Notes T Kelly

Unramified Brauer classes on cyclic covers of $\mathbb{P}^2$. (Viray)

The Brauer group

field F

$$\mathrm{Br}\,F = \frac{\{\text{c.s.a.} /F\}}{A \sim B \text{ if } M_n(A) \simeq M_n B} = H^2(\mathrm{Gal}(F^s/F), F^{s\times})$$

X geom irred. scheme / field char 0.

$$\mathrm{Br}_{Az} X = \frac{\{\text{Azumaya Alg}\}}{\text{Morita Equiv. } \mathcal{A} \sim \mathcal{A}' \text{ if}}$$

$\exists$ loc. coh $\mathcal{O}_X$-module $\mathcal{E}$ $\mathcal{A} \otimes \mathrm{End}(\mathcal{E}) \simeq \mathcal{A}' \otimes \mathrm{End}(\mathcal{E}$

$\mathrm{Br}_{coh} X = H^2_{et}(X, \mathbb{G}_m)$.

$$\mathrm{Br}_{unram} X = \bigcap_{v \in X^{(1)}} \ker\left(\mathrm{Br}(k(X)) \xrightarrow{\partial_v} H^1(\kappa(v), \mathbb{Q}/\mathbb{Z})\right)$$

X smooth, proj then $\mathrm{Br}_{Az} = \mathrm{Br}_{coh} = \mathrm{Br}_{unr} =: \mathrm{Br}\,X$.

$X/\mathbb{C}$ $\quad H^3(X, \mathbb{Z}) = 0$

then $\mathrm{Br}\,X = \dfrac{H^2(X, \mathbb{Z})}{c_1(\mathrm{Pic}\,X)} \otimes \mathbb{Q}/\mathbb{Z}$.

<u>Applications</u> X birational invt of sm proj vars.

(Used by Artin-Mumford to exhibit a unirational var w/ nontriv Br. $\Rightarrow$ non rational)

- Arithmetic

  Manin: func of Br grp + CFT

  $\Rightarrow$ obstruction to local global princ

Viray 2

- Moduli of stable sheaves on K3's.
  obs to mod. space being fine is $\alpha \in \mathrm{Br}\, M$.

Want: explicit repr. of Br classes: eg. as an Azumaya alg. GSA/S, $\underset{\text{Brauer}}{\text{Severi-}}$ bundle.

Thm (van Geemen) (double cover of $\mathbb{P}^2$ branched over a sextic C)
Let $X/\mathbb{C}$ be a deg 2 K3 of Pic rk 1.
Then $\exists$ a natural isom
$$\mathrm{Hom}(\underset{H^2_{\bullet}(X,\mathbb{Z})}{\underset{\cap\mid}{NS(X)^{\perp}}}, \mathbb{Q}/\mathbb{Z})[2] \xrightarrow{\sim} \frac{\mathrm{Pic}\, C}{K_C}[2]$$
Moreover, $\forall\, \alpha \in \mathrm{Br}\, X[2]$, $\exists$ explicit geom const of Az_α which involves one of
1.) deg 8 K3 (3 quadrics in $\mathbb{P}^5$)
2.) double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ branched over $(2,2)$
3.) cubic 4fold cont plane.

Rmks 1) Geom construction Loses group structure.
2.) Proof involves classification of lattices.

Thm (C. Ingalls, A. Obus, E Ozman, V Appendix H. Thomas)
$\pi: X \to \mathbb{P}^2_{/k = k^s}$ branched over smooth curve C of deg $2d$, char $k \neq 2$.

then $\exists$ an exact sequence of groups

$$0 \to \frac{\operatorname{Pic} X}{\mathbb{Z}H + 2\operatorname{Pic} X} \to \frac{\operatorname{Pic} C}{K_C}[2] \xrightarrow{\mathcal{A}_X} \operatorname{Br} X[2] \to 0$$

Moreover, $\exists$ explicit geom const of $\mathcal{A}_X$ which over an open subset of param. involv.

1.) $(2,1)$ hypersurface in $\mathbb{P}^{2d-1} \times \mathbb{P}^2$
2.) $(2,2)$ hypersurface in $\mathbb{P}^{d-1} \times \mathbb{P}^2$
3.) cubic $(2d-2)$-fold containing $(d-4)$ linspace.

Thm (Catanese/$\mathbb{C}$, Beauville)

$C \subseteq \mathbb{P}^2$ smooth, $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} \simeq \mathcal{O}_C(\varepsilon)$

$\varepsilon = 0$ or 1

then $\exists$ exact seq.

$$0 \to \bigoplus \mathcal{O}_{\mathbb{P}^2}(b_i) \xrightarrow{M} \bigoplus \mathcal{O}_{\mathbb{P}^2}(-a_i) \to \mathcal{L} \to 0$$

where M is a symm matrix

$\Rightarrow$ ($\det M = C$, $\deg m_{ij} = 2d - a_i - a_j$)

Prop (H Thomas)

For a general curve, C, $\deg M$.

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & & & \end{pmatrix}, \begin{pmatrix} 2 & \cdots & 2 \\ \vdots & & \\ 2 & & \end{pmatrix}, \begin{pmatrix} 3 & 2 & \cdots & 2 \\ 2 & 1 & \cdots & 1 \\ \vdots & \vdots & & \\ 2 & 1 & & \end{pmatrix}$$

vG $(2,2)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$

$$\begin{tikzcd} & (2,2)\text{ hypersurface} \arrow[dl] \arrow[dr, "\substack{\text{conic bundle}\\ \text{over } \mathbb{P}^2 \text{ w/ deg. locus} = C}"] & & \mathcal{A} \arrow[d] \\ \mathbb{P}^2_{s,t,u} & & \mathbb{P}^2_{x,y,z} & X \arrow[l, leftrightarrow] \end{tikzcd}$$

$\mathcal{A}$ smooth conic bundle $/X$.

Thm 2 (IOOV)

$\exists$ exact seq and comm diagram

$$\begin{tikzcd} 0 \arrow[r] & \dfrac{\operatorname{Pic} X}{\mathbb{Z}H + 2\operatorname{Pic} X} \arrow[r] & \dfrac{\operatorname{Pic} \tilde{C}}{K_C}[2] \arrow[r, "\psi"] \arrow[d, hook, "\phi"'] & \operatorname{Br} X[2] \arrow[r] \arrow[d, hook] & 0 \\ & & \operatorname{Br} \underbrace{\mathbb{P}^2 \setminus (C \cup L)}_{\mathcal{U}} \arrow[r] & \operatorname{Br} k(x)[2] & \end{tikzcd}$$

Pf uses $\operatorname{Br}_{unram}$

So $\operatorname{Br}_{unram} \mathcal{U}[2] = \ker \left(\begin{array}{c} H^1(k(C), \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \times H^1(k(L), \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \\ \downarrow \\ \bigoplus_{p \in C \cup L} \mathbb{Z}/2 \end{array} \right)$