The Euclidean Distance Degree of an Algebraic Variety

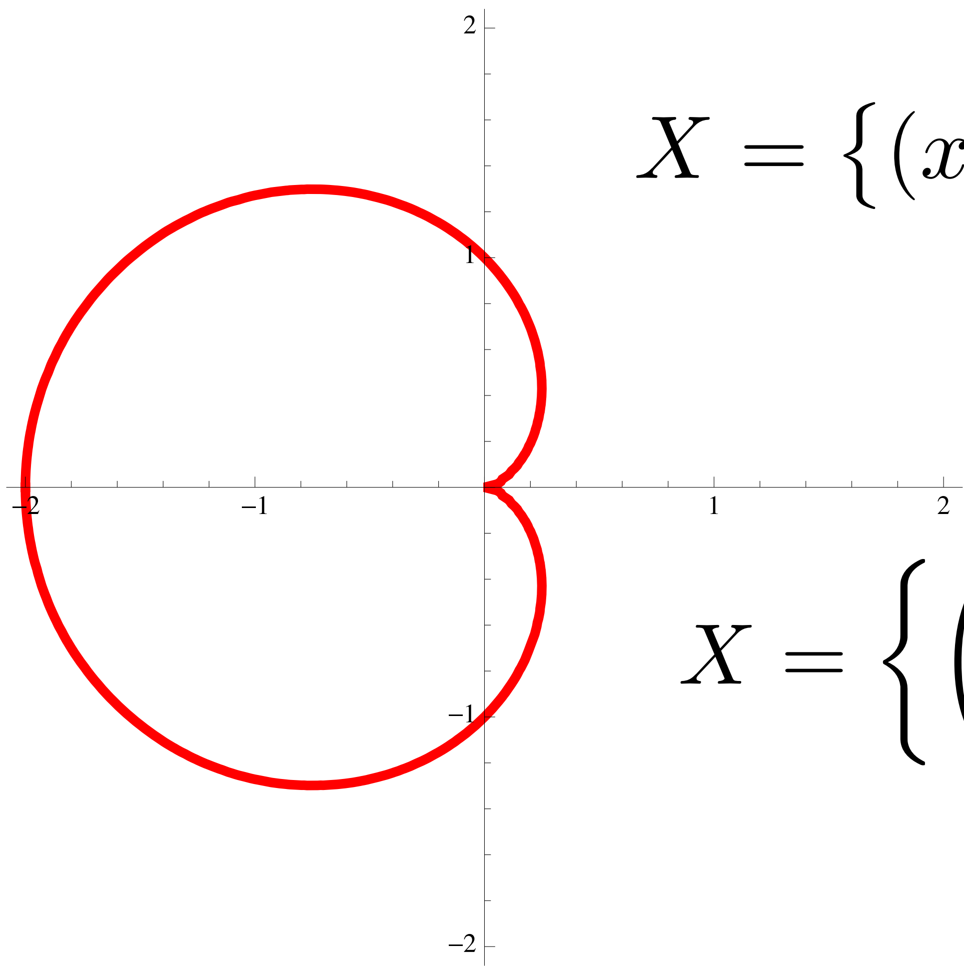
Rekha R Thomas
(University of Washington, Seattle)

Joint work with:

Jan Draisma, Emil Horobet *(TU Eindhoven)*
Giorgio Ottaviani *(U Firenze)*
Bernd Sturmfels *(UC Berkeley)*

Many models in science and engineering are expressed as the set of real solutions to a system of polynomial equations in several variables.

Such a solution set is a (real) algebraic variety $X \subseteq \mathbb{R}^n$



$$X = \{(x, y) \in \mathbb{R}^2 \,:\, (x^2 + y^2 + x)^2 = x^2 + y^2\}$$

implicit representation

$$X = \left\{ \left(\frac{2t^2 - 2}{(1 + t^2)^2}, \frac{-4t}{(1 + t^2)^2} \right) \,:\, t \in \mathbb{R} \right\}$$

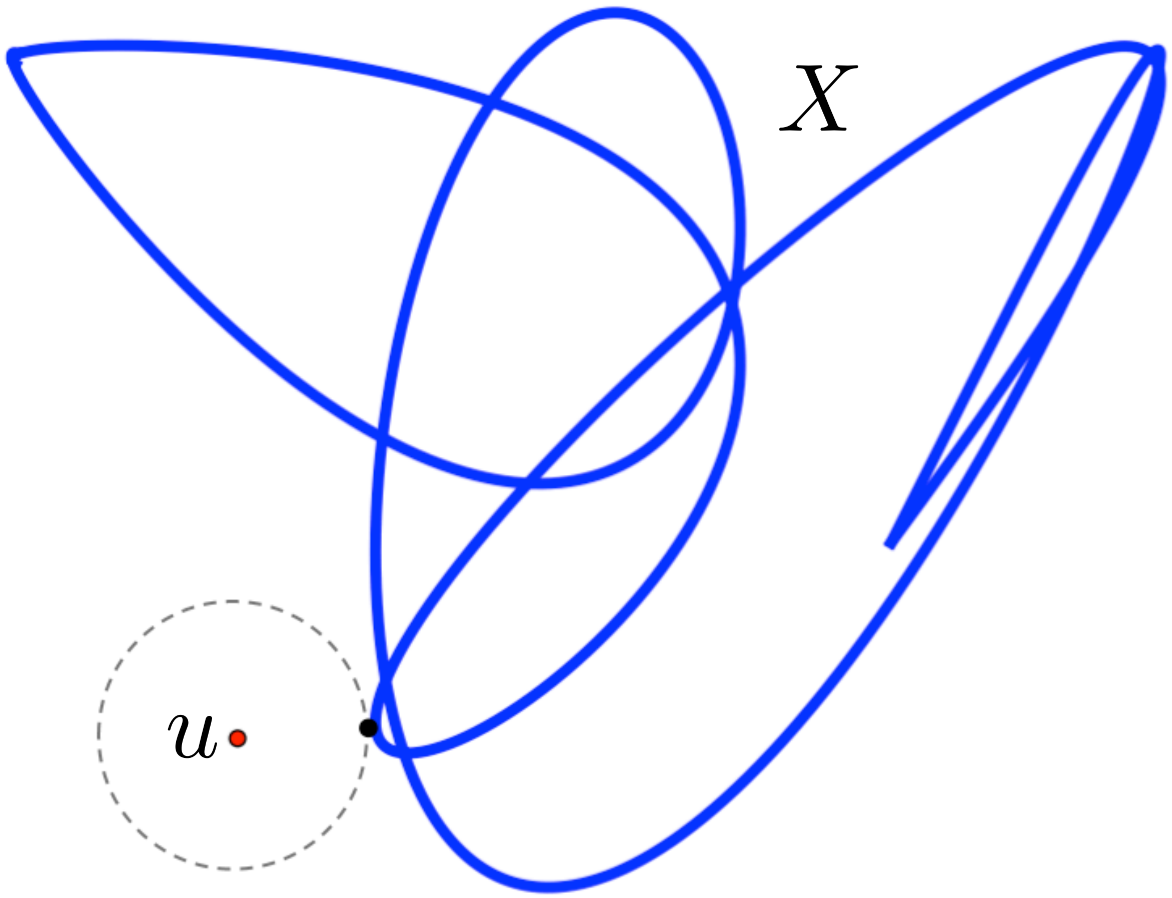
parametric representation

... and a common problem is to minimize the Euclidean distance to the variety from a given data point.

$u \in \mathbb{R}^n$ data point

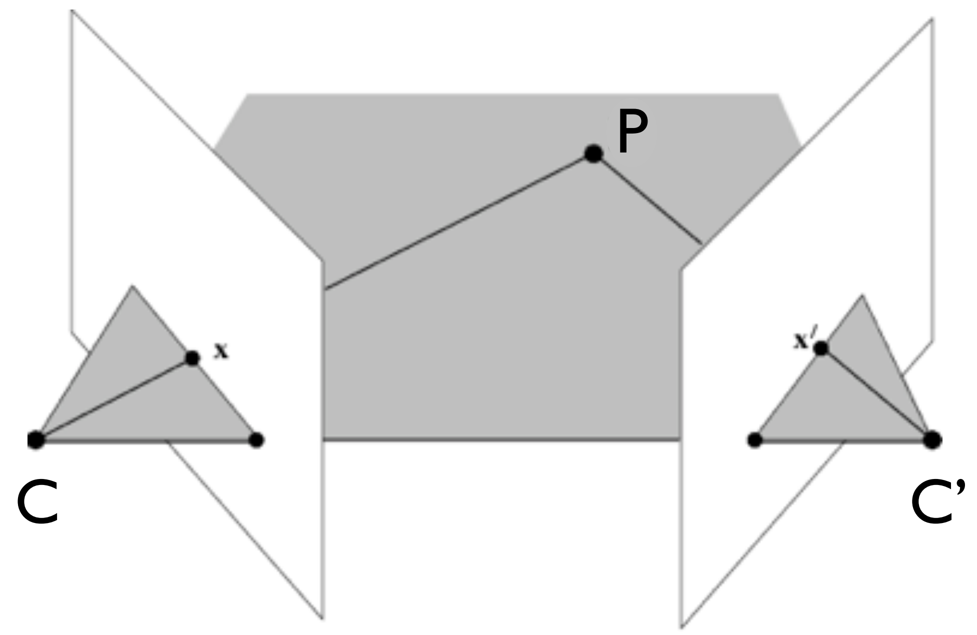
$X \subseteq \mathbb{R}^n$ real algebraic variety

$$u^* = \operatorname{argmin} \ d_u(x) = \|u - x\|^2$$
$$\text{s.t.} \quad x \in X$$



3D Reconstruction in Computer Vision

Want to recover a 3D point P from noisy images of it, u_i in $n \geq 2$ cameras.



camera i:

$$A_i : \quad \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$$
$$\bar{P} \mapsto A_i\bar{P}$$

image in camera i: $x_i = \Pi A_i \bar{P}$ *(dehomogenize)*

Under Gaussian noise, MLE of u:

$$\begin{array}{ll} \text{argmin} & \sum_{i=1}^{n} \|x_i - u_i\|^2 \\ \text{s.t.} & x_i = \Pi A_i \bar{P} \;\; \forall\, i = 1, \ldots, n \end{array}$$

$$\nabla d_u = 2(u - x) \qquad X = \{x \in \mathbb{R}^n \ : \ f_1(x) = \cdots f_s(x) = 0\}$$

KKT: $$\nabla d_u = \sum \lambda_i \nabla f_i$$

$$\mathcal{C}_u = \left\{ x \in \mathbb{C}^n \ : \ \begin{array}{l} f_1(x) = \cdots = f_s(x) = 0 \\ x \text{ smooth} \\ u - x \perp T_x(X) \end{array} \right\}$$

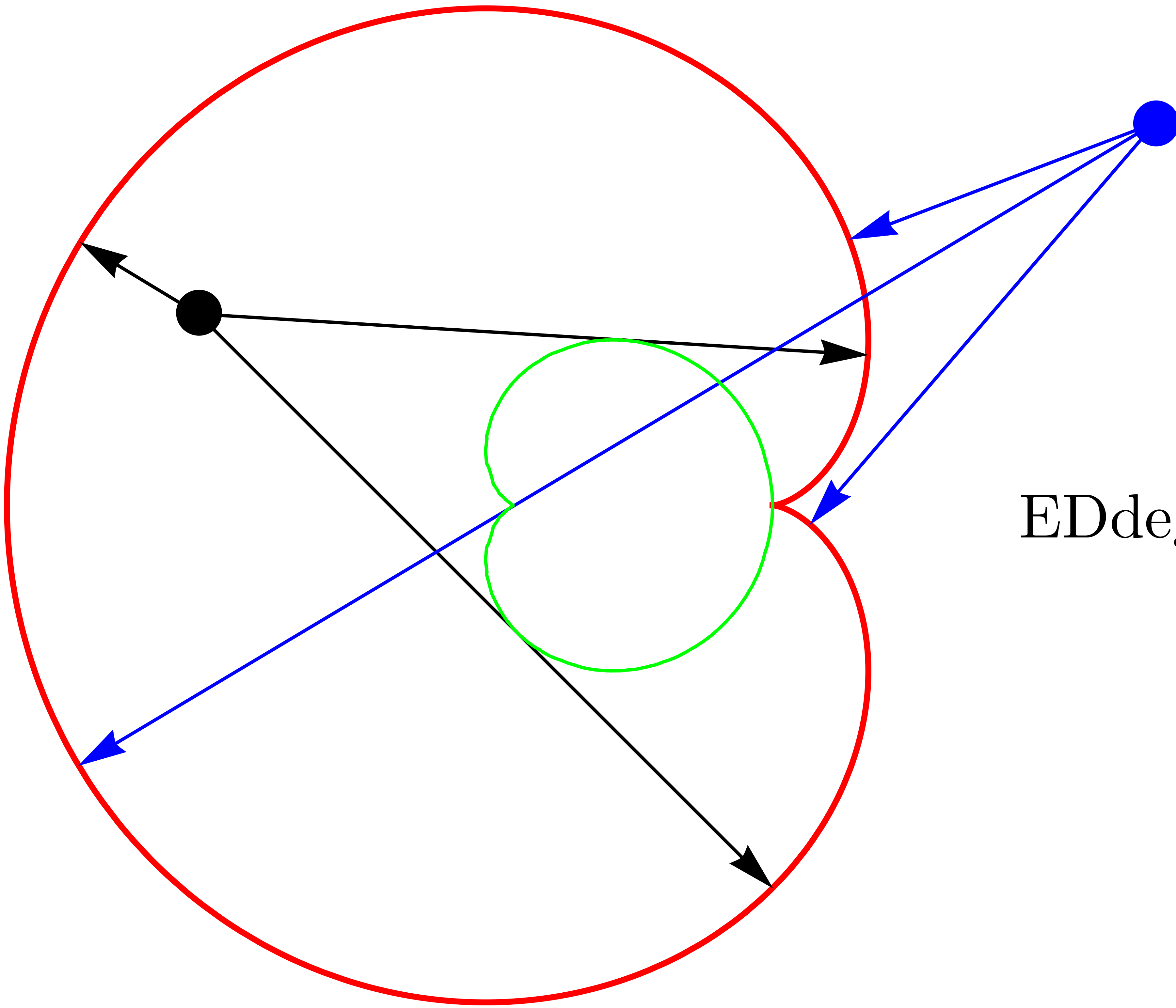
tangent space of X at x

$$\boxed{\text{Euclidean distance degree of } X := |\mathcal{C}_u|}$$

!!! constant for generic u !!!

measure of the algebraic complexity of the ED problem on X

Cardiod: $X = \{(x,y) \in \mathbb{R}^2 \, : \, (x^2+y^2+x)^2 = x^2+y^2\}$



$\mathrm{EDdegree}(X) = 3$

Curves in the plane

$$X = \{(x, y) \in \mathbb{R}^2 \, : \, f(x, y) = 0\} \qquad \text{degree}(f) = d$$

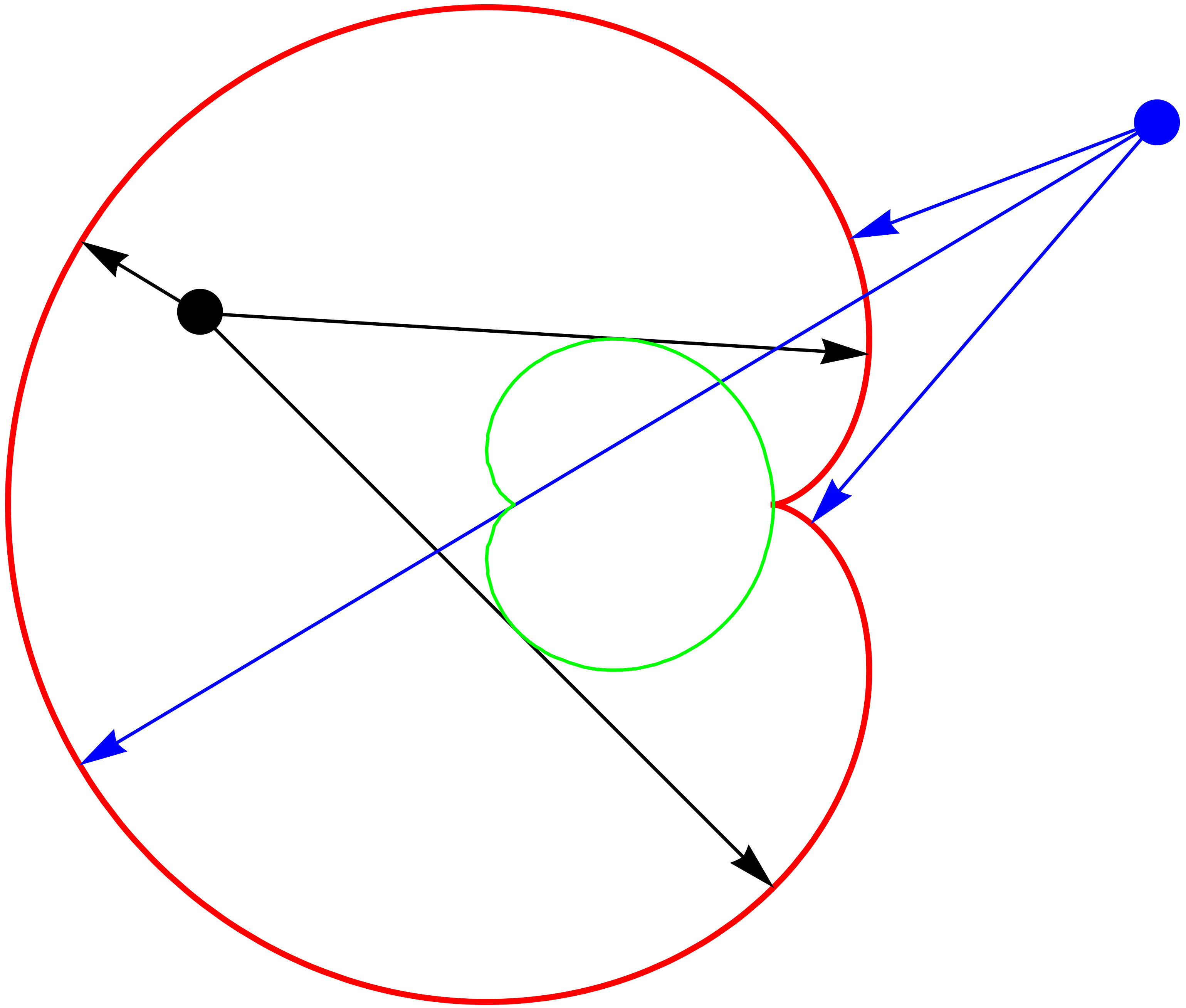
$(u, v) \in \mathbb{R}^2$ **data point**

need to solve: $$f(x, y) = \det \begin{pmatrix} u - x & v - y \\ \partial f / \partial x & \partial f / \partial y \end{pmatrix} = 0$$

expect d^2 *complex solutions (Bezout's Thm)*

$$X \text{ general plane curve of degree } d \Rightarrow \text{ EDdegree}(X) = d^2$$

Cardiod: $X = \{(x,y) \in \mathbb{R}^2 \,:\, (x^2+y^2+x)^2 = x^2+y^2\}$

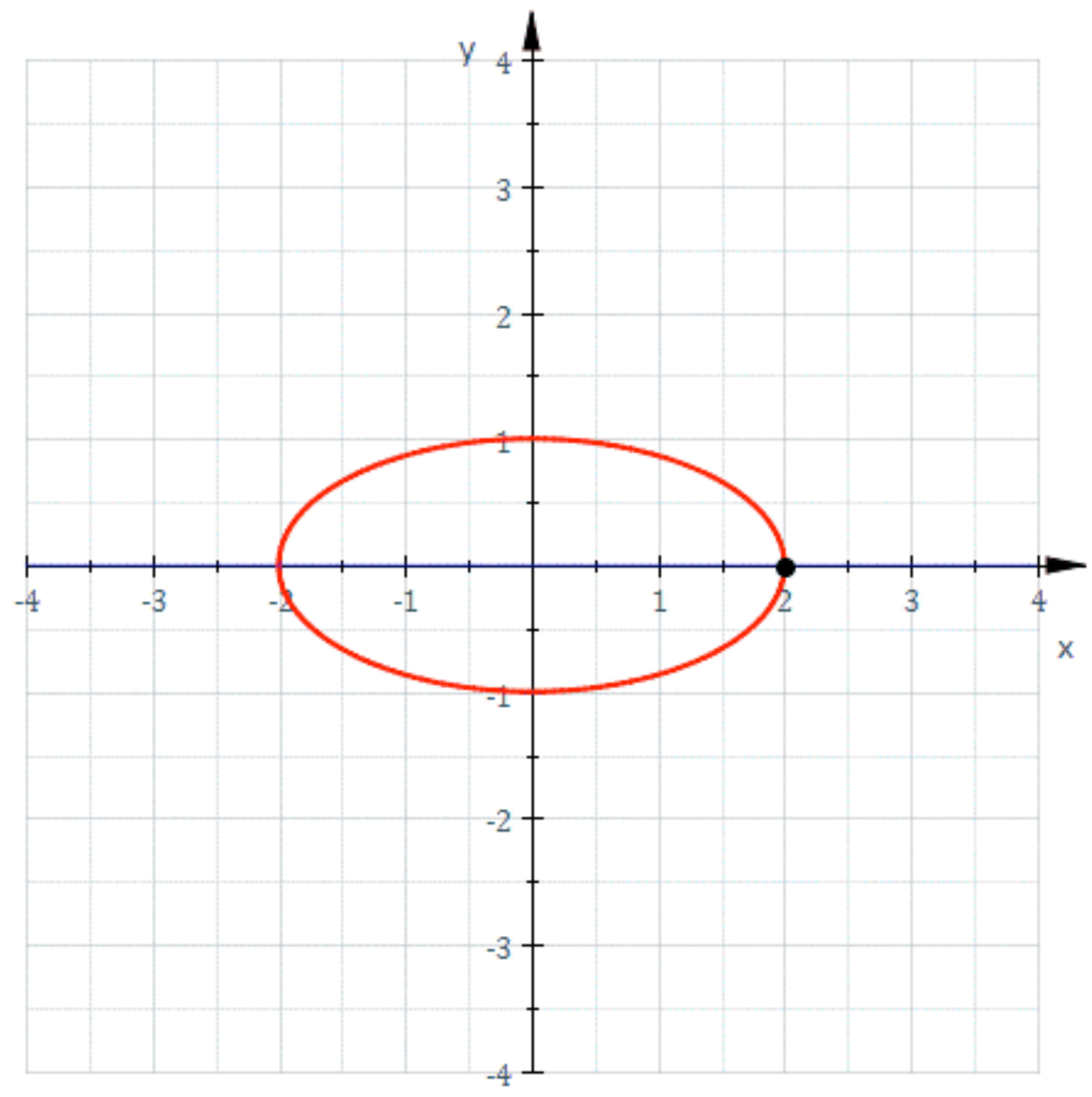


$$\mathrm{EDdegree}(X) = 3$$

ED discriminant :

$$\{(u,v) \in \mathbb{R}^2 \,:\, 27u^4 + 54u^2v^2 + 27v^4 + 54u^3 + 54uv^2 + 36u^2 + 9v^2 + 8u \,=\, 0\}$$

ED discriminant of an ellipse



http://en.wikipedia.org/wiki/File:Evolute1.gif

EDdegree in Applications

Linear Regression: $\boxed{\mathrm{EDdegree}(\text{linear space}) = 1}$

$$x \in \text{ linear space}, \quad u - x \perp \text{ linear space}$$

unique solution

Eckart-Young Theorem: Fix $r \leq s \leq t$

$X = \{s \times t \text{ matrices of rank } \leq r\}$

$$\boxed{\mathrm{EDdegree}(X) = \binom{s}{r}}$$

SVD $\quad U = T_1 \cdot \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_s) \cdot T_2$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_s$$

EYT $\quad U^* = T_1 \cdot \mathrm{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0) \cdot T_2$

critical points of d_U :

$$T_1 \cdot \mathrm{diag}(0, \ldots, 0, \sigma_{i_1}, 0, \ldots, 0, \sigma_{i_r}, 0, \ldots, 0) \cdot T_2$$

$I = \{i_1 < \ldots < i_r\}$ — all r-element subsets of $\{1, \ldots, s\}$.

General Bound: $X \subset \mathbb{C}^n, \quad \operatorname{codim}(X) = c$

cut out by $f_1, f_2, \ldots, f_c, \ldots, f_s$

of degrees $d_1 \geq d_2 \geq \cdots \geq d_c \geq \cdots \geq d_s \quad \Rightarrow$

$$\boxed{\text{EDdegree}(X) \leq d_1 d_2 \cdots d_c \cdot \sum_{i_1+i_2+\cdots+i_c \leq n-c} (d_1-1)^{i_1}(d_2-1)^{i_2}\cdots(d_c-1)^{i_c}.}$$

Equality for generic $\underbrace{\text{complete intersection of codimension } c.}_{c=s}.$

Quadratic programs: X cut out by c quadratic polynomials

$\Rightarrow \text{EDdegree}(X) \leq 2^c\binom{n}{c}$

Parametric Version: X parametrized by rational functions

$$\begin{array}{rccc} \psi : & \mathbb{R}^m & \to & \mathbb{R}^n \\ & (t_1, \ldots, t_m) & \mapsto & (\psi_1(t), \ldots, \psi_n(t)) \end{array}$$

$$D_u(t) = \sum_{i=1}^{n} (\psi_i(t) - u_i)^2$$ squared distance function from u

get critical points by solving: $$\frac{\partial D_u}{\partial t_1} = \cdots = \frac{\partial D_u}{\partial t_m} = 0.$$

get an upper bound on EDdegree by Bezout's Theorem

Computer vision example: *(Stewenius et al)*

n	2	3	4	5	6	7
ED	6	47	148	336	638	1081

Conjecture:

$$\frac{9}{2}n^3 - \frac{21}{2}n^2 + 8n - 4.$$

Hurwitz stability

$$x(z) = x_0 z^n + x_1 z^{n-1} + x_2 z^{n-2} + \cdots + x_{n-1} z + x_n, \quad x_i \in \mathbb{R}$$

stable *if each complex root has negative real part*.

Hurwitz test:

$$H_5 = \begin{pmatrix} x_1 & x_3 & x_5 & 0 & 0 \\ x_0 & x_2 & x_4 & 0 & 0 \\ 0 & x_1 & x_3 & x_5 & 0 \\ 0 & x_0 & x_2 & x_4 & 0 \\ 0 & 0 & x_1 & x_3 & x_5 \end{pmatrix}.$$

Hurwitz determinants

$$\bar{\Gamma}_n = \frac{1}{x_n} \det(H_n)$$

$$\Gamma_n = \bar{\Gamma}_n \big|_{x_0=1}$$

$$x(z) \text{ stable} \iff \text{all leading principal minors of } H_n \text{ are positive}$$

	$\mathrm{ED}(\Gamma_n)$	$\mathrm{ED}(\bar{\Gamma}_n)$
$n = 2m+1$	$8m-3$	$4m-2$
$n = 2m$	$4m-3$	$8m-6$

Critical formations on a line (Cayley-Menger variety)

$X \subset \mathbb{R}^{\binom{p}{2}}$ parametrized by $d_{ij} = (z_i - z_j)^2$

squared distances among p points on a line

X cut out by 2×2 minors of:

$$\begin{bmatrix} 2d_{1p} & d_{1p}+d_{2p}-d_{12} & d_{1p}+d_{3p}-d_{13} & \cdots & d_{1p}+d_{p-1,p}-d_{1,p-1} \\ d_{1p}+d_{2p}-d_{12} & 2d_{2p} & d_{2p}+d_{3p}-d_{23} & \cdots & d_{2p}+d_{p-1,p}-d_{2,p-1} \\ d_{1p}+d_{3p}-d_{13} & d_{2p}+d_{3p}-d_{23} & 2d_{3p} & \cdots & d_{3p}+d_{p-1,p}-d_{3,p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{1p}+d_{p-1,p}-d_{1,p-1} & d_{2p}+d_{p-1,p}-d_{2,p-1} & d_{3p}+d_{p-1,p}-d_{3,p-1} & \cdots & 2d_{p-1,p} \end{bmatrix}$$

$$\mathrm{EDdegree}(X) = \begin{cases} \frac{3^{p-1}-1}{2} & \text{if } p \equiv 1,2 \mod 3 \\ \frac{3^{p-1}-1}{2} - \frac{p!}{3((p/3)!)^3} & \text{if } p \equiv 0 \mod 3 \end{cases}$$

refines results by Anderson & Helmke

Average (real) EDdegree

Equip data space with a probability measure ω

Example: standard multivariate Gaussian

$$\text{aEDdegree}(X,\omega) \;:=\; \int_{\mathbb{R}^n} \#\{\text{real critical points of } d_u \text{ on } X\} \cdot |\omega|$$

expected number of real critical points of d_u

Can compute this integral sometimes via numerical integration

Hurwitz

n	$\mathrm{ED}(\Gamma_n)$	$\mathrm{ED}(\bar{\Gamma}_n)$	$\mathrm{aED}(\Gamma_n)$	$\mathrm{aED}(\bar{\Gamma}_n)$
3	5	2	1.162...	2
4	5	10	1.883...	2.068...
5	13	6	2.142...	3.052...
6	9	18	2.416...	3.53...
7	21	10	2.66...	3.742...

Ellipse

aEDdegree $= 3.04658...$

Cardiod

aEDdegree $= 2.8375$

Rank one tensors

Format	aEDdegree	EDdegree
$2 \times 2 \times 2$	4.2891...	6
$2 \times 2 \times 2 \times 2$	11.0647...	24
$2 \times 2 \times n, \ \ n \geq 3$	5.6038...	8
$2 \times 3 \times 3$	8.8402...	15
$2 \times 3 \times n, \ \ n \geq 4$	10.3725...	18
$3 \times 3 \times 3$	16.0196...	37
$3 \times 3 \times 4$	21.2651...	55
$3 \times 3 \times n, \ \ n \geq 5$	23.0552...	61

Duality

$X \subset \mathbb{C}^n$ *cone (over a projective variety)*

$X^* \subset \mathbb{C}^n$ *dual variety*

$$X^* := \overline{\{y \in \mathbb{C}^n \mid \exists\, x \in X_{\text{smooth}} \, : \, y \perp T_x X\}}$$

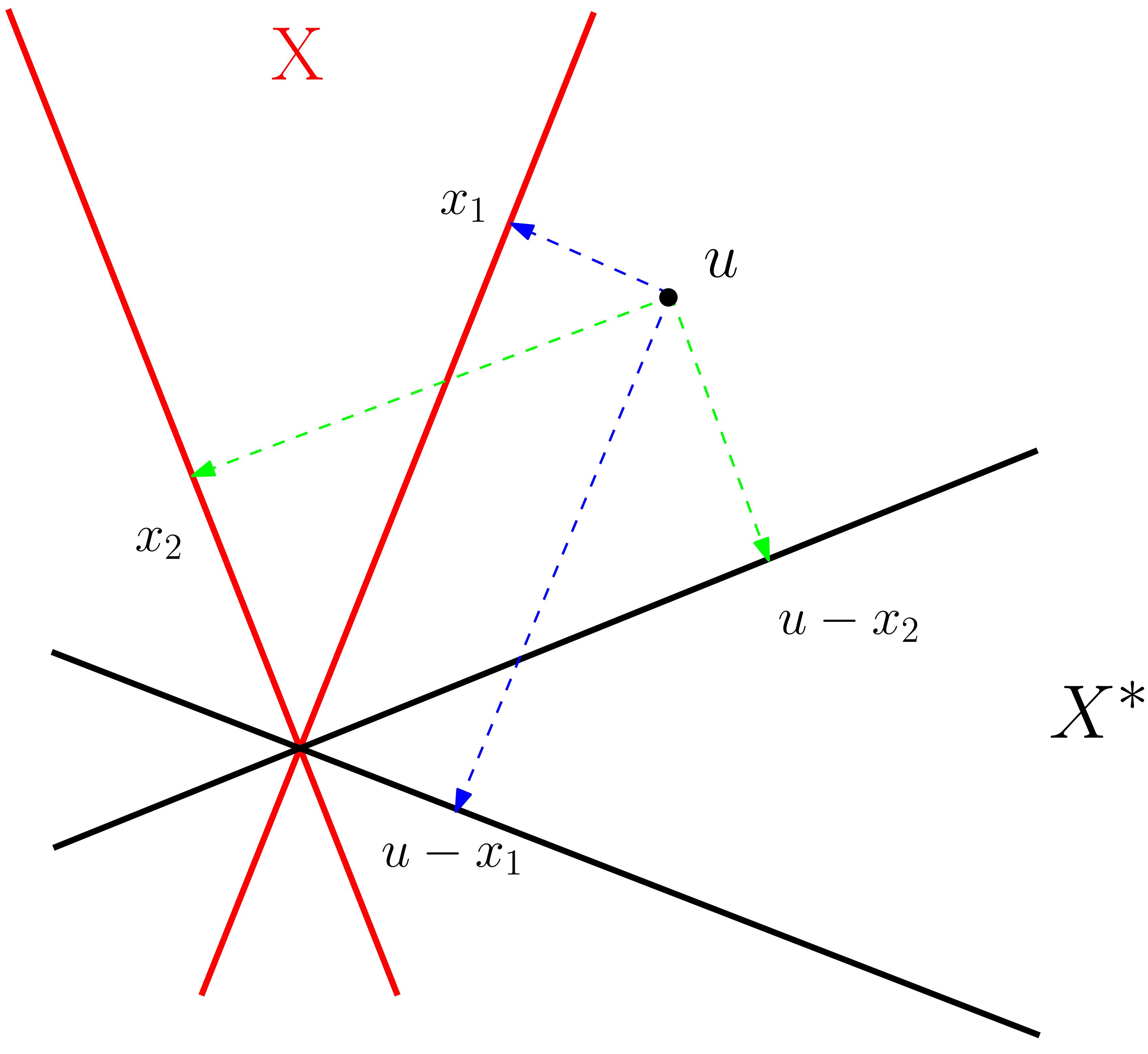
Example $X_r = \{s \times t \text{ matrices of rank } \leq r\}$

$$X_r^* = X_{s-r} = \{s \times t \text{ matrices of rank } \leq s - r\}$$

$$\text{EDdegree}(X_r) = \binom{s}{r} = \binom{s}{s-r} = \text{EDdegree}(X_{s-r})$$

$$U_I = T_1 \cdot \text{diag}(\sigma_i \, : \, i \in I) \cdot T_2 \;\leftrightarrow\; U_{I^c} = T_1 \cdot \text{diag}(\sigma_i \, : \, i \in I^c) \cdot T_2$$

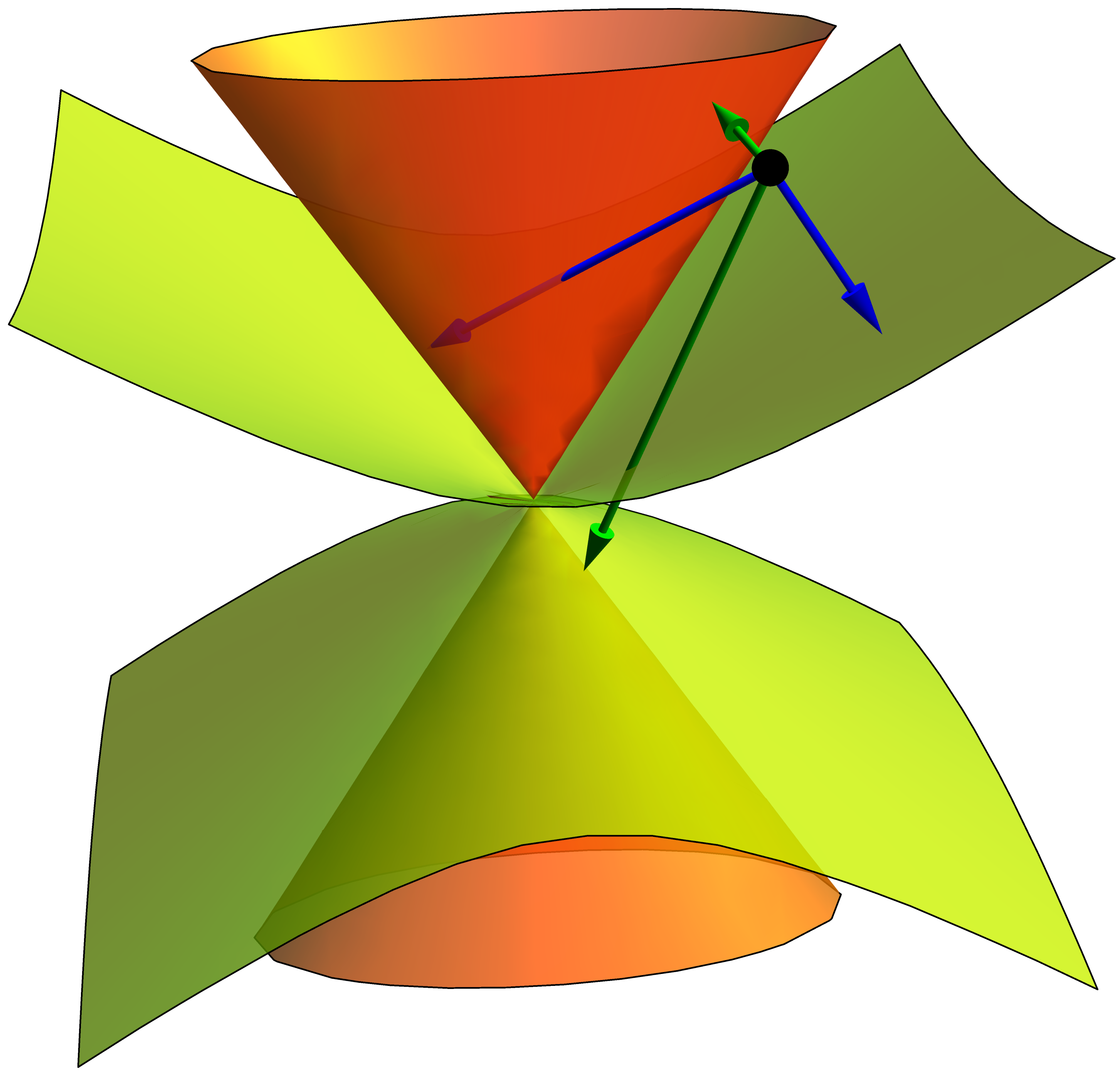
$$U = U_I + U_{I^c}$$


X
x_1
u
x_2
$u - x_2$
X^*
$u - x_1$

set $Y := X^*$

Theorem (Duality): Let $u \in \mathbb{C}^n$ be generic.

- $\mathrm{EDdegree}(X) = \mathrm{EDdegree}(Y)$
- $x \mapsto u - x$ is a bijection between critical points of d_u on $X \& Y$
- If $u \in \mathbb{R}^n$ map sends
  real critical points to real critical points
- $\mathrm{aEDdegree}(X, \omega) = \mathrm{aEDdegree}(Y, \omega)$ for all ω
- The map is proximity-reversing: the closer a real critical point x is to the data point u, the further $u - x$ is from u.



Algebraic Formulas (under various assumptions)

- $$\boxed{\begin{aligned}\mathrm{EDdegree}(X) &= \delta_0(X) + \cdots + \delta_{n-2}(X)\\ &= \delta_{n-2}(Y) + \cdots + \delta_0(Y) \;=\; \mathrm{EDdegree}(Y)\end{aligned}}$$

  $\delta_i(X), \delta_i(Y)$ invariants of the conormal variety

  $$\mathcal{N}_{X,Y} := \overline{\{(x,y) \in \mathbb{C}^n \times \mathbb{C}^n \mid y \in Y_{\mathrm{reg}} \text{ and } x \perp T_y Y\}}.$$

polar classes

- $$\boxed{\mathrm{EDdegree}(X) \;=\; \sum_{i=0}^{m} (-1)^i \cdot (2^{m+1-i} - 1) \cdot \deg(c_i(X))}$$

  $c_i(X)$ ith Chern class of the cotangent bundle of X

Chern classes

- X smooth irreducible curve of degree d and genus g

  $$\boxed{\mathrm{EDdegree}(X) \;=\; 3d + 2g - 2}$$

- $X \subset \mathbb{P}^{n-1}$, dim $= m$, smooth toric variety

  $$\boxed{\mathrm{EDdegree}(X) \;=\; \sum_{j=0}^{m} (-1)^{m-j} \cdot (2^{j+1} - 1) \cdot V_j}$$

sums of normalized volumes of j-faces

Hilbert & Cohn-Vossen: *Anschauliche Geometrie*,
Springer-Verlag, Berlin 1932

"The simplest curves are the planar curves. Among them the simplest one is the line.
The next simplest is the circle.
After that comes the parabola,
and finally, general conics."

Hilbert & Cohn-Vossen: *Anschauliche Geometrie*,
Springer-Verlag, Berlin 1932

"The simplest curves are the planar curves. Among them the simplest one is the line (EDdegree 1). The next simplest is the circle (EDdegree 2). After that comes the parabola (EDdegree 3), and finally, general conics (EDdegree 4)."

THANK YOU