1 Spectral Graph Theory

We assume the graph has n vertices and m edges. The vertex set is V and the edge set is E.

1.1 Courant-Fischer Theorem

1. **Eigenvalue version**: Let A be a symmetric matrix with eigenvalues $\lambda_1 \le \cdots \le \lambda_n$, then

$$\lambda_i = \min_{\substack{\text{subspace } W\subseteq\mathbb{R}^n\\ \dim(W)=i}} \max_{\substack{x\in W\\ x\neq \mathbf{0}}} \frac{x^\top A x}{x^\top x} = \max_{\substack{\text{subspace } W\subseteq\mathbb{R}^n\\ \dim(W)=n+1-i}} \min_{\substack{x\in W\\ x\neq \mathbf{0}}} \frac{x^\top A x}{x^\top x}.$$

2. **Eigenbasis version**: Let A be a symmetric matrix with eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ and corresponding orthonormal eigenvectors $x_1,\ldots,x_n$, then

$$\lambda_i = \min_{\substack{x\perp x_1,\ldots x_{i-1}\\ x\neq \mathbf{0}}} \frac{x^\top A x}{x^\top x} = \max_{\substack{x\perp x_{i+1},\ldots x_n\\ x\neq \mathbf{0}}} \frac{x^\top A x}{x^\top x}.$$

Note that we also have $\lambda_i = \frac{x_i^\top A x_i}{x_i^\top x_i}$.

Applying Courant-Fischer theorem, we have $\lambda_2 x^\top x \le x^\top L x \le \lambda_n x^\top x$ for all $x \perp \mathbf{1}$, as $\mathbf{1}$ is the eigenvector of L corresponding to eigenvalue 0. For connected graphs, $\lambda_2 > 0$.

1.2 PSD Order (Loewner Order)

Defined only for symmetric matrices: $A \preceq B$ iff for all $x \in \mathbb{R}^n$, we have $x^\top A x \le x^\top B x$. We also define $G \preceq H$ for two graphs G and H iff $L_G \preceq L_H$. We always have $G \succeq H$ if H is a subgraph of G.

Properties:
1. If $A \preceq B$ and $B \preceq C$, then $A \preceq C$.
2. If $A \preceq B$, then $A + C \preceq B + C$ for any symmetric C.
3. If $A \preceq B$ and $C \preceq D$, then $A + C \preceq B + D$.
4. If $A \succ 0$ and $\alpha \ge 1$, then $\frac{1}{\alpha}A \preceq A \preceq \alpha A$.
5. If $A \preceq B$, then $\lambda_i(A) \le \lambda_i(B)$ for all i. Proof by Courant-Fischer theorem. The converse is not true.

1.3 Bounding the λ_2 and λ_n

1.3.1 Test Vector Method

Since $\lambda_2 \le \frac{y^\top L y}{y^\top y}$ for any $y \perp \mathbf{1}$, we can upper bound the λ_2 by any **test vector** y. Similarly, we can lower bound the λ_n by test vectors by $\lambda_n \ge \frac{y^\top L y}{y^\top y}$.

1. For a complete graph K_n, $L = n\mathbf{I} - \mathbf{1}\mathbf{1}^\top$ and for any $x \perp \mathbf{1}$ we have $x^\top L x = n x^\top x$. Therefore, $\lambda_2(K_n) = \cdots = \lambda_n(K_n) = n$ and any $x \perp \mathbf{1}$ is an eigenvector.
2. For a path graph P_n, let $x(i) = n + 1 - 2i$ be the test vector which satisfies $x \perp \mathbf{1}$, we get $\lambda_2(P_n) \le \frac{12}{n^2}$. Let $x(1) = -1$, $x(n) = 1$ and $x(i) = 0$ for other i to be the test vector, we get $\lambda_n(P_n) \ge 1$.
3. For a complete binary tree T_n (depth equals zero for a single root), let $x(i) = 0$ for all non-leaf nodes, $x(i) = -1$ for even-numbered leaf nodes and $x(i) = 1$ for odd-numbered leaf nodes be the test vector, we get $\lambda_n(T_n) \ge 1$. Let $x(1) = 0$, $x(i) = 1$ for the left subtree of the root and $x(i) = -1$ for the right subtree of the root be the test vector, we get $\lambda_2(T_n) \le \frac{2}{n-1}$.

1.3.2 Consequences of PSD Order

Since $x^\top(D-A)x = \sum_{(u,v)} w(u,v)(x(u)-x(v))^2 \ge 0$ and $x^\top(D+A)x = \sum_{(u,v)} w(u,v)(x(u)+x(v))^2 \ge 0$, we have $D \succeq A$ and $D \succeq -A$. In addition, we have $D \preceq (\max D_{i,i})\boldsymbol{I}$. Therefore, we have $L = D - A \succeq 2D \succeq (2\max D_{i,i})\boldsymbol{I}$, which implies $\lambda_n \le 2\max D_{i,i}$ for any graph. For unit-weight graphs, this means $\lambda_n \le 2\max \operatorname{degree}(v)$. The bound is tight for a single-edge graph.

To get lower bounds of $\lambda_2(H)$, we first establish $f(n)H \succeq G$ for some G with known lower bounds on $\lambda_2(G)$. Usually $G = K_n$ because $\lambda_2(K_n) = n$. Then it follows that $\lambda_2(H) \ge \lambda_2(G)/f(n)$.

1. **Path Graph** P_n: Let $G_{i,j}$ denote a unit-weight graph consisting of one edge (i,j) and P_n be the path graph connecting 1 and n. Then $(n-1)P_n \succeq G_{1,n}$. Proof follows from applying Cauchy-Schwartz for $\delta_i := x(i+1) - x(i)$. For weighted paths, we have $G_{1,n} \preceq \left(\sum_{i=1}^{n-1}\frac{1}{w_i}\right)\sum_{i=1}^{n-1} w_i G_{i,i+1}$.
Applying path inequality, we have $K_n = \sum_{i<j} G_{i,j} \preceq \sum_{i<j}(j-i)P_{i,j} \preceq \sum_{i<j}(j-i)P_n \preceq n^3 P_n$, which implies $\lambda_2(P_n) \ge \lambda_2(K_n)/n^3 = 1/n^2$.
2. **Any unit-weight graph** G: Define the diameter D of a graph G to be the maximum length of the shortest paths between any two nodes. Let $G^s_{i,j}$ be the shortest path from i to j. Applying path inequality, we have $K_n = \sum_{i<j} G_{i,j} \preceq \sum_{i<j} D G^s_{i,j} \preceq \sum_{i<j} DG \preceq n^2 DG$, which implies $\lambda_2(G) \ge \frac{1}{nD}$.
3. **Complete Binary Tree** T_n: Define G_e be the single-edge graph with edge e, and $T_{i,j}$ be the unique path between i and j. Applying the weighted path inequality, we have $K_n = \sum_{i<j} G_{i,j} \preceq \sum_{i<j}\left(\left(\sum_{e\in T^{i,j}}\frac{1}{w(e)}\right)\left(\sum_{e\in T^{i,j}} w(e)G_e\right)\right) \preceq \left(\max_{i<j}\sum_{e\in T^{i,j}}\frac{1}{w(e)}\right)\left(\sum_{i<j}\sum_{e\in T^{i,j}} w(e)G_e\right)$. For e connecting level i and $i+1$ for $i \in [d-1]$, we set $w(e) = 2^i$. Then $\max_{i<j}\sum_{e\in T^{i,j}}\frac{1}{w(e)} \le 4$. Since the number of occurrence of e in $T^{i,j}$ for any $i<j$ is upper bounded by $n^2 2^{-i}$, we have $\sum_{i<j}\sum_{e\in T^{i,j}} w(e)G_e \preceq \sum_e w(e) n^2 2^{-i} G_e = \sum_e n^2 G_e = n^2 T_n$. Therefore, $K_n \preceq 4n^2 T_n$, which implies $\lambda_2(T_n) \ge \frac{1}{4n}$.

2 Conductance

Definitions:
1. **Conductance of a vertex subset**: Given $\emptyset \subset S \subset V$, the conductance $\phi(S) := \phi(S) = \frac{|E(S, V\setminus S)|}{\min\{\mathbf{vol}(S), \mathbf{vol}(V\setminus S)\}}$, where $\mathbf{vol}(S) := \sum_{v\in S}\operatorname{degree}(v)$. Define $\mathbf{1}_S$ to be the n-dimensional vector with only 1 for the vertices of S and 0 for the vertices of $V \setminus S$. Assuming $\mathbf{vol}(S) \le \mathbf{vol}(V)/2$, thus $|E(V, V\setminus S)| = \sum_{(u,v)\in E}(\mathbf{1}_S(u) - \mathbf{1}_S(v))^2 = \mathbf{1}_S^\top L \mathbf{1}_S$ and $\mathbf{vol}(S) = \mathbf{1}_S^\top D \mathbf{1}_S$. Then $\phi(S) = \frac{\mathbf{1}_S^\top L\mathbf{1}_S}{\mathbf{1}_S^\top D\mathbf{1}_S}$.
2. **Conductance of a graph**: The conductance $\phi(G) := \min_{\emptyset\subset S\subset V}\phi(S) = \min_{\substack{\emptyset\subset S\subset V\\ \mathbf{vol}(S)\le\mathbf{vol}(V)/2}}\phi(S)$.
3. **ϕ-expander**: For any $\phi \in (0,1]$, we call a graph G to be a ϕ-expander if $\phi(G) \ge \phi$.
4. **ϕ-expander decomposition of quality** q: A partition $\{X_i\}$ of the vertex set V is called a ϕ-expander decomposition of quality q if (1) each induced graph $G[X_i]$ is a ϕ-expander, and (2.i) #edges not contained in any $G[X_i]$ is at most $q\cdot\phi\cdot m$. The second condition is equivalent to (2.ii) The partition removes at most $q\cdot\phi\cdot m$ edges.
5. **Normalized Laplacian**: We define the *normalized Laplacian* to be $N := D^{-1/2} L D^{-1/2}$. N is still PSD and the first eigenbasis is $D^{1/2}\mathbf{1}$. By Courant-Fischer theorem, $\lambda_2(N) = \min_{x\perp D^{1/2}\mathbf{1}}\frac{x^\top N x}{x^\top x} = \min_{z\perp d}\frac{z^\top L z}{z^\top D z}$.

2.1 Cheeger's Inequality

Notice that the $\lambda_2(N)$ has similar forms to $\phi(G)$. Cheeger's inequality aims to bound $\phi(G)$ by $\lambda_2(N)$.

Cheeger's Inequality: $\frac{\lambda_2(N)}{2} \le \phi(G) \le \sqrt{2\lambda_2(N)}$.

The lower bound is proved by restricting the minimum in $\lambda_2(N)$ to be $z_S = \mathbf{1}_S - \alpha\mathbf{1}$ for some α such that $z_S \perp d$. The upper bound is proved by constructing S for any $z \perp d$ such that $\frac{\mathbf{1}_S^\top L\mathbf{1}_S}{\mathbf{1}_S^\top D\mathbf{1}_S} \le \sqrt{2\frac{z^\top L z}{z^\top D z}}$.

3 Random Walk