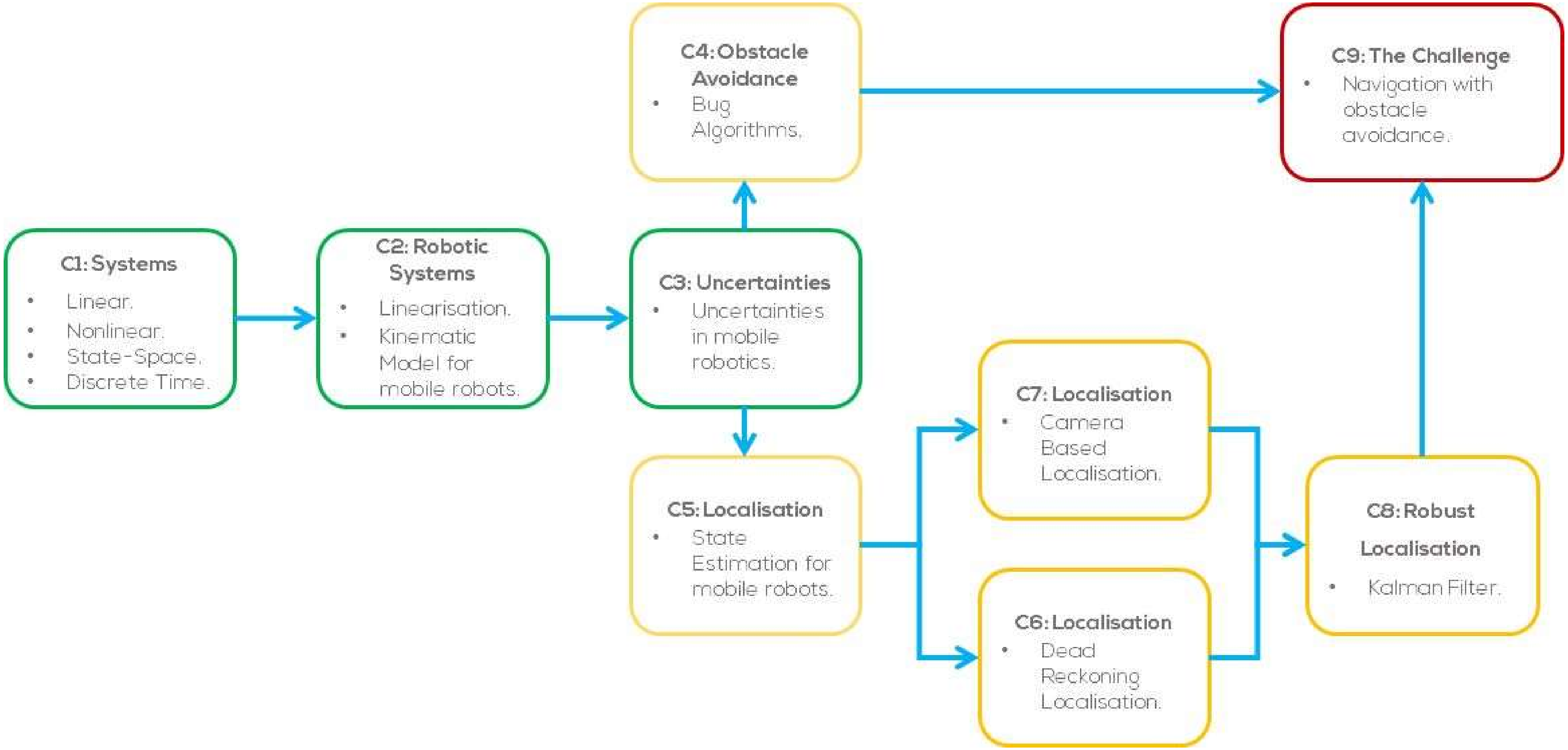
Integration of Robotics and Intelligent Systems

{Learn, Create, Innovate};

The roadmap


C1: Systems
• Linear.
• Nonlinear.
• State-Space.
• Discrete Time.
C2: Robotic Systems
• Linearisation.
• Kinematic Model for mobile robots.
C3: Uncertainties
• Uncertainties in mobile robotics.
C4: Obstacle Avoidance
• Bug Algorithms.
C5: Localisation
• State Estimation for mobile robots.
C6: Localisation
• Dead Reckoning Localisation.
C7: Localisation
• Camera Based Localisation.
C8: Robust Localisation
• Kalman Filter.
C9: The Challenge
• Navigation with obstacle avoidance.

Systems general aspects

In general, the notion of **system** is used in many fields of activity. With this notion, we want to delimitate a form of existence in a well-defined space.

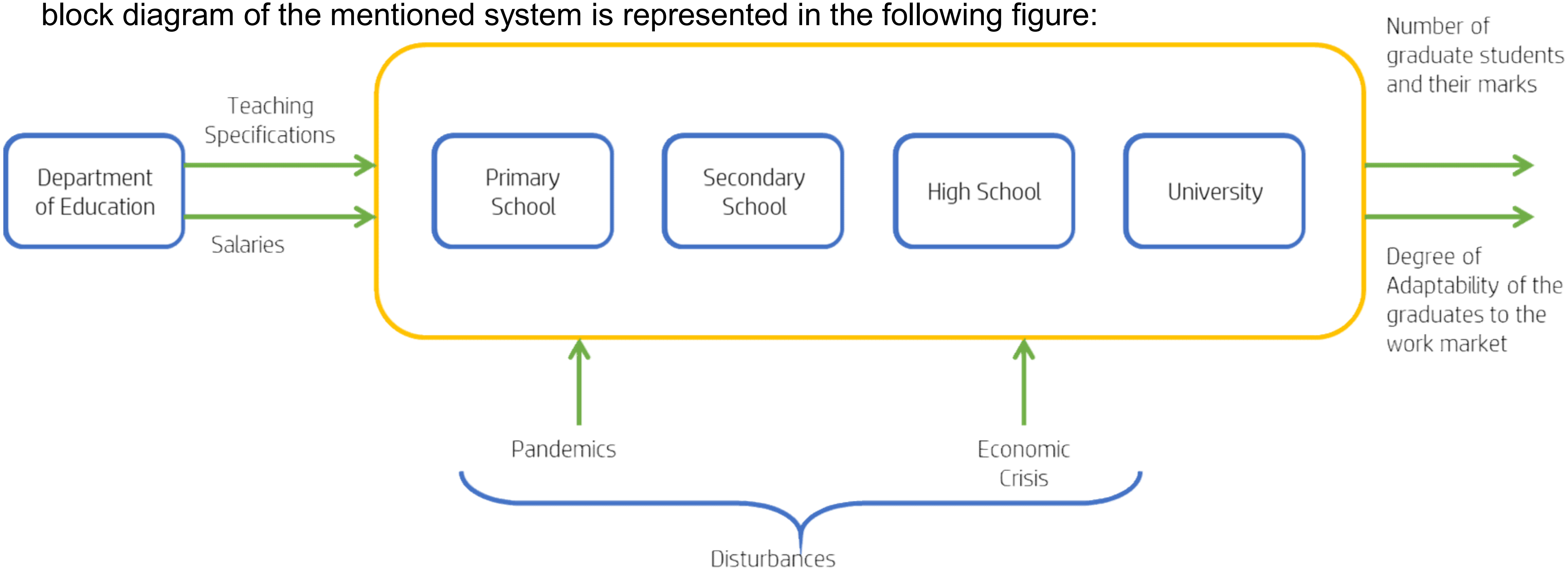
Some examples of dynamical systems: the democratic system, the Mexican education system, the nervous system, the automatic temperature regulation system, a mobile robot, a robotic manipulator, etc.

The notion of **system** helps us, in a first instance, to delimitate: a state management mechanism, a way of education at the national level, part of the components that contribute to the integration of the human body in the environment, the elements necessary to obtain a constant temperature in an enclosure, the elements of a mobile robot, the elements of a robotic manipulator, etc.

Generally speaking, a dynamic system is well structured having multiple connections between its component parts. Another characteristic is that its elements are structured according to the same criteria or in order to achieve the same goal. In many situations a system can contain subsystems which in turn can be regarded as independent systems.

Systems general aspects

For a better understanding, consider one of the examples stated above, namely the **education system**. The block diagram of the mentioned system is represented in the following figure:



It should be emphasized that this representation does not illustrate in detail all elements of the education system. On the other hand, the educations system has many other inputs, outputs and disturbances.

Systems general aspects

Aspects regarding the dynamic nature of the education system:

1) the output performance depends on the structure of the teaching specifications over a period of time that includes the current year and also a number of previous years

2) the salary has an important contribution to the quality of the teaching. Good salary attracts good teachers

3) the relations between schools and universities have a decisive role in terms of the quality and continuity of the educational process

4) the professionalism of the academics

5) the facilities and the labs of each education institution contribute to student formation

6) another aspect regarding the output performance is related to the presence of disturbances, which can have negative consequences if a rejection mechanism is not applied.

Dynamical systems

A dynamical system consists of two elements:

1. A non-empty space $\mathcal{D}$ (for instance $\mathbb{R}^2$)
2. A map from this space and the time into the same space $f : \mathcal{D} \times \mathbb{R} \rightarrow \mathcal{D}$

A dynamical system is described by the following differential equation (ODE – ordinary differential equation):

$$\dot{x}(t) = f(x(t), t) \qquad (1)$$

This can be seen as a geometrical concept. In other words, for every point of the space $x \in \mathcal{D}$, the function $f(x, t)$ provides the information about the evolution of the system at the instant t. Given an initial condition, the trajectory of the state follows the field of velocities $f = f(x, t)$. When the function f does not depend on time, i.e., $f = f(x)$, then the system is said to be a time-invariant system.

Dynamical systems

A very important and useful concept of a dynamical system is the state of the system. It can be defined as the minimal information that determines the future of the system.

Q: Which are the states for Educational systems illustrated previously?

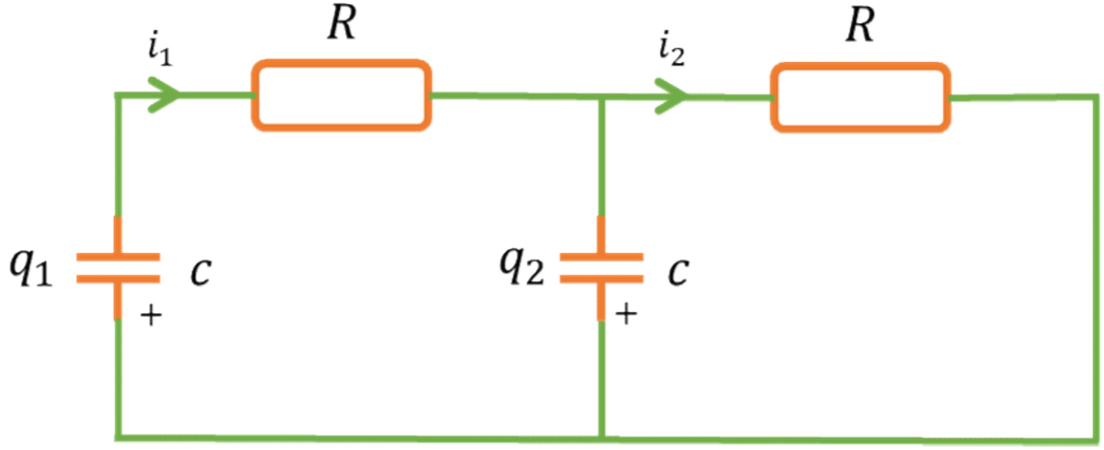
For the dynamical system (1), the state of the system is given by $x(t) \in \mathcal{D}$. Hence the state of the system is a point in the space $\mathcal{D}$. This course will focus on dynamical systems with this property.

The dynamical systems can be classified in autonomous systems and non-autonomous systems. This classification is given by the source of energy which determine the future evolution of the system. For instance, the system (1) is isolated from the rest of the Universe, its evolution only depends on itself, and we say that the system (1) is an autonomous system.

Q: The Educational systems illustrated previously is autonomous or non-autonomous?

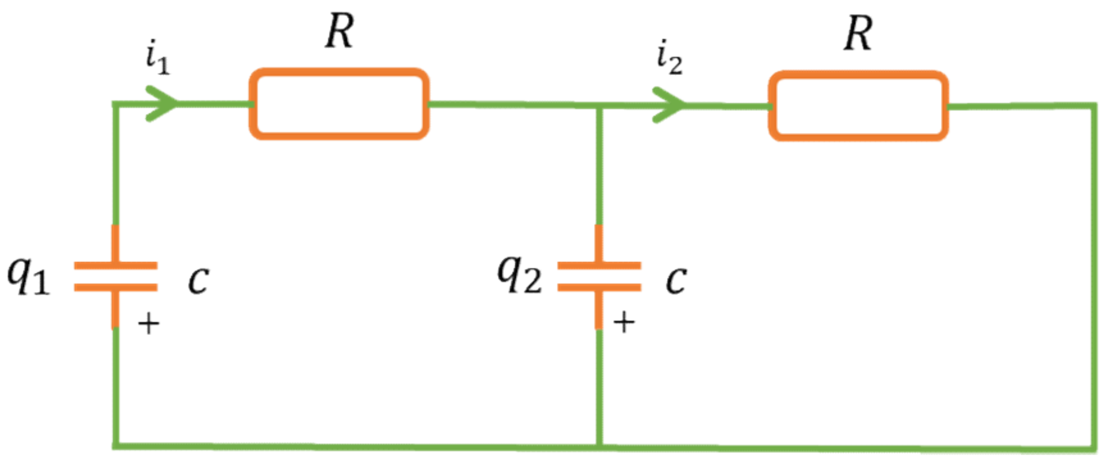
Worked example

Let us consider the following electrical circuit:



Q: Which are the states (the set of coordinates) which can describe the dynamics of this electrical circuit?

Dynamical systems



Q: Which are the states (the set of coordinates) which can describe the dynamics of this electrical circuit?

A: The dynamics of the circuit can be described using infinite set of coordinates, but two sets are straightforward: the changes at the capacitors $q = (q_1, q_2)$ *and the current* $i = (i_1, i_2)$. *In this example, we are going to model the same circuit using both sets of coordinates.*

Dynamical systems

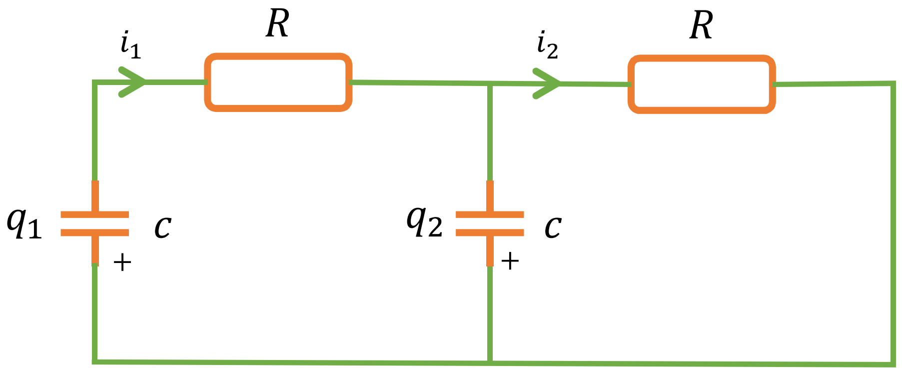
- **Using q:**

Applying Kirchoff's Voltage Law (KVL) on the left:

$$\sum_i V_i^{left} = \frac{1}{c}q_1 + i_1 R - \frac{1}{c}q_2 = 0 \Rightarrow$$
$$\Rightarrow i_1 = -\frac{1}{cR}q_1 + \frac{1}{cR}q_2 \tag{2}$$

And using KVL on the right:

$$\sum_i V_i^{right} = i_2 R + \frac{1}{c}q_2 = 0 \Rightarrow$$
$$\Rightarrow i_2 = -\frac{1}{cR}q_2 \tag{3}$$



Dynamical systems

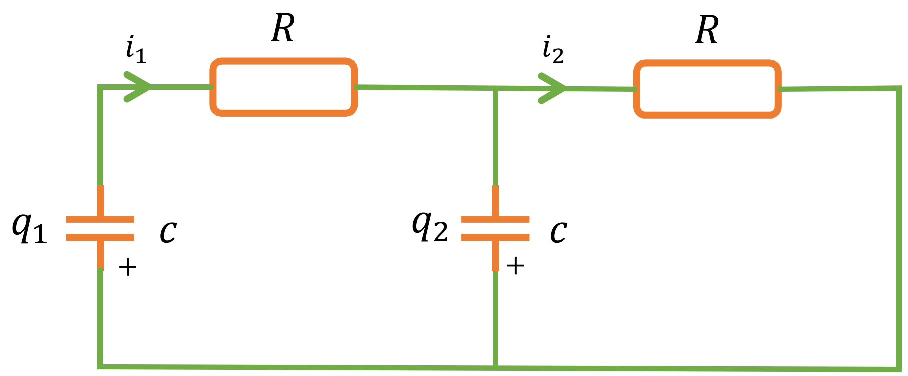
Moreover, both charges and currents are related as follows:

$$\dot{q}_1 = i_1 = -\frac{1}{cR}q_1 + \frac{1}{cR}q_2 \qquad (4)$$

$$\dot{q}_2 = i_2 - i_1 = -\frac{1}{cR}q_2 + \frac{1}{cR}q_1 - \frac{1}{cR}q_2 \qquad (5)$$

Or equivalently, the matrix form:

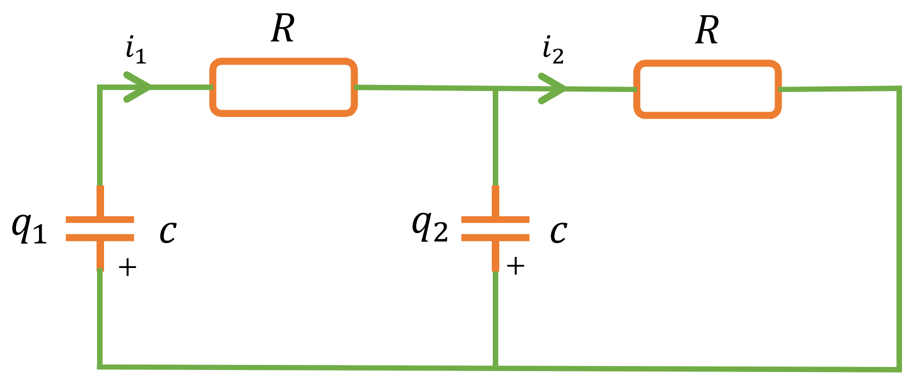
$$\dot{\boldsymbol{q}} = \begin{bmatrix} -\frac{1}{Rc} & \frac{1}{Rc} \\ \frac{1}{Rc} & -\frac{2}{Rc} \end{bmatrix} \boldsymbol{q} \qquad (6)$$



Dynamical systems

- **Using i:**

The time derivatives of (3) and (2) are given by:



$$\dot{i}_2 R + \frac{1}{c}\dot{q}_2 = 0 \Rightarrow \dot{q}_2 = -Rc\dot{i}_2 \qquad (7)$$

$$\frac{1}{c}\dot{q}_1 + \dot{i}_1 R - \frac{1}{c}\dot{q}_2 = 0 \Rightarrow \dot{q}_1 = -Rc\dot{i}_1 - Rc\dot{i}_2 \qquad (8)$$

Dynamical systems

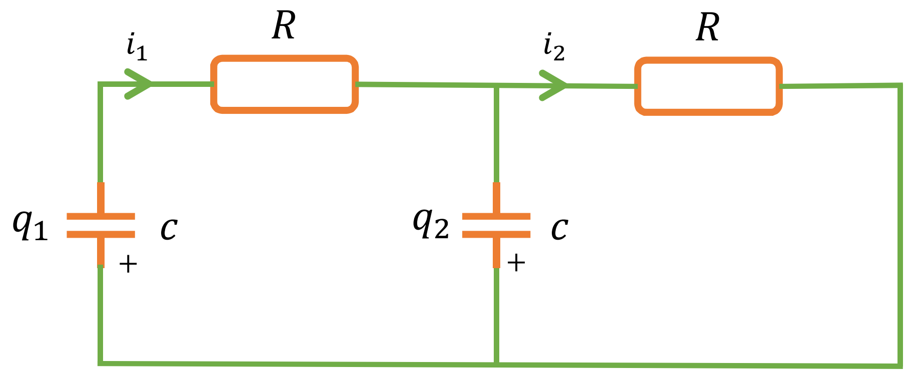
The dynamical equations in the capacitors can be written as:

$$i_1 = \dot{q}_1 = -Rc\dot{i}_1 - Rc\dot{i}_2 \tag{9}$$

$$i_2 - i_1 = \dot{q}_2 = -\frac{1}{cR}q_2 + \frac{1}{cR}q_1 - \frac{1}{cR}q_2 \tag{10}$$

Reordering the above equation, we get the result in the matrix form:

$$\dot{\boldsymbol{i}} = \begin{bmatrix} -\frac{2}{Rc} & \frac{1}{Rc} \\ \frac{1}{Rc} & -\frac{1}{Rc} \end{bmatrix} \boldsymbol{i} \tag{11}$$



Q: This electrical circuit is an autonomous systems or a non-autonomous system?

State-space representation of a linear system

The key point in control engineering and system theory is *interaction*. We are interested in studying the dynamical evolution of interconnected systems. In particular, feedback systems are the most important for us as robotics and control engineers. Therefore, we would like to model our system as a dynamical system including explicitly input u and output y:

$$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^{n_x}, \qquad u \in \mathbb{R}^{n_u} \tag{12}$$

$$y = h(x,u) \qquad y \in \mathbb{R}^{n_y} \tag{13}$$

where n_x is the number of state coordinates, n_u is the number of inputs, n_y is the number of outputs.

This representation of a system is very general and most real systems can be modelled by (12) and (13). The equations are referred to as the system equation and the output equation, respectively.

In contrast with the transfer function representation of a system, the state-space representation is not limited to linear systems.

State-space representation of a linear system

The general definition of a dynamical system can be used to describe the behaviour of a linear system as follows:

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^{n_x}, \qquad u \in \mathbb{R}^{n_u} \tag{14}$$

$$y = Cx + Du \qquad y \in \mathbb{R}^{n_y} \tag{15}$$

where $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$, and $D \in \mathbb{R}^{n_y \times n_u}$.

Equations (14) and (15) are said to be the state-space representation of a linear system. In short, we will say that the four matrices (A, B, C, D) represent a time-invariant linear (LTI) system.

For systems with single input and output, i.e., $n_u = n_y = 1$, B is a column vector, C is a row vector and D is a number. These systems are referred to as Single-Input Single-Output (SISO). Systems with several inputs and several outputs, i.e., $n_u > 1$, $n_y > 1$, are referred to as Multiple-Input Multiple-Output (MIMO).

State-space representation of a linear system

For today lecture we restrict our attention to SISO systems.

Any ordinary differential equation in the form:

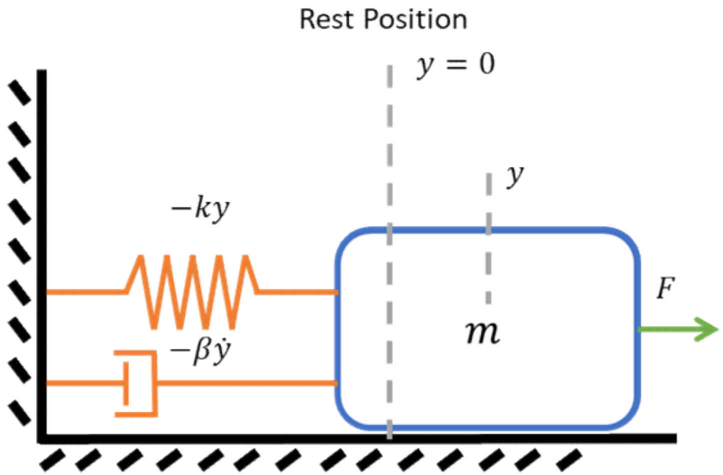
$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u \qquad (16)$$

with $m < n$ has an equivalent state-space representation.

State-space representation of a linear system

Worked example

Let us consider an Ideal Mass-Spring-Damper system where an external force F is applied on the mass. The output of the system is the position of the mass y.

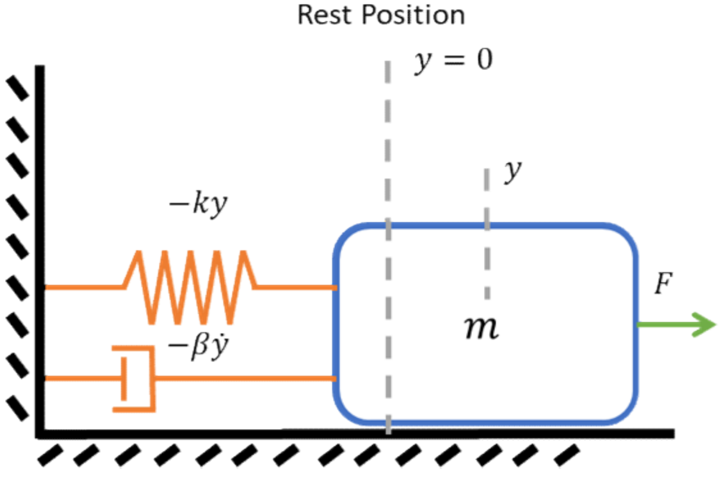


Q: Which are the states (the set of coordinates) which can describe the dynamics of this mechanical system?

State-space representation of a linear system

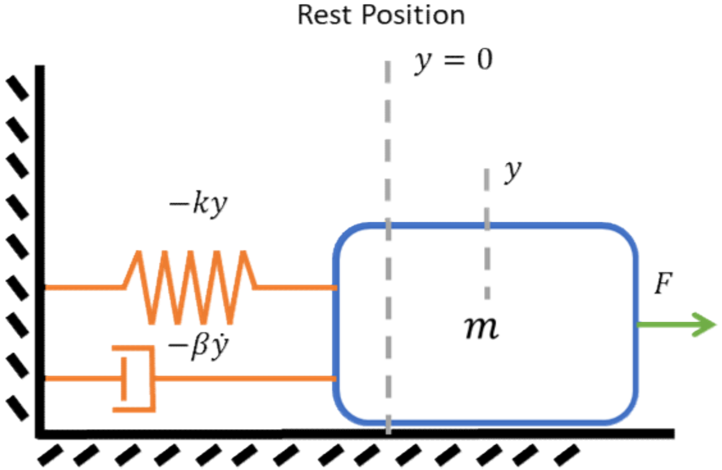
Applying Newton's second law, the dynamics of the system are given by:

$$\sum_i F_i = ma = m\,\ddot{y}$$



State-space representation of a linear system

There are three forces in the direction of y: the spring force $(-ky)$, the damper force $(-\beta\,\dot{y})$, and the external force (F).



$$F + (-ky) + (-\beta\,\dot{y}) = m\,\ddot{y} \tag{17}$$

$$\ddot{y} + \frac{\beta}{m}\,\dot{y} + \frac{k}{m}y = \frac{F}{m} \tag{18}$$

Let us define the set of states as:

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned} \tag{19}$$

State-space representation of a linear system

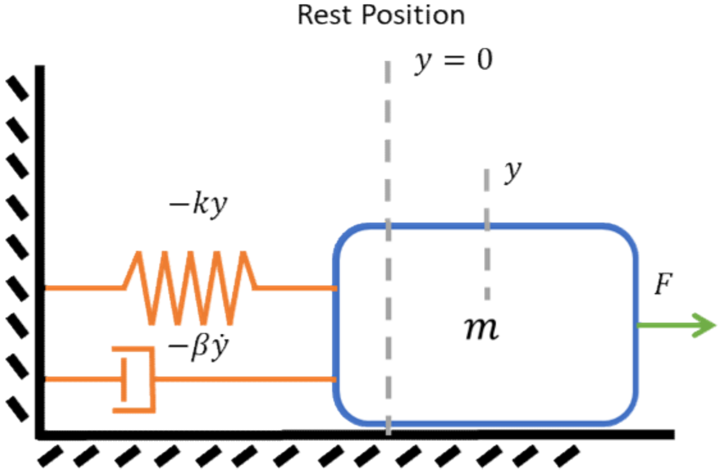
Then we can find a state-space representation of this system. From the definition of both coordinates, it is trivial that $\dot{x}_1 = x_2$, then (18) can be rewritten in term of x_1, x_2, and $\dot{x}_2$.

$$\dot{x}_2 + \frac{\beta}{m}x_2 + \frac{k}{m}x_1 = \frac{F}{m} \tag{20}$$

As a result, the system is described by two first order differential equations:

$$\dot{x}_1 = x_2 \tag{21}$$

$$\dot{x}_2 = -\frac{\beta}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}F \tag{22}$$



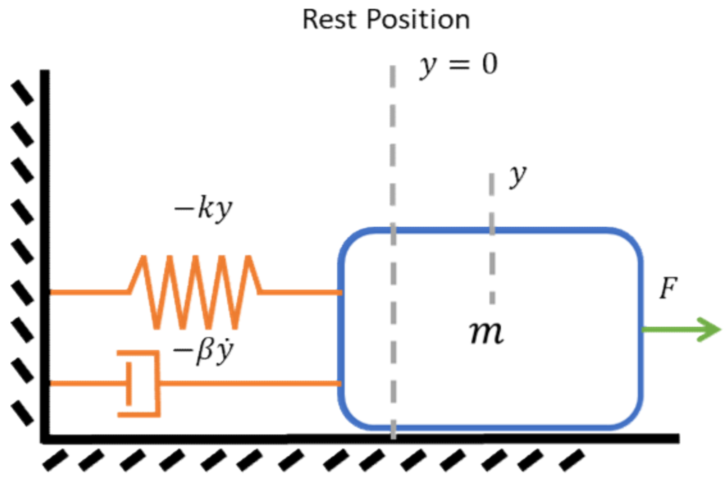
State-space representation of a linear system

We rewrite these two equations using matrices and the state $\boldsymbol{x} = (x_1, x_2)$.

$$\dot{x}_1 = 0x_1 + x_2 + 0F \tag{23}$$

$$\dot{x}_2 = -\frac{\beta}{m}\,x_2 - \frac{k}{m}x_1 + \frac{1}{m}F \tag{24}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix}F$$



Using (19), the output equation is given by:

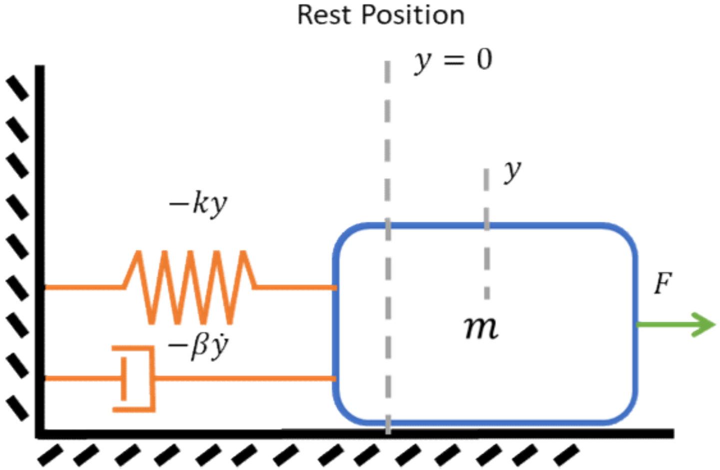
$$y = x_1 + 0x_2 + 0F = [1 \quad 0]\boldsymbol{x} + 0F \tag{25}$$

State-space representation of a linear system

In summary, the state-space representation of an ideal mass-spring-damper is given by:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$



State-space representation of a linear system

Observation: The state-space is a mathematical description of the system that could be different to the real space where the system performs is trajectory. In the example of the mechanical systems mass-spring-damper, the system moves along the y-axis, which is an one-dimensional space. However, the state of the system, as a mathematical concept, evolves on a two-dimensional space.

As we have stated, there is a state-space representation of all ODEs. Usually, we prefer to express differential equations in the Laplace domain, and we speak about *transfer functions*.

Any transfer function can be represented by infinite state-space representations. Three representations are very important: controller canonical form, observer canonical form, and modal form.

Controller canonical form

Definition: Consider the system described by:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_{n-1}\frac{d^{n-1}u(t)}{dt^{n-1}} + b_{n-2}\frac{d^{n-2}u(t)}{dt^{n-2}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t) \qquad (26)$$

or equivalently:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \qquad (27)$$

$where\ b_i \neq 0\ for\ at\ least\ one\ 1 \geq i < n.$

Controller canonical form

Simplest case: Let us consider the case where the ODE does not contain derivatives of the input:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = u \tag{28}$$

or equivalently:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{29}$$

Controller canonical form

Let us define the set of state coordinates as:

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \frac{dy(t)}{dt} = \dot{x}_1(t), \\ &\vdots \\ x_n(t) &= \frac{d^{n-1}y(t)}{dt^{n-1}} = \dot{x}_{n-1}(t). \end{aligned} \tag{30}$$

Then, substituting the above states in (28), if follows:

$$\dot{x}_n(t) + a_{n-1}x_n(t) + \cdots + a_1x_2(t) + a_0x_1(t) = u(t) \tag{31}$$

or:

$$\dot{x}_n(t) = -a_{n-1}x_n(t) - \cdots - a_1x_2(t) - a_0x_1(t) + u(t) \tag{32}$$

Controller canonical form

The controller canonical form of (28) or (29) is given by:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{33}$$

Controller canonical form

Exercise: Consider the problem of a ball in free fall:

$$m\frac{d^2y(t)}{dt^2} = -mg$$

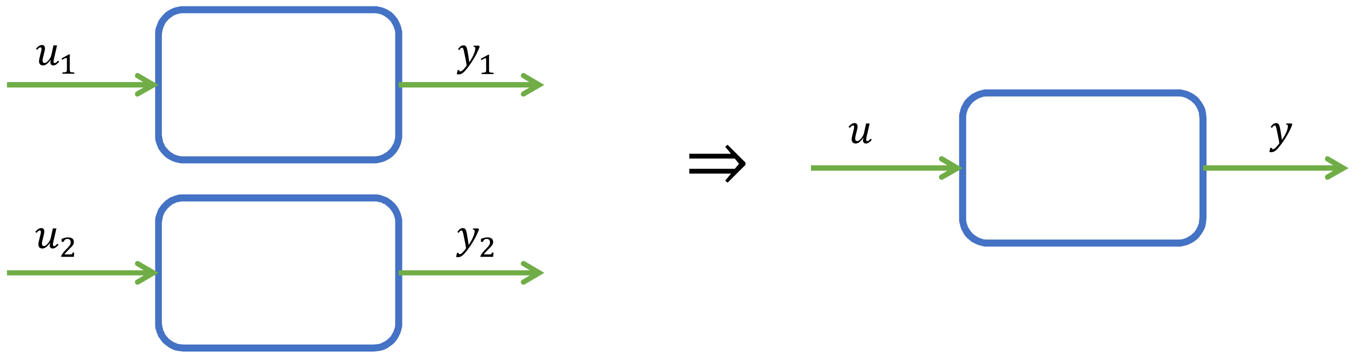
Which are the states for this system?

Nonlinear systems

Compared to linear systems, nonlinear systems have a much rich behaviour. A nonlinear system can have a particular behaviour for a specific region of the space, but a completely different behaviour for other regions.

The fundamental reason is that for linear systems (linear ODEs) the superposition principle holds. There are several formulations for the superposition principle.

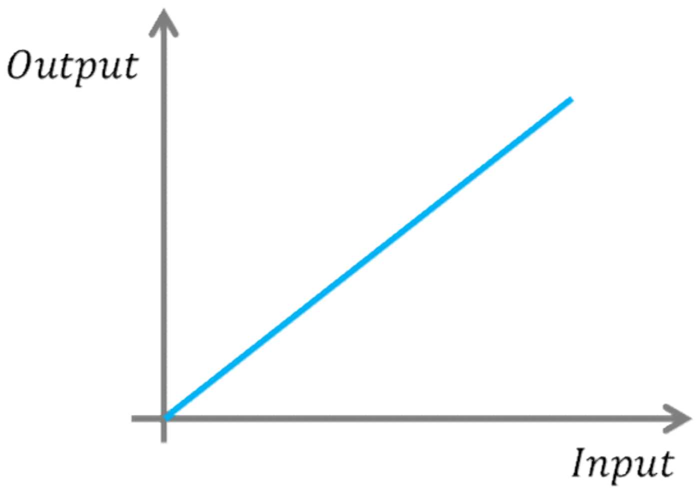
If input u_1 produces output y_1, and input u_2 produces output y_2, then input $\mathrm{u} = u_1 + u_2$ produces output $\mathrm{y} = y_1 + y_2$.



Nonlinear systems

Definition: A system is said to be linear if it satisfies the superposition principle. Consider a system with inputs u_1 and u_2 and outputs $y_1 = f(u_1)$ and $y_2 = f(u_2)$.

$$f(a_2 u_1 + a_2 u_2) = a_1 f(u_1) + a_2 f(u_2)$$



Discrete-time dynamic models

A digital computer by its very nature deals internally with discrete-time data or numerical values of functions at equally spaced intervals determined by the sampling period. Thus, discrete-time models such as *difference equations* are widely used in computer control applications. One way a continuous-time dynamic model can be converted to discrete-time form is by employing a finite difference approximation.

Consider a nonlinear differential equation

$$\frac{dy(t)}{dt} = f(y, u)$$

where y is the output variable and u is the input variable.

Discrete-time dynamic models

This equation can be numerically integrated (for instance using Euler method) by introducing a finite difference approximation for the derivative.

For example, the frst-order, backward difference approximation to the derivative at $t = k\Delta t$ is:

$$\frac{dy(t)}{dt} \cong \frac{y(k) - y(k-1)}{\Delta t}$$

where Δt is the integration interval (the control engineers name it *sampling time*) specified by the user and $y(k)$ denotes the values of $y(k)$ at $t = k\Delta t$.

Discrete-time dynamic models

So,

$$\frac{y(k) - y(k-1)}{\Delta t} \cong f\left(y(k-1), u(k-1)\right)$$

or:

$$y(k) = y(k-1) + \Delta t\, f\left(y(k-1), u(k-1)\right)$$

This is a first-order difference equation that can be used to predict $y(k)$ based on information at the previous time step $(k-1)$. This type of expression is called a recurrence relation.

Discrete-time dynamic models

For higher-order ODEs, we can use a generalisarion of the Euler method that we used for solving first-order ODEs. To illustrate the method, let us consider a 2^{nd} order ODE:

$$\frac{d^2\, y(t)}{dt^2} = f(t, y, \frac{dy(t)}{dt})$$

or:

$$\ddot{y} = f(t, y, \dot{y})$$

Discrete-time dynamic models

For discretization, the idea is to write the second order system (ODE) as a system of two first order systems (ODEs) and then apply Euler's method to the first order equations.

So, as we did before, we'll define a new variable:

$$\begin{cases} y = x_1 \\ \dot{y} = x_2 = \dot{x}_1 \end{cases}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(t, x_1, x_2) \end{cases}$$

We need initial conditions:

$$\begin{cases} x_1(t_0) = 0 \\ x_2(t_0) = 0 \end{cases}$$

For Newton the initial conditions are the initial position and initial velocity.

Discrete-time dynamic models

Now, the idea is to solve both x_1 and x_2 simultaneously using Euler's method for both first order ODEs:

$$\begin{cases} \dfrac{x_1(k) - x_1(k-1)}{\Delta t} = x_2(k-1) \\ \dfrac{x_2(k) - x_2(k-1)}{\Delta t} = f(t, x_1, x_2) \end{cases}$$

$$\begin{cases} x_1(k) = x_1(k-1) + \Delta t\, x_2(k-1) \\ x_2(k) = x_2(k-1) + \Delta t\, f((k-1), x_1, x_2) \end{cases}$$

This can be generalized to third order ODEs, or fourth order ODEs, as well as *n* order ODEs.