Measurement uncertainties

Alfredo A. Louro

1 *Introduction*

Every value that we measure has an associated **uncertainty**. Say we measure the diameter of a disk with a ruler. You would think that after making the measurement as carefully as we can we would know the value of the diameter. But we don't know it with absolute certainty. Several things might have happened: We might not have laid down the ruler exactly along a diameter; the markings on the ruler may not be exactly right; and finally, our ruler has millimeter divisions, so we can't really measure with a higher precision than half a millimeter.

It gets more interesting. If our goal is to calculate the area of the disk based on our measurement of the diameter, we have to square the diameter, divide by 4 and multiply by π. What is the uncertainty in the calculated value of the area, given that we start off with an uncertain value of the diameter?

These are the questions that we shall address in this paper. There are some variations on the theme of how to treat measurement uncertainties, so it's good that the International Standardization Organization (ISO), the same people that have given us standard formats for everything ranging from film speed[1] to paper sizes[2] to how to write the date and time[3], have also standardized the procedure for estimating and reporting measurement uncertainties. The ISO standard is called GUM, Guide to Uncertainties in Measurement, and we will follow it in this paper.

[1] See the ISO setting on your camera. The higher the ISO number, the faster the film, so the shorter exposure time is needed.

[2] ISO paper sizes are based on the SI system, and they are widely used in Europe. A4 is the most commonly used size, similar to letter size in North America.

[3] At the time of writing this it is 09-12-23 21:51.

A caveat: Some of the rules that follow may seem arbitrary, and you may wonder where they come from. Without diving into probability theory, this is the best we can do for now. In the future you will see a much better justification for these rules.

2 *Measurements and uncertainties*

The model we are working with is this: When we measure a quantity X, the result we get is the sum of two contributions: The "true value", which is what we would like to know, and an **uncertainty**, which is a random variable, meaning that if we repeat the measurement, we get a different value of the uncertainty. This is why we can never know the "true value" of something (assuming it exists!). It is hidden by a random uncertainty.

So the best we can do is to try to constrain the unknown true value to within an interval, with a certain probability. If we can't measure the true value of g, at least we would like to say something like "with 95% probability, g lies somewhere between 9.805 and 9.815". So if we were to measure g repeatedly, 95% of the time we would get a value within that interval.

We will call x the *measured value assigned to the quantity* X, and $u(x)$ the *standard uncertainty of measurement associated with the measured value* x. Very often we will be concerned with the *relative standard uncertainty of measurement* $w(x) = u(x)/x$. $w(x)$ is usually expressed in percent form.

3 *Estimating the uncertainty in a measurement*

When you measure a quantity repeatedly, one of two things may happen: The measurement may yield different values each time – a "Type A" measurement – or they may yield the same value each time – a "Type B" measurement.

Statistical methods are used to estimate the uncertainties in a Type A measurement. We will not pursue this further in this work.

In fact, usually we won't even have the luxury of making repeated measurements. One measurement will have to do, and this also falls into the Type B category.

There are a couple of ways of estimating uncertainties in a Type B measurement. First, it's possible that the uncertainty is given by the manufacturer. For example, resistors like the one shown in Figure 1 have some coloured bands that are not just for decoration. The three leftmost bands are colour-coded to give the value of the resistance, and the fourth band on the right gives the relative uncertainty in the value of the resistance. In this case, a gold band tells us that the relative uncertainty is 5%.

Figure 1: A resistor

In the case of a measuring instrument, the manufacturer's data sheet may also provide an estimate of the relative uncertainty for measurements made with that instrument.

The other way to estimate the measurement uncertainty is using the smallest interval that the instrument measures. For instance, a ruler may have marks spaced by a millimeter. In that case, we give the measurement to the nearest millimeter (e.g. 93 mm), and the uncertainty as 0.5 mm. In general, for an analog instrument the rule would be to use half the smallest division.

If we are using a digital instrument, the uncertainty is half the last significant digit. Thus, if your voltmeter is showing 9.75 Volts, the uncertainty is estimated to be 0.005 Volts.

4 Uncertainty propagation

Very often, we measure a quantity to calculate something. For example, the volume of a cylinder of diameter D and length L is

$$V = \pi \frac{D^2}{4} L \tag{1}$$

We measure the diameter and the length with their uncertainties, $u(D)$ and $u(L)$; what is the resulting uncertainty in the volume $u(V)$?

According to GUM, if we have a function $y(x_1, x_2, \cdots, x_N)$ where the x_i are measured with uncertainties $u(x_i)$, the uncertainty in the calculated value y is given by

$$u(y) = \sqrt{c_1^2 u(x_1)^2 + c_2^2 u(x_2)^2 + \cdots c_N^2 u(x_N)^2} \tag{2}$$

where the c_i are called **sensitivity coefficients**, because they express how sensitive y is to uncertainties in each of the variables.

The sensitivity coefficients are just the **partial derivatives** of y with respect to each x_i:

$$c_i = \frac{\partial y}{\partial x_i} \tag{3}$$

For an explanation of the concept of partial derivative, please see appendix A before continuing.

You can see why the c_i are called sensitivity coefficients. Since they are multiplying the respective uncertainties, they determine how sensitive the measurement of y is to uncertainties in the measurement of each variable. If a sensitivity coefficient is very high compared with the others, that tells us that the corresponding variable weighs very heavily in overall uncertainty. In that case, we may want to measure that particular variable with a more precise instrument, to reduce its uncertainty.

In our example of the cylinder, V is a function of D and L, with

$$\frac{\partial V}{\partial D} = \frac{\pi}{2} D L \tag{4}$$

$$\frac{\partial V}{\partial L} = \pi \frac{D^2}{4} \tag{5}$$

Then the uncertainty in V is given by

$$u(V) = \sqrt{\left(\frac{\pi}{2} D L\right)^2 u(D)^2 + \left(\pi \frac{D^2}{4}\right)^2 u(L)^2} \tag{6}$$

5 Sums and products

The uncertainty propagation formula takes on a particularly simple form when the function is a sum or a product of several variables.

If

$$y = A_1 x_1 + A_2 x_2 + \cdots + A_N x_N \tag{7}$$

then each $c_i = A_i$, and

$$u(y) = \sqrt{A_1^2 u(x_1)^2 + A_2^2 u(x_2)^2 + \cdots + A_N^2 u(x_N)^2} \tag{8}$$

For example, the perimeter of a rectangle of sides a and b is $P = 2a + 2b$. The uncertainty in the perimeter $u(P)$ is

$$u(P) = \sqrt{4u(a)^2 + 4u(b)^2} \tag{9}$$

More interesting perhaps is the case where

$$y = A x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N} \tag{10}$$

Here the sensitivity coefficients are given by

$$\frac{\partial y}{\partial x_i} = n_i \frac{y}{x_i} \tag{11}$$

so that

$$u(y) = \sqrt{n_1^2 \left(\frac{y}{x_1}\right)^2 u(x_1)^2 + n_2^2 \left(\frac{y}{x_2}\right)^2 u(x_2)^2 + \cdots n_N^2 \left(\frac{y}{x_N}\right)^2 u(x_N)^2} \tag{12}$$

Now divide by y, and remember that $u(y)/y$ is the relative uncertainty $w(y)$, and in the same way $u(x_i)/x_i$ is the relative uncertainty $w(x_i)$:

$$w(y) = \sqrt{n_1^2 w(x_1)^2 + n_2^2 w(x_2)^2 + \cdots + n_N^2 w(x_N)^2} \tag{13}$$

So in the case of a product, it is most convenient to deal with the relative uncertainties. Recall the volume of the cylinder discussed earlier:

$$V = \pi \frac{D^2}{4} L \tag{14}$$

Applying equation (13), we find the relative uncertainty in the volume to be

$$w(V) = \sqrt{4w(D)^2 + w(L)^2} \tag{15}$$

Notice how simple and yet informative this result is. It is clear that the relative uncertainty in the diameter weighs significantly more (four times more to be exact) than the relative uncertainty in the length. Maybe we should measure the diameter with special care.

Very often the expressions we have to deal with are products of powers of variables. If that's the case, then we should look for the relative uncertainty right away.

6 *An electrical example*

One more example. We are presented with a resistor and a voltage source, and asked to measure the power dissipated by the resistor. The value of the resistance and its relative uncertainty is encoded on the resistor itself. The value of the voltage across the resistor is measured with a voltmeter. Find the relative uncertainty and the standard uncertainty of the power.

Here is a checklist of the procedure:

1. Write the calculated quantity as a function of the measured variables.

$$P = \frac{V^2}{R} \tag{16}$$

2. Measure and estimate the uncertainty or the relative uncertainty of each of the measured variables. Give the uncertainties to at most 2 significant figures.

   The value of the resistance and its relative uncertainty is encoded on the resistor: $120\Omega \pm 5\%$. The voltage is measured with a voltmeter that provides 4 significant figures, so we take half the smallest significant figure as the uncertainty: (6.750 ± 0.0005) V. Since the power is a product of powers of V and R, we will need the relative uncertainties. The relative uncertainty of the resistance is already given. For the voltage we calculate it as

$$w(V) = \frac{u(V)}{V} = 7.4 \times 10^{-3}\% \tag{17}$$

3. Write the error propagation formula for the quantities involved in the problem, and calculate the desired uncertainty.

$$w(P) = \sqrt{4w(V)^2 + w(R)^2} = 5\% \tag{18}$$

   Notice that this is essentially the same as the relative uncertainty of the resistance! The precision of the voltmeter is such that the uncertainty of the voltage is negligible by comparison. If we wanted a more precise result, we would concentrate our efforts on measuring the resistance with a smaller uncertainty, rather than merely reading the label.

4. Calculate the desired quantity itself and report the result. Make sure that the value of the quantity is not given with more significant figures than is warranted by the experimental uncertainty.

   If I enter the measured values of R and V in my calculator and calculate the power, it gives me a value of 0.3796875 Watts. But are all those digits really justified? The relative uncertainty is 5%, or about

0.02 Watts. So already the 9 in 0.379 is a lie! We report the final result as

$$P = 0.38 \text{ W} \pm 5\% \tag{19}$$

or

$$P = (0.38 \pm 0.02) \text{ W} \tag{20}$$

A Partial derivatives

The partial derivative, as you can imagine, is an extension of the concept of derivative to functions of more than one variable. Think of such a function as $y = f(x_1, x_2, \dots x_N)$. The partial derivative with respect to x_i is the answer to the question "How does y change when I change the variable x_i *only*?". It is calculated by pretending that all the other variables in the expression for y are constant, reducing the problem to one variable only. The notation for the partial derivative with respect to x_i is

$$\frac{\partial f}{\partial x_i} \tag{21}$$

For example, say $y = 3x_1^4/x_2$. The partial derivatives of y with respect to x_1 and x_2 are

$$\frac{\partial f}{\partial x_1} = \frac{12x_1^3}{x_2} \tag{22}$$

$$\frac{\partial f}{\partial x_2} = -\frac{3x_1^4}{x_2^2} \tag{23}$$

Exercises

1. The volume of a sphere is

$$V = \frac{4}{3}\pi R^3$$

If the measured radius of a sphere is $R = 3.0$ cm, with a relative uncertainty $w(R) = 2\%$, calculate its volume V, with its relative uncertainty $w(V)$.

2.(a) Alice measures the length of a sheet of paper with a ruler whose smallest division is 1 mm. The measured length is $L = 27.9$ cm. What is the standard uncertainty in the length, $u(L)$? What is the relative uncertainty in the length, $w(L)$?

(b) Alice then measures the width of the sheet of paper with the same ruler, and finds $W = 21.6$ cm. With these data, she calculates the area of the sheet of paper, $A = LW$. What is the calculated value of A? What are the sensitivity coefficients $\partial A/\partial L$ and $\partial A/\partial W$? Finally, using the error propagation formula, what is the standard uncertainty in the area, $u(A)$? What is the relative uncertainty in the area, $w(A)$?

3.(a) Bob measures the focal length f of a converging lens by placing an object at a distance s from the lens, and measuring the distance s' to a screen on the other side of the lens, where a sharp image is formed. According to the thin lens equation,

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$$

For an object distance $s = (40.0 \pm 0.1)$ cm, Bob finds $s' = (13.5 \pm 0.1)$ cm. Calculate the focal length f and its standard uncertainty.

(b) Bob finds the spec sheet for the lens, where it says that the focal length is 10 cm. Does this agree with his measured value of f, within the experimental uncertainty?