Zero-shot Generalization in Inventory Management:
Train, then Estimate and Decide

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Abstract

Deploying deep reinforcement learning (DRL) in real-world inventory management presents challenges, including dynamic environments and uncertain problem parameters, e.g. demand and lead time distributions. These challenges highlight a research gap, suggesting a need for a unifying framework to model and solve sequential decision-making under parameter uncertainty. We address this by exploring an underexplored area of DRL for inventory management: training generally capable agents (GCAs) under zero-shot generalization (ZSG). Here, GCAs are advanced DRL policies designed to handle a broad range of sampled problem instances with diverse inventory challenges. ZSG refers to the ability to successfully apply learned policies to unseen instances with unknown parameters without retraining.

We propose a unifying Super-Markov Decision Process formulation and the Train, then Estimate and Decide (TED) framework to train and deploy a GCA tailored to inventory management applications. The TED framework consists of three phases: training a GCA on varied problem instances, continuously estimating problem parameters during deployment, and making decisions based on these estimates. Applied to periodic review inventory problems with lost sales, cyclic demand patterns, and stochastic lead times, our trained agent, the Generally Capable Lost Sales Network (GC-LSN) consistently outperforms well-known traditional policies when problem parameters are known. Moreover, under conditions where demand and/or lead time distributions are initially unknown and must be estimated, we benchmark against online learning methods that provide worst-case performance guarantees. Our GC-LSN policy, paired with the Kaplan-Meier estimator, is demonstrated to complement these methods by providing superior empirical performance.

Keywords: Inventory control, deep reinforcement learning, forecasting, zero-shot generalization.

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1 Introduction

The digital revolution, characterized by unprecedented availability of historical data and advances in computational capabilities, has led to a paradigm shift in operations management research toward data-driven methodologies (Mišić and Perakis 2020). For example, in inventory management, where simple and well-analyzed heuristics are widely adopted, there is a growing interest in developing and applying machine learning (ML) (Qi et al. 2023) and deep reinforcement learning (DRL) algorithms (Gijsbrechts et al. 2022). This evolution promises more sophisticated and dynamic solutions to complex inventory practices where heuristics are still the norm (Boute et al. 2022).

However, deploying data-driven ML applications in real-life operations involving sequential decision-making presents significant challenges. One major challenge is the unpredictable, dynamic and constantly changing nature of operational environments (Amiri et al. 2023, Cheung et al. 2023), requiring trained policies to be robust and adaptable to similar yet unseen conditions during deployment (cf. Kirk et al. 2023). For instance, global industries frequently encounter shifts in demand patterns and supply uncertainties, highlighting the limitations of relying on swathes of historical data and algorithms trained on them (Gong and Simchi-Levi 2023).

A related challenge involves the decision-makers' (DMs) often incomplete understanding of the problem environment, henceforth referred to as the problem parameters or *parameterization*, that influence state transitions. Here, the system's *state* (cf. Puterman 2014) refers to the current status of all relevant variables at a given point in time, and *state transitions* are the result of actions taken, leading to a new state. (For example, in the lost sales inventory system (cf. Zipkin 2008), the *state transitions* correspond to the changes to on-hand and pipeline inventory and costs incurred in each period when a certain demand occurs and a certain order is placed. The problem *parameters* influence state transition probabilities and costs, and include the applicable demand and lead time distributions, the holding cost and penalty costs.)

Clearly, the optimal actions in a problem depend on the problem parameters. Thus, for effective deployment of ML/DRL algorithms in real-world settings, it is crucial to estimate and adapt to the true problem parameters. Unpredictable changes, such as shifts in demand distributions and the introduction of new products, can limit the applicability of historical data. Additionally, DMs may face difficulties in accurately observing their current state, for example, due to inaccurate inventory recordings (Bai et al. 2023b). Consequently, DMs are often compelled to re-estimate problem parameters based on recent observations, and such estimates must be frequently updated

to avoid decision making based on outdated information.

In most applications of ML/DRL to inventory management, policies are trained for each specific set of parameters (or *instance*) of the decision problem (see e.g. Oroojlooyjadid et al. 2021, Gijsbrechts et al. 2022, Maichle et al. 2024, Wang and Minner 2024). However, when deploying such models in a practical setting, the need to remodel and retrain a policy whenever the estimated problem parameters are updated represents a significant computational challenge. This challenge becomes even more pressing when considering that DMs typically need to make decisions across many tasks in each decision period, such as managing replenishments for thousands of products, and that training a single model can take hours (see Gijsbrechts et al. 2022).

These challenges reveal a significant research gap, indicating the need for a unifying framework to model and solve the sequential decision-making problems when problem parameters are uncertain and frequently updated. In this paper, we address this challenge by exploring an underexplored area of DRL for inventory management problems: training generally capable agents (GCAs) under zero-shot generalization (ZSG). In our context, *GCAs* refer to advanced DRL policies that perform well across a broad range of sampled problem instances with diverse inventory challenges while *ZSG* means that a GCA can be directly applied to new instances with unknown parameters that it was never explicitly trained on. Motivated by the desire to bring DRL applications closer to practical inventory applications, we propose a unifying MDP formulation and a solution framework to train and deploy a GCA tailored to inventory management problems. In particular, our contributions can be summarized as follows.

Firstly, to formalize the problem setting, we propose Super-Markov Decision Processes as a generalization of Markov Decision Processes (MDPs) (Puterman 2014, Porteus 2002). These latter models are traditionally used to model individual problem instances in inventory management. In contrast, our proposed Super-Markov Decision Process (Super-MDP) encompasses all MDP instances that can potentially arise from specific parameterizations of a class of inventory problems, treating each instance as a particular representation within the broader Super-MDP framework. Thus, Super-MDPs can unify the sequential decision-making process governed by the DM's ignorance on the applicable problem parameters.

Moreover, we propose the *Train, then Estimate and Decide (TED)* framework to train GCAs based on the Super-MDP formulation, and to enable their deployment to unknown environments. To *train* a GCA, we observe that while problem parameters such as true demand and lead time distributions are unknown in typical inventory applications, DMs have domain knowledge that

typically allows them to determine a (possible wide) *range* in which the true problem parameters are expected to reside. We later call this range as *probable parameter space.* As a result, we can formulate a Super-MDP tailored to the resulting class of problem instances that the DM needs to address. To effectively *train* a GCA for this Super-MDP, we sample problem instances from that class, and train a policy using DRL to select actions depending on the dynamic problem state as well as the problem parameters, i.e. we learn a *single* policy that is applicable for all instances in the class. During the *Estimate* phase, we continuously collect data while simultaneously estimating problem parameters, thus reducing the uncertainty caused by the policies' incomplete knowledge of the environment. For the *Decide* phase, we note that a given set of problem parameter estimates yields a specific instance of an inventory problem, and we take decisions for that instance by deploying the trained GCA based on these parameter estimates without retraining (ZSG).

The TED framework builds upon a rich tradition in *classical* inventory management, where problem specific policies (e.g. base-stock policies, (s, S) policies, capped base-stock policies etc.) are optimized based on *estimated* problem parameters (e.g. forecasted demand distribution, see Silver et al. 2016, Axsäter 2015), and the resulting decisions are deployed in practice. Training new DRL policies whenever parameter estimates are updated is prohibitively time-consuming especially in case of many products, and the TED framework overcomes this limitation by training a *single* policy (our GCA) for a wide range of parameters, and as such it is readily applicable without retraining after updating parameter estimates, enabling data-driven decision making for complex decision problems.

To demonstrate the effectiveness of our TED framework, we apply it to a class of inventory problems involving periodic review, and further characterized by lost sales, and possibly by cyclic demand patterns and stochastic lead times. For this class of inventory problems problem, we train a GCA during the Train phase, named *Generally Capable Lost Sales Network (GC-LSN)*. To validate it, GC-LSN is rigorously benchmarked versus the base-stock and capped base-stock policies on a wide range of problems assuming full availability of the parameters of each problem instance, both for GC-LSN and for the benchmarks. We find that GC-LSN consistently outperforms these well-performing benchmarks. Moreover, as part of the TED framework, we test GC-LSN under conditions where demand and lead time distributions are initially *unknown* and must be estimated. For the Estimate phase, we adopt the non-parametric Kaplan-Meier estimator (Kaplan and Meier 1958) to estimate the demand distribution and construct a relative frequency distribution for the lead time distribution. Our experiments show that TED outperforms a range of benchmarks specif-

ically designed for online learning (Lyu et al. 2024, Agrawal and Jia 2022, Zhang et al. 2020). To our knowledge, TED stands uniquely as the only general-purpose algorithmic framework capable of addressing these diverse inventory challenges collectively, particularly when key information such as demand and lead time distributions are initially unknown and when data is initially limited and censored.

We make the implementation code of TED publicly available, including a documentation and, importantly, the trained model weights of our generally capable agent GC-LSN. This can be accessed through our GitHub repository [1]. Our objective in making the code and the trained agent available is twofold: academically, to support the principles of open science and enhance the reproducibility of reinforcement learning algorithms, as emphasized by Henderson et al. (2018); and managerially, to provide a cutting-edge, ready-to-deploy lost sales inventory policy; practitioners facing similar inventory challenges may find this resource valuable.

This paper is organized as follows: §2 reviews the literature, §3 provides preliminaries. We formulate the problem description in §4 while §5 introduces the TED framework. §6 discuses the periodic review inventory control problem. Experimental setup and numerical results are presented in §7, while §8 concludes the paper.

2 Literature Review

Our research intersects with three expanding bodies of literature. Initially, we explore the concept of training GCAs under *ZSG*—the fundamental approach of this paper. Subsequently, we examine the prevalent ideas in *online learning for inventory systems*, which represent the conventional methodology for addressing the leading application of this study: periodic review inventory control under conditions of lost sales with limited and censored data. Finally, we review *ML for operations management*, specifically DRL, which serves as the main methodological tool within our TED framework for training GCAs.

2.1 Zero-shot generalization

ZSG represents an emerging frontier in artificial intelligence, characterized by its ability to handle tasks in unseen contexts (parameters) without prior direct training. This approach is crucial for sequential decision-making problems where adaptability in variable environments is key. The field

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[1]https://github.com/tarkantemizoz/DynaPlex

gained significant momentum following the introduction of Contextual Markov Decision Processes (CMDPs) by Hallak et al. (2015), where the aim is to train agents that can competently navigate any possible context. For an extensive review of ZSG in this domain, we direct readers to Kirk et al. (2023) and the references therein.

According to Kirk et al. (2023), ZSG concept can be roughly split into two categories: independent and identically distributed (iid) ZSG, where training and testing environments are drawn from the same distribution, and out-of-distribution ZSG, where test environments differ fundamentally from those encountered during training. In this paper, we focus on iid ZSG, preparing agents for robust performance across diverse yet distributionally similar instances.

ZSG research also differentiates based on the observability of the environment's parameters: one setting where the DM can directly observe parameters (e.g., Prasanna et al. 2024), and another where the parameters must be inferred through environmental interactions (e.g., Sodhani et al. 2022). Our study aligns with the latter. However, unlike Sodhani et al. (2022), which focuses on rich, high-dimensional data with many irrelevant features, such as visual inputs for robot manipulation, our focus is on operations management, where parameter estimation and translation into specific problem instances are relatively straightforward, emphasizing the development of scalable, high-performing policies.

Despite its importance, ZSG in inventory management remains relatively underexplored. Notable early work bridging this gap includes Batsis and Samothrakis (2024), Akkerman et al. (2024), and van der Haar (2023), which suggest potential for developing GCAs in this area. A key distinction of our work from those applications in inventory management is the introduction of the unifying Super-MDP[2] formulation of the problem instances, which enables modeling cases where true problem parameters are initially unknown. This uncertainty necessitates the Estimate phase in our TED framework, allowing for real-time parameter estimation and policy adaptation. In contrast, prior applications assume that all true parameters are known in advance.

2.2   Online learning for inventory systems

In real-life scenarios involving inventory management, DMs frequently encounter unknown dynamics in demand and supply. As a result, a significant segment of literature has explored online

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[2]Our Super-MDPs are similar to CMDPs; however, in the inventory domain, the DMs typically leverage domain knowledge to estimate likely ranges for the problem parameters. This expertise informs the construction of a probable parameter space, where each specific parameter estimate within this space defines a unique problem instance, i.e., an MDP. This approach forms the basis for our subsequent developments and results, enabling the effective application of ZSG to inventory management problems.

learning strategies tailored to managing lost sales inventory control with non-zero lead times. This body of work has primarily focused on optimizing policy parameters (e.g. base-stock levels) in online learning, and has recently achieved breakthroughs in the form of worst-case regret bounds for parameter optimization within various heuristic policy classes, including results for base-stock policies (Agrawal and Jia 2022), capped-base stock policies (Lyu et al. 2024), and constant order policies (Chen et al. 2023a). Of course, adoption of heuristics limits decision makers to the performance of those heuristics for various cases. For instance, constant order policies perform poorly when the long-run average cost does not remain insensitive to lead times (Bai et al. 2023a), while appropriate heuristics for the case of cyclic demand patterns are missing (Gong and Simchi-Levi 2023).

Our study introduces a new approach to online decision making using an *offline* learning setting. Indeed, our GC-LSN is trained prior to being confronted with any specific instance: we adopt a unified Super-MDP formulation and the TED framework for training GCAs. Within TED, we propose to subsequently feed the trained policy network with a relatively straightforward parameter estimation procedure (i.e., based on the Kaplan-Meier estimator (Kaplan and Meier 1958)), and focus on empirical performance in numerical experiments. Our approach is demonstrated to complement the online learning literature with a method that performs very well empirically under a wide range of assumptions. The resulting general policy provides a unique tool for benchmarking a variety of online inventory control strategies for a broad range of problems, both in cases where there are benchmarks with worst-case performance guarantees available, and in cases where such benchmarks are not yet available.

2.3   Machine learning in operations management

ML has extensive applications across various fields of operations management, including supply chain management, revenue management, and healthcare (Mišić and Perakis 2020). Typically, ML approaches adhere to the Predict-Then-Optimize (PTO) paradigm, where ML models forecast future scenarios, such as customer demands, which are then used to inform optimization decisions (Elmachtoub and Grigas 2021). However, forecasting becomes particularly challenging in environments where the future is highly stochastic and decisions have far-reaching consequences (Sinclair et al. 2023).

An alternative to the PTO paradigm is DRL, which seeks to optimize decisions directly (Boute et al. 2022). DRL has been effectively applied in various domains, including multi-echelon supply chains

(Harsha et al. 2021, Oroojlooyjadid et al. 2021), lost sales inventory control (Gijsbrechts et al. 2022, Temizöz et al. 2023), and large-scale online retail operations (Madeka et al. 2022, Andaz et al. 2024, Liu et al. 2023). Nevertheless, the effectiveness of DRL often hinges on the availability of accurate uncertainty distributions (see, e.g., Temizöz et al. 2023) or a substantial dataset of historical transactions that are applicable for the task to be learned (see, e.g., Madeka et al. 2022). In scenarios where data is limited and/or censored, and in cases where problem parameters are not fully observable, the utility of the previous DRL applications may diminish.

The TED framework aims to fill this gap by expanding the application of DRL to include settings with scarce, censored and evolving data, thereby broadening the utility of these algorithms in practical operations management.

3 Preliminaries

This section introduces the fundamental concepts and notation necessary to define the problem setting and the TED framework presented in the subsequent sections. We begin with an introduction to MDPs, a common modeling framework for sequential decision-making processes (Puterman 2014).

Definition 1 (Markov Decision Process). *A Markov Decision Process is represented by the tuple $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, f, C, \mathbf{s}_0 \rangle$. Here, $\mathcal{S}$ and $\mathcal{A}$ are finite sets of states and actions, respectively. The transition function $f : \mathcal{S} \times \mathcal{A} \to \mathcal{S}$ maps each state-action pair $(\mathbf{s}, a)$ to a subsequent state $\mathbf{s}'$. The cost function $C : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ assigns a (bounded) cost $c = C(\mathbf{s}, a)$ to each state-action combination, while the initial state $\mathbf{s}_0$ is either deterministically or randomly set.*

Many common operational problems can be modeled as MDPs. For instance, consider a single-item periodic review inventory control problem with zero lead time. In this system, the current inventory position represents the state, the order decision constitutes the action, and state transitions are determined by external demand realizations and the chosen actions. However, the dynamics and outcomes of such systems are governed by external parameters, which are typically assumed to be static and fixed within the MDP framework. In this context, the dynamics and outcomes are influenced by the demand distribution ζ, the lead time distribution τ, the holding cost h, and the penalty cost for unmet demand p. We refer to all these factors as the parameterization of the MDP, or more generally, as the parameterization of the decision problem:

Definition 2 (Parameterization and Parameter Space). *The **parameterization** of a problem consists of exogenous parameters that completely determine the system's dynamics and outcomes. The parameter space is the set of all possible parameterizations of a decision problem, denoted as $\mathcal{P}$. It comprises each combination of parameters $\mathbf{p}$, with $\mathbf{p} \in \mathcal{P}$.*

Parameters can either be directly observed or estimated based on data. In our example, we assume that the holding cost h and penalty cost p are directly observable, while the demand distribution $\boldsymbol{\zeta}$ and lead time distribution $\boldsymbol{\tau}$ are typically estimated from historical data. The parameter space, denoted as $\mathcal{P}$, is defined by parameters organized in the vector $\mathbf{p} = (h, p, \boldsymbol{\zeta}, \boldsymbol{\tau})$, and encompasses all possible combinations of these elements.

For clarity, we denote an MDP associated with a specific parameterization as $\mathcal{M}^{\mathbf{p}} = \langle \mathcal{S}^{\mathbf{p}}, \mathcal{A}^{\mathbf{p}}, f^{\mathbf{p}}, C^{\mathbf{p}}, \mathbf{s}_0^{\mathbf{p}} \rangle$ and refer to such an MDP as a *problem instance*. However, to simplify notation, we often omit explicit references to this parameter dependence, even though every MDP is inherently tied to a specific parameterization.

A policy $\pi^{\mathbf{p}}$ for an MDP $\mathcal{M}^{\mathbf{p}}$ with parameterization $\mathbf{p}$ is defined as a function that maps states to actions, i.e., $\pi^{\mathbf{p}} : \mathcal{S}^{\mathbf{p}} \to \mathcal{A}^{\mathbf{p}}$. Our analysis focuses on minimizing the undiscounted average cost over an infinite horizon, aiming to identify the optimal policy $\pi^{\mathbf{p}*}$ that minimizes long-term costs:

$$\pi^{\mathbf{p}*} = \arg\min_{\pi^{\mathbf{p}}} \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[\sum_{t=0}^{T-1} C^{\mathbf{p}}(\mathbf{s}_t, \pi^{\mathbf{p}}(\mathbf{s}_t)) \right],$$

where $\mathbf{s}_t$ is the state at time t and $\pi^{\mathbf{p}}(\mathbf{s}_t)$ is the action taken in state $\mathbf{s}_t$ according to policy $\pi^{\mathbf{p}}$. The expectation $\mathbb{E}$ accounts for the stochastic processes governing the state transitions. Throughout this paper, finding policy $\pi^{\mathbf{p}*}$ for $\mathcal{M}^{\mathbf{p}}$ is referred as solving a task with a parameterization $\mathbf{p}$. While the solution methodologies for finding $\pi^{\mathbf{p}*}$ are vast (Puterman 2014), we are interested in finding a single policy (a GCA π_S such that $\pi_S : \mathcal{S} \times \mathcal{P} \to \mathcal{A}$) that is robust to changes in $\mathbf{p}$, and especially in a policy that can perform well when $\mathbf{p}$ is unknown.

4   Problem Description

In this section, we formally define the problem setting of this paper, using the periodic review inventory control problem as our primary example to illustrate the associated decision-making challenges.

Let $\mathcal{D}$ represent a class of sequential decision-making problems, i.e. periodic review inventory

control problem, influenced by a predefined parameter space $\mathcal{P}$. Within $\mathcal{D}$, the DM is tasked with managing n independent and distinct tasks, each requiring the solution of an MDP ($\mathcal{M}^{\mathbf{p}}$) tailored to its respective parameterization $\mathbf{p} \in \mathcal{P}$. These parameters may be subject to external factors and can change over time. For instance, the DM may need to make replenishment decisions for multiple products over a designated time period, where the demand and/or lead time distributions are both censored and non-stationary. Consequently, the DM may encounter three common challenges (decision contexts) when managing the sequential decision-making problems within $\mathcal{D}$:

- *(Scalability-1)* The DM must make decisions for n independent tasks, each characterized by different parameterizations. This necessitates solving at least n distinct MDPs, each tailored to its specific parameterization, and employing n different decision-making policies. As n increases, this approach becomes time-consuming and impractical.

- *(Non-stationarity-2)* The parameters (parametrization) within a specific task may evolve over time, requiring the DM to solve new MDPs for updated parameter combinations. For example, an increase in product demand could render the existing replenishment strategy ineffective. Additionally, new tasks with unknown parameterizations from the established parameter space may emerge at any point, such as through the introduction of new products.

- *(Obscurity-3)* The DM often lacks direct observation of the true parameters and must rely on inferences drawn from potentially limited and censored real-time data. For instance, while the actual demand distribution for a product remains unknown, the DM might use an empirical distribution constructed from observed sales data to inform replenishment decisions. As more sales data becomes available, estimates of the distribution must be updated, necessitating the addition of new parameterizations.

To address the challenges associated with solving $\mathcal{D}$, our primary objective is to train a GCA that can perform effective real-time decision-making across problems with diverse parameterizations without requiring additional training, achieving what is known as ZSG. This agent should be applicable to the three decision contexts outlined above: (1) context involving many independent decisions, (2) context with non-stationary parameter settings requiring policies to adapt to new problem parameterizations, and (3) context where the true parameterization is unknown but can be estimated based on observations. Since context (1) involves applying the GCA in a scenario with perfect parameter estimates and context (2) typically results in unknown parameters that require

estimation, we emphasize that contexts (1) and (2) can be viewed as special cases of context (3). Therefore, our framework and analysis primarily focus on context (3).

In our setting, we assume that the DM does not know when or how the parameterization will change. Without this knowledge, the DM cannot utilize probabilistic information to anticipate and optimize for potential parameter changes. This scenario leads to the following assumption:

Assumption 1 (Decision Maker's Optimization Rationale). *The decision maker assumes stationarity of the problem parameters and aims to optimize the policy accordingly, pursuing optimal decision-making as long as the problem's parameterization remains unchanged.*

Assumption 1 enables the decomposition of the problem into a sequence of independent and stationary MDPs, each characterized by a distinct parameterization. This approach is common in inventory management, where the DMs typically base their optimization strategies on their current estimates of problem parameters (Axsäter 2015). In fact, it is akin to a rolling horizon strategy: the parameterization is assumed to be stationary for the time being, while at each time step, changes in parameters are monitored, and parameters are potentially re-estimated.

The ability to decompose the problem $\mathcal{D}$ into independent and stationary MDPs motivates our definition of the Super-MDP. The Super-MDP can be understood as a population of all MDPs related to our decision problem $\mathcal{D}$ and thus formally defines the problem $\mathcal{D}$ as follows:

Definition 3 (Super-Markov Decision Process). *A Super-Markov Decision Process is defined by the tuple $\mathcal{M}_\mathcal{S} = (\mathcal{P}, \mathcal{S}, \mathcal{A}, \mathcal{H}, \mathcal{F})$, where:*

- *$\mathcal{P}$ represents the parameter space, containing the true problem parameterizations $\mathbf{p} \in \mathcal{P}$.*

- *$\mathcal{S}$ denotes the finite state space, encompassing all possible states $\mathbf{s} \in \mathcal{S}$.*

- *$\mathcal{A}$ indicates the finite action space, where each action $a \in \mathcal{A}$, and $\mathcal{A} = \{0, 1, \ldots, m\}$.*

- *$\mathcal{H}$ is the distribution over the parameter space $\mathcal{P}$, generating problem parameterizations $\mathbf{p} \sim \mathcal{H}$.*

- *$\mathcal{F}$ is the mapping function that relates each parameterization $\mathbf{p}$ to the corresponding elements of $\mathcal{M}^\mathbf{p}$ (i.e., $f^\mathbf{p}$, $C^\mathbf{p}$, $\mathbf{s}_0^\mathbf{p}$). Both the state space and action space are defined universally across parameterizations, such that $\mathcal{S}^\mathbf{p} \subseteq \mathcal{S}$ and $\mathcal{A}^\mathbf{p} \subseteq \mathcal{A}$.*

Note that the true parameter space $\mathcal{P}$ and the distribution of parameterizations $\mathcal{H}$ may be unknown. We later define and utilize a probable parameterization space and establish a parameter sampling function to conceptualize the Super-MDP.

The definition of the Super-MDP aligns with the objective of training a GCA. If a policy is a GCA, it should generate effective actions for a problem instance generated by $\mathcal{H}$ from the parameter space. A policy for a Super-MDP, denoted as π_S, can be defined as a function from state-parameterization pairs to actions, $\pi_S : \mathcal{S} \times \mathcal{P} \to \mathcal{A}$. Our objective is to identify a jointly optimal policy π_S^*, defined as:

$$\pi_S^* = \arg\min_\pi \mathbb{E}_{\mathbf{p} \sim \mathcal{H}} \left[\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[\sum_{t=0}^{T-1} C^{\mathbf{p}}(\mathbf{s}_t, \pi(\mathbf{s}_t, \mathbf{p})) \right] \right].$$

Since finding the optimal policy for even moderately sized MDPs is intractable, we must rely on approximation methods. Let $\bar{C}_\pi$ denote the expected per-period costs of policy π_S for a Super-MDP over parameterizations $\mathbf{p} \in \mathcal{P}$ with $\mathbf{p} \sim \mathcal{H}$. With a GCA, we aim to achieve an approximately optimal policy $\hat{\pi}_S$ such that $\bar{C}\pi_S^* + \epsilon \geq \bar{C}\hat{\pi}_S$, where ϵ is the approximation error. To achieve such a policy, we first develop the TED approach, which we detail next.

5 Train, then Estimate and Decide

Achieving an approximately optimal policy $\hat{\pi}_S$ for a Super-MDP requires a solution approach capable of simultaneously addressing MDPs with varying parameterizations, thereby obtaining a GCA. Our primary strategy involves separating the training and deployment phases of this policy, ensuring that it does not require retraining when the problem's parameterization changes, thus achieving ZSG. We propose such a solution within our *Train, then Estimate and Decide-TED* framework.

In the *Train* phase, we first construct the Super-MDP for the problem class $\mathcal{D}$. Since the true parameter space $\mathcal{P}$ and the distribution over the parameter space $\mathcal{H}$ are unknown, we define a probable parameter space $\hat{\mathcal{P}}$ and an associated distribution $\hat{\mathcal{H}}$. The policy (GCA) is parameterized as a neural network, and we sample from the probable parameter space for training (see Figure 1, left part of the Train box) using $\hat{\mathcal{H}}$. The policy is trained for each sampled parameterization under the assumption that the parameterization remains fixed during training (see Figure 1, right part of the Train box). The parameterization of the problem is incorporated into the policy (the neural network) as input features, enabling the neural network to generalize to unseen parameterizations
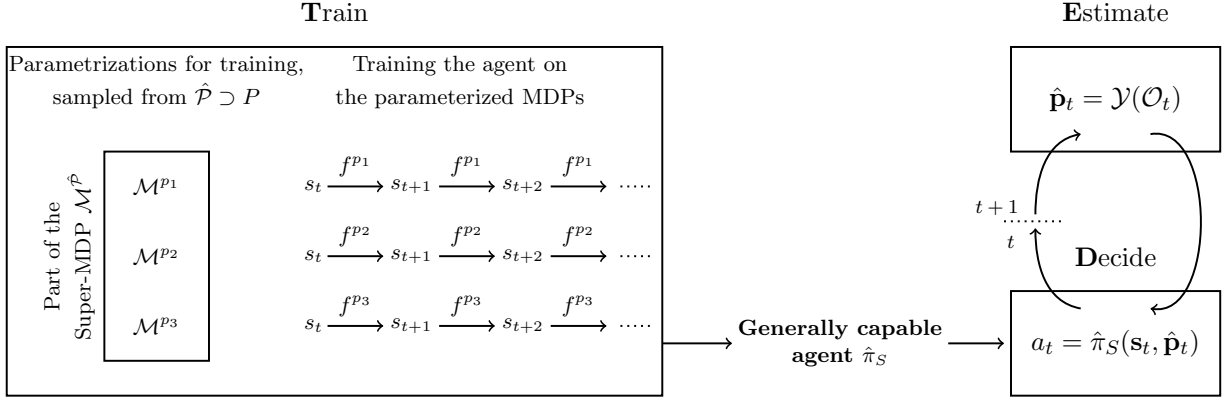
Figure 1: Train, then Estimate and Decide framework for solving sequential decision-making problems with dynamic parameter estimation

during the deployment phase, which comprises the *Estimate* and *Decide* steps.

In the *Estimate* phase, we estimate the parameterization $\hat{\mathbf{p}}_t$ of the actual system at time t using a function $\mathcal{Y}$ based on the observations collected up to time t ($\mathcal{O}_t$). In the *Decide* phase, this parameterization estimate is fed into the pre-trained policy along with the current state. The policy then outputs the action to be taken at time t (see right part of Figure 1).

The main novelty of this approach is that the training step, during which the policy is optimized, precedes the estimation step. This contrasts with current prevalent approaches, which first estimate parameters and then optimize based on parameter estimates, see Agrawal and Jia (2022), Lyu et al. (2024), Chen et al. (2023a) for similar applications covering online learning in inventory management.

We provide detailed descriptions of the different phases in the subsequent sections.

5.1 Train

The Train phase occurs before the DM begins observing data and making decisions. This phase aims to develop a GCA that can make effective decisions for (unseen) problem instances in $\mathcal{P}$ without further training. To achieve this, the phase involves the formulation of a Super-MDP and the implementation of a learning algorithm to train the GCA for the formulated Super-MDP. We identify key challenges and conditions for both components to enhance the GCA's ability to generalize to unseen instances.

Construction of the Super-MDP for training

The DM initiates the Train phase by first defining a Super-MDP for a designated problem class $\mathcal{D}$. As outlined in the problem description, the true parameters may be obscure and unknown, making both the parameter space $\mathcal{P}$ and the probability distribution $\mathcal{H}$ uncertain. To address this, we make two key assumptions. First, we assume that the DM can construct a probable parameter space $\hat{\mathcal{P}}$ that fully contains the true parameter space, i.e., $\mathcal{P} \subseteq \hat{\mathcal{P}}$. This assumption is reasonable because the DM possesses a higher-level understanding of the operational environment, and parameters are often naturally bounded (e.g., non-negativity of demand). Secondly, the distribution of parameters $\mathcal{H}$ is less critical at this stage, as we will estimate the actual parameters during the Estimate phase. Instead, we define a distribution $\hat{\mathcal{H}}$ over the probable parameter space $\hat{\mathcal{P}}$ to sample parameters for training.

Remark 1. *To train a generally capable agent for the Super-MDP $\mathcal{M}_{\mathcal{S}} = (\mathcal{P}, \mathcal{S}, \mathcal{A}, \mathcal{H}, \mathcal{F})$, we employ a more general Super-MDP $\mathcal{M}_{\mathcal{S}} = (\hat{\mathcal{P}}, \mathcal{S}, \mathcal{A}, \hat{\mathcal{H}}, \mathcal{F})$, where $\hat{\mathcal{P}}$ is the probable parameter space that fully contains the true parameter space, i.e., $\mathcal{P} \subseteq \hat{\mathcal{P}}$, and $\hat{\mathcal{H}}$ is the probable parameter sampling distribution ensuring uniform covarage of $\mathcal{P}$.*

Figure 2 illustrates an example of constructing $\hat{\mathcal{P}}$ and $\hat{\mathcal{H}}$, where $\hat{\mathcal{P}}$ is smooth and bounded, and $\hat{\mathcal{H}}$ samples parameterizations uniformly from $\hat{\mathcal{P}}$.
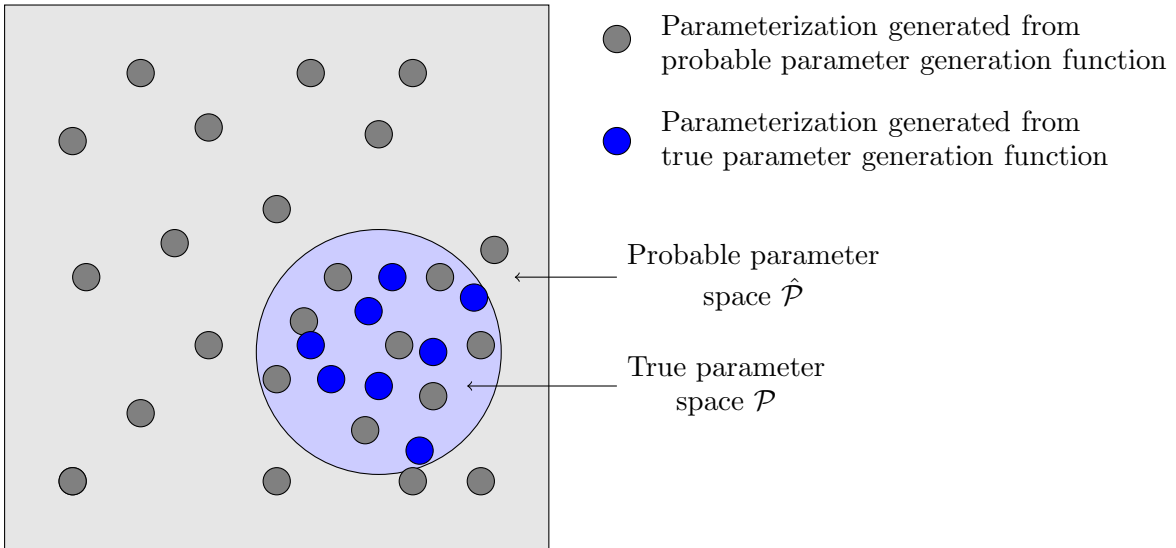

Figure 2: Construction of $\hat{\mathcal{P}}$ and $\hat{\mathcal{H}}$. As the DM samples more parameterizations from $\hat{\mathcal{H}}$, the probable parameter space can more effectively approximate the true parameter space, potentially improving ZSG by ensuring comprehensive coverage.

The construction of the probable parameter space $\hat{\mathcal{P}}$ and the probability distribution $\hat{\mathcal{H}}$ over that space may significantly impact the generalization performance of the GCA. By defining $\hat{\mathcal{P}}$ as a superset of the true parameter set $\mathcal{P}$, we ensure that reasonably similar parameterizations are available for sampling. However, even for similar parameterizations (as defined by a distance metric), the resulting costs can vary substantially. To address this, we define the Lipschitz conditions for Super-MDPs, which, when satisfied, control the distance in transitions and costs for two similar parameterizations.

Definition 4 (Lipschitz Super-Markov Decision Processes). *Let $\mathcal{M}_\mathcal{S} = (\hat{\mathcal{P}}, \mathcal{S}, \mathcal{A}, \hat{\mathcal{H}}, \mathcal{F})$ be a given Super-Markov Decision Process, and $d(\cdot, \cdot)$ a distance metric defined over the parameter space. We call $\mathcal{M}_\mathcal{S}$ a Lipschitz Super-Markov Decision Process with smoothness factors L_f and L_r, if for any two parametrizations $\mathbf{p_i}, \mathbf{p_j} \in \hat{\mathcal{P}}$, we satisfy the following constraints:*

$$\forall (\mathbf{s}, a), \quad W(f^{\mathbf{p_i}}(\mathbf{s}, a), f^{\mathbf{p_j}}(\mathbf{s}, a)) \leq L_f \, d(\mathbf{p_i}, \mathbf{p_j}),$$

$$\forall (\mathbf{s}, a), \quad \|C^{\mathbf{p_i}}(\mathbf{s}, a) - C^{\mathbf{p_j}}(\mathbf{s}, a)\| \leq L_r \, d(\mathbf{p_i}, \mathbf{p_j}).$$

Here W denotes the Wasserstein distance (Villani 2008) between two transition probability distributions.

These relationships indicate that similar parameterizations (i.e., small $d(\mathbf{p}_i, \mathbf{p}_j)$) result in minor differences in state transitions and costs when the same actions are taken in the same states. Conversely, more substantial differences between parameterizations (e.g., larger parameter differences) lead to greater variations, though these are controlled by the Lipschitz constants L_f and L_r.

The Lipschitz property for Super-MDPs, as defined in Definition 4, has important implications for the ZSG capabilities of a GCA. Specifically, the Lipschitz conditions enforce a smoothness constraint on how transition probabilities and reward functions vary with the problem parameters. This smoothness ensures predictable and continuous system behavior as parameters change. Consequently, if an agent learns a decision-making policy from collected problem instances within a parameter space, it can effectively anticipate and perform well for closely related yet unseen problem instances (we will present a quantitative analysis for this in §5.2). This extrapolation of learned behaviors to new instances is supported by the Lipschitz guarantee that new instances will not drastically differ in dynamics and rewards, thereby diminishing the need for additional training. As a result, a policy effective in one instance is likely to succeed in similar ones, thereby enhancing the GCA's performance in a ZSG setting (see Xu and Mannor 2012, for a related discussion).

To ensure the Lipschitz property, the DM can adopt several strategies when constructing the probable parameter space and the corresponding distribution over parameterizations. A key approach involves avoiding the definition of discontinuous and non-smooth parameterizations. Parameterizations that induce highly non-linear dependencies—where minor variations in problem parameters lead to significant changes in transition probabilities or costs—may compromise the Lipschitz continuity of a Super-MDP. Furthermore, the variability or spread of the parameters plays a crucial role in affecting Lipschitz continuity, emphasizing the critical importance of designing the parameter distribution function carefully.

In summary, during the *Train* phase, constructing a probable parameter space $\hat{\mathcal{P}}$ and a parameter distribution function $\hat{\mathcal{H}}$ that support a Lipschitz Super-MDP can *enhance the GCA's ability to generalize across unseen problem instances.*

The algorithm and training

We now explore algorithmic strategies for training GCAs under ZSG and utilize DRL as our primary methodological tool due to its proven success in handling large-scale sequential decision-making problems. According to Kirk et al. (2023), supported by numerous references in their paper, methods addressing ZSG challenges in DRL can be categorized into three main areas: DRL-specific improvements, enhancing similarity between training and testing data, and addressing differences between training and testing environments. We begin by examining DRL-specific strategies for an algorithm adoption.

The DRL domain has advanced significantly since the seminal introduction of the Deep Q-Network (DQN) algorithm by Mnih et al. (2013). However, designing an algorithm for training GCAs under ZSG conditions remains challenging, as highlighted by Kirk et al. (2023). They identify several factors that can enhance DRL algorithms' ZSG capabilities, including: 1) refining the policy network by iteratively distilling it into a new network with reinitialized weights to mitigate non-stationarity effects, 2) distinguishing between policy and value networks to allow targeted optimization of each component, 3) employing model-based approaches to reduce sample complexity and improve learning efficiency.

Given these considerations, the Deep Controlled Learning (DCL) algorithm (Temizöz et al. 2023) emerges as a promising candidate for our application. DCL is specifically designed for environments heavily influenced by exogenous stochastic factors—common in operations management problems—and employs a model-based approximate policy iteration approach. This method itera-

tively refines policies by reinitializing the weights of a neural network policy at each step, thereby bypassing the need for a value network during policy optimization through efficient simulations.

In summary, DCL trains a given policy (denoted by $\pi_{S,0}$ or simply π_0) through successive rounds of approximate policy improvement, framing reinforcement learning as a classification problem. Each iteration involves training a neural network using a dataset of state and estimated optimal action pairs (obtained via simulations), with the estimated optimal actions serving as labels. Since DCL does not inherently support the Super-MDP formulation for training GCAs, we extend it following the strategies outlined by Kirk et al. (2023). Algorithm 1 details this adaptation, referred to as Deep Controlled Learning for Super-MDPs (Super-DCL), which is tailored for training GCAs within a Super-MDP framework.

Algorithm 1 Deep Controlled Learning for Super-Markov Decision Processes

1: **Input**: Super-MDP model: $\mathcal{M}_{\mathcal{S}} = (\hat{\mathcal{P}}, \mathcal{S}, \mathcal{A}, \hat{\mathcal{H}}, \mathcal{F})$, initial policy: π_0, neural network structure: N_θ, number of approximate policy iterations: n, number of samples to be collected: N, number of threads: w, number of samples collected for a specific parameterization: R, length of the warm-up period: L, simulation budget per state-action pairs: M, depth of the simulations: H, maximum number of promising actions for simulations: P.
2: **for** $i = 0, 1, \ldots, n-1$ **do**
3:     $\mathcal{K}_i = \{\}$, the dataset
4:     **for each** $thread = 1, \ldots, w$ **do in parallel**
5:         **for** $j = 1, \ldots, \lceil N/w/R \rceil$ **do**
6:             Sample parameterization $\mathbf{p}^j \sim \hat{\mathcal{H}}$, construct $\mathcal{M}^{\mathbf{p}^j} = (\mathcal{S}, \mathcal{A}, f^{\mathbf{p}^j}, C^{\mathbf{p}^j}, \mathbf{s}_0^{\mathbf{p}^j})$ by $\mathcal{F}(\mathbf{p}^j)$
7:             **for** $k = 1, \ldots, R$ **do**
8:                 $\mathbf{s}_1 = (\mathbf{s}_{l+1} = f^{\mathbf{p}^j}(\mathbf{s}_l, \pi_i(\mathbf{s}_l, \mathbf{p}^j))$ **for** $l = 0, \ldots, L-1$, with $\mathbf{s}_0 = \mathbf{s}_0^{\mathbf{p}^j}$)
9:                 Find estimated optimal action $\hat{\pi}_i^+(\mathbf{s}_k, \mathbf{p}^j) = Simulator(\mathcal{M}^{\mathbf{p}^j}, \mathbf{s}_k, \pi_i, M, H, P)$
10:                Add $((\mathbf{s}_k, \mathbf{p}^j), \hat{\pi}_i^+(\mathbf{s}_k, \mathbf{p}^j))$ to the data set $\mathcal{K}_i$
11:                $\mathbf{s}_{k+1} = f^{\mathbf{p}^j}(\mathbf{s}_k, \hat{\pi}_i^+(\mathbf{s}_k, \mathbf{p}^j))$
12:             **end for**
13:         **end for**
14:     **end for**
15:     $\pi_{i+1} = Classifier(N_\theta, \mathcal{K}_i)$
16: **end for**
17: **Output**: $\pi_1, \ldots, \pi_n$

We now discuss the strategies to enhance the ZSG capability of the Super-DCL algorithm: enhancing similarity between training and testing data, and addressing differences between training and testing environments. Kirk et al. (2023) emphasize that increasing the similarity between training and testing data can substantially reduce the generalization gap and improve performance during testing. This similarity can be achieved by designing the training environment to closely mimic the testing environment or by ensuring that the testing environment lies within the range

of environments encountered during training. Such alignment can be facilitated by employing *extensive sampling* during training and ensuring *domain randomization*, which involves generating diverse parameterizations.

The potential benefits of extensive sampling and domain randomization are illustrated in Figure 2. For the DM, utilizing a large dataset and randomizing parameters during training broadens the coverage of the true, unknown parameter space, thereby enhancing ZSG performance. To support this approach, we introduce a hyperparameter R in the Super-DCL algorithm, which triggers the sampling of new parameterizations after R samples have been collected for a specific parameterization (Line 7).

Another method to improve the ZSG of a GCA involves training it to recognize and exploit differences among the environments it encounters. By incorporating parameters as features during training (Line 10), the GCA learns to identify and adapt to the dynamics of each problem instance. This strategy allows the agent to effectively apply these learned distinctions during testing, thereby *bridging the gap between the training and testing environments* (Kirk et al. 2023). Such awareness during the learning process serves as an *inductive bias*, enabling the agent to accurately identify and respond to environmental characteristics at test time.

The main novelty in Super-DCL lies in moving beyond training an MDP for a fixed parameterization $\mathbf{p}^j$ to varying the parameterization using previously defined sampling function $\hat{H}$ and reinitializing the system after R sampling periods with the fixed parameterization $\mathbf{p}^j$ (Lines 5-11). Moreover, the policy π not only takes states as input but also incorporates the parameterization used to generate the training samples (Line 10). For a comprehensive explanation of DCL, including the details of the sampling process, parallelization among threads (Line 4), the warm-up step for sampling a state (Line 8), and the $Simulator$ (Line 9) and $Classifier$ (Line 15) algorithms, we direct readers to Temizöz et al. (2023).

In conclusion, we can *train a GCA to solve the formulated Super-MDP and achieve ZSG* by adhering to the Lipschitz conditions during the formulation of a Super-MDP—through the rigorous construction of the probable parameter space and the parameter generation function—and by utilizing the Super-DCL algorithm for training. In this process, the final iteration of the Super-DCL algorithm yields the policy π_n, which serves as an approximation of the optimal policy ($\hat{\pi}_S$).

5.2 Estimate

We now assume that the policy $\hat{\pi}_S$ (trained in the Train phase) is a GCA such that $\bar{C}\pi_S^* + \epsilon \geq \bar{C}\hat{\pi}_S$. Hence, this GCA can produce effective decisions for the parameterizations generated by $\hat{\mathcal{H}}$ within the probable parameter space, $\hat{\mathcal{P}}$, which is assumed to cover all parameterizations relevant to our decision problem ($\mathcal{P} \subseteq \hat{\mathcal{P}}$). However, in our original decision problem $\mathcal{D}$, the true parameterizations remain obscure, potentially affecting the GCA's effectiveness. To mitigate this, the Estimate phase of the TED framework focuses on the online adaptation of the trained GCA by dynamically estimating the true problem parameters as the DM collects observations from the environment. As part of this phase, we first concentrate on the example techniques for online adaption, then we will analyze the impact of estimation accuracy on GCA's performance.

Online adaptation of the trained GCA

In the Estimate phase, the TED framework encourages the use of any effective technique to accurately determine the true parameters. For instance, in inventory management problems, demand may be censored in the event of stockouts—only sales data is accessible, not the actual demand. This limitation often leads to a chronic underestimation of true demand, potentially causing continuous understocking (Huh and Rusmevichientong 2009). To address this issue, specialized techniques such as the Kaplan-Meier estimator (Kaplan and Meier 1958) are necessary for accurately estimating the demand distribution. We will utilize this estimator in our numerical experiments later.

In contrast, other parameters, such as the distribution of lead times, may be estimated through alternative methods. One such method involves constructing a relative frequency distribution based on the observed values of lead times. This approach allows for the empirical estimation of the probability distribution by normalizing the frequency of each observed value by the total number of observations. It is crucial for the DM to apply the most appropriate estimation techniques for each parameter to ensure the GCA can perform effectively in real-time scenarios.

Impact of the estimator accuracy on GCA's performance

For a decision made in the decision problem $\mathcal{D}$ at time t, let $\mathbf{p} \in \mathcal{P}$ denote the true parameterization of $\mathcal{D}$, and let $\hat{\mathbf{p}}_t \in \hat{\mathcal{P}}$ represent the parameter estimates. Let $\mathcal{O}_t$ denote the observations collected up to time t. We define the estimator of the parameterization at time t as $\hat{\mathbf{p}}_t := \mathcal{Y}(\mathcal{O}_t)$, where

$\mathcal{Y}$ represents a general function that maps the observations collected until time t to the probable parameter space $\hat{\mathcal{P}}$.

For a general estimator, we now define how its accuracy—i.e., the similarity of the parameter estimates to the true parameterization—impacts the performance of the GCA. To formalize the similarity between the parameter estimates and the true parameterization, we introduce the *Parameterization Distance*, which measures the distance between parameterizations in terms of their influence on the dynamics of the decision-making process.

Definition 5 (Parameterization Distance)**.** *The Parameterization Distance $d_P(\mathbf{p}_i, \mathbf{p}_j)$ measures the distance in state transitions and costs between two tasks $\mathcal{M}^{\mathbf{p}_i}$ and $\mathcal{M}^{\mathbf{p}_j}$ generated with the parameterizations $\mathbf{p}_i$ and $\mathbf{p}_j$ by the mapping function $\mathcal{F}$ of a Lipschitz-Super-Markov Decision Process $\mathcal{M}_S$. For any two parameterizations $\mathbf{p}_i, \mathbf{p}_j \in \hat{\mathcal{P}}$, the parameterization distance is defined as:*

$$d_P(\mathbf{p}_i, \mathbf{p}_j) := \max_{s,a} \left(|C^{\mathbf{p}_i}(\mathbf{s}, a) - C^{\mathbf{p}_j}(\mathbf{s}, a)| + W(f^{\mathbf{p}_i}(\mathbf{s}, a), f^{\mathbf{p}_j}(\mathbf{s}, a)) \right),$$

where W denotes the Wasserstein distance (Villani 2008) between two transition probability distributions. Note that the right hand side of the definition is bounded because of the Lipschitz conditions, see Definition 4.

Let the DM solve a given task over a finite horizon T. Let $\bar{C}_T^{\pi}(\mathbf{s}, \mathbf{p})$ denote the expected per-period cost of a policy π starting from state $\mathbf{s}$ under parameterization $\mathbf{p}$ over T periods of time. We aim to bound the cost associated with inaccurately estimating the true parameterization. We begin by defining an assumption regarding the parameterization distance between the true parameters and their estimates over time.

Assumption 2 (Consistency of the Estimator)**.** *We assume that the estimator $\mathcal{Y} : \mathcal{O}_t \to \hat{\mathcal{P}}$ is a* consistent *estimator, i.e. for any two decision epochs t_i and t_j with $t_i < t_j$, it holds that $d_P(\mathbf{p}, \hat{\mathbf{p}}_{t_j}) < d_P(\mathbf{p}, \hat{\mathbf{p}}_{t_i})$.*

Assumption 2 states that the estimator $\mathcal{Y}$ provides increasingly accurate estimates of the true parameterization as the number of observations grows. We can now formulate the cost differences between taking an action based on the estimated parameterization $\hat{\mathbf{p}}_t$ and taking an action with knowledge of the true parameterization $\mathbf{p}$. This formulation expresses the cost difference when $\hat{\mathbf{p}}_t$ is employed in a task with true parameterization $\mathbf{p}$ over T periods:

Theorem 1 (Bound on Cost Difference Due to Estimation Error). *Let π be a policy for a Lipschitz Super-Markov Decision Process with a probable parameter space $\hat{\mathcal{P}}$. While the parameterization distance between the true parameters and the parameter estimates decreases over time, the difference in the corresponding expected per-period costs is bounded by the sum of parameterization distances over the horizon T.*

$$|\bar{C}_T^\pi(\mathbf{s}, \hat{\mathbf{p}}) - \bar{C}_T^\pi(\mathbf{s}, \mathbf{p})| \leq \sum_{t=1}^{T} d_P(\mathbf{p}, \hat{\mathbf{p}}_t),$$

where $\bar{C}_T^\pi(\mathbf{s}, \hat{\mathbf{p}})$ denotes the expected per-period cost of a policy π starting from state $\mathbf{s}$ under estimated parameterizations $\hat{\mathbf{p}}_t$ from period 1 to T.

Proof. This theorem is an application of the Simulation Lemma in Kearns and Singh (2002) for the Super-MDPs, the proof can be found in Appendix A. $\square$

In essence, Theorem 1 posits that the GCA with an estimated parameterization $\hat{\mathbf{p}} \in \hat{\mathcal{P}}$ performs comparably to the same GCA with the true parameterization $\mathbf{p} \in \mathcal{P}$, owing to the bounded parametrization distance between the estimated and true parameters. This finding underscores the importance of a well-functioning estimator function, $\mathcal{Y}$. It suggests that with an effective parameter estimation mechanism, a GCA can attain high performance for unseen instances without the need for retraining, and this level of performance will increase as long as the estimator function $\mathcal{Y}$ is consistent and fed with more observations.

5.3 Decide

The *Decide* phase represents decision-making steps within the TED framework. During this phase, the DM utilizes the trained GCA, $\hat{\pi}_S$, to make decisions. Specifically, for any task, the DM provides the current state of the environment, $\mathbf{s}$, along with the latest parameter estimates, $\hat{\mathbf{p}}$, to the GCA. The GCA then maps this pair to an appropriate action, $\hat{\pi}_S(\mathbf{s}, \hat{\mathbf{p}}) = a, a \in \mathcal{A}$. This phase capitalizes on the conditions defining the training strategies employed during the *Train* phase and the accuracy of the parameter estimates refined in the *Estimate* phase. If these conditions are met, the GCA can be expected to exhibit ZSG capabilities and maintain effectiveness in real-time decision-making.

However, challenges arise when the DM has no observations available to estimate the parameters, such that $\mathcal{O} = \emptyset$. In such scenarios, the GCA, which relies on parameter estimates to map states to actions, faces limitations. Simply selecting an arbitrary parametrization from the probable parameter space $\hat{\mathcal{P}}$ may not be ideal. To address this, we propose a more strategic approach:

formulating and solving a robust optimization problem over the probable parameter space. This method aims to enhance the resilience of the decision-making process by optimizing performance across a range of possible parameterizations in the absence of direct observational data.

We define the robust optimization model $\mathcal{Q}$, where the objective is to select a parameterization $\hat{\mathbf{p}}$ that minimizes the expected cumulative cost over a horizon T given the worst-possible parameter realization $\mathbf{p}$, assuming no initial observations are available and a given policy π is used. Here, T covers the periods where the DM does not have any observation to estimate the parameters. For instance, the DM can only observe the demands after lead time if there is no stock available initially. The detailed formulation of $\mathcal{Q}$ is as follows:

- **Decision Variable**:

$$\hat{\mathbf{p}} \in \hat{\mathcal{P}}$$

- **Objective Function**:

$$\min_{\hat{\mathbf{p}} \in \hat{\mathcal{P}}} \max_{\mathbf{p} \in \hat{\mathcal{P}}} \mathbb{E} \left[\sum_{t=0}^{T-1} C(\mathbf{s}_t, \pi(\mathbf{s}_t, \hat{\mathbf{p}})) \mid \mathbf{p}, \pi \right]$$

where $C(\mathbf{s}_t, \pi(\mathbf{s}_t, \hat{\mathbf{p}}))$ represents the cost at time t with state $\mathbf{s}_t$, parameter estimate $\hat{\mathbf{p}}$ and action $\pi(\mathbf{s}_t, \hat{\mathbf{p}})$, given the true parametrization $\mathbf{p}$ and a policy π.

- **Constraints**:

$$\mathbf{s}_{t+1} = f^{\mathbf{P}}(\mathbf{s}_t, \pi(\mathbf{s}_t, \hat{\mathbf{p}}))$$

with $\mathbf{s}_t \in \mathcal{S}$ and $\pi(\mathbf{s}_t, \hat{\mathbf{p}}) \in \mathcal{A}$ for all $t \in \{0, \ldots, T-1\}$, ensuring that actions taken are feasible for the estimated parameters.

This robust optimization model $\mathcal{Q}$ is designed to mitigate risk in the face of uncertainty about the true parameterization $\mathbf{p}$ during initial decision periods without any observations. Therefore, solution to $\mathcal{Q}$ can guide the DM when the DM lacks parameter estimates due to the absence of data.

6   Periodic Review Inventory Control

We will test our framework by examining its application on a broad class of discrete-time, periodic-review inventory problems characterized by lost sales.

Overview

In general, for problems in this class, the sequence of events for any given time point t unfolds as follows. Let OH_t denote the on-hand inventory at the beginning of the period t. At the start of period t, a new shipment of inventory q_t is delivered and added to OH_t. Then, the DM reviews the inventory and places a new order a_t. If the supplier allows for instant delivery, a portion of this order, q_t^0, may be immediately added to the on-hand inventory. The rest of the orders will be received according to a lead time distribution. The random demand D_t is realized, and the DM observes the sales, which are the minimum of D_t and the available inventory $OH_t + q_t + q_t^0$. The unmet demands are lost. At the end of period t, the DM incurs costs based on the unsatisfied demands and the on-hand inventory they have. In particular, the cost at the end of period is: $C_t = h(OH_t + q_t + q_t^0 - D_t)^+ + p(D_t - (OH_t + q_t + q_t^0))^+$, where h is the holding cost of an item in inventory, and p is the penalty cost for each unmet demand. Note that this cost is not always observable, and Lyu et al. (2024) have instead adopted maximizing the profit (G_t) as an objective - this objective is adapted to our case as $G_t = p\min(D_t, OH_t + q_t + q_t^0) - h(OH_t + q_t + q_t^0 - D_t)^+$ when the demand distribution is unknown and censored.

Demand and supply processes

We next briefly explain the details of the demand and supply processes that determine the amounts D_t, q_t and q_t^0. The demand D_t at each time step t is considered a discrete random variable within the bounds of $[0, \infty)$ and follows a structured pattern. We model this demand as occurring in cycles of constant length K, where K may span from 1 to K_{max}. The demands are independent across the periods but not identically distributed within a cycle, allowing for a variety of distribution patterns. Mathematically, the demands D_t can be grouped into K subsets according to the remainder when t is divided by K (i.e., $t \mod K$), with each subset of demands being generated independently from its specific distribution (see Gong and Simchi-Levi 2023, for a similar demand process analysis).

The supply process in our inventory system, determining the amounts q_t and q_t^0, is influenced by stochastic lead times. To model this process, we reference two prevalent approaches in the literature: orders can cross, where an order that is placed after another order may be received before that order (order crossover, see Stolyar and Wang 2022), and orders cannot cross (see Kaplan (1970)). In the former case, the lead times are i.i.d., while in the latter case orders are received in the same sequence in which they were placed. We incorporate both cases as they have practical relevance; see Andaz et al. (2024) for order crossing, and Bai et al. (2023a) for the latter case and a general analysis on stochastic lead times.

Super-MDP formulation

Our problem class include instances with a wide range of cost parameters, demand distributions (including cyclic demand), and lead time distributions (with or without order crossover). When constructing a Super-MDP for this problem during the Train phase, we focus on strategies to maintain the Lipschitz conditions of the Super-MDP (see Definition 4). The probable parameter space and the associated parameter generation function are created accordingly. We define bounds for the cost parameter p and parameters related to demand and lead time distributions, and we let demand and lead time distributions arise from two-moment fits (following Adan et al. 1995) and other techniques. For details, we refer to Appendix B, which contains a detailed description of the construction Super-MDP as well as the subsequent training of the GCA.

In the Estimate phase, we utilize the Kaplan-Meier estimator (Kaplan and Meier 1958) for estimating the demand distribution based on censored data, and we construct an empirical relative frequency distribution for estimating the lead time distribution. For further details we refer to Appendix C, which discusses Estimate and Decide phases for our problem as well as our strategy for adopting a parametrization when no observation is available for the unknown parameters.

7   Experiments

We implement Super-DCL algorithm to train the GCA for addressing the periodic review inventory control problem discussed in §6. The algorithm is programmed in C++20 and executed on five AMD EPYC 9654 processors, each with 192 hardware threads. The GCA, herein referred to as the *Generally Capable Lost Sales Network - GC-LSN*, underwent a training duration of $\sim$ 16 hours. Information on the hyperparameters used in the Super-DCL algorithm can be found in Appendix D. The features used in GC-LSN are described in Appendix E, and consist of problem parameter estimates as well as a state space representation. The trained GC-LSN represents a single generally applicable policy that has been employed for all numerical experiments reported in this section.

For the numerical results, we organize our problem instances into groups referred to as *Cases*. Each *Case* comprises a collection of instances that share similar inventory challenges but differ in their parameter settings. Specifically: *Case 1* - iid demand with deterministic lead times, *Case 2* - cyclic demand patterns with deterministic lead times and *Case 3* - stochastic lead times.

Case 1 includes 320 problem instances where demand in each period is iid, and lead times are deterministic. These instances are structured using a full factorial design with the following

parameter ranges: mean demand $\mu \in \{3.0, 5.0, 7.0, 10.0\}$, penalty cost $p \in \{9.0, 39.0, 69.0, 99.0\}$, deterministic lead time $\in \{2, 4, 6, 8, 10\}$. For each mean demand value, the standard deviation is selected such that the demand distribution follows one of the following: binomial distribution (very low variance), Poisson distribution (low variance), negative binomial distribution (high variance) and geometric distribution (very high variance).

Case 2 encompasses 243 problem instances where demand follows cyclic patterns, and lead times remain deterministic. The parameter ranges for these instances are: $K \in \{3, 5, 7\}$, $p \in \{9.0, 39.0, 69.0\}$, lead time $\in \{3, 6, 9\}$. Details regarding the specific demand distributions within each cycle can be found in Appendix F.

Case 3 consists of 240 problem instances with stochastic lead times. These instances are further divided into two conditions: order crossover - in half of the instances, orders can cross; sequential orders - in the remaining instances, orders are received in the exact sequence they were placed. For each condition, we adopt 10 distinct lead time distributions. The ranges of other parameters are: $K \in \{1, 3, 5, 7\}$, $p \in \{9.0, 39.0, 69.0\}$. For details regarding the lead time and demand distributions, we refer Appendix F.

We report the numerical results in two parts. We first investigate the performance of GC-LSN compared to classical benchmarks when the problem parameters are considered *known*. The main purpose of these experiments is to validate that GC-LSN is a competitive policy under a wide range of inventory challenges. Then, we will move to the case where the parameters of the problem instances, demand and/or lead time distributions are unknown, must be estimated, and hence GC-LSN is fed with *estimated* parameters.

7.1 Benchmarking performance when problem parameters are known

When problem parameters are known, we bypass the estimation step of the TED framework and can directly apply GC-LSN to each instance. Since there is no tailored approach capable of handling the diverse inventory challenges present in *Case 2* and *Case 3*—where even effective heuristics for systems with cyclic demands are lacking (Gong and Simchi-Levi 2023)—we adopt the base-stock policy and capped base-stock policy (Xin 2021) as our benchmarks. We conduct a simulation-based optimization of their parameters (ie. base-stock level) for each instance, assuming full knowledge of the true problem parameters (demand and lead time distributions), thereby placing the benchmarks in the same informational position as GC-LSN for this set of experiments.

When comparing GC-LSN against the benchmarks, we report results in terms of relative cost

improvements, referred to as *average cost regret*, defined as: $Regret(\hat{\pi}_S, \pi_B)_{C,T} = \frac{\bar{C}_T^{\hat{\pi}_S} - \bar{C}_T^{\pi_B}}{\bar{C}_T^{\pi_B}}$. Here, $\bar{C}_T^{\hat{\pi}_S}$ denotes the average per period cost of GC-LSN over T periods, and $\bar{C}_T^{\pi_B}$ denotes the average per period cost of the benchmark policy π_B over the same T periods. A negative regret indicates that GC-LSN outperforms the benchmark.

We obtain unbiased estimators of the average costs for both GC-LSN and the benchmarks per period through simulations. Each evaluation consists of 1000 runs, with each run spanning $T = 5000$ periods and initiating after a warm-up period of 100 periods. The results are statistically significant, with the half-width of a 95% confidence interval being less than 1% of the corresponding cost value.

	Case 1				Case 2			Case 3	
	$p = 9$	$p = 39$	$p = 69$	$p = 99$	$K = 3$	$K = 5$	$K = 7$	$l = 0$	$l = 1$
Base-stock policy	−7.4%	−3.3%	−2.3%	−1.7%	−8.8%	−7.2%	−7.8%	−2.6%	−1.4%
Capped base-stock policy	−0.9%	−0.9%	−0.7%	−0.5%	−4.1%	−4.1%	−4.4%	−1.3%	−1.3%

Table 1: Average cost regret of GC-LSN - the lower the better

Table 1 presents the average cost regret of GC-LSN compared to the base-stock policy and the capped base-stock policy. For Case 1, we find that GC-LSN consistently outperforms both benchmark policies across varying penalty costs (p). We observe that the average regret increases with higher penalty costs, which is expected since both benchmark policies are asymptotically optimal as the penalty cost grows. For Case 2, GC-LSN consistently outperforms both base stock and capped base stock policies by a larger margin with the difference being more pronounced when demands exhibit cyclic patterns, probably because for those cases the benchmarks are less effective. For Case 3, GC-LSN again demonstrates superior performance by achieving negative regret compared to the benchmark policies. This highlights GC-LSN's ability to effectively utilize complete information in managing inventory under stochastic lead times. These results validate GC-LSN as a GCA, effectively managing inventory under varied inventory challenges.

7.2 Benchmarking performance when problem parameters are unknown

In Section 7.1, we demonstrated GC-LSN's performance for complex decision making problems where parameters are known. We next move to the even more challenging case of *unknown* problem parameters that need to be estimated. In particular, the DM is confronted with a periodic review lost sales inventory system with known cost parameters, but must make decisions *without*

information on the true demand and lead time distributions. Moreover, when demand exceeds on-hand inventory, only the actual sales are recorded, and hence the true demand realizations remain unknown in case of stock-outs. We employ GC-LSN based on *estimated* problem parameters (using methods discussed in Section 6).

When reporting the results, we obtain unbiased estimators of the average profits per period (when the demand distribution is unknown, because the DM can only observe the sales data, cf. Section 6) and average costs per period (when only the lead time distribution is unknown) of policies through simulations. Each evaluation is comprised of 1000 runs, with each run spanning $200, 500, 1000$ or 2000 periods - without any warm-up period, to assess how well our agent adapt to unknown distribution conditions as it gathers more observations.

In this section, we will consider **clairvoyant** base-stock and **clairvoyant** capped base-stock policies. By "clairvoyant," we mean that the parameters of these policies (ie. base-stock level) are optimized assuming full knowledge of the true problem parameters, similar to the experiments in Section 7.1. When comparing GC-LSN against these benchmarks, we report relative improvements in terms of costs and profits. When the demand distribution is unknown, we use average profit regret $Regret(\hat{\pi}_S, \pi_B)_{G,T} = \frac{\bar{G}_T^{\pi_B} - \bar{G}_T^{\hat{\pi}_S}}{\bar{G}_T^{\pi_B}}$. Here, $\bar{G}_T^{\hat{\pi}_S}$ denotes the average profit of GC-LSN over T periods, and $\bar{G}_T^{\pi_B}$ denotes the average profit of the benchmark policy π_B over the same T periods. When only the lead time distribution is unknown, we will utilize average cost regret $Regret(\hat{\pi}_S, \pi_B)_{C,T} = \frac{\bar{C}_T^{\hat{\pi}_S} - \bar{C}_T^{\pi_B}}{\bar{C}_T^{\pi_B}}$.

7.2.1 Benchmarking where online learning algorithms are applicable

GC-LSN is very broadly applicable, and our rational for selecting this particular setting is that it is the focus of several studies that focus on online learning for inventory control; see Section 2.2. Such studies provide us with policies that are designed for decision making in online settings where demand distributions are unknown, and that hence are suitable for benchmarking the full TED framework.

To this end, we adopt the test bed and experiment configuration proposed by Lyu et al. (2024), and consider the six learning algorithms tested in that study as a benchmark (this includes the state-of-the-art methods proposed by Lyu et al. (2024) as well Agrawal and Jia (2022) and Zhang et al. (2020)). We report the results of the best performing benchmark for each instance; the acronym π_B will be used to refer to this benchmark. In the test bed, we vary the penalty cost $p \in \{5, 10\}$. We vary the lead time as $Lt \in \{1, 3, 5, 7\}$. The demand distribution is either geometric (very high

variance) or Poisson (low variance), both with a mean of 10. Note that neither GC-LSN nor any of the benchmarks were specifically trained for or geared towards this specific demand distribution.

Lt	Policy	Period	Poisson distribution with mean 10							
			$p = 5.0$				$p = 10.0$			
			200	500	1000	2000	200	500	1000	2000
1	GC-LSN	Average Profit	**42.7**	**43.1**	**43.2**	**43.3**	**91.1**	**91.6**	**91.8**	**91.9**
	π_B	Average Profit	41.9	42.6	42.9	43.1	89.4	90.3	90.8	91.0
3	GC-LSN	Average Profit	**41.6**	**42.1**	**42.3**	**42.5**	**88.4**	**89.5**	**89.9**	**90.1**
	π_B	Average Profit	40.2	41.2	41.6	41.8	86.5	87.8	88.4	88.8
5	GC-LSN	Average Profit	**40.7**	**41.5**	**41.8**	**42.0**	**86.6**	**88.2**	**88.8**	**89.2**
	π_B	Average Profit	38.6	40.2	40.8	41.1	84.2	86.3	87.2	87.9
7	GC-LSN	Average Profit	**39.9**	**41.1**	**41.6**	**41.8**	**85.0**	**87.3**	**88.2**	**88.6**
	π_B	Average Profit	36.7	39.1	40.0	40.5	82.5	85.5	86.7	87.3

Lt	Policy	Period	Geometric distribution with mean 10							
			$p = 5.0$				$p = 10.0$			
			200	500	1000	2000	200	500	1000	2000
1	GC-LSN	Average Profit	**27.2**	**27.5**	**27.7**	**27.8**	68.1	69.0	69.6	69.9
	π_B	Average Profit	26.7	27.2	27.3	27.5	**69.4**	**69.7**	**69.9**	69.9
3	GC-LSN	Average Profit	**25.5**	**25.8**	**26.0**	**26.1**	**64.4**	**65.6**	**66.2**	**66.6**
	π_B	Average Profit	24.1	24.9	25.2	25.5	64.1	65.5	65.1	65.4
5	GC-LSN	Average Profit	**24.7**	**25.3**	**25.6**	**25.8**	**62.4**	**63.8**	**64.6**	**65.0**
	π_B	Average Profit	22.7	23.8	24.3	24.6	61.0	62.6	63.1	63.5
7	GC-LSN	Average Profit	**23.8**	**24.7**	**25.0**	**25.2**	**60.4**	**62.4**	**63.4**	**63.9**
	π_B	Average Profit	21.0	22.7	23.7	24.2	58.2	60.3	61.2	61.9

Table 2: Average profit of the GC-LSN agent compared to the best reported learning algorithm result, π_B, in the test instances taken from Lyu et al. (2024). We highlight the largest profit for each problem instance in **bold**.

Table 2 displays the average profits of GC-LSN and the best benchmark for each instance π_B across various horizon lengths. The results indicate that GC-LSN consistently outperforms the best existing learning algorithms for lost sales in nearly all instances, without necessitating additional training or simulations. As expected, the performance of GC-LSN improves with longer horizons. This improvement is attributed to the enhanced accuracy of parameter estimates, which benefit from an increased number of observations. Kaplan-Meier estimator is apparently well-suited to this task when employed alongside GC-LSN.

To gain insights into how quickly the TED framework can learn an effective policy through demand distribution estimation, we compare it against benchmarks that have full information about the demand distribution, clairvoyant base-stock and clairvoyant capped base-stock policies. In Table 3, we report the average profit regret of GC-LSN against these benchmarks using the same

test bed employed by Lyu et al. (2024).

Policy	Benchmark	Period	200	500	1000	2000	5000
GC-LSN	Clairvoyant base stock policy	Regret %	-0.9%	-1.3%	-1.5%	-1.6%	-1.7%
	Clairvoyant capped base stock policy	Regret %	2.5%	1.6%	1.2%	1.0%	0.8%

Table 3: Average profit regret -the lower the better- of GC-LSN when compared to the clairvoyant base stock policy and capped base stock policy. The parameters of these policies are optimized assuming full information regarding the true problem parameters, whereas GC-LSN is fed with parameter estimates to make decisions.

One would expect this regret to be a non-negative percentage since the benchmark policies have an informational advantage over GC-LSN. However, this is not always the case, as demonstrated by the results in Table 3. Notably, GC-LSN, based on demand estimates, outperforms the clairvoyant base-stock policy within 200 periods, despite being at a significant information disadvantage. Remarkably, while most online learning algorithms focus on converging the performance of the base-stock policy, GC-LSN surpasses it within 200 periods. Compared to the capped base-stock policy, the regret is initially 2.5% and decreases as the number of periods increases, thanks to the Kaplan-Meier estimator.

7.2.2 Benchmarking where there is no online learning algorithm available

Lastly, we evaluate the performance of GC-LSN on our extensive test beds—*Case 1*, *Case 2*, and *Case 3* when the problem parameters are unknown. It is important to note that there are no well-performing online learning algorithms available for *Case 2* and *Case 3*. For instance, Gong and Simchi-Levi (2023) investigates online learning for cyclic demands in lost sales inventory control with zero lead times, while Chen et al. (2023b) explores online learning for stochastic lead times, but studies constant order policy, which performs poorly when the long-run average cost is sensitive to lead times.

Given the absence of effective online learning algorithms for these cases, we adopt clairvoyant base-stock and clairvoyant capped base-stock policies as our benchmarks. These policies are optimized under the assumption that the true demand and lead time distribution parameters are known. For Case 1 and Case 2, we assume unknown demand distribution while for Case 3 we first consider unknown demand distribution with known lead time distribution, then unknown lead time distribution with known demand distribution, and lastly unknown demand and lead time distributions. This comprehensive approach allows us to assess the performance of GC-LSN across varied inventory challenges under different conditions of parameter uncertainty.

			C.v. base-stock policy				C.v. capped base-stock policy			
			200	500	1000	2000	200	500	1000	2000
Case 1	Unknown demand	$p = 9$	3.6	0.1	−1.1	−1.7	7.0	2.9	1.5	0.8
		$p = 39$	1.2	0.4	0.0	−0.2	1.8	0.7	0.4	0.1
		$p = 69$	1.0	0.4	0.1	0.0	1.2	0.5	0.3	0.1
		$p = 99$	0.8	0.3	0.1	0.0	1.0	0.4	0.2	0.1
Case 2	Unknown demand	$K = 3$	1.6	−0.4	−1.1	−1.5	3.5	1.2	0.4	0.0
		$K = 5$	2.2	0.3	−0.4	−0.8	3.5	1.3	0.5	0.1
		$K = 7$	2.7	0.4	−0.4	−0.9	4.2	1.6	0.7	0.2
Case 3	Unknown demand	$l = 0$	0.7	−0.1	−0.5	−0.7	1.4	0.5	0.1	−0.1
		$l = 1$	2.3	1.0	0.5	0.2	2.1	1.0	0.5	0.3
	Unknown lead time	$l = 0$	−1.6	−2.1	−2.3	−2.4	−0.6	−0.9	−1.1	−1.1
		$l = 1$	3.5	0.9	−0.2	−0.8	0.4	−0.5	−0.9	−1.1
	Unknown demand - lead time	$l = 0$	0.8	−0.1	−0.4	−0.6	1.5	0.5	0.1	−0.1
		$l = 1$	3.0	1.3	0.7	0.3	2.7	1.2	0.7	0.4

Table 4: Average profit regret (for unknown demands) and average cost regret (for unknown lead time and known demands) -the lower the better for both (% is omitted for saving space)- of GC-LSN when compared to clairvoyant (c.v.) base stock policy and clairvoyant (c.v.) capped base stock policy. The parameters of these policies are optimized assuming full information regarding the true problem parameters, whereas GC-LSN is fed with parameter estimates to make decisions.

Table 4 presents the average profit regret and average cost regret of GC-LSN compared to the clairvoyant base-stock and capped base-stock policies. We observe that the performance of GC-LSN improves as the number of periods increases, benefiting from more accurate estimates of demand and lead time distributions over time. In several instances, GC-LSN, which relies on parameter estimates, outperforms both the clairvoyant base-stock and capped base-stock policies. Notably, GC-LSN maintains strong performance even when both demand and lead time distributions are unknown. These results underscore the efficacy of GC-LSN as a GCA under ZSG conditions. This capability is particularly evident in experimental setups where no other solution approaches are available, highlighting GC-LSN's robustness and adaptability in managing inventory challenges with incomplete information.

8   Conclusion

In this study, we develop a solution framework designed to train generally capable agents that can make effective real-time decisions in operations management problems. These agents are trained to handle ZSG settings, where decision-makers' estimates about the environmental factors might change unpredictably, yet the agents do not require retraining. We introduce the concept of *Super-*

Markov Decision Processes to address the formulation challenges encountered in these settings. Furthermore, our *Train, then Estimate and Decide* framework trains generally capable agents and allow them to adapt in real-time for effective decision-making. In numerical experiments focusing on the periodic review inventory control problem, our trained generally capable agent, the *Generally Capable Lost Sales Network*, outperforms the benchmarks.

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A   Proof of Theorem 1

Theorem 1 (Bound on Cost Difference Due to Estimation Error). *Let π be a generally capable agent for a Lipschitz Super-Markov Decision Process with a probable parameter space $\hat{\mathcal{P}}$. As the parameterization distance between the true parameters and the parameter estimates decreases over time, the difference in the corresponding expected per-period costs is bounded by the sum of parameterization distances over the horizon T.*

$$|\bar{C}_T^{\pi}(\mathbf{s}, \hat{\mathbf{p}}) - \bar{C}_T^{\pi}(\mathbf{s}, \mathbf{p})| \leq \sum_{t=1}^{T} d_P(\mathbf{p}, \hat{\mathbf{p}}_t),$$

where $\bar{C}_T^{\pi}(\mathbf{s}, \hat{\mathbf{p}})$ denotes the expected per-period cost of a policy π starting from state $\mathbf{s}$ under estimated parameterizations $\hat{\mathbf{p}}_t$ from period 1 to T.

Proof. This theorem is an application of the Simulation Lemma in Kearns and Singh (2002) for the Super-MDPs. The Simulation Lemma is a foundational result that provides a bridge between the behavior of a reinforcement learning policy in a simulated environment and its performance in the true environment. Here, it implies that if the estimated parameters closely approximate the true parameters, any policy that is optimal (or near-optimal) in the simulated environment will also be near-optimal in the true environment. Therefore, our GCA can achieve zero-shot generalization, performing effectively even without precise knowledge of the true parameters.

Lobel and Parr (2024) developed an optimal tightness bound for the Simulation Lemma, addressing both discounted and undiscounted cases. Our proof is based on their results. They define two MDPs, $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, f, C \rangle$ and $\hat{\mathcal{M}} = \langle \mathcal{S}, \mathcal{A}, \hat{f}, \hat{C} \rangle$, which share a state-action space but differ in their transition and reward functions. Without loss of generality, we extend this concept by defining a Super-MDP $\mathcal{M}_S$ that encompasses these MDPs under distinct parameterizations $\mathbf{p}$ and $\hat{\mathbf{p}}$, hence yielding different cost and transition dynamics. In Appendix B, they show:

$$|V_T^{\pi}(\mathbf{s}, \mathbf{p}) - V_T^{\pi}(\mathbf{s}, \hat{\mathbf{p}})| \leq T(\epsilon_r + \epsilon_f(T-1)/4),$$

where $V_T^{\pi}(\mathbf{s}, \mathbf{p})$ represents the expected cumulative cost obtained by following policy π from state s over T time steps, with ϵ_r representing the discrepancy in cost functions and ϵ_f representing the discrepancy in state transitions. Dividing both sides by T provides the bound on the expected

per-period cost difference between $\mathbf{p}$ and $\hat{\mathbf{p}}$:

$$|\bar{C}_T^\pi(\mathbf{s}, \mathbf{p}) - \bar{C}_T^\pi(\mathbf{s}, \hat{\mathbf{p}})| \leq (\epsilon_r + \epsilon_f(T-1)/4).$$

Applying the Lipschitz continuity constraints (Definition 4) and parameterization distance (Definition 5), we can relate ϵ_r and ϵ_f to these distances:

$$(\epsilon_r + \epsilon_f(T-1)/4) \leq \sum_{t=1}^{T} d_P(\mathbf{p}, \hat{\mathbf{p}}_t),$$

which completes the proof. While Lobel and Parr (2024) provides a tighter static bound, our dynamic bound—based on the parameterization distance between true and estimated parameters—demonstrates that as the parameter estimates become more accurate, the marginal increase in the bound diminishes over time.

$\square$

B   Train phase for periodic review inventory control

We first explain the details for constructing a Super-MDP $\mathcal{M}_\mathcal{S} = (\hat{\mathcal{P}}, \mathcal{S}, \mathcal{A}, \hat{\mathcal{H}}, \mathcal{F})$ for the periodic review inventory control problem. We then define the initial policy used in Super-DCL algorithm.

Probable parameter space $\hat{\mathcal{P}}$

Recall that the parameters $\mathbf{p}$ related to the inventory problem discussed in §6 are: $\mathbf{p} = (h, p, \boldsymbol{\zeta}, \boldsymbol{\tau})$, where h denotes the holding costs, p the penalty costs, $\boldsymbol{\zeta}$ the specifics of the demand distribution and $\boldsymbol{\tau}$ the specifics of the lead time distribution.

For the cost-related parameters h and p, we maintain h as a constant, set to $h = 1$, while p is variable, ranging within $[p_{min}, p_{max}]$, where both p_{min} and p_{max} are predefined scalars.

Concerning the demand distribution, we adopt a structure capable of accommodating the cyclic nature of demand. Define j as the subset of demands generated from the same distribution in the cycles, where $j = t \mod K$, K is the cycle length, $K < K_{max}$, and t is time period. The mean demand for each subset is denoted by μ_j, ranging from $[\mu_{min}, \mu_{max}]$, and the standard deviation of the demand by σ_j, which varies between $[\sigma_{min}(\mu_j), \mu_j \cdot 2]$. The demand distributions are assumed to be fitted based on the first two moments, following the methodology specified by Adan et al. (1995). The function $\sigma_{min}(\mu_j)$ provides the minimum feasible standard deviation for fitting such a distribution depending on the mean demand (see van Hezewijk et al. 2024) while $\mu_j \cdot 2$ represents

the upper limit, indicative of a highly variable distribution. Let $\mathbf{D}_{\mu_j,\sigma_j}$ denote the fitted demand distribution. The realized demand at period t is distributed according to $\mathbf{D}_{\mu_j,\sigma_j}$ with $j = t \mod K$. We construct $\boldsymbol{\zeta}$ to encapsulate the mean and standard deviation of the demand for each cycle period, defined as $\boldsymbol{\zeta} = (\mu_0, \sigma_0, \ldots, \mu_{K-1}, \sigma_{K-1})$. This structure ensures that $\boldsymbol{\zeta}$ captures the sufficient statistics of the cyclic demand distribution.

We assume the lead time for a placed order takes values between 0 and L_{max}. Let l denotes whether the orders cross or not, such that $l = \mathbf{1}_{\{ordercrossing\}}$. Let p_j denote the probability of an order placed in period t will be received in period $t + j$, and we define $\mathbf{L} = \{p_{L_{max}}, p_{L_{max}-1} \ldots, p_0\}$, such that $\sum_{j=0}^{j=L_{max}} p_j = 1.0$. We construct the lead time parameter as $\boldsymbol{\tau} = (l, \mathbf{L})$.

Table 5 demonstrates the bounds we adapted for the probable parameter space of periodic review inventory control problem.

h	p_{min}	p_{max}	μ_{min}	μ_{max}	K_{max}	L_{max}
1.0	2.0	100.0	2.0	12.0	7	10

Table 5: Bounds for the probable parameter space for periodic review inventory control problem.

Parameter generation function $\hat{\mathcal{H}}$

We now focus on the parameter generation function $\hat{\mathcal{H}}$, which is used each time a new parameterization is generated, see Line 6 of the Algorithm 1. Our aim is to create a comprehensive function that can uniformly cover the probable parameter space $\hat{\mathcal{P}}$, hence the true parameter space $\mathcal{P}$. We assume uniform distribution for penalty cost p, demand cycle length K, and the mean demand and standard deviation of the demand for each subset of a cycle, μ_j and σ_j, such that $p \sim U(p_{min}, p_{max})$, $K \sim U[1, K_{max}]$, and $\mu_j \sim U(\mu_{min}, \mu_{max})$ and $\sigma_j \sim U(\sigma_{min}(\mu_j), \mu_j \cdot 2)$.

For the lead time distribution, we first determine minimum lead time l_{min} and maximum lead time l_{max}, such that $l_{min} \sim U[0, L_{max}]$ and $l_{max} \sim U[l_{min}, L_{max}]$. In case, $l_{min} = l_{max}$, the lead time is deterministic. If the lead times are stochastic, we assume order crossing with probability 0.5. We set $p_j = 0.0$ for $j < l_{min}$ and $j > l_{max}$. We determine the remaining individual lead time probabilities in one of the following ways with equal probability: 1) we assign the probabilities uniformly to $p_{l_{min}}, .., p_{l_{max}}$, 2) we fit a truncated discrete distribution between $p_{l_{min}}$ and $p_{l_{max}}$ using the same method we use for demand distribution (see Adan et al. 1995)) with mean $(p_{l_{min}} + p_{l_{min}})/2$, 3) we assign random probabilities while ensuring that the total probabilities sum to 1.0 and each of p_j, $l_{min} \leq j \leq l_{max}$, is assigned a value.

The configurations adapted for designing the Super-MDP for the periodic review inventory con-

trol problem align with existing literature (Zipkin 2000). These configurations may also facilitate the maintenance of the Lipschitz conditions (see Definition 4) by constructing a smooth and continuous probable parameter space $\hat{\mathcal{P}}$, employing a parameter generation function $\hat{\mathcal{H}}$ which uniformly covers $\hat{\mathcal{P}}$, and through the use of a linear cost function.

State and action spaces, $\mathcal{S}$ and $\mathcal{A}$

We define other Super-MDP elements for the periodic review inventory control model, and continue with state and action spaces, $\mathcal{S}$ and $\mathcal{A}$. We first define what a state composed of. Recall that OH_t denotes the on-hand inventory at the beginning of period t. Let the vector $\mathbf{x_t}$ denote the pipeline inventory, the orders that have not been received yet, and o_t denotes the number of to-be-received orders placed t periods ago. So we can define $\mathbf{x_t}$ as $\mathbf{x_t} = \{o_{t-L_{max}}, o_{t-L_{max}-1}, \ldots, o_{t-1}\}$. The pair $(OH_t, \mathbf{x_t})$ denotes the state of the system in the period t.

We construct a state space $\mathcal{S}$ covering all possible state spaces related to parameterizations $\mathcal{S}^{\mathbf{P}}$, such that $\mathcal{S}^{\mathbf{P}} \subseteq \mathcal{S}$. Specifically, we set $\mathcal{S} = \mathcal{S}^{\mathbf{p}_{max}}$, where $\mathbf{p}_{max}$ is the parameterization with largest penalty cost, demand distribution with the largest mean and standard deviation and longest lead time.

Actions represent the amount ordered by the DM. For the action space $\mathcal{A} = \{0, 1, \ldots, m\}$, we consider the same parameterization $\mathbf{p}_{max}$, and the maximum order quantity m corresponds to the single-period newsvendor fractile bound (see Zipkin 2008).

State transitions, costs and initial state for a given parameterization, $f^{\mathbf{P}}$, $C^{\mathbf{P}}$ and $s_0^{\mathbf{P}}$

We analyze how the state $\mathbf{s}_t$ transitions to a new state $\mathbf{s}_{t+1}$. We update the on-hand inventory as $OH_{t+1} = \max\{0, OH_t + q_t + q_t^0 - D_t\}$. Let $q_{t,j}$ denote the order received in the period t that is placed j periods earlier, $1 \geq j \geq L_{max}$ and $q_t = \sum_{j=1}^{j=L_{max}} = q_{t,j}$. We update the the pipeline vector $\mathbf{x}_{t+1} = \{o_{t-Lmax-1} - q_{t,Lmax-1}, o_{t-Lmax-2} - q_{t,Lmax-2}, \ldots, o_{t-1} - q_{t,1}, a_t - q_t^0\}$. Thus, we can write the state transition matrix as follows:

$$
\begin{aligned}
f^{\mathbf{P}}(\mathbf{s}_t, a_t) &= (OH_{t+1}, \mathbf{x}_{t+1}) \\
&= (\max\{0, OH_t + q_t + q_t^0 - D_t\}, \{o_{t-Lmax-1} - q_{t,Lmax-1}, \ldots, o_{t-1} - q_{t,1}, a_t - q_t^0\}).
\end{aligned}
$$

Determining the exact values of q_t and q_t^0 requires some straightforward but cumborsome formulation, readers may refer to Bai et al. (2023a) and Chen et al. (2023b).

After receiving the orders, placing new orders and observing the demand, the system incurs

costs at the end of a period with the following formulation:

$$C^{\mathbf{P}}(\mathbf{s}_t, a_t) = h \cdot OH_{t+1} + p \cdot (D_t - (OH_t + q_t + q_t^0))^+, \tag{1}$$

where the values OH_t, OH_{t+1}, q_t and q_t^0 inherently depend on $\mathbf{s}_t$ and a_t, and the demand D_t is drawn from the distribution $\mathcal{D}_{\mu_j, \sigma_j}$, with $j = t \mod K$.

For the initial states $\mathbf{s}_0^{\mathbf{P}}$, we follow the online learning in inventory systems literature and consider deterministic initial states. Specifically, we assume initial on-hand inventory $OH_0 = 0$ and there is no order in the pipeline. The system starts with the DM taking the first action after observing the empty inventory.

Initial policy for Super-DCL algorithm π_0

Super-DCL algorithm requires an initial policy as an input which it will iteratively refine to approximate the optimal policy. Let $I_{\max}^{\mathbf{P}}$ denote the newsvendor fractile for cumulative demand over lead time, which bounds the optimal inventory position (see Zipkin 2008) under parameterization $\mathbf{p}$. Then π_0 will be a capped base-stock policy with base-stock level $I_{\max}^{\mathbf{P}}$ with the maximum order quantity $m_{\mathbf{s}}^{\mathbf{P}}$, which is the single-period newsvendor fractile bound (Zipkin 2008).

Given the constructed Super-MDP formulation for the periodic review inventory control problem and the initial policy π_0, the DM can train a GCA using Super-DCL algorithm. Readers may refer to Appendix D for the set of hyperparameters and neural network structure required by the algorithm.

C Estimate and Decide phases for periodic review inventory control

We explain how the DM interacts with the environment, collects observations, use them for estimating the parameters, and take decisions with the trained GCA. To this end, we present a set of assumptions:

Assumption 3 (Decision Maker's interaction with the environment). *The following assumptions guide us through Estimate and Decide phases for the numerical experiments on our inventory problem. They are also commonly made in theory and in practice, see the related online inventory literature in §2.*

- *The decision-maker has direct information on holding cost h and the penalty cost p, so there is no estimation process for them.*

- *For parameters for which no direct information is available - the demand and lead time distributions, the decision-maker initially has no observation.*

- *The decision-maker knows the cycle length K for the demands.*

- *The decision-maker knows whether the lead times are deterministic (if so it is assumed to be known and orders do not cross, $l = 0$) or stochastic. If stochastic, the decision-maker knows whether the orders cross or not.*

- *For each received order, the decision-maker has full information on when that particular order is placed: they can observe the individual lead times. It means that the pipeline inventory vector $\mathbf{x}$ can be updated accordingly.*

We start with the decision-making strategy of the DM when they do not have any observations, and hence, no parameter estimates are available. Following the guidelines in §5.3, we aim to have a robust approach and minimize the costs in the worst case scenario. Without explicitly formulate and solve the robust optimization problem $\mathcal{Q}$, we greedily adopt a parameterization with deterministic demands equal to μ_{max} until the first sale or positive on-hand inventory (in case the demand distribution is unknown) and largest deterministic lead time L_{max} until we receive the first order arrival (in case the lead time distribution is unknown). This parametrization worked well in numerical experiments.

When the DM starts collecting observations, related unknown parameters are estimated. For the unknown demand distribution, we utilize the Kaplan-Meier (KM) estimator (Kaplan and Meier 1958). While initially proposed for estimating the survival function from lifetime data, KM estimator is also adopted in inventory problems. Interested readers may refer to Huh et al. (2011), Lyu et al. (2024) for the usage of the KM estimator to build an empirical cumulative distribution function (CDF) from the censored demand observations. While we depart from them in having cyclic demand distribution, the procedure stays the same as the DM knows the cycle length K, see Assumption 3. So we repeat the process for each subset of demands within a cycle. We then derive estimated mean demand $\hat{\mu}$ and standard deviation of the demand $\hat{\sigma}$ from the empirical CDF to create and update demand distribution estimate $\hat{\boldsymbol{\zeta}} = (\hat{\mu}_0, \hat{\sigma}_0, \ldots, \hat{\mu}_{K-1}, \hat{\sigma}_{K-1})$.

The estimation for the lead time probabilities is rather straightforward. In particular, the DM builds an empirical relative frequency distribution regarding how many periods it takes for an order to be received, and calculates the corresponding estimates for lead time probabilities

$\hat{\mathbf{L}} = \{\hat{p}_{L_{max}}, \hat{p}_{L_{max}-1} \ldots, \hat{p}_0\}$. This estimation process is the same regardless of whether the order cross or not, as that information is contained in l of the parameter $\hat{\boldsymbol{\tau}} = (l, \hat{\mathbf{L}})$.

D   Experimental Setup

Table 6 presents the set of hyperparameters used to train Super-DCL algorithm in our numerical experiments. For a comprehensive overview of the hyperparameters utilized, we direct readers to Temizöz et al. (2023). We employed the identical neural network architecture described in their study and adhered to the selection criteria for sampling and simulation-related hyperparameters outlined therein. To this end, we increased the number of samples N and the number of approximate policy iteration n for performance gains, and decreased the depth of the simulations H and simulation budget per state-action pairs M for computational efficiency. We refrained from further hyperparameter tuning. Departing from that study, we introduce two new hyperparameters: R, the number of samples collected for a parameterization; P, the maximum number of promising actions for simulations (see Danihelka et al. 2022, for an example usage).

Sampling and Simulation		Neural Network Structure N_θ	
Depth of the simulations:	$H = 21$	Number of layers:	4
Simulation budget per state-action pairs:	$M = 500$	Number of neurons:	{ 256, 128, 128, 128 }
Number of samples:	$N = 5000000$	Optimizer:	Adam
Length of the warm-up period:	$L = 100$	Mini-batch size:	$MiniBatchSize = 1024$
Number of samples collected for a parametrization:	$R = 100$	Maximum epoch:	100
Maximum number of promising actions:	$P = 16$	Early stopping patience	15
Number of approximate policy iterations: $n = 5$			

Table 6: The hyperparameters of the Super-DCL algorithm used in the experiments.

E   Featurization of GC-LSN

Below we present the features for training GC-LSN. Using GC-LSN, the DM can take actions for any product at time t with the estimated parameters. The features contains the following elements:

1. l, whether the orders can cross.

2. p, penalty cost for each unmet demand.

3. $OH_t + q_t$, on-hand inventory at the beginning of period t + received orders in period t before taking an action.

4. Pipeline inventory vector, $\{o_{t-Lmax-1} - q_{t,Lmax-1}, o_{t-Lmax-2} - q_{t,Lmax-2}, \ldots, o_{t-1} - q_{t,1}\}$.

5. $\hat{\mathbf{L}}$, estimated lead time probabilities.

6. K, demand cycle length.

7. $\hat{\boldsymbol{\zeta}}_t^f$, features for estimated demand distribution statistics.

Note that features 5 and 7 are based on estimations. If the corresponding true parameters are known, we use them directly. Since the input size of the neural networks stays the same during training and demand cycle lengths K vary across different parameterizations, we adapt $\hat{\boldsymbol{\zeta}}_t^f$, estimated demand distribution statistics (mean and standard deviation) for the next K_{max} periods. Hence, regardless of the cycle length, $|\hat{\boldsymbol{\zeta}}_t^f|$ stays the same.

F   Instances of Case 2 and Case 3

Case 2

For Case 2, we have the following mean demands according to varying cycle length:

- For $K = 3$, we have: $\{\mu_1 = 2.5, \mu_2 = 4.5, \mu_3 = 3.0\}, \{\mu_1 = 9.0, \mu_2 = 11.0, \mu_3 = 9.5\}, \{\mu_1 = 3.0, \mu_2 = 6.0, \mu_3 = 10.0\}$

- For $K = 5$, we have: $\{\mu_1 = 4.0, \mu_2 = 2.5, \mu_3 = 3.0, \mu_4 = 4.5, \mu_5 = 3.0\}, \{\mu_1 = 7.5, \mu_2 = 11.5, \mu_3 = 10.0, \mu_4 = 9.5, \mu_5 = 8.5\}, \{\mu_1 = 11.0, \mu_2 = 3.0, \mu_3 = 5.0, \mu_4 = 8.0, \mu_5 = 6.0\}$

- For $K = 7$, we have: $\{\mu_1 = 3.0, \mu_2 = 4.0, \mu_3 = 2.0, \mu_4 = 3.5, \mu_5 = 4.5, \mu_6 = 2.5, \mu_7 = 4.0\}, \{\mu_1 = 11.0, \mu_2 = 10.0, \mu_3 = 9.0, \mu_4 = 10.0, \mu_5 = 11.0, \mu_6 = 8.5, \mu_7 = 9.5\}, \{\mu_1 = 3.0, \mu_2 = 5.0, \mu_3 = 5.0, \mu_4 = 7.0, \mu_5 = 7.0, \mu_6 = 10.0, \mu_7 = 10.0\}$

For each mean demand unit, we adapt three different standard deviation combinations: low, high and mix. For each combination, the corresponding standard deviations are chosen according to following distributions:

- For low standard deviation combination, we have: $\sigma_1 - Poisson, \sigma_2 - Binomial, \sigma_3 - Poisson, \sigma_4 - Binomial, \sigma_5 - Poisson, \sigma_6 - Binomial, \sigma_7 - Poisson$.

- For high standard deviation combination, we have: $\sigma_1 - Geometric, \sigma_2 - NegativeBinomial, \sigma_3 - Geometric, \sigma_4 - NegativeBinomial, \sigma_5 - Geometric, \sigma_6 - NegativeBinomial, \sigma_7 - Geometric$.

- For mix standard deviation combination, we have: $\sigma_1 - Poisson, \sigma_2 - Geometric, \sigma_3 - NegativeBinomial, \sigma_4 - Binomial, \sigma_5 - Poisson, \sigma_6 - Geometric, \sigma_7 - NegativeBinomial.$

We then fit a demand distribution by using the two moments as described by Adan et al. (1995).

Case 3

For Case 3, we consider the following demand distributions: for $K = 1$, $\mu_1 = 5.0$; for $K = 3$, $\mu_1 = 8.0, \mu_2 = 10.0, \mu_3 = 6.0$; for $K = 5$, $\mu_1 = 11.0, \mu_2 = 3.0, \mu_3 = 5.0, \mu_4 = 8.0, \mu_5 = 6.0$; for $K = 7$, $\mu_1 = 3.0, \mu_2 = 5.0, \mu_3 = 5.0, \mu_4 = 7.0, \mu_5 = 7.0, \mu_6 = 10.0, \mu_7 = 10.0$. The corresponding standard deviations are chosen according to following distributions: $\sigma_1 - Poisson, \sigma_2 - Geometric, \sigma_3 - NegativeBinomial, \sigma_4 - Binomial, \sigma_5 - Poisson, \sigma_6 - Geometric, \sigma_7 - NegativeBinomial$. We then fit a demand distribution by using two moments as described by Adan et al. (1995).

When $l = 0$, we have the following lead time probabilities $\mathbf{L}$:

- $\mathbf{L} = \{0.0, 0.1, 0.18, 0.216, 0.2016, 0.1512, 0.09072, 0.042336, 0.0145152, 0.00326592, 0.00036288\}$

- $\mathbf{L} = \{0.0, 0.0, 0.2, 0.32, 0.288, 0.1536, 0.0384, 0.0, 0.0, 0.0, 0.0\}$

- $\mathbf{L} = \{0.0, 0.0, 0.0, 0.0, 0.1, 0.27, 0.378, 0.2268, 0.0252, 0.0, 0.0\}$

- $\mathbf{L} = \{0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.1, 0.27, 0.378, 0.252\}$

- $\mathbf{L} = \{0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.2, 0.32, 0.288, 0.1536, 0.0384\}$

- $\mathbf{L} = \{0.0, 0.0, 0.0, 0.0, 0.05, 0.095, 0.171, 0.2052, 0.21546, 0.184338, 0.079002\}$

- $\mathbf{L} = \{0.0, 0.0, 0.0, 0.0, 0.1, 0.225, 0.3375, 0.253125, 0.0759375, 0.0084375, 0.0\}$

- $\mathbf{L} = \{0.05, 0.095, 0.171, 0.2052, 0.21546, 0.184338, 0.079002, 0.0, 0.0, 0.0, 0.0\}$

- $\mathbf{L} = \{0.0, 0.0, 0.0, 0.0, 0.0, 0.5, 0.25, 0.125, 0.0625, 0.03125, 0.03125\}$

- $\mathbf{L} = \{0.0, 0.0, 0.0, 0.3, 0.49, 0.189, 0.021, 0.0, 0.0, 0.0, 0.0\}$

When $l = 1$, we have the following lead time probabilities $\mathbf{L}$:

- $\mathbf{L} = \{0.0, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1\}$

- $\mathbf{L} = \{0.0, 0.0, 0.2, 0.2, 0.2, 0.2, 0.2, 0.0, 0.0, 0.0, 0.0\}$

- $\mathbf{L} = \{0.0, 0.0, 0.0, 0.0, 0.1, 0.2, 0.3, 0.3, 0.1, 0.0, 0.0\}$

- **L** = {0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.1, 0.2, 0.3, 0.4}

- **L** = {0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.2, 0.2, 0.2, 0.2, 0.2}

- **L** = {0.0, 0.0, 0.0, 0.0, 0.05, 0.05, 0.1, 0.1, 0.15, 0.25, 0.3}

- **L** = {0.05, 0.05, 0.1, 0.1, 0.15, 0.25, 0.3, 0.0, 0.0, 0.0, 0.0}

- **L** = {0.0, 0.0, 0.0, 0.0, 0.1, 0.15, 0.25, 0.25, 0.15, 0.1, 0.0}

- **L** = {0.0, 0.0, 0.0, 0.0, 0.0, 0.5, 0.0, 0.0, 0.0, 0.0, 0.5}

- **L** = {0.0, 0.0, 0.0, 0.3, 0.4, 0.2, 0.1, 0.0, 0.0, 0.0, 0.0}

This figure "Parametrizations.jpeg" is available in "jpeg" format from:

http://arxiv.org/ps/2411.00515v1