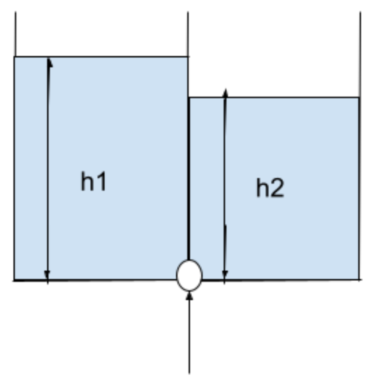
Controller Design on the Blue Roof

Contents

I. Two Water Tanks Analysis

The schematic of the two water tanks, connected via a cylindrical tube, is shown as:



where h_1, h_2 are the water levels, r is radius of the circular tube, d is the length of the tube.
Now, given a certain h_1, h_2 pair, we want to compute the flow rate of the water through the tube, and then derive a transition matrix of this simple system.

Suppose $h_1 > h_2$, we can compute the flow rate Q as:

$$h_1 - h_2 = \frac{128\mu dQ}{\pi\rho g r^4} \tag{1}$$

assuming $\mu = 0.001 kg/m \cdot s$, $\rho = 998 kg/m^3$.
Thus

$$\begin{aligned} Q &= \frac{\pi\rho g r^4}{128\mu d}(h_1 - h_2) \\ &= k(h_1 - h_2) \end{aligned} \tag{2}$$

Due to symmetry, when $h_2 > h_1$, the water flows at the same rate, it is just a matter of the sign. Given this relation, we can construct a transition matrix.

Suppose we are given a time interval T that is small enough, we can ignore the error produced by discretising the time, and it follows:

$$\begin{aligned} h_1[t+1] &= h_1[t] + QT \\ &= h_1[t] + kT(h_2[t] - h_1[t]) \\ &= (1-kT)h_1[t] + kTh_2(t) \end{aligned} \tag{3}$$

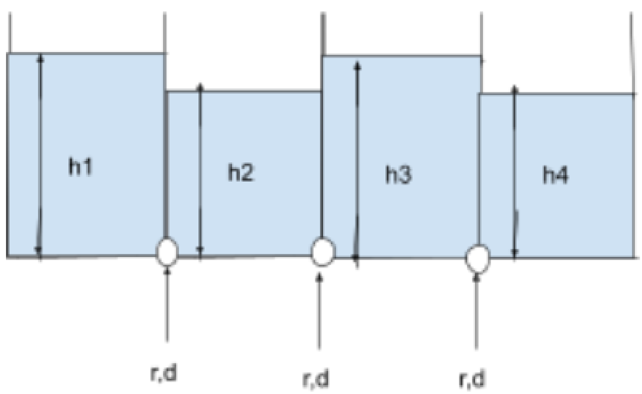
The transitional matrix can be constructed as:

$$\begin{bmatrix} h_1[t+1] \\ h_2[t+1] \end{bmatrix} = \begin{bmatrix} 1-kT & kT \\ kT & 1-kT \end{bmatrix} \begin{bmatrix} h_1[t] \\ h_2[t] \end{bmatrix} \tag{4}$$

II. Four Water Tanks Analysis

A real picture of a blue-roof is shown here:

Our application consists of four water tanks connected in parallel with identical geometry, thus the constant k is the same for all tubes. As shown here:



Their transitional matrix can be constructed similarly as above.

$$\begin{aligned} \begin{bmatrix} h_1[t+1] \\ h_2[t+1] \\ h_3[t+1] \\ h_4[t+1] \end{bmatrix} &= \begin{bmatrix} h_1[t] + kT(h_2[t] - h_1[t]) \\ h_2[t] + kT(h_1[t] - h_2[t]) + kT(h_3[t] - h_2[t]) \\ h_3[t] + kT(h_2[t] - h_3[t]) + kT(h_4[t] - h_3[t]) \\ h_4[t] + kT(h_3[t] - h_4[t]) \end{bmatrix} \\ &= \begin{bmatrix} 1-kT & kT & 0 & 0 \\ kT & 1-2kT & kT & 0 \\ 0 & kT & 1-2kT & kT \\ 0 & 0 & kT & 1-kT \end{bmatrix} \begin{bmatrix} h_1[t] \\ h_2[t] \\ h_3[t] \\ h_4[t] \end{bmatrix} \end{aligned} \tag{5}$$

III. Control schemes

A. Theory

Digital instruments are installed on all three of the tubes, they can perform two actions: open and close. Therefore the control actions are chosen from the 8-element set

$$C = \{000, 001, 010, 011, 100, 101, 110, 111\} \tag{6}$$

The fact that only 8 actions can be taken greatly simplified our analysis. The control purpose is to maintain the water level around the reference water levels $(h_1, h_2, h_3, h_4) = (a_1, a_2.a_3.a_4)$. Therefore, the stage cost is obvious

$$c(h_1, h_2, h_3, h_4) = ||h - a|| = \sqrt{(h_1 - a_1)^2 + (h_2 - a_2)^2 + (h_3 - a_3)^2 + (h_4 - a_4)^2} \tag{7}$$

Then, at each time $t = N$, given a set of water levels $(h_1[t], h_2[t], h_3[t], h_4[t])$, we wish to choose a control from C such that $c(h_1[t+1], h_2[t+1], h_3[t+1], h_4[t+1])$ is minimized. Since C is a small finite list, we can solve this by iterating over C.

If different control is chosen, the transitional matrix is different. For example, if the control $\{0, 1, 0\}$ is chosen, the transitional matrix is simply

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-kT & kT & 0 \\ 0 & kT & 1-kT & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Other 7 matrices can be simply observed using the matrix obtained in section 2.

B. Algorithm

The above procedure can be summarized as the pseudo code above, given the matrices corresponding to each control is $A000, A001, A010, A011, A100, A101, A110, A111$, and the reference level is $a = (a1, a2, a3, a4)$.

Algorithm 1 Choice of Optimal Controller in a Markov Way

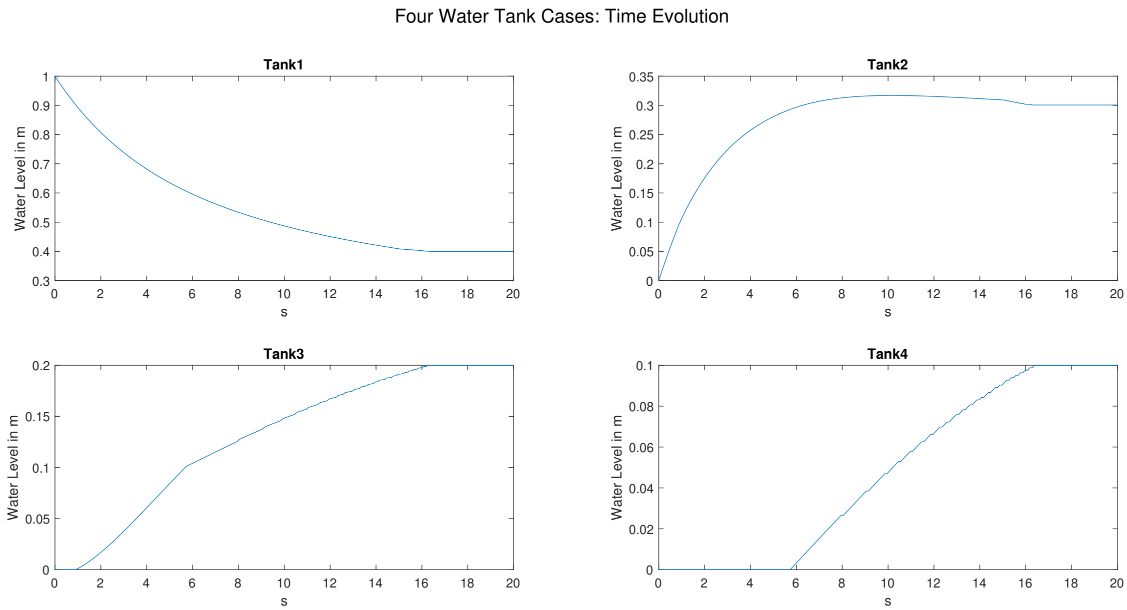
while $t \leq$ TimeEnd/Interval **do**
  Compute nextxxx $=$ Axxx $* [h1(t); h2(t); h3(t); h4(t)]$ for eight different xxx
  Compute dxxx $=$ norm$(h(t) - a)$ for eight different xxx
  Find minimum norm and minimizer.
  Set next state using the minimizer.
end while

IV. Numerical Simulation

We let the parameters be: $\mu = 0.001, \rho = 998, g = 9.81, r = 0.01, d = 0.02$. We choose a time interval of $T = 0.1$s, and a simulation interval $[0, 20s]$. And the eight matrices are contained in the file def-matrices.m.

A. Deterministic Case

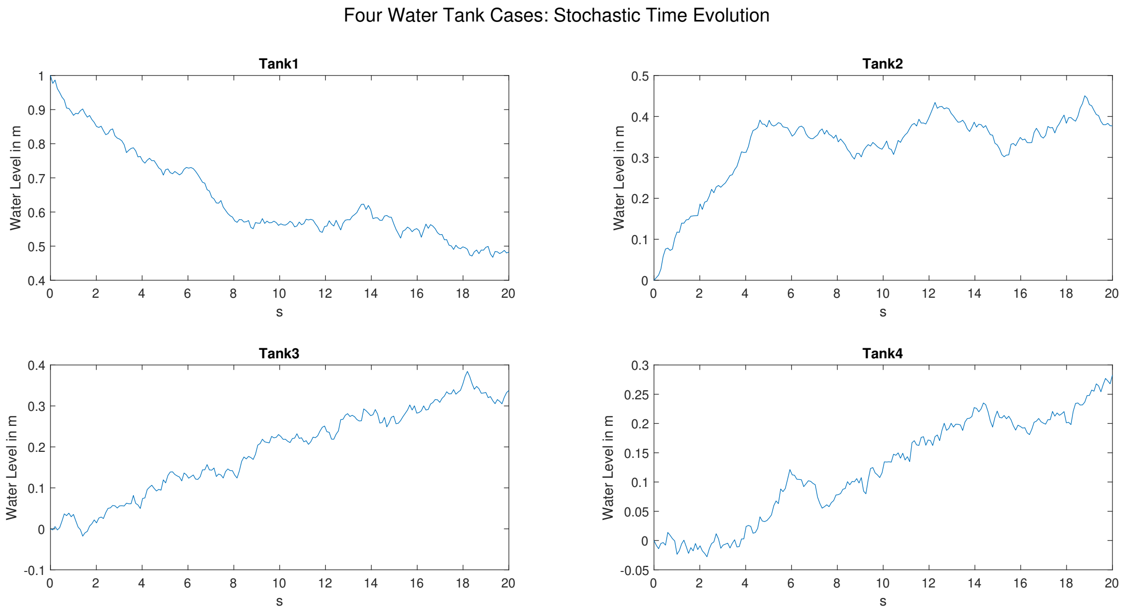
Suppose we are given the initial condition $h = (1m, 0, 0, 0)$, that is, all water is in the first tank. And we wish to reach the desired levels $a = (0.4m, 0.3m, 0.2m, 0.1m)$. Applying the above algorithm, we obtain the resultant time evolution. One can



see the characteristics of this type of control: as we get further from the tank 1, which is full of water at time $t = 0$, the process of rising is more smooth. As we get closer to tank 1, more overshoot and non-smoothness will occur. And it is easy to see all water levels converge to their desired level.

B. Stochastic Case

Using the initial condition and the reference level similar to above. However, this time we are assuming a gaussian random vector disturbance with $\mu = 0.01, \sigma^2 = 0.001$ for all four states. The resultant time evolution is shown as. The behavior is not



as satisfactory as in the deterministic case, but one can see all water levels still converge to their desired levels, the order of the tank levels are perserved.