Integració de poliacions singulars
usant carmins

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#### Outline:

- Intro to regular foliations

  Gnoupoids, holonomy, paths (Haefliger, Molino, Ehresmann) Books

- singular foliations (Androuldakis, Skandalis, Debond)
- holonomy, paths for sing. foliations (A, S, D, Fernandez, Crainic, Villatoro Zambon, G)

#### Regular Foliations

#### Infinitessimal definition

A regular foliation on a manifold M is a collection of vector fields FCX(M) such that for each point pEM There is some coordinates  $(x_1, \dots, x_n): \bigcup \longrightarrow \mathbb{R}^n$ with  $F |_{\mathcal{U}} = \operatorname{Span}_{C^{\infty}(\mathcal{U})} \left( \frac{1}{3} \times_{1} , \dots, \frac{1}{3} \times_{K} \right)$ 

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Global definition A regisol. on a misd. M Is an equivalent relation ~ (or a partition) s.t.  $\forall p \in M \exists coord (L \rightarrow \mathbb{R}^n \\ \downarrow \uparrow \downarrow (x_1, 1, -, x_n(q))$ 

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In U

 $x_{k+1}(q_1) = x_{k+1}(q_2)$ 
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#### Flow box theorem

The vactor fields

 $\chi_{n},\chi_{k}\in\mathcal{X}(M)$ 

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b)  $[Y_i, Y_j]$   $= 0 \quad \forall \quad 1 \leq j, i \leq k$ 

#### Frobenius theorem

A regular foliation of dim=Kis equivalent to a collection of vector fields  $F \subseteq K(M)$ s.t.

a)  $F(p) = \{Y(p): Y \in F\} \subset T_pM$ is a subvertor space of dim=K  $\forall p \in M$ 

b) [天/天] S 天



Troupoids 15 a small category where arrows are invertible, this is · Set of points M & set of arrows G · Maps 5:4 > M, t:4 -> M 0: G5x+6 -> G, e:M>G V: d → d a) o is associative b) for yell elyof=40ely=4 C) V(q) 0 9 = C(x), 90 V(p) = C(y)

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Example Let M be a set with an egital. ~ Take G=Graph (v)= } (y,x) & MXM: Y~X} S(4,x) = x, t(4,x) = 4(z,y) o(J,x) = (Z,x)e(x) = (x, x)

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b) for  $y \leftarrow x \in M$   $e(y) \circ y = y \circ e(y) = y$ 90 V(p)= C(y) C) V(q) 0 4 = e(x),

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In M= RK x Rh-k as points

Take G= { Paths constant in Rh-k}/Hon

$$S(x) = 8(0)$$
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- e(x) constant path on x

V(8) inverse parametrization

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is a collection of v.f. F=Xc(M)

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#### Results

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- · Paths/Homotopy is a Lie groupoid

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Smoth maps to a sing fol in a mfd M a map  $Y:V_{SR}^{n} \to F$  is smooth if  $Y = M = 1 \text{ high d} \cup M \text{ with}$  $Y = \sum_{i \in I} f_{i}(t,q) Y_{i}(q)$ 

y q∈U, I pinte, Y; ∈F& f; ∈C(VxM)

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A F-homotopy between paths in F (Yo, Yo) & (Y, Y) is a smooth map  $Y: [0,1]^2 \longrightarrow F$ 8: [0,1]2 -> M (5,0) = 8,(0) = 8, (0)  $V(0,t,q) = V_0(t,q) & Y_{(1,t,q)} = Y_1(t,q)$   $V(0,t) = V_0(t) & V(1,t) = V_1(t)$ 

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A F-homotopy between paths in F (Yo, Yo) & (Y, Y) is a smooth map  $\gamma: [0,1] \longrightarrow F$  $\gamma: [0,1]^2 \longrightarrow M \qquad \gamma(s,0) = \gamma(0)$  $Y_{(0,t,q)} = Y_{o(t,q)} & Y_{(1,t,q)} = Y_{1}(t,q)$   $Y_{(0,t)} = Y_{o(t)} & Y_{(1,t)} = Y_{1}(t,q)$  $(5,t,q)^{2} \frac{\partial}{\partial t} \chi(t,t)$ The vector field W(s,t,q) = d +,5 s disfy W(s,1,8(s,1) ∈ Ix(s,1) F ¥ s∈[0,1]

#### Frobenius Theorem

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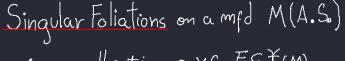
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