What is...a higher dimensional stable family?

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Compact moduli spaces

Motivation: Construct a modular compactification of the moduli space of canonically polarized manifolds of dimension n.

Remark

For $n = 1$, this is the moduli space of stable curves.

Goal: Construct a moduli space of canonically polarized *stable varieties* of dimension n.

Stable varieties

A possible way to construct moduli spaces:

- $Z \hookrightarrow \mathbb{P}^N = \mathbb{P}(H^0(Z, \omega_Z^{\otimes m}))$, N m are independent of Z.
- $T \subset \mathcal{H}ilb_h(\mathbb{P}^N)$ locus of *stable* varieties
- $M_h := {}^{T}\!/\mathrm{PGL}_{N+1}$

What is. . . *stable*?

Difficulty

For curves we can use GIT, but in higher dimensions it is not known that GIT would give the same compactification for different m's. In other words, the notion of stable could depend on the choice of m.

Stable curves

C is *stable* if

- C has only ordinary double points (a.k.a., nodes), and
- C has only finitely many automorphisms.

or equivalently,

C is *stable* if

- C has only ordinary double points (a.k.a., nodes), and
- ω_C is ample.

Stable reduction (of curves)

Instead of trying to generalize the conditions defining stable curves directly to higher dimensions, let us consider the reason stable curves work.

A smooth family over a non-compact base may be compactified étale locally with no worse than stable fibers.

The special fibers are uniquely determined by the general fiber.

Key: uniqueness of specialization.

Uniqueness of specialization

Consider two families of stable curves:

$$\begin{tikzcd} X^1 \arrow[dr, "f^1"'] & & X^2 \arrow[dl, "f^2"] \\ & B & \end{tikzcd}$$

Assume that the general fibers are isomorphic: $X^1_{\mathrm{gen}} \simeq X^2_{\mathrm{gen}}$.
Uniqueness of specialization means that then the families are isomorphic: $X^1 \simeq X^2$.

Stable families of curves

Why does uniqueness of specialization hold for stable families?

or more generally,

What makes stable families the "right" choice?

Let's assume (for simplicity) that we are working in characteristic 0.

- There exists a log resolution of singularities $\phi : Y \to X$ such that the composition $g = f \circ \phi$ has reduced fibers

  (We will deal with the importance of this condition later),
- $K_{X/B}$ is f-ample, and
- X has Du Val singularities.

X has *Du Val singularities* if for any resolution $\phi : Y \to X$,

$$K_Y \sim \phi^* K_X + (\text{effective}).$$

Canonical singularities

The higher dimensional analogue of *Du Val* is *canonical*.
The definition is essentially the same.

X has *canonical singularities* if for a resolution $\phi : Y \to X$,

$$K_Y \sim \phi^* K_X + (\text{effective}).$$

Corollary

If X has canonical singularities and $\phi : Y \to X$ is a resolution of singularities, then

$$\mathscr{O}_X(mK_X) \simeq \phi_* \mathscr{O}_Y(mK_Y).$$

for any $m \geq 0$.

Towards stable families

Let $f : X \to B$ be a flat, projective morphism.
For simplicity, (always) assume that B is a smooth curve and that

(i) $K_{X/B}$ is f-ample, and
(ii) X has canonical singularities.

Consider a resolution of singularities of X:

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[rr, bend left, "g"] & X \arrow[r, "f"] & B \end{tikzcd}$$

By (i), $X \simeq \mathrm{Proj}_B(\oplus_{m\geq 0} f_* \mathscr{O}_X(mK_{X/B}))$.

By (ii) and the previous corollary, $\mathscr{O}_X(mK_X) \simeq \phi_* \mathscr{O}_Y(mK_Y)$.

Hence, $X \simeq \mathrm{Proj}_B(\oplus_{m\geq 0} g_* \mathscr{O}_Y(mK_{Y/B}))$.

Towards stable families

Consider two families $f^i : X^i \to B$ for $i = 1, 2$ such that

- $K_{X^i/B}$ is f^i-ample, and
- X^i has canonical singularities.

$$\begin{tikzcd} X^1 \arrow[dr, "f^1"'] & & X^2 \arrow[dl, "f^2"] \\ & B & \end{tikzcd}$$

Assume that the general fibers are isomorphic: $X^1_{\mathrm{gen}} \simeq X^2_{\mathrm{gen}}$ and consider a common resolution of singularities.

Towards stable families

Assume that the general fibers are isomorphic: $X^1_{\mathrm{gen}} \simeq X^2_{\mathrm{gen}}$ and consider a common resolution of singularities:

$$\begin{tikzcd} & Y \arrow[dl, "\phi^1"'] \arrow[dd, "g"] \arrow[dr, "\phi^2"] & \\ X^1 \arrow[dr, "f^1"'] & & X^2 \arrow[dl, "f^2"] \\ & B. & \end{tikzcd}$$

By the previous observation,

$$X^1 \simeq \mathrm{Proj}_B(\oplus_{m\geq 0} g_* \mathscr{O}_Y(mK_{Y/B})) \simeq X^2$$

Uniqueness of specialization

Let $f : X \to B$ be a flat, projective morphism. If

(i) $K_{X/B}$ is f-ample, and

(ii) X has canonical singularities,

then the general fiber of f determines its special fibers and hence families that satisfy these conditions also satisfy *uniqueness of specialization*.

For stable curves we had one more condition:

(iii) There exists a log resolution of singularities $\phi : Y \to X$ such that the composition $g = f \circ \phi$ has reduced fibers.

This will help us determine the definition of a *stable variety* in higher dimensions.

What is... stable? – First approximation

Let $f : X \to B$ be a flat, projective morphism such that

(i) $K_{X/B}$ is f-ample,

(ii) X has canonical singularities, and

(iii) there exists a log resolution of singularities $\phi : Y \to X$ such that the composition $g = f \circ \phi$ has reduced fibers.

This is what a stable *family* should be, but what is a stable *variety*?

What restrictions do these conditions impose on the singularities of the fibers of f?

Stable varieties

Let $f : X \to B$ be as above and
$\phi : Y \to X$ the resolution guaranteed by (iii).
Let $b \in B$ and $X_b = f^{-1}(b)$ (reduced, as we are defining stable *varieties*), $Y_b = g^{-1}(b)$, and $\widetilde{X}_b := \phi_*^{-1} X_b$, the strict transform of X_b.
By (iii), Y_b is reduced, so

$$\phi^* X_b = Y_b = \widetilde{X}_b + E_b.$$

where E_b is the reduced exceptional divisor of the morphism
$\phi_b : Y_b \to X_b$.
Equivalently we have

$$\widetilde{X}_b = \phi^* X_b - E_b.$$

Next we will compute the canonical divisors of X_b and Y_b.

- X has canonical singularities, so
$$K_Y \sim \phi^* K_X + (\text{effective}).$$
- By adjunction,
$$K_{X_b} \sim (K_X + X_b)\big|_{X_b},$$
and
$$K_{\widetilde{X}_b} \sim (K_Y + \widetilde{X}_b)\big|_{\widetilde{X}_b},$$
- so
$$\begin{aligned} K_{\widetilde{X}_b} &\sim (\phi^* K_X + (\text{effective}) + \widetilde{X}_b)\big|_{\widetilde{X}_b} \\ &\sim (\phi^* K_X + (\text{effective}) + \phi^* X_b - E_b)\big|_{\widetilde{X}_b} \\ &\sim (\phi^*(K_X + X_b) + (\text{effective}) - E_b)\big|_{\widetilde{X}_b} \\ &\sim \phi^* K_{X_b} + (\text{effective}) - (\text{reduced}) \end{aligned}$$

Corollary

X_b has log canonical singularities.

Stable varieties – normal case

A stable variety should have

- log canonical singularities, and
- an ample canonical divisor

X_b has *log canonical singularities* if for a resolution $\phi : \widetilde{X}_b \to X_b$,

$$K_{\widetilde{X}_b} \sim \phi^* K_{X_b} + (\text{effective}) - (\text{reduced}).$$

<u>**Issue**</u>: One **better not** assume that the fibers are normal.

Stable varieties – non-normal case

X is a stable variety if

- X has semi log canonical singularities, and
- K_X is an ample $\mathbb{Q}$-Cartier divisor.

What is a stable family?

A minimal guess:
A stable family is a flat family of stable varieties.

Remark

If one traces the computation that led to log canonical singularities backwards, then one will discover that most of the conditions on the family can be reproduced from the conditions on the fiber.
Unfortunately this is not good enough.

An important example

- Let $S \subset L := \mathbb{P}^5$ be a rational quartic scroll, i.e., the image of $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$ embedded by the global sections of $\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,2)$.
- Embed $L \subset \mathbb{P}^6$ as a hyperplane. Let $Q = \overline{C(S)} \subset \mathbb{P}^6$ be the projectivized cone over S with vertex $P \in \mathbb{P}^6$. Then $S = Q \cap L$.
- Let $H \simeq \mathbb{P}^5 \subset \mathbb{P}^6$ be a general hyperplane containing P, and let $R := (S \cap H) \subset (L \cap H) \simeq \mathbb{P}^4$ be a rational normal curve.
- Let $T = \overline{C(R)} \subset H$ be the projectivized cone over R with vertex $P \in H$. Clearly, $T = (Q \cap H) \subset Q$ is a hyperplane section.
- Let V be a Veronese surface, i.e., the image of the 2-uple embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. Assume that $V \subset L \simeq \mathbb{P}^5 \subset \mathbb{P}^6$.
- Let $Q' = \overline{C(V)}$ be the projectivized cone over V with vertex $P \in \mathbb{P}^6$, $R' := (V \cap H) \subset (L \cap H) \simeq \mathbb{P}^4$, and $T' = \overline{C(R')} \subset H$ the projectivized cone over R' with vertex $P \in H$.
- Then $V = Q' \cap L$, $T' = Q' \cap H$, and R' is another rational normal curve.

An important example (continued)

Both S and T are hyperplane sections of Q:

$$S = Q \cap L \qquad T = Q \cap H.$$

Therefore, there exists a flat family over a smooth curve with total space birational to Q that has both S and T as fibers.

Similarly, both V and T' are hyperplane sections of Q':

$$V = Q' \cap L \qquad T' = Q' \cap H.$$

Therefore, there exists a flat family over a smooth curve with total space birational to Q' that has both V and T' as fibers.

Since $T \simeq T'$ this implies that $V \simeq \mathbb{P}^2$ can be deformed to $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$. But this could not happen through a smooth family! For instance, $K_V^2 = 9$ and $K_S^2 = 8$, so their Hilbert polynomials are different.

Underlying Reason

Hilbert polynomials correspond to line bundles and in order for them to be equal in a flat family, there has to be a global line bundle restricting to the one used on the fibers.
On the family coming from Q there is no global line bundle that restricts to a pluricanonical bundle on all the fibers.

Stable families

Viehweg's condition

For a stable family $f : X \to B$ require that $K_{X/B}$ be $\mathbb{Q}$-Cartier.

Lemma (Easy)

If $mK_{X/B}$ is Cartier, then $\mathscr{O}_X(mK_{X/B})\big|_{X_b} \simeq \mathscr{O}_X(mK_{X_b})$

In order to force the moduli functor to be of finite type one needs to fix the Hilbert polynomial of a fixed line bundle on the stable varieties one is studying. In other words, to make the $\mathbb{Q}$-Cartier assumption reasonable, one has to fix the multiple of $K_{X/B}$ that one requires to be Cartier. This multiple will be called the *index* of f. By the lemma this means that the same multiple makes K_{X_b} Cartier. In other words the fibers of a family of index m are stable varieties of index m.

However, there is still a glitch.

Example

Suppose that

- $mK_{X/B}$ is Cartier for some $m > 0$,
- qK_{X_b} is Cartier for some $0 < q < m$ for every $b \in B$, but
- $qK_{X/B}$ is not Cartier.

This means that $f : X \to B$ is a family of stable varieties of *index* m but not of *index* q, even though the fibers of the family are stable varieties of *index* q.

Difficulty

This example shows that under the assumptions we have made so far the moduli space of stable varieties of index m may have a different scheme structure than the moduli space of stable varieties of index q even along the locus of those varieties that have index q.

Towards Kollár's condition

Difficulty

A stable variety of index q may degenerate to one with index $m > q$ so in order to obtain a compact moduli space we need to allow families of larger index than the index of the members.

At the same time, different choices of index may lead to different scheme structure near the point representing a fixed stable variety, even if all the nearby points have the same index.

In other words, even if all small deformations of a given stable variety of index q also have index q, we may be forced to allow families of these stable varieties of larger index, which might lead to different scheme structure on the moduli space depending on the actual index.

Kollár's condition

Kollár's condition

For a stable family $f : X \to B$, in addition to Viehweg's condition, that is, requiring that $K_{X/B}$ be $\mathbb{Q}$-Cartier, also require that

$$\mathscr{O}_X(qK_{X/B})\big|_{X_b} \simeq \mathscr{O}_X(qK_{X_b}) \qquad \forall q$$

Lemma (Easy)

If $\mathscr{O}_X(qK_{X/B})\big|_{X_b} \simeq \mathscr{O}_X(qK_{X_b})$ and $\mathscr{O}_X(qK_{X_b})$ is a line bundle, then $\mathscr{O}_X(qK_{X/B})$ is a line bundle in a neighborhood of X_b.

Corollary

Kollár's condition resolves the problem of different indices. For a stable family $f : X \to B$ of arbitrary index, if qK_{X_b} is a Cartier divisor, then qK_X is a Cartier divisor near X_b and hence the scheme structure of the moduli space near the point $[X_b]$ is independent of the index of the families considered.

Stable varieties and stable families

X is a *stable variety* if

- X has semi log canonical singularities, and
- K_X is an ample $\mathbb{Q}$-Cartier divisor.

A *stable family* is a flat family, $f : X \to B$, of stable varieties that satisfies Kollár's condition:

- $K_{X/B}$ is $\mathbb{Q}$-Cartier, and
- $\mathscr{O}_X(qK_{X/B})\big|_{X_b} \simeq \mathscr{O}_X(qK_{X_b})$ for every $q \in \mathbb{N}$.

Functoriality of stable families

Example (Kollár)

In characteristic $p > 0$, there exists a family that satisfies Viehweg's condition but not Kollár's condition.

Open Problem

It is still an open problem whether a similar example exists in characteristic 0.

In fact, it turns out that until very recently even the absolute simplest case of $q = 1$ was not known.

Theorem (Kollár-K)

For a flat family $f : X \to B$ of stable varieties,

$$\mathscr{O}_X(K_{X/B})\big|_{X_b} \simeq \mathscr{O}_X(K_{X_b}).$$

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Remark

The same statement is true if all fibers are Cohen-Macaulay (proved by Grothendieck), but (semi) log canonical singularities are not necessarily Cohen-Macaulay.

Question (Kollár)

How general can f be so it would still satisfy the above restriction property? For instance, would having normal, or S_2 fibers be enough to imply this?

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How general can f be so it would still satisfy the above restriction property? For instance, would having normal, or S_2 fibers be enough to imply this?

Theorem (Patakfalvi)

For any n there exists a flat projective family $f : X \to B$ such that

- *B is a smooth curve,*
- *$X_{general}$ is smooth of dimension n,*
- *$X_{special}$ has a single isolated singular point,*
- *$X_{special}$ is normal and S_{n-1}, but $\mathcal{O}_X(K_{X/B})\big|_{X_b} \not\simeq \mathcal{O}_X(K_{X_b})$.*

The state of art about compact moduli spaces

- There is no published result in the literature yet, but one should be coming soon. (So, I will not state a formal theorem here).
- Compact moduli spaces of canonically polarized stable varieties exist.
- This result is the culmination of a lot of work by many people. Top contributors: Kollár, Viehweg, Alexeev (and many others).
- Still open: compact moduli of stable pairs: (X, Δ).