Control in The Presence of Uncertainty

Tracking and Regulation

Dispense del Corso di Controllo Robusto e Adattativo

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The Regulator Theory

Consider a linear system affected by disturbances and such that its output should asymptotically track a certain, pre-specified, reference signal.

In what follows, we discuss this control problem and present possible solutions.

To begin with, consider a system to be controlled described by equations of the form

$$\dot{x} = Ax + Bu + Pd, \qquad e = Cx + Qd, \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $e(t) \in \mathbb{R}^p$, $d(t) \in \mathbb{R}^r$, and A, B, P, C and Q matrices of appropriate dimensions and with constant entries.

The signal $d(t)$, denoted exogeneous signal, is in general composed of two components: the former models a set of disturbances acting on the system to be controlled, the latter a set of reference signals.

In what follows we assume that the exogeneous signal is generated by a linear system, denoted exosystem, described by the equation

$$\dot{d} = Sd, \tag{2}$$

with S a matrix with constant entries. Note that, under this assumption, it is possible to generate, for example, constant or polynomial references/disturbances and sinusoidal references/disturbances with any given frequency.

The variable $e(t)$, denoted tracking error, is a measure of the error between the ideal behaviour of the system and the actual behaviour.

Ideally, the variable $e(t)$ should be regulated to zero, that is it should converge asymptotically to zero, despite the presence of the disturbances.

If this happens we say that the tracking error is regulated to zero, that is it converges asymptotically to zero, hence the disturbances do not affect the asymptotic behaviour of the system and the output $Cx(t)$ asymptotically tracks the reference signal $-Qd(t)$.

In general, the tracking error does not naturally converge to zero, hence it is necessary to determine an input signal $u(t)$ which *drives* it to zero.

The simplest possible way to construct such an input signal is to assume that it is generated via static feedback of the state $x(t)$ of the system to be controlled and of the state $d(t)$ of the exosystem, that is

$$u = Kx + Ld. \tag{3}$$

In practice, it is unrealistic to assume that both $x(t)$ and $d(t)$ are measurable, hence it may be more natural to assume that the input signal $u(t)$ is generated via dynamic feedback of the error signal only, that is it is generated by the system

$$\dot{\chi} = F\chi + Ge, \qquad u = H\chi, \tag{4}$$

with $\chi(t) \in I\!R^{\nu}$, for some $\nu > 0$, and F, G and H matrices with constant entries.

Definition (Full information regulator problem)

Consider the system (1), driven by the exosystem (2) and interconnected with the controller (3). The full information regulator problem is the problem of determining the matrices K and L of the controller such that[a]

(S) the system

$$\dot{x} = (A + BK)x$$

is asymptotically stable;

(R) all trajectories of the system

$$\dot{d} = Sd, \qquad \dot{x} = (A + BK)x + (BL + P)d, \qquad e = Cx + Qd, \tag{5}$$

are such that

$$\lim_{t \to \infty} e(t) = 0.$$

[a] (S) stands for stability and (R) for regulation.

Definition (Error feedback regulator problem)

Consider the system (1), driven by the exosystem (2) and interconnected with the controller (4). The error feedback regulator problem is the problem of determining the matrices F, G and H of the controller such that

(S) the system

$$\dot{x} = Ax + BH\chi, \qquad \dot{\chi} = F\chi + GCx,$$

is asymptotically stable;

(R) all trajectories of the system

$$\dot{d} = Sd, \quad \dot{x} = Ax + BH\chi + Pd, \quad \dot{\chi} = F\chi + G(Cx + Qd), \quad e = Cx + Qd, \tag{6}$$

are such that

$$\lim_{t\to\infty} e(t) = 0.$$

Assumption (1)

The matrix S of the exosystem has all eigenvalues with non-negative real part.

Assumption (2)

The system (1) *with $d = 0$ is reachable.*

Assumption (1) implies that there are no initial conditions $d(0)$ such that the signal $d(t)$ converges (asymptotically) to zero. This assumption is not restrictive. In fact, disturbances converging to zero do not have any effect on the asymptotic behaviour of the system, and references which converge to zero can be tracked simply by driving the state of the system to zero, i.e. by stabilizing the system.

Assumption (2) implies that it is possible to arbitrarily assign the eigenvalues of the matrix $A + BK$ by a proper selection of K. Note that, in practice, this assumption can be replaced by the weaker assumption that the system (1) with $d = 0$ is stabilizable.

The Full Information Regulator Problem

We now present a preliminary result which is instrumental to derive a solution to the full information regulator problem.

Lemma

Consider the full information regulator problem. Suppose Assumption (1) holds. Suppose, in addition, that there exists matrices K and L such that condition (S) holds.

Then condition (R) holds if and only if there exists a matrix $\Pi \in \mathbb{R}^{n \times r}$ such that the equations

$$\Pi S = (A + BK)\Pi + (P + BL), \qquad 0 = C\Pi + Q, \tag{7}$$

hold.

Proof.

Consider the system (5) and the coordinates transformation

$$\hat{d} = d, \qquad \hat{x} = x - \Pi d,$$

where Π is the solution of the equation[a]

$$\Pi S = (A + BK)\Pi + (P + BL).$$

Note that, by condition (S) and Assumption (1), there is a unique matrix Π which solves this equation.

□

[a] This equation is a so-called Sylvester equation. The Sylvester equation is a (matrix) equation of the form $A_1 X = X A_2 + A_3$, in the unknown X. This equation has a unique solution, for any A_3, if and only if the matrices A_1 and A_2 do not have common eigenvalues.

Proof.

In the new coordinates $\hat{x}$ and $\hat{d}$ the system is described by the equations

$$\dot{\hat{d}} = S\hat{d}, \qquad \dot{\hat{x}} = (A + BK)\hat{x}, \qquad e = C\hat{x} + (C\Pi + Q)\hat{d}.$$

Note now that, by condition (S) $\lim_{t\to\infty} \hat{x}(t) = 0$, hence condition (R) holds, by Assumption (1), if and only if

$$C\Pi + Q = 0.$$

In summary, under the state assumptions, condition (R) holds if and only if there exists a matrix Π such that equations (7) hold. □

We are now ready to state and prove the result which provides conditions for the solution of the full information regulator problem.

Theorem

Consider the full information regulator problem. Suppose Assumptions (1) and (2) hold.

There exists a full information control law described by the equation (3) *which solves the full information regulator problem if and only if there exist two matrices* Π *and* Γ *such that the equations*

$$\Pi S = A\Pi + B\Gamma + P, \qquad 0 = C\Pi + Q \tag{8}$$

hold.

The Full Information Regulator Problem

Proof.

(Necessity) Suppose there exist two matrices K and L such that conditions (S) and (R) of the full information regulator problem hold.

Then, by Lemma 3, there exists a matrix Π such that equations (7) hold.

As a result, the matrices Π and $\Gamma = K\Pi + L$ are such that equations (8) hold.

$\square$

The Full Information Regulator Problem

Proof.

(Sufficiency) The proof of the sufficiency is constructive. Suppose there are two matrices Π and Γ such that equations (8) hold.

The full information regulator problem is solved selecting K and L as follows.

The matrix K is any matrix such that the system

$$\dot{x} = (A + BK)x$$

is asymptotically stable. By Assumption (2) such a matrix K does exist.

The matrix L is selected as

$$L = \Gamma - K\Pi.$$

This selection is such that condition (S) of the full information regulator problem holds, hence to complete the proof we have only to show that, with K and L as selected above, the equations (7) hold.

This is trivially the case. In fact, replacing L in (7) yields the equations (8), which hold by assumption.

As a result, also condition (R) of the full information regulator problem holds. □

The Full Information Regulator Problem

The proof of Theorem 4 implies that a controller (it is not the only one) which solves the full information regulator problem is described by the equation

$$u = Kx + (\Gamma - K\Pi)d,$$

with K such that a stability condition holds, and Π and Γ such that equations (8) hold.

By Assumption (2) the stability condition can be always satisfied.

As a result, the solution of the full information regulator problem relies upon the existence of a solution of equations (8).

The FBI Equations

Equations (8), known as the Francis-Byrnes-Isidori (FBI) equations, are linear equations in the unknown Π and Γ, for which the following statement holds.

Lemma (Hautus)

The equations (8)*, in the unknown* Π *and* Γ*, are solvable for any* P *and* Q *if and only if*

$$\operatorname{rank}\begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} = n + p, \tag{9}$$

for all s *which are eigenvalues of the matrix* S*.*

The equations (8) can be rewritten in compact form as

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} S = \begin{bmatrix} -P \\ -Q \end{bmatrix},$$

which is a so-called generalized Sylvester equation.

The FBI Equations

For single-input, single-output systems (i.e. $m = p = 1$) the condition expressed by Lemma 5 has a very simple interpretation.

In fact, the complex number s such that

$$\operatorname{rank}\begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} < n + 1$$

are the zeros of the system

$$\dot{x} = Ax + Bu \qquad\qquad y = Cx,$$

which coincides with the roots of the numerator polynomial of the transfer function

$$W(s) = C(sI - A)^{-1}B,$$

that is the zeros of $W(s)$.

This implies that, for single-input, single-output systems the full information regulator problem is solvable if and only if the eigenvalues of the exosystem are not zeros of the transfer function of the system (1), with input u, output e and $d = 0$.

The Error Feedback Regulator Problem

To provide a solution to the error feedback regulator problem we need to introduce a new assumption.

Assumption (3)

The system

$$\begin{bmatrix} \dot{x} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}, \qquad e = \begin{bmatrix} C & Q \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} \tag{10}$$

is observable.

Note that Assumption (3) implies observability of the system

$$\dot{x} = Ax, \qquad y = Cx. \tag{11}$$

To prove this property note that observability of the system (10) implies that

$$\operatorname{rank}\begin{bmatrix} C & Q \\ CA & \star \\ \vdots & \vdots \\ CA^{n+r-1} & \star \end{bmatrix} = n + r.$$

This, in turn, implies

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n+r-1} \end{bmatrix} = n$$

and, by Cayley-Hamilton Theorem,

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n,$$

which implies observability of system (11).

Similarly to what discussed in the case of Assumption (2), Assumption (3) can be replaced by the weaker assumption that the system (10) is detectable.

We are now ready to state and prove the result which provides conditions for the solution of the error feedback regulator problem.

Theorem

Consider the error feedback regulator problem. Suppose Assumptions (1), (2) and (3) hold.

There exists an error feedback control law described by the equation (4) *which solves the full information regulator problem if and only if there exist two matrices* Π *and* Γ *such that the equations*

$$\Pi S = A\Pi + B\Gamma + P, \qquad 0 = C\Pi + Q \tag{12}$$

hold.

Theorem 6 can be alternatively stated as follows.

Consider the error feedback regulator problem. Suppose Assumptions (1), (2) and (3) hold. Then the error feedback regulator problem is solvable if and only if the full information regulator problem is solvable.

Proof.

(Necessity) The proof of the necessity is similar to the proof of the necessity of Theorem 4, hence omitted.

(Sufficiency) The proof of the sufficiency is constructive. Suppose there are two matrices Π and Γ such that equations (12) hold.

Then, by Theorem 4 the full information control law

$$u = Kx + (\Gamma - K\Pi)d,$$

with K such that the system $\dot{x} = (A + BK)x$ is asymptoticaly stable, solves the full information regulator problem.

This control law is not implementable, because we only measure e. However, by Assumption (3), it is possible to build asymptotic estimates ξ and δ of x and d, hence implement the control law

$$u = K\xi + (\Gamma - K\Pi)\delta. \tag{13}$$

□

Proof.

To this end, consider an observer described by the equation

$$\begin{bmatrix} \dot{\xi} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \begin{bmatrix} \xi \\ \delta \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \left(\begin{bmatrix} C & Q \end{bmatrix} \begin{bmatrix} \xi \\ \delta \end{bmatrix} - e \right) + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K & \Gamma - K\Pi \end{bmatrix} \begin{bmatrix} \xi \\ \delta \end{bmatrix}.$$

Note that the estimation errors $e_x = x - \xi$ and $e_d = d - \delta$ are such that

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_d \end{bmatrix} = \left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \begin{bmatrix} C & Q \end{bmatrix} \right) \begin{bmatrix} e_x \\ e_d \end{bmatrix}, \tag{14}$$

hence, by Assumption (3), there exist G_1 and G_2 that assign the eigenvalues of this error system.

□

The Error Feedback Regulator Problem

Proof.

Note now that the control law (13) can be rewritten as

$$u = Kx + (\Gamma - K\Pi)d - (Ke_x + (\Gamma - K\Pi)e_d),$$

hence the control law is composed of the full information control law, which solves the considered regulator problem, and of an additive disturbance which decays exponentially to zero.

Such a disturbance does not affect the regulation requirement, provided the closed-loop system is asymptotically stable.

To complete the proof we need to show that condition (S) holds. For, note that, in the coordinates x, e_x and e_d the closed-loop system, with $d = 0$, is described by the equations

$$\begin{bmatrix} \dot{x} \\ \dot{e}_x \\ \dot{e}_d \end{bmatrix} = \begin{bmatrix} A + BK & -BK & -B(\Gamma - K\Pi) \\ 0 & A + G_1 C & P + G_1 Q \\ 0 & G_2 C & S + G_2 Q \end{bmatrix} \begin{bmatrix} x \\ e_x \\ e_d \end{bmatrix}. \tag{15}$$

Recall that the matrices G_1 and G_2 have been selected to render system (14) asymptotically stable, and that K is such that the system $\dot{x} = (A + BK)x$ is asymptotically stable. As a result, system (15) is asymptotically stable. □

The proof of Theorem 6 implies that a controller (it is not the only one) which solves the error feedback regulator problem is described by equations of the form (4) with $\chi = \left[\begin{array}{cc} \xi' & \delta' \end{array} \right]'$,

$$F = \left[\begin{array}{cc} A + G_1 C + BK & P + G_1 Q + B(\Gamma - K\Pi) \\ G_2 C & S + G_2 Q \end{array} \right],$$

$$G = -\left[\begin{array}{c} G_1 \\ G_2 \end{array} \right], \qquad H = \left[\begin{array}{cc} K & \Gamma - K\Pi \end{array} \right], \tag{16}$$

K, G_1 and G_2 such that a stability condition holds, and Π and Γ such that equations (12) hold.

This controller, and in particular the matrix F, possesses a very interesting property.

Theorem (Internal model property)

The matrix F in equation (16) is such that

$$F\Sigma = \Sigma S,$$

for some matrix Σ of rank r. In particular, any eigenvalue of S is also an eigevalue of F.

Proof.

Let

$$\Sigma = \begin{bmatrix} \Pi \\ I \end{bmatrix}$$

and note that $\mathrm{rank}\Sigma = r$, by construction, and that

$$\begin{aligned} F\Sigma &= \begin{bmatrix} A\Pi + G_1 C\Pi + BK\Pi + P + G_1 Q + B(\Gamma - K\Pi) \\ -G_2 C\Pi + S - G_2 Q \end{bmatrix} \\ &= \begin{bmatrix} (A\Pi + B\Gamma + P) + G_1(C\Pi + Q) \\ S - G_2(C\Pi + Q) \end{bmatrix} = \begin{bmatrix} \Pi S \\ S \end{bmatrix} = \Sigma S, \end{aligned}$$

hence the first claim.

To prove the second claim, let λ be an eigenvalue of S and v the corresponding eigenvector.

Then $Sv = \lambda v$, hence

$$F\Sigma v = \Sigma S v = \lambda \Sigma v,$$

which shows that λ is an eigenvalue of F with eigenvector Σv, and this proves the second claim. □

It is possible to prove that the property highlighted in Proposition 7 is shared by all error feedback control laws which solve the considered regulation problem, and not only the proposed controller.

This property, which is often referred to as the internal model principle, can be interpreted as follows.

The control law solving the regulator problem has to *contain* a copy of the exosystem, i.e. it has to be able to generate, when $e = 0$, a copy of the exogeneous signal.

Exercise (Assignment 7)

Consider the model of an inverted pendulum on a cart described by the equations

$$M\ddot{s} + F\dot{s} - \mu = d_1, \qquad \ddot{\phi} - \frac{g}{L}\sin\phi + \frac{1}{L}\ddot{s}\cos\phi = 0,$$

where $s(t)$ is the displacement of the pivot, $\phi(t)$ is the angular rotation of the pendulum, $\mu(t)$ is the external force exerted on the cart and $d_1(t)$ is an external disturbance acting on the cart.

$M = 1\,kg$ is the mass of the cart, $L = 1\,m$ is the effective pendulum length, $F = 1\,kg\,s^{-1}$ is a friction coefficient, and $g = 9.81\,m\,s^{-1}$ is the gravitational acceleration.

A1) *Compute all the equilibrium points of the system for $\mu(t) = d_1(t) = 0$.*

A2) *Write the equations of the linearised system around the equilibrium $\phi = s = \dot{\phi} = \dot{s} = 0$.*

A3) *Express the obtained linear dynamics in the standard state space form*

$$\dot{x} = Ax + Bu + Pd_1, \qquad y = Cx,$$

where $x(t) = [s(t),\ \dot{s}(t),\ \phi(t),\ \dot{\phi}(t)]'$, $u(t) = \mu(t)$ and $y(t) = [s(t),\ \phi(t)]$.

Exercise (Assignment 6 – cont'd.)

A4) *Show that the pair (A, B) is controllable.*

A5) *Consider the linear system in part A3). Assume that d_1 is an unknown constant. The control law should be designed in such a way that the effect of the disturbance d_1 is asymptotically rejected, and the first output $s(t)$ asymptotically tracks the reference signal $d_2(t) = \alpha \sin \omega t$. Pose this problem as a regulator problem.*
(Note that this is a non-standard regulation problem, since the regulation condition involves only part of the measured output.)

A6) *Consider the regulator problem determined in part A5). Show that the problem is solvable by means of a full information control law.*

A7) *Consider the regulator problem determined in part A5). Show that the problem is solvable by means of an error feedback control law.*
(Since the problem is not in the standard form discussed in the lectures, you should use the output $y(t)$ to verify the observability condition and the error signal $e(t) = s(t) - d_2(t)$ to assess the regulation requirement.)

Exercise (Assignment 6 – cont'd.)

B1) *Assume* $d_1(t)$ *is a square wave of amplitude* 0.5 *and period* 50 s, $\alpha = 1$ *and* $\omega = 0.1$. *Design a full information control law solving the regulator problem posed in part A5).*

B2) *Display plots of* $y(t)$ *and* $u(t)$ *for* $x(0) = 0$. *These plots should show that the regulation goal has been achieved. Discuss why the disturbance* $d_1(t)$ *does not affect the output* $y(t)$.

B3) *Assume that the full information control law designed in part B1) is used to control the nonlinear model of the pendulum. Display plots of* $y(t)$ *and* $u(t)$ *for the nonlinear system, from the initial state* $x(0) = 0$. *Is the regulation goal achieved? Discuss why the disturbance* $d_1(t)$ *does not affect the output* $y(t)$ *and why the regulation goal is only* approximately *achieved.*

B4) *Repeat steps B1) to B3) with* $\omega = 1$ *and* $\omega = 10$. *Discuss the obtained results.*

Conclusions

We have solved the tracking and regulation problem for general linear systems and have shown that this problems can be solved provided a linear matrix equation is solvable.

The proposed solution is robust, since both (S) and (R) holds even if the data of the problem undergo (small) perturbations.

The proposed approach extends naturally to nonlinear systems, for which the FBI equation is a partial differential equation subject to an algebraic constraint.