1 Convexity

TBD.

2 Spectral Graph Theory

We assume the graph has n vertices and m edges.

2.1 Courant-Fischer Theorem

1. **Eigenvalue version**: Let A be a symmetric matrix with eigenvalues $\lambda_1 \le \cdots \le \lambda_n$, then

$$\lambda_i = \min_{\substack{\text{subspace } W \subseteq \mathbb{R}^n \\ \dim(W)=i}} \max_{\substack{x \in W \\ x \neq \mathbf{0}}} \frac{x^\top A x}{x^\top x} = \max_{\substack{\text{subspace } W \subseteq \mathbb{R}^n \\ \dim(W)=n+1-i}} \min_{\substack{x \in W \\ x \neq \mathbf{0}}} \frac{x^\top A x}{x^\top x}.$$

2. **Eigenbasis version**: Let A be a symmetric matrix with eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ and corresponding orthonormal eigenvectors $x_1, \ldots, x_n$, then

$$\lambda_i = \min_{\substack{x \perp x_1, \ldots x_{i-1} \\ x \neq \mathbf{0}}} \frac{x^\top A x}{x^\top x} = \max_{\substack{x \perp x_{i+1}, \ldots x_n \\ x \neq \mathbf{0}}} \frac{x^\top A x}{x^\top x}.$$

Note that we also have $\lambda_i = \frac{x_i^\top A x_i}{x_i^\top x_i}$.

Applying Courant-Fischer theorem, we have $\lambda_2 x^\top x \le x^\top L x \le \lambda_n x^\top x$ for all $x \perp \mathbf{1}$, as $\mathbf{1}$ is the eigenvector of L corresponding to eigenvalue 0. For connected graphs, $\lambda_2 > 0$.

2.2 PSD Order (Loewner Order)

Defined only for symmetric matrices: $A \preceq B$ iff for all $x \in \mathbb{R}^n$, we have $x^\top A x \le x^\top B x$. We also define $G \preceq H$ for two graphs G and H iff $L_G \preceq L_H$. We always have $G \succeq H$ if H is a subgraph of G.

Properties:

1. If $A \preceq B$ and $B \preceq C$, then $A \preceq C$.
2. If $A \preceq B$, then $A + C \preceq B + C$ for any symmetric C.
3. If $A \preceq B$ and $C \preceq D$, then $A + C \preceq B + D$.
4. If $A \succ 0$ and $\alpha \ge 1$, then $\frac{1}{\alpha} A \preceq A \preceq \alpha A$.
5. If $A \preceq B$, then $\lambda_i(A) \le \lambda_i(B)$ for all i. Proof by Courant-Fischer theorem. The converse is not true.

2.3 Bounding the λ_2 and λ_n

2.3.1 Test Vector Method

Since $\lambda_2 \le \frac{y^\top L y}{y^\top y}$ for any $y \perp \mathbf{1}$, we can upper bound the λ_2 by any **test vector** y. Similarly, we can lower bound the λ_n by test vectors by $\lambda_n \ge \frac{y^\top L y}{y^\top y}$.

1. For a complete graph K_n, $L = n\boldsymbol{I} - \mathbf{1}\mathbf{1}^\top$ and for any $x \perp \mathbf{1}$ we have $x^\top L x = n x^\top x$. Therefore, $\lambda_2(K_n) = \cdots = \lambda_n(K_n) = n$ and any $x \perp \mathbf{1}$ is an eigenvector.
2. For a path graph P_n, let $x(i) = n + 1 - 2i$ be the test vector which satisfies $x \perp \mathbf{1}$, we get $\lambda_2(P_n) \le \frac{12}{n^2}$. Let $x(1) = -1$, $x(n) = 1$ and $x(i) = 0$ for other i to be the test vector, we get $\lambda_n(P_n) \ge 1$.
3. For a complete binary tree T_n (depth equals zero for a single root), let $x(i) = 0$ for all non-leaf nodes, $x(i) = -1$ for even-numbered leaf nodes and $x(i) = 1$ for odd-numbered leaf nodes be the test vector, we get $\lambda_n(T_n) \ge 1$. Let $x(1) = 0$, $x(i) = 1$ for the left subtree of the root and $x(i) = -1$ for the right subtree of the root be the test vector, we get $\lambda_2(T_n) \le \frac{2}{n-1}$.

2.3.2 Consequences of PSD Order

Since $x^\top (D - A) x = \sum_{(u,v)} w(u,v)(x(u) - x(v))^2 \ge 0$ and $x^\top (D + A) x = \sum_{(u,v)} w(u,v)(x(u) + x(v))^2 \ge 0$, we have $D \succeq A$ and $D \succeq -A$. In addition, we have $D \preceq (\max D_{i,i})\boldsymbol{I}$. Therefore, we have $L = D - A \succeq 2D \succeq (2\max D_{i,i})\boldsymbol{I}$, which implies $\lambda_n \le 2\max D_{i,i}$ for any graph. For unit-weight graphs, this means $\lambda_n \le 2\max \text{degree}(v)$. The bound is tight for a single-edge graph.

To get lower bounds of $\lambda_2(H)$, we first establish $f(n)H \succeq G$ for some G with known lower bounds on $\lambda_2(G)$. Usually $G = K_n$ because $\lambda_2(K_n) = n$. Then it follows that $\lambda_2(H) \ge \lambda_2(G)/f(n)$.

1. **Path Graph** P_n: Let $G_{i,j}$ denote a unit-weight graph consisting of one edge (i,j) and P_n be the path graph connecting 1 and n. Then $(n-1)P_n \succeq G_{1,n}$. Proof follows from applying Cauchy-Schwartz for $\delta_i := x(i+1) - x(i)$. For weighted paths, we have $G_{1,n} \preceq \left(\sum_{i=1}^{n-1} \frac{1}{w_i}\right) \sum_{i=1}^{n-1} w_i G_{i,i+1}$.
   Applying path inequality, we have $K_n = \sum_{i<j} G_{i,j} \preceq \sum_{i<j} (j-i) P_{i,j} \preceq \sum_{i<j} (j-i) P_n \preceq n^3 P_n$, which implies $\lambda_2(P_n) \ge \lambda_2(K_n)/n^3 = 1/n^2$.
2. **Any unit-weight graph** G: Define the diameter D of a graph G to be the maximum length of the shortest paths between any two nodes. Let $G_{i,j}^s$ be the shortest path from i to j. Applying path inequality, we have $K_n = \sum_{i<j} G_{i,j} \preceq \sum_{i<j} D G_{i,j}^s \preceq \sum_{i<j} D G \preceq n^2 D G$, which implies $\lambda_2(G) \ge \frac{1}{nD}$.
3. **Complete Binary Tree** T_n: Define G_e be the single-edge graph with edge e, and $T_{i,j}$ be the unique path between i and j. Applying the weighted path inequality, we have $K_n = \sum_{i<j} G_{i,j} \preceq \sum_{i<j} \left(\left(\sum_{e \in T^{i,j}} \frac{1}{w(e)} \right) \left(\sum_{e \in T^{i,j}} w(e) G_e \right) \right) \preceq \left(\max_{i<j} \sum_{e \in T^{i,j}} \frac{1}{w(e)} \right) \left(\sum_{i<j} \sum_{e \in T^{i,j}} w(e) G_e \right)$. For e connecting level i and $i+1$ for $i \in [d-1]$, we set $w(e) = 2^i$. Then $\max_{i<j} \sum_{e \in T^{i,j}} \frac{1}{w(e)} \le 4$. Since the number of occurrence of e in $T^{i,j}$ for any $i < j$ is upper bounded by $n^2 2^{-i}$, we have $\sum_{i<j} \sum_{e \in T^{i,j}} w(e) G_e \preceq \sum_e w(e) n^2 2^{-i} G_e = \sum_e n^2 G_e = n^2 T_n$. Therefore, $K_n \preceq 4n^2 T_n$, which implies $\lambda_2(T_n) \ge \frac{1}{4n}$.