Control in The Presence of Uncertainty

Introduction and Basic Tools

Dispense del Corso di Controllo Robusto e Adattativo

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Recommended Reading

- L_2-Gain and Passivity Techniques in Nonlinear Control, A. van der Schaft (Chapter 3)
- Feedback Control Theory, J. Doyle, B. Francis, A. Tannenbaum (Chapter 2)
- Robust Adaptive Control, J. Sun and P.A. Ioannou (Chapters 1, 2, and 4)
- Systems and Control Theory – An Introduction, A. Astolfi (Section 4.7)
- Nonlinear Systems, H. Khalil (Chapter 6)
- Nonlinear and Adaptive Control with Applications, A. Astolfi, D. Karagiannis, R. Ortega (Chapter 3)
- Nonlinear and Adaptive Control Design, M. Krstic, I. Kanellakopoulos, P. Kokotovic (Chapter 3)
- Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems, A. Isidori, A. Astolfi

Uncertainty

Control systems are designed so that certain signals, such as tracking errors and actuator inputs, do not exceed pre-specified levels.

Hindering the achievement of this goal are the presence of

- uncertainty about the plant to be controlled, since the mathematical models that we use in representing a physical system is an idealizations/approximation;
- unmodeled dynamics that produce uncertainty, usually at high frequency or because of neglected dynamics;
- errors in measuring signals (sensors can measure signals only to a certain accuracy);
- external (exogeneous) disturbances and *references*;
- noise and stochastic perturbations;
- ....

Depending on the type of uncertainty, and on the way it is modeled, one can follow several different analysis and design approaches.

Controlling Uncertainty

Adaptive control is a control method which incorporates some sort of in-built mechanism (an adaptation mechanism) for adjusting the controller characteristic as a function of a parameterized model of the plant.

Robust control is a control design method which guarantees performance in the presence of bounded (in some sense) uncertainty relying on a worst-case approach.

The regulator theory provides a control design method which presupposes that external disturbances can be described via a (known) dynamic model.

Stochastic control provides a control design method which presupposes that uncertainty can be modeled via probabilistic quantities (moments: mean, variance, ...).

One could clearly consider a combination of these approaches to develop, for example, robust adaptive controllers, an adaptive regulator theory, a robust stochastic control, ...

Underlying Principles and Tools

The **certainty equivalence principle** stipulates that one could replace *true values* with estimated values, provided the the estimation error has specific favorable properties, such as boundedness and/or asymptotic convergence.

The **internal model principle** stipulates that one has to incorporate in the feedback path a suitably reduplicated model of the dynamic structure of the exogenous signals.

The **gradient method** is an optimization method which generates a search direction along which a cost can be (continuously) reduced.

Dissipativity theory provides a framework for the analysis and design of control systems using an input-output description based on energy considerations.

The notion of **parameterization** provides a method to describe uncertainty in dynamical systems as the (linear) combination of known, measured, signals and unknown quantities.

The course requires a good understanding of basic notions such as controllability and observability, realization, minimality, stability, Lyapunov functions, norms.

It also requires some familiarity with linear algebra, basic inequalities and basic calculus.

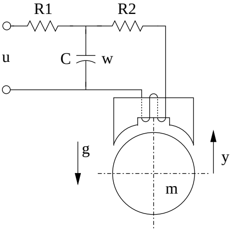
Each lecture is supported by exercises. Some of these exercises may require, in the final part of the course, the use of Matlab/Simulink (or any equivalent SW).

Some of the assignments contribute to the exam.

The course covers the basics of adaptive control for linear (and nonlinear) systems, robust control in the so-called H_∞ sense, and linear regulator theory for continuous-time systems.

Example: A Magnetic Levitation System

Consider a magnetic levitation system consisting of an iron ball in a vertical magnetic field created by a single electromagnet.



The system is described by the model (with domain of validity $-\infty < \xi_1 < 1$, that is when $\xi_1 = 1$ the ball touches the electromagnet)

$$\dot{\xi}_1 = \frac{1}{m}\xi_2, \qquad \dot{\xi}_2 = \frac{1}{2k}\xi_3^2 - mg, \qquad \dot{\xi}_3 = -\frac{1}{k}R_2\left(1-\xi_1\right)\xi_3 + w,$$

where ξ_1 describes the ball position, ξ_2 describes the ball momentum, ξ_3 describes the flux in the inductance, and w describes the voltage applied to the electromagnet.

k is a positive constant that depends on the number of coil turns.

Example: A Magnetic Levitation System

In low-power applications it is typical to neglect the dynamics of the actuator, hence it is assumed that w is the manipulated variable.

In medium-to-high power applications the voltage w is generated using a rectifier that includes a capacitance. The dynamics of this actuator can be described by the RC circuit shown in the figure, where the actual control voltage is described by u, while R_1 and C model the parasitic resistance and capacitance, respectively.

The model of the levitated ball system, including the actuator dynamics, is given by the equations

$$
\begin{aligned}
\dot{x}_1 &= \frac{1}{m} x_2, \\
\dot{x}_2 &= \frac{1}{2k} x_3^2 - mg, \\
\dot{x}_3 &= -\frac{1}{k} R_2 \left(1 - x_1\right) x_3 + x_4, \\
\dot{x}_4 &= -\frac{1}{Ck} \left(1 - x_1\right) x_3 - \frac{1}{R_1 C} x_4 + \frac{1}{R_1 C} u,
\end{aligned}
$$

where $x_1 = \xi_1$, $x_2 = \xi_2$, $x_3 = \xi_3$ and we have added the co-ordinate $x_4 = w$ that represents the voltage across the capacitor.

Example: Thyristor-Controlled Series Capacitor (TCSC)

Consider the (normalised) averaged model of a TCSC

$$\begin{aligned} \dot{x}_1 &= x_2 u, \\ \dot{x}_2 &= -x_1 u - \theta_1 x_2 - \theta_2, \end{aligned} \tag{1}$$

where x_1 and x_2 describe the dynamic phasors of the capacitor voltage, and $u > 0$ is the control signal which is directly related to the thyristor firing angle.

θ_1 and θ_2 are unknown positive parameters representing the nominal action of the control and one component of the phasor of the line current.

The control objective is to drive the state to $x^* = [-1, 0]^\top$ with a positive control action.

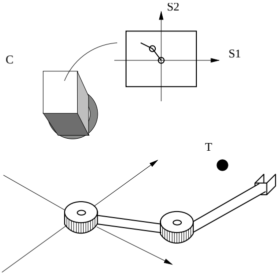
The *known-parameters controller* $u = \theta_2$ achieves the desired objective: the time derivative of $V(x) = \frac{1}{2}(x_1 + 1)^2 + \frac{1}{2}x_2^2$ along the trajectories of (1) is $\dot{V} = -\theta_1 x_2^2$.

This device is used in Flexible AC Transmission Systems (FACTS) to regulate the power flow in a distribution line.

Example: Visual Servoing

Consider the visual servoing of a planar two-link robot manipulator in the so-called fixed-camera configuration where the camera orientation and the scale factor are unknown.

The control goal is to place the robot end-effector in some desired position by using a vision system equipped with a fixed camera perpendicular to the plane where the robot evolves.



We model the action of the camera as a static mapping from the joint positions $q \in \mathbb{R}^2$ to the position (in pixels) of the robot tip in the image output, denoted $x \in \mathbb{R}^2$.

This mapping is described by

$$x = ae^{J\theta}\left(k(q) - \vartheta_1\right) + \vartheta_2, \tag{2}$$

where θ is the orientation of the camera with respect to the robot frame, $a \geq a_m > 0$ and ϑ_1 and ϑ_2 denote intrinsic camera parameters (scale factors, focal length and centre offset).

$k : \mathbb{R}^2 \to \mathbb{R}^2$ defines the robot direct kinematics and

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad e^{J\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Invoking standard time-scale separation arguments we can assume that the robot is described by a simple integrator $\dot{q} = v$, where $v \in \mathbb{R}^2$ are the joint velocities.

The direct kinematics yield $\dot{k} = \mathcal{J}(q)\dot{q}$, where $\mathcal{J}(q) = \dfrac{\partial k}{\partial q}$ is the analytic robot Jacobian.

Differentiating (2) and replacing the latter expression yields the dynamic model

$$\dot{x} = ae^{J\theta}u, \tag{3}$$

where $u = \mathcal{J}(q)v$ is a new input.

The problem is to find a control law u such that $x(t)$ asymptotically tracks a reference trajectory $x^* = x^*(t)$ in spite of the lack of knowledge of a and θ.

Note that, if θ were known, a stabilising control law for system (3) could be obtained, under some assumptions, without the knowledge of the uncertain parameter a.

Exercise

Show that the feedback

$$\upsilon(x,\theta) = -\frac{1}{a_m} e^{-J\theta}\left(\tilde{x} - \dot{x}^*\right),$$

where $\tilde{x} = x - x^*$ is the tracking error, yields the closed-loop error dynamics

$$\dot{\tilde{x}} = -\frac{a}{a_m}\left(\tilde{x} - \dot{x}^*\right) - \dot{x}^*,$$

the trajectories of which converge to zero if either $a/a_m = 1$ or $|\dot{x}^*(t)|$ goes to zero.

Consider the problem of estimating the unknown frequencies and bias of the signal

$$y(t) = E_0 + \sum_{i=1}^{n} E_i \sin(\omega_i t + \phi_i),$$

with known $n \geq 1$ and unknown bias E_0, angular frequencies ω_i, amplitudes E_i, and phases ϕ_i, for $i = 1, \cdots, n$.

For $n = 1$ and $E_0 = 0$ the measured signal

$$y(t) = E_1 \sin(\omega_1 t + \phi_1)$$

may be regarded as the output y of the system

$$\dot{y} = x, \qquad \dot{x} = -\theta_1 y,$$

where $\theta_1 = \omega_1^2 > 0$ is unknown and x is unmeasurable.

Example: Frequency Estimation

Exercise

One could consider the frequency estimation problem, in the case $n = 1$, as the problem of designing an observer for the state of the partially unknown system

$$\begin{bmatrix} \dot{y} \\ \dot{x} \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\theta_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ x \\ \theta_1 \end{bmatrix},$$

with output y.

Show that the system is not observable. Explain how to construct an estimate of θ_1.

Signal Norms

The typical objective in a control system is to make some output behave in a desired way by manipulating some input (and in some robust way).

This is expressed stating that the output of the system has to be small (in some sense to be specified) and that the control effort should not be excessive (again in some sense to be specified). Similarly, one could require that the effect of disturbances be small (...) and/or that the state stays close (...) to some equilibrium, or operating condition.

To make the above statements precise one has to clarify what is "small" for a signal?

There are several ways in which one could define the size of a signal, that is of a function which for every $t \geq 0$ returns a vector in $\mathbb{R}^n$, that is

$$r : \mathbb{R}^{\geq 0} \to \mathbb{R}^n.$$

Note that one could consider also functions defined for all $t \in \mathbb{R}$.

Signal Norms

In control theory there are very specific norms/measures which are widely used, namely

- 1-norm: $\|r\|_1 = \int_0^{\infty} |r(\tau)| d\tau = \int_0^{\infty} \sum_{i=1}^{n} |r_i(\tau)| d\tau;$

- 2-norm: $\|r\|_2 = \sqrt{\int_0^{\infty} r^\top(\tau) r(\tau) d\tau} = \sqrt{\int_0^{\infty} \sum_{i=1}^{n} r_i^2(\tau) d\tau};$

- p-norm: $\|r\|_p = \left(\int_0^{\infty} |r(\tau)|^p d\tau\right)^{1/p} = \left(\int_0^{\infty} \sum_{i=1}^{n} |r_i(\tau)|^p d\tau\right)^{1/p};$

- ∞-norm: $\|r\|_\infty = \max_{i=1}^{n} \sup_{\tau} |r_i(\tau)|.$

We say that $r \in L_p$ (for $p \geq 1$, or $p = \infty$) if its p-norm is finite.

Exercise

Consider the signal $s(t) = \dfrac{1}{t+1}$.

Show that $s \in L_2 \cap L_\infty$ *and that* $s \notin L_1$. *Show also that* $s \in L_p$, *for every* $p > 1$.

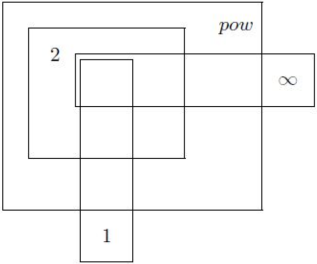
Exercise

Consider scalar signals, that is $n = 1$.

Show that if $\|r\|_1$ *and* $\|r\|_\infty$ *are well-defined, that is bounded, then* $\|r\|_2 \leq \|r\|_1 \|r\|_\infty$.

A power signal is a signal such that $\lim_{T\to\infty} \frac{1}{T}\int_0^T r(\tau)^2 d\tau = pow(r)$ *is finite.*

Show that if r *has bounded 2-norm then it is a power signal with* $pow(r) = 0$, *and that if* r *is a power signal and it has a well-defined* ∞*-norm, then* $pow(r) \leq \|r\|_\infty$.



System Gain

Signal norms can be used to define the notion of *gain* for systems. This is achieved using the notion of induced norm.

A system has finite p-gain (including $p = \infty$) if every input signal u with bounded p-norm generates an output signal y with bounded p-norm and

$$\sup_{u \in L_p} \frac{\|y\|_p}{\|u\|_p} < \infty.$$

This is a very abstract notion which applies to general dynamical systems.

For linear systems one could be more specific and consider *mixed* norms, for example one could consider the gain resulting from the use of L_2 input signals and the ∞-norm of the output (or vice versa).

One could also consider special input signals, such as the Dirac impulse, and the 2- or ∞-norms of the output signal, yielding the so-called 2- and ∞-norm of the system.

System Gain

Finally one could derive frequency domain characterizations of the notion of gain.

For example the induced L_2 norm of a linear, SISO, time-invariant, asymptotically stable, system equals the distance, in the complex plane, from the origin to the farthest point on the Nyquist plot of the associated transfer function $G(s)$ (more on this later).

It also equals the peak value of the Bode magnitude plot of the frequency response of the system, that is (note the inconsistent use of the ∞-norm)

$$\|G\|_\infty = \sup_\omega |G(j\omega)|.$$

Exercise

Show that the ∞-norm of the transfer function $G(s) = \dfrac{as+1}{bs+1}$, with $a \geq 0$ and $b > 0$ is

$$\|G\|_\infty = \begin{cases} a/b & \textit{if } a \geq b \\ 1 & \textit{if } a < b \end{cases}$$

For linear, strictly proper, systems existence of one norm implies, and is implied, by existence of any norm. This is not the case for nonlinear systems, that may have an L_2 gain, but not an L_∞ gain, for example.

Lyapunov Stability

To study the qualitative behavior of a system, hence to describe the properties of its trajectories for all t and for $t \to \infty$ we consider the notion of stability.

This notion allows studying the behavior of trajectories close to an equilibrium point or to a given motion.

Definition

Consider a dynamical system with state x and an equilibrium point x_e. Let x_0 denote the state of the system at $t = 0$.

The equilibrium x_e is stable in the sense of Lyapunov if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\|x_0 - x_e\| < \delta$$

implies

$$\|x(t) - x_e\| < \epsilon$$

for all $t \geq 0$.

Exercise

Consider the continuous-time system

$$\dot{x}_1 = \psi(t, x_1, x_2)x_2, \qquad \dot{x}_2 = -\psi(t, x_1, x_2)x_1,$$

with $\psi(t, x_1, x_2) > 0$ *for all* t, x_1, *and* x_2.

Show that the system has a unique equilibrium point at the origin.

Show that the equilibrium point is stable (exploit the relation $x_1\dot{x}_1 + x_2\dot{x}_2 = 0$*).*

Definition

An equilibrium point x_e is attractive if all trajectories with initial conditions in some neighborhood of x_e converge to x_e.

Definition

Consider a dynamical system with state x and an equilibrium point x_e.

The equilibrium x_e is *asymptotically stable* if it is *stable* and *attractive.*

Theorem

Consider the nonlinear autonomous system described by the equation

$$\dot{x} = f(x)$$

with $x(t) \in X \subset \mathbb{R}^n$. *Assume* $x_e = 0$ *is an equilibrium point, that is* $f(0) = 0$. *Suppose that*

- *either the eigenvalues of* $\frac{\partial f}{\partial x}(0)$ *are in the left half of the complex plane;*
- *or there exists a positive definite (differentiable) function* $V : x \to R^{\geq 0}$ *(that is* $V(0) = 0$ *and* $V(x) > 0$ *for all* $x \neq 0$ *in a neighborhood of the origin) such that* $V_x f(x) < 0$ *for all* $x \neq 0$ *in a neighborhood of the origin.*

Then the equilibrium x_e *is (locally) asymptotically stable.*

Barbalat's Lemma has been commonly employed in control to conclude that a bounded signal converges asymptotically to zero provided some conditions hold.

Lemma (Barbalat)

Suppose $r \in L_\infty \cap L_2$ and $\dot{r} \in L_\infty$. Then $\lim_{t\to\infty} r(t) = 0$.

The requirement $r \in L_\infty$ is not necessary (although this condition is often present), that is $r \in L_2$ and $\dot{r} \in L_\infty$ imply $\lim_{t\to\infty} r(t) = 0$.

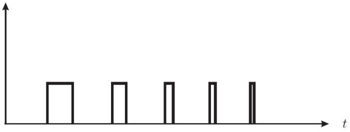
One could alternatively require that $r \in L_\infty \cap L_p$, for some $p \geq 1$.

The requirement that $\dot{r} \in L_\infty$ cannot be relaxed.

Barbalat's Lemma is an alternative to LaSalle Invariance Principle and does not require time-invariance.

Barbalat's Lemma seems somewhat intuitive: how can a function, the integral of which in the interval $[0, \infty)$ is finite, not converge to zero?

One such a function is composed of a series of rectangular pulses, of areas 1, 1/2, 1/4, ..., occurring every second. The integral of the function, over $[0, \infty)$ is equal to 2, yet the function does not converge to zero.



One could argue that such a function is discontinuous: we can smooth out all corners making the function differentiable and reach the same conclusion.

Note, however, that the considered function has unbounded derivative, in fact to reach the same level from zero and back over a shorter and shorter time period it requires a larger and larger time derivative.

Exercise

Consider the system

$$\dot{x}_1 = -x_1 + x_2\ \psi(t), \qquad \dot{x}_2 = -x_1\ \psi(t),$$

with ψ a generic function of t, and the positive definite and radially unbounded function

$$V(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right).$$

By computing the time derivative of V along the trajectories of the system show that if $\psi(t)$ is bounded for all t then x_1 converges asymptotically to zero.

Explain why LaSalle Invariance Principle cannot be used to conclude convergence of x_1 to zero and discuss what is the asymptotic behaviour of x_2.

The Gradient Method

Consider the problem of minimizing a function $f : \mathbb{R}^n \to \mathbb{R}$. Assume the function is continuously differentiable and let ∇f be the gradient of f, that is

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Note that ∇f is a column vector. The points such that $\nabla f(x) = 0$ are called stationary points of the function f.

Lemma

Consider a continuously differentiable function f and the dynamical system $\dot{x} = -\nabla f(x)$.

All bounded trajectories converge to a stationary point of the function f.

Moreover

- *if f is radially unbounded then all trajectories are bounded;*
- *if f is convex then all trajectories converge to the (set of) global minimizer(s) of f.*

Exercise

Consider the quadratic function $f(x) = \frac{1}{2}x^\top Qx + c^\top x + d$*, with* $Q = Q^\top$*.*

Assume that $Q > 0$*, that is* f *is convex.*

Show that the (linear) gradient system $\dot{x} = -(Qx + c)$ *has a unique equilibrium which is (globally) asymptotically stable.*

Exercise

Consider again the quadratic function $f(x) = \frac{1}{2}x^\top Qx + c^\top x + d$*, with* $Q = Q^\top$*.*

Assume that $Q > 0$*, that is* f *is convex.*

Show that the discrete-time (linear) gradient system $x_{k+1} = x_k - \alpha_k(Qx_k + c)$*, with* $\alpha_k > 0$*, has a unique equilibrium (regardless of the value of* α_k*) which is (globally) asymptotically stable provided* $\alpha_k > 0$ *is sufficient small.*

Provide an estimate for the largest value of α_k *guaranteeing asymptotic stability as a function of the eigenvalues of* Q*.*

The theory of dissipative dynamical systems allows studying stability properties for interconnected, possibly large-scale, systems using a simple energy-based perspective.

Consider a nonlinear system Σ described by equations of the form

$$\dot{x} = f(x, u), \qquad y = h(x, u),$$

where $x(t) \in X \subset \mathbb{R}^n$ is the state, $u(t) \in U \subset \mathbb{R}^m$ is the input, $y(t) \in Y \subset \mathbb{R}^p$ is the output, and f and h are smooth mappings.

On the space $U \times Y$ of the external variables, that is the input and output variables, define a function

$$s : U \times Y \to \mathbb{R},$$

called the *supply rate*.

Dissipativity

Definition

The system Σ is said to be *dissipative* with respect to the supply rate s if there exists a function $S : X \rightarrow R^{\geq 0}$, called the *storage function*, such that, for all $t_1 \geq t_0$ and all input signal u

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(\tau), y(\tau))d\tau, \tag{4}$$

where $x(t_1)$ is the state of the system at time t_1 resulting from the initial state $x(t_0)$ and the input function $u_{[t_0,t_1]}$.

If equation (4) holds with the equality sign, for all $x(t_0)$, $t_1 \geq t_0$, and all $u_{[t_0,t_1]}$, then σ is called *lossless* with respect to Σ.

The inequality (4) is called *dissipation inequality*. It expresses the fact that the *stored energy* $S(x(t_1))$ in the system at any future time t_1 is at most equal to the sum of the stored energy $S(x(t_0))$ at the present time t_0 and the totally externally supplied energy, given by the integral term, over the period $[t_0, t_1]$.

In a dissipative system there can be no internal *creation of energy*: only internal dissipation of energy is possible.

Consider again the dissipation inequality (4), that is

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(u(\tau, y(\tau))d\tau.$$

Suppose that the storage function S is differentiable, divide the above inequality by $t_1 - t_0$ and consider the limit as $t_1 - t_0 \to 0$. This yields the so-called *differential dissipation inequality*

$$S_x f(x, u) \leq s(u, h(x, u)). \tag{5}$$

One important choice of supply rate is (in the case in which the system has the same number of input and output signals)

$$s(u, y) = u^\top y. \tag{6}$$

Suppose Σ is dissipative with respect to the supply rate (6). Then for some function $S \geq 0$, and for all $x(0)$, $T \geq 0$ and all input signals $u_{[0,T]}$,

$$\int_0^T u(\tau)^\top y(\tau) d\tau \geq S(x(T)) - S(x(0)) \geq -S(x(0)),$$

or, equivalently,

$$-\int_0^T u(\tau)^\top y(\tau) d\tau \leq S(x(0)),$$

which means that the maximum amount of energy that can be extracted is bounded by a finite constant (which depends on the initial condition).

We call such a system a *passive system*.

Passivity

Definition

The system Σ is

- *passive* if it is dissipative with respect to the supply rate $s(u, y) = u^\top y$;
- *strictly input passive* if there exists a constant $\delta > 0$ such that it is dissipative with respect to the supply rate $s(u, y) = u^\top y - \delta \|u\|^2$;
- *strictly output passive* if there exists a constant $\epsilon > 0$ such that it is dissipative with respect to the supply rate $s(u, y) = u^\top y - \epsilon \|y\|^2$;
- *conservative* if it is lossless with respect to the supply rate $s(u, y) = u^\top y$.

The Hill-Moylan and KYP Conditions

Consider now systems Σ_a which are affine in the input, that is systems Σ described by the equations

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x). \tag{7}$$

Theorem (Hill-Moylan)

Consider the affine system Σ_a. Suppose the system is passive with a differentiable storage function S. Then

$$S_x f(x) \leq 0, \qquad S_x g(x) = h^\top(x).$$

Theorem (Kalman-Yakubovic-Popov)

Consider the affine system Σ_a. Suppose the system is linear, that is $\dot{x} = Ax + Bu$, $y = Cx$, and the system is passive with a quadratic storage function $S(x) = \frac{1}{2}x^\top Px$, $P = P^\top \geq 0$. Then

$$A^\top P + PA \leq 0, \qquad PB = C^\top.$$

Theorem

Consider the affine system Σ_a.

Suppose the system is strictly output passive with a differentiable storage function S.

Then

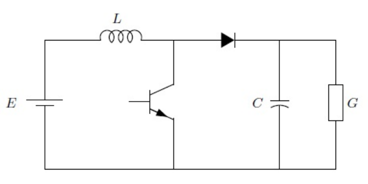
$$S_x f(x) \le -\epsilon h^\top(x) h(x), \qquad S_x g(x) = h^\top(x).$$

Exercise

Consider an ideal DC-DC boost converter described by the equations

$$L\dot{x}_1 = -sx_2 + E, \qquad C\dot{x}_2 = sx_1 - Gx_2, \qquad y = x_1,$$

with $L > 0$, $C > 0$, $G > 0$, $E > 0$ *and* $s \in \{0, 1\}$ *a switching signal.*



Show that the system, with input E *and output* y *is passive with storage function*

$$S(x_1, x_2) = \frac{L}{2}x_1^2 + \frac{C}{2}x_2^2.$$

Positive Realness

Consider a linear system

$$\dot{x} = Ax + Bu, \qquad y = Cx,$$

and let $G(s) = C(sI - A)^{-1}B$ be the associated transfer function. Assume the system is reachable and observable. Then the following statements are equivalent.

- The system is passive with a quadratic storage function $S(x) = \frac{1}{2}x^\top Px$, $P = P^\top > 0$.
- There exist matrices $P = P^\top > 0$ and L such that $A^\top P + PA = -L^\top L$ and $PB = C^\top$.
- All poles of $G(s)$ have non-positive real part; for all ω such that $j\omega$ is not a pole of $G(s)$ the matrix $G(j\omega) + G^\top(-j\omega)$ is positive semi-definite; any imaginary pole $j\omega$ of $G(s)$ is simple and $\lim_{s \to j\omega} (s - j\omega)G(s)$ is positive semi-definite and Hermitian.

A transfer function $G(s)$ which satisfies the last condition is called positive real.

Definition

A transfer function $G(s)$ is strictly positive real if the transfer function $G(s - \epsilon)$ is positive real for some $\epsilon > 0$.

For single-input, single-output, linear reachable and observable systems one has

$$G(j\omega) + G^\top(-j\omega) = 2Re[G(j\omega)],$$

which gives a condition on the Nyquist plot, that is the Nyquist plot of $G(j\omega)$ lies in the closed right-half of the complex plane.

Note that this is possible only if the relative degree of the transfer function is either zero (which requires a nonzero D matrix) or one.

Corollary

A single-input, single-output, linear reachable and observable system is strictly positive real if, and only if, all poles of $G(s)$ have negative real part; $Re[G(j\omega)] > 0$, for all $\omega \in [0, \infty)$ and $G(\infty) > 0$ or $\lim_{\omega \to \infty} \omega^2 Re[G(j\omega)] > 0$.

Positive Realness

Exercise

Show that the transfer function

- $G(s) = \dfrac{1}{s}$ *is positive real;*
- $G(s) = \dfrac{1}{s+a}$, *with* $a > 0$, *is strictly positive real;*
- $G(s) = \dfrac{1}{s+s+1}$, *is not positive real;*
- $G(s) = \dfrac{s}{s^2+\omega^2}$, *with* $\omega > 0$, *is positive real.*

Exercise

Show that the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -a x_1^3 - b x_2 + u, \qquad y = x_2,$$

is passive for all $a > 0$ *and* $k \geq 0$. *(Use* $V = \dfrac{a}{4}x_1^4 + \dfrac{1}{2}x_2^2$ *as storage function.)*

L_2-gain

A second important choice of supply rate is ($\gamma > 0$)

$$s(u, y) = \frac{1}{2}\left(\gamma^2 \|u\|^2 - \|y\|^2\right). \tag{8}$$

Suppose Σ is dissipative with respect to the supply rate (8). Then for some function $S \geq 0$, and for all $x(0)$, $T \geq 0$ and all input signals $u_{[0,T]}$,

$$\frac{1}{2}\int_0^T \left(\gamma^2 \|u(\tau)\|^2 - \|y(\tau)\|^2\right) d\tau \geq S(x(T)) - S(x(0)) \geq -S(x(0)),$$

hence

$$\int_0^T \|y(\tau)\|^2 \leq \gamma^2 \int_0^T \|u(\tau)\|^2 + 2S(x(0)),$$

that is the system Σ has L_2-gain smaller or equal to γ (the quantity $2S(x(0))$ is often referred to as *bias*).

Definition

The system Σ has L_2-gain smaller or equal to γ if it is dissipative with respect to the supply rate $s(u, y) = \frac{1}{2}\left(\gamma^2 \|u\|^2 - \|y\|^2\right)$.

Theorem

Consider the affine system Σ_a.

Suppose the system has L_2-gain smaller or equal to $\gamma > 0$ with a differentiable storage function S.

Then

$$S_x\left(f(x) + g(x)u\right) \leq \frac{1}{2}\left(\gamma^2 \|u\|^2 - \|h(x)\|^2\right).$$

L_2-gain

The differential dissipation inequality of the L_2-gain can be rewritten as

$$S_x\left(f(x)+g(x)u\right)-\frac{1}{2}\gamma^2\|u\|^2+\frac{1}{2}\|h(x)\|^2\leq 0.$$

This can be *checked* maximizing with respect to u, giving the maximizer

$$u^\star(x)=\frac{1}{\gamma^2}g^\top(x)S_x^\top,$$

hence, the partial differential Hamilton-Jacobi inequality

$$S_x f(x)+\frac{1}{2}\frac{1}{\gamma^2}S_x g(x)g^\top(x)S_x^\top+\frac{1}{2}h^\top(x)h(x)\leq 0.$$

Theorem

The system Σ_a has L_2-gain smaller or equal to $\gamma>0$ with a differentiable storage function if and only if there exists a differential solution $S\geq 0$ of the HJ inequality.

L_2-gain

For linear systems one could check the L_2-gain condition by means of a quadratic storage function $S(x) = \frac{1}{2}x^\top Px$, $P = P^\top \geq 0$.

Theorem

Consider the system

$$\dot{x} = Ax + Bu, \qquad y = Cx.$$

Suppose the system is observable.

The following statements are equivalent.

(i) *The system has L_2 -gain smaller then $\gamma > 0$.*

(ii) *There exists a positive definite matrix $P = P^\top$ such that*

$$A^\top P + PA + P\frac{BB^\top}{\gamma^2}P + C^\top C \leq 0 \quad (ARI)$$

Theorem

(iii) *All eigenvalues of A have negative real part and the Hamiltonian matrix*

$$H = \begin{bmatrix} A & \dfrac{BB^\top}{\gamma^2} \\ -C^\top C & -A^\top \end{bmatrix}$$

has no eigenvalues on the imaginary axis.

(iv) *There exists a matrix $X = X^\top > 0$ solving the LMI*

$$\begin{bmatrix} A^\top X + XA & XB & C^\top \\ B^\top X & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} < 0.$$

Proof.

We have already proved (i) $\Leftrightarrow$ (ii) and (iv) $\Leftrightarrow$ (ii) is easily obtained with a Schur complement argument.

We prove that (iii) $\Leftrightarrow$ (i). To begin with note that $J^{-1}HJ = -H^\top$, with $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, hence H is similar to minus its transpose, which means that if λ is an eigenvalue of H than also $-\lambda$ is.

Write

$$H = L + MN = \begin{bmatrix} A & 0 \\ -C^\top C & -A^\top \end{bmatrix} + \begin{bmatrix} \frac{B}{\gamma} \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{B^\top}{\gamma} \end{bmatrix}$$

and note that L does not have eigenvalues on the imaginary axis. $\square$

L_2-gain

Suppose now that H has an eigenvalue on the imaginary axis, say $j\omega$. Then there exists a vector v such that

$$Lv + MNv = j\omega v \quad \Leftrightarrow \quad (j\omega I - L)v = MNv \quad \Leftrightarrow \quad v = (j\omega I - L)^{-1} MNv,$$

hence (setting $w = Nv$)

$$w = N(j\omega I - L)^{-1} Mw.$$

Let $G(s) = C(sI - A)^{-1}B$ and note that

$$N(j\omega I - L)^{-1} M = \frac{G(-j\omega)^\top G(j\omega)}{\gamma^2}$$

hence

$$\|w\|^2 = \frac{\|G(j\omega)w\|^2}{\gamma^2} \quad \Leftrightarrow \quad \|G(j\omega)\| = \gamma$$

implying that the L_2-gain of the system is exactly equal to γ, that is a contradiction. $\square$

Passivity and L_2-gain – Linear Systems

For single-input, single-output linear systems we have (partially) established the following properties.

- Passivity is equivalent to positive realness, that is the Nyquist plot of $G(j\omega)$ lies in the closed right-half of the complex plane.
- Passivity implies stability, minimum phaseness, and relative degree one (or zero).
- *Strict* passivity implies asymptotic stability, minimum phaseness, and relative degree one (or zero).
- L_2-gain smaller than $\gamma > 0$ is equivalent to asymptotic stability and the property that the Nyquist plot of $G(j\omega)$ lies inside a circle centered at the origin and with radius γ.
- If $G(s)$ is a *strictly* passive transfer function then $\tilde{G}(s) = \dfrac{G(s) - 1}{G(s) + 1}$ has L_2-gain smaller than 1.

Stability of Dissipative Systems

Theorem

Consider the system Σ. Suppose

- *the system is dissipative with a differentiable storage function S;*
- *the system has an equilibrium at $x_\star$ for $u = 0$ and $h(x_\star, 0) = 0$;*
- *$x_\star$ is a local strict minimizer of S;*
- *the supply rate is such that $s(0, y) < 0$, for all $y \neq 0$.*

Then $x_\star$ is a stable equilibrium point of the system $\dot{x} = f(x, 0)$.

Suppose, in addition, that the only solution of $\dot{x} = f(x, 0)$ such that $y(t) = 0$ for all t is $x = x_\star$. Then the equilibrium $x_\star$ of the system $\dot{x} = f(x, 0)$ is asymptotically stable.

Definition

The system Σ_a is zero-state observable (detectable, resp.) if $u(t) = 0$ and $y(t) = 0$, for all $t \geq 0$, imply $x(t) = 0$ for all $t \geq 0$ ($\lim_{t \to \infty} x(t) = 0$, resp.).

Lemma

Consider the system Σ_a and assume $x = 0$ is an equilibrium.

Let $S \geq 0$ be a differentiable storage function such that

- *either $S_x f(x) \leq -\epsilon h^\top(x) h(x)$, for some $\epsilon > 0$;*
- *or $S_x f(x) = 0$ and $S_x g(x) = h^\top(x)$.*

Suppose Σ_a is zero-state observable.

Then $S(x) > 0$ for all $x \neq 0$.

Proof.

Consider system Σ_a with $u = 0$. Let $S \geq 0$ be such that $S_x f(x) \leq -\epsilon h^\top(x)h(x)$.

Integrating the differential dissipation inequality yields

$$S(x(T)) - S(x(0)) \leq -\epsilon \int_0^T \|y(t)\|^2 dt \quad \xRightarrow{S(x(T))>0} \quad S(x(0)) \geq \epsilon \int_0^T \|y(t)\|^2 dt.$$

Suppose now that $S(x(0)) = 0$. This yields $y(t) = 0$, for all $t \geq 0$, and thus $x(0) = 0$, showing that $S(x) > 0$ for all $x \neq 0$.

Consider system Σ_a with $u = -y$. Let $S \geq 0$ be such that $S_x f(x) = 0$ and $S_x g(x) = h^\top(x)$.

The differential dissipation inequality is now

$$S_x(f(x) + g(x)u) = -h^\top(x)h(x),$$

from which the claim directly follows by integration. $\square$

Stabilization of Passive Systems

Consider system Σ_a and suppose the system is passive with a differentiable storage function $S > 0$, that is

$$S_x f(x) \leq 0, \qquad S_x g(x) = h^\top(x) \quad \Rightarrow \quad \dot{S} \leq u^\top y.$$

Setting $u = -ky + v$ yields $\dot{S} \leq v^\top y - k\|y\|^2$, that is a strictly output passive system.

The closed-loop system can be interpreted as resulting from the interconnection of the passive system Σ_a and the strictly input passive static system

$$\tilde{u} = k\tilde{y}, \qquad k > 0,$$

with input $\tilde{u}$ and output $\tilde{y}$, via the interconnection equations

$$u = -\tilde{u} + v, \qquad \tilde{y} = y.$$

Corollary

Consider the passive system Σ_a. Assume that S is differentiable and positive definite at $x^\star = 0$. Assume, moreover, that the system is zero state detectable.

Then the feedback $u = -ky$, with $k > 0$, renders the zero equilibrium of the closed-loop system asymptotically stable.

Consider the model of a fully-actuated rigid robot given by the equation

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u,$$

where $q(t)$ describes the generalized positions, $M(q) = M^\top(q) > 0$ is the inertia matrix, $C(q,\dot{q})\dot{q}$ accounts for the Coriolis terms, $g(q)$ describes the action of gravity and it is such that $\dfrac{\partial V}{\partial q} = g(q)$, with V the potential energy function, and u models the external torques applied to the robot.

Define the storage function

$$S(q,\dot{q}) = \frac{1}{2}\dot{q}^\top M(q)\dot{q} + V(q) - V(0)$$

and assume that $V(q) - V(0)$ is positive definite around 0 and that $\det\left(\dfrac{\partial g}{\partial q}(0)\right) \neq 0$.

Recall that $\dfrac{\dot{M}}{2} - C(q,\dot{q})$ is skew symmetric, that is $v^\top\left(\dfrac{\dot{M}}{2} - C(q,\dot{q})\right)v = 0$ for all v.

Exercise (Robustness of a PD controller)

Consider the robot model under the indicated assumptions and let $y = \dot{q}$.

Show that the model is passive (lossless) and zero state observable.

Let $u = -K_P q + v$, *with* $K_P = K_P^\top > 0$. *Show that the resulting closed-loop system is passive (lossless) with respect to a modified storage function.*

Let $u = -K_P q - K_D \dot{q} + v$, *with* $K_P = K_P^\top > 0$ *and* $K_D = K_D^\top > 0$. *Show that the resulting closed-loop system is strictly output passive and that the zero equilibrium is locally asymptotically stable.*

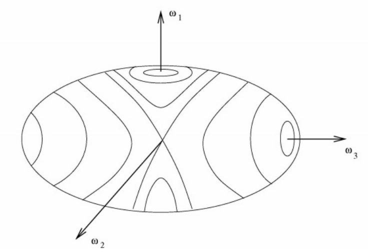
Show that if $q = 0$ *is a stationary point of the potential energy then the function* $V(q) - V(0) + \frac{1}{2} q^\top K_P q$ *is positive definite, provided* $K_P = K_P^\top > 0$ *is sufficiently large.*

How can the proposed PD controller be modified to stabilize points which are not stationary points of the potential energy?

How can the considered controller be implemented without velocity measurements?

Exercise

Consider a rigid body in an inertial reference frame and let ω_1, ω_2 and ω_3 denote the angular velocity components along a body fixed reference frame having the origin at the center of gravity and consisting of the three principal axes.



The Euler's equations with two independent controls aligned with two principal axes are

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + u_1, \qquad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 + u_2, \qquad I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2,$$

where $I_1 > 0$, $I_2 > 0$ and $I_3 > 0$ denote the principal moments of inertia and u_1 and u_2 the control torques. Assume $I_i \neq I_j$, for $i \neq j$.

Exercise (cont'd)

Recall that the kinetic energy of the rigid body is

$$K = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

and that the modulo of the angular momentum vector is

$$J = \frac{1}{2}\left(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2\right).$$

Determine two output signals y_1 and y_2 such that the system, with input $u = (u_1, u_2)$ and output $y = (y_1, y_2)$ is passive.

Show that the system, with the determined output signals, is not zero-state observable.

Let $u = -ky$, with $k > 0$. Discuss the properties of the closed-loop system.

Consider two nonlinear systems described by equations of the form

$$\Sigma_1 \left\{ \begin{array}{lcl} \dot{x}_1 & = & f_1(x_1, u_1), \\ y_1 & = & h_1(x_1, u_1), \end{array} \right. \qquad \Sigma_2 \left\{ \begin{array}{lcl} \dot{x}_2 & = & f_2(x_2, u_2), \\ y_2 & = & h_2(x_2, u_2). \end{array} \right.$$

Suppose "$U_1 = Y_2$" and "$U_2 = Y_1$" and that both Σ_1 and Σ_2 are passive (or strictly output passive), with storage functions S_1 and S_2, that is

$$S_1(x_1(t_1)) \leq S_1(x_1(t_0)) + \int_{t_0}^{t_1} u_1^\top(\tau) y_1(\tau) d\tau,$$

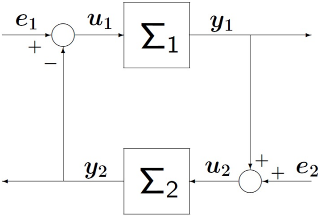
and

$$S_2(x_2(t_1)) \leq S_2(x_2(t_0)) + \int_{t_0}^{t_1} u_2^\top(\tau) y_2(\tau) d\tau.$$

Consider the (negative) feedback interconnection described by the equations

$$u_1 = e_1 - y_2, \qquad u_2 = e_2 + y_1,$$

with e_1 and e_2 new external signals.



$$S_1(x_1(t_1)) \leq S_1(x_1(t_0)) + \int_{t_0}^{t_1} (e_1(\tau) - y_2(\tau))^\top (\tau) y_1(\tau) d\tau$$

$$+$$

$$S_2(x_2(t_1)) \leq S_2(x_2(t_0)) + \int_{t_0}^{t_1} (e_2(\tau) + y_1(\tau))^\top (\tau) y_2(\tau) d\tau$$

$$=$$

$$\overbrace{S_1(x_1(t_1)) + S_2(x_2(t_1))}^{S(x(t_1))} \leq \overbrace{S_1(x_1(t_0)) + S_2(x_2(t_0))}^{S(x(t_0))} + \int_{t_0}^{t_1} \overbrace{e_1^\top(\tau) y_1(\tau) + e_2^\top(\tau) y_2(\tau)}^{e^\top(\tau) y(\tau)} d\tau.$$

Theorem (The Passivity Theorem)

Consider Σ_1 and Σ_2 interconnected by the equations $u_1 = e_1 - y_2$ and $u_2 = e_2 + y_1$.

(i) *Suppose Σ_1 and Σ_2 are passive (strictly output passive, resp.). Then the interconnected system with input (e_1, e_2) and output (y_1, y_2) is passive (strictly output passive, resp.).*

(ii) *Suppose Σ_1 and Σ_2 are passive with differentiable storage functions S_1 and S_2 having strict minimizer at $x_1^\star$ and $x_2^\star$. Then the equilibrium $(x_1^\star, x_2^\star)$ of the interconnected system with $e_1 = 0$ and $e_2 = 0$ is stable.*

(iii) *Suppose Σ_1 and Σ_2 are strictly output passive with differentiable storage functions S_1 and S_2 having strict minimizer at $x_1^\star$ and $x_2^\star$ and are zero-state detectable. Then the equilibrium $(x_1^\star, x_2^\star)$ of the interconnected system with $e_1 = 0$ and $e_2 = 0$ is asymptotically stable.*

The Passivity Theorem

Exercise

Prove claim (iii) of the Passivity Theorem by showing that the interconnected system is strictly output passive with $\epsilon = \min(\epsilon_1, \epsilon_2)$*.*

Exercise

Consider the systems, with $k_1 > 0$ *and* $k_2 > 0$*,*

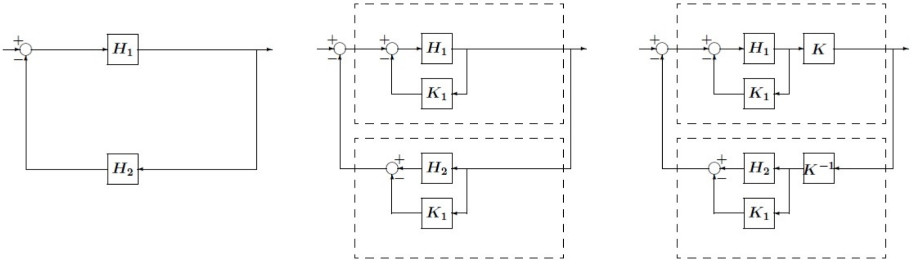
$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - k_1 x_2 + u_1, \qquad y_1 = x_2,$$

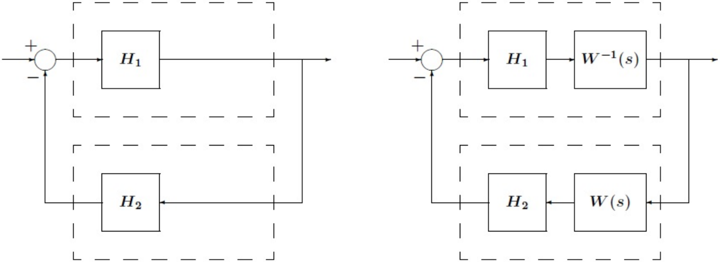
$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = -z_1 - k_2 z_2^3 + u_2, \qquad y_2 = z_2.$$

Show that both systems are strictly output passive and zero-state observable.

Hence show that the zero equilibrium of the interconnected system obtained setting $u_1 = -y_2$ *and* $u_2 = y_1$ *is asymptotically stable.*

The Passivity Theorem allows studying interconnected systems by means of *loop transformations* which modify passivity properties of the individual subsystems, but not of the overall system.





$$H_1 \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 + u \\ y_1 = x_1 \end{cases} \qquad H_2 \left\{ \; y_2 = k_2 u_2, \quad k_2 > 0 \right. \qquad W(s) = \frac{1}{s+1}$$

The essential ingredient of the Passivity Theorem is that the interconnection $u_1 = e_1 - y_2$ and $u_2 = e_2 + y_1$ is *neutral* with respect to the passivity supply rate, that is

$$u_1^\top y_1 + u_2^\top y_2 = (e_1 - y_2)^\top y_1 + (e_2 + y_1)^\top y_2 = e_1^\top y_1 + e_2^\top y_2.$$

This suggests that one could consider more complex interconnections. For example, consider k systems Σ_i, with input u_i and output y_i which are dissipative with respect to the supply rates $s_i(u_i, y_i)$ with storage functions S_i.

Consider now an interconnection equation $u = Fe + Gy$, with $u = (u_1, \cdots, u_k)$, $y = (y_1, \cdots, y_k)$, and $e = (e_1, \cdots, e_k)$ new external signals, and assume that

$$s_1(u_1, y_1) + s_2(u_2, y_2) + \cdots + s_k(u_k, y_k) \leq s(Fe + Gy, y)$$

for some function s. Then the interconnected system is dissipative with respect to the supply rate s and storage function $S = S_1 + S_2 + \cdots S_k$.

Theorem

Consider k linear, single-input, single-output, systems with transfer functions $G_i(s)$.

Suppose all systems are passive and the interconnection equation is

$$u = e - Hy,$$

with $u = (u_1, \cdots, u_k)$, $y = (y_1, \cdots, y_k)$, and $e = (e_1, \cdots, e_k)$ new external signals.

Suppose that there exists a matrix $P = \mathrm{diag}(P_1, P_2, \cdots, P_k)$, with $P_i > 0$, for all i, such that

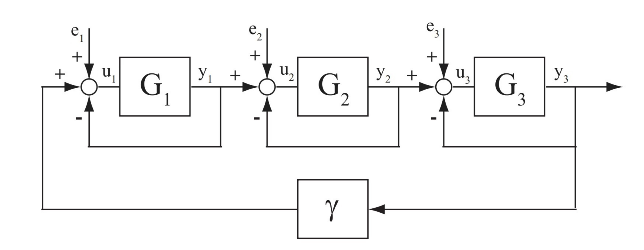
$$PH + H^\top P > 0.$$

Then the interconnected system is stable.

Exercise

Consider three linear, single-input, single-output, systems with transfer functions $G_1(s)$, $G_2(s)$ and $G_3(s)$. Suppose the systems are passive and are interconnected via the equation (recall that $u = (u_1, u_2, u_3)$, $y = (y_1, y_2, y_3)$, and $e = (e_1, e_2, e_3)$)

$$u = e - Hy, \qquad H = \begin{bmatrix} 1 & 0 & -\gamma \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$



Exercise (cont'd)

Let, without loss of generality, $P = \mathrm{diag}(1, P_2, P_3)$ *and note that*

$$PH + H^\top P > 0 \quad \Leftrightarrow \quad \begin{array}{l} P_2(4 - P_2) > 0 \\ -2P_2^2 P_3 + 2P_2 P_3 \gamma - 2P_2 \gamma^2 + 8P_2 P_3 - 2P_3^2 > 0 \end{array}$$

Setting, for example, $P_2 = 3$ *satisfies the first condition and yields the condition*

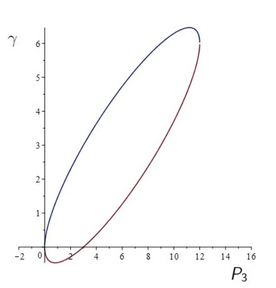
$$-2P_3^2 + 6P_3\gamma - 6\gamma^2 + 6P_3 > 0$$

Note now that the set of points such that

$$-2P_3^2 + 6P_3\gamma - 6\gamma^2 + 6P_3 = 0$$

is an ellipse, the interior points of which satisfy the inequality condition.

As a result, one could identify values of γ *such that the stability condition hold.*



The Small Gain Theorem

Consider now the systems Σ_1 and Σ_2 and assume they have L_2-gain smaller or equal to $\gamma_1 > 0$ and $\gamma_2 > 0$ with storage functions S_1 and S_2, respectively, that is

$$S_1(x_1(t_1)) - S_1(x_1(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} \gamma_1^2 \|u_1(\tau)\|^2 - \|y_1(\tau)\|^2 d\tau,$$

and

$$S_2(x_2(t_1)) - S_2(x_1(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} \gamma_2^2 \|u_2(\tau)\|^2 - \|y_2(\tau)\|^2 d\tau.$$

Consider the feedback interconnection described by the equations $u_1 = \pm y_2$ and $u_2 = \pm y_1$ and assume $\gamma_1\gamma_2 < 1$.

Note that the *small gain condition* $\gamma_1\gamma_2 < 1$ implies that there exists α such that

$$\gamma_1 < \alpha < \frac{1}{\gamma_2} \quad \Leftrightarrow \quad \alpha^2 > \gamma_1^2 \ \text{ and } \ 1 > \alpha^2\gamma_2^2$$

The Small Gain Theorem

$$S_1(x_1(t_1)) - S_1(x_1(t_0)) \leq \frac{1}{2}\int_{t_0}^{t_1} \gamma_1^2\|y_2(\tau)\|^2 - \|y_1(\tau)\|^2 d\tau$$

$$+$$

$$\alpha^2 \times \qquad S_2(x_2(t_1)) - S_2(x_1(t_0)) \leq \frac{1}{2}\int_{t_0}^{t_1} \gamma_2^2\|y_1(\tau)\|^2 - \|y_2(\tau)\|^2 d\tau$$

$$=$$

$$\overbrace{S_1(x_1(t_1)) + \alpha^2 S_2(x_2(t_1))}^{S(x(t_1))} - \overbrace{S_1(x_1(t_0)) + \alpha^2 S_2(x_2(t_0))}^{S(x(t_0))} \leq$$

$$\leq \frac{1}{2}\int_{t_0}^{t_1} [\alpha^2\gamma_2^2 - 1]\|y_1(\tau)\|^2 + [\gamma_1^2 - \alpha^2]\|y_2(\tau)\|^2 d\tau$$

$$\leq -\frac{1}{2}\int_{t_0}^{t_1} \epsilon_1\|y_1(\tau)\|^2 + \epsilon_2\|y_2(\tau)\|^2 d\tau$$

for some $\epsilon_i > 0$.

The Small Gain Theorem

Theorem (The Small Gain Theorem)

Consider Σ_1 and Σ_2 interconnected by the equations $u_1 = \pm y_2$ and $u_2 = \pm y_1$.

Suppose Σ_1 and Σ_2 have L_2-gain smaller or equal to $\gamma_1 > 0$ and $\gamma_2 > 0$ and $\gamma_1 \gamma_2 < 1$.

Suppose, in addition, that S_1 and S_2 are differentiable and have strict local minimizer at $x_1^\star = 0$ and $x_2^\star = 0$ and that are zero-state detectable.

Then the equilibrium $(x_1^\star, x_2^\star) = (0, 0)$ of the interconnected system is asymptotically stable.

Corollary (Robust Stability)

Consider the affine system Σ_a and assume $x = 0$ is an equilibrium.

Assume the system has L_2-gain smaller or equal to $\gamma > 0$ with a differentiable storage function having a strict minimizer at $x = 0$.

Consider the perturbed model described by the equation $\dot{x} = f(x) + g(x)\Delta h(x)$.

Then the zero equilibrium of the perturbed model is locally asymptotically stable for all Δ such that

$$\sigma_{\max}(\Delta) < \frac{1}{\gamma}.$$

Corollary

Consider a system Σ. If Σ is strictly output passive, then it has finite L_2-gain.

Exercise

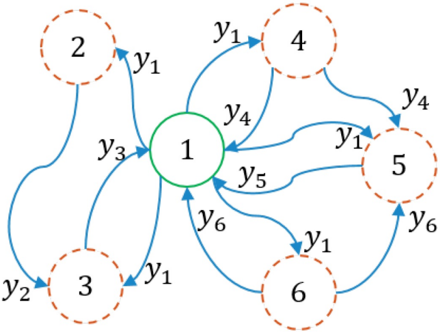
Prove the above Corollary.

The Small Gain Theorem for Large Scale Systems

A major disadvantage of the small gain theorem is that it is formulated for the feedback interconnection of only two systems.

Complex systems, however, result from the interconnection of several systems which form what we could call a network of interconnected systems.

One could provide a small gain theorem for large scale interconnected systems in the same spirit as the passivity theorem that we have presented.



Consider a network of k interconnected dynamical systems in which each node, a dynamical system denoted as Σ_i, has $k-1$ inputs $u_{ji}(t) \in \mathbb{R}$, one output $y_i(t) \in \mathbb{R}$, and satisfies the L_2-like differential dissipation inequality

$$\dot{V}_i \leq -a_i y_i^2 + \sum_{j=1}^{i-1} b_{ji} u_{ji}^2 + \sum_{j=i+1}^{k} b_{ji} u_{ji}^2, \tag{9}$$

with respect to a differentiable storage function V_i and $a_i > 0$ and $b_{ji} \geq 0$.

The nodes are interconnected via the equations $u_{ji} = y_j$, for all $b_{ji} \neq 0$, whereas $b_{ji} = 0$ means that Σ_j is not connected to Σ_i, that is $u_{ji} = 0$.

In practice one is interested in the dissipation inequality for the overall network system.

More specifically, one may wonder whether there exist positive scaling coefficients c_1, $\cdots$, c_k such that the storage function of the overall network system, defined as

$$V = \sum_{i=1}^{k} c_i V_i,$$

satisfies the dissipation inequality $\dot{V} \leq 0$ or

$$\dot{V} \leq -W(y) \leq 0. \tag{10}$$

Definition (*Z-matrix and M-matrix*)

A matrix M is a *Z-matrix* if all its off-diagonal elements are non-positive: $(M)_{ij} \leq 0$, $i \neq j$.

An *M-matrix* is a Z-matrix with all eigenvalues having nonnegative real part.

Theorem

Let M be a Z-matrix. Then the following conditions are equivalent.

(C1) *M is a non-singular M-matrix.*

(C2) *All principal minors of M are positive.*

(C3) *All leading principal minors of M are positive.*

(C4) *There exists a vector $v > 0$ such that $Mv > 0$.*

Note: the notation $v > 0$ used for a vector v means that all the component of v are positive, that is $v_i > 0$, for all i.

Theorem

These exists a vector of scaling coefficients $c = [c_1, \ldots, c_k]^\top > 0$ *such that* V *satisfies condition* (10) *if*

$$M = \begin{bmatrix} a_1 & -b_{12} & \cdots & -b_{1k} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -b_{k1} & \cdots & -b_{k(k-1)} & a_k \end{bmatrix} \tag{11}$$

is a non-singular M-matrix.

Proof.

Note that

$$\dot{V} = \sum_{i=1}^{k} c_i \dot{V}_i \leq -\phi^\top Mc \leq 0, \tag{12}$$

where $\phi = [y_1^2, \ldots, y_k^2]^\top$.

As a result, as long as there exists a vector $c > 0$ such that $Mc > 0$ the dissipation inequality $\dot{V} \leq -\phi^\top Mc = -W(y) \leq 0$ holds.

The conclusion is therefore obtained invoking condition (C4). □

The Small Gain Theorem for Large Scale Systems

The condition expressed by the theorem is not generic: it is straightforward to build networks for which it is not satisfied.

One such a network is a simple-loop containing two scalar nodes that violate the small-gain condition, that is either the condition $\frac{b_{21}}{a_1}\frac{b_{12}}{a_2} < 1$ or, equivalently, the condition $a_1 a_2 - b_{12} b_{21} > 0$.

The small-gain analysis for the above example reveals the fact that if one is allowed to adjust the coefficients a_i's arbitrarily, one can always enforce condition (10), provided that there is a distributed controller on each of the nodes of the network to make the a_i's tunable design parameters.

In practice, however, this is not always feasible, for example because of dynamics that cannot be controlled, or economical concerns do not allow using as many distributed controllers as nodes.

How many controllers are needed to enforce condition (10) and where these should be placed considering the structure of the network?

How to tune the design parameters a_i's of those nodes that can be actively controlled?

Definition

A node Σ_{i_a} is called an *active node* if it satisfies the dissipation inequality (9) with an adjustable $a_{i_a} \in [\underline{a}_{i_a}, +\infty)$, with $\underline{a}_{i_a} > 0$, where i_a is the node index of the *active node*.

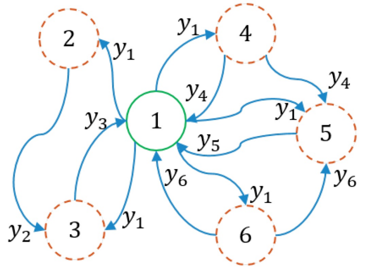
Theorem

The matrix (11) *can be made a non-singular M-matrix by adjusting the parameters* a_{i_a} *provided that every directed cycle of the underlying directed graph describing the network contains at least one vertex corresponding to an active node.*

The above theorem provides a sufficient design condition which only relies on the network structure.

Exercise

Consider the network control system in the figure below in which a solid green circle indicates an active node *and a red dashed circle indicates a* non-active *node.*



Show that the small gain, network-based, design condition holds.

Parametric Models

The availability of mathematical models is essential for the design of adaptive estimators and controllers.

For simplicity we focus on models for linear, time-invariant, systems, since this allows using both state space, and frequency-domain (Laplace-domain) tools and formalism. Most of the ideas and tools can be however used also for classes of nonlinear systems.

In addition, we restrict our attention almost exclusively to single-input, single-output (SISO) systems, to avoid the additional complication arising from handling matrices of transfer functions.

The essential feature that we plan to exploit is the possibility to separate measured signals/well-modelled systems from unknown parameters and states/unmodelled dynamics to obtain parametric models with special structure and properties.

Consider a SISO linear system described by the equations

$$\dot{x} = Ax + Bu, \qquad y = Cx,$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, and A, B and C matrices of appropriate dimension.

The triple (A, B, C) is described by $n^2 + 2n$ coefficients which are often referred to as plant parameters. Note, however, that if the coordinates x are selected such that the system is in reachability or observability canonical form then n^2 parameters are either 0 or 1, thus leaving $2n$ parameters to specify the system.

These $2n$ parameters are the coefficients of the numerator and denominator polynomials of the transfer function

$$\frac{y(s)}{u(s)} = G(s) = C(sI - A)^{-1}B = \frac{Z(s)}{R(s)}, \tag{13}$$

where

$$Z(s) = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0, \qquad R(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0.$$

Finally, in the Laplace-domain, the model of the system is

$$y(s) = \frac{Z(s)}{R(s)}u(s) + C(sI - A)^{-1}x_0, \qquad x_0 = x(0).$$

A given input-output model can be described by more than $2n$ parameters: in this case we say that the model is over-parameterized or that the parameterization is non-minimal.

For example, the model

$$y(s) = \frac{Z(s)}{R(s)} \, \frac{\Lambda(s)}{\Lambda(s)} u(s),$$

with $\Lambda(s)$ any Hurwitz polynomial of degree $r \geq 1$, has the same input-output properties of the considered system and it is therefore over-parameterized.

Note, however, that the responses to a nonzero initial state are not the same and that any state-space realization of order $n + r$, with $r \geq 1$, is non-minimal.

Parameterizations are strongly linked to specific problems: some parameterizations may be more convenient than others. In particular, a parameterization that is particularly useful is one in which parameters are lumped together and separated from signals, since both in parameter estimation and adaptive control the parameters are the unknown (constant) to be estimated from measurements of input and output signals.

The model (13) can be described by the n-th order differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots a_0 y = b_{n-1}u^{(n-1)} + b_{n-2}y^{(n-2)} + \cdots b_0 u, \tag{14}$$

where the notation $\xi^{(r)}$ indicates the r-th order time derivative of the signal ξ.

Let

$$\theta = \begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 & a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix}^\top = \begin{bmatrix} \theta_1^\top & \theta_2^\top \end{bmatrix}^\top$$

and

$$Y = \begin{bmatrix} u^{(n-1)} & \cdots & u & -y^{(n-1)} & \cdots & -y \end{bmatrix}^\top = \begin{bmatrix} \alpha_{n-1}^\top(s)u & -\alpha_{n-1}^\top(s)y \end{bmatrix}^\top$$

with $\alpha_i(s) = \begin{bmatrix} s^i & s^{i-1} & \cdots & 1 \end{bmatrix}^\top$.

The differential equation (14) can be written as

$$y^{(n)} = \theta^\top Y \tag{15}$$

which is linear in θ, a property that is instrumental to develop adaptive estimation and control schemes.

The parameterized model (15) allows estimating θ from measurements of $y^{(n)}$ and Y.

In practice, however, it is not reasonable to assume that derivatives of u and y are measured, or can be computed. One way to avoid measuring derivatives, or performing differentiation, is to *filter* each side of (15) with an n-order stable filter $\dfrac{1}{\Lambda(s)}$ yielding

$$\boxed{z = \theta^\top \phi = \theta_1^\top \phi_1 + \theta_2^\top \phi_2,} \tag{16}$$

where

$$z = \frac{1}{\Lambda(s)} y^{(n)} = \frac{s^n}{\Lambda(s)} y, \qquad \phi = \begin{bmatrix} \phi_1^\top & \phi_2^\top \end{bmatrix}^\top = \begin{bmatrix} \dfrac{\alpha_{n-1}^\top(s)}{\Lambda(s)} u & -\dfrac{\alpha_{n-1}^\top(s)}{\Lambda(s)} y. \end{bmatrix}^\top$$

and

$$\Lambda(s) = s^n + \lambda_{n-1} s^{n-1} + \cdots + \lambda_0$$

any Hurwitz polynomial.

Note that the signals z and ϕ can be generated by filtering the input u and output y with stable filters with transfer functions $\frac{s^i}{\Lambda(s)}$, with $i = 0, 1, \cdots, n$.

Note now that
$$\Lambda(s) = s^n + \lambda^\top \alpha_{n-1}(s),$$
with $\lambda = \begin{bmatrix} \lambda_{n-1} & \cdots & \lambda_0 \end{bmatrix}^\top$, hence
$$z = \frac{s^n}{\Lambda(s)} y = \frac{\Lambda(s) - \lambda^\top \alpha_{n-1}(s)}{\Lambda(s)} y = y - \lambda^\top \frac{\alpha_{n-1}(s)}{\Lambda(s)} y$$
yielding the linearly parameterized model
$$\boxed{y = z + \lambda^\top \frac{\alpha_{n-1}(s)}{\Lambda(s)} y = \theta_1^\top \phi_1 + (\theta_2 - \lambda)^\top \phi_2 = \begin{bmatrix} \theta_1^\top & (\theta_2 - \lambda)^\top \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \theta_\lambda^\top \phi,}$$
which shows a *separation* between measured and filtered signals and unknown parameters.

To obtain a state-space representation of the linearly parameterized model define

$$\Lambda_c = \begin{bmatrix} -\lambda_{n-1} & -\lambda_{n-2} & \cdots & -\lambda_1 & -\lambda_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad \ell = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and note that

$$\ell\Lambda(s) = (sI - \Lambda_c)\alpha_{n-1}(s) \quad \Leftrightarrow \quad (sI - \Lambda_c)^{-1}\ell = \frac{\alpha_{n-1}(s)}{\Lambda(s)} = \frac{\alpha_{n-1}(s)}{\det(sI - \Lambda_c)}.$$

$(sI - \Lambda_c)^{-1}\ell$ is the transfer function of a system with "A" matrix equal to Λ_c, "B" matrix equal to ℓ, and "C" matrix equal to I.

Therefore a state space realization of the system with input u and outputs y and z is

$$
\begin{aligned}
\dot{\phi}_1 &= \Lambda_c \phi_1 + \ell u, \\
\dot{\phi}_2 &= \Lambda_c \phi_2 - \ell y, \\
y &= \theta_1^\top \phi_1 + (\theta_2 - \lambda)^\top \phi_2, \\
z &= y + \lambda^\top \phi_2.
\end{aligned}
$$

This state space representation is non-minimal: it is of order $2n$ to describe an n-dimensional system. The transfer function from u to y is

$$\frac{y(s)}{u(s)} = \frac{Z(s)}{R(s)} \frac{\Lambda(s)}{\Lambda(s)} = \frac{Z(s)}{R(s)}.$$

This state space representation has the same input-output behavior of the original plant, provided all initial states are equal to zero, that is $x(0) = 0$, $\phi_1(0) = \phi_2(0) = 0$.

Non-zero values of $x(0)$ can be accounted for using the model

$$y(s) = \frac{Z(s)}{R(s)}u(s) + C(sI - A)^{-1}x_0, \qquad\qquad x_0 = x(0)$$

and repeating the same procedure. This yields the state space representation

$$\begin{aligned}
\dot{\phi}_1 &= \Lambda_c\phi_1 + \ell u, \\
\dot{\phi}_2 &= \Lambda_c\phi_2 - \ell y, \\
y &= \theta_1^\top\phi_1 + (\theta_2 - \lambda)^\top\phi_2 + \eta_0, \\
z &= y + \lambda^\top\phi_2 + \eta_0,
\end{aligned}$$

with η_0 generated by the autonomous system

$$\dot{\omega} = \Lambda_c\omega, \qquad \eta_0 = C_0\omega, \qquad \omega(0) = B_0x(0),$$

where B_0 and C_0 are such that $C_0\mathrm{adj}(sI - \Lambda_c)B_0 = C\mathrm{adj}(sI - A)$.

Non-zero initial conditions generate exponentially decaying terms in the outputs y and z.

Exercise

Consider the nonlinear scalar system

$$\dot{x} = a_0 f(x,t) + b_0 g(x,t) + c_0 u,$$

with a_0, b_0 *and* c_0 *scalar constants.*

Suppose that for any $x(0)$ *and* u *the state* x *is well defined for all* $t \geq 0$.

Suppose, in addition, that x *and* u *are measured and that* f *and* g *are known.*

Show that, by filtering each side, one can obtain a parametric model of the form

$$z = W_f(s)\theta^\top \phi,$$

with $z = sW_f(s)x$ *and* $W_f(s)$ *any strictly proper and stable transfer function.*

Bilinear Parametric Models

Consider a linear SISO system with input-output behaviour described by the relation

$$y(s) = k_0 \frac{Z_0(s)}{R(s)} u(s),$$

where k_0 is a scalar, $R(s)$ is a monic polynomial of degree n, and $Z_0(s)$ is a monic and Hurwitz polynomial of degree $m < n$.

Assume that $Z_0(s)$ and $R(s)$ satisfy the Diophantine equation

$$k_0 Z_0(s) P(s) + R(s) Q(s) = Z_0(s) A(s),$$

in the unknown polynomials

$$Q(s) = s^{n-1} + q^\top \alpha_{n-2}(s), \qquad q \in \mathbb{R}^{n-1},$$

$$P(s) = p^\top \alpha_{n-1}(s), \qquad p \in \mathbb{R}^n$$

where $A(s)$ is monic and Hurwitz polynomial of degree $2n - m - 1$.

The coefficient k_0 is often referred to as the high-frequency gain of the system.

The Diophantine equation relates k_0, $Z_0(s)$ and $R(s)$ to $P(s)$, $Q(s)$ and $A(s)$ which typically arise in control design. In particular the roots of $Z_0(s)A(s)$ are the poles of the closed-loop system obtained from the unity feedback interconnection of the plant with transfer function $k_0 \dfrac{Z_0(s)}{R(s)}$ with the controller with transfer function $\dfrac{P(s)}{Q(s)}$.

This interpretation justifies the assumptions that $Z_0(s)$ is Hurwitz, that is the system is minimum-phase, and $A(s)$ is also Hurwitz.

The Diophantine equation thus allows, for a given $A(s)$, computing the controller transfer function $\dfrac{P(s)}{Q(s)}$ which assign partially user-selected closed-loop poles.

In the case in which the plant parameters are not know one could use the Diophantine equation to directly estimate the controller, rather than estimating the plan parameter (from which one could then derive a controller).

These two approaches, that is estimating the plant parameters to construct the controller on the basis of the estimated parameters and estimating directly the controller parameters, give rise to what are known as direct and indirect adaptive schemes, respectively.

The plant equation gives

$$R(s)y(s) = k_0 Z_0(s)u(s) \quad \overset{\times Q(s)}{\Longrightarrow} \quad Q(s)R(s)y(s) = k_0 Z_0(s)Q(s)u(s)$$

whereas the Diophantine equation can be rewritten as

$$Q(s)R(s) = Z_0(s)\left(A(s) - k_0 P(s)\right).$$

Thus

$$Z_0(s)\left(A(s) - k_0 P(s)\right) y(s) = k_0 Z_0(s)Q(s)u(s)$$

and, since $Z_0(s)$ is Hurwitz,

$$A(s)y(s) = k_0 P(s)y(s) + k_0 Q(s)u(s)$$

and

$$\boxed{A(s)y(s) = k_0\left[p^\top \alpha_{n-1}(s)y(s) + q^\top \alpha_{n-2}(s)u(s) + s^{n-1}u(s)\right].}$$

The parametric model

$$\boxed{A(s)y(s) = k_0 \left[p^\top \alpha_{n-1}(s) y(s) + q^\top \alpha_{n-2}(s) u(s) + s^{n-1} u(s) \right].}$$

can be filtered with the stable filter $A(s)$ to yield

$$y(s) = k_0 \left[p^\top \frac{\alpha_{n-1}(s)}{A(s)} y(s) + q^\top \frac{\alpha_{n-2}(s)}{A(s)} u(s) + \frac{s^{n-1}}{A(s)} u(s) \right],$$

which we can rewrite as

$$\boxed{y = k_0 \left(\theta^\top \phi + z_0 \right)}$$

where

$$\theta = \begin{bmatrix} q^\top & p^\top \end{bmatrix}^\top, \qquad \phi = \begin{bmatrix} \frac{\alpha_{n-2}^\top(s)}{A(s)} u(s) & \frac{\alpha_{n-1}^\top(s)}{A(s)} y(s) \end{bmatrix}^\top, \qquad z_0 = \frac{s^{n-1}}{A(s)} u(s).$$

In the parametric model

$$\boxed{y = k_0\left(\theta^\top \phi + z_0\right)}$$

ϕ and z_0 can be generated by filtering the input and output signals. Thus, the only unknown are θ and k_0.

If k_0 is known, it could be absorbed in ϕ to give a model that is linear in θ.

If k_0 is unknown then the model is *bilinear* in the unknown parameters θ and k_0.

As for the first, linear, parameterization, this approach can be used to derive bilinear models for some classes of nonlinear systems.

We have proposed two parameterizations which rely on measurement of input and output signals to generate a non-minimal description of a system which is linear or bilinear in the unknown parameters.

There are several alternative parameterizations that one could consider, all based on similar ideas, such as the filtering of the available signals. Not all parameterizations yield models which are linear/bilinear in the unknown parameters.

The proposed parameterizations do not take into account any a priori information on the parameters, such as boundedness. For example, if one wishes to parameterize the transfer function

$$G(s) = \frac{1}{s^2 + as + 1}, \qquad a \in [a_{min}, a_{max}],$$

with $a_{min} > 0$, then one has to resort to alternative methods or to settle for a very conservative description of the system.

If the uncertainty is dynamic, for example one wishes to parameterize the transfer function

$$G(s) = e^{-\tau s}\frac{1}{s+1}, \qquad \tau \in [0, \bar{\tau}],$$

with $\bar{\tau} > 0$, then, again, an alternative parametric description has to be considered, since the uncertainty cannot be captured by an unknown parameter.

Consider a linear, single-input, single-output, system with transfer function $G(s)$.

The transfer function $G(s)$ describes the nominal model, whereas the actual system is described by the perturbed model

$$\tilde{G}(s) = \Big(1 + \Delta(s) W_2(s)\Big)\, G(s),$$

where $W_2(s)$ is a fixed, stable, transfer function and $\Delta(s)$ is a *variable*, stable, transfer function such that $\|\Delta(s)\|_\infty \leq 1$.

The idea behind this uncertainty model is that $\Delta(s)W_2(s)$ is the normalized plant perturbation away from 1, that is

$$\frac{\tilde{G}(s)}{G(s)} - 1 = \Delta(s)W_2(s) \qquad \Rightarrow \qquad \left|\frac{\tilde{G}(j\omega)}{G(j\omega)} - 1\right| \leq |W_2(j\omega)|, \quad \forall \omega.$$

Recall that the ∞-norm of a stable transfer function $G(s)$ is the induced L_2-norm, hence dissipative systems theory and the Small Gain Theorem can be used to study properties of a system subject to a perturbation satisfying an ∞-norm bound.

The perturbation $\Delta(s)$ is allowable if $G(s)$ and $\tilde{G}(S)$ have the same unstable poles.

The inequality

$$\left|\frac{\tilde{G}(j\omega)}{G(j\omega)} - 1\right| \leq W_2(j\omega)|, \quad \forall\omega$$

describes a disk in the complex plane: for each frequency ω the point $\dfrac{\tilde{G}(j\omega)}{G(j\omega)}$ lies in a disk with center 1 and radius $|W_2(j\omega)|$.

In general, $|W_2(j\omega)|$ is an increasing function of ω, since uncertainty increases as the frequency increases.

The role of Δ is to account for phase uncertainty and to act as scaling factor on the magnitude of the perturbation.

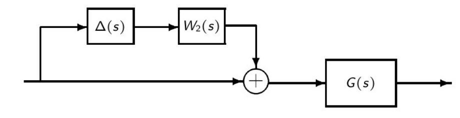
The uncertainty model is characterized by the nominal plant $G(s)$ and the weight $W_2(s)$.

Typically the nominal plant is the plant in which parameters have been set to the nominal value and *dynamic perturbations*, such as high frequency modes or delays, have been neglected.

The weight is more difficult to determine and has to be identified on a case by case basis.

Exercise

Show that the perturbed model $\tilde{G}(s)$ can be described via the block diagram below.



Exercise

Consider the transfer function

$$\tilde{G}(s) = \frac{1}{s^2 + as + 1}, \qquad a \in [0.1, 10].$$

Let the nominal model be

$$G(s) = \frac{1}{s^2 + \frac{10+0.1}{2} s + 1}.$$

Compute the weight W_2.

Exercise

Consider a nominal plant with transfer function

$$G(s) = \frac{1}{s^2}$$

and a perturbed plant with transfer function

$$\tilde{G}(s) = e^{-\tau s}\,\frac{1}{s^2},$$

with $\tau \in [0, 0.1]$*. This time-delay can be regarded as a multiplicative perturbation provided the weight* $W_2(s)$ *satisfies the condition*

$$|e^{-j\omega\tau} - 1| \leq |W_2(j\omega)|, \quad \forall\omega, \quad \forall\tau \in [0, 0.1].$$

Show, using Matlab or a similar SW, that the weight $W_2(s) = \dfrac{0.21s}{0.1s+1}$ *satisfies this condition.*

Multiplicative Perturbations

Exercise

Consider a nominal plant with transfer function $G(s) = \dfrac{k_0}{s-2}$ *and a perturbed plant with transfer function* $\tilde{G}(s) = \dfrac{k}{s-2}$. *with* $k \in [0.1, 10]$. *Let* $W_2(s)$ *be constant. Determine* k_0 *to minimize the weight* $W_2(s)$.

Exercise

Consider, again, the family of plants with transfer function

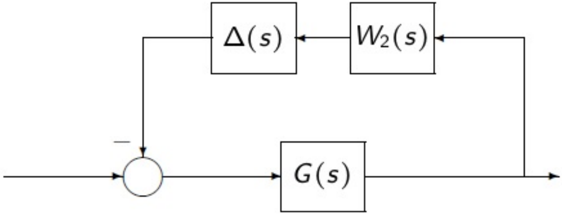
$$G(s) = \frac{1}{s^2 + as + 1}, \qquad a \in [a_{min}, a_{max}],$$

with $a_{min} > 0$. *Let* $\bar{a} = \frac{a_{min}+a_{max}}{2}$.
Show that the family can be described by the model

$$\tilde{G}(s) = \frac{G(s)}{1 + \Delta(s)W_2(s)G(s)}$$

with $G(s) = \dfrac{1}{s^2 + \bar{a}s + 1}$, $|\Delta| \leq 1$ *and* $W_2(s) = \dfrac{a_{max} - a_{min}}{2}$.

As demonstrated by the last exercise, not all families of perturbed models are best represented by a multiplicative perturbation. An alternative family of perturbations is given by the so-called feedback perturbation, as represented in the figure below.



This describes an alternative parameterization of a family of models which relies on a feedback structure, with the perturbation $\Delta(s)$ and the weight $W_2(s)$ stable and $\Delta(s)$ such that $\|\Delta(s)\|_\infty \leq 1$.

Exogeneous Signals

Consider the scenario in which the system to be controlled may be affected by disturbances, and the output of the system does not have to be regulated to zero, but should asymptotically track a certain, prespecified, reference signal.

This scenario can be modelled introducing a signal $d(t)$, denoted exogeneous signal, which is in general composed of two components: the former models a set of disturbances acting on the system to be controlled, the latter a set of reference signals.

Classically, it is assumed that the exogeneous signal is generated by a linear system, denoted exosystem, described by the equation

$$\dot{d} = Sd, \tag{17}$$

with S a matrix with constant entries.

Under this assumption it is possible to generate, for example, constant or polynomial references/disturbances and sinusoidal references/disturbances with any given frequency.

In addition, the exosystem is linearly parameterized in the matrix S, hence adaptive estimators can be used to estimate some properties of the exogeneous signal..

Conclusions

We have introduced the basic tools for the analysis and design of adaptive systems and robust control systems.

We will see that the properties of passivity and L_2-gain allows formulating and solving adaptive stabilization, robust stabilization and robust performance problems.

Finally, the parameterization of external disturbances and reference via an exosystem allows solving exact tracking and regulation problems with internal stability.