

# Quantization (Deformation)

Introduction

Alfonso Garmendia

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# Quantization

Classical M ← → Quantum M

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Classical M

# Quantization

← →

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# Quantization

1

Classical M

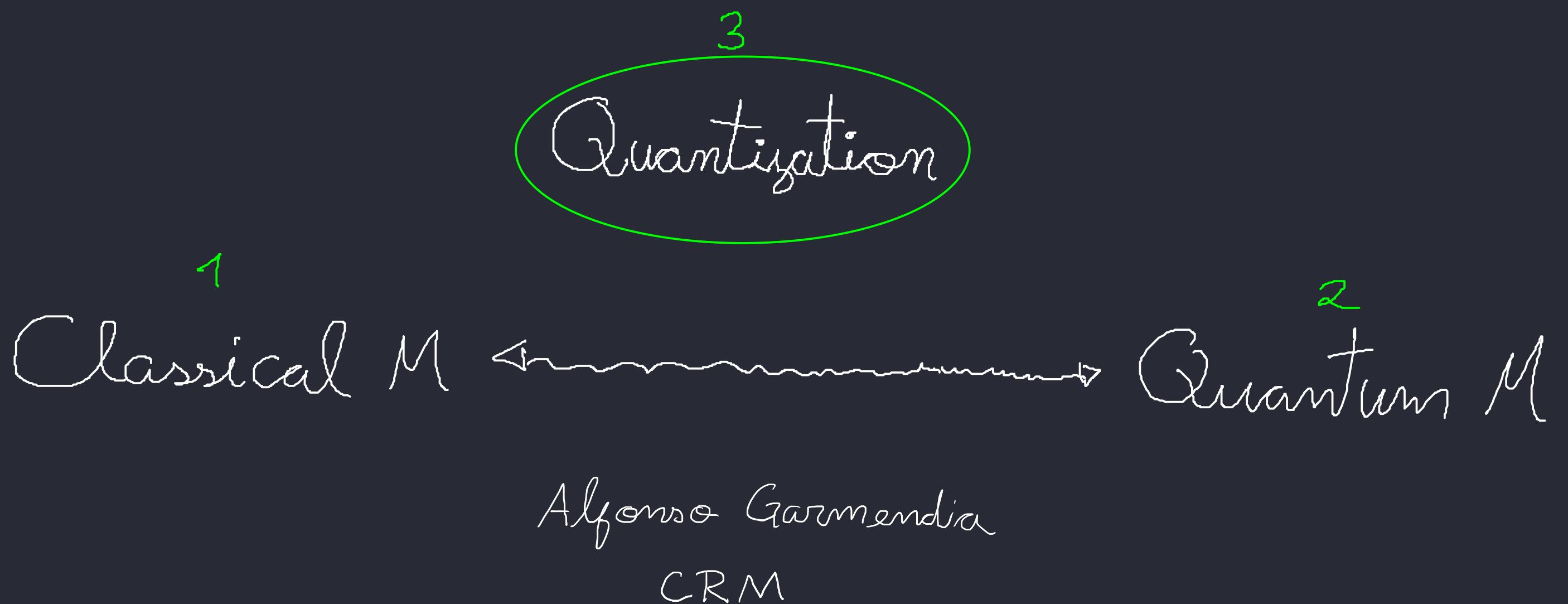


2

Quantum M

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3

# Quantisation

(Deformation)<sup>4</sup>

1

# Classical N

2

Quantum M

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# 1 Classical Mechanics

## The Model

- A Poisson manifold  $(M, \{ -, - \})$
- A Hamiltonian  $H \in C^\infty(M)$

⇒ The evolution equation is

$$\frac{d}{dt} f = \{ H, f \}$$

2

# 1 Classical Mechanics

⇒

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■

## The main example

- $M = \mathbb{R}^{2n}$  coordinates  $(p_1, \dots, p_n, q_1, \dots, q_n)$

$$\{ f, g \} = \sum_i \left( \frac{\partial}{\partial p_i} f \right) \left( \frac{\partial}{\partial q_i} g \right) - \left( \frac{\partial}{\partial p_i} g \right) \left( \frac{\partial}{\partial q_i} f \right)$$

- $H(p, q) = \frac{\|p\|^2}{2} + V(q)$

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$$H(p, q) = \frac{\|p\|^2}{2} + V(q)$$

⇒ The evolution equation is equivalent to

$$\frac{dp_i}{dt} = \{ H, p_i \} = - \frac{\partial H}{\partial q_i}$$

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} Hamilton-Jacobi  
equations  
(Euler-Lagrange)

if these equations aren't familiar,

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Newton's Eq in  $\mathbb{R}^n$

$$q_i = - \partial_{q_i} V(q)$$

Force

Acceleration





# Quantum Mechanics

## The Model

- A Hilbert space  $L^2$  & states
- a subalgebra  $A \subseteq B(L)$  of observables
- Schrödinger operator  $\hat{H}$  on  $L$

⇒ The evolution equation is

$$i\hbar \frac{\partial}{\partial t} f = \hat{H}(f)$$

for states

$$i\hbar \frac{d}{dt} G = [\hat{H}, G]$$

for observables □

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- $L \subset L^2(\mathbb{R}^n)$  &  $A \subseteq \mathcal{B}(L)$  (Let it be vacue)

$$\hat{H} = -\frac{\hbar^2}{2} \Delta + V \leftarrow V \in C^\infty(M)$$

$\downarrow \quad \partial_{q_1}^2 + \dots + \partial_{q_n}^2$  the laplacian

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Schrödinger's equation

\* For observables Note that

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$$J_t(Qf) = (\partial_t Q)f + Q(\partial_t f) \quad \leftarrow \text{Leibniz}$$

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$$J_t(Qf) = (\partial_t Q)f + Q(\partial_t f) \quad \leftarrow \text{Leibniz}$$

$$\stackrel{!}{=} \hat{H}(Qf) = (\partial_t Q)f + \frac{1}{i\hbar} Q(\hat{H}f)$$

$$\Rightarrow i\hbar(\partial_t Q)(f) = [\hat{H}, Q](f) \quad \text{if for observables}$$





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## Example: Quantization example: Harmonic Oscillator

Quantum

for  $f \in C^\infty(\mathbb{R} \times \mathbb{R})$

$$i\hbar \frac{d}{dt} f = -\frac{\hbar^2}{2} \frac{d^2}{dq^2} f - \frac{q^2}{2} f$$

Schrödinger's

Classic

$$\frac{d^2}{dt^2} q = q^2 \quad \text{for } q \in C^\infty(\mathbb{R})$$

Newton's

3

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$$\text{Let } p := \frac{dq}{dt}$$

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Hamilton-Jacobi eq for

$$H(p, q) = \frac{p^2}{2} - \frac{q^2}{2} \quad \text{in } M = \mathbb{R}^2$$

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$$a_{(p, q)} = p + q \quad \& \quad a_{(p, q)}^+ = -p + q$$

$\Rightarrow$

$$\{H, a\} = p + q = a$$

$$\{H, a^+\} = p - q = -a^+$$



3 Example: Quantization example: Harmonic Oscillator

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Schrödinger's

Classic

$$\frac{d^2 q}{dt^2} = q^2 \quad \text{for } q \in C^\infty(\mathbb{R})$$

Newton's

Symmetries:

$$\hat{a} = i\hbar \frac{d}{dq} + q \quad | \quad \hat{a}^\dagger = -i\hbar \frac{d}{dq} + q$$

$$[\hat{H}, \hat{a}] = i\hbar \hat{a} \quad | \quad [\hat{H}, \hat{a}^\dagger] = -i\hbar \hat{a}^\dagger$$

Let  $f_0$  an eigenvector for  $\hat{H} \Rightarrow H(f_0) = \lambda f_0$

Hamilton-Jacobi eq for

$$\Rightarrow \frac{dP}{dt} = q^2$$

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$$\text{Let } f_0 \text{ an eigenvector for } \hat{H} \Rightarrow \hat{H}(f_0) = \lambda f_0$$

$$\Rightarrow \hat{H}(\hat{a} f_0) = (\lambda + i\hbar)(\hat{a} f_0)$$

$$\hat{H}(\hat{a}^\dagger f_0) = (\lambda - i\hbar)(\hat{a}^\dagger f_0)$$

$\hat{a}$  creation

$\hat{a}^\dagger$  destruction

Hermite Polynomials

Classic

$$\frac{d^2}{dt^2} q = q^2 \quad \text{for } q \in C^\infty(\mathbb{R})$$

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$$\text{let } p := \frac{d}{dt} q$$

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### 3 Quantization Dream

Poisson Mfd  $M \leftarrow \begin{cases} Q \\ J \\ A \\ N \\ T \end{cases} \rightarrow$  Hilbert space  $L_M$   
any function  $f \in C^\infty(M) \leftarrow$  operator  $\hat{f} : L_M \rightarrow L_M$

such that:

- (1)  $f \rightarrow \hat{f}$  is  $C$ -linear
- (2)  $[\hat{f}, \hat{g}] = i\hbar \{f, g\}$
- (3)  $\frac{\|P\|^2}{2} + V(g) = -\frac{\hbar^2}{2} \Delta + V$

- (4) for  $P$  a power series in  $\mathbb{R}$

$$\widehat{P(f)} = P(\hat{f})$$

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such that:

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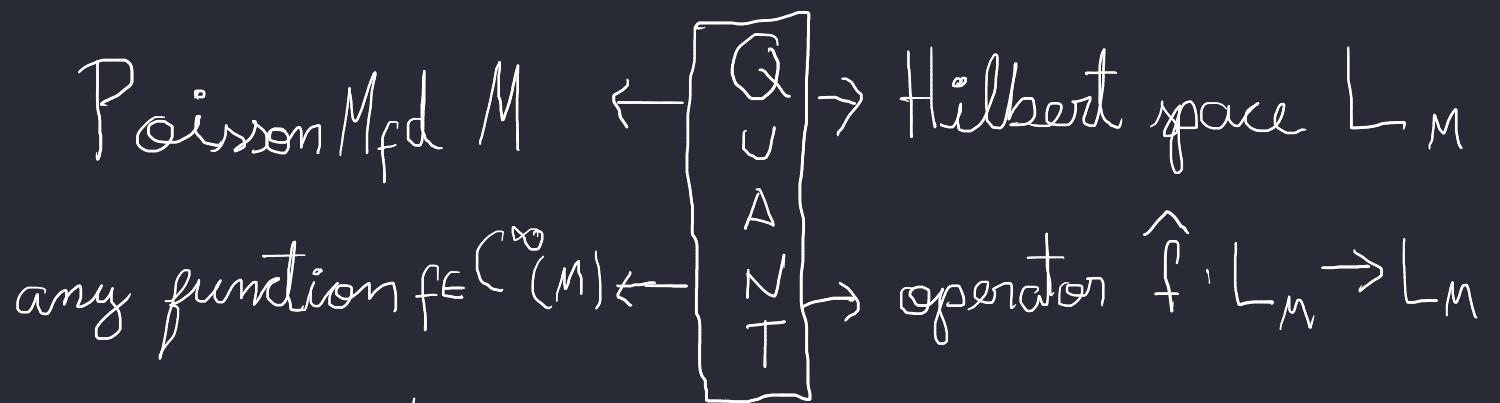
(4) for  $P$  a power series in  $\mathbb{R}$

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Why we want this? (1) convenience

(2) if  $\partial_t f = \{H, f\}$  } solutions to equations  
 $\Rightarrow \partial_t \hat{f} = \frac{1}{i\hbar} [\hat{H}, \hat{f}]$  } symmetries to symmetries

### 3 Quantization Dream



such that:

$$\textcircled{1} \quad f \rightarrow \hat{f} \text{ is } \mathbb{C}\text{-linear}$$

$$\textcircled{2} \quad [\hat{f}, \hat{g}] = i\hbar \{f, g\}$$

$$\textcircled{3} \quad \frac{\|P\|^2}{2} + V(g) = -\frac{\hbar^2}{2} \Delta + V$$

\textcircled{4} for  $P$  a power series in  $\mathbb{R}$

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Why we want this? ① convenience

$$\textcircled{2} \quad \begin{aligned} & \text{if } \partial_t f = \{H, f\} \\ & \Rightarrow \partial_t \hat{f} = \frac{1}{i\hbar} [\hat{H}, \hat{f}] \end{aligned} \quad \left. \begin{array}{l} \text{solutions to solutions} \\ \text{symmetries to symmetries} \end{array} \right\}$$

③ It fits in the known examples

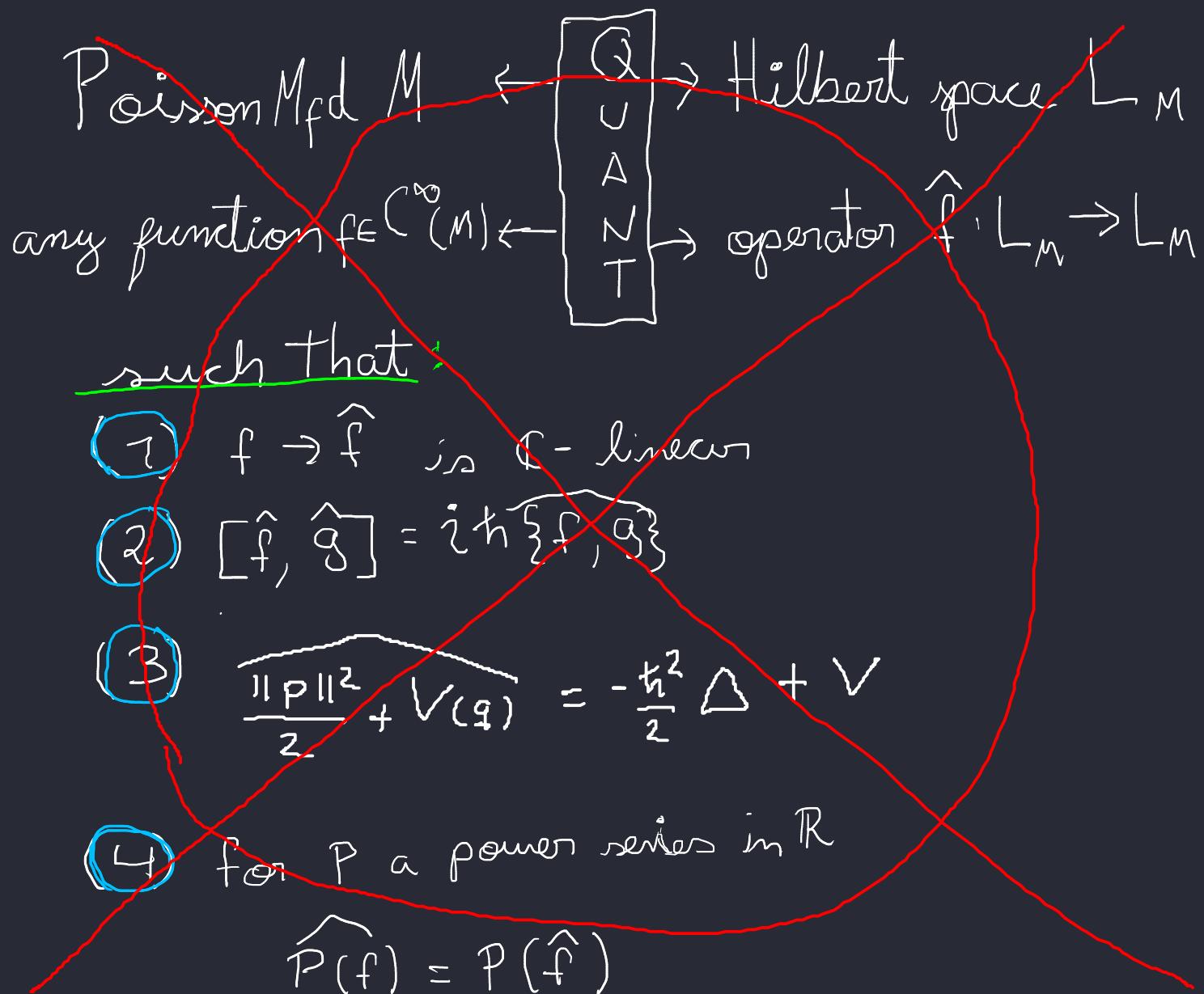
④ It is very convenient. For example to solve

$$\partial_t f = \hat{g} f$$

The natural solution is:  $f = e^{t\hat{g}}$  if exists

and it would because  $e^{t\hat{g}} = \widehat{e^{tg}}$

### 3 Quantization Dream



Why we want this?

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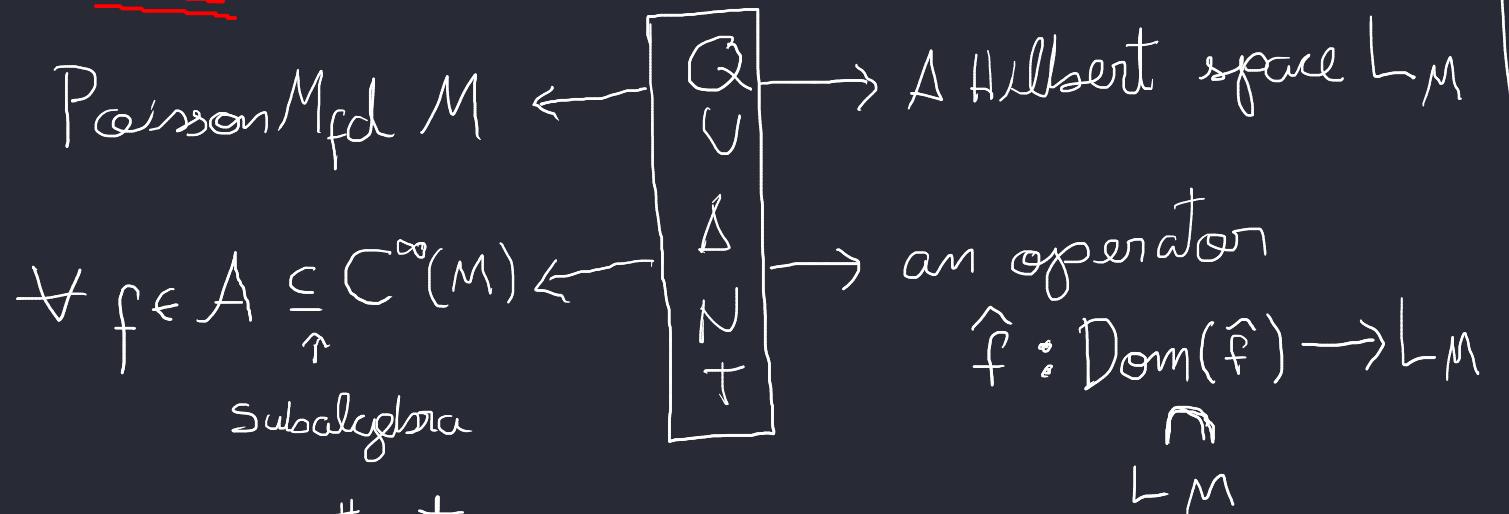
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\* PROBLEM! This Dream is a lie!

### 3 Real Quantization (still using $\mathbb{C}$ )



such that

1  $f \mapsto \hat{f}$  is  $\mathbb{C}$ -linear

2  $[\hat{f}, \hat{g}] = i\hbar \overbrace{\{f, g\}} + \mathcal{O}(\hbar^2)$

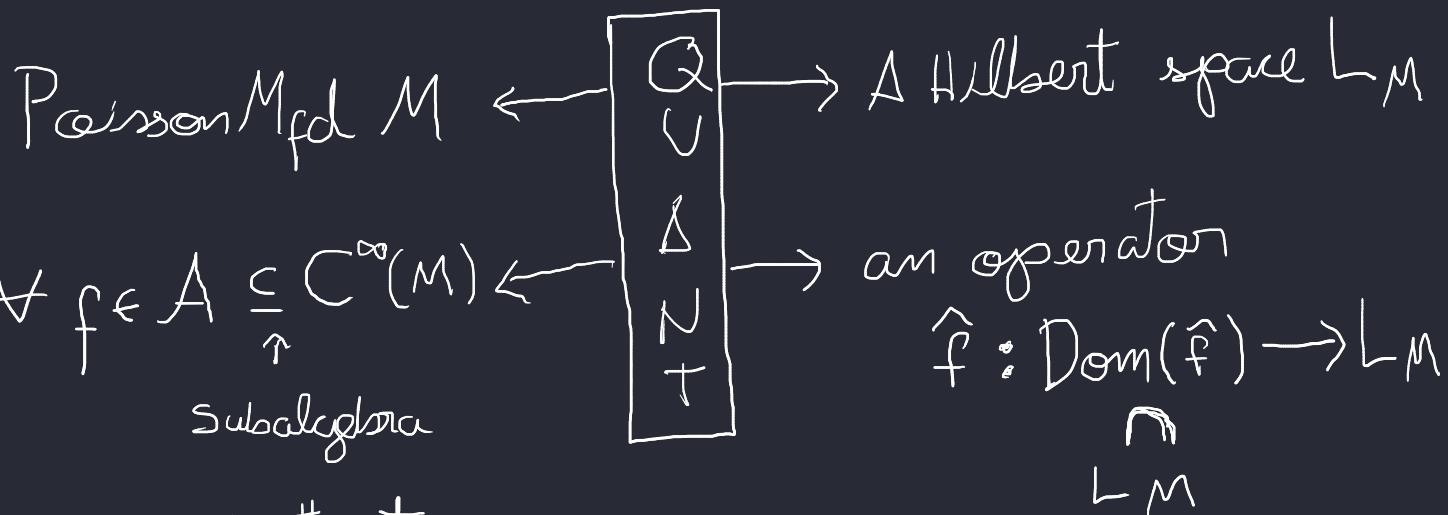
3  $\overbrace{\frac{\|P\|^2}{2} + V(q)} = -\frac{\hbar^2}{2} \Delta + V$

4 for  $P$  a power series in  $\mathbb{R}$

$$P(\hat{f}) = \widehat{P(f)} + \mathcal{O}(\hbar)$$



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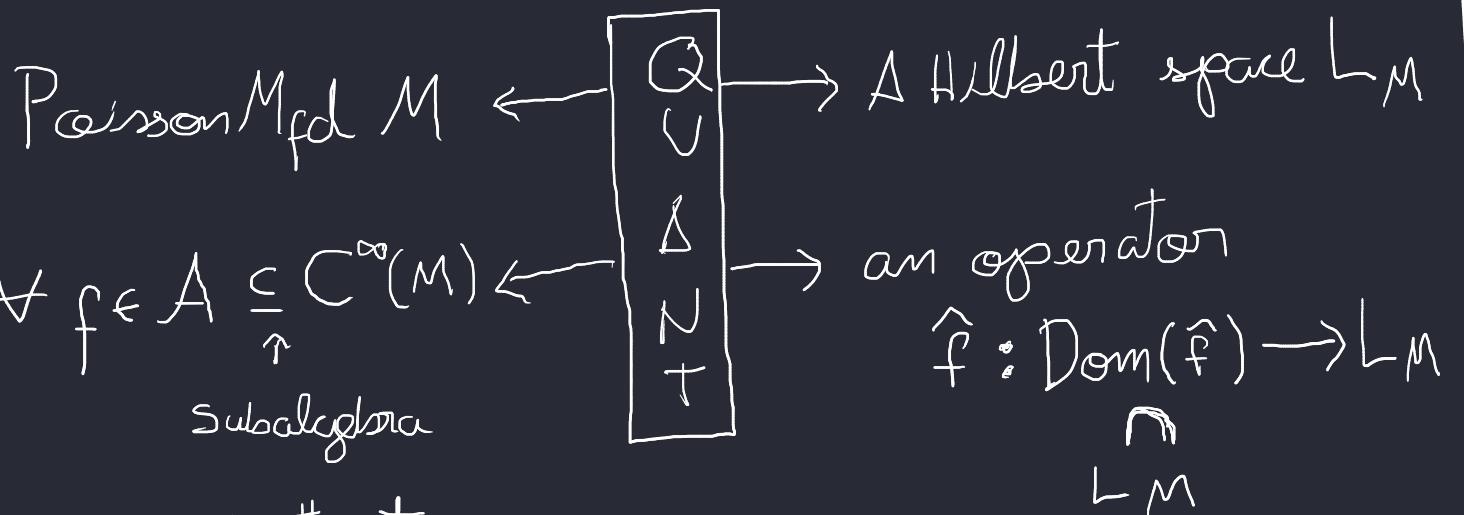
### Example $M = \mathbb{R}^{2n}$

in  $\mathbb{R}^n$  there is the Fourier transform  
 for any  $f \in C_c^\infty(\mathbb{R}^n)$  it is

$$\mathcal{F}(f)(p) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} \langle p, q' \rangle} f(q') dq'$$

it satisfy:

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• it satisfy:  $\mathcal{F}(\partial_{q_i} f)(P) = -\frac{i}{\hbar} P_i \mathcal{F}(f)(P)$

• Therefore:  $\mathcal{F}\left(-\frac{\hbar^2}{2} \Delta(f)\right) = \frac{\|P\|^2}{2} \mathcal{F}(f)(P)$

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Poisson Mfd  $M \leftarrow \begin{bmatrix} Q \\ V \\ \Delta \\ N \\ T \end{bmatrix} \rightarrow$  A Hilbert space  $L_M$

$\forall f \in A \subseteq C^\infty(M) \leftarrow$  an operator  
 $\hat{f} : \text{Dom}(\hat{f}) \rightarrow L_M$   
 Subalgebra

such that

1  $f \mapsto \hat{f}$  is  $\mathbb{C}$ -linear

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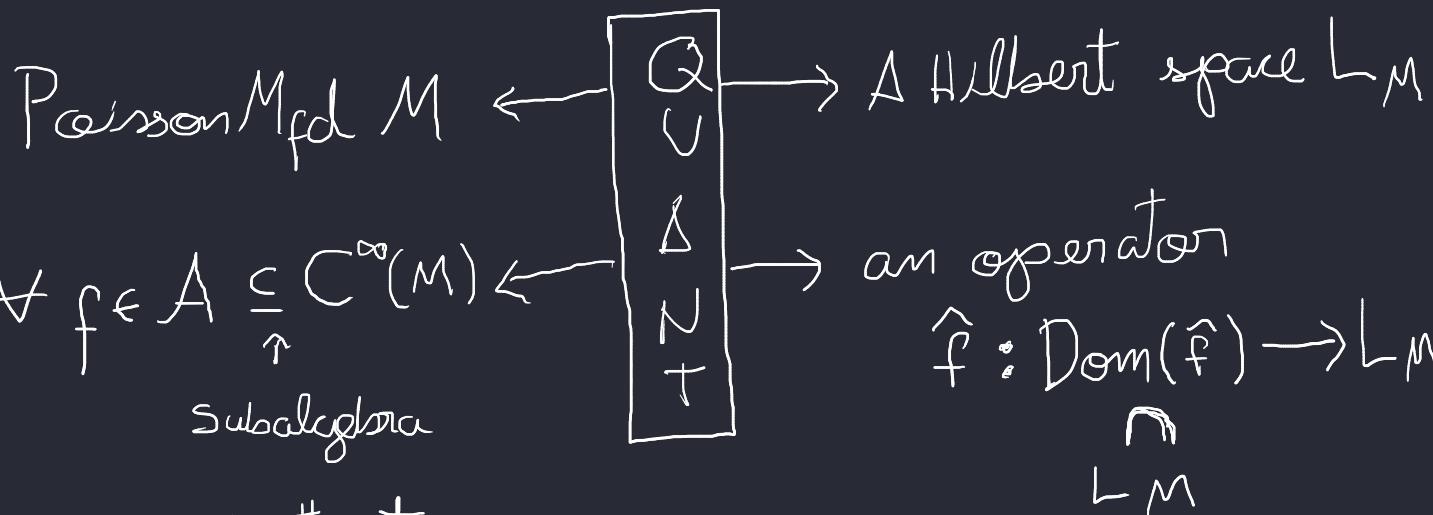
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- it satisfy:  $\mathcal{F}(d_{q_i} f)(P) = -\frac{i}{\hbar} P_i \mathcal{F}(f)(P)$
- Therefore:  $\mathcal{F}\left(-\frac{\hbar^2}{2} \Delta(f)\right) = \frac{\|P\|^2}{2} \mathcal{F}(f)(P)$
- This help us rewrite the Schrödinger operator as:

$$\begin{aligned} \hat{H}(f)(q) &= \mathcal{F}^{-1} \left( \mathcal{F} \left( -\frac{\hbar^2}{2} \Delta(f) + Vf \right) \right) q \\ &= \iint e^{Bla(q-q')} \underbrace{H_{(P,q')} f(q') dq' dP}_{H_{(P,q)}} \end{aligned}$$

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- Therefore:  $\mathcal{F}\left(-\frac{\hbar^2}{2} \Delta(f)\right) = \frac{\|P\|^2}{2} \mathcal{F}(f)(P)$
- This help us rewrite the Schrödinger operator as:

$$\begin{aligned} \hat{H}(f)(q) &= \mathcal{F}^{-1} \left( \mathcal{F} \left( -\frac{\hbar^2}{2} \Delta(f) + V \right) \right) q \\ &= \iint e^{B\alpha(q-q')} H_{(P,q)} f(q') dq' dP \end{aligned}$$



## 4 Formal Deformation Quantization

### Idea

Poisson  $M \xrightarrow{DQ} C^*$ -Algebra  $A_M$

$f \in C^\infty(M) \mapsto a_f \in A_M$

Not DQ  
Gelfand-Naimark-Segal  
Rep theory  $\xrightarrow{\text{Hilbert } L_M}$

$\xrightarrow{\quad} \hat{f}_M : \text{Dom}(\hat{f}) \rightarrow L_M$

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### How

- $f \in M \xrightarrow{DQ} (C^\infty(M)[[\hbar]], *)$   
Power series | Not usual product
- $f \in C^\infty(M) \mapsto \hat{f} = f + O(\hbar) + O(\hbar^2) + \dots$
- $f * g = f \cdot g + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \dots + O(\hbar^2)$  ①

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such that:

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This gives some restrictions

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- There is a cohomology theory  
the Hochschild Cohomology  
controlling the  $B_i$  in formula ①  
such that  $*$  is associative
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etc...  
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This gives some restrictions

- There is a cohomology theory the Hochschild Cohomology controlling the  $B_i$  in formula (1) such that  $*$  is associative
- $B_1(f, g) - B_1(g, f) = i \{f, g\}$
- Theorems of Kontsevich-Tamarkin-Weil-Fedosov etc...  
There exist a product  $*$  for any Poisson  $M$   
 $*$  is unique up to something non-trivial
- There is no explicit construction of  $*$  for any  $M$  yet.

### Quantization<sup>3</sup> (Deformation)<sup>4</sup>

Classical  $M \longleftrightarrow$  Quantum  $M$   
Alfonso Garmendia  
CRM

## 1 Classical Mechanics

### The Model

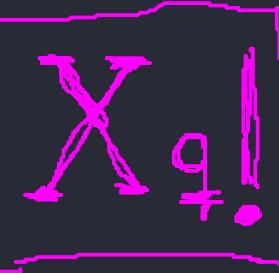
- A Poisson manifold  $(M, \{ \cdot, \cdot \})$
- A Hamiltonian  $H \in C^\infty(M)$
- ⇒ The evolution equation is  

$$\frac{d}{dt} f = \{ H, f \}$$

## 2 Quantum Mechanics

### The Model

- A Hilbert space  $L^2$  & states
  - a subalgebra  $A \subseteq B(L)$  of observables
  - Schrödinger operator  $\hat{H}$  on  $L^2$
- ⇒ The evolution equation is
- $$i\hbar \frac{d}{dt} f = \hat{H}(f) \quad i\hbar \frac{d}{dt} Q = [\hat{H}, Q]$$
- for states                                    for observables



## 3 Example: Quantization example: Harmonic Oscillator

Quantum for  $f \in C^\infty(\mathbb{R} \times \mathbb{R})$   $i\hbar \frac{d}{dt} f = -\frac{\hbar^2}{2} \frac{d^2 f}{dq^2} - \frac{q^2 f}{2}$  [Schrödinger's]

Symmetries:

$$\Rightarrow \hat{a} = i\hbar \frac{d}{dq} + q \quad \hat{a}^\dagger = -i\hbar \frac{d}{dq} + q$$

$$[\hat{H}, \hat{a}] = i\hbar \hat{a} \quad [\hat{H}, \hat{a}^\dagger] = -i\hbar \hat{a}^\dagger$$

Let  $f_0$  an eigenvector for  $\hat{H} \Rightarrow \hat{H}(f_0) = \lambda f_0$

$$\Rightarrow \hat{H}(\hat{a} f_0) = (\lambda + i\hbar)(\hat{a} f_0) \quad \text{so } \hat{a} f_0 \text{ & } \hat{a}^\dagger f_0 \text{ are eigenvectors}$$

$$\hat{H}(\hat{a}^\dagger f_0) = (\lambda - i\hbar)(\hat{a}^\dagger f_0)$$

$\hat{a}$  creation       $\hat{a}^\dagger$  destruction

Hermite polynomials

Classic  $\frac{d^2 q}{dt^2} = q^2 \text{ for } q \in C^\infty(\mathbb{R})$  [Newton's]

Let  $p := \frac{dq}{dt}$   $\left\{ \begin{array}{l} \text{Hamilton-Jacobi eq for} \\ \Rightarrow \frac{dp}{dt} = q^2 \end{array} \right.$

$$H(p, q) = \frac{p^2}{2} - \frac{q^2}{2} \text{ in } M = \mathbb{R}^2$$

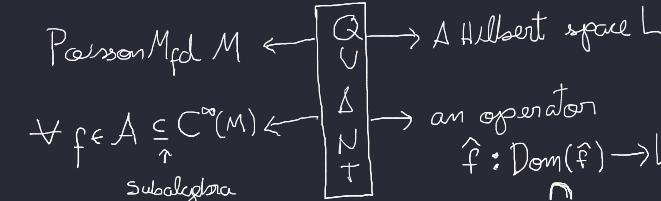
This system have the symmetries

$$a_{(p,q)} = p + q \quad \& \quad a_{(p,q)}^\dagger = -p + q$$

$$\{ H, a \} = p + q = a$$

$$\{ H, a^\dagger \} = p - q = -a^\dagger$$

## 3 Real Quantization (still using $\mathbb{C}$ )



such that

- $f \mapsto \hat{f}$  is  $\mathbb{C}$ -linear
- $[\hat{f}, \hat{g}] = i\hbar \{ f, g \} + \mathcal{O}(\hbar^2)$
- $\widehat{\left( \frac{1}{2} P^2 + V(q) \right)} = -\frac{\hbar^2}{2} \Delta + V$
- for  $P$  a power series in  $\mathbb{R}$

$$P(\hat{f}) = \widehat{P(f)} + \mathcal{O}(\hbar)$$

## Example $M = \mathbb{R}^{2n}$

in  $\mathbb{R}^n$  there is the Fourier transform  
for any  $f \in C_c^\infty(\mathbb{R}^n)$  it is

$$\mathcal{F}(f)(p) = \int_{\mathbb{R}^n} e^{-i\langle p, q \rangle} f(q) dq$$

it satisfy:  $\mathcal{F}(i\hbar \frac{d}{dt} f)(p) = -\frac{i}{\hbar} p; \mathcal{F}(f)(p)$

Therefore:  $\mathcal{F}\left(-\frac{\hbar^2}{2} \Delta(f)\right) = \frac{\|P\|^2}{2} \mathcal{F}(f)(p)$

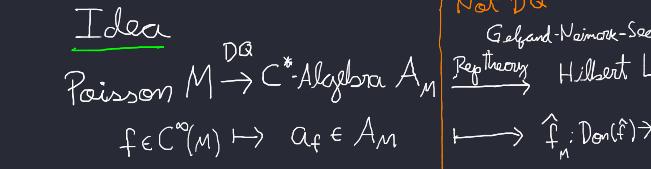
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$$\hat{H}(f)(q) = \mathcal{F}^{-1}\left(\mathcal{F}\left(-\frac{\hbar^2}{2} \Delta(f) + V_f\right)\right)q$$

$$= \iint e^{i\langle p, q - q' \rangle} \underline{H(p, q')} f(q') dp dq'$$

This gives some restrictions

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