1, that is forced with another frequency ω_2. E.g. a laterally forced pendulum.

Parametric resonance

Occurs when a system with natural frequency ω_1 has a parameter that is modified with a different frequency ω_2. For example a vertical pendulum where the gravity is changed sinusoidally.

L10: Circle Maps: 2005 Q4 b,c,d; 2006 Q3 c)

- Circle maps, are like any other map (e.g. Logistic map $x_{n+1} = rx_n(1-x_n)$), but they give rise to Arnold Tongues which was presented in the previous lecture.
- The 'standard circle map' is of the form $\theta_{n+1} = \theta_n + \alpha - K\sin(\theta_n) \mod 2\pi$ or it can be written as $\phi_{n+1} = \phi_n + \Omega - \frac{K}{2\pi}\sin(2\pi\phi_n) \mod 1$ if you let $\theta_n = 2\pi\phi_n$ and $\alpha = 2\pi\Omega$
  - The map is invertible if $K < 1$. For $K = 0$ you have rigid rotation and rotation number $\rho = \Omega$.

- For $0 < K < 1$ the curve of the roation number $\rho(\Omega)$ is called the Devil's Staircase (derivative is zero almost everywhere). Also known as *Farey tree*, because nearby rotation numbers obey Farey arithmetic $(n/m + p/q = (n+m)/(p+q))$. For each rational $\Omega = p/q$ there is an interval of Ω-values where $\rho = p/q$, and you get mode (same as phase I presume)-locked motion with a stable and unstable $p : q$ orbit. There is also a non-zero measure of Ω-values with irrational rotation number, and you get quasi-periodic motion (known as KAM theorem, apparently).
- Arnold tongue with root at $\alpha 2\pi p/q$ is phased locked rotation number $\rho = p/q$.
- Smaller q implies thinner tongue. For small $K = \epsilon$, $\alpha = 2\pi p/q \pm O(\epsilon^{q-1})$

- The rigid rotation map $\theta_{n+1} = \theta_n + \alpha \mod 2\pi$ with $\alpha \in \mathbb{R}$. The rotation number is $\rho = \alpha/2\pi$.

Skill: If you are asked to find the value of α in the circle map above that corresponds to a certain rotation number, do the following:

- Set $\alpha = 2\pi p/q + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \ldots$. And apply Rotation number theorem 2, to get the map $f^{(q)}(\theta) = \theta + 2\pi p$.

- Cancel terms where possible, apply double angle formula and then use a taylor expansion $\cos(x) = 1$ and $sin(x) = x$ to simplify further. First try and (split $\theta + \alpha$, against $\epsilon \sin\theta$), and then use taylor expansion, then get rid of π by changing the signs and then redo sin/cos expansion and taylor expansion. Finally determine α_1 and α_2 by solving $O(\epsilon)$ and $O(\epsilon^2)$.

Rotation Number

Definition wrong in notes apparently, so need to find question and solution on it.

Rotation Number Theorems

1) Provided the circle map is smooth and invertible, then the rotation number ρ is independent of the initial condition θ_0. Something missing after this in the notes I think...
2) The rotation number $\rho(f) = p/q$ (i.e. is rational) if and only if there is a $q-$ periodic orbit that winds round the circle p times: $f^{(q)}(x_0) = x + 2\pi p$, for some $x_0 \in [0, 2\pi)$. There is a proof for this (L10, that you'll have to go over) - when you get it...

- $f^{(q)}(\theta)$ above implies computed the map of the map q times. e.g. $f^{(2)} = f(f(\theta))$. It lets you find the fixed points that have the rotation number $\rho = p/q$.

L11: Synchronisation of coupled oscillators

- Kuramoto Model for N oscillators with weak coupling: $\dot{\phi}_i = \omega_i + \sum_{j=1}^{N} K_{ij} f(\phi_j - \phi_i)$. Require $f(\phi + 2\pi) = f(\phi)$ and $f(-\phi) = -f(\phi)$. f is usually sin().

- For $N = 2$ case, synchronicity requires that $K > 0$ is large enough and $\Delta\omega = \omega_2 - \omega_1$.

4 Global Bifurcations and Chaos L12-L16

L12: Homoclinic Bifurcations: Poincare map: 2005 Q3b - d; 2007 Q4;

- We are now dealing with global bifurcations.

- Key skill of this lecture is constructing a Poincare map

Homoclinic orbit/trajectory

A homoclinic orbit is a trajectory of a dynamical system that tends to the same invariant set (equilibrium, fixed point, periodic orbit etc.) as time tends to positive and negative infinity, i.e. $t \to \pm\infty$. A **heteroclinic** orbit is a trajectory that tends to two different invariant sets as $t \to +\infty$ and $t \to -\infty$. Homoclinic orbits to equilibria are of codimension-one and their existence (if the intersection of the stable and unstable manifold are 90 degrees tangential) is in it self a bifurcation! The equilibria from which a homoclinic orbit is always a saddle-point I believe.

- Importand aside: The leading eigenvalue is defined as the eigenvalue closest to the imaginary axis. For homoclinic bifurcations, the position of this eigenvalue, be it positive, negative, complex, etc..., determines the features of the bifurcation.

Shilnikov's Tame Homoclinic Bifurcation Theorem

A non-degenerate homoclinic orbit to a saddle or saddle-focus with a real determining eigenvalue implies a unique periodic orbit bifurcates from the homoclinic orbit with period $T \to \infty$

Shilnikov's Chaotic Homoclinic Bifurcation Theorem

An infinite number of periodic orbits exist in a neighbourhood of a saddle-focus homoclinic orbit with **complex determining eigenvalues**. Moreover, there are infinitely many multiple-pulse homoclinic orbits at nearby parameter-values.

- Because Poincare maps can be used to explore homoclinic bifurcations, Alan defines stable and unstable manifolds for maps again: The stable manifold $W^s(\hat{x})$ of a hyperbolic fixed point $\hat{x}$ is the set of all solutions that tend to $\hat{x}$ as time $n \to \infty$. The unstable manifold $W^u(\hat{x})$ is the set of solutions that tend to $\hat{x}$ as $n \to \infty$.

- Stable/unstable manifolds for ODEs are one dimension more than the manifolds of the Poincaré section, and thus so is the homoclinic orbit.
- A transverse intersection (not tangent) between stable and unstable manifolds leads to structurally stable homoclinic orbit - it's existance does not represent a bifurcation.
- **Homoclinic Tangle**: In essence it could be described as two transverse homoclinic orbits that don't meet in a loop but instead wrap around each other (at attracting and repelling fixed points). The area of the loops are always constant and the get longer and thinner closer to the saddle-point. 2008 Q2d);

Determing the Poincare sections:

- Define the Poincare sections, each should fix one variable.
- Try to solve the ODEs and find the equation for the time of flight $t = ...$
- $\prod_{loc}$ is the map from one section to another via the saddle point, $\prod_{glob}$ is the map from from one section to another away from the saddle.
- It looks quite complicated but if you consider the intersections as vectors and see how they change, you can find the mappings.

L13: Horseshoe map 2006 Q4 a,b); 2016 Q4 b);

- One can draw shaded regions in a homoclinic tangle that shows equivalent behaviour to the Horseshoe map.
- The horseshoe map maps vertical and horizontal lines onto a box. Each horizontal and vertical line can be represented by $s_i = \{0, 1\}$. For horizontal boxes i is positive and so is the mapping f^i, for vertical boxes its the other way round. e.g. $s_1 s_2 = 01$ means we are looking at the second line in a set of four lines, so the mapping f has only occurred twice, i.e. we are about to go into f^3.
- The result is a cantor set of infinite intersections of horizontal and vertical strips. This is the invariant set of the map f, and can be expressed by a bi-infinite symbolic sequence.
- The horseshoe map shows chaotic behaviour:
  - 1) There are infinitely many periodic orbits
  - 2) Sensitive dependence on initial conditions
  - 3) A trajectory exists that visits arbitrary close to every point.

* Consider all finite strings in order: $0, 1, 00, 01, 10, 11, ...$ then you can create a string that alternates these left to right around the shift point: ...10000.10111... then this string is also an invariant set that contains **all** possible finite length strings and therefore gets arbitrarily close to all points in the invariant set under appropriate iterates of f.

- A sequence of lines/intersections can be expressed in bi-infinite symbolic sequences as $...s_3s_2s_1s_0 \cdot s_{-1}s_{-2}s_{-3}...$ and moving the dot turns the horseshoe map into a shift map.

L 14: Chaotic Attractors:

- From now on assuem that invariant sets A are **compact**, which means they are both closed and bounded (i.e. it cannot escape to infinity).

Chaotic invariant set

A compact invariant set A is called chaotic if it satisfies three additional conditions:
1. Sensitive dependence on initial conditions: There exists an $\epsilon > 0$ such that, for any $x \in A$, and any neighbourhood $U \subset A$ of x, there exists a $y \in U$ and a $t > 0$ such that $|\phi_t(x) - \phi_t(y)| > \epsilon$.
2. There exists a dense trajectory that eventually visits arbitrarily close to every point of the attractor: There exists an $x \in A$ such that for each point $y \in A$ and each $\epsilon > 0$ there exists a t ($t > 0$ or $t < 0$) such that $|\phi_t(x) - y| < \epsilon$. (called topological transitivity sometimes apparently)
3. There is a countable infinity of periodic orbits within the dynamics.

- A chaotic invariant set that is also an attractor is called a strange attractor (like the Lorenz equations).

L15: Homoclinic tangencies 2008 Q2 d);

- The invariant set of the Smale Horseshoe is unstable. This is caused by the transverse intersection between the stable and unstable manifolds, which is structurally stable. The bifurcation event is a homoclinic tangency, where the stable and unstable manifolds become tangent.

- The horseshoe map is a necessary consequence of the homoclinic tangle, but its the homoclinic tangency that actually creates it.

- Each of the infinitely many periodic orbits are created via a fold bifurcation, creating one stable and one unstable orbit. The stable orbits go unstable in periodic doubling cascades, leading to chaos. The Hénon map provides a good normal form for this.

- The Hénon map is a simple discrete-time analogy of the forced duffing equation: $x_{n+1} = 1 - ax_n^2 + by_n$ and $y_{n+1} = x_n$. Determinant of the Jacobian is $|J| = -b$.
  - Fixed point given by $x^* = y^* = \frac{b-1 \pm \sqrt{(b-1)^2+4a}}{2a}$
  - Has a period-doubling bifurcation ($\lambda = -1$).
  - For $-\infty < a < 1.55$, the dynamics contain Smale Horseshoe. For the 'classic values' $a - 1.4, b = 0.3$, stable and unstable manifolds intersect transversely and you get the famous Henon map with fractal behaviour.
  - It also shows chaotic behaviour for certain parameter values.

L16: Quantifying Chaos:

- This goes into quantifying how 'unpredictable' the dynamics within a chaotic attractor is: Liapunov exponent. Then it looks at fractals and their dimensions.

Better definition for Lyapunov exponents in 2006 Q4c) perhaps.

Liapunov Exponent

Small perturbations to $x(t)$ are governed by the linear variational equations $\dot{y} = Df(x(t))y$. The solution with $y(0) = e$, where e is a unit vector, we call $y_e(t)$. Solutions are generally exponentials, so we define the rate of expansion up to time t as $\lambda_t(x(t), e) = \frac{1}{t} log||y_e(t)||$. For the trajectory $x(t)$ in the direction e the Liapunov Exponent is defined as $\hat{\lambda}(x(t), e) = lim_{t\to\infty}\lambda_t(x(t), e)$. A spectrum of Liapunov exponents can be obtained by considering all possible initial vectors e.
Condition for a chaotic attractor: at least one positive λ.
Relation to Floquet Multipliers (FM) for a periodic orbit: For a periodic orbit, the FM equals $e^{\lambda T}$, where T is the period of the orbit, and λ is the LP exponent.

- For a stable periodic orbit, there is one LP and is equal to zero. For quasi-periodic motion (tori) there are two LP exponents, one equals zero, the other is negative. For a chaotic attractor there is at least one positive LP, implying sensitivity to initial conditions.
- Because one cannot compute the infinite limit of the LP one can use the Finite-Time-Liapunov Exponent (FTLE) $= \frac{1}{T} log|eig\Phi(T)|$, where $\Phi(T)$ is the fundamental solution matrix of the variational equation $\dot{z} = Df(x(t))z$, such that $z(t) = \Phi(t)z(0)$.

Fractals: 2016 Q4 d) 2011 Q4d);

- They don't necessarily have integer dimensions. Like the Koch curve has dimension 1.268.
- Definition: A fractal dimension is the dimension of a set whose dimension is not necessarily an integer.

- The Lorenz attractor has fractal dimension between 2 and 3 because each wing is a cantor set of infinite separate wings. The cantor set has dimensions between zero and one and the wings are two dimensional so the fractal dimension lies betwen 2 and 3 for the Lorenz attractor.

Box counting fractal dimension

The fractal dimension of a set A embedded within an m-dimensional space is defined by considering covering the object with m-dimensional hypercubes ('boxes') of size δ. Let N be the number of such boxes required to cover A. The fractal dimension is then defined as:

$$D_B = lim_{\delta \to 0} \frac{log(N)}{log(1/\delta)} = lim_{N \to \infty} \frac{log(N)}{log(1/\delta)}$$

Fractal

A fractal is a set with non-integer fractal dimension.

- Strange attractors are typically multi-fractal (D_B does not converge uniformly). One has a spectrum of different fractal dimensions depending on the length scale one looks at.
- The Hénon map for $a = 1.4, b = 0.3$ has fractals inside it. Other fractal sets include the famous Mandelbrot set: $z_{n+1} = z^2 + c$, $c \in \mathbb{C}$.