Control in The Presence of Uncertainty

Adaptive Estimation and Control

Dispense del Corso di Controllo Robusto e Adattativo

A.A. 2019/2020

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Parameter Estimation

In the first part of the course we have introduced some basic analysis and design tools and we have seen how it is possible to construct models of linear, and some nonlinear, dynamical systems which are parameterized in terms of parameters and available, measured, signals.

In what follows we assume that the parameters are constant, that is their time derivative is identically equal to zero. Occasionally we may discuss the implication of the fact that parameters may vary with time.

The availability of dynamic models allows exploiting simple time-domain and frequency-domain techniques to (asymptotically) estimate the parameters using dynamical systems: the so-called adaptive estimators.

The use of dynamical systems as mechanism to perform the estimate lends itself to the on-line implementation of the proposed estimation schemes.

The essential idea behind any parameter estimation method is the comparison between the observed system response $y(t) = y(\theta, t)$ and the response $\hat{y}(\hat{\theta}, t)$ of a parameterized model, the structure of which is the same as the model that has generated $y(t)$.

The *comparison* is used to continuously update the parameter estimate $\hat{\theta}$ in such a way that $y(t)$ and $\hat{y}(\hat{\theta}, t)$ are close, for t sufficiently large.

Note that, without additional assumptions, it is in general not possible to conclude that $y(\theta, t) = \hat{y}(\hat{\theta}, t)$, for all $t \geq 0$, implies $\hat{\theta}(t) = \theta$, or even that $\hat{\theta}(t)$ is close to θ.

The on-line estimation procedure thus involves three steps:

- model parameterization;
- generation of the update law for $\hat{\theta}$;
- design of the input signal to guarantee convergence of $\hat{\theta}(t)$ to θ.

Note that the final step can be accomplished only if one is allowed to run experiments on the plant. However, we will see that while parameter convergence is important in estimation problem it is actually not important in adaptive control, since one could achieve adaptive stabilization without recovering the actual value of the parameter θ.

Parameter Estimation – One Unknown Parameter

Consider the simplest possible linear, SISO, plant, namely the plant described by the input-output relation

$$y(t) = \theta u(t),$$

with input $u \in L_\infty$, output $y(t)$ ($\in L_\infty$), and unknown scalar parameter θ.

Assume that u and y are measured. The goal of a parameter estimation algorithm is to generate an estimate for θ. If the measured data were noise-free, one could simply write

$$\hat{\theta}(t) = \frac{y(t)}{u(t)},$$

provided $u(t) \neq 0$, for all t. This naive approach is undesirable since the division may be ill-conditioned and the effect of measurement noise is amplified by the division. Note that one could consider alternative versions of this naive approach, such as

$$\hat{\theta}(t) = \frac{W(s)\,y}{W(s)\,u},$$

where $W(s)$ is any L_∞ stable transfer function (note that one could also use other operators, such as a moving average, or a windowed integrator or a time delay).

Recall that the notation $W(s)\,y$ is a shorthand for $\mathcal{L}^{-1}\big(W(s)\mathcal{L}(y(t))\big)$.

To generate a recursive, that is dynamic, scheme we construct a predicted output as

$$\hat{y}(t) = \hat{\theta}(t)u(t)$$

and the prediction, or estimation, error

$$\epsilon_1 = y - \hat{y} = y - \hat{\theta}u = (\theta - \hat{\theta})u = -\tilde{\theta}u,$$

where $\tilde{\theta} = \hat{\theta} - \theta$ is the parameter estimation error. Note that ϵ_1 is measurable and $\dot{\tilde{\theta}} = \dot{\hat{\theta}} = w$, where w is to be designed to provide the on-line parameter update law.

The update law $\dot{\hat{\theta}} = w$ is obtained by minimizing a cost criteria involving ϵ_1 with respect to $\hat{\theta}$. The simplest possible cost is

$$J(\hat{\theta}) = \frac{\epsilon_1^2}{2} = \frac{(y - \hat{\theta}u)^2}{2},$$

which can be continuously minimized using the gradient system

$$\dot{\hat{\theta}} = -\gamma\nabla J(\hat{\theta}) = \gamma(y - \hat{\theta}u)u = \gamma\epsilon_1 u = -\gamma\tilde{\theta}u^2,$$

where $\gamma > 0$ is the so-called adaptation gain.

To assess the properties of the proposed adaptation mechanism consider the Lyapunov function

$$V(\tilde{\theta}) = \frac{\tilde{\theta}^2}{2\gamma}$$

and recall that $\dot{\tilde{\theta}} = -\gamma u^2 \tilde{\theta}$, hence

$$\dot{V} = -u^2 \tilde{\theta}^2 = -\epsilon_1^2 \le 0,$$

which implies that the equilibrium $\tilde{\theta} = 0$ is stable and that $\epsilon_1 \in L_2$. In addition, since $u \in L_\infty$, $\dot{\tilde{\theta}} \in L_\infty \cap L_2$. This, however, does not guarantee convergence of $\dot{\tilde{\theta}}$ and ϵ_1 to zero.

If $\dot{u} \in L_\infty$ then $\dot{\epsilon}_1 \in L_\infty$ and $\ddot{\tilde{\theta}} \in L_\infty$, hence $\lim_{t \to \infty} \epsilon_1(t) = 0$ and $\lim_{t \to \infty} \dot{\tilde{\theta}}(t) = 0$.

Can we conclude convergence of $\tilde{\theta}(t)$ and, if so, what is its limit?

Recall that, along the trajectories of the system, $V(\tilde{\theta}(t))$ is non-increasing and it is bounded from below, since it is a positive definite function. Hence it has a limit, that is

$$\lim_{t\to\infty} V(\tilde{\theta}(t)) = V_\infty \qquad \Rightarrow \qquad \lim_{t\to\infty} \hat{\theta}(t) = \theta \pm \sqrt{2\gamma V_\infty}.$$

This implies that $\hat{\theta}(t)$ converges to the true value θ if and only if $V_\infty = 0$.

This condition is not easy to verify, hence a different type of analysis is required.

Recall that $\dot{\tilde{\theta}} = -\gamma u^2 \tilde{\theta}$, hence

$$\tilde{\theta}(t) = e^{-\gamma \int_0^t u^2(\tau)d\tau} \tilde{\theta}(0).$$

As a result $\tilde{\theta}(t)$ converges to zero if and only if $u \notin L_2$. Note also that the estimate improves monotonically over time, that is $|\tilde{\theta}(t)|$ is monotonically non-increasing.

To have exponential convergence of $\tilde{\theta}(t)$ to zero one needs the more restrictive condition

$$0 < \alpha_0 T_0 \le \int_t^{t+T_0} u^2(\tau)d\tau,$$

for all $t \ge 0$, and some $\alpha_0 > 0$ and $T_0 > 0$.

Consider the scalar system

$$\dot{x} = -ax + bu,$$

where the parameters $a > 0$ and b are constant and unknown, and u and x are available for measurement. The condition $a > 0$ implies that for any $u \in L_\infty$ one has $x \in L_\infty$.

Our objective is to generate update laws, to update the parameter estimates $\hat{a}$ and $\hat{b}$, driven by the estimation error $\epsilon_1 = x - \hat{x}$, where $\hat{x}$ is an estimate of the state resulting from the use of $\hat{a}$ and $\hat{b}$ (consistently with a certainty equivalence architecture).

The state estimate $\hat{x}$ is generated by a *copy* of the equation of the system, that is

$$\dot{\hat{x}} = -\hat{a}\hat{x} + \hat{b}u. \qquad \text{(P)}$$

This equation is known as the *parallel model*. Alternatively, one could rewrite the plant as

$$\dot{x} = -a_m x + (a_m - a)x + bu \qquad \Leftrightarrow \qquad x(s) = \frac{1}{s + a_m}\left[(a_m - a)x + bu\right],$$

yielding the co-called *series parallel model*:

$$\dot{\hat{x}} = -a_m\hat{x} + (a_m - \hat{a})x + \hat{b}u \qquad \Leftrightarrow \qquad \hat{x}(s) = \frac{1}{s + a_m}\left[(a_m - \hat{a})x + \hat{b}u\right]. \qquad \text{(SP)}$$

The estimation error $\epsilon_1 = x - \hat{x}$ satisfies the differential equation

$$\dot{\epsilon}_1 = -a\epsilon_1 + \tilde{a}\hat{x} - \tilde{b}u$$

for the parallel model (P), and the differential equation

$$\dot{\epsilon}_1 = -a_m\epsilon_1 + \tilde{a}x - \tilde{b}u.$$

for the series parallel model (SP).

Consider now the storage function $S(\epsilon_1) = \frac{\epsilon_1^2}{2}$ and note that

$$\dot{S} = -a\epsilon_1^2 + \epsilon_1\tilde{a}\hat{x} - \epsilon_1\tilde{b}u = -a\epsilon_1^2 + \begin{bmatrix} \epsilon_1\hat{x} & -\epsilon_1 u \end{bmatrix}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix},$$

for the parallel model (P), and

$$\dot{S} = -a_m\epsilon_1^2 + \epsilon_1\tilde{a}x - \epsilon_1\tilde{b}u = -a_m\epsilon_1^2 + \begin{bmatrix} \epsilon_1 x & -\epsilon_1 u \end{bmatrix}\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix},$$

for the series parallel model (SP).

The time derivatives of the storage function reveal that the (P) and (SP) estimation error equations describe *passive systems* with input $(\tilde{a}, \tilde{b})$ and outputs $(\epsilon_1\hat{x}, -\epsilon_1 u)$ and $(\epsilon_1 x, -\epsilon_1 u)$, respectively.

Interconnecting each of these passive systems to the passive system $\dot{\xi} = v$, with $\xi(t) \in I\!R^2$ and $v(t) I\!R^2$ and storage function $S_\xi(\xi) = \frac{1}{2}\xi^\top\xi$, via the interconnection equations (note the negative feedback, consistent with the Passivity Theorem)

$$v = \begin{bmatrix} \epsilon_1\hat{x} \\ -\epsilon_1 u \end{bmatrix}, \qquad \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = -\xi, \qquad \text{(P)}$$

$$v = \begin{bmatrix} \epsilon_1 x \\ -\epsilon_1 u \end{bmatrix}, \qquad \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = -\xi, \qquad \text{(SP)}$$

yields

$$\dot{\epsilon}_1 = -a\epsilon_1 + \tilde{a}\hat{x} - \tilde{b}u, \qquad \dot{\tilde{a}} = -\epsilon_1\hat{x}, \qquad \dot{\tilde{b}} = \epsilon_1 u, \qquad \text{(P)}$$

and

$$\dot{\epsilon}_1 = -a_m\epsilon_1 + \tilde{a}x - \tilde{b}u, \qquad \dot{\tilde{a}} = -\epsilon_1 x, \qquad \dot{\tilde{b}} = \epsilon_1 u. \qquad \text{(SP)}$$

These equations are not implementable, because $\tilde{a}$ and $\tilde{b}$ are not known.

They, however, allow implementing the update laws (recall that $\dot{\tilde{a}} = \dot{\hat{a}}$ and $\dot{\tilde{b}} = \dot{\hat{b}}$)

$$\dot{\hat{a}} = -\epsilon_1 \hat{x}, \qquad \dot{\hat{b}} = \epsilon_1 u, \qquad (P)$$

$$\dot{\hat{a}} = -\epsilon_1 x, \qquad \dot{\hat{b}} = \epsilon_1 u. \qquad (SP)$$

For both systems the overall storage function is

$$S_T(\epsilon_1, \tilde{a}, \tilde{b}) = \frac{1}{2}\left(\epsilon_1^2 + \tilde{a}^2 + \tilde{b}^2\right)$$

and it is such that $\dot{S}_T = -a\epsilon_1^2 \le 0$ (P) and $\dot{S}_T = -a_m\epsilon_1^2 \le 0$ (SP) (recall that $a > 0$ and $a_m > 0$ and note that the (SP) system is passive *by design*).

We conclude that $\epsilon_1 \in L_2 \cap L_\infty$, $\tilde{a} \in L_\infty$, $\tilde{b} \in L_\infty$, and $\tilde{x} \in L_\infty$, provided $u \in L_\infty$. These imply

$$\lim_{t\to\infty} \epsilon_1(t) = 0, \qquad \lim_{t\to\infty} \dot{\hat{a}}(t) = 0, \qquad \lim_{t\to\infty} \dot{\hat{b}}(t) = 0, \qquad \lim_{t\to\infty} \left(\tilde{a}^2(t) + \tilde{b}^2(t)\right) = 2\lim_{t\to\infty} S(t),$$

but do not imply convergence of $\tilde{a}$ and/or $\tilde{b}$, which requires additional assumptions on u.

This procedure is called *signal chasing* and it is widely used in adaptive systems.

To summarize, we have seen two schemes for estimating the parameters of the plant

$$\dot{x} = -ax + bu,$$

with $a > 0$ and $u \in L_\infty$, namely

(P)

$$\dot{\hat{x}} = -\hat{a}\hat{x} + \hat{b}u$$
$$\dot{\hat{a}} = -(x - \hat{x})\hat{x}$$
$$\dot{\hat{b}} = (x - \hat{x})u$$

(SP)

$$\dot{\hat{x}} = -a_m\hat{x} + (a_m - \hat{a})x + \hat{b}u$$
$$\dot{\hat{a}} = -(x - \hat{x})x$$
$$\dot{\hat{b}} = (x - \hat{x})u$$

Note that the update laws can be scaled, by multiplying the right-hand sides by constants $\gamma_1 > 0$ and $\gamma_2 > 0$ and exploiting the storage function $S_T(\epsilon_1, \tilde{a}, \tilde{b}) = \frac{1}{2}\left(\epsilon_1^2 + \frac{\tilde{a}^2}{\gamma_1} + \frac{\tilde{b}^2}{\gamma_2}\right)$.

The two schemes differ in their performance in the presence of noise. If x is corrupted by noise, that is one measures $x + \nu$ then the scheme (SP) may generate *biased* estimates because of the presence of the x^2 term in the $\dot{\hat{a}}$ equation.

Exercise (Assignment 1)

Consider the system

$$\dot{x} = -ax + bu.$$

Let $a = 1$ *and* $b = 2$. *Run simulations for the adaptive estimation algorithms based on the (P) and (SP) models.*

Evaluate the performance of the algorithms for various input signals, as a function of the adaptive gains, and of the presence, or otherwise, of noise on the measurements.

Suppose that a *is slowly time varying, that is* $a = 1 + 0.1 \sin\left(\frac{2\pi t}{24 \times 3600}\right)$. *This is the case, for example, if* a *changes during the day as a function of the ambient temperature.*

Assess the performance of the designed adaptive estimation algorithms in this scenario.

Parameter Estimation – The General Case

Consider the linear system

$$\dot{x} = A_p x + B_p u,$$

with state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$, both available for measurement, and $A_p \in \mathbb{R}^{n\times n}$ and $B_p \in \mathbb{R}^{n\times m}$ unknown. Suppose that A_p is asymptotically stable and $u \in L_\infty$.

Define the parallel model (P)

$$\dot{\hat{x}} = \hat{A}_p \hat{x} + \hat{B}_p u,$$

with state $\hat{x}(t) \in \mathbb{R}^n$, where $\hat{A}_p(t) \in \mathbb{R}^{n\times n}$ and $\hat{B}_p(t) \in \mathbb{R}^{n\times m}$ are the estimates of A_p and B_p at time t to be generated by update laws.

Similarly we could consider the series parallel model (SP)

$$\dot{\hat{x}} = A_m \hat{x} + (\hat{A}_p - A_m)x + \hat{B}_p u.$$

In what follows we focus on the parallel model: similar considerations can be carried out for the series parallel model.

The estimation error equation for the parallel model is

$$\dot{\epsilon}_1 = A_p \epsilon_1 - \tilde{A}_p \hat{x} - \tilde{B}_p u,$$

where $\epsilon_1 = x - \hat{x}$, $\tilde{A}_p = \hat{A}_p - A_p$ and $\tilde{B}_p = \hat{B}_p - B_p$.

Consider the storage function $S(\epsilon_1) = \epsilon_1^\top P \epsilon_1$, where $P = P^\top > 0$ is the solution of the Lyapunov equation $A_p^\top P + PA_p = -I$, and note that

$$\dot{S} = \dot{\epsilon}_1^\top P \epsilon_1 + \epsilon_1 P \dot{\epsilon}_1 = \epsilon_1^\top (A_p^\top P + PA_p)\epsilon_1 - 2\epsilon_1^\top P \tilde{A}_p \hat{x} - 2\epsilon_1^\top P \tilde{B}_p u.$$

Recall the properties of the trace operator (A and B matrices, v and w vectors)

$$\mathrm{tr}(AB) = \mathrm{tr}(BA), \qquad \mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B), \qquad \mathrm{tr}(v\, w^\top) = w^\top v.$$

As a result

$$\dot{S} = -\epsilon_1^\top \epsilon_1 - 2\ \mathrm{tr}(\tilde{A}_p^\top P \epsilon_1 \hat{x}^\top) - 2\ \mathrm{tr}(\tilde{B}_p^\top P \epsilon_1 u^\top),$$

which suggests that the estimation error system is passive from the input $(\tilde{A}_p, \tilde{B}_p)$ to the output $(-P\epsilon_1 \hat{x}^\top, -P\epsilon_1 u^\top)$.

Consider now the system

$$\dot{\Xi} = W,$$

with $\Xi(t) \in R^{n\times q}$ and $W(t) \in I\!R^{n\times q}$, with $q = n$ or $q = m$, and the storage function

$$S_{\Xi} = \mathrm{tr}\left(\frac{\Xi^\top P \Xi}{\gamma}\right).$$

A direct computation yields

$$\dot{S}_{\Xi} = 2\ \mathrm{tr}\left(\frac{\Xi^\top P \dot{\Xi}}{\gamma}\right)$$

which suggests that the system is passive from the input $P\dot{\Xi} = PW$ to the output Ξ.

Exploiting the two passivity properties we conclude that the update laws

$$\dot{\hat{A}}_p = \gamma_1 \epsilon_1 \hat{x}^\top, \qquad \dot{\hat{B}}_p = \gamma_2 \epsilon_1 u^\top,$$

with $\gamma_1 > 0$ and $\gamma_2 > 0$ yield

$$\epsilon_1 \in L_2 \cap L_\infty, \qquad \hat{A}_p \in L_\infty, \qquad \hat{B}_p \in L_\infty, \qquad \hat{x} \in L_\infty, \qquad \lim_{t\to\infty} \epsilon_1 = 0.$$

Exercise

Show that the function

$$\mathrm{tr}\left(\frac{\Xi^\top M \Xi}{\gamma}\right)$$

with $M = M^\top > 0$ *is positive definite in* Ξ*, that is the function is zero for* $\Xi = 0$ *and positive otherwise.*

Exercise

Consider the series parallel model and derive the update laws for the estimates $\hat{A}_p$ *and* $\hat{B}_p$*.*

Show that, since the matrix A_m *is known, one could use the storage function*

$$S_\Xi = \mathrm{tr}\left(\frac{\Xi^\top \Xi}{\gamma}\right).$$

Explain why this storage function cannot be used to derive an update law in the case in which the parallel model is used.

Parameter Estimation – Sufficiently Rich Signals

Consider again the scalar system

$$\dot{x} = -ax + bu,$$

with a and b constant and unknown. Assume that $a > 0$ and that $u \in L_\infty$ and let $x(0)$ be the initial state.

Consider the abstract problem of determining for which classes of inputs it is possible, regardless of any specific algorithm, estimate a and b by processing input/output data, that is measurements of u and x.

Clearly, if $u = 0$ one has $x(t) = e^{-at}x(0)$, which shows that the measured signals do not carry any information on b and the information on a disappears exponentially fast. In principle, one could obtain a from the relation

$$a = -\frac{1}{T} \ln \frac{x(t)}{x(t-T)} > 0,$$

for $t \geq T$, where $T > 0$ is a user selected parameter. This relation becomes rapidly numerically ill-conditioned, it is extremely sensitive to measurement noise, and cannot be used if $x(0) = 0$.

Assume now that $u = u_0$, with u_0 constant. Then

$$x(t) = e^{-at}\left(x(0) - \frac{b}{a}u_0\right) + \frac{b}{a}u_0,$$

which reveals that one could estimate b/a (the DC gain of the system). Again one could estimate a using delayed measurements of $x(t)$ in what would be a non-robust scheme.

Assume, finally, that $u(t) = \sin\omega_0 t$, with $\omega_0 \neq 0$. Then

$$x(t) = A\sin(\omega_0 t + \phi) + exp, \qquad A = \frac{|b|}{|j\omega_0 + a|}, \qquad \phi = (sign(b) - 1)\frac{\pi}{2} - tan^{-1}\frac{\omega_0}{a},$$

where "exp" indicates exponentially decaying terms, which reveals that one could determine a and b from measurements of the steady state response of the system.

This example demonstrates that the on-line estimate of a and b is possible only if the output signal carries sufficient information on the parameters and, in turn, this information is the result of a particular selection of input signal.

Definition

A signal $u : I\!R^{\geq 0} \to I\!R$ is sufficient rich of order n if it contains at least $\frac{n}{2}$ distinct frequencies.

Exercise

Show that the signal $u(t) = \sum_{i=1}^{m} A_i \sin \omega_i t$*, with* $A_i \neq 0$*, for all* i*, and* $\omega_i \neq \omega_k$*, for all* $i \neq k$*, is sufficiently rich of order* n*, for any* $n \leq 2m$*.*

A more general definition of richness that includes signals which are not linear combination of sinusoids can also be given.

Definition

A signal $u : \mathbb{R}^{\geq 0} \to \mathbb{R}^m$ is said to be stationary if the limit

$$R_u(t) = \lim_{T \to \infty} \int_{t_0}^{t_0+T} u(\tau) u^\top (t+\tau) d\tau$$

exists, uniformly in t_0. The matrix $R_u(t)$ is called the auto-covariance of u.

The matrix $R_u(t)$ is positive semi-definite and it is Fourier transformable, that is

$$S_u(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} R_u(\tau) d\tau,$$

where $S_u(\omega)$ is the so-called spectral measure of u, is well defined.

Definition

A stationary signal $u : \mathbb{R}^{\geq 0} \to \mathbb{R}$ is sufficiently rich of order n if the support of the spectral measure $S_u(\omega)$ of u contains at least n points.

Parameter Estimation – Sufficiently Rich Signals

Definition

A signal $u : \mathbb{R}^{\geq 0} \to \mathbb{R}^m$ is *persistently exciting* if there exist a constant t_0 and positive constants T and α such that

$$\frac{1}{T}\int_t^{t+T} u(\tau)u^\top(\tau)d\tau \geq \alpha I > 0,$$

for all $t \geq t_0$.

Consider now the equation

$$\phi = H(s)u,$$

with $H(s)$ a proper and stable transfer function and $\phi(t) \in \mathbb{R}^n$.

Theorem

Let $u : \mathbb{R}^{\geq 0} \to \mathbb{R}$ be a stationary signal. Assume that $H(j\omega_1), \cdots, H(j\omega_n)$ are linearly independent (on $\mathbb{C}^n$) for all $\omega_i \in \mathbb{R}$, with $\omega_i \neq \omega_k$ for $i \neq k$. Then ϕ is persistently exciting if and only if u is sufficiently rich of order n.

Exercise

Consider the linear, single-input, single-output, system described by the relation $y(s) = kG(s)u(s)$*, with*

$$G(s) = \frac{s^2 + 1}{(s+1)^3},$$

and k *an unknown parameter.*

Show that the system can be parameterized as

$$y = \theta\,\phi,$$

where $\theta = k$ *and* $\phi = G(s)u$*.*

Show that if u *is sufficiently rich of order* 1 *then* ϕ *is not persistently exciting, while it is persistently exciting if* u *is sufficiently rich of order* $n \geq 2$*.*

Theorem

Consider the system

$$\dot{x} = A_p x + B_p u,$$

with $x(t) \in \mathbb{R}^n$ *and* $u(t) \in \mathbb{R}^m$. *Assume that the state* x *is measurable, that* $u \in L_\infty$, *and that* A_p *and* B_p *are constant matrices to estimate. Assume, in addition, that* A_p *is asymptotically stable and that the pair* (A_p, B_p) *is reachable.*

Consider the (P) and the (SP) parameter estimation algorithm.

Let $m = 1$. *If the input* u *is sufficiently rich of order* $n + 1$ *then the estimates* $\hat{A}_p$ *and* $\hat{B}_p$ *converge exponentially fast to the unknown parameters* A_p *and* B_p.

Let $m > 1$. *If each input* u_i *is sufficiently rich of order* $n + 1$ *and uncorrelated, that is each* u_i *contains different angular frequencies, then the estimates* $\hat{A}_p$ *and* $\hat{B}_p$ *converge exponentially fast to the unknown parameters* A_p *and* B_p.

The foregoing parameter estimation algorithms requires full state measurements. In practice this is unfeasible, hence one has to develop estimation methods which rely solely on input and output signals.

To this end consider the linear parametric model

$$z = W(s)\theta^\top \psi,$$

derived on the basis of input-output signals, and assume that $W(s)$ is stable and $\psi \in L_\infty$.

Since θ is a constant vector, for any transfer function $L(s)$ one could write the alternative, equivalent from an input-output perspective, model

$$z = W(s)L(s)\theta^\top \phi,$$

where $\phi = L^{-1}(s)\psi$.

Suppose $L(s)$ is such that $L^{-1}(s)$ is stable and proper and $W(s)L(s)$ is strictly proper and SPR. Note that it is always possible to manipulate the model to satisfy the above conditions: this may require the redefinition of the filtered signal z and ϕ.

Let $\hat{\theta}(t)$ be the estimate of θ at time t. Then the estimate of z is given by

$$\hat{z} = W(s)L(s)\hat{\theta}^\top \phi$$

and the estimation error is defined as

$$\epsilon = z - \hat{z} = W(s)L(s)(-\tilde{\theta}^\top \phi).$$

Let

$$\dot{e} = Ae + B(-\tilde{\theta}^\top \phi), \qquad\qquad \epsilon = Ce,$$

be a state-space representation of the *error* equation, *i.e.* $W(s)L(s) = C(sI - A)^{-1}B$.

By SPR-ness of the transfer function $W(s)L(s)$ there esist matrices $P = P^\top > 0$ and $L = L^\top > 0$ such that

$$A^\top P + PA = -L, \qquad PB = C^\top.$$

Consider now the storage function $S(e) = \dfrac{1}{2}e^\top Pe$ and note that

$$\dot{S} = -\frac{1}{2}e^\top Le + e^\top PB(-\tilde{\theta}^\top \phi) = -\frac{1}{2}e^\top Le + \epsilon(-\tilde{\theta}^\top \phi), = -\frac{1}{2}e^\top Le + (-\epsilon\phi^\top)\tilde{\theta}$$

which shows that the *error* system is strictly passive from the input $\tilde{\theta}$ to the output $-\epsilon\phi$.

Consider now the interconnection of the error system with the passive system

$$\dot{\xi} = \Gamma w$$

where $\Gamma = \Gamma^\top > 0$, via the interconnection equations

$$w = -\epsilon\phi^\top, \qquad \tilde{\theta} = -\xi.$$

These yield

$$\dot{\tilde{\theta}} = -\dot{\xi} = -\Gamma(-\epsilon\phi^\top) = \Gamma\phi\epsilon \qquad \Longrightarrow \qquad \dot{\hat{\theta}} = \Gamma\phi\epsilon.$$

One can therefore conclude that $e \in L_\infty$, $\epsilon \in L_\infty$, $\hat{\theta} \in L_\infty$ and $e \in L_2$.

Theorem

Suppose that $\dot{\phi} \in L_\infty$ and that ϕ is persistently exciting. Then $\hat{\theta}(t)$ converges to θ exponentially fast.

Exercise

Consider the system

$$y(s) = \frac{b_1 s + b_0}{(s+1)(s+2)} u(s),$$

where b_1 and b_2 have to be estimated.

Show that the system can be described by the parametric model

$$y = \frac{1}{(s+1)(s+2)} \theta^\top \psi = W(s)\theta^\top \psi,$$

where $\theta^\top = [b_1 \quad b_0]$ and $\psi = [\dot{u} \quad u]^\top$.

Select $L(s) = s + 2$ and show that $W(s)L(s)$ is SPR and that the parameterized model can be written as

$$y = \frac{1}{s+1} \theta^\top \phi.$$

Determine explicitly the term ϕ and show that is it is measurable.

Parameter Estimation – Gradient Identifiers

An alternative to the class of parameter identifiers which rely on the interconnection of passive systems can be obtained exploiting the gradient method. The idea of the gradient estimators is to develop an algebraic estimation error equation from which one could define a cost function to be continuously minimized.

Consider again the parameterized model

$$z = W(s)\theta^\top \psi$$

and note that, since θ is constant, one could write the equivalent model

$$z = \theta^\top \phi,$$

where $\phi = W(s)\psi$. This justifies the definition of the estimate $\hat{z} = \hat{\theta}^\top \phi$, of the estimation error

$$\epsilon = z - \hat{z} = \theta^\top \phi - \hat{\theta}^\top \phi = -\tilde{\theta}^\top \phi,$$

and of the instantaneous cost function

$$J(\hat{\theta}) = \frac{\epsilon^2}{2} = \frac{(z - \hat{\theta}^\top \phi)^2}{2}.$$

Note that other cost functions could be considered, for example a cost computed on the basis of a weighted integral of the square of the estimation error.

Note that the cost function $J(\hat{\theta})$ is quadratic in the unknown $\hat{\theta}$, hence it is (globally) convex and the gradient method provides a globally converging (under proper assumptions) minimization algorithm.

The algorithm is obtained by defining a trajectory that evolves on the basis of the differential equation

$$\dot{\hat{\theta}} = -\Gamma \nabla J(\hat{\theta}) = \Gamma(z - \hat{\theta}\phi)\phi = \Gamma\phi\epsilon,$$

where $\Gamma = \Gamma^\top > 0$.

The gradient method yields an adaptive law which has the same from as that obtained using the SPR design. This is the consequence of the fact that one could take $L(s) = W^{-1}(s)$ in the SPR design.

Theorem

Consider the gradient algorithm and assume $\phi \in L_\infty$. Then $\epsilon \in L_\infty \cap L_2$, $\hat{\theta} \in L_\infty$, $\dot{\hat{\theta}} \in L_\infty$, and $\dot{\hat{\theta}} \in L_2$. In addition, if ϕ is persistently exciting then $\hat{\theta}(t)$ converges to θ exponentially.

Proof.

We prove only the first claim. To begin with note that, since θ is constant, one has

$$\dot{\tilde{\theta}} = \Gamma\phi\epsilon.$$

Consider now the Lyapunov function $V(\tilde{\theta}) = \dfrac{\tilde{\theta}^\top \Gamma^{-1}\tilde{\theta}}{2}$ and note that $\dot{V} = \tilde{\theta}^\top \phi\epsilon = -\epsilon^2 \le 0$, from which we prove the claim. $\square$

Interestingly, the convergence proof of the gradient method does not use $J(\hat{\theta})$ as Lyapunov function, which may therefore be increasing. Explain why this is the case!

One major disadvantage of the gradient algorithm, hence also of the algorithm based on the SPR design, is that the parameter estimates are coupled, that is one can only prove that the function $V(\tilde{\theta}(t))$ is non-increasing, whereas one may be interested in a parameter-wise property, that is the property that each $|\tilde{\theta}_i(t)|$ be non-increasing. Before addressing this issue, we show that the gradient identifiers can incorporate prior knowledge on the parameters.

Suppose, now, that some a priori information on the parameter is available and that this can be expressed as $\theta \in \mathcal{S}$, where

$$\mathcal{S} = \{\theta \in \Theta \mid g(\theta) \leq 0\}.$$

Suppose, in addition, that $\mathcal{S}$ is a convex set.

The idea of the so-called gradient projection method is to modify the gradient-based adaptive scheme to guarantee that the parameter estimates, if initialized in S, that is $\hat{\theta}(0) \in \mathcal{S}$, provides an estimate that remains in $\mathcal{S}$ for all $t \geq 0$.

Let $\mathcal{S}^0$ be the interior of $\mathcal{S}$ and $\delta\mathcal{S}$ the boundary of $\mathcal{S}$.

To this end, let

$$\dot{\hat{\theta}} = \begin{cases} \Gamma\phi\epsilon, & \text{if } \hat{\theta}(t) \in \mathcal{S}^0 \text{or } \hat{\theta}(t) \in \delta\mathcal{S} \text{ and } (\Gamma\phi\epsilon)^\top \nabla g \leq 0, \\ \Gamma\phi\epsilon - \Gamma \dfrac{\nabla g \nabla^\top g}{\nabla g \Gamma \nabla^\top g} \Gamma\phi\epsilon, & \text{otherwise.} \end{cases}$$

Note that the *projection* term can be used with any parameter estimation algorithm.

Theorem

The gradient algorithm with projection retains all the properties established for the algorithm without projection and in addition it guarantees that $\hat{\theta}(t) \in \mathcal{S}$ *for all* $t \geq 0$, *provided* $\hat{\theta}(0) \in \mathcal{S}$.

Proof.

Note that $\hat{\theta}(t)$ can *escape* the set $\mathcal{S}$ only through its boundary $\delta\mathcal{S}$. Note now that on $\delta\mathcal{S}$ one has $\dot{\hat{\theta}}^\top \nabla g \leq 0$, that is the vector $\dot{\hat{\theta}}$ points either inside $\mathcal{S}$ or along the tangent plane to $\delta\mathcal{S}$ at any point of $\delta\mathcal{S}$. The conclusion therefore follows by convexity of $\mathcal{S}$.

Consider now the same Lyapunov function used for the method without project and note that the *projection* term is such that (recall that $\dot{\tilde{\theta}} = \dot{\hat{\theta}}$ and that $\tilde{\theta}^\top \nabla g = (\hat{\theta} - \theta)^\top \nabla g$)

$$- \underbrace{\tilde{\theta}^\top \Gamma^{-1} \Gamma \nabla g}_{\geq 0 \text{ by convexity of } \mathcal{S}} \times \underbrace{\frac{1}{\nabla g \Gamma \nabla^\top g} \nabla^\top g \Gamma \phi \epsilon}_{\geq 0 \text{ by definition}} \leq 0,$$

therefore the term introduced by the project can only make $\dot{V}$ more negative. □

The method of Dynamic Regression Extension and Mixing (DREM) is a method which allows obtaining decoupled estimators for each component of the parameter vector θ. This is achieved by filtering and post-processing the measured signals.

To begin with consider the parametric model $z = \theta^\top \phi$, with $\theta \in I\!R^q$.

The first step in DREM is to introduce a linear, single-input, q-output, bounded-input bounded-output stable filter $\mathcal{H}$ and to define the vector $\mathcal{Z}$ and the matrix $\Phi(t) \in I\!R^{q\times q}$ as

$$\mathcal{Z} = \mathcal{H}z, \qquad \Phi = \mathcal{H}\phi^T.$$

These signals satisfy the q relations

$$\mathcal{Z} = \Phi\theta.$$

Recall now that for any square matrix M one has $\mathrm{adj}\{M\}M = \det M\ I$. Hence

$$\mathrm{adj}\{\Phi\}\mathcal{Z} = \mathrm{adj}\{\Phi\}\Phi\ \theta = \det\Phi\ \theta$$

yielding, by setting $\tilde{\mathcal{Z}} = \mathrm{adj}\{\Phi\}\mathcal{Z}$, the q decoupled equations

$$\tilde{\mathcal{Z}}_i = \det\Phi\ \theta_i.$$

Parameter Estimation – DREM

Theorem

Consider the scalar equations $\tilde{\mathcal{Z}}_i = \det\Phi\ \theta_i$ *and the gradient estimator*

$$\dot{\hat{\theta}}_i = \gamma_i \det\Phi \left(\tilde{\mathcal{Z}}_i - \det\Phi\ \hat{\theta}_i\right),$$

with $\gamma_i > 0$*. Then the parameter estimation error* $\tilde{\theta}$ *is such that*

$$\dot{\tilde{\theta}}_i = -\gamma_i\ (\det\Phi)^2\,\tilde{\theta}_i \quad \Rightarrow \quad |\tilde{\theta}_i(t_1)| \leq |\tilde{\theta}_i(t_0)|, \quad t_1 \geq t_0.$$

Moreover

$$\lim_{t\to\infty} \tilde{\theta}_i(t) = 0 \quad \Leftrightarrow \quad \det\Phi \notin L_2.$$

The filter $\mathcal{H}$ can be constructed considering q filters of the form

$$H_i(s) = \frac{b_i}{s + \tau_i}, \quad \text{or} \quad H_i(s) = e^{-\tau_i s},$$

with $b_i \neq 0$, for all $i \in [1, q]$, $\tau_i > 0$, for all $i \in [1, q]$, and $\tau_i \neq a_j$ for all $[1, q] \ni i \neq j \in [1, q]$.

Exercise (Assignment 2)

Consider the system $\dot{x} = -ax + bu$, with $a > 0$, $b \neq 0$ and a and b unknown.

Determine a parameterization of the considered system and design a DREM parameter identifier and a gradient identifier.

Run simulations for the DREM and gradient estimators assuming $a = 0.4$, $b = 0.4$, and selecting, for the DREM estimator,

$$H_1(s) = \frac{1}{s+1}, \qquad H_2(s) = \frac{2}{s+2}.$$

Consider the cases $u(t) = 10$ and $u(t) = 10 \sin \frac{5t}{2}$. Compare the performance of the estimators in terms of transient response, speed of response and overall performance. Plot the phase portraits of the parameter estimation errors and comment on the results.

Consider the bilinear parametric model

$$y = k_0(\theta^\top \phi + z_0)$$

and assume, for simplicity, that the sign of k_0 is known and positive. The more general case in which the sign is unknown requires some modification and the introduction of what is called a Nussbaum-type function. Suppose in addition that all signals are bounded.

One naive way to deal with the parameter identification problem for the bilinear model is to rewrite it as

$$y = k_0\theta^\top \phi + k_0 z_0 = \begin{bmatrix} k_0\theta^\top & k_0 \end{bmatrix} \begin{bmatrix} \phi \\ z_0 \end{bmatrix}$$

and regard this as a re-parameterized linear model with parameters $\theta_1 = k_0\theta$ and $\theta_2 = k_0$. One can therefore generate estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ_1 and θ_2 and then derive estimates for θ and k_0 as

$$\hat{\theta} = \frac{\hat{\theta}_1}{\hat{\theta}_2}, \qquad \hat{k}_0 = \hat{\theta}_2.$$

This approach requires the knowledge of a lower bound on k_0 and the use of a projection-based algorithm, hence may not provide the best solution.

Consider, instead, the SPR design and rewrite the model as

$$z = W(s)L(s)k_0\left(\theta^\top\psi + z_1\right),$$

where $z_1 = L^{-1}(s)z_0$ and $\psi = L^{-1}(s)\phi$ and $L(s)$ is such that $L^{-1}(s)$ is proper and stable and $W(s)L(s)$ is strictly proper and SPR.

The estimate $\hat{z}$ of z and the estimation error are defined as

$$\hat{z} = W(s)L(s)\hat{k}_0\left(\hat{\theta}^\top\psi + \hat{z}_1\right), \qquad \epsilon = z - \hat{z}.$$

Note now that

$$\epsilon = W(s)L(s)\left(k_0\theta^\top\psi - \hat{k}_0\hat{\theta}^\top\psi - \tilde{k}_0 z_1\right),$$

which can be rewritten as

$$\epsilon = W(s)L(s)\left(-k_0\tilde{\theta}^\top\psi - \tilde{k}_0(\hat{\theta}^\top\psi + z_1)\right),$$

where $\tilde{k}_0 = \hat{k}_0 - k_0$ and $\tilde{\theta} = \hat{\theta} - \theta$.

Let

$$\dot{e} = Ae + B\left(-k_0\tilde{\theta}^\top\psi - \tilde{k}_0(\hat{\theta}^\top\psi + z_1)\right), \qquad \epsilon = Ce,$$

be a state-space representation of the error equation, *i.e.* $W(s)L(s) = C(sI - A)^{-1}B$.

By SPR-ness of the transfer function $W(s)L(s)$ there exist, similarly to the case of the SPR design for the linear model, matrices $P = P^\top > 0$ and $L = L^\top > 0$ such that $A^\top P + PA = -L$ and $PB = C^\top$.

One could therefore select the storage function $S(e) = \frac{1}{2}e^\top Pe$ and note that

$$\dot{S} = -\frac{1}{2}e^\top Le + e^\top PB(-k_0\tilde{\theta}^\top\psi - \tilde{k}_0(\hat{\theta}^\top\psi + z_1)) = -\frac{1}{2}e^\top Le + (-k_0\psi^\top\epsilon)\tilde{\theta} + ((-\hat{\theta}^\top\psi - z_1)\epsilon)\tilde{k}_0,$$

which shows that the error system is strictly passive from the inputs $\tilde{\theta}$ and $\tilde{k}_0$ to the outputs $-k_0\psi\epsilon$ and $(-\hat{\theta}^\top\psi - z_1)\epsilon$.

Observe that the first output is not measurable, since it contains the unmeasured parameter k_0. One, could however still use a passive interconnection using the trivial observation that $k_0\,\dfrac{1}{k_0} = 1$ and making sure that the passive system used to interconnect the first output to the first input is *scaled* by $\dfrac{1}{k_0}$.

To this end consider the systems

$$\dot{\xi} = \frac{1}{k_0}\Gamma w, \qquad \dot{\eta} = \gamma\omega,$$

with $\Gamma = \Gamma^\top > 0$ and $\gamma > 0$.

These are passive from the input w to the output ξ and from the input ω to the output η. Note that the ξ system is not realizable since $\frac{1}{k_0}$ is unknown.

Consider now the interconnection equations

$$w = -k_0\psi\epsilon, \qquad \tilde{\theta} = -(\xi), \qquad \omega = -\hat{\theta}^\top\psi - z_1, \qquad \tilde{k}_0 = -(\omega).$$

Note that even if one output signal and the ξ system are not realizable, their interconnection is, leading to the parameter update laws

$$\dot{\hat{\theta}} = \Gamma\psi\epsilon, \qquad \dot{\hat{k}}_0 = \gamma\epsilon(\hat{\theta}\psi + z_1).$$

By the Passivity Theorem we conclude stability of the interconnected system, thus $e \in L_\infty$, $\tilde{\theta} \in L_\infty$, $\tilde{k}_0 \in L_\infty$ and $e \in L_2$, $\dot{\hat{\theta}} \in L_2$, and $\dot{\hat{k}}_0 \in L_2$. Finally, if $\dot{\psi} \in L_\infty$, ψ is persistently exciting, and $\omega \in L_2$ then $\hat{\theta}(t)$ converges to θ and $\hat{k}_0$ converges to a constant.

Normalization

We have, up to now, discussed several parameter estimation methods which require boundedness of the *external* signals, that is the the input and output signals. This is achieved assuming that the input is in L_∞ and the system is stable.

We now consider how such parameter estimation methods could be modified to be used with signals which are not necessarily bounded and with systems which may be unstable. Considering these scenario is crucial to be able to exploit parameter estimation algorithms in a closed-loop configuration to control potentially unstable systems, that is systems for which bounded properties of the signals have to be proved and cannot be assumed.

We consider only two simple examples which contain all the essential ingredients of a process, called normalization, which allows obtaining bounded/converging estimates even if the measured signals are not bounded. The normalization can then be used in more involved schemes with minor modifications.

Consider the linear, SISO, plant described by the relation

$$y(t) = \theta u(t),$$

with input u, output y and unknown θ. When all signals are bounded, the estimate of θ has been obtained considering an estimated output $\hat{y} = \hat{\theta} u$ and the cost function

$$J(\hat{\theta}) = \frac{(y - \hat{y})^2}{2} = \frac{(y - \hat{\theta} u)^2}{2}.$$

If u and y are not in L_∞ then the minimization problem $\min_{\hat{\theta}} J(\hat{\theta})$ does not make sense.

This issue can be solved defining normalized measurements

$$\bar{u} = \frac{u}{m}, \qquad \bar{y} = \frac{y}{m},$$

where $m > 0$ is the so-called normalizing signal, defined as $m^2 = 1 + n_s^2$, with the signal n_s selected such that $\frac{u}{m} \in L_\infty$, for example $n_s = u$, yielding $m^2 = 1 + u^2$.

Since $\bar{u} \in L_\infty$ and $\bar{y} \in L_\infty$, we can now follow the same approach that we have used in the case in which all signal were bounded defining the estimated output $\hat{\bar{y}} = \hat{\theta}\bar{u}$, and the estimation error $\bar{\epsilon}_1 = \bar{y} - \hat{\bar{y}} = \bar{y} - \hat{\theta}\bar{u}$.

The adaptive law is now determined by solving the well-posed optimization problem

$$\min_{\hat{\theta}} J(\hat{\theta}) = \min_{\hat{\theta}} \frac{(\bar{y} - \hat{\theta}\bar{u})^2}{2} = \min_{\hat{\theta}} \frac{(y - \hat{\theta}u)^2}{2m^2},$$

which yields the gradient-based update law

$$\dot{\hat{\theta}} = \gamma\bar{\epsilon}_1\bar{u} = \gamma\frac{\epsilon_1}{m^2}u = \gamma\epsilon u,$$

where $\epsilon = \dfrac{\epsilon_1}{m^2} = -\tilde{\theta}\dfrac{u}{m^2}$ is referred to as the normalized estimation error.

Note that $\dot{\tilde{\theta}} = \dot{\hat{\theta}}$, hence letting $V(\tilde{\theta}) = \dfrac{\tilde{\theta}^2}{2\gamma}$ yields

$$\dot{V} = -\tilde{\theta}^2\bar{u}^2 = -\epsilon^2 m^2 \leq 0.$$

We conclude that $\tilde{\theta} \in L_\infty$, $\hat{\theta} \in L_\infty$, and $\epsilon m \in L_2$. Since $\bar{u} \in L_\infty$, then $\epsilon \in L_\infty$ and $\epsilon m \in L_\infty$, which imply that $\dot{\tilde{\theta}} \in L_2 \cap L_\infty$ and that

$$\dot{\overbrace{\epsilon m}} \in L_\infty.$$

In conclusion

$$\lim_{t\to\infty} \epsilon m = 0, \qquad \lim_{t\to\infty} \dot{\tilde{\theta}} = 0,$$

which means that even in the case of unbounded signals it is possible to develop an adaptive estimator that guarantees bounded parameter estimate and a speed of adaptation that is bounded in an L_2 and L_∞ sense.

Selecting $n_s = 0$, that is $m = 1$, yields the un-normalized update law, from which we can establish bounded parameter estimate, but we cannot guarantee boundedness of $\dot{\tilde{\theta}}$ in an L_p sense, unless $u \in L_\infty$.

Normalization – The Scalar Case

To illustrate how the normalization process can be used to estimate the parameters of a dynamical system consider the system (for simplicity we have selected the "b" matrix to be equal to 1)

$$\dot{x} = -ax + u,$$

which we reparameterize as

$$x = \frac{1}{s + a_m}\left[(a_m - a)x + u\right],$$

with $a_m > 0$. We do not assume that $u \in L_\infty$, but only that it is piece-wise continuous, and a is arbitrary (that is, not positive).

The objective is to estimate a using measurement of x and u. To this end, consider an estimate $\hat{x}$ of x defined as

$$\hat{x} = \frac{1}{s + a_m}\left[(a_m - \hat{a})x + u\right],$$

and the estimation error ϵ implicitly defined as

$$\epsilon = x - \hat{x} - \frac{1}{s + a_m} n_s^2 \epsilon = \frac{1}{s + a_m}\left(\tilde{a}x - n_s^2 \epsilon\right) \qquad \Leftrightarrow \qquad \dot{\epsilon} = -a_m \epsilon + \tilde{a}x - n_s^2 \epsilon$$

where n_s is the normalizing signal to be defined.

Consider now the storage function $S(\epsilon) = \frac{1}{2}\epsilon^2$ and note that

$$\dot{S} = -a_m\epsilon^2 - n_s^2\epsilon^2 + \epsilon x\tilde{a},$$

which reveals a passivity property from the input $\tilde{a}$ to the output ϵx.

Consider now the negative feedback interconnection of the estimation error system with the passive system $\dot{\xi} = w$, via the interconnection equations $w = \epsilon x$ and $\tilde{a} = -\xi$.

By the Passivity Theorem we conclude that the overall system has a storage function $S_T(\epsilon, \tilde{a}) = \dfrac{1}{2}\epsilon^2 + \dfrac{1}{2}\tilde{a}^2$, which is such that

$$\dot{S}_T = -a_m\epsilon^2 - n_s^2\epsilon^2 \leq 0.$$

As a result $\epsilon \in L_\infty$, $\tilde{a} \in L_\infty$, $\epsilon \in L_2$, and $n_s\epsilon \in L_2$. Let now $m^2 = 1 + n_s^2$ and assume that $\dfrac{x}{m} \in L_\infty$ (for example one could select $n_s = x$, yielding $m^2 = 1 + x^2$). As a result one could write

$$\dot{\tilde{a}} = \dot{\hat{a}} = -\epsilon x = -\epsilon m\frac{x}{m},$$

which implies that $\dot{\tilde{a}} \in L_2$ (and $\dot{\hat{a}} \in L_2$).

Normalization

The signal n_s is used to *normalize* the effect of unbounded x, yielding a bounded speed of adaptation regardless of boundedness of x.

The normalizing term $-n_s^2\epsilon$ is similar to a nonlinear damping term, which is used in some nonlinear control design methods to strengthen the negativity of the time derivative of a Lyapunov function.

Despite the normalization, it is not possible to establish the property that $\dot{\tilde{a}} \in L_\infty$, which is essential to prove convergence of $\tilde{a}(t)$.

The normalization procedure can be used with any of the schemes that we have seen. For example, in the SPR design one has to define the normalized estimation error

$$\epsilon = z - \hat{z} - W(s)L(s)n_s^2\epsilon$$

and in the gradient method one has to select the normalized estimation error/cost

$$\epsilon = \frac{z - \hat{z}}{1 + n_s^2}, \qquad J(\hat{\theta}) = (1 + n_s^2)\frac{\epsilon^2}{2},$$

where, in both cases, $m^2 = 1 + n_s^2$ is such that the ratio of the *regressor* with m is bounded.

Consider the SISO system

$$\dot{x} = Ax + bu, \qquad y = cx,$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}$ and output $y(t) \in \mathbb{R}$. Assume, for simplicity, that u is a piecewise continuous and bounded function of time and A is asymptotically stable. Assume, in addition, that the plant is reachable and observable.

Consider the problem of estimating, simultaneously, the parameters of the system and the states from measurements of the input and output signals.

Observe that this problem is partially ill-posed, since we cannot estimate the matrices A, b and c, but only the parameters of the transfer function $W(s) = c(sI - A)^{-1}b$. The reason for this is that there are infinitely many, algebraically equivalent, state-space representations giving the same transfer function.

This shortcoming, however, allows selecting a particular structure, minimally parameterized, which facilitates the solution of the simultaneous state and parameter estimation problem.

Assume that the triple (A, b, c) is in observer form, that is it is such that

$$A = \left[\begin{array}{c|cccc} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{array}\right], \qquad b = \left[\begin{array}{c} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{array}\right], \qquad c = \left[\begin{array}{ccccc} 1 & 0 & \cdots & 0 & 0 \end{array}\right],$$

and note that

$$\dot{x} = Ax + bu = (S - ac)x + bu = Sx - ay + bu,$$

where $a = \left[\begin{array}{ccccc} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{array}\right]^\top$, and S is the upper shift matrix.

Recall, finally, that the vectors a and b identify the transfer function of the system, that is

$$c(sI - A)^{-1} b = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots a_0}.$$

Adaptive Luenberger Observers

The adaptive Luenberger observer is given by the system

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{b}u + L(y - \hat{y}), \qquad \hat{y} = c\hat{x},$$

with $\hat{A} = S - \hat{a}c$, $L = a^\star - \hat{a}$, $a^\star$ selected such that $S - a^\star c$ is Hurwitz, and $\hat{a}$ and $\hat{b}$ estimates of a and b,respectively.

A wide class of adaptive law can be used to generate $\hat{a}$ and $\hat{b}$, for example starting from the parametric model $z = \theta^\top \phi$, where z and ϕ are filtered, measured, signals obtained from u and y, and $\theta = \begin{bmatrix} a^\top & b^\top \end{bmatrix}^\top$.

Theorem

An adaptive observer formed by the combination of the proposed adaptive Luenberger observer with any parameter estimation law (which uses normalization if required) based on the plan parametric model $z = \theta^\top \phi$ is such that

- *all signals are in L_∞;*
- *the output estimation error $\tilde{y} = y - \hat{y}$ converges to zero as $t \to \infty$;*
- *if u is sufficiently rich of order $2n$ then the state estimation error $x - \hat{x}$ and the parameter estimation error $\theta - \hat{\theta}$ converge exponentially to zero.*

Consider the problem of estimating the amplitude, phase and frequency of the signal

$$y(t) = E_1 \sin(\omega_1 t + \phi_1).$$

As already discussed, the signal y can be regarded as the output of the system

$$\dot{y} = x, \qquad \dot{x} = -\theta_1 y,$$

with $\theta_1 = \omega_1^2$, and E_1 and ϕ_1 related to the initial states $y(0)$ and $x(0)$.

This problem can be solved using the parameter estimation methods and the adaptive Luenberger observer discussed in the previous lectures.

We present, herein, an alternative approach based on what is called the I&I (Immersion and Invariance) approach, which regards parameter and state estimation problems in a unified way and as reduced order observer design problems.

To apply the I&I methodology define the state and parameter estimation errors

$$z_x = k_1 y + \hat{x}(k_1^2 + \theta_1) - x, \qquad z_\theta = y\hat{x} + \hat{\theta}_1 - \theta_1,$$

with $k_1 > 0$.

These error variables differ from the classical error variables $\hat{x} - x$ and $\hat{\theta} - \theta$ in that they contain additional *injection* terms.

Consider now the dynamics of the estimation error variables, that is the equations

$$\dot{z}_x = k_1 x + \dot{\hat{x}}(k_1^2 + \theta_1) + \theta_1 y, \qquad \dot{z}_\theta = x\hat{x} + y\dot{\hat{x}} + \dot{\hat{\theta}}_1.$$

Using the relations

$$x = -z_x + k_1 y + \hat{x}(k_1^2 + \theta), \qquad \theta = -z_\theta + y\hat{x} + \hat{\theta}_1,$$

yields

$$\dot{z}_x = -k_1 z_x + (k_1^2 + \theta_1)(y + k_1\hat{x} + \dot{\hat{x}}), \qquad \dot{z}_\theta = -\hat{x} z_x - \hat{x}^2 z_\theta - \Delta + \dot{\hat{\theta}}_1,$$

where $\Delta = \Delta(y, \hat{x}, \hat{\theta}_1)$ is a function of known signals.

Selecting

$$\dot{\hat{x}} = -k_1 \hat{x} - y, \qquad \dot{\hat{\theta}}_1 = \Delta(y, \hat{x}, \hat{\theta}_1),$$

yields the state and parameter estimation error system

$$\begin{bmatrix} \dot{z}_x \\ \dot{z}_\theta \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ -\hat{x} & -\hat{x}^2 \end{bmatrix} \begin{bmatrix} z_x \\ z_\theta \end{bmatrix}.$$

Let $V(z_x, z_\theta) = \dfrac{z_x^2}{2k_1} + \dfrac{z_\theta^2}{2}$. Then $\dot{V} = -z_x^2 - z_\theta z_x \hat{x} - z_\theta^2 \hat{x}^2 < 0$, for all $(z_x, z_\theta \hat{x}) \neq (0,0)$.

As a result $z_x \in L_\infty \cap L_2$, $z_\theta \in L_\infty$, $z_\theta \hat{x} \in L_2$ and, since $\hat{x} \in L_\infty$, $\dot{z}_x \in L_\infty$ and $\overset{\cdot}{\overbrace{z_\theta \hat{x}}} \in L_\infty$, z_x converges (exponentially) to zero and $z_\theta \hat{x}$ converges to zero.

We conclude that, provided $E_1 \neq 0$ and $\omega_1 \neq 0$, $\theta_{est} = y\hat{x} + \hat{\theta}_1$ is an asymptotic estimate of $\theta_1 = \omega^2$ and $x_{est} = k_1 y + \hat{x}(k_1^2 + \theta_{est})$ is an asymptotic estimate of x.

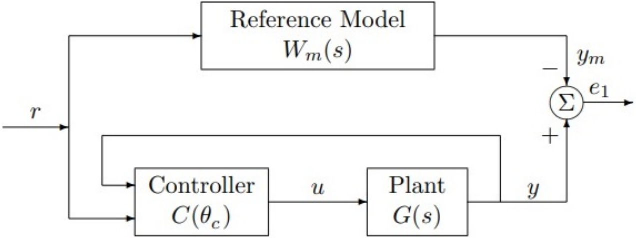
Exercise (Assignment 3)

To study the performance of the I&I frequency estimator consider the two signals $y_1(t) = \sin 100t$ *and* $y_2(t) = \sin t$.

Implement the I&I frequency estimator and run simulations showing the effectiveness of the approach. Experiment with different values of the gain k_1 *and discuss what is a good selection for such a gain.*

Suppose that the measured signals are perturbed by a periodic disturbance, that is one measures $y_1 + d$, *with* $d = 0.001 \sin 50t$. *Run simulations for this scenario and discuss the impact that the disturbance has on the convergence of the frequency estimates.*

Model Reference Adaptive Control (MRAC) is one of the main adaptive control approaches. The basis structure of a MRAC scheme is described in the figure below.



The reference model is selected to generate the desired output trajectory that the plant output has to follow. The error signal e_1 describes the mismatch between the desired behaviour and the actual behaviour. The closed-loop system is composed of a *standard* feedback loop containing the plant and a parameterized controller together with an adjustment/learning mechanism which generates on-line the controller parameter θ_c.

Model Reference Adaptive Control

MRAC could be direct or indirect and with, or without, normalization.

In direct MRAC the parameter vector of the controller is updated directly by an adaptation mechanism, whereas in indirect MRAC the parameter vector of the controller is obtained via some algebraic equations that relate the controller parameters with the plant parametesr, which are estimated on-line.

In direct and indirect MRAC with/without normalization the controller is combined with a parameter estimation algorithms with/without normalization.

This design procedure allows using a wide class of adaptive laws relying, for example, on the gradient descent or on the SPR design.

In the case of un-normalized MRAC the controller is modified to give an error equation the form of which allows using passivity arguments. As a result, the design stage is more complex, but the study of the stability properties relies purely on passivity arguments.

Consider the scalar system

$$\dot{x} = ax + u,$$

with a an unknown constant. The control objective is to determine a function $u = u(x, t)$ such that the state of the closed-loop system converges asymptotically to zero for any initial condition $x(0)$.

To this end let $a_m > 0$ be a parameter selected by the designer and note that selecting $u = -kx$, with $k = a + a_m$ yields the closed-loop system $\dot{x} = -a_m x$, that is the closed-loop system behaves like the *model* $\dot{x}_m = -a_m x_m$.

The control law $u = -kx = -(a + a_m)x$ is often referred to as the known parameter controller and it is clearly non-implementable. As a result, one has to replace k with an estimate $\hat{k}$, computed by viewing the problem as an on-line parameter estimation problem, yielding the control law

$$u = -\hat{k}x.$$

Note that this control law is nonlinear in the *state* $(x, \hat{k})$, despite the fact that the underlying system is linear.

To obtain the *update law* for $\hat{k}$ rewrite the system as

$$\dot{x} = ax - kx + kx + u = (a-k)x + kx + u = -a_m x + kx + u$$

or, equivalently, as

$$x = \frac{1}{s+a_m}(kx+u).$$

These equations provide two equivalent parameterizations of the system in terms of the controller parameter k.

Since x and u are measured and a_m is known one could consider a wide class of parameter identifiers. The one that generates the simplest possible update law, yielding a simple stability proof, is the one based on the use of the un-normalized SPR design.

Since $\dfrac{1}{s+a_m}$ is SPR, one could directly apply the SPR design with the state estimate

$$\hat{x} = \frac{1}{s+a_m}(\hat{k}x+u) = \frac{1}{s+a_m}(0),$$

where the last equality has been obtained noting that $u = -\hat{k}x$. Selecting $\hat{x}(0) = 0$, one then has $\hat{x}(t) = 0$, for all $t \geq 0$.

Defining the regulation error $\epsilon_1 = x - \hat{x} = x$ and letting $\tilde{k} = \hat{k} - k$ yields

$$\dot{\epsilon}_1 = -a_m \epsilon_1 - \tilde{k}x, \qquad \epsilon_1 = \frac{1}{s + a_m}(-\tilde{k}x).$$

Selecting the storage function $S = \dfrac{\epsilon_1^2}{2}$ yields

$$\dot{S} = -a_m \epsilon_1^2 - \tilde{k}\epsilon_1 x,$$

which reveals a passivity property from the input $\tilde{k}$ to the output $-\epsilon_1 x = -x^2$.

Consider now the interconnection with the passive system $\dot{\xi} = \gamma v$ with output ξ and $\gamma > 0$, via the interconnection equations $v = -\epsilon_1 x$, $\tilde{k} = -\xi$, yielding the update law

$$\dot{\tilde{k}} = \gamma \epsilon_1 x = \gamma x^2.$$

The interconnected system can be studied with the storage function $S_T(\epsilon_1, \xi) = \dfrac{\epsilon_1^2}{2} + \dfrac{\xi^2}{2\gamma}$, yielding

$$\dot{S}_T = -a_m \epsilon_1^2 \leq 0.$$

As a result $\epsilon_1 \in L_\infty \cap L_2$, $\hat{k} \in L_\infty$ and $\dot{\epsilon_1} \in L_\infty$. This implies that $\lim_{t\to\infty} \epsilon_1(t) = \lim_{t\to\infty} x(t) = 0$.
Finally, boundedness of $\hat{k}$ and convergence of x imply $\lim_{t\to\infty} \dot{\hat{k}}(t) = 0$ and $\lim_{t\to\infty} u(t) = 0$.

Note that, even if we are able to establish adaptive regulation (to zero), we are not able to prove convergence of $\hat{k}$ to $k = a + a_m$, that is the closed-loop system does not behave as the model $\dot{x}_m = -a_m x_m$. Lack of parameter convergence to the actual value is not crucial in adaptive regulation: as this example demonstrates adaptive regulation can be achieved regardless of the convergence of the estimated parameter to its true value.

For this scalar example one could integrate the system and obtain the relation

$$\lim_{t\to\infty} \hat{k}(t) = \hat{k}_\infty = a + \sqrt{\gamma x^2(0) + (\hat{k}(0) - a)^2} \geq a,$$

showing that the $\hat{k}(t)$ converges to a stabilizing gain (for almost all initial conditions), the value of which depends upon the initial conditions and the adaptive gain γ. The selection of $\hat{k}(0)$ and γ affects the value of $\hat{k}_\infty$ and the transient response of $\hat{k}$ and of x.

Consider now a slightly more general problem, that is the problem of designing an adaptive control law such that the state of the system

$$\dot{x} = ax + bu \qquad \Leftrightarrow \qquad x = \frac{b}{s-a}u,$$

with a and b unknown constants, tracks the state of the (known) reference model

$$\dot{x}_m = -a_m x_m + b_m r \qquad \Leftrightarrow \qquad x_m = \frac{b_m}{s+a_m}r,$$

with $a_m > 0$, for any bounded, piecewise continuous, reference signal r, and it is such that all closed-loop signals are bounded. To achieve the considered control objective one could select the non-implementable *known parameter controller*

$$u = -kx + lr,$$

with k and l such that

$$\frac{x(s)}{r(s)} = \frac{bl}{s-a+bk} = \frac{b_m}{s+a_m} = \frac{x_m(s)}{r(s)},$$

that is $k = \dfrac{a_m + a}{b}$ and $l = \dfrac{b_m}{b}$, which are well defined provided $b \neq 0$ (*i.e.* the system is controllable).

The known parameter controller guarantees that, for any initial condition $x(0)$ of the system and any initial condition $x_m(0)$ of the model $\lim_{t\to\infty} |x(t) - x_m(t)| = 0$, exponentially.

In an adaptive scheme, the known parameter controller is replaced by the controller

$$u = -\hat{k}x + \hat{l}r,$$

with $\hat{k}$ and $\hat{l}$ to be computed on-line from the available signals. To design the update laws for $\hat{k}$ and $\hat{l}$ rewrite the system as

$$\dot{x} = -a_m x + b_m r + b(kx - lr + u) \qquad \Leftrightarrow \qquad x = \overbrace{\frac{b_m}{s + a_m} r}^{x_m} + \frac{b}{s + a_m}(kx - lr + u),$$

and note that the error variable $e = x - x_m$ is such that

$$e = \frac{b}{s + a_m}(kx - lr + u).$$

The equation describing the error variable is in the form of a bilinear parametric model, hence we could use the corresponding parameter estimation method.

Define the estimate $\hat{e}$ of e as

$$\hat{e} = \frac{\hat{b}}{s + a_m}(\hat{k}x - \hat{l}r + u) = \frac{1}{s + a_m}(0),$$

where the last equality has been obtained recalling that $u = -\hat{k}x + \hat{l}r$. This equation implies that the estimation error $\epsilon_1 = e - \hat{e}$ coincides with the tracking error e, that is there is no need to generate $\hat{e}$.

As a result, defining $\tilde{k} = \hat{k} - k$ and $\tilde{l} = \hat{l} - l$ and recalling that $\epsilon_1 = e$ yields

$$\dot{\epsilon}_1 = -a_m\epsilon_1 + b(-\tilde{k}x + \tilde{l}r) \qquad \Leftrightarrow \qquad \epsilon_1 = e = \frac{b}{s + a_m}(-\tilde{k}x + \tilde{l}r).$$

Similarly to the regulation problem, consider the storage function $S = \frac{\epsilon_1^2}{2}$ and note that

$$\dot{S} = -a_m\epsilon_1^2 + \epsilon_1 b(-\tilde{k}x + \tilde{l}r) = -a_m\epsilon_1^2 + |b| \begin{bmatrix} -\epsilon_1 x\,\mathrm{sign}(b) & \epsilon_1 r\,\mathrm{sign}(b) \end{bmatrix} \begin{bmatrix} \tilde{k} \\ \tilde{l} \end{bmatrix},$$

which reveals a passivity property from the inputs $\tilde{k}$ and $\tilde{l}$ to the outputs $-\epsilon_1 x\,\mathrm{sign}(b)$ and $\epsilon_1 r\,\mathrm{sign}(b)$.

The passivity property suggests the selection of the update laws

$$\dot{\tilde{k}} = \gamma_1 \epsilon_1 x \operatorname{sign}(b) = \dot{\hat{k}}, \qquad \dot{\tilde{l}} = -\gamma_2 \epsilon_1 r \operatorname{sign}(b) = \dot{\hat{l}},$$

with $\gamma_1 > 0$ and $\gamma_2 > 0$, which are implementable provided the sign of b is known.

It follows that $\epsilon_1 \in L_\infty \cap L_2$, $\tilde{k} \in L_\infty$, and $\tilde{l} \in L_\infty$. Thus $x \in L_\infty$, $u \in L_\infty$, $\dot{\epsilon}_1 \in L_\infty$, which imply that $\lim_{t\to\infty} \epsilon_1(t) = 0$.

We have therefore established that (provided the sign of b is known) the proposed adaptive control law is such that all closed-loop signals are bounded and the state x of the system tracks, asymptotically, the state x_m of the model, for any bounded, piecewise continuous, signal r.

This asymptotic tracking result does not imply that $\hat{k}$ and $\hat{l}$ converge to k and l, respectively, unless the signal r is sufficiently rich of order 2 (in which case one has exponential convergence of $\hat{k}$ and $\hat{l}$ to k and l, respectively.

The assumption that the sign of b is known can be relaxed with a slightly more elaborate construction, which we do not discuss for brevity.

To understand the role of the assumption that the unknown parameters are constant, consider the adaptive regulation problem for the system

$$\dot{x} = a(t)x + u,$$

where $a(t)$ is an unknown (non-constant) parameter which is such that $a(t)$ is bounded for all $t \geq 0$. Suppose in addition that there exists an unknown l_a such that, for some known constant $\Delta_a > 0$,

$$|a(t) - l_a| \leq \Delta_a.$$

One could attempt to study this regulation problem using the same storage function used in the constant parameter case, that is $S_T = \frac{x^2}{2} + \frac{\tilde{a}^2}{2}$, where $\tilde{a} = \hat{a} - a$, the time derivative of which along the trajectories of the system with state x and $\tilde{a}$ is

$$\dot{S}_T = ax^2 + xu + \tilde{a}\dot{\tilde{a}} = \hat{a}x^2 - \tilde{a}x^2 + xu + \tilde{a}\dot{\hat{a}} - \tilde{a}\dot{a}.$$

The selection $u = -\hat{a}x - x$ and $\dot{\hat{a}} = x^2$, yields

$$\dot{S}_T = -x^2 - \tilde{a}\dot{a} \leq -x^2 + |\tilde{a}||\dot{a}|,$$

which contains a term upon which we have no control.

The main reason why this approach fails is that we use a in the storage function, via the term $\tilde{a}^2$. This is used to guarantee boundedness of $\hat{a}$, without guaranteeing convergence of $\hat{a}$ to a, thus replacing a with another constant, for example l_a, may achieve the same objective without yielding any *undesired* time derivative.

Consider therefore the modified storage function

$$\tilde{S}_T = \frac{x^2}{2} + \frac{(\hat{a} - l_a)^2}{2}$$

yielding (note that $\hat{a} - l_a = -(a - \hat{a}) + (a - l_a)$)

$$\dot{\tilde{S}}_T = \hat{a}x^2 + xu + \left[(a - \hat{a})x^2 - (a - \hat{a})\dot{\hat{a}}\right] + (a - l_a)\dot{\hat{a}}.$$

This suggests the selection

$$u = -x - \hat{a}x - \Delta_a x, \qquad \dot{\hat{a}} = x^2,$$

which gives $\dot{\tilde{S}}_T \leq -x^2$, hence $x \in L_\infty \cap L_2$, $\hat{a} \in L_\infty$, $\dot{x} \in L_\infty$, thus $\lim_{t \to \infty} x(t) = 0$.

Note that the control law does not use l_a, pretty much like a standard adaptive law, and uses solely a bound on the maximum variation of a and no bound on $\dot{a}$.

The proposed design, known as the congelation of variable design, can be given a passivity interpretation similarly to the standard design.

Let $u = -x - \hat{a}x - \Delta_a x$. Consider the closed-loop system

$$\dot{x} = -x + (a - l_a - \Delta_a)x - (\hat{a} - l_a)x,$$

with $\hat{a} - l_a$ regarded as an input, and the storage function $S(x) = \frac{x^2}{2}$. Note that

$$\dot{S} = -x^2 + \overbrace{(a - l_a - \Delta_a)}^{\leq 0} x^2 - (\hat{a} - l_a)x^2 \leq -x^2 - (\hat{a} - l_a)x^2,$$

which suggests that the system is passive from the input $\hat{a} - l_a$ to the output $-x^2$ and justifies the use of the parameter update law

$$\overbrace{\hat{a} - l_a}^{\cdot} = \dot{\hat{a}} = x^2.$$

The use of the *congealed* variable l_a allows defining a new passive interconnection, from which an update law can be designed. This passivity property cannot be enforced with respect to the *input variable* $\hat{a} - a$, because of the non-zero time derivative of a.

An alternative to the classical MRAC design is offered by the so-called I&I adaptive control approach, in which the adaptive control problem is regarded as a partial state feedback control problem.

To illustrate the I&I approach, consider the system

$$\dot{x} = ax + u,$$

with a an unknown constant.

Consider now the extended system, with states x and a and control u, described by the equations

$$\dot{a} = 0, \qquad \dot{x} = ax + u.$$

The control objective is to design u, as functions of x only, such that the equilibrium $(\theta, 0)$ is stable and x converges to zero with *target dynamics* $\dot{x}_m = -a_m x_m$, with $a_m > 0$.

The control objective can be achieved selecting the *known parameter controller*

$$u = -a_m x - ax.$$

In the spirit of designing a reduced order observer for the state a, define an extended system

$$\dot{a} = 0, \qquad \dot{x} = ax + u, \qquad \dot{\hat{a}} = w,$$

with

$$u = -a_m x - a_{est} x$$

and w such that a can be asymptotically reconstructed from x and $\hat{a}$, that is there exists a function $a_{est} = a_{est}(x, \hat{a})$ which provides some sort of estimate of a.

Define, consistently with the standard reduced order observer design procedure, the error variable

$$z = \overbrace{\hat{a} + \beta(x)}^{a_{est}} - a,$$

with β to be selected, and note that

$$\dot{z} = \dot{\hat{a}} + \frac{\partial \beta}{\partial x}(ax - a_m x - a_{est} x) = \dot{\hat{a}} + \frac{\partial \beta}{\partial x}(-zx - a_m x) = \left(\dot{\hat{a}} - \frac{\partial \beta}{\partial x} a_m x\right) - \left(\frac{\partial \beta}{\partial x} x\right) z.$$

Selecting $\dot{\hat{a}} = a_m \frac{\partial \beta}{\partial x} x$ yields

$$\dot{z} = -\left(\frac{\partial \beta}{\partial x} x\right) z.$$

The function β has to be selected to render the equilibrium $z = 0$ stable. This is achieved selecting β such that $\frac{\partial \beta}{\partial x} x > 0$ for all $x \neq 0$, for example

$$\beta(x) = \frac{x^2}{2}, \qquad \text{or} \qquad \frac{1}{2}\log(1 + x^2).$$

Note that the first selection yields the *classical* update law, whereas the second one yields a normalized update law.

Using the first selection of β yields the closed-loop system, in the a, x and z coordinates,

$$\dot{a} = 0, \qquad \dot{x} = -a_m x - xz, \qquad \dot{z} = -x^2 z.$$

We conclude that $z \in L_\infty$, $xz \in L_2$, $x \in L_\infty$ and $\lim_{t\to\infty} x(t) = 0$, and the x variables *achieves the target dynamics* with an L_2 perturbation.

Note that the update law and the parameter estimates are, respectively,

$$\dot{\hat{a}} = a_m x^2, \qquad a_{est} = \hat{a} + \frac{x^2}{2}.$$

Exercise (Assignment 4)

Consider the system

$$\dot{x} = ax + u,$$

with a constant and unknown. Use, in the simulations, $a = 1$ *and* $a_m = 1$.

Design a (classical) MRAC adaptive controller and an I&I adaptive controller to achieve adaptive regulation (with at least two selections of the function β *for the I&I design).*

Compare the performance of the resulting closed-loop systems in terms of speed of response, control amplitude, and in the presence of measurement noise, *that is the measured variable is* $x + d$, *with* $d(t) = 0.1 \sin \dfrac{t}{5}$.

Finally, compare the performance of the resulting controller in the case in which the parameter is given by

$$a = 1 + \frac{1}{10} \sin 10t, \qquad a = 1 + 10 \sin \frac{t}{10}.$$

Consider the linear system

$$\dot{x} = Ax + Bu,$$

with state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$, both available for measurement, and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ unknown. Suppose that the system is reachable. The control objective is to determine u such that all closed-loop signals are bounded and the plant state follows the state x_m of the reference model

$$\dot{x}_m = A_m x_m + B_m r,$$

with $A_m \in \mathbb{R}^{n\times n}$ Hurwitz, $B_m \in \mathbb{R}^{n\times m}$, and $r(t) \in \mathbb{R}^m$ a bounded and piecewise continuous reference signal. If the matrices A and B were known one could select

$$u = -Kx + Lr,$$

yielding the closed-loop system

$$\dot{x} = (A - BK)x + BLr,$$

with K and L such that

$$A - BK = A_m, \qquad BL = B_m.$$

This implies that the input-state behaviour of the closed-loop system coincides with the input-state behaviour of the model. In general it may be impossible to find matrices K and L satisfying the above *matching* conditions and existence of such matrices may rely on a particular structure for A and/or B.

Suppose that K and L satisfying the matching conditions exist. Then one could select

$$u = -\hat{K}x + \hat{L}r,$$

with $\hat{K}$ and $\hat{L}$ estimates of K and L, respectively.

Using these definitions the system can be rewritten as

$$\dot{x} = Ax + B(-\hat{K}x + \hat{L}r) = (A_m + BK)x + BLr + B(-\hat{K}x + \hat{L}r) - BLr = A_m x + B_m r - B\tilde{K}x + B\tilde{L}r,$$

with $\tilde{K} = \hat{K} - K$ and $\tilde{L} = \hat{L} - L$. Let $e = x - x_m$ and note that

$$\dot{e} = A_m e + B(-\tilde{K}x + \tilde{L}r),$$

which depends on the unknown B. In the scalar case we have exploited the fact that the sign of B is, by assumption, known. This assumption can be extended to the general case assuming that L is symmetric and positive (similarly if it is negative) definite and rewriting the error system as

$$\dot{e} = A_m e + B_m L^{-1}(-\tilde{K}x + \tilde{L}r).$$

Consider now the storage function $S(e) = e^\top Pe$, with $P = P^\top > 0$, such that

$$A_m^\top P + PA_m = -Q,$$

with $Q = Q^\top > 0$, and note that

$$\dot{S} = -e^\top Qe + 2e^\top PB_m L^{-1}(-\tilde{K}x + \tilde{L}r) = -e^\top Qe - 2\mathrm{tr}\,(\tilde{K}^\top L^{-1} B_m^\top Pex^\top) + 2\mathrm{tr}\,(\tilde{L}^\top L^{-1} B_m^\top er^\top).$$

To define the update laws consider the storage function

$$S_{update} = \mathrm{tr}\,\left(\tilde{K}^\top L^{-1}\tilde{K} + \tilde{L}^\top L^{-1}\tilde{L}\right)$$

and note that

$$\dot{S}_{update} = 2\mathrm{tr}(\tilde{K}^\top L^{-1}\dot{\tilde{K}} + \tilde{L}^\top L^{-1}\dot{\tilde{L}}).$$

Both systems, that is the error system and the system defining the update laws, are passive with respect to properly selected input and output signals.

Interconnecting the systems in a negative feedback interconnection yields the update laws

$$\dot{\tilde{K}} = \dot{\hat{K}} = B_m^\top P e x^\top, \qquad \dot{\tilde{L}} = \dot{\hat{L}} = -B_m^\top P e r^\top,$$

hence, by the Passivity Theorem,

$$\dot{S}_T = \dot{S} + \dot{S}_{update} = -e^\top Q e \leq 0.$$

We conclude that $e \in L_\infty \cap L_2$, $\tilde{K} \in L_\infty$, $\tilde{L} \in L_\infty$, hence $\dot{\tilde{K}} \in L_\infty$ and $\dot{\tilde{L}} \in L_\infty$. These imply that $\dot{e} \in L_\infty$. As a result $\lim_{t\to\infty} e(t) = 0$.

The assumption that L is positive definite has strong implications on the structure of B and on the selection of B_m. In fact, if $L = L^\top > 0$ then

$$BLB^\top = B_m B^\top = B B_m^\top \geq 0,$$

which imposes non-generic symmetry and positivity conditions on $B_m B^\top$. Note that $BLB^\top$ cannot be strictly positive definite unless $m = n$ and $\mathrm{rank} B = n$.

We have introduced several examples to illustrate the design of MRAC schemes for systems in which the whole state vector is measurable.

We now consider the more realistic and challenging problem in which only the output of the system is available for measurement. For simplicity we focus, however, on SISO systems.

Consider the SISO system (the plant)

$$\dot{x} = Ax + Bu, \qquad y = Cx,$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, and $y(t) \in \mathbb{R}$, and A, B and C matrices of appropriate dimensions, and its associated transfer function

$$G(s) = C(sI - A)^{-1}B = k\frac{Z(s)}{R(s)},$$

with $Z(s)$ and $R(s)$ monic polynomials and k constant.

The reference model, selected by the designer, is described by the equations

$$\dot{x}_m = A_m x_m + B_m r, \qquad\qquad y_m = C_m x_m,$$

with $x_m(t) \in \mathbb{R}^{n_m}$, $y_m(t) \in \mathbb{R}$ and $r(t) \in \mathbb{R}$ a bounded and piecewise continuous function of time.

The transfer function of the model is

$$W_m(s) = C_m(sI - A_m)^{-1}B_m = k_m \frac{Z_m(s)}{R_m(s)},$$

with $Z_m(s)$ and $R_m(s)$ monic polynomials.

To begin with we consider the problem of designing a controller such that the output of the system, in closed-loop with such a controller, tracks the output of the reference model for any given reference r, pretending that all parameters are known.

Plant assumptions

- $Z(s)$ is a monic Hurwitz polynomial of degree m.
- An upper bound N of the degree n of $R(s)$ is known.
- The relative degree of the system, that is $rd = n - m$, is known.
- The sign of the high frequency gain k is known (assume it is positive).

Reference model assumptions

- $Z_m(s)$ and $R_m(s)$ are monic Hurwitz polynomials of degree m_m and n_m, respectively and $n_m \leq N$.
- The relative degree of the model, that is $rd_m = n_m - m_m$, is such that $rd_m = rd$.

The plant is minimum phase and can be unstable; pole-zero cancellations are allowed. Minimum phaseness is required because the controller cancels the zeroes of the plant.

The assumptions on the relative degree and on the high frequency gain can be relaxed.

A trivial design (recall that all parameters are known) is obtained selecting

$$u = C(s)r = \frac{k_m}{k}\,\frac{Z_m(s)}{Z(s)}\,\frac{R(s)}{R_m(s)}\,r,$$

which yields the closed-loop system

$$y = W_m(s)r.$$

This design is viable only if $R(s)$ is Hurwitz. It is however undesired because of its open-loop structure.

Consider, instead, the feedback control law

$$u = \theta_1^\top \frac{\alpha_{N-2}(s)}{\Lambda(s)} u + \theta_2^\top \frac{\alpha_{N-2}(s)}{\Lambda(s)} y + \theta_3 y + cr,$$

where θ_1, θ_2, θ_3 and c are constants, $\Lambda(s) = \Lambda_0(s)Z_m(s)$ is Hurwitz, monic, and of degree $N-1$ (hence $\Lambda_0(s)$ is monic, Hurwitz, and of degree $N-1-m_m$), and

$$\alpha_{N-2}(s) = \begin{bmatrix} s^{N-2} & s^{N-3} & \cdots & 1 \end{bmatrix}^\top$$

if $N \geq 2$, otherwise $\alpha_{N-2}(s) = 0$.

The controller parameter

$$\theta = \begin{bmatrix} \theta_1^\top & \theta_2^\top & \theta_3 & c \end{bmatrix} = \begin{bmatrix} \bar{\theta}^\top & c \end{bmatrix}$$

is to be selected such that the closed-loop transfer function equals $W_m(s)$ and the closed-loop system is stable.

The closed-loop transfer function is

$$\frac{y(s)}{r(s)} = G_{cl}(s) = \frac{c\,k\,Z(s)\,\Lambda^2(s)}{\Lambda(s)[(\Lambda(s) - \theta_1^\top \alpha_{N-2}(s))R(s) - kZ(s)(\theta_2^\top \alpha_{N-2}(s) + \theta_3 \Lambda(s))]}.$$

The control objective is therefore achieved if

$$\frac{c\,k\,Z(s)\,\Lambda^2(s)}{\Lambda(s)[(\Lambda(s) - \theta_1^\top \alpha_{N-2}(s))R(s) - kZ(s)(\theta_2^\top \alpha_{N-2}(s) + \theta_3 \Lambda(s))]} = k_m \frac{Z_m(s)}{R_m(s)}.$$

The degree of the denominator of $G_{cl}(s)$ is $n + 2N - 2$ and the degree of the denominator of $W_m(s)$ is $n_m \leq N$, hence $n + 2N - 2 - n_m$ pole-zero cancellations have to occur.

Such cancellations are stable since $Z(s)$ and $\Lambda(s)$ are Hurwitz.

Let $c = \dfrac{k_m}{k}$ and recall that $\Lambda(s) = \Lambda_0(s)Z_m(s)$. Then the matching condition becomes

$$\Big(\Lambda(s) - \theta_1^\top \alpha_{N-2}(s)\Big)R(s) - kZ(s)\Big(\theta_2^\top \alpha_{N-2}(s) + \theta_3\Lambda(s)\Big) = Z(s)\Lambda_0(s)R_m(s),$$

or

$$\theta_1^\top \alpha_{N-2}(s)R(s) + k\Big(\theta_2^\top \alpha_{N-2}(s) + \theta_3\Lambda(s)\Big)Z(s) = \Lambda(s)R(s) - Z(s)\Lambda_0(s)R_m(s).$$

Equating the coefficients of equal powers of s yields a set of linear equations of the form

$$S\bar{\theta} = p,$$

where $S \in \mathbb{R}^{(N+n-1)\times(2N-1)}$ depends on the coefficients of $R(s)$, $kZ(s)$ and $\Lambda(s)$, and $p \in \mathbb{R}^{N+n-1}$ depends on the coefficients of $\Lambda(s)R(s) - Z(s)\Lambda_0(s)R_m(s)$.

The existence of a solution $\bar{\theta}$ depends on the properties of the matrix S: one could have infinitely many solutions, one solution, or no solution.

Note, finally, that the controller parameters are nonlinear functions of the coefficients of $Z(s)$ and $R(s)$. In the special case in which $N = n$ and $rd = 1$ then $\bar{\theta}$ is a linear function of the coefficients of $Z(s)$ and $R(s)$.

Lemma

Consider the equation $S\bar{\theta} = p$. The equation has always at least one solution. In addition, if $R(s)$ and $Z(s)$ are coprime and $N = n$ then the solution is unique.

The developed *known parameter* controller assigns the closed-loop poles at the roots of the polynomial $Z(s)\Lambda_0(s)R_m(s)$ and changes the high frequency gain from k to k_m.

The transfer function matching is achieved by cancelling the zeros of the plant and replacing these by those of the reference model, that is by designing $\Lambda(s) = \Lambda_0(s)Z_m(s)$. This cancellation is feasible by the assumptions on the polynomials $Z(s)$ and $\Lambda_0(s)Z_m(s)$.

In our analysis the initial conditions have been neglected: for zero initial conditions of the plant and the model we have that $y(t) = y_m(t)$, for all $t \geq 0$ and all bounded and piecewise continuous reference input $r(t)$.

For nonzero initial conditions, because all cancellations occur in $\mathbb{C}^-$, one has that

$$\lim_{t\to\infty} (y(t) - y_m(t)) = 0,$$

exponentially.

A state-space realization of the controller is given by

$$\dot{\omega}_1 = F\omega_1 + gu, \qquad \dot{\omega}_2 = F\omega_2 + gy, \qquad u = \theta^\top \omega,$$

where $\omega_1(t) \in \mathbb{R}^{N-1}$, $\omega_2(t) \in \mathbb{R}^{N-1}$, $\theta = \begin{bmatrix} \theta_1^\top & \theta_2^\top & \theta_3 & c \end{bmatrix}^\top$, $\omega = \begin{bmatrix} \omega_1^\top & \omega_2^\top & y & r \end{bmatrix}^\top$,

$$F = \begin{bmatrix} -\lambda_{N-2} & -\lambda_{N-3} & \cdots & -\lambda_1 & -\lambda_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad g = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and

$$\Lambda(s) = s^{N-1} + \lambda_{N-2}s^{N-2} + \cdots + \lambda_1 s + \lambda_0 = \det(sI - F).$$

Note that F and g are such that

$$(sI - F)^{-1}g = \frac{\alpha_{N-2}(s)}{\Lambda(s)}.$$

Exercise

Consider a linear SISO system with transfer function

$$G(s) = -2\frac{s+5}{s^2-2s+1}$$

and the reference model

$$y_m = \frac{3}{s+3}r.$$

Note that $N = n = 2$ *and* $rd = rd_m = 1$. *Let* $\Lambda(s) = s + 1 = \Lambda_0(s)$ *and*

$$u = \theta_1 \frac{1}{s+1}u + \theta_2 \frac{1}{s+1}y + \theta_3 y + cr.$$

Show that $c = -3/2$ *and that the equation* $S\bar{\theta} = p$ *is*

$$\begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 6 \\ 1 & -10 & -10 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 12 \\ -14 \end{bmatrix},$$

hence write a state-space realization of the controller achieving the model matching *requirement.*

Exercise

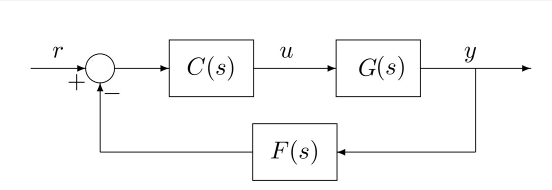
Show that an alternative equivalent state-space realization of the controller is

$$\dot{\omega}_1 = F^\top \omega_1 + \theta_1 u, \qquad \dot{\omega}_2 = F^\top \omega_2 + \theta_2 y, \qquad u = g^\top \omega_1 + g^\top \omega_2 + \theta_3 y + cr.$$

Exercise

Show that the controller can be implemented as in the diagram below, where

$$C(s) = c\frac{\Lambda(s)}{\Lambda(s) - \theta_1^\top \alpha_{N-2}(s)}, \qquad F(s) = -\frac{1}{c}\,\frac{\theta_2^\top \alpha_{N-2}(s) + \theta_3 \Lambda(s)}{\Lambda(s)}.$$



To obtain an adaptive version of the considered controller we follow an approach proposed by Feuer and Morse and focus mainly on the case $rd = 1$, since the complexity of the design increases with the relative degree.

In addition, we assume that the reference model is SPR.

Recalling the structure of the known parameter controller, it is reasonable to parameterize the adaptive controller as

$$\dot{\omega}_1 = F\omega_1 + gu, \qquad \dot{\omega}_2 = F\omega_2 + gy, \qquad u = \hat{\theta}^\top \omega,$$

where $\hat{\theta}$ is an estimate of θ to be computed online with an adaptive estimation algorithm.

To design the estimation algorithm consider a state-space realization of the closed-loop system, that is

$$\dot{\xi} = \overbrace{\begin{bmatrix} A & 0 & 0 \\ 0 & F & 0 \\ gC & 0 & F \end{bmatrix}}^{A_{cl}} \xi + \overbrace{\begin{bmatrix} B \\ g \\ 0 \end{bmatrix}}^{B_{cl}} (\hat{\theta}^\top \omega), \qquad y = \overbrace{\begin{bmatrix} C & 0 & 0 \end{bmatrix}}^{C_{cl}} \xi,$$

with $\xi = \begin{bmatrix} x^\top & \omega_1^\top & \omega_2^\top \end{bmatrix}^\top$.

MRAC for SISO Systems

This can be re-written as

$$\dot{\xi} = \underbrace{A_{cl}\xi + B_{cl}\theta^\top\omega} + B_{cl}(u - \theta^\top\omega), \qquad y = C_{cl}\xi,$$

$$\parallel$$

$$\dot{\xi} = \overbrace{A_{cl}^\star\xi + c\,B_{cl}\,r} + B_{cl}(u - \theta^\top\omega), \qquad y = C_{cl}\xi,$$

with $A_{cl}^\star$ such that

$$c\ C_{cl}(sI - A_{cl}^\star)^{-1}B_{cl} = W_m(s).$$

With some manipulation we can therefore obtain an *error equation* of the form

$$e_1 = W_m(s)\rho(u - \theta^\top\omega),$$

with $\rho = 1/c$ and $e_1 = y - y_m$, that is a bilinear parametric model, yielding the parameter update law

$$\dot{\hat{\theta}} = -\Gamma e_1\omega,$$

with $\Gamma = \Gamma^\top > 0$.

Theorem

Consider the MRAC scheme under the stated assumptions. Then the following holds.

- *All signals are bounded and the tracking error* $e_1 = y - y_m$ *converges asymptotically to zero for any bounded and piecewise continuous reference* r.
- *If* r *is sufficiently rich of order* $2N$, $\dot{r} \in L_\infty$ *and* $Z(s)$ *and* $R(s)$ *are coprime then the parameter error* $\tilde{\theta}$ *and the tracking error* e_1 *converge to zero exponentially.*

Exercise (Assignment 5)

Consider the second order system with transfer function

$$G(s) = k\frac{s + b_0}{s^2 + a_1 s + a_0},$$

where k, $b_0 > 0$, a_1 *and* a_0 *are unknown constants, and the reference model*

$$y_m = \frac{1}{s+1} r.$$

Let $\Lambda(s) = s + 2$.

Verify that all assumptions for the design of a MRAC are met.

Perform simulations of the system in closed-loop with the MRAC assuming that $b_0 = 2$, $a_1 = 5$, $a_2 = -10$ *and* $k = 1$ *and for* $r = E_1 \sin \omega_1 t + E_2 \sin \omega_2 t$, *with* $E_1 \neq 0$, $E_2 \neq 0$, $\omega_1 \neq \omega_2$, .

The relative degree one assumption is key to be able to exploit the SPR design method for the parameter identification.

In general, regardless of the value of the relative degree, one always obtains the error equation

$$e_1 = W_m(s)\rho(u - \theta^\top \omega).$$

To exploit the SPR design one rewrites this equation as

$$e_1 = W_m(s)(s + p_0)\, \cdots\, (s + p_{rd-1})\rho(u_f - \theta^\top \omega_f),$$

where

$$u_f = \frac{1}{(s + p_0)\, \cdots\, (s + p_{rd-1})} u, \qquad \omega_f = \frac{1}{(s + p_0)\, \cdots\, (s + p_{rd-1})} \omega,$$

and $p_0, \ldots, p_{rd-1}$ are positive constants to be selected either to render

$$W_m(s)(s + p_0)\, \cdots\, (s + p_{rd-1})$$

SPR or to allow applicability of the SPR adaptive estimation algorithm for bilinearly parameterized models.

Adaptive integrator backstepping is a powerful adaptive control design methodology which allows solving linear and nonlinear adaptive control problems in a modular way. While we do not discuss in-depth such a methodology, we provide some examples to understand its basic principles and the class of problems to which it can be applied.

Consider, to begin with, the adaptive stabilization problem for the system

$$\dot{x}_1 = x_2 + \phi_1(x_1), \qquad \dot{x}_2 = \theta\phi_2(x) + u,$$

where θ is an unknown constant, $x_1(t) \in I\!R$, $x_2(t) \in I\!R$, and $u(t) \in I\!R$. In addition $\phi_1(0) = 0$ and $\phi_2(0) = 0$, that is the origin is an equilibrium of the system for $u = 0$.

If θ were known we could proceed as follows. Pretend that x_2 is a virtual control and design the *partial* stabilizer

$$x_2^\star = -x_1 - \phi_1(x_1) = \alpha_1(x_1).$$

This suggests that a way to address the problem is to *control* x_2 to be as close as possible to $x_2^\star$ by using the control u. This can be achieved selecting the Lyapunov function

$$V(x_1, x_2) = \frac{x_1^2}{2} + \frac{(x_2 - \alpha_1(x_1))^2}{2},$$

which *penalizes* the *distance* of x_1 from zero and of x_2 from $x_2^\star$.

Adaptive Backstepping

Along the trajectories of the system one has

$$
\begin{aligned}
\dot{V} &= \underbrace{x_1(x_2 + \phi_1(x_1))}_{\Vert} + (x_2 - \alpha_1(x_1))(\theta\phi_2(x) + u - \dot{\alpha}_1(x_1)) \\
&= \overbrace{-x_1^2 + x_1(x_2 - \alpha_1(x_1))} + (x_2 - \alpha_1(x_1))(\theta\phi_2(x) + u - \dot{\alpha}_1(x_1))
\end{aligned}
$$

which suggests the selection

$$u = -\theta\phi_2(x) + \dot{\alpha}_1(x_1) - x_1 - (x_2 - \alpha_1(x_1)),$$

yielding

$$\dot{V} = -x_1^2 - (x_2 - \alpha_1(x_1))^2 = -2V.$$

Note that $\dot{\alpha}_1(x_1)$ is known, since it is a function of x_1 and x_2. Replacing the known parameter controller with

$$u = -\hat{\theta}\phi_2(x) + \dot{\alpha}_1(x_1) - x_1 - (x_2 - \alpha_1(x_1))$$

and using V as a storage function yields

$$\dot{V} = -2V - \tilde{\theta}(x_2 - \alpha_1(x_1))\phi_2(x),$$

which reveals a passivity property from $\tilde{\theta}$ to $-(x_2 - \alpha_1(x_1))\phi_2(x)$.

Exploiting the passivity property we conclude that the update law

$$\dot{\tilde{\theta}} = \dot{\hat{\theta}} = \gamma(x_2 - \alpha_1(x_1))\phi_2(x) = \gamma \begin{bmatrix} 0 & \phi_2(x) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 - \alpha_1(x_1) \end{bmatrix}$$

with $\gamma > 0$, yields a closed-loop adaptive system such that $x_1 \in L_\infty \cap L_2$, $x_2 \in L_\infty \cap L_2$, $\tilde{\theta} \in L_\infty$, $\dot{x}_1 \in L_\infty$, $\dot{x}_2 \in L_\infty$, hence $\lim_{t\to\infty} x_1(t) = 0$ and $\lim_{t\to\infty} x_2(t) = 0$.

This adaptive design is simple because the system satisfies the so-called matching condition: the parametric uncertainty is matched with the control, that is it can be directly *canceled* by the control signal, provided all parameters are known.

We now illustrate how the proposed ideas can be extended to the case of the so-called extended matching, that is to the case in which the parametric uncertainty is *one integrator away* from the controller.

Consider the system

$$\dot{x}_1 = x_2 + \theta\phi_1(x_1), \qquad \dot{x}_2 = u,$$

with $\phi_1(0) = 0$ and θ an unknown constant. If θ were known we could repeat the previous procedure and construct a stabilizing controller on the basis of

$$x_2^\star = -x_1 - \theta\phi_1(x_1) = \alpha_1(x_1, \theta).$$

However, since θ is not know, we cannot complete the procedure as α_1 is also a function of θ. To circumvent this issue we proceed as follow.

Pretend, again, that x_2 is the control signal and note that an adaptive controller for the system

$$\dot{x}_1 = x_2 + \theta\phi_1(x_1),$$

is given by

$$x_2^\star = -x_1 - \hat{\theta}_1\phi_1(x_1) = \alpha_1(x_1, \hat{\theta}_1), \qquad \dot{\hat{\theta}}_1 = -\gamma x_1\phi_1(x_1),$$

with $\gamma > 0$, and where we have used the notation $\hat{\theta}_1$ to indicate that this is the estimate of θ at the first step of backstepping.

This is a *dynamic* controller that has to be *backstepped* through one integrator.

Consider now the differential equation for the *variable*

$$z_2 = x_2 - \alpha_1(x_1, \hat{\theta}_1),$$

that is

$$\dot{z}_2 = u - \frac{\partial \alpha_1}{\partial x_1}(x_2 + \theta\phi_1(x_1)) - \frac{\partial \alpha_1}{\partial \hat{\theta}_1}(-\gamma x_1 \phi_1(x_1)) = u - Z_{20}(x_1, z_2, \hat{\theta}_1) - Z_{21}(x_1, \hat{\theta}_1)\theta,$$

and the candidate storage function

$$S(x_1, z_2, \hat{\theta}_1) = \frac{x_1^2}{2} + \frac{z_2^2}{2} + \frac{(\hat{\theta}_1 - \theta)^2}{2}$$

yielding $\dot{S} = -x_1^2 + z_2\dot{z}_2 = -x_1^2 + z_2\left(u - Z_{20}(x_1, z_2, \hat{\theta}_1) - Z_{21}(x_1, \hat{\theta}_1)\theta\right)$.

One could attempt to use the estimate $\hat{\theta}_1$ to eliminate the θ dependent term in $\dot{S}$ by means of the control law

$$u = -z_2 + Z_{20}(x_1, z_2, \hat{\theta}_1) + Z_{21}(x_1, \hat{\theta}_1)\hat{\theta}_1$$

yielding

$$\dot{S} = -x_1^2 - z_2^2 - z_2 Z_{21}(x_1, \hat{\theta}_1)(\hat{\theta}_1 - \theta),$$

which contains a term upon which we have no control.

There are two ways in which this issue can be resolved. The simplest possible approach is to consider a second estimate $\hat{\theta}_2$ for the parameter θ and the control law

$$u = -z_2 + Z_{20}(x_1, z_2, \hat{\theta}_1) + Z_{21}(x_1, \hat{\theta}_1)\hat{\theta}_2$$

yielding

$$\dot{S} = -x_1^2 - z_2^2 - z_2 Z_{21}(x_1, \hat{\theta}_1)(\hat{\theta}_2 - \theta),$$

that is a passivity property for the x_1, z_2, $\hat{\theta}_1$ system from the input $\hat{\theta}_2 - \theta$ to the output $-z_2 Z_{21}(x_1, \hat{\theta}_1)$, from which we can design the update law

$$\dot{\hat{\theta}}_2 = \gamma z_2 Z_{21}(x_1, \hat{\theta}_1).$$

This approach has the clear disadvantage that one has to estimate θ twice, that is one has to use the so-called over-parameterization.

The second solution relies on the fact that one could use the storage function

$$S(x_1, z_1, \theta_1) = \frac{x_1^2}{2} + \frac{z_2^2}{2} + \frac{(\hat{\theta} - \theta)^2}{2}$$

without committing the update law for $\hat{\theta}$ at the first step. Interestingly, the resulting update law is the sum of the two update laws designed in the over-parameterization approach, that is

$$\dot{\hat{\theta}} = -\gamma(\phi_1(x_1)x_1 - Z_{21}(x_1, \hat{\theta})z_2).$$

Conclusions

We have solved adaptive parameter estimation problems introducing several alternative methods which differ in terms of the underlying structure of the model and of the information pattern. We have also seen how these methods can be modified to accommodate unbounded signals.

We have studied simultaneous state and parameter estimation problems introducing an adaptive version of Luenberger observer and discussing the I&I methodology on a specific example.

We have discussed in detail the MRAC problem providing examples, highlighting the impact that non-constant parameters have on the analysis and design, and proposing three design approaches: the classical MRAC design, the congelation of variable design, and the I&I design.

We have presented the simplest possible *embodiment* of adaptive integrator backstepping. We have also discussed how it may result in over-parameterization and how this shortcoming can be resolved.