PB-groupoids vs VB-groupoids

Francesco Cattafi and Alfonso Garmendia

arXiv:2406.06259

1. Classical PB-VB correspondence

In differential geometry, the notions of principal bundles and vector bundles over a smooth manifold M are intimately related by a standard correspondence.

$$\{\text{ Principal } \mathrm{GL}(k)\text{-bundles over } M\} \longleftrightarrow \{\text{ Vector bundles of rank } k \text{ over } M\}$$

This correspondence can be seen in many general textbooks but we will review it here for the sake of self-containment:

- Given a vector bundle $E \to M$ of rank k, let
$$P := \{\mathrm{Frames}(E_x) \ : \ x \in M\} \cong \{\text{ordered basis of } E_x \ : \ x \in M\}.$$
The Lie group $\mathrm{GL}(k)$ acts canonically in P, and for any $x \in M$ and $p, q \in P_x$ there is a unique $A \in \mathrm{GL}(k)$ such that $p = qA$, implying that P is a principal $\mathrm{GL}(k)$-bundle over M.
- Given a principal $\mathrm{GL}(k)$-bundle $P \to M$, let
$$E := (P \times \mathbb{R}^k)\big/\big\{(p, v) \sim (pA, A^{-1}v)\, \forall A \in \mathrm{GL}(K)\big\}.$$

This correspondence, is the base for many results in mathematics, particularly for the study of connections, gauge theories and geometric structures.

2. Lie groupoids and VB-groupoids

For proofs and more info about these objects see [2]. A Lie groupoid can be seen in different ways. This poster will see them as a manifold with a set of relations. To use the standard notation for these objects, the manifold will be called the base and the relations will be called arrows.

Definition 2.1. *A Lie groupoid is a pair of manifolds, $\mathcal{G}$ called the set of arrows and M called the base, together with smooth maps:*

- *surjective submersions: $s\colon \mathcal{G} \to M$ called source and $t\colon \mathcal{G} \to M$ called target.*
- *$m\colon \mathcal{G}_{\mathbf{s}}\times_{\mathbf{t}} \mathcal{G} \to \mathcal{G}, (g_2, g_1) \mapsto g_2 \circ_m g_1$ called composition.*
- *$u\colon M \to \mathcal{G}$ called units and $\tau\colon \mathcal{G} \to \mathcal{G}; g \mapsto g^\tau$ called inverse.*

such that, for all $(g_2, g_1) \in \mathcal{G}_{\mathbf{s}}\times_{\mathbf{t}} \mathcal{G}$ there is $s(g_2 \circ_m g_1) = s(g_1)$, $t(g_2 \circ_m g_1) = t(g_2)$; m is associative, the image of u is a set of units for m, and τ assigns to each arrow an inverse for m.

Example 2.2. *Let $\pi\colon M \to N$ be a surjective submersion. The fiber product groupoid is given by M as the base and by $M \times_\pi M$ as the arrows. The maps are given by:*

$$\begin{array}{ccc} s(x,y) = y & & t(x,y) = x \\ u(x) = (x,x) & (x,y) \circ_m (y,z) = (x,z) & (x,y)^\tau = (y,x) \end{array}.$$

Following the logic of manifold with a set of relations, we can consider vector bundles over these as vector bundles on the manifold ans on the set of relations.

Definition 2.3. *A* **VB-groupoid of rank** (l, k) *is a commutative diagram*

$$\begin{tikzcd} E_{\mathcal{G}} \arrow[r, "\pi_{\mathcal{G}}"] \arrow[d, shift right, "\widetilde{t}"'] \arrow[d, shift left, "\widetilde{s}"] & \mathcal{G} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ E_M \arrow[r, "\pi_M"] & M \end{tikzcd}$$

such that

- *$\pi_{\mathcal{G}} : E_{\mathcal{G}} \to \mathcal{G}$ is a vector bundle of rank $l + k$ and $\pi_M : E_M \to M$ is a vector bundle of rank k;*
- *$\mathcal{G} \rightrightarrows M$ and $E_{\mathcal{G}} \rightrightarrows E_M$ are Lie groupoids;*
- *the structure maps $(\widetilde{s}, s), (\widetilde{t}, t), (\widetilde{m}, m), (\widetilde{u}, u), (\widetilde{\tau}, \tau)$ are morphisms of vector bundles.*

Example 2.4. *Given any Lie groupoid $\mathcal{G} \rightrightarrows M$, the tangent manifold of the arrows $T\mathcal{G} \rightrightarrows TM$ is a VB-groupoid over $\mathcal{G} \rightrightarrows M$.*

Example 2.5. *Given any VB groupoid $E_{\mathcal{G}} \rightrightarrows E_M$ over a Lie groupoid $\mathcal{G} \rightrightarrows M$ there is a vector bundle over M called the core, and given by the set*

$$C_M = u^* \ker(\widetilde{s}) = \{v \in \ker(\widetilde{s})_{u(x)} \subset (E_{\mathcal{G}})_{u(x)} \ : \ x \in M\}.$$

Moreover there is a canonical Lie groupoid structure in the dual spaces of $E_{\mathcal{G}}$ and C_M making it a VB-groupoid $E^*_{\mathcal{G}} \rightrightarrows C^*_M$ over $\mathcal{G} \rightrightarrows M$.

In [1] we described the VB-PB correspondence for VB-groupoids. Future works may investigate gauge theories, connections and G-structures.

3. VB-groupoids and Poisson geometry

Let us state a list of facts to motivate the relation between VB-groupoids and Poisson geometry. These facts are part of the folklore knowledge but it can be traced to the work "Groupoïdes Symplectiques" of Coste, Dazord and Weinstein in 1987.

- Any Lie groupoid has an asociated Lie algebroid. This construction is given by the core of the tangent Lie groupoid. More precisely, the Lie algebroid associated to $\mathcal{G} \rightrightarrows M$ is the core of $T\mathcal{G} \rightrightarrows TM$ given by $C_M = u^* \ker(\widetilde{s} = ds)$.
- Any Poisson manifold (M, π) has an associated Lie algebroid $\pi^\sharp\colon T^*M \to TM$.
- For any manifold M the cotangent T^*M is a symplectic manifold and therefore a Poisson manifold. For a Lie groupoid $\mathcal{G} \rightrightarrows M$ the cotangent of $\mathcal{G}$ i.e. $T^*\mathcal{G}$ is a symplectic manifold.
- One of the basic examples of Poisson manifolds is given by the dual of a Lie algebra $\mathfrak{g}$. In this case, linear functions on $\mathfrak{g}^*$ corresponds to elements in $x, y \in \mathfrak{g}$ and for any $\alpha \in \mathfrak{g}^*$ there is $\{x, y\}\alpha = \alpha([x, y])$. There is a similar construction for the dual of a Lie algebroid, so dual of Lie algebroids are also Poisson manifolds. For the Lie algebroid TM we get the symplectic (therefore Poisson) manifold of T^*M.
- The dual of the tangent groupoid $T^*\mathcal{G} \rightrightarrows C^*_M$ is the symplectic groupoid integrating the Poisson manifold C^*_M.
- In particular for a Lie algebra $\mathfrak{g}$ with Lie groupo G the Lie groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ is symplectic integrating $\mathfrak{g}^*$.

4. Lie 2-groupoids and $\mathrm{GL}(l, k)$

In general, since we will consider different groupoid structures on the same space, we adopt the notation $s_{ij}, t_{ij}, m_{ij}, u_{ij}, \tau_{ij}$ for the structure maps of a Lie groupoid $\mathcal{G}_i \rightrightarrows \mathcal{G}_j$.

Definition 4.1. *(see [3]) A* **Lie 2-groupoid** *$\mathcal{G}_2 \rightrightarrows \mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ is a double Lie groupoid where the base groupoid $\mathcal{G}_0 \rightrightarrows M = \mathcal{G}_0$ is the unit groupoid. In other words it is a commutative diagram of Lie groupoids*

$$\begin{tikzcd} \mathcal{G}_2 \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & & \mathcal{G}_1 \arrow[dl, shift left] \arrow[dl, shift right] \\ & \mathcal{G}_0 & \end{tikzcd}$$

such that where the following three conditions are satisfied:

1. all the source and targets maps are Lie groupoid morphisms;

2. the interchange law

$$(g_1 \circ_{m_{20}} g_2) \circ_{m_{21}} (g_3 \circ_{m_{20}} g_4) = (g_1 \circ_{m_{21}} g_3) \circ_{m_{20}} (g_2 \circ_{m_{21}} g_4)$$

holds for all $g_i \in \mathcal{G}_2$ such that the compositions above make sense;

We will focus in a particular Lie 2-groupoid which we describe here below:

Definition-Example 4.2. *For any pair (l, k) of natural numbers the* ***general linear 2-groupoid*** *of rank (l, k), denoted by $\mathrm{GL}(l, k)$, is the Lie 2-groupoid with*

$$\mathrm{GL}(l,k)_2 := \left\{ \left(d, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix}\right) \in \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \times \mathrm{GL}(l+k) \ : \ (I_l + Jd) \in \mathrm{GL}(l) \text{ and } (I_k + dJ) \in \mathrm{GL}(k) \right\}$$

$$\mathrm{GL}(l,k)_1 := \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k) \times \mathrm{GL}(l) \times \mathrm{GL}(k),$$
$$\mathrm{GL}(l,k)_0 := \mathrm{Hom}(\mathbb{R}^l, \mathbb{R}^k).$$

- *The groupoid structure on $\mathrm{GL}(l,k)_2 \rightrightarrows \mathrm{GL}(l,k)_0$ is the unique one with source and target maps $t_{20}, s_{20}\colon \mathrm{GL}(l,k)_2 \to \mathrm{GL}(l,k)_0$ given by:*
$$((I + dJ)B)^{-1}dA \xleftarrow{t_{20}} \left(d, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix}\right) \xmapsto{s_{20}} d \ ,$$
and composition map $m_{20} : (\mathrm{GL}(l,k)_2)\ {}_{s_{20}}\!\times_{t_{20}} (\mathrm{GL}(l,k)_2) \to \mathrm{GL}(l,k)_2$ given by the matrix multiplication:
$$\left(\left(d, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix}\right), \left(t_{20}\left(d, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix}\right), \begin{pmatrix} A' & J'B' \\ 0 & B' \end{pmatrix}\right)\right) \xmapsto{m_{20}} \left(d, \begin{pmatrix} AA' & (AJ'B^{-1} + J)BB' \\ 0 & BB' \end{pmatrix}\right)$$
- *The groupoid structure on $\mathrm{GL}(l,k)_2 \rightrightarrows \mathrm{GL}(l,k)_1$ is the unique one with source and target maps $t_{21}, s_{21}\colon \mathrm{GL}(l,k)_2 \to \mathrm{GL}(l,k)_1$ given by:*
$$(d\,,\ A\,,\ (I + dJ)B) \xleftarrow{t_{21}} \left(d\,, \begin{pmatrix} A & JB \\ 0 & B \end{pmatrix}\right) \xmapsto{s_{21}} (d\,,\ (I + Jd)^{-1}A\,,\ B) \ ,$$
and composition map $m_{21} : (\mathrm{GL}(l,k)_2)\ {}_{s_{21}}\!\times_{t_{21}} (\mathrm{GL}(l,k)_2) \to \mathrm{GL}(l,k)_2$ given by:
$$\left(\left(d, \begin{pmatrix} A & J(I + dJ')B' \\ 0 & (I + dJ')B' \end{pmatrix}\right), \left(d, \begin{pmatrix} (I + Jd)^{-1}A & J'B' \\ 0 & B' \end{pmatrix}\right)\right) \xmapsto{m_{21}} \left(d, \begin{pmatrix} A & (JdJ' + J + J')B' \\ 0 & B' \end{pmatrix}\right)$$
- *The groupoid $\mathrm{GL}(l,k)_1 \rightrightarrows \mathrm{GL}(l,k)_0$ is given by the action groupoid of the canonical left action of $\mathrm{GL}(l) \times \mathrm{GL}(k)$ on $\mathrm{GL}(l,k)_0$.*

5. Frame principal bundle of a VB-groupoid

Definition 5.1. *(see [1]) Let $E_{\mathcal{G}} \rightrightarrows E_M$ be a VB-groupoid of rank (l, k) over $\mathcal{G} \rightrightarrows M$. A frame $\phi_g \in \mathrm{Fr}(E_{\mathcal{G}})$ is called an* **s-bisection frame** *if the following two conditions holds:*

1. $\phi_g|_{\mathbb{R}^l \times \{0\}}$ takes value in $\ker(\widetilde{s}_g)$;

2. $\phi_g|_{\{0\} \times \mathbb{R}^k}$ is a bisection frame (i.e. its image is transverse to $\ker(\widetilde{t}_g)$).

$$\mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}} := \{\phi_g : \mathbb{R}^{l+k} \to (E_{\mathcal{G}})_g \mid \phi_g \text{ s-bisection frame}\} \subseteq \mathrm{Fr}(E_{\mathcal{G}}).$$

This set of s-bisection frames of $E_{\mathcal{G}}$ is called the ***s-bisection frame bundle*** *of $E_{\mathcal{G}}$.*

Proposition 5.2. *(see [1]) There is a groupoid structure on $\mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}}$ over $\mathrm{Fr}(C_M) \times_M \mathrm{Fr}(E_M)$.*

Proposition 5.3. *(see [1]) There is a canonical principal $\mathrm{GL}(l,k)_1 \rightrightarrows \mathrm{GL}(l,k)_0$ action on $\mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}}$, and a canonical $\mathcal{G}(l,k)_1 \rightrightarrows \mathrm{GL}(l,k)_0$ action on $\mathrm{Fr}(C_M) \times_M \mathrm{Fr}(E_M)$ extending to a Lie 2-groupoid action:*

$$\begin{tikzcd} \mathrm{GL}(l,k)_2 \arrow[d, shift right, "\mathbf{t}"'] \arrow[d, shift left, "\mathbf{s}"] & \mathrm{Fr}(E_{\mathcal{G}})^{\mathrm{sbis}} \arrow[l, bend right] \arrow[r, "\Pi_{\mathcal{G}}"] \arrow[d, shift right, "\mathbf{t}"'] \arrow[d, shift left, "\mathbf{s}"] \arrow[dl, dotted] & \mathcal{G} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ \mathrm{GL}(l,k)_1 \arrow[d, shift right, "\mathbf{t}"'] \arrow[d, shift left, "\mathbf{s}"] & (\mathrm{Fr}(C_M) \times_M \mathrm{Fr}(E_M)) \arrow[l, bend right] \arrow[r, "\Pi_M"] \arrow[dl, dotted] & M \\ \mathrm{GL}(l,k)_0 & & \end{tikzcd}$$

We called PB-groupoid a diagram as above, with a principal Lie 2-groupoid action on a Lie groupoid. Then we get a correspondence:

$$\{\ \mathrm{GL}(l,k)\text{-PB-groupoids over } \mathcal{G} \rightrightarrows M\} \longleftrightarrow \{\text{ VB-groupoids of rank } (l,k) \text{ over } \mathcal{G} \rightrightarrows M\}$$

References

[1] Francesco Cattafi and A. G. PB-groupoids vs VB-groupoids arXiv:2406.06259

[2] Kirill C.H. Mackenzie General theory of Lie groupoids and Lie algebroids. *London Mathematical Society.*, Lecture notes Series Vol 213, 2005.

[3] Matias del Hoyo and Davide Stefani The general linear 2-groupoid *Pacific J.Math* 298.1 pp.33-57, 2019.